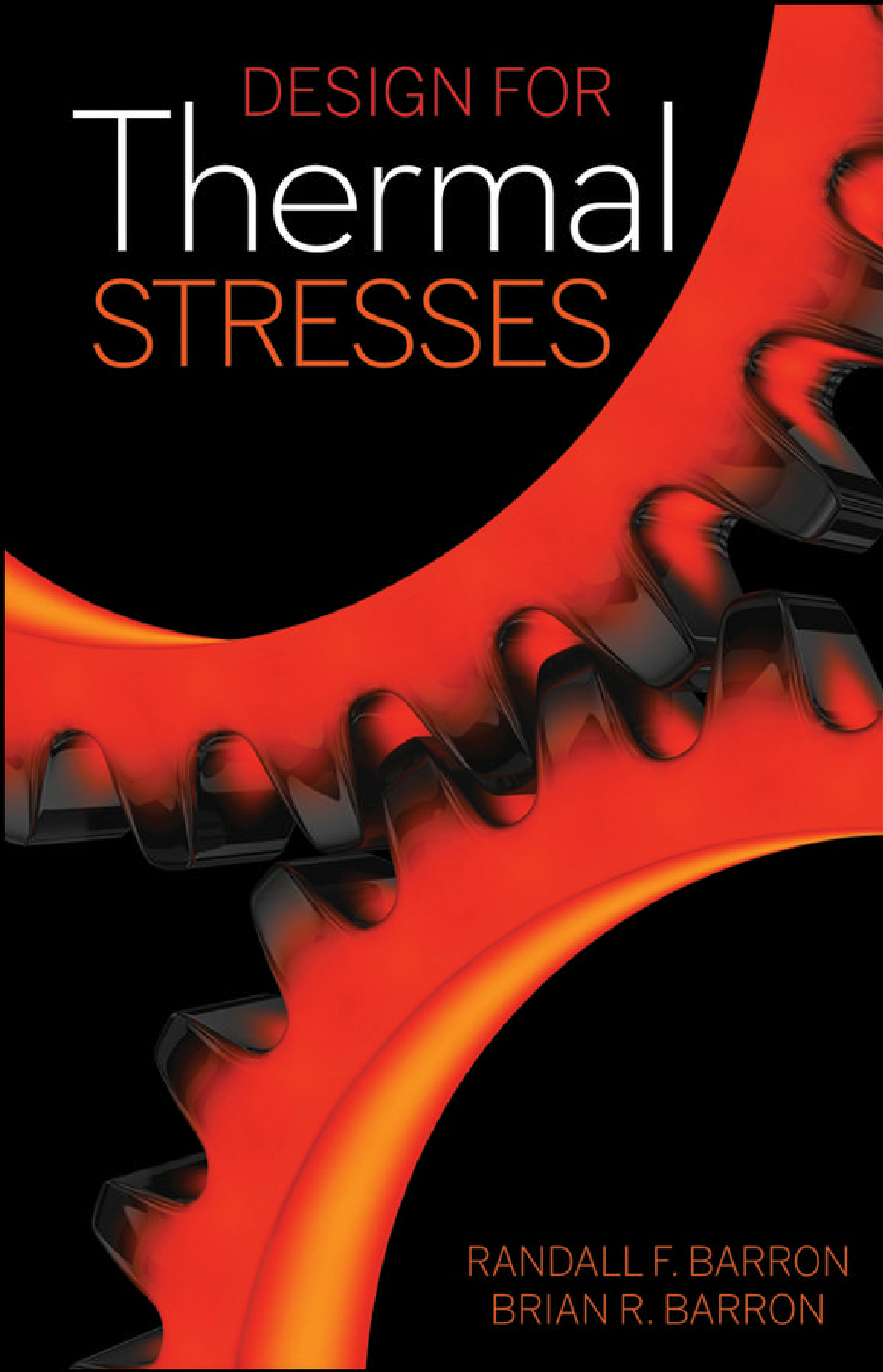
DESIGN FOR Thermal STRESSES



RANDALL F. BARRON
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CONTENTS

Preface **xi**

Nomenclature **xiii**

1 Introduction **1**

1.1 Definition of Thermal Stress 1

1.2 Thermal–Mechanical Design 3

1.3 Factor of Safety in Design 4

1.4 Thermal Expansion Coefficient 7

1.5 Young's Modulus 11

1.6 Poisson's Ratio 13

1.7 Other Elastic Moduli 14

1.8 Thermal Diffusivity 16

1.9 Thermal Shock Parameters 17

1.10 Historical Note 19

Problems 23

References 25

2 Thermal Stresses in Bars **26**

2.1 Stress and Strain 26

2.2 Bar between Two Supports 27

2.3 Bars in Parallel 32

2.4 Bars with Partial Removal of Constraints 35

2.5 Nonuniform Temperature Distribution 43

2.6 Historical Note 52

Problems 53

References 58

3 Thermal Bending **59**

3.1 Limits on the Analysis 59

3.2 Stress Relationships 60

3.3 Displacement Relations 64

3.4 General Thermal Bending Relations 65

3.5 Shear Stresses 67

3.6 Beam Bending Examples 69

3.7 Thermal Bowing of Pipes 97

3.8 Historical Note 108

Problems 110

References 117

4 Thermal Stresses in Trusses and Frames **118**

4.1 Elastic Energy Method 118

4.2 Unit-Load Method 123

4.3 Trusses with External Constraints 129

4.4 Trusses with Internal Constraints 132

4.5 The Finite Element Method 142

4.6 Elastic Energy in Bending 153

4.7 Pipe Thermal Expansion Loops 158

4.8 Pipe Bends 172

4.9 Elastic Energy in Torsion 178

4.10 Historical Note 185
Problems 186
References 195

5 Basic Equations of Thermoelasticity 197
5.1 Introduction 197
5.2 Strain Relationships 198
5.3 Stress Relationships 203
5.4 Stress–Strain Relations 206
5.5 Temperature Field Equation 208
5.6 Reduction of the Governing Equations 212
5.7 Historical Note 215
Problems 217
References 220

6 Plane Stress 221
6.1 Introduction 221
6.2 Stress Resultants 222
6.3 Circular Plate with a Hot Spot 224
6.4 Two-Dimensional Problems 239
6.5 Plate with a Circular Hole 247
6.6 Historical Note 256
Problems 257
References 262

7 Bending Thermal Stresses in Plates 264
7.1 Introduction 264
7.2 Governing Relations for Bending of Rectangular Plates 265
7.3 Boundary Conditions for Plate Bending 273
7.4 Bending of Simply-Supported Rectangular Plates 277
7.5 Rectangular Plates with Two-Dimensional Temperature Distributions 283
7.6 Axisymmetric Bending of Circular Plates 287

7.7 Axisymmetric Thermal Bending Examples 292

7.8 Circular Plates with a Two-Dimensional Temperature Distribution 305

7.9 Historical Note 310

Problems 312

References 315

8 Thermal Stresses in Shells 317

8.1 Introduction 317

8.2 Cylindrical Shells with Axisymmetric Loading 319

8.3 Cooldown of Ring-Stiffened Cylindrical Vessels 329

8.4 Cylindrical Vessels with Axial Temperature Variation 336

8.5 Short Cylinders 344

8.6 Axisymmetric Loading of Spherical Shells 350

8.7 Approximate Analysis of Spherical Shells under Axisymmetric Loading 357

8.8 Historical Note 371

Problems 373

References 377

9 Thick-Walled Cylinders and Spheres 378

9.1 Introduction 378

9.2 Governing Equations for Plane Strain 379

9.3 Hollow Cylinder with Steady-State Heat Transfer 384

9.4 Solid Cylinder 388

9.5 Thick-Walled Spherical Vessels 397

9.6 Solid Spheres 402

9.7 Historical Note 411

Problems 412

References 415

10 Thermoelastic Stability **416**

10.1 Introduction 416

10.2 Thermal Buckling of Columns 416

10.3 General Formulation for Beam Columns 420

10.4 Postbuckling Behavior of Columns 423

10.5 Lateral Thermal Buckling of Beams 426

10.6 Symmetrical Buckling of Circular Plates 432

10.7 Thermal Buckling of Rectangular Plates 437

10.8 Thermal Buckling of Cylindrical Shells 450

10.9 Historical Note 454

Problems 455

References 460

Appendix A Preferred Prefixes in the SI System of Units **461**

Appendix B Properties of Materials at 300 K **462**

Appendix C Properties of Selected Materials as a Function of Temperature **464**

C.1 Properties of 2024-T3 Aluminum 464

C.2 Properties of C1020 Carbon Steel 465

C.3 Properties of 9% Nickel Steel 465

C.4 Properties of 304 Stainless Steel 466

C.5 Properties of Beryllium Copper 466

C.6 Properties of Titanium Alloy 467

C.7 Properties of Teflon 467

References 468

Appendix D Bessel Functions **469**

D.1 Introduction 469

D.2 Bessel Functions of the First Kind 470

D.3 Bessel Functions of Noninteger Order 470
D.4 Bessel Functions of the Second Kind 472
D.5 Bessel's Equation 474
D.6 Recurrence Relationships for $J_n(x)$ and $Y_n(x)$ 475
D.7 Asymptotic Relations and Zeros for $J_n(x)$ and $Y_n(x)$ 476
D.8 Modified Bessel Functions 477
D.9 Modified Bessel Equation 478
D.10 Recurrence Relations for the Modified Bessel Functions 479
D.11 Asymptotic Relations for $I_n(x)$ and $K_n(x)$ 480
References 483

Appendix E Kelvin Functions **485**

E.1 Introduction 485
E.2 Kelvin Functions 486
E.3 Differential Equation for Kelvin Functions 490
E.4 Recurrence Relationships for the Kelvin Functions 491
E.5 Asymptotic Relations for the Kelvin Functions 492
E.6 Zeros of the Kelvin Functions 493

Appendix F Matrices and Determinants **494**

F.1 Determinants 494
F.2 Matrices 499
References 504

Index **505**

PREFACE

Situations involving thermal stresses arise in many engineering areas, from aerospace structures to zirconium-clad nuclear fuel rods. It is important for the engineer to recognize the importance of alleviation of thermal stresses and to have the tools to carry out this task. For example, in the design of cryogenic fluid transfer systems, the vacuum-jacketed transfer lines must accommodate differential contractions as large as an inch or a couple of centimeters when the inner pipe is cooled from ambient temperature to the cryogenic temperature range. In process industry systems, such as shell-and-tube heat exchangers, differential thermal expansion between the tube and the heat exchanger shell would result in mechanical failure if design approaches were not use to alleviate this situation. Even the seemingly mundane or "everyday" problem of buckling of concrete slabs in highways due to summer heating is a thermal stress problem.

Thermal stresses arise in each of these systems when the system components undergo a change in temperature while the component is mechanically constrained and not free to expand or contract with the temperature change. Often thermal stresses cannot be changed by "making the part bigger." Thermal stresses arise as a result of constraints and thermal stresses may be controlled safely by reducing the extent of the restraint. For example, flexible expansion bellows are used in cryogenic fluid transfer lines to reduce the constraint and forces between the cold inner line and the warm outer vacuum jacket to acceptable levels. The problem of thermal buckling of concrete slabs can be alleviated by providing a sufficiently large gap to allow some unconstrained expansion of the highway slab.

Some of the treatments of thermal stresses concentrate on the *analysis* of the thermomechanical system, which is an important consideration. However, the design process involves an interaction of both the *analysis* and *synthesis*

processes, and not simply an application of “formulas.” This design process is emphasized in this text. Example problems are included to illustrate the application of the principles for practicing engineers and student study, and homework problems are included to allow practice in applying the principles.

This text evolved from the authors’ academic and industrial experiences. One of us (RFB) has taught senior undergraduate and graduate level mechanical engineering courses in the areas of thermal stresses, directed MS and PhD thesis and dissertation research projects, including studies of thermal stresses in the thermal shroud of a space environmental simulation chamber, and conducted continuing education courses involving thermal stress applications for practicing engineers for more than 3 decades. He has first-hand industrial experience in the area of thermal stress design, including application of thermal design principles in design and manufacture of cryogenic liquid storage and transfer systems (dewars and vacuum-jacketed transfer lines), heat exchangers, and space environmental simulation chambers. BRB has conducted research involving the development of a hybrid finite element and finite difference numerical technique for solving thermal problems involving ultrashort laser pulses in layered media, in which thermal stresses can present severe design challenges.

The first part of the text (Chapters 1–4) covers thermal stress design in bars, beams, and trusses, which involves a “strength-of-materials” approach. Both analytical and numerical design methods are presented. The second part of the book (Chapters 5–9) covers more advanced thermal design for plates, shells, and thick-walled vessels, which involves a “theory of elasticity” approach. The final chapter (Chapter 10) covers the problem of thermal buckling in columns, beams, plates, and shells. Material on thermal viscoelastic problems (creep, etc.) is not included because of space limitations. The material included in the appendixes includes a discussion of the SI units used for quantities in thermal stress problems, tables of material properties relating to thermal stresses, brief coverage of mathematical functions (Bessel and Kelvin functions), and the characteristics of matrices and determinants required for design and analysis of plates and shells.

The book is written for use in junior- or senior-level undergraduate engineering elective courses in thermal design (mechanical, chemical, or civil engineering) and for graduate-level courses in thermal stresses. The proposed text is intended for use as a textbook for these classes, and a sufficient number of classroom-tested homework problems are included for a one-semester course in thermal stress design.

In addition, the book is intended for use by practicing engineers in the process industries, cryogenic and space-related fields, heat exchanger industries, and other areas where consideration of thermal stresses is an important part of the design problem. Detailed example problems are included with emphasis on practical engineering systems. The proposed text would serve as a reference and a source of background material to help engineers accomplish their design tasks.

Our most heartfelt thanks and appreciation is extended to each of our spouses, Shirley and Kitty, who generously gave their support and encouragement during the months of book preparation.

NOMENCLATURE

a	linear dimension; plate or shell radius, m or in.
A	area, m^2 or in^2
b	linear dimension; hot spot radius or plate radius, m or in.
ber(x), bei(x)	Kelvin functions
B	bulk modulus, Pa or lb_f/in^2 (psi); constant in eq. (6-142); shallow shell parameter, eq. (8-176)
B_0, B_1,	Fourier coefficients, eq. (7-154)
B_1, B_2	constants, eq. (9-31)
$\mathbf{B}$	direction cosine matrix, eq. (4-19)
$\mathbf{B}^{-1}$	inverse of the direction cosine matrix, eq. (4-23)
det $\mathbf{B}$	determinant of the direction cosine matrix, eq. (4-24)
$\mathrm{Bi} = h_c b/k_t$	Biot number, dimensionless
c	specific heat, J/kg-K or Btu/lb_m-°F
c_v	specific heat at constant volume, J/kg-K or Btu/lb_m-°F
$C_1, C_2, \ldots$	constants of integration
D	flexural rigidity, eq. (5-78), N-m or lb_f-in.
D_i	pipe inside diameter, m or in
D_o	pipe outside diameter, m or in.
$D_m = D_o - t$	pipe mean diameter, m or in.
$\mathbf{D}^{(e)}$	global element displacement matrix, eq. (4-46)
e	electron charge (Chapter 1), C
e_m	elongation due to mechanical loads, m or in.
e_t	thermal strain parameter, dimensionless
E	Young's modulus, Pa or lb_f/in^2 (psi)
f_s	factor of safety, dimensionless

f_1, f_2, f_3, f_4	factors defined by eq. (3-161), dimensionless
$f_1(\beta L)$, $f_2(\beta L)$, etc.	functions defined by eq. (8-106), etc., dimensionless
f_j	dummy forces in the jth member caused by a unit load, dimensionless
F	force, N or lb_f
$\mathrm{Fo} = \kappa t/b^2$	Fourier number, dimensionless
F_1	end support reaction for a beam, N or lb_f
F_T	thermal force, eq. (2-78), N or lb_f
$\mathbf{F}$	force matrix, eq. (4-20)
g_T	temperature gradient, eq. (6-119), °C/m or °F/in.
G	shear modulus, Pa or lb_f/in^2 (psi)
h	bar or beam dimension, plate thickness, m or in.
h_c	convective heat transfer coefficient, W/m^2-°C or Btu/hr-ft^2-°F
H	height of a pipe expansion loop, m or in.
I	area moment of inertia, m^4 or in^4.
$I_0(x)$	modified Bessel function of the first kind and order 0.
$I_1(x)$	modified Bessel function of the first kind and order 1.
I_{yz}	product of inertia, m^4 or in^4.
$j_{0,1}$, $j_{1,1}$, etc.	zeros of the Bessel functions of order 0, order 1, etc.
J	polar moment of inertia, m^4 or in^4
$J_0(x)$	Bessel function of the first kind and order 0.
$J_1(x)$	Bessel function of the first kind and order 1.
k	foundation modulus, Pa or lb_f/in^2; buckling parameter, eq. (10-17), eq. (10-42a)
k_0	buckling parameter, eq. (10-104), dimensionless
k_1	thermal buckling parameter, eq. (10-109), dimensionless
$\mathrm{ker}(x)$, $\mathrm{kei}(x)$	Kelvin functions
k_{sp}	spring constant, N/m or lb_f/in.
k_t	thermal conductivity, W/m-K or Btu/hr-ft-°F
k_θ	rotational spring constant, eq. (7-38a), N/rad or lb_f/rad
$\mathbf{k}^{(e)}$	element spring constant matrix, eq. (4-51)
K	constant in potential energy curve (Chapter 1), N/m-kg or lb_f/in.-lb_m; extensional rigidity, eq. (6-83), N/m or lb_f/in.
K_b	coefficient, eq. (10-8), dimensionless
K_n	buckling factor, eq. (10-97), N/m or lb_f/in.
K_T	thermal factor, eq. (10-94), N/m or lb_f/in.
$K_0(x)$	modified Bessel function of the second kind and order 0.
$K_1(x)$	modified Bessel function of the second kind and order 1.
K_1, K_2	dimensionless factors for pipe expansion loops, eqs. (4-91) and (4-92)
K_a, K_b, K_c	factors in eq. (2-58), dimensionless
K_{fl}	flexibility factor, eq. (4-101), dimensionless
K_{si}	stress intensity factor, eq. (4-104), dimensionless

K_y, K_z	factors given by eqs. (3-16) and (3-17), N-m or lb_f-in.
$\mathbf{K}^{(e)}$	element stiffness matrix, eq. (4-58)
L	length, m or in.; linear operator, eq. (8-134)
ΔL	change in length, m or in.
m_j	dummy bending moment in the jth member caused by a unit load, dimensionless
m_r	dummy unit bending moment, dimensionless
$m_{j,r}$	dummy bending moment in the jth member caused by a unit load, dimensionless
m_t	dummy torque resulting from the application of a unit load, dimensionless
$m_{t,r}$	dummy torque resulting from the application of a unit moment, dimensionless
M	bending moment, N-m or lb_f-in.; fin parameter, eq. (6-60), m^{-1} or in^{-1}
M_r	bending stress resultant (moment per unit length) in the radial direction, N or lb_f
M_t	torque, m-N or lb_f-in.
M_T	thermal bending moment, N-m or lb_f-in.; thermal stress resultant, N or lb_f
M_x	bending stress resultant (moment per unit length) in the x-direction, N or lb_f
M_y	bending stress resultant (moment per unit length) in the y-direction, N or lb_f
M_θ	bending stress resultant (moment per unit length) in the polar direction, N or lb_f
M_ϕ	bending stress resultant (moment per unit length) in the azimuth direction, N or lb_f
n	integer index
N_0	mechanical edge force per unit length, N/m or lb_f/in.
N_{cr}^*	value of force N_0 that would result in buckling for isothermal conditions, eq, (10-102), N/m or lb_f/in.
N_r	radial membrane stress resultant, eq. (6-2), N/m or lb_f/in.
N_T	thermal stress resultant, eq. (6-5), N/m or lb_f/in.
$N_{T,cr}$	critical or buckling thermal stress resultant, eq. (10-128), N/m or lb_f/in.
N_x	x-membrane stress resultant, eq. (6-79a), N/m or lb_f/in.
N_{xy}	shear membrane stress resultant, eq. (6-79c), N/m or lb_f/in.
N_y	y-membrane stress resultant, eq. (6-79b), N/m or lb_f/in.
N_θ	polar membrane stress resultant, eq. (6-3), N/m or lb_f/in.
p	pressure, Pa or lb_f/in^2 (psi)
P	external force, N or lb_f
P_e	reaction force, N or lb_f
q	transverse applied load per unit length on beams, N/m or lb_f/in.

q_a	heat flux per unit outside area of a pipe, W/m^2 or Btu/hr-ft^2
q_g	energy dissipation rate per unit volume, W/m^3 or Btu/hr-in^3.
Q	heat transfer rate, W or Btu/hr
Q_r	shear stress resultant (force per unit length) in the r-direction, N/m or lb$_f$/in.
Q_x	shear stress resultant (force per unit length) in the x-direction, N/m or lb$_f$/in.
Q_y	shear stress resultant (force per unit length) in the y-direction, N/m or lb$_f$/in.
r	radial coordinate, m or in.
r_o	equilibrium atomic spacing (Chapter 1), m or in.
r_g	radius of gyration, m or in.
R	radius of curvature, m or in.
R_c	corner reaction force, N or lb$_f$
R_m	mean radius of a pipe bend or elbow, m or in.
s	arc length, m or in.
s_m	stress, eq. (5-70), Pa or psi
S_f	fatigue strength, Pa or lb$_f$/in^2 (psi)
S_u	ultimate strength, Pa or lb$_f$/in^2 (psi)
S_y	yield strength, Pa or lb$_f$/in^2 (psi)
$\mathbf{S}$	matrix defined by eq. (4-53)
t	pipe wall thickness, m or in.; time, s or hr
T	temperature, K or °C or °F
T_0	temperature in the stress-free condition, K or °C or °F
$\mathbf{T}$	transformation matrix, eq. (4-47)
$\Delta T = T - T_0$,	temperature change, K or °C or °F
$\Delta T_1 = T_1 - T_0$,	temperature change, K or °C or °F
$\Delta T_{cr} = T_{cr} - T_0$,	critical or buckling temperature change, °C or °F
ΔT^*_{cr}	critical or buckling temperature change for zero N_0, °C or °F, eq. (10-109)
TSP	thermal shock parameter, K-m/s$^{1/2}$ or °F-ft/hr$^{1/2}$
TSR	thermal stress ratio, K or °F
u	displacement in the x-direction, m or in.
$\mathbf{u}^{(e)}$	local element displacement matrix, eq. (4-46)
U	potential energy (Chapter 1); internal energy, J/kg or Btu/lb$_m$ (or ft-lb$_f$)
U_c	complementary strain energy, internal energy, J/kg or Btu/lb$_m$ (or ft-lb$_f$)
U_s	strain energy, J/kg or Btu/lb$_m$ (or ft-lb$_f$)
v	displacement in the y-direction, m or in.
$\overline{\mathrm{v}}$	average velocity, m/s or in./sec.
V	volume (Chapter 1), m^3 or in^3; transverse shear force in beams, N or lb$_f$.; total rotation for a spherical shell eq. (8-123)
$V(x, y)$	potential function, N-m or lb$_f$-in.

V_{rg}	volume of the stiffening ring, m^3 or in^3
V_{sh}	volume of the shell segment between the stiffening rings, m^3 or in^3.
V_T	thermal shear force, eq. (3-42), N or $\mathrm{lb_f}$.
V_x	effective shear force per length, eq. (7-35), N/m or $\mathrm{lb_f}$/in.
ΔV	volume change (Chapter 1), m^3 or in^3
w	displacement in the z-direction
W	work done by a mechanical force, N-m or ft-$\mathrm{lb_f}$; width of a pipe expansion loop, m or in.
W_c	complementary work done by a mechanical force, N-m or ft-$\mathrm{lb_f}$
W_{vc}	volume-change work, J = N-m or ft-$\mathrm{lb_f}$
x	coordinate, m or in.
$x = \sqrt{2}\lambda\phi$	shallow shell parameter, eq. (8-172)
y	coordinate, m or in.
Y	dimensionless stress function, eq. (4-94)
$Y_0(x)$	Bessel function of the second kind and order 0.
$Y_1(x)$	Bessel function of the second kind and order 1.
$Y(x)$	displacement function, eq. (7-45)
z	coordinate, m or in.
z_b	width coordinate for a beam, m or in.
Z	atomic valence (Chapter 1), dimensionless

GREEK LETTERS

α	linear thermal expansion coefficient, K^{-1} or $^\circ\mathrm{F}^{-1}$
β	reciprocal relaxation length parameter (Chapter 3), m^{-1} or in^{-1}; shell parameter, m^{-1} or in^{-1}, eq. (8-25); dimensionless parameter in eq. (10-72).
β_m	Fourier series coefficient, eq. (7-50)
β_t	volumetric thermal expansion coefficient, K^{-1} or $^\circ\mathrm{F}^{-1}$
γ_G	Grüneisen constant, eq. (1-6), dimensionless
γ	shear strain, dimensionless
γ_P	parameter defined by eq. (3-136), m^{-3} or in^{-3}
Γ_y	quantity defined by eq. (3-37)
δ	gap width, m or in.; deflection of a spherical shell in planes of constant θ, eq. (8-148), m or in.
δ_{cr}	critical gap width, m or in.
δ_{ij}	Kroneker delta, eq. (5-37)
δ_P	deflection parameter, eq. (3-139), m or in.
Δ	displacement, m or in.
ε	extensional strain, dimensionless
ε_b	bending strain component, dimensionless

ε_m	mechanical strain, dimensionless
ε_0	permittivity of free space, 8.8542×10^{-12} F/m (Chapter 1); strain at the centroid axis (Chapter 3), dimensionless
ε_x, ε_y, ε_z	extensional strains in the x-, y-, and z-directions, dimensionless
ζ	liquid fill parameter, dimensionless
$\eta = H/L$	dimensionless parameter, eq. (4-90)
θ	polar angle coordinate, rad
θ_q	thermal parameter defined by eq. (3-175), dimensionless
θ_1, θ_2	angles, rad
κ	thermal diffusivity, m^2/s or ft^2/hr
κ_x	curvature in the x-direction, eq. (7-15a), m^{-1} or in^{-1}
κ_y	curvature in the xy-direction, eq. (7-15b), m^{-1} or in^{-1}
κ_{xy}	twist (curvature), eq. (7-15c), m^{-1} or in^{-1}
λ	direction cosine, eq. (4-47); eigenvalue, eq. (9-48); spherical shell parameter, eq. (8-139)
λ_L	Lamé constant, Pa or lb_f/in^2 (psi)
λ_m	mean free path (Chapter 1), m or in.
λ_0	spherical dome parameter, eq. (8-144), m^{-1} or in^{-1}
λ_p	factor defined by eq. (4-102), dimensionless
μ	Poisson's ratio, dimensionless; direction cosine, eq. (4-47)
$\xi = W/L$	dimensionless parameter, eq. (4-90)
ξ_c	coefficient of constraint, dimensionless
ρ	density, kg/m^3 or lb_m/in^3
σ	stress, Pa or lb_f/in^2 (psi)
Σ	stress parameter defined by eq. (3-156), Pa or lb_f/in^2 (psi)
τ	shear stress, Pa or lb_f/in^2 (psi)
τ_{rg}	time constant for the stiffening ring, s
τ_{sh}	time constant for the shell, s
ϕ	location angle for a pipe, rad; azimuth angle coordinate, rad
ϕ_o	liquid fill angle for a pipe, rad
Φ	Goodier displacement function, eq. (6-88), m^2 or in^2
ψ	angle, rad
Ψ	Airy stress function, eq. (6-103), N-m or lb_f-in
$\varphi_1(x)$, $\varphi_2(x)$, etc.	shallow shell functions, eqs. (8-187a) through (8-205).
ω	rotation, rad
ω_r	rotation in the r-direction, rad
ω_x	rotation in the x-direction, rad
ω_y	rotation in the y-direction, rad

1

INTRODUCTION

1.1 DEFINITION OF THERMAL STRESS

Thermal stresses are stresses that result when a temperature change of the material occurs in the presence of constraints. Thermal stresses are actually mechanical stresses resulting from forces caused by a part attempting to expand or contract when it is constrained.

Without constraints, there would be no thermal stresses. For example, consider the bar shown in Figure 1-1. If the bar were subjected to a temperature change ΔT of 20°C and the ends were free to move, the stress in the bar would be zero. On the other hand, if the same bar were subjected to the same temperature change and the ends were rigidly fixed (no displacement at the ends of the bar), stresses would be developed in the bar as a result of the forces (tensile or compressive) on the ends of the bar. These stresses are called thermal stresses.

There are two types of constraints as far as thermal stresses are concerned: (a) external constraints and (b) internal constraints. *External constraints* are restraints on the entire system that prevent expansion or contraction of the system when temperature changes occur. For example, if a length of pipe were fixed at two places by pipe support brackets, this constraint would be an external one.

Internal constraints are restraints present within the material because the material expands or contracts by different amounts in various locations, yet the material must remain continuous. Suppose the pipe in the previous example were simply supported on hangers, and the inner portion of the pipe were suddenly heated 10°C warmer than the outer surface by the introduction of a hot liquid into the pipe, as shown in Figure 1-2. If the outer surface remains at the initial

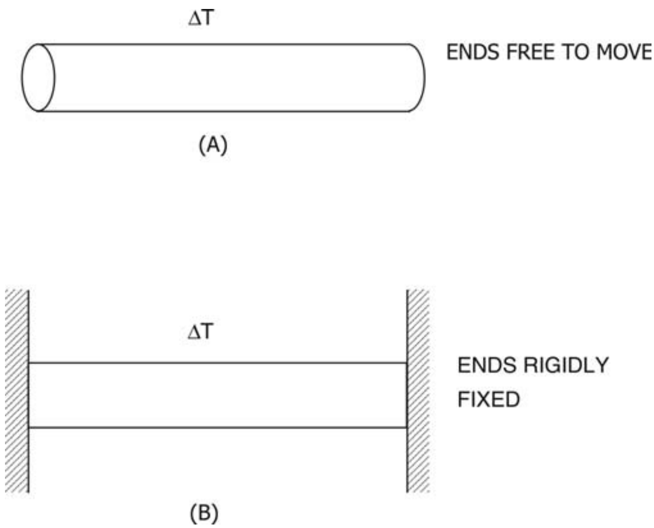

Figure 1-1. Illustration of external constraints. (A) No constraint—the bar is free to expand or contract. Thermal stresses are not present. (B) External constraint—the bar has both ends rigidly fixed and no motion is possible. Thermal stresses are induced when the bar experiences a change in temperature.

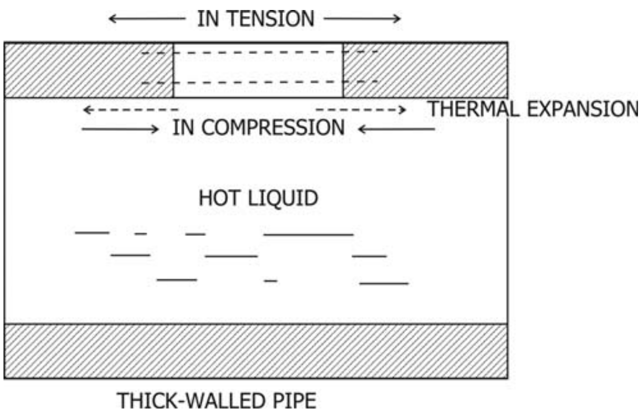

Figure 1-2. Internal constraints. The inner surface is heated by the fluid and tends to expand, but the outer (cool) surface constrains the free motion. Thermal stresses are induced by this constraint.

temperature, the outer layers would not expand, because the outer temperature did not change, whereas the inner layers would tend to expand due to a temperature change. Thermal stresses will arise in this case because the inner layer of material and outer layer of material are not free to move independently. This type of constraint is an internal one.

1.2 THERMAL–MECHANICAL DESIGN

The design process involves more than "solving the problem" in a mathematical manner [Shigley and Mischke, 1989]. Ideally, there would be no design limitations other than safety. However, usually multiple factors must be considered when designing a product. A general design flowchart is shown in Figure 1-3.

Initially, there is usually a perceived need for a product, process, or system. The specifications for the item required to meet this need must be defined. Often this specification process is called *preliminary design*. The input and output quantities, operating environment, and reliability and economic considerations must be determined. For example, anticipated forces that would be applied to the system must be specified.

After the design problem has been defined, the next step involves an interaction between synthesis, analysis, and optimization. Generally, there are many possible design solutions for a given set of specifications. (Not everyone drives the same model of car, for example, although all car models provide a solution to the problem of transportation from one place to another.) Various components for a

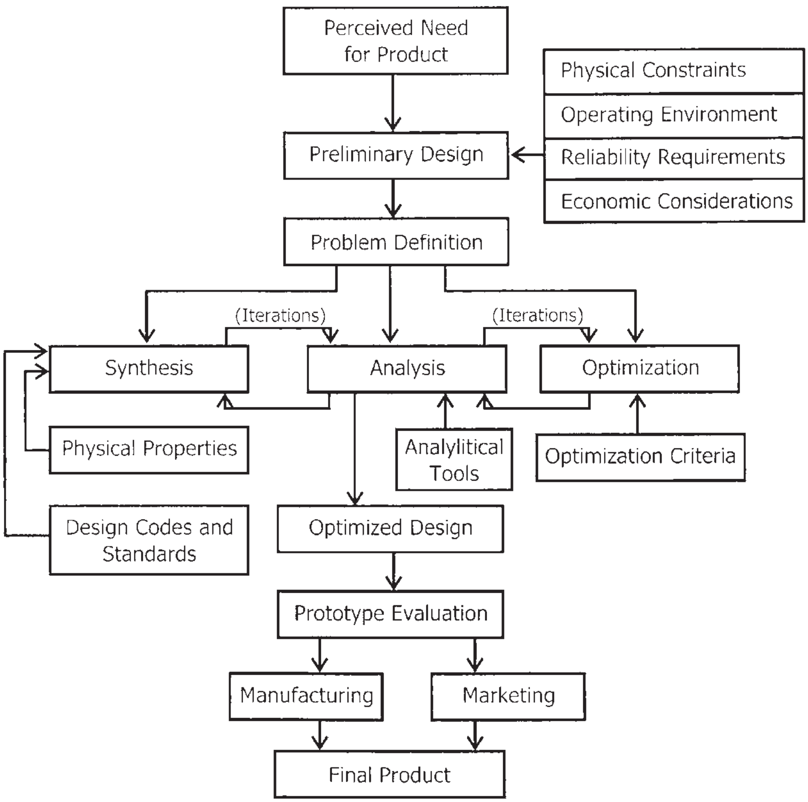

Figure 1-3. General design flowchart.

system may be proposed or synthesized. An abstract or mathematical model is developed for the analysis of the system. The results of the analysis may be used to synthesize an improved approach to the design solution. Based on specific criteria defining what is meant by the "best" system, the optimum or best system is selected to meet the design criteria.

In many cases, the optimal design emerging from the synthesis/analysis design phase is evaluated or tested. A prototype may be constructed and subjected to conditions given in the initial specifications for the system. After the evaluation phase has been completed successfully, the design then moves into the manufacturing and marketing arena.

When including consideration of thermal stresses in the design process, there are many cases in which the stresses are weakly dependent or even independent of the dimensions of the part. In these cases, the designer has at least three alternatives to consider: (a) materials selection, (b) limitation of temperature changes, and (c) relaxation of constraints.

For identical loading and environmental conditions, different materials will experience different thermal stresses. For example, a bar of 304 stainless steel, rigidly fixed at both ends, will experience a thermal stress that is about eight times that for Invar under the same conditions. Many factors in addition to thermal stresses dictate the final choice of materials in most design situations. Cost, ease of fabrication, and corrosion resistance are some of these factors. The designer may not have complete freedom to select a material based on thermal stress considerations alone.

A reduction of the temperature change will generally reduce thermal stresses. For a bar with rigidly fixed ends, if the temperature change is 50°C instead of 100°C, the thermal stress will be reduced to one-half of the thermal stress value for the larger temperature difference. In some steady-state thermal conditions, the temperature change of the part may be reduced by using thermal insulation. The design temperature change is often determined by factors that cannot be changed by the designer, however.

In many cases, the most effect approach to limit thermal stresses in the design stage is to reduce or relax the constraints on the system. The system may be made less constrained by introducing more flexible elements. This approach will be illustrated in the following chapters.

1.3 FACTOR OF SAFETY IN DESIGN

In general, a part is designed such that it does not fail, except under desired conditions. For example, fuses must fail when a specified electric current is applied so that the electrical system may be protected. On the other hand, the wall thickness for a transfer line carrying liquid oxygen is selected such that the pipe does not rupture during operation of the system.

One issue in the design process is the level at which the part would tend to fail. This issue is addressed in the *factor of safety* f_s. It is defined as the ratio of the failure parameter of the part to the design value of the same parameter. The first

decision that the designer must make is to define what constitutes "failure" for the component or system under consideration. There are several failure criteria, including

(a) breaking (rupture) of the part
(b) excessive permanent deformation (yielding) of the part
(c) breaking after fluctuating loads have been applied for a period of time (fatigue)
(d) buckling (elastic instability)
(e) excessive displacement or vibration
(f) intolerable wear of the part
(g) excessive noise generation by the part

The selection of the proper failure criteria is often the key to evaluating and planning for safety considerations.

If the failure criterion is the breaking or rupture of the part when stress is applied and the temperature is not high enough for creep effects to be significant, the failure parameter would be the *ultimate strength* S_u for the material. On the other hand, if the failure criterion is yielding, then the *yield strength* S_y would be the failure parameter selected. In either case, the design parameter would be the maximum applied stress σ for the part. The factor of safety may be written as follows, for these cases:

$$f_s = \frac{S_u}{\sigma} \quad \text{or} \quad f_s = \frac{S_y}{\sigma} \tag{1-1}$$

The factor of safety may be defined in a similar manner for the other failure criteria.

The factor of safety may be prescribed, as is the case for such codes as the *ASME Code for Unfired Pressure Vessels, Section VIII, Division 1*, in which the factor of safety for design of cylindrical pressure vessels is set at 3.5. When the factor of safety is not prescribed, the designer must select it during the early stages of the design process. It is generally not economical to use a factor of safety that assures that absolutely no failure will occur under the worst possible combination of conditions. As a result, the selection of the factor is often based on the experience of the designer in related design situations.

In general, the value of the factor of safety reflects uncertainties in many factors involved in the design. Some of these uncertainties are as follows:

(a) Scatter (uncertainty) in the material property data
(b) Uncertainty in the maximum applied loading
(c) Validity of simplifications (assumptions) in the model used to estimate the stresses or displacements for the system
(d) The type of environment (corrosive, etc.) to which the part will be exposed
(e) The extent to which initial stresses or deformations may be introduced during fabrication and assembly of the system

One of the more important factors in selection of the factor of safety is the extent to which human life and limb would be endangered if a failure of the system did occur or the possibility that failure would result in costly or unfavorable litigation.

The probabilistic or reliability-based design method [Shigley and Mischke, 1989] attempts to reduce the uncertainty in the design process; however, the disadvantage of this method lies in the fact that there is uncertainty in the "uncertainty" (probabilistic) data and the data is not extensive.

The uncertainty in the value of the strength parameter (ultimate or yield strength) may be alleviated somewhat by understanding the causes of the scatter in the data for the strength parameter. The values of the ultimate and yields strengths reported in the literature are generally average or mean values. In this case, 50 percent of the data lies above the mean value and 50 percent of the data lies below the reported value. A 1-in-2 chance would be excellent odds for a horse race, but this is not what one would likely employ in the design of a mechanical part. The value for the strength for which the probability of encountering a strength less than this value may be found from the normal probability distribution tables, if the standard deviation $\hat{\sigma}_S$ is known from the strength data. The ultimate strength for this case is given by the following expression:

$$S_u = k_S \overline{S}_u \tag{1-2}$$

The quantity $\overline{S}_u$ is the average ultimate strength, and the factor k_S is defined by

$$k_S = 1 - F_p(\hat{\sigma}_S/\overline{S}_u) \tag{1-3}$$

Values for the probability factor F_p are given in Table 1-1. Similar expressions may be used for the yield strength and fatigue strength.

Information on the standard deviation for the strength data is not readily available for all materials. If no specific standard deviation data are available, the following approximation may be used for the ratio $(\hat{\sigma}_S/\overline{S})$: 0.05 for ultimate

TABLE 1-1. Probability Factor F_p for Various Probabilities of Survival

Survival Rate[a]	Failure Rate[b]	Probability Factor[c], F_p
0.900	0.100	1.282
0.950	0.050	1.645
0.975	0.025	1.960
0.990	0.010	2.33
0.999	0.001	3.09
0.9999	0.0001	3.72

[a] The *survival rate* is the probability that the actual strength value is not less than the S value given by eq. (1-2).
[b] The *failure rate* is (1 − survival rate) or the probability that the actual strength is less than the S value given by eq. (1-2).
[c] F_p is used in eq. (1-3).

strength; 0.075 for yield strength; and 0.10 for fatigue strength or endurance limit. The designer has the task of deciding what risk is acceptable for the minimum strength used in the design.

The reliability of the maximum anticipated loading (either mechanical or thermal) used in the design affects the value of the factor of safety selected. If there are safeguards (pressure relief valves, for example) on the system to prevent the loading from exceeding a selected level, then the factor of safety may be smaller than for the case in which the loading is more uncertain.

The validity of the mathematical model (set of assumptions or simplifications) used in the design has a definite influence on the factor of safety selected. It may be noted that a very complicated numerical analysis (or, as is commonly stated, a "sophisticated" analysis) is not precisely accurate, despite the opinions of some overly enthusiastic novice computer analysist. The estimated uncertainty in the analysis may be used as a guide in selecting the factor of safety.

Example 1-1 304 stainless steel is to be used in a design. A factor of safety of 2.5 is selected, based on yielding as the failure criterion. It is desired that the uncertainty (failure rate) for the yield strength be 0.1%, and the standard deviation for the yield strength data is 7.5 percent of the mean yield strength. Determine the stress to be used in the design.

The average yield strength for 304 stainless steel is found in Appendix B:

$$\overline{S}_y = 232\,\text{MPa} \qquad (33{,}600\,\text{psi})$$

For a 0.1 percent failure rate or 99.9 percent survival rate, the probability factor from Table 1-1 is $F_p = 3.09$. The factor k_S may be found from eq. (1-3):

$$k_S = 1 - (0.075)(3.09) = 0.7683$$

The yield strength value that will be exceeded 99.9 percent of the time for 304 stainless steel is found from eq. (1-2):

$$S_y = k_S \overline{S}_y = (0.07683)(232) = 178.2\,\text{MPa} \qquad (25{,}850\,\text{psi})$$

The design stress is found from the definition of the factor of safety:

$$\sigma\,(\text{design}) = \frac{S_y}{f_s} = \frac{178.2}{2.5} = 71.3\,\text{MPa} \qquad (10{,}340\,\text{psi})$$

1.4 THERMAL EXPANSION COEFFICIENT

One of the important material properties related to thermal stresses is the *thermal expansion coefficient*. There are generally two thermal expansion coefficients that we will consider: (a) the *linear* thermal expansion coefficient, α, and (b) the *volumetric* thermal expansion coefficient, β_t.

The linear thermal expansion coefficient is defined as the fractional change in length (or any other linear dimension) per unit change in temperature while the stress on the material is kept constant. The following is the mathematical definition of the linear thermal expansion coefficient:

$$\alpha = \frac{1}{L}\left(\frac{\partial L}{\partial T}\right)_\sigma \tag{1-4}$$

Usually, the linear thermal expansion coefficient is measured under conditions of zero applied stress σ. Values for the linear thermal expansion coefficient for several engineering materials are given in Appendix B. Values for the linear thermal expansion coefficient as a function of temperature for several metals are presented in Appendix C.

The volumetric thermal expansion coefficient is defined as the fractional change in volume per unit change in temperature while the pressure (all-around stress) is held constant. The following is the mathematical definition of the volumetric thermal expansion coefficient:

$$\beta_t = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p \tag{1-5}$$

For an isotropic material (properties the same in all directions), the two thermal expansion coefficients are related by the following simple relation:

$$\beta_t = 3\alpha \tag{1-6}$$

The variation of the thermal expansion coefficient with temperature may be understood by considering the intermolecular forces of the material [Kittel, 1966]. The intermolecular potential energy curve for a pair of atoms, as shown in Figure 1-4, is not symmetrical. As the atom acquires more energy (or as the temperature is increased), the mean spacing of the two atoms becomes larger, i.e., the material expands.

If the potential energy curve were symmetric, for example, if $U = \frac{1}{2}K(r - r_0)^2$, then the positions of the two atoms at the extreme positions r_1 and r_2 for a given energy E are

$$r_1 = r_0 - \sqrt{2U/K} \qquad \text{and} \qquad r_2 = r_0 + \sqrt{2U/K} \tag{1-7}$$

The quantity r_0 is the equilibrium spacing at $T = 0$. The average spacing of the two atoms for a symmetrical energy curve is

$$r_{\text{ave}} = \tfrac{1}{2}(r_1 + r_2) = r_0 = \text{constant} \tag{1-8}$$

Because the equilibrium spacing remains constant, independent of the energy level, for a pair of atoms with a symmetrical energy curve, there would be no thermal expansion for this material, because, although the atoms would move farther apart as the temperature is increased, their average spacing would remain unchanged.

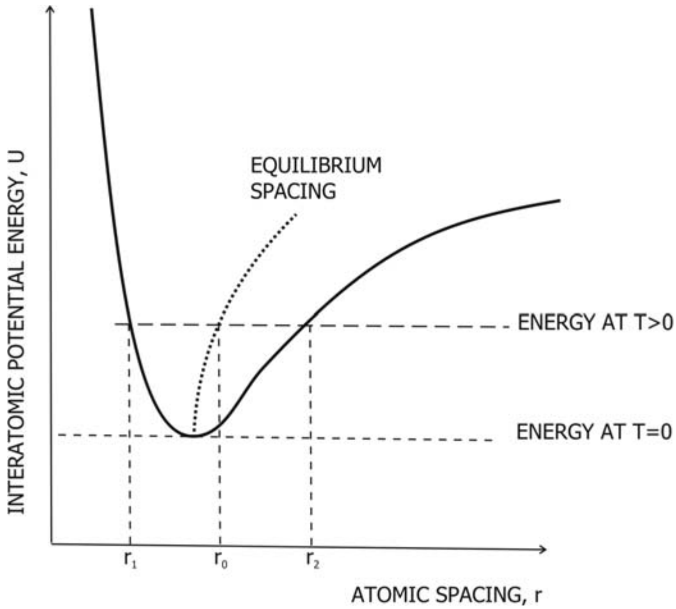

Figure 1-4. Interatomic potential energy curve for the potential energy between two atoms.

The actual potential energy curve is asymmetrical about the equilibrium spacing at absolute zero; therefore, the equilibrium spacing of the atoms increases as the temperature of the material is increased. The rate at which the mean spacing of the atoms changes increases as the energy or temperature is increased. This results in an increase of the thermal expansion coefficient as the temperature is increased. The thermal expansion coefficient approaches a value of zero as the material temperature approaches absolute zero, as required by the third law of thermodynamics [McClintock et al., 1984].

For crystalline solids, the specific heat of the material is dependent on the vibrational energy of the atoms. Since the thermal expansion coefficient is also associated with interatomic vibrational energy, one might except to find a relationship between these two properties. This interdependence is given by the Grüneisen relationship [Yates, 1972]:

$$\beta_t = \frac{\gamma_G c_v \rho}{B} \tag{1-9}$$

or

$$\alpha = \frac{\gamma_G c_v \rho (1 - 2\mu)}{E} \tag{1-10}$$

The quantity c_v is the specific heat at constant volume, ρ is the material density, and B is the isothermal bulk modulus, discussed in Section 1.6. The quantity E is Young's modulus, which is directly related to the bulk modulus by eq. (1-20).

TABLE 1-2. Values of the Grüneisen Constant for Selected Materials at Ambient Temperature

Material	Lattice Structure	Grüneisen Constant, γ_G
Aluminum, Al	FCC	2.17
Copper, Cu	FCC	1.96
Gold, Au	FCC	2.40
Lead, Pb	FCC	2.73
Nickel, Ni	FCC	1.88
Palladium, Pa	FCC	2.23
Platinum, Pt	FCC	2.54
Silver, Ag	FCC	2.40
Iron, Fe	BCC	1.60
Molybdenum, Mo	BCC	1.57
Tantalum, Ta	BCC	1.75
Tungsten, W	BCC	1.62
Cobalt, Co	HCP	1.87
Zinc, Zn	HCP	2.01
Bismuth, Bi	Rhombic	1.14
Tin, Sn	BC tetra	2.14

Note. FCC, face-centered-cubic; BCC, body-centered-cubic; HCP, hexagonal close-packed.

The parameter γ_G is the *Grüneisen constant* [Grüneisen, 1926]. Some typical values for the Grüneisen constant are given in Table 1-2.

The bulk modulus and density for a metal are not strongly dependent on temperature. If the Grüneisen constant were truly independent of temperature (it does actually depend on temperature in certain temperature ranges), then eq. (1-9) indicates that the thermal expansion coefficient would vary in the same manner with temperature as the specific heat does. The temperature variation of some metals is given in Appendix C. For a pure crystalline solid at low temperatures, the thermal expansion coefficient is proportional to T^3. At temperatures around ambient temperature and above ambient temperature, the thermal expansion coefficient is practically proportional to temperature, and is much less dependent on temperature than is the case at very low temperatures.

There are some cases, particular at cryogenic temperatures, that the thermal expansion coefficient cannot be treated as a constant, within acceptable accuracy. The cryogenic temperature range is defined [Scott, 1959] as temperatures less than 123 K or −150°C (−238°F). In this temperature region, we may use the *thermal strain parameter* e_t, defined by the following expression:

$$e_t(T) = \int_0^T \alpha \, dT \tag{1-11}$$

The average thermal expansion coefficient between two temperature limits, T_1 and T_2, is given by

$$\overline{\alpha} = \frac{e_t(T_2) - e_t(T_1)}{T_2 - T_1} \tag{1-12}$$

The thermal strain parameter e_t is tabulated in Appendix C for some metals. The parameter may be found for other materials by (a) fitting the thermal expansion coefficient to an analytical expression, using the least-squares curve-fitting technique, and then carrying out the integration analytically, or (b) carrying out the integration of the tabular or experimental thermal expansion coefficient data numerically.

Example 1-2 The density of silver at 300 K (80°F) is 10,500 kg/m^3 (0.379 lb$_m$/in^3), and the bulk modulus for silver is 92.82 GPa (13.46 × 10^6 psi). The vibrational energy contribution to the specific heat (Debye specific heat) is 0.216 kJ/kg-K (0.0517 Btu/lb$_m$–°F) [Gopal, 1966]. It may be noted that the total specific heat for silver is 0.236 kJ/kg-K (0.0564 Btu/lb$_m$–°F). Determine the linear thermal expansion coefficient from the Grüneisen relationship.

The value of the Grüneisen constant is found from Table 1-2 for silver:

$$\gamma_G = 2.40$$

The volumetric thermal expansion coefficient is found from eq. (1-9):

$$\beta_t = \frac{(2.40)(0.216 \times 10^3)(10{,}500)}{(92.82 \times 10^9)} = 58.6 \times 10^{-6}\ \text{K}^{-1}$$

The linear thermal expansion coefficient is found from eq. (1-6):

$$\alpha = (58.6 \times 10^{-6})/(3) = 19.5 \times 10^{-6}\ \text{K}^{-1} \qquad (10.8 \times 10^{-6}\ {}^\circ\text{F}^{-1})$$

The measured value for the linear thermal expansion coefficient is in excellent agreement with this calculated value [Corruccini and Gniewek, 1961]:

$$\alpha(\text{measured}) = 19.3 \times 10^{-6}\text{K}^{-1}$$

1.5 YOUNG'S MODULUS

Young's modulus gives a measure of the flexibility of a material, so this is another material property of importance in determining thermal stresses. Young's modulus is usually measured under isothermal (constant temperature) conditions. The mathematical definition of Young's modulus (specifically, the *tangent* modulus) is

$$E = \left(\frac{\partial \sigma}{\partial \varepsilon}\right)_T \qquad (1\text{-}13)$$

The stress level σ is below the proportional limit for the material. The quantity ε is the mechanical strain caused by the stress σ. Values for Young's modulus for several materials are given in Appendix B.

Young's modulus is related primarily to the forces between atoms in a material. A typical interatomic force curve is shown in Figure 1-5. The value of

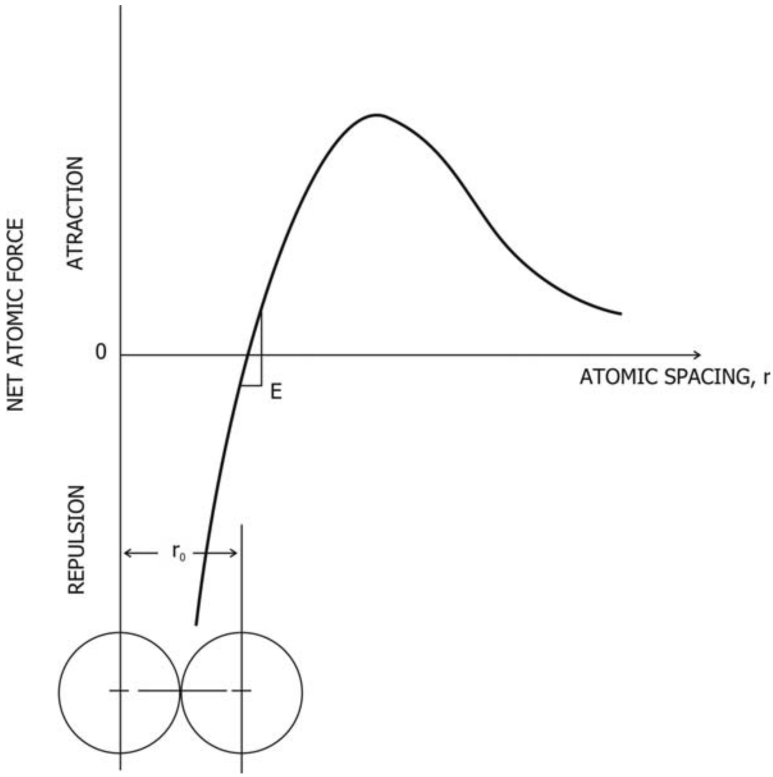

Figure 1-5. Interatomic force curve for the force between two atoms. Young's modulus is related to the slope of this curve.

Young's modulus is determined from the slope of the interatomic force curve at the equilibrium spacing r_0 of the atoms. One theoretical relationship for Young's modulus is as follows [Ruoff, 1973]:

$$E = \frac{9Z^2e^2}{16\pi\varepsilon_0 r_0^4} \tag{1-14}$$

The quantity Z is the valence of the atomic ion, e is the electron charge ($e = 0.1601 \times 10^{-18}$ C), ε_0 is the permittivity of free space ($\varepsilon_0 = 8.8542 \times 10^{-12}$ F/m), and r_0 is the equilibrium spacing of the atoms.

Example 1-3 The equilibrium spacing of the silver atoms in the metal is 0.288 nm, and the valence of silver is $+1$. Estimate the value of Young's modulus for silver.

Using these values in eq. (1-14), we find the following value of Young's modulus:

$$E = \frac{(9)(1)^2(0.1601 \times 10^{-18})^2}{(16\pi)(8.8542 \times 10^{-12})(0.288 \times 10^{-9})^4} = 75.3 \times 10^9 \text{ Pa} = 75.3 \text{ GPa}$$

The experimental value of Young's modulus for silver is 72.4 GPa (10.6×10^6 psi) [Bolz and Tuve, 1970].

1.6 POISSON'S RATIO

When the atoms of a material are pulled apart by a force applied in a certain direction, there is a corresponding contraction of the material in the lateral direction, perpendicular to the applied force. *Poisson's ratio* μ is the magnitude of the ratio of the lateral strain to the strain in the direction of the applied force. For a force applied in the x-direction, Poisson's ratio may be written as follows:

$$\mu = \frac{-\varepsilon_y}{\varepsilon_x} \tag{1-15}$$

The quantity ε_y is the mechanical strain in the y-direction when a force is applied in the x-direction, and ε_x is the mechanical strain in the x-direction (the direction of the applied force). The negative sign is introduced because the strain in the transverse direction will be a contraction (negative strain) if the force causes an elongation (positive strain) in the x-direction. Numerical values of Poisson's ratio for several materials are given in Appendix B.

The effect of application of a tensile force on the volume of a material may be examined. Suppose we have a bar with a length L, and cross-sectional dimensions $a \times b$. The initial volume V_0, with the bar unloaded, is

$$V_0 = Lab$$

The final volume V_1, after a tensile load has been applied in the lengthwise direction, is related to the original dimensions and Poisson's ratio:

$$V_1 = (L + \varepsilon_x L)(a + \varepsilon_y a)(b + \varepsilon_z b)$$

If we introduce Poisson's ratio from eq. (1-15) and expand the expression for the final volume, we obtain

$$V_1 = Lab(1 + \varepsilon_x)(1 - \mu\varepsilon_x)(1 - \mu\varepsilon_x) = V_0(1 + \varepsilon_x)(1 - 2\mu\varepsilon_x + \mu^2\varepsilon_x^2)$$

The fractional change in volume may be found as follows, where we have omitted the terms involving ε_x^2 and ε_x^3, because these values are negligible for small strains:

$$\frac{\Delta V}{V} = \frac{V_1 - V_0}{V_0} = \varepsilon_x(1 - 2\mu) \tag{1-16}$$

For a homogeneous isotropic material, the value of Poisson's ratio is between zero and $\frac{1}{2}$. If Poisson's ratio were greater than $\frac{1}{2}$, a pressure applied to the material would cause the volume of the material to increase, and this behavior is not observed in engineering materials. For a material with Poisson's ratio $\mu = \frac{1}{2}$, the volume does not change as a tensile or compressive force is applied.

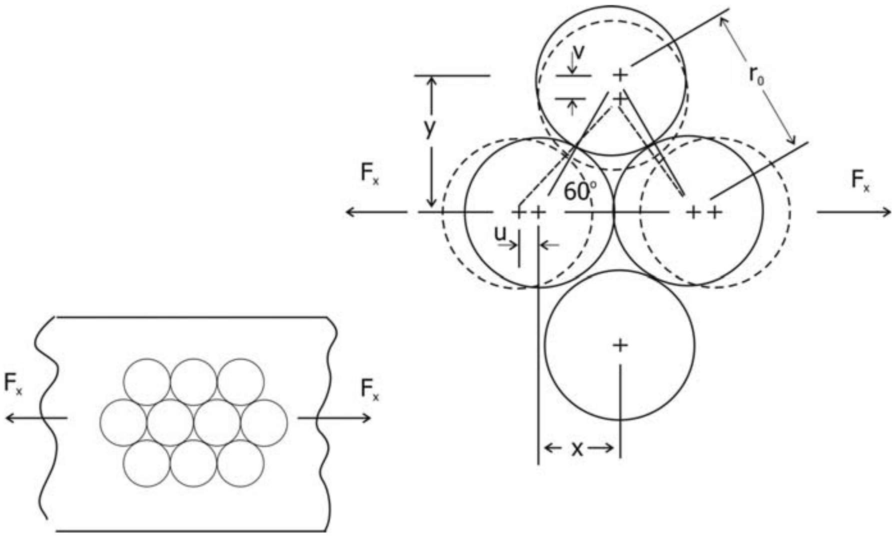

Figure 1-6. Poisson's ratio for a material with a face-centered-cubic or hexagonal close-packed lattice structure.

Poisson's ratio is a property that depends primarily on the geometry or arrangement of the atoms in the material. Because of this characteristic, Poisson's ratio is practically independent of temperature. It may be shown that Poisson's ratio for a metal having a face-centered-cubic (FCC) or hexagonal close-packed (HCP) lattice arrangement should be $\mu = \frac{1}{3}$. From Figure 1-6, we observe the following:

$$x^2 + y^2 = r_0^2 = \text{constant}$$

For small displacements u and v, the displacements may be found as follows:

$$2xu + 2yv \approx 2xdx + 2ydy = 0$$

or

$$\frac{v}{u} = \frac{y\varepsilon_y}{x\varepsilon_x} = \frac{-x}{y}$$

Poisson's ratio is defined by eq. (1-15):

$$\mu = \frac{-\varepsilon_y}{\varepsilon_x} = \left(\frac{x}{y}\right)^2 = \tan^2(30^\circ) = \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}$$

1.7 OTHER ELASTIC MODULI

In addition to Young's modulus and Poisson's ratio, several other elastic moduli have been defined. For an isotropic material, only two of the elastic moduli are

independent. In this text, we will usually choose Young's modulus and Poisson's ratio as the independent properties.

The *modulus of elasticity in shear* G is defined as the ratio of the shearing stress τ to the shear strain γ for a material in the elastic region (stresses less than the proportional limit). This property is also called the *shear modulus* and the *modulus of rigidity*:

$$G = \frac{\tau}{\gamma} \tag{1-17}$$

The shear modulus is related to Young's modulus and Poisson's ratio for an isotropic material by the following relationship:

$$G = \frac{E}{2(1+\mu)} \tag{1-18}$$

For a material with Poisson's ratio $\mu = \frac{1}{3}$, the shear modulus is $G = \frac{3}{8}E$.

The *isothermal bulk modulus* B is defined as the change in pressure per unit volumetric strain (change in volume per unit volume) of a material under constant-temperature conditions. The bulk modulus has also been called the *volume modulus of elasticity*:

$$B = V\left(\frac{\partial p}{\partial V}\right)_T \tag{1-19}$$

The bulk modulus is related to Young's modulus and Poisson's ratio for an isotropic material by the following relationship:

$$B = \frac{E}{3(1-2\mu)} \tag{1-20}$$

For a material with Poisson's ratio $\mu = \frac{1}{3}$, the bulk modulus is $B = E$. Note that the bulk modulus is infinite for a material having a Poisson's ratio $\mu = \frac{1}{2}$. As mentioned in Section 1.6, materials having a Poisson's ratio of $\frac{1}{2}$ experience no volume change (zero volumetric strain) when a pressure is applied.

To obtain a relationship between B, G, and E, let us combine eqs. (1-18) and (1-20) as follows:

$$\frac{1}{B} + \frac{3}{G} = \frac{3(1-2\mu)}{E} + \frac{(3)(2)(1+\mu)}{E} = \frac{9}{E} \tag{1-21}$$

The *Lamé elastic constant* λ_L is defined by the following relationship:

$$\lambda_L = \frac{\mu E}{(1+\mu)(1-2\mu)} = \frac{3\mu B}{1+\mu} \tag{1-22}$$

For a material with Poisson's ratio $\mu = \frac{1}{3}$, the Lamé constant is $\lambda_L = \frac{3}{4}E$. Using eq. (1-18), we may write the following relationship for the Lamé constant in terms of the shear modulus and the bulk modulus for any value of Poisson's ratio:

$$\lambda_L + \tfrac{2}{3}G = B \tag{1-23}$$

Values for the elastic moduli may be found from the data in Appendix B and the relationships given in this section.

Example 1-4 Determine the elastic moduli for 304 stainless steel at 300 K (80°F). Young's modulus and Poisson's ratio are found from Appendix B for 304 stainless steel: $E = 193$ MPa (28.0×10^6 psi) and $\mu = 0.305$.

The shear modulus is found from eq. (1-18):

$$G = \frac{(193)}{(2)(1 + 0.305)} = 73.9 \text{ GPa} \qquad (10.7 \times 10^6 \text{ psi})$$

The bulk modulus is found from eq. (1-20):

$$B = \frac{(193)}{(3)[1 - (2)(0.305)]} = 165 \text{ GPa} \qquad (23.9 \times 10^6 \text{ psi})$$

Finally, the Lamé constant is found from eq. (1-22):

$$\lambda_L = \frac{(0.305)(193)}{(1 + 0.305)[1 - (2)(0.305)]} = 115.7 \text{ GPa} \qquad (16.8 \times 10^6 \text{ psi})$$

1.8 THERMAL DIFFUSIVITY

In many situations involving thermal stresses, transient or time-dependent temperature distributions are involved. In these cases, the temperature distribution and the thermal stress distribution are dependent on a material property called the *thermal diffusivity* κ. The thermal diffusivity is defined in terms of the material thermal conductivity k_t, density ρ, and specific heat c:

$$\kappa \equiv \frac{k_t}{\rho c} \tag{1-24}$$

The units for the thermal diffusivity in the SI system are {m^2/s}, and typical units in the conventional system are {ft^2/hr}.

The value of the thermal diffusivity gives a measure of how rapidly energy may be conducted into a solid material. A large value of thermal diffusivity means that energy may diffuse rapidly into the material, and steep temperature gradients (large temperature changes over small distances) will not be developed. This behavior tends to result in lower thermal stresses in the transient situation than the case of a material with a small thermal diffusivity.

One relationship for the thermal diffusivity of a solid material is as follows [Berman, 1976]:

$$\kappa = \tfrac{1}{3}\overline{v}\lambda_m \tag{1-25}$$

The quantity $\overline{v}$ is the velocity of the "energy carriers" (electrons, lattice vibrational waves or phonons, etc.), and λ_m is the average distance traveled by the carriers

between collisions, or the *mean free path* for the energy carriers. For metals at ambient temperature and higher, the thermal diffusivity is relatively constant with temperature change. At very low temperatures, the thermal diffusivity of metals is strongly dependent on temperature and varies as T^{-3} to T^{-4}. The temperature dependence of the thermal diffusivity of some selected materials is displayed in Appendix C.

1.9 THERMAL SHOCK PARAMETERS

Thermal shock occurs when a material is subjected to rapidly changing temperatures in the environment around the material. Some examples of thermal shock situations include space vehicle reentry into the atmosphere, start-up of a cold automobile engine, and quenching of a metal part. Under identical environmental conditions, some materials are more resistant to thermal shock than others. Brittle materials exhibit small mechanical strains before rupture, so thermal shock can be a serious problem for such materials. Ductile materials can withstand larger mechanical strains before rupture; however, thermal shock may cause yielding for ductile materials. In addition, repeated thermal shock can result in a thermal fatigue failure for ductile materials.

The *strength–weight ratio* S_y/ρ is an important parameter in selection of materials to withstand a specified tensile load for minimum weight of the part. Similarly, a thermal shock parameter would be a convenient material property to assist the designer in selection of materials that would resist thermal shock for a given temperature change. Schott and Winkelmann suggested one of the original thermal shock parameters in 1894 [Richards, 1961]:

$$\text{TSP} = \frac{S_u\sqrt{\kappa}}{\alpha E} \tag{1-26}$$

The quantity S_u is the ultimate tensile strength of the material, and κ is the thermal diffusivity for the material.

A material with a high value of ultimate tensile strength would be able to withstand a higher stress level than a material with a low ultimate tensile strength. A material with a low thermal expansion coefficient α would develop smaller thermal strains (and correspondingly lower thermal stresses) than a material that expands by a large amount when the material temperature is changed. A material with a small Young's modulus E would be more flexible and able to accommodate thermal strains better than a material with a large Young's modulus. Finally, a material with a large thermal diffusivity κ would tend to develop smaller temperature gradients than a material with small κ, because thermal energy can be spread out throughout the high-κ material more rapidly.

In summary, a material that would have good thermal shock resistance should have a large ultimate tensile strength S_u, a small thermal expansion coefficient α, a small Young's modulus E, and a large thermal diffusivity κ. These characteristics are brought together in the thermal shock parameter TSP. A material having a

TABLE 1-3. Values of the Thermal Shock Parameter TSP and Thermal Stress Ratio TSR for Several Materials at 300 K (27°C or 80°F)

Material	αE, MPa/K	TSR $= S_u/\alpha E$, K	TSP $= S_u \kappa^{1/2}/\alpha E$, K-m/s$^{1/2}$
Aluminum, 2024-T3	1.652	280	1.94
Aluminum, 3003-H12	1.553	83.7	0.68
Aluminum, 6061-T6	1.615	192	1.57
Beryllium copper	2.208	521	2.59
Brass, 70/30	1.210	291	1.70
Bronze, UNS-22000	1.833	202	0.60
Copper/10% Ni	2.009	179	0.67
Inconel, 600	2.795	229	0.68
Invar	0.387	1660	2.81
Monel, K-500	2.516	380	0.84
Gray cast iron, Class 20	0.880	160	0.65
Gray cast iron, Class 40	1.339	218	0.88
Steel, C1020, annealed	2.440	180	0.77
Steel, 4340	2.397	422	1.19
Steel, 9% Ni	2.230	384	1.03
Stainless steel, 304	3.088	167	0.34
Stainless steel, 416	1.980	258	0.69
Titanium, Ti–5Al–2.5Sn	1.066	819	1.57
Concrete, 1:2½:3¼	0.224	7.7[a]	0.006[a]
Glass, silicate 7740	0.282	35.4[a]	0.02[a]
Glass, Pyrex	0.205	134[a]	0.09[a]
Nylon	0.252	241	0.33
Teflon	0.050	366	0.13

[a] Strength values in tension.

large value of TSP would have good thermal shock resistance. The values for the thermal shock parameter for several materials are listed in Table 1-3.

Under steady-state conditions, the transient thermal properties do not influence the thermal stresses. In these cases, the *thermal stress ratio TSR* is an important material property for use in assessing the material resistance to thermal stresses [Gatewood, 1957]:

$$\mathrm{TSR} = \frac{S_u}{\alpha E} \tag{1-27}$$

Values for the thermal stress ratio for several materials are also tabulated in Table 1-3. Generally, a material with a large thermal stress ratio will have good resistance to thermal stresses.

Example 1-5 Tubes made of red brass (UNS-C2300, 85% Cu, 15% Zn) having a 05105 temper are to be used in a steam condenser. The ultimate tensile strength for the material is 305 MPa (44,200 psi), and Young's modulus for the red brass is 90 GPa (13×10^6 psi). The thermal expansion coefficient for the material is 18×10^{-6} K^{-1} (10×10^{-6} °F^{-1}), and the thermal diffusivity is 18.0 mm^2/s.

Determine the thermal stress ratio and thermal shock parameter for the red brass tubing.

The thermal stress ratio is found from eq. (1-27):

$$\mathrm{TSR} = \frac{(305)(10^6)}{(18)(10^{-6})(90)(10^9)} = 188.3\ \mathrm{K}$$

The thermal shock parameter is found from eq. (1-26):

$$\mathrm{TSP} = (188.3)(18.0 \times 10^{-6})^{1/2} = 0.799\ \text{K-m/s}^{1/2}$$

It is noted from Table 1-3 that these values are slightly lower than the corresponding values for 70/30 brass.

Example 1-6 In the design of a heat exchanger, the engineer has a choice of the following materials for use as the heat exchanger tubing: red brass, copper (Cu/10% Ni), and aluminum (2024-T3). Which material should be selected from a thermal stress resistance standpoint?

From Table 1-3 and Example 1.5, we find the following values for the thermal stress ratio and thermal shock parameter:

Material	TSR, K	TSP, K-m/s$^{1/2}$
Aluminum, 2024-T3	280	1.94
Red brass	188	0.80
Copper (Cu/10% Ni)	179	0.67

When the fluid is suddenly introduced into the heat exchanger, originally at ambient temperature, the tubing may experience thermal shock. The aluminum has the largest TSP, so aluminum would be the best material for thermal shock resistance. In addition, the TSR for aluminum is largest of the three materials, so aluminum would also be best for steady-state thermal stress resistance.

We may conclude that the engineer should select aluminum (2024-T3) as the best of the three materials from a thermal stress standpoint.

1.10 HISTORICAL NOTE

People have known about thermal stresses from the time that the first person broke a clay vessel by heating the vessel too rapidly. It wasn't until the 1800s, however, that the first analytical analysis was made for thermal stresses [Timoshenko, 1983].

Robert Hooke (1635–1703) worked with Robert Boyle on perfecting an air pump at Oxford. Boyle recommended Hooke as the curator of the experiments of the Royal Society in England, of which Hooke was a charter member. In the

1670s Hooke conducted experiments with elastic bodies, and in 1678 he published the first technical paper in which elastic properties of materials were examined. Based on his experiments with springs and other elastic bodies, Hooke concluded in his paper "De Potentiâ Restitutiva" ("Of Springs") in 1678: "It is very evident that the Rule or Law of Nature in every springing body is, that the force or power thereof to restore itself to its natural position is always proportional to the distance or space it is removed therefrom, whether it be by rarefaction, or the separation of the parts the one from the other, or by Condensation, or crowding of those parts together." In less formal words, Hooke's law may be stated in the form: "There is a linear relationship between the force and deformation for bodies at stresses below the proportional limit." This principle is the beginning point for all elastic analyses, including thermoelastic analysis.

Thomas Young (1778–1829) (Figure 1-7) originally studied medicine and received his doctor's degree from Göttingen University in 1796. A few years later while at Cambridge (in 1796) he became interested in the physical sciences, including acoustics and optics. In 1802 he was appointed a professor of natural philosophy (the forerunner of today's physics and other scientific areas) by the Royal Institution. Many of his main contributions to mechanics of materials were presented in his course on natural philosophy during the year he taught at the Royal Institution. He introduced the concept of the modulus of elasticity, which is called *Young's modulus* today (although Young's definition was somewhat different from that used now). In his lecture notes entitled *A Course of Lectures on Natural Philosophy and the Mechanical Arts*, published in 1807,Young stated: "The modulus of elasticity of any substance is a column of the same substance, capable of producing a pressure on its base which is to the weight causing a

Figure 1-7. Thomas Young (From S. P. Timoshenko, 1983. Used by permission of Dover Publications, Inc.)

certain degree of compression as the length of the substance is to the diminution of its length." The modulus defined by Young was essentially the product of Young's modulus and the cross-sectional area in present-day terminology.

C.L.M.H. Navier (1785–1836) published a book on strength of materials in 1826, in which he defined the modulus of elasticity for tension or compression as the ratio of the force per unit cross-sectional area to the elongation per unit length. This modulus is the property that is denoted as Young's modulus today. Navier actually measured the Young's modulus for the iron that was used in the construction of the Pont des Invalides in Paris.

S. D. Poisson (1781–1840) (Figure 1-8) taught mathematics at the École Polytechnique, and he applied his mathematical skills in solving several problems involving the theoretical strength of materials. He was interested in the theory of elasticity based on molecular force considerations. In his memoir, *Mémoire sur l'équilibre et le mouvement des corps élastiques*, published in 1829, Poisson applies general elasticity equations that he had developed to isotropic materials. He found that, for simple tension of a rod or bar, the axial elongation produces a lateral contraction. Poisson's relationships yielded a value of the ratio of lateral strain to axial strain, called *Poisson's ratio* today, to have a universal value of $\mu = \frac{1}{4}$. As discussed in Section 1.6, Poisson's ratio for isotropic materials may have values between 0 and $\frac{1}{2}$.

Gabriel Lamé (1795–1870) (Figure 1-9) graduated from the École Polytechnique in 1818 and worked with a then-new Russian engineering school, the Institute of Engineers of Ways of Communication in St. Petersburg. Lamé taught applied mathematics and physics at the school and helped with the design of

Figure 1-8. S. D. Poisson (From S. P. Timoshenko, 1983. Used by permission of Dover Publications, Inc.)

Figure 1-9. Gabriel Lamé (From S. P. Timoshenko, 1983. Used by permission of Dover Publications, Inc.)

several suspension bridges built in the St. Petersburg area. In 1852, Lamé produced the first book on the theory of elasticity, entitled *Leçons sur la Théorie Mathématique de l'Élasticité des Corps Solides.* He concluded that to define the elastic properties of an isotropic material, only two different elastic constants were required. In Lamé's general elasticity equations, he selected the two constants as Lamé's elastic modulus λ_L and the modulus of elasticity in shear G.

J. M. Constant Duhamel (1797–1872) also graduated from Éçole Polytechnique in 1816 and, after studying for law and teaching mathematics in some other schools, he joined the faculty at Éçole Polytechnique in 1830. After publishing several papers in the area of conduction heat transfer, Duhamel made some basic contributions to the theory of elasticity. In 1835 he published the paper "Mémoir sur le calcul des actions moléculaires développées par les changements du température dans les corps solids." In this paper he developed the basic partial differential equations for stress equilibrium conditions, including stresses produced by temperature variation. This paper was the first to give an analytical treatment of thermal stresses.

Duhamel applied the general equations to several problems of practical interest. He obtained a solution for the stress distribution in the wall of a hollow spherical vessel and a hollow cylindrical vessel in which the temperature varies across the wall of the vessel. Duhamel was one of the first investigators to use the principle of superposition in thermal stress analysis.

Franz Neumann (1798–1895) (Figure 1-10) began service in the German Army when he was only 17 years old. After a year in the military service, he returned Berlin to complete his high school education and to study at Berlin University. He studied mineralogy and was awarded a faculty position to teach

Figure 1-10. Franz Neumann (From S. P. Timoshenko, 1983. Used by permission of Dover Publications, Inc.)

mineralogy at the University of Königsberg after receiving his doctorate. In 1843, Neumann published an extensive memoir dealing with double refraction of light, in which he presents the basic principles used in experimental photoelastic stress analysis. Based on stress equilibrium equations similar to those developed by Duhamel, Neuman solved the stress distribution problem for a sphere with radially varying temperature. He also conducted photoelastic tests and experimentally measured the thermal stresses in the sphere, and found that the experimental and theoretical data were in satisfactory agreement. This is the first time that thermal stresses were measured in the laboratory. Neumann analyzed the problem of thermal stresses in circular plates in which the temperature varies in the radial direction in the plate, but is constant in the thickness direction. He also solved the plate thermal bending problem for a circular plate in which the temperature was a function of the axial coordinate only.

PROBLEMS

1-1. A part is to be constructed using 9-percent nickel steel, for which the average ultimate strength is 856 mPa (124,200 psi) and the standard deviation of the ultimate strength data is 42.9 MPa (6220 psi). If an ultimate strength of 756 MPa (109,600 psi) is used in the design, determine the probability that the actual ultimate strength of the material used is less than 756 MPa.

1-2. In a certain design, 6061-T6 aluminum is used, for which the average yield strength is 275 MPa (39,890 psi) and the standard deviation for the yield

strength data is 20 MPa (2900 psi). A factor of safety of 1.50, based on the yield strength for a 99.99 percent survival rate, is to be used in the design. Determine the design stress that should be used.

1-3. At room temperature, Young's modulus and Poisson's ratio for constantan (cupronickel, 55% Cu/45% Ni) are $E = 165$ GPa (23.9×10^6 psi) and $\mu = 0.325$. The specific heat and density for constantan are $c_v = 0.409$ kJ/kg-K (0.0977 Btu/lb$_m$-°F) and $\rho = 8922$ kg/m^3 (0.322 lb$_m$/in^3) at room temperature. Determine the linear thermal expansion coefficient α for constantan at room temperature, if the Grüneisen constant for constantan is $\gamma_G = 1.91$.

1-4. Niobium is a material used in superconducting magnets, which operate at cryogenic temperatures. The properties of niobium at 4.2 K (−269°C or 7.6°R or −452°F) are density, 8580 kg/m^3 (0.310 lb$_m$/in^3); Young's modulus, 68.9 GPa (10.0×10^6 psi); Poisson's ratio, 0.270; and Grüneisen constant, 1.57. The specific heat of niobium for temperatures below 22 K is given by

$$c_v = C_0(T/\theta_D)^3$$

The factor $C_0 = 20.921$ kJ/kg-K, T is the absolute temperature, and $\theta_D = 265$ K (477°R) = the *Debye temperature* for niobium. Determine the linear thermal expansion coefficient for niobium at 4.2 K.

1-5. Using the data from Problem 1-4, determine the total change in length of a niobium wire having an initial length of 800 m (2625 ft) when the wire is cooled from 20.3 to 4.2 K. The properties, except the thermal expansion coefficient, may be treated as constant.

1-6. Platinum has a valence of +2, and the equilibrium spacing of the atoms in platinum is 0.28 nm. Determine the value of Young's modulus for platinum predicted by the theoretical expression, eq. (1-14). How does this value compare with the measured value of Young's modulus for platinum, $E = 146.9$ GPa?

1-7. Determine the shear modulus, bulk modulus, and Lamé constant for (a) 2024-T3 aluminum and (b) C1020 steel.

1-8. The shear modulus and Lamé constant for platinum are $G = 55.5$ GPa (8.05×10^6 psi) and $\lambda_L = 103.0$ GPa (14.94×10^6 psi). Determine the value of Young's modulus and Poisson's ratio for platinum.

1-9. In a particular application involving transient thermal stresses, it is desired to select the best material for thermal shock resistance from the following: (a) 3003-H12 aluminum, (b) copper/10% nickel, and (c) 304 stainless steel. Determine the thermal shock parameter for each of these materials and select the one that would have the best shock resistance characteristics.

1-10. In a steady-state situation involving thermal stresses, the designer has a choice of the following three materials: C1020 carbon steel, annealed;

6061-T6 aluminum; and a nickel alloy ($\alpha = 11.3 \times 10^{-6}\,\mathrm{K}^{-1} = 6.28 \times 10^{-6}\,{}^\circ\mathrm{F}^{-1}$; $E = 220\,\mathrm{GPa} = 31.9 \times 10^6\,\mathrm{psi}$; $S_u = 955\,\mathrm{MPa} = 138{,}500\,\mathrm{psi}$). Determine the thermal stress ratio for each material and select the one that would have the best thermal stress resistance.

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2

THERMAL STRESSES IN BARS

2.1 STRESS AND STRAIN

From a basic viewpoint, *stress* is defined as a force per unit area on which the force acts. From an engineering viewpoint, there are two types of stresses: (a) direct or extensional stresses, in which the force acts normal or perpendicular to the surface, and (b) shear stresses, in which the force acts tangential or parallel to the surface. We will use the symbol σ to denote the direct stresses and the symbol τ to denote the shear stresses. Tensile stresses will be considered as positive stresses, and compressive stresses will be considered as negative stresses. Because thermal deformations do not result in thermal shearing stresses for simple systems, we will consider only direct stresses in this chapter.

Again, from a basic viewpoint, *strain* is defined as a deformation per unit length. There are also two types of strain, from an engineering standpoint: (a) direct or extensional strains, in which the deformation is a change in length of the material, and (b) shearing strains, in which the deformation is a distortion of the shape of the material. We will use the symbol ε to denote direct strains and the symbol γ to denote shearing strains. If the material elongates or stretches, the corresponding direct strain will be considered as positive strain, and if the material contracts, the corresponding direct strain will be considered as negative strain.

In the presence of temperature changes, the strain of a material is generally a function of stress and temperature, $\varepsilon = \varepsilon(\sigma, T)$. Applying the chain rule of calculus, we may express the differential direct strain as

$$d\varepsilon = \left(\frac{\partial \varepsilon}{\partial \sigma}\right)_T d\sigma + \left(\frac{\partial \varepsilon}{\partial T}\right)_\sigma dT \tag{2-1}$$

According to eq. (1-13), the first term is related to Young's modulus. The subscript on the partial derivatives denotes the physical quantity that is held constant in the evaluation of the derivative. For materials with stresses less than the elastic limit,

$$E = \left(\frac{\partial \sigma}{\partial \varepsilon}\right)_T = \frac{\sigma}{\varepsilon_m} \tag{2-2}$$

The quantity ε_m is the mechanical strain, or strain due to the application of a force. Because the direct strain is the change in length per unit length, we may use eq. (1-4) to evaluate the second term in eq. (2-1):

$$\alpha = \frac{1}{L}\left(\frac{\partial L}{\partial T}\right)_\sigma = \left(\frac{\partial \varepsilon}{\partial T}\right)_\sigma \tag{2-3}$$

The differential of the direct strain may be written in the following form, using the information from eqs. (2-2) and (2-3):

$$d\varepsilon = \frac{d\sigma}{E} + \alpha\, dT \tag{2-4}$$

By integrating eq. (2-4) from the temperature T_0, for which the stress is zero, to any value of temperature T, we obtain the stress–strain–temperature relationship (modified Hooke's law) for simple one-dimensional systems:

$$\varepsilon = \frac{\sigma}{E} + \int_{T_0}^{T} \alpha\ dT = \frac{\sigma}{E} + [e_t(T) - e_t(T_0)] \tag{2-5}$$

The quantity e_t is the thermal strain parameter defined by eq. (1-11). If the thermal expansion coefficient α can be approximated as constant, eq. (2-5) may be written in the following form:

$$\varepsilon = \frac{\sigma}{E} + \alpha\ \Delta T \tag{2-6}$$

The quantity $\Delta T = (T - T_0)$.

The first term on the right side of eq. (2-6) is the *mechanical strain*, or strain produced by the application of a force, and the second term is the *thermal strain*, or strain produced by a temperature change.

We may solve for the stress from eq. (2-6) to obtain the alternate expression for the stress–strain–temperature relationship:

$$\sigma = E\varepsilon - \alpha E\ \Delta T \tag{2-7}$$

2.2 BAR BETWEEN TWO SUPPORTS

Let us consider one of the basic problems involving thermal stresses and external constraints. As shown in Figure 2-1, a bar is rigidly attached between two supports and the bar undergoes a uniform temperature change $\Delta T = T - T_0$. At the initial temperature T_0, the stress in the bar is zero. The stress in the bar may be found directly from eq. (2-7), because the change in length of the bar ΔL is zero

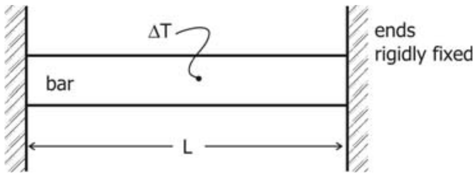

Figure 2-1. Bar of length L rigidly attached between two supports.

(because the ends are rigidly fixed) and the corresponding strain $\varepsilon = \Delta L/L$ is also zero:

$$\sigma = -\alpha E \Delta T \tag{2-8}$$

We note from eq. (2-8) that the thermal stress in the bar, in this case, is independent of the dimensions of the bar. We cannot make the bar bigger to reduce the stress. The reaction of the bar on the supports $P = \sigma A$, where A is the cross sectional area of the bar. If the cross section were made larger, the reaction at the supports would be increased, but the stress in the bar would remain constant.

Keeping in mind that there would be no thermal stresses if there were no constraint, we would anticipate that the designer could control thermal stresses by relaxing the constraints on the bar. One method of relaxing the external constraint is to allow the bar to move somewhat by providing a gap of width δ between the end of the bar and one support, as shown in Figure 2-2.

If the thermal deformation $\Delta L_t = \alpha L \, \Delta T$ is less than the gap width δ, the end of the bar does not come in contact with the support. In this case, the external constraint is zero, and no thermal stresses are developed:

$$\text{For} \quad \alpha \, L \, |\Delta T| \le \delta: \qquad \sigma = 0 \tag{2-9}$$

On the other hand, if $\alpha L |\Delta T|$ is greater than the gap width, the strain of the bar is the gap width (change in length of the bar) divided by the bar length:

$$\text{For} \quad \alpha L \, |\Delta T| > \delta: \qquad \varepsilon = \pm \delta / L \tag{2-10}$$

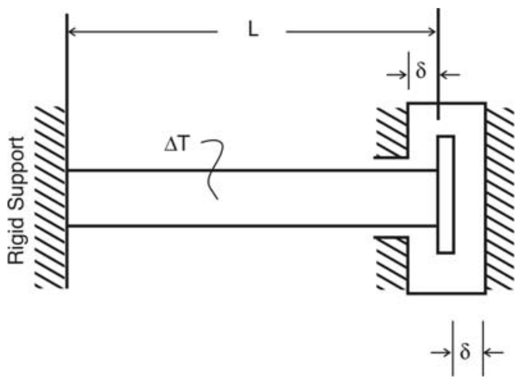

Figure 2-2. Bar with a gap of width δ at one end.

The upper sign (+) applies if the temperature change is positive; the lower sign (−) applies if the temperature change is negative. This expression may be used in eq. (2-7) to obtain the relationship for the stress in this case:

$$\sigma = \pm\frac{E\delta}{L} - \alpha E\Delta T \tag{2-11}$$

or

$$\sigma = -\alpha E\Delta T\left[1 - \left(\frac{\delta/L}{\alpha\,|\Delta T|}\right)\right] \tag{2-12}$$

The designer may limit the thermal stresses by appropriate choice of the gap width δ.

Another technique for relaxing the constraints is to place a flexible element (a spring, for example) at one end of the bar, as shown in Figure 2-3. The total change in length of the bar ΔL may be found from eq. (2-6):

$$\Delta L = \varepsilon L = \frac{\sigma L}{E} + \alpha L\Delta T \tag{2-13}$$

Let us consider the case in which a linear spring is used at the end of the bar. The spring force F_{sp} is proportional to the displacement of the spring, with the constant of proportionality defined as the *spring constant* k_{sp}. The force in the spring is considered as zero when the bar temperature is T_0:

$$F_{sp} = k_{sp}\Delta L = -F_{bar} = \sigma A \tag{2-14}$$

Using eq. (2-14) to eliminate the change in length of the bar in eq. (2-13), we obtain the following relationship:

$$-\frac{\sigma A}{k_{sp}} = \frac{\sigma L}{E} + \alpha L\Delta T \tag{2-15}$$

Solving for the stress from eq. (2-15), we obtain

$$\sigma = -\frac{\alpha E\Delta T}{1 + \left(AE/k_{sp}L\right)} \tag{2-16}$$

The designer may limit the thermal stresses by appropriate choice of the spring constant k_{sp}. We note that, if the spring constant k_{sp} is zero (zero spring force), then the thermal stress is zero, because the bar is unconstrained. On the other

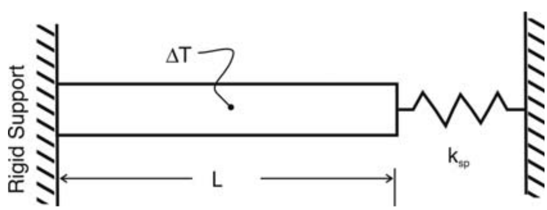

Figure 2-3. Bar with a spring element with a spring constant k_{sp}.

hand, if the spring constant is infinite (zero displacement for the spring), then the thermal stress given by eq. (2-16) reduces to that given by eq. (2-8).

Example 2-1 A 100-mm nominal (4-in. nominal) SCH 40 pipe is used in a steam system. The pipe's outside diameter is 114.3 mm (4.500 in.), its wall thickness is 6.0 mm (0.237 in.), and the length of the pipe between supports is 2.540 m (100 in.). The pipe material is a chromium–molybdenum alloy steel having the following properties: ultimate strength, 400 MPa (58,000 psi); yield strength, 270 MPa (39,200 psi); Young's modulus, 200 GPa (29×10^6 psi); and thermal expansion coefficient, $12.5 \times 10^{-6}\ \mathrm{K}^{-1}$. The allowable thermal stress for the pipe is 100 MPa (14,500 psi). The pipe is initially stress-free at a temperature of 20°C (68°F), and is finally heated to a uniform temperature of 200°C (392°F). Determine the thermal stress if the pipe is rigidly fixed at each end. What gap size would be required to limit the thermal stress to 100 MPa? If a bellows joint were used as the spring element, what spring constant would be required to limit the stress to the allowable value of 100 MPa?

The thermal stress is given by eq. (1-8) for the pipe rigidly fixed between two supports:

$$\sigma = -\alpha E \Delta T = -(12.5)(10^{-6})(200)(10^9)(200 - 20)$$

$$\sigma = -450 \times 10^6\ \mathrm{Pa} = -450\ \mathrm{MPa} \qquad (-65{,}300\ \mathrm{psi}) \qquad \text{(compression)}$$

This stress exceeds the allowable stress of 100 MPa. In fact, this stress is higher than the ultimate strength of the material (400 MPa).

We may use eq. (2-12) to determine the gap size required to limit the stress to the design value in compression. Let us first calculate the dimensionless quantity,

$$\frac{\sigma}{\alpha E \Delta T} = \frac{(-100)(10^6)}{(12.5)(10^{-6})(200)(10^9)(180)} = -0.2222$$

Then,

$$\frac{\delta}{\alpha L\,|\Delta T|} = 1 - 0.2222 = 0.7778$$

The required gap size is

$$\delta = (0.7778)(12.5)(10^{-6})(2.540)(180) = 4.45 \times 10^{-3}\ \mathrm{m}$$

$$\delta = 4.45\ \mathrm{mm} \qquad (0.175\ \mathrm{in.})$$

This is a practical gap size for a sliding joint, as illustrated in Figure 2-2.

The cross-sectional area of the pipe is

$$A = \pi(D_o - t)t = (\pi)(0.1143 - 0.0060)(0.006) = 20.41 \times 10^{-4}\ \mathrm{m}^2 = 20.41\ \mathrm{cm}^2$$

The axial force on the pipe is found as follows:

$$F = \sigma A = (100)(10^6)(20.41)(10^{-4}) = 204{,}100\ \mathrm{N} = 204.1\ \mathrm{kN} \quad (45{,}880\ \mathrm{lb_f})$$

This is a fairly large reaction force; however, the force is well below the critical axial force to cause buckling of the pipe, for the length given. The area moment of inertia for the pipe is given by

$$I = \frac{\pi}{8}[D_o^2 - 2(D_o - t)t](D_o - t)t \tag{2-17}$$

$$I = \frac{\pi}{8}[(0.1143)^2 - (2)(0.1143 - 0.0060)(0.0060)](0.1143 - 0.0060)(0.0060)$$

$$I = 3.002 \times 10^{-6}\,\text{m}^4 = 300.2\,\text{cm}^4$$

The critical or buckling axial force for a column with one fixed end and one "pinned" end is given by

$$F_{\text{cr}} = \frac{2\pi^2 EI}{L^2} = \frac{(2\pi^2)(200)(10^9)(3.002)(10^{-6})}{(2.54)^2} = 1.837 \times 10^6\ \text{N}$$

$$F_{\text{cr}} = 1.837\ \text{MN} \quad (413{,}000\ \text{lb}_\text{f})$$

The force required to buckle the pipe is about (1837/204.1) = 9 times the actual force.

It would be practical, in this problem, to eliminate the thermal stress altogether by making the gap somewhat larger. Using eq. (2-9), the minimum gap size for zero thermal stress is

$$\delta = \alpha L\,|\Delta T| = (12.5)(10^{-6})(2.540)(180) = 5.72 \times 10^{-3}\,\text{m}$$

$$\delta = 5.72\,\text{mm} \qquad (0.225\,\text{in.})$$

The constraint on the pipe could also be reduced by using a flexible bellows in the pipe. The maximum spring constant for the bellows may be found from eq. (2-16):

$$\frac{\sigma}{\alpha E \Delta T} = -0.2222 = -\frac{1}{1 + \left(AE/k_{\text{sp}}L\right)}$$

$$\frac{AE}{k_{\text{sp}}L} = \frac{1}{0.2222} - 1 = 3.500$$

The maximum value for the spring constant of the bellows is

$$k_{\text{sp}} = \frac{(20.41)(10^{-4})(200)(10^9)}{(3.50)(2.540)} = 45.92 \times 10^6\ \text{N/m}$$

$$k_{\text{sp}} = 45.92\ \text{MN/m} \qquad (262{,}200\ \text{lb}_\text{f}/\text{in.})$$

This is a fairly stiff spring constant. A typical commercially available pipe bellows with one corrugation for a 100-mm nominal pipe would have a spring constant of about 0.440 MN/m (2510 lb_f/in.). If we were to use a bellows with

this spring constant, the thermal stress would be almost negligible:

$$\frac{AE}{k_{sp}L} = \frac{(20.41)(10^{-4})(200)(10^9)}{(0.440)(10^6)(2.540)} = 365.2$$

Using eq. (2-16),

$$\frac{\sigma}{\alpha E \Delta T} = -\frac{1}{1 + 365.2} = -0.00273$$

The thermal stress with the more flexible bellows is

$$\sigma = (-0.00273)(450)(10^6) = -1.229 \times 10^6\,\text{Pa} = -1.229\,\text{MPa} \quad (-178.2\,\text{psi})$$

2.3 BARS IN PARALLEL

Let us consider the system shown in Figure 2-4, in which two bars are attached to each other at one end and attached to a support at the other end. Initially, the system is at a uniform temperature T_0, and the stress is zero with the external load P_e not applied. Finally, the bars are heated or cooled such that the final bar temperatures T_1 and T_2 are uniform and the external load is applied in the positive direction indicated in Figure 2-4. The temperature changes are defined as follows:

$$\Delta T_1 = T_1 - T_0 \qquad \text{and} \qquad \Delta T_2 = T_2 - T_0$$

The bars are constrained such that both experience the same change in length. Using eq. (2-6) to evaluate the strain for each bar and noting that the strain is the change in length per unit overall length, we may write the following expressions:

$$\Delta L_1 = \frac{\sigma_1 L_1}{E_1} + \alpha_1 L_1 \Delta T_1 = \Delta L_0 \tag{2-18}$$

$$\Delta L_2 = \frac{\sigma_2 L_2}{E_2} + \alpha_2 L_2 \Delta T_2 = \Delta L_0 \tag{2-19}$$

The quantity ΔL_0 is the overall change in length for the system.

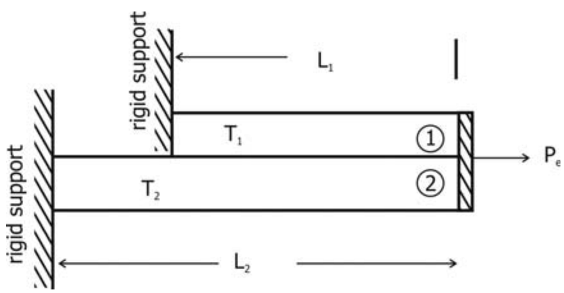

Figure 2-4. Two bars in parallel, attached at one end to each other.

The force balance applied at the end of the system is

$$P_e = \sigma_1 A_1 + \sigma_2 A_2 \tag{2-20}$$

The following expressions may be written from eqs. (2-18) and (2-19):

$$\sigma_1 A_1 = (A_1 E_1/L_1)\Delta L_0 - A_1 E_1 \alpha_1 \Delta T_1 \tag{2-21}$$

$$\sigma_2 A_2 = (A_2 E_2/L_2)\Delta L_0 - A_2 E_2 \alpha_2 \Delta T_2 \tag{2-22}$$

Making these substitutions into eq. (2-20), the force-balance expression, we obtain

$$P_e = [(A_1 E_1/L_1) + (A_2 E_2/L_2)]\Delta L_0 - (A_1 E_1 \alpha_1 \Delta T_1 + A_2 E_2 \alpha_2 \Delta T_2) \tag{2-23}$$

We may solve for the overall change in length for the system:

$$\Delta L_0 = \frac{(A_1 E_1 \alpha_1 \Delta T_1 + A_2 E_2 \alpha_2 \Delta T_2) + P_e}{(A_1 E_1/L_1) + (A_2 E_2/L_2)} \tag{2-24}$$

The stress in each bar may be written, using eqs. (2-18) and (2-19), as

$$\sigma_1 = -E_1(\alpha_1 \Delta T_1 - \Delta L_0/L_1) \tag{2-25}$$

$$\sigma_2 = -E_2(\alpha_2 \Delta T_2 - \Delta L_0/L_2) \tag{2-26}$$

If we substitute the expression for the change in length of the system ΔL_0 from eq. (2-24) into eqs. (2-25) and (2-26), we obtain the final expressions for the stress in each bar:

$$\sigma_1 = -E_1\left[\frac{\alpha_1 \Delta T_1 - \alpha_2 \Delta T_2(L_2/L_1) - (P_e L_2/L_1 A_2 E_2)}{1 + (A_1 E_1 L_2/A_2 E_2 L_1)}\right] \tag{2-27}$$

$$\sigma_2 = -E_2\left[\frac{\alpha_2 \Delta T_2 - \alpha_1 \Delta T_1(L_1/L_2) - (P_e L_1/L_2 A_1 E_1)}{1 + (A_2 E_2 L_1/A_1 E_1 L_2)}\right] \tag{2-28}$$

From the force-balance expression, eq. (2-20), we find the following relationship between the stress in each bar:

$$\sigma_2 = (P_e/A_2) - \sigma_1(A_1/A_2) \tag{2-29}$$

The case for more than two bars in parallel may be analyzed in a similar manner. The overall deflection of the end of the bar system consisting of N bars in parallel is given by the following, which is a generalization of eq. (2-24):

$$\Delta L_0 = \frac{P_e + \sum_{j=1}^{N} \alpha_j E_j A_j \Delta T_j}{\sum_{j=1}^{N} (E_j A_j/L_j)} \tag{2-30}$$

The stress in the nth bar is given by an expression similar to that given in eqs. (2-25) and (2-26):

$$\sigma_n = -E_n(\alpha_n \Delta T_n - \Delta L_0/L_n) \qquad (n = 1, 2, \ldots, N) \tag{2-31}$$

Example 2-2 Two 304 stainless steel pipes are concentric, as shown in Figure 2-5, and are attached at both ends. The inner pipe (1) is a 100-mm (4-in.) nominal SCH 10 pipe (outside diameter, 114.3 mm or 4.500 in.; wall thickness, 3.05 mm or 0.120 in.) with a length of 5.00 m (16.40 ft). The outer pipe (2) is a 150-mm (6-in.) nominal SCH 10 pipe (outside diameter, 168.3 mm or 6.625 in.; wall thickness, 3.40 mm or 0.134 in.) with a length of 5.00 m also. The maximum allowable stress for the material is 130 MPa (18,850 psi), Young's modulus is 198 GPa (28.7×10^6 psi), and the thermal expansion coefficient is $14.8 \times 10^{-6}\ \mathrm{K}^{-1}$ ($8.22 \times 10^{-6}\ {}^\circ\mathrm{F}^{-1}$). Both pipes are initially stress-free at a temperature of 20°C (68°F), and the outer pipe is warmed to 25°C (77°F). Determine the lowest temperature to which the inner pipe may be cooled by a liquid flowing through it such that the design stress level is not exceeded. There is no external force applied at the ends of the pipe system.

The cross-sectional areas for each pipe are

$$A_1 = \pi(D_{o1} - t_1)t_1 = (\pi)(0.1143 - 0.00305)(0.00305)$$

$$A_1 = 10.66 \times 10^{-4}\ \mathrm{m}^2 = 10.66\ \mathrm{cm}^2\ (1.542\ \mathrm{in}^2)$$

$$A_2 = (\pi)(0.1683 - 0.00340)(0.00340) = 17.61 \times 10^{-4}\ \mathrm{m}^2 = 17.61\ \mathrm{cm}^2 (2.730\ \mathrm{in}^2)$$

The ratio term is

$$\frac{A_1 E_1 L_2}{A_2 E_2 L_1} = \frac{A_1}{A_2} = \frac{10.66}{17.61} = 0.6052$$

We note, in this case, that $(\Delta T_1 - \Delta T_2) = [(T_1 - T_0) - (T_2 + T_0)] = (T_1 - T_2)$. Because the two pipes are constructed of the same material and have the same length, eq. (2-27) reduces to the following, for this problem:

$$\sigma_1 = -\frac{E\alpha(T_1 - T_2)}{1 + (A_1/A_2)}$$

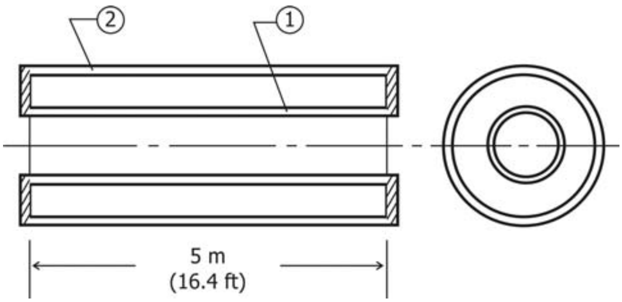

Figure 2-5. Sketch for Example 2-2. Inner pipe, element 1; outer pipe, element 2. Both ends of the circular pipes are attached.

The magnitude of the force in each pipe is the same; therefore, the highest stress will occur in the pipe with the smallest cross-sectional area, which is the inner pipe in this case. Also, if the inner pipe is cooled, it will experience a tensile stress (positive) and the outer pipe will experience a compressive stress (negative):

$$\sigma_1 = +130 \times 10^6 = -\frac{(198)(10^9)(14.8)(10^{-6})(T_1 - T_2)}{1 + 0.6052}$$

$$= -(1.826)(10^6)(T_1 - T_2)$$

$$T_1 - T_2 = -\frac{130}{1.826} = -71.2^\circ\text{C}$$

The lowest temperature for the inner pipe is

$$T_1 = 25^\circ - 71.2^\circ = -46.2^\circ\text{C}\ (-51.2^\circ\text{F})$$

The corresponding stress in the outer pipe is found from the force-balance expression, eq. (2-20):

$$\sigma_2 = -\sigma_1(A_1/A_2) = -(130)(10^6)(0.6052)$$

$$\sigma_2 = -78.7 \times 10^6\ \text{Pa} = -78.7\ \text{MPa} \quad (-11{,}400\ \text{psi})$$

2.4 BARS WITH PARTIAL REMOVAL OF CONSTRAINTS

As for the case of a single bar with thermal stresses, the thermal stresses in a system of bars may be controlled by partial relaxation of the constraints between the bars. The partial relaxation of the constraints may be achieved by providing an expansion gap or by providing a flexible element (spring) in series with one of the components of the system.

2.4.1 Bars with Expansion Gaps

Let us consider the system shown in Figure 2-6, in which two bars are connected at one end and fixed at the other ends. A gap of total width 2δ is provided in member 1.

The change in length of each bar due to thermal effects is

$$\Delta L_{1,t} = \alpha_1 L_1 \Delta T_1 \qquad \text{and} \qquad \Delta L_{2,t} = \alpha_2 L_2 \Delta T_2 \tag{2-32}$$

The critical gap size δ_{cr} is the gap size for which the two bars begin to move as a unit:

$$\alpha_1 L_1 \Delta T_1 = \alpha_2 L_2 \Delta T_2 \pm \delta_{cr} \tag{2-33}$$

The critical gap size is given by

$$\delta_{cr} = |\alpha_2 L_2 \Delta T_2 - \alpha_1 L_1 \Delta T_1| \tag{2-34}$$

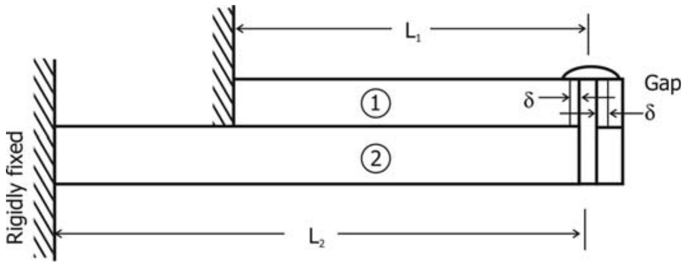

Figure 2-6. Two bars with a gap of width δ in one member.

If the actual gap size is greater than the critical gap size, the bars are actually unconstrained, and the thermal stress is zero.

$$\text{For} \quad \delta \geq \delta_{cr}: \qquad \sigma_1 = \sigma_2 = 0 \tag{2-35}$$

On the other hand, if the actual gap size is less than the critical value, the bars will be partially constrained:

$$\Delta L_1 = \Delta L_2 \pm \delta = \Delta L_0 \pm \delta \tag{2-36}$$

The quantity ΔL_0 is the overall displacement for the system of bars. The upper sign (+) applies when the thermal expansion for bar 1 is greater than the thermal expansion of bar 2, $(\alpha_1 L_1 \Delta T_1) > (\alpha_2 L_2 \Delta T_2)$, and the lower sign (−) applies when $(\alpha_1 L_1 \Delta T_1) < (\alpha_2 L_2 \Delta T_2)$.

Equations (2-18) and (2-19) may be used to evaluate the change in length for each bar:

$$\Delta L_1 = \frac{\sigma_1 L_1}{E_1} + \alpha_1 L_1 \Delta T_1 = \Delta L_0 \pm \delta \tag{2-37}$$

$$\Delta L_2 = \frac{\sigma_2 L_2}{E_2} + \alpha_2 L_2 \Delta T_2 = \Delta L_0 \tag{2-38}$$

Solving for the product of stress and cross-sectional area from eqs. (2-37) and (2-38), we obtain

$$\sigma_1 A_1 = \frac{A_1 E_1 \Delta L_0}{L_1} - A_1 E_1 \alpha_1 \Delta T_1 \pm \frac{A_1 E_1 \delta}{L_1} \tag{2-39}$$

$$\sigma_2 A_2 = \frac{A_2 E_2 \Delta L_0}{L_2} - A_2 E_2 \alpha_2 \Delta T_2 \tag{2-40}$$

For this problem, let us consider the case in which there is no external load applied. The force-balance equation for this case is

$$F_{net} = 0 = \sigma_1 A_1 + \sigma_2 A_2 \tag{2-41}$$

Making the substitutions from eqs. (2-39) and (2-40), we obtain

$$\left(\frac{A_1E_1}{L_1} + \frac{A_2E_2}{L_2}\right)\Delta L_0 - (A_1E_1\alpha_1\Delta T_1 + A_2E_2\alpha_2\Delta T_2) \pm \frac{A_1E_1\delta}{L_1} = 0$$

The overall change in length for the system may be found:

$$\Delta L_0 = \frac{A_1E_1\alpha_1\ \Delta T_1 + A_2E_2\alpha_2\ \Delta T_2}{A_1E_1/L_1 + A_2E_2/L_2} \pm \frac{A_1E_1\delta/L_1}{A_1E_1/L_1 + A_2E_2/L_2} \tag{2-42}$$

We observe that the first term on the right side of eq. (2-42) is the same (with $P_e = 0$) as the expression given in eq. (2-24) for complete constraint.

The thermal stresses for this case may be found from eqs. (2-37) and (2-38):

$$\sigma_1 = E_1\left[\frac{\Delta L_0}{L_1} - \alpha_1\ \Delta T_1 \pm \frac{\delta}{L_1}\right] \tag{2-43}$$

$$\sigma_2 = E_2\left[\frac{\Delta L_0}{L_2} - \alpha_2\ \Delta T_2\right] \tag{2-44}$$

The analysis may be extended to the case of N bars in parallel, with bar 1 having partial constraint. The critical gap size for the general case is

$$\delta_{\text{cr}} = \left|\Delta L_0^* - \alpha_1\Delta T_1\right| \tag{2-45}$$

The quantity ΔL_0^* is the overall displacement for the system of N bars with member 1 not present. Using eq. (2-24), we get

$$\Delta L_0^* = \frac{\sum A_jE_j\alpha_j\Delta T_j}{\sum A_jE_j/L_j} \tag{2-46}$$

If the actual gap is larger than the critical gap, bar 1 is unconstrained, and the stress in bar 1, σ_1, is zero. The stress in the other bars is given by

$$\sigma_n = -E_n(\alpha_n\ \Delta T_n - \Delta L_0^*/L_n) \qquad (n = 2, 3, \ldots, N) \tag{2-47}$$

If the actual gap is smaller than the critical gap size, the system will be partially constrained. The overall change in length of the system for this case is given by

$$\Delta L_0 = \frac{\sum\left(A_jE_j\alpha_j\ \Delta T_j\right) \pm (A_1E_1\delta/L_1)}{\sum A_jE_j/L_j} \tag{2-48}$$

The upper sign (−) is used if $(\alpha_1L_1\ \Delta T_1) > \Delta L_0^*$, and the lower sign (+) is used if $(\alpha_1L_1\Delta T_1) < \Delta L_0^*$. The thermal stress in each bar may be evaluated from the

following expression, for the case in which the actual gap is smaller than the critical gap size:

$$\sigma_1 = E_1 \left[\frac{\Delta L_0}{L_1} - \alpha_1 \ \Delta T_1 \pm \frac{\delta}{L_1} \right] \tag{2-49}$$

$$\sigma_n = E_n \left[\frac{\Delta L_0}{L_n} - \alpha_n \ \Delta T_n \right] \qquad (n = 2, 3, \ldots, N) \tag{2-50}$$

Example 2-3 Suppose the inner pipe in Example 2-2 must be cooled to −180°C (−292°F). Determine the size of the expansion gap in the inner pipe to limit the thermal stress to 130 MPa (18,850 psi).

If no gap were used in the system, the thermal stress in the inner pipe would be as follows:

$$(\sigma_1)_{\text{ng}} = -\frac{E\alpha(T_1 - T_2)}{1 + (A_1/A_2)} = -\frac{(198)(10^9)(14.8)(10^{-6})(-180 - 25)}{1 + 0.6052}$$

$$(\sigma_1)_{\text{ng}} = +3.74 \times 10^9 \text{ Pa} = 3740 \text{ MPa} \qquad (542{,}000 \text{ psi})$$

This stress is almost 29 times the allowable stress (130 MPa); therefore, the expansion gap is needed.

The lengths and materials of both members are the same, so member 1 will contract more than member 2, because $\Delta T_1 = -200^\circ\text{C} < \Delta T_2 = +5^\circ\text{C}$. The resulting stress in member 1 will be tensile (positive). The overall change in length may be found from eq. (2-43), using the lower (−) sign:

$$\frac{(130)(10^6)}{(198)(10^9)} = \frac{\Delta L_0 - \delta}{L_1} - (14.8)(10^{-6})(-180 - 20)$$

$$\frac{\Delta L_0 - \delta}{L_1} = 0.0657 \times 10^{-3} - 2.960 \times 10^{-3} = -2.303 \times 10^{-3}$$

Using eq. (2-42), we obtain

$$\frac{\Delta L_0 - \delta}{L} = \frac{\alpha(A_1 \Delta T_1 + A_2 \Delta T_2) + A_1(\delta/L)}{A_1 + A_2} - \frac{\delta}{L}$$

$$\frac{\Delta L_0 - \delta}{L} = \frac{\alpha(A_1 \Delta T_1 + A_2 \Delta T_2) - A_2(\delta/L)}{A_1 + A_2}$$

$$\times (-2.303)(10^{-3})(10.66 + 17.61) = (14.8)(10^{-6})[(10.66)(-200)$$

$$+ (17.61)(5)] - 17.61\delta$$

$$\delta = \frac{-0.03025 + 0.06511}{17.61} = 0.00198 \text{ m} = 1.98 \text{ mm } (0.078 \text{ in.})$$

The critical gap size is given by eq. (2-34):

$$\delta_{\text{cr}} = \alpha L \, |\Delta T_2 - \Delta T_1| = (14.8)(10^{-6})(5.00) \, |5 + 200| = 0.01517 \text{ m} = 15.17 \text{ mm}$$

The actual gap is smaller than the critical gap. The stress in the outer line may be evaluated from eq. (2-41), the force-balance expression:

$$\sigma_2 = -\sigma_1(A_1/A_2) = -(130)(10.66/17.61) = -78.7\,\text{MPa}\ (-11{,}410\,\text{psi})$$

2.4.2 Bars with Spring Elements

Partial relaxation of the constraints in a system may be achieved by providing a flexible element or spring in series with one of the members. Let us consider the system shown in Figure 2-7, in which a spring element having a spring constant k_{sp} is connected to member 1.

The overall change in length for the system is equal to the total change in length for member 2, in this case, $\Delta L_0 = \Delta L_2$. The overall change in length for the system is also equal to the change in length of member 1 plus the change in length of the spring element:

$$\Delta L_0 = \Delta L_1 + \Delta L_{sp} \tag{2-51}$$

The change in length of the spring is related to the force on the spring and the spring constant:

$$\Delta L_{sp} = \frac{\sigma_1 A_1}{k_{sp}} \tag{2-52}$$

The change in length for each member may be found from eq. (2-6) and the definition of strain. For member 2,

$$\Delta L_2 = \frac{\sigma_2 L_2}{E_2} + \alpha_2 L_2 \Delta T_2 = \Delta L_0 \tag{2-53}$$

Similarly, the change in length for member 1 may be found from eq. (2-6):

$$\Delta L_1 = \frac{\sigma_1 L_1}{E_1} + \alpha_1 L_1 \Delta T_1 = \Delta L_0 - \frac{\sigma_1 A_1}{k_{sp}} \tag{2-54}$$

or

$$\Delta L_0 = \sigma_1 A_1 \left(\frac{L_1}{A_1 E_1} + \frac{1}{k_{sp}} \right) + \alpha_1 L_1 \Delta T_1 \tag{2-55}$$

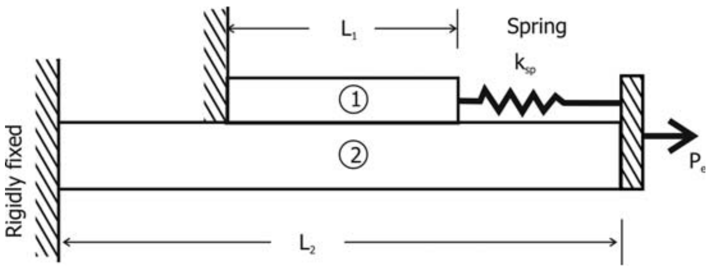

Figure 2-7. Two bars with a spring element in series with one bar (member 1).

If we apply a force balance to the system, we obtain the following relationship:

$$\sigma_1 A_1 + \sigma_2 A_2 = P_e \tag{2-56}$$

Using eqs. (2-53) and (2-55) to eliminate the stresses, the following expression is obtained for the overall change in length for the system:

$$\Delta L_0 = \frac{\left[\dfrac{\alpha_1 L_1 \Delta T_1}{(L_1/A_1E_1) + (1/k_{\text{sp}})} + \dfrac{\alpha_2 L_2 \Delta T_2}{\left(\dfrac{L_2}{A_2E_2}\right)}\right] + P_e}{\dfrac{1}{\left(\dfrac{L_1}{A_1E_1}\right) + \dfrac{1}{k_{\text{sp}}}} + \dfrac{A_2E_2}{L_2}} \tag{2-57}$$

This expression may be simplified to the following form:

$$\frac{\Delta L_0}{L_1} = \frac{\alpha_1 \Delta T_1 + K_a \alpha_2 \Delta T_2 + K_b \left(\frac{P_e}{A_1E_1}\right)}{K_c} \tag{2-58}$$

The K factors in eq. (2-58) are defined as follows:

$$K_a = \left(\frac{A_2E_2}{A_1E_1}\right)\left[1 + \left(\frac{A_1E_1}{k_{\text{sp}}L_1}\right)\right] \tag{2-59}$$

$$K_b = 1 + \left(\frac{A_1E_1}{k_{\text{sp}}L_1}\right) \tag{2-60}$$

$$K_c = K_a \left(\frac{L_1}{L_2}\right) \tag{2-61}$$

The stress in member 2 may be found from eq. (2-53) after the overall change in length for the system has been determined:

$$\sigma_2 = E_2 \left(\frac{\Delta L_0}{L_2} - \alpha_2 \Delta T_2\right) \tag{2-62}$$

The stress in member 1 may be evaluated by eq. (2-55) or from the force balance expression, eq. (2-56):

$$\sigma_1 = E_1 \left[\frac{\frac{\Delta L_0}{L_1} - \alpha_1 \Delta T_1}{K_b}\right] \tag{2-63}$$

or

$$\sigma_1 = -\sigma_2 \left(\frac{A_2}{A_1}\right) + \frac{P_e}{A_1} \tag{2-64}$$

The effectiveness of the spring element in reducing stresses is dependent on the ratio of the "spring constant" (A_1E_1/L_1) for the member (1) connected to

the flexible element and the spring constant of the flexible element k_{sp}, expressed in eq. (2-60). If the ratio given by the second term in eq. (2-60) is very large or if the spring is extremely flexible ($k_{sp} \to 0$), we note that the stress in member 1 approaches zero, according to eq. (2-63). The change in length for the system, according to eq. (2-57) with $k_{sp} = 0$, is

$$\frac{\Delta L_0}{L_2} \to \alpha_2 \Delta T_2 + \frac{P_e}{A_2 E_2} \qquad (\text{for } k_{sp} \to 0) \tag{2-65}$$

Making this substitution into eq. (2-62), we find that the stress in member 2 for an extremely flexible spring is the stress produced by the external load only:

$$\sigma_2 \to \frac{P_e}{A_2} \qquad (\text{for } k_{sp} \to 0) \tag{2-66}$$

On the other hand, if the spring constant ratio is extremely small or if the spring is extremely rigid ($k_{sp} \to \infty$), the behavior of the system approaches the case of two members rigidly attached to each other. The expression for the overall change in length, eq. (2-57), approaches the same as that given by eq. (2-24).

Example 2-4 A vacuum-jacketed transfer line for liquid oxygen is shown in Figure 2-8. The outer line (1) is a 150-mm nominal (6-in. nominal) SCH 10 pipe with an outside diameter of 168.3 mm (6.625 in.), a wall thickness of 3.4 mm (0.134 in.), and a solid cross-sectional area of 17.63 cm^2 (2.733 in^2). The inner line (2) is a 100-mm nominal (4-in. nominal) SCH 10 pipe with an outside diameter of 114.3 mm (4.500 in.), a wall thickness of 3.1 mm (0.120 in.), and a solid cross-sectional area of 10.65 cm^2 (1.651 in^2). The length of the outer line is 12.00 m (39.37 ft), and the length of the inner line is 12.20 m (40.03 ft). The system is stress-free at a temperature of 20°C (68°F). Under operating conditions, the temperature of the inner line is −180°C (−292°F), and the temperature of the outer line is +30°C (86°F). Both lines are constructed of 304 stainless steel, with Young's modulus, 198 MPa (28.7 × 10^6 psi) and thermal expansion coefficient, 14.8 × 10^{-6} K^{-1} (8.22 × 10^{-6} °F^{-1}). Because of other loadings, it is desired to limit the thermal stresses for the system to 16 MPa (2320 psi) by placing an expansion bellows in the outer line. Determine the required spring constant k_{sp} for the bellows.

The maximum thermal stress for $P_e = 0$ will generally occur for the member having the smallest cross-sectional area, which is the inner line, in this case. The inner line will tend to contract; therefore, the thermal stress in the inner line will be tensile:

$$\sigma_2 = +16 \text{ MPa}$$

The thermal stress in the outer line may be found from eq. (2-64), the force-balance expression:

$$\sigma_1 = -\sigma_2 \left(\frac{A_2}{A_1}\right) = -(16)\left(\frac{10.65}{17.63}\right) = -9.665 \text{ MPa} \quad (-1400 \text{ psi})$$

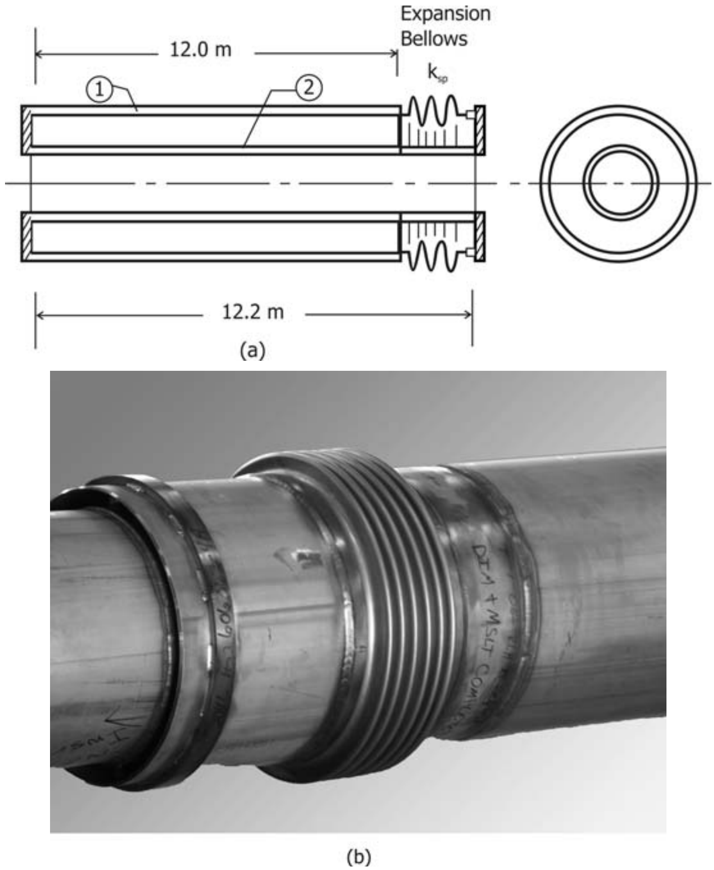

Figure 2-8. Vacuum-jacketed LOX transfer line with an expansion bellows in the outer line, Example 2-4: (a) schematic; (b) commercial cryogenic transfer line, showing the expansion bellows in the outer line. [Used by permission of Eden Cryogenics, Plain City, OH]

The overall change in length for the system may be found from eq. (2-53):

$$\Delta L_0 = (12.20)\left[\frac{(16)(10^6)}{(198)(10^9)} + (14.8)(10^{-6})(-180 - 20)\right]$$

$$\Delta L_0 = (12.20)(-0.002879) = -0.035126 \text{ m} = -35.126 \text{ mm } (-1.383 \text{ in.})$$

For this case, eq. (2-55) may be used also for the overall change in length:

$$-0.035126 = (-9.665)(10^6)(17.63)(10^{-4})\left[\frac{(12.00)}{(17.63)(10^{-4})(198)(10^9)} + \frac{1}{k_{sp}}\right]$$
$$+ (14.8)(10^{-6})(12.00)(30 - 20)$$

$$\frac{-0.035126 - 0.001776}{-17{,}040} = 2.1656 \times 10^{-6} = 0.034377 \times 10^{-6} + \frac{1}{k_{sp}}$$

The required value (maximum) for the spring constant of the bellows may be found:

$$k_{\mathrm{sp}} = \frac{1}{(2.1656 + 0.03438)(10^{-6})} = 454.5 \times 10^3 \text{ N/m}$$

$$k_{sp} = 454.5 \text{ kN/m} \quad (2595 \text{ lb}_\text{f}/\text{in.})$$

We may check the result by using eq. (2-63) to calculate the stress in the outer member (1):

$$K_b = 1 + \left[\frac{(17.63)(10^{-4})(198)(10^9)}{(454.5)(10^3)(12.00)}\right] = 1 + 64.00 = 65.00$$

$$\sigma_1 = (198)(10^9)\left[\frac{(-0.035126/12.00) - (14.8)(10^{-6})(10)}{65.00}\right] = -9.667 \times 10^6 \text{ Pa}$$

The axial force in the outer line, member 1, is

$$F_1 = \sigma_1 A_1 = (-9.665)(10^6)(17.63)(10^{-4}) = -17.04 \times 10^3 \text{ N}$$

$$F_1 = -17.04 \text{ kN } (3830 \text{ lb}_\text{f})$$

The change in length for the bellows may be found from the force-deflection relationship for the "spring":

$$\Delta L_{\mathrm{sp}} = \frac{F_1}{k_{\mathrm{sp}}} = \frac{(-17.04)(10^3)}{(454.5)(10^3)} = -0.0375 \text{ m} = -37.5 \text{ mm } (-1.48 \text{ in.})$$

2.5 NONUNIFORM TEMPERATURE DISTRIBUTION

In the previous sections of this chapter, we have considered thermal stresses caused by temperature changes that are uniform along the length of the member and across the cross section of the member. Let us now consider the effect of temperature variation within the member or members.

2.5.1 Temperature Variation in the Lengthwise Direction

Let us consider the case in which the temperature varies along the length of a bar, but is constant across any cross section. The coordinate x is measured from one end (fixed) of the bar, and the temperature change is $\Delta T = T(x) - T_0$, where T_0 is the temperature of the bar under stress-free conditions.

At any location x along the bar, the displacement of one side of a differential section of the bar is denoted by u, as shown in Figure 2-9. If the displacement is not constant, the displacement of the other side of the differential element at

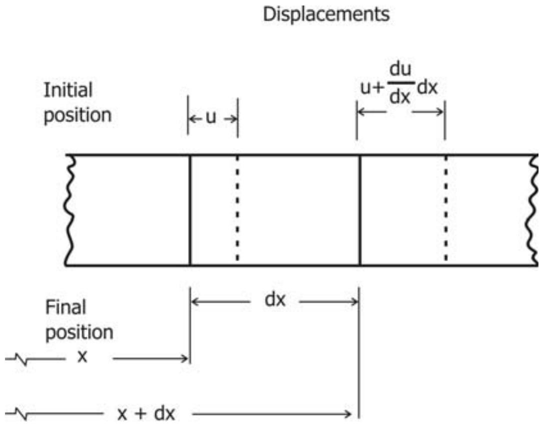

Figure 2-9. Displacements in a bar with a nonuniform axial temperature distribution.

$x + dx$ is $[u + (du/dx)dx]$. The strain for the differential element may be found from the definition of strain: change in length per unit original length:

$$\varepsilon = \frac{\left(u + \dfrac{du}{dx}dx\right) - u}{dx} = \frac{du}{dx} \tag{2-67}$$

The stress–strain–temperature relationship, eq. (2-6), applies for a particular location, whether the strain is uniform or not:

$$\varepsilon = \frac{\sigma}{E} + \alpha \Delta T = \frac{du}{dx} \tag{2-68}$$

If we apply a force balance to the element, in the absence of traction or shear forces on the surface of the bar, we obtain

$$\sigma A - \left(\sigma + \frac{d\sigma}{dx}dx\right) A = 0$$

Therefore,

$$\frac{d\sigma}{dx} = 0 \qquad \text{or,} \qquad \sigma = \text{constant} \tag{2-69}$$

The strain relationship, eq. (2-68), may be integrated to obtain the displacement at any point along the bar, with the condition that the displacement is zero at one end of the bar (*at* $x = 0$):

$$u = \frac{\sigma}{E}x + \int_0^x \alpha \Delta T(x)\, dx \tag{2-70}$$

If the thermal expansion coefficient may be considered as constant, eq. (2-70) may be written as

$$u(x) = \frac{\sigma}{E}x + \alpha \int_0^x \Delta T(x)\, dx \tag{2-71}$$

The displacement at the other end of the bar (*at* $x = L$) may be written in the following form:

$$u(L) = \frac{\sigma}{E}L + \alpha L\,(\Delta T)_{\text{ave}} \tag{2-72}$$

The temperature difference in the second term of eq. (2-72) is the temperature change averaged along the length of the bar:

$$(\Delta T)_{\text{ave}} = \frac{1}{L}\int_0^L \Delta T(x)\,dx \tag{2-73}$$

Note that the overall strain for the bar ε_0 is defined as

$$\varepsilon_0 = \frac{u(L)}{L} = \frac{\sigma}{E} + \alpha\,(\Delta T)_{\text{ave}} \tag{2-74}$$

This expression given in eq. (2-74) is identical in form as the expression for uniform temperature change, eq. (2-6). This result implies that the expressions developed previously for a uniform temperature change may also be used to determine the thermal stress for the case in which the temperature varies along the length of the bar, if the average temperature change for the bar is used in this case.

Example 2-5 Conduction heat transfer takes place along a rod that is rigidly fixed between two surfaces. The rod has a 6-mm (0.236-in.) diameter and is 300 mm (11.81 in.) long. The rod material is 4340 alloy steel, with a Young's modulus of 214 GPa (31.0×10^6 psi) and thermal expansion coefficient of $11.2 \times 10^{-6}\ \text{K}^{-1}$ ($6.22 \times 10^{-6}\ {}^\circ\text{F}^{-1}$). The rod is stress-free at 25°C (77°F). One end of the rod ($x = 0$) is maintained at 25°C, and the other end ($x = L$) is maintained at 95°C (203°F). The temperature distribution along the length of the rod is given by

$$T(x) = T_0 + (T_1 - T_0)(x/L)$$

The quantity T_1 is the temperature of the rod at $x = L$, and T_0 is the temperature at $x = 0$. Determine the thermal stress in the rod.

The average temperature along the length of the rod may be determined:

$$(\Delta T)_{\text{ave}} = \frac{1}{L}\int_0^L (T_1 - T_0)\left(\frac{x}{L}\right)dx = (T_1 - T_0)\frac{x^2}{2L^2}\bigg|_0^L = \frac{1}{2}(T_1 - T_0)$$

$$(\Delta T)_{\text{ave}} = \tfrac{1}{2}(95^\circ - 25^\circ) = 35^\circ C$$

The displacement of the bar at $x = L$ is zero, so we may use eq. (2-72) to determine the stress in the bar:

$$\sigma = -\alpha E\,(\Delta T)_{\text{ave}} = -(11.2)(10^{-6})(214)(10^9)(35^\circ)$$

$$\sigma = -83.89 \times 10^6\ \text{Pa} = -83.89\ \text{MPa} \qquad (-12{,}170\ \text{psi})$$

The displacement along the length of the bar may be found from eq. (2-71):

$$u(x)/L = \left(\frac{\sigma}{E}\right)\left(\frac{x}{L}\right) + \frac{1}{2}\alpha L(T_1 - T_0)\left(\frac{x}{L}\right)^2$$

We note that the first term on the right side is the displacement (negative) caused by the compressive mechanical stress due to the force on the ends of the bar, and the second term is the displacement (positive) caused by the thermal expansion of the bar as it is heated. Using the expression for the thermal stress, the displacement may be written in the following form:

$$\frac{u(x)}{L} = \frac{1}{2}\alpha(T_1 - T_0)\left(\frac{x}{L}\right)\left[1 - \left(\frac{x}{L}\right)\right] = -0.000784\left(\frac{x}{L}\right)\left[1 - \left(\frac{x}{L}\right)\right]$$

If the temperature change were uniform, then it may be shown that the displacement would be zero at all points along the bar, because the displacement due to mechanical load would be exactly offset by the thermal expansion. Since the temperature is not uniform in this example, the mechanical and thermal effects are offset only at each end of the bar.

2.5.2 Temperature Variation across the Cross Section

Let us consider the situation in which the temperature in the bar is uniform along the length of the bar (in the x-direction), but the temperature may vary across the cross section (in the y-direction) at any location along the length of the bar. The case in which the bar temperature varies in both the x- and y-directions involves two-dimensional stresses and strains, which we will consider in a later chapter.

In addition, we will consider the case for which the temperature distribution is symmetrical about the centroid axis of the cross section or the case for which the bar is externally constrained such that it cannot bend. We define a symmetrical temperature distribution as one for which the following integral across the cross section is zero:

$$\int_A \alpha E \Delta T(y) y \, dA \equiv 0 \tag{2-75}$$

The coordinate y is measured from the centroid axis of the cross section. If the temperature distribution is not symmetrical or if the bar is not constrained in

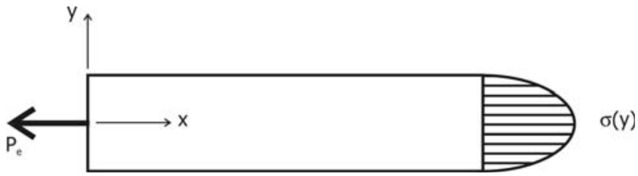

Figure 2-10. Bar with a symmetrical transverse (across the thickness) temperature variation.

bending, transverse deformations and bending stresses will be present. The effect of bending is considered in Chapter 3.

If we apply a force balance for the system shown in Figure 2-10, we obtain the following relationship:

$$P_e = \int_A \sigma \, dA \tag{2-76}$$

The temperature does not change along the length of the bar, in this case, so the total strain ε_0 is constant. Using the stress given by eq. (2-8), we may expand eq. (2-76) as follows:

$$P_e = \int_A E\varepsilon_0 \, dA - \int_A \alpha E \Delta T \, dA \tag{2-77}$$

Let us define the "thermal force" F_T as

$$F_T = \int_A \alpha E \Delta T \, dA \tag{2-78}$$

If we make this substitution into eq. (2-77) and note that the total strain ε_0 is not a function of the cross-sectional area, we obtain

$$P_e + F_T = \varepsilon_0 \int_A E \, dA \tag{2-79}$$

For the case in which the material properties α and E are not functions of temperature, eqs. (2-78) and (2-79) may be written as

$$F_T = \alpha E \int_A \Delta T \, dA = \alpha E A \Delta T_m \tag{2-80}$$

$$\varepsilon_0 = \frac{P_e + F_T}{EA} \tag{2-81}$$

The quantity ΔT_m is the mean temperature change across the cross section:

$$\Delta T_m = \frac{1}{A} \int_A \Delta T \, dA \tag{2-82}$$

We may use eq. (2-8) to evaluate the stress at any location across any cross section not near the ends of the bar:

$$\sigma(y) = E\varepsilon_0 - \alpha E \Delta T(y) = \frac{P_e}{A} + \alpha E \left[\Delta T_m - \Delta T(y)\right] \tag{2-83}$$

It would be in order to note at this point that the loading at the ends of the bar may not be uniform, and the stress distribution at the ends of the bar may not be exactly given by eq. (2-83). We may use eq. (2-83) for the stress at locations a distance on the order of the thickness of the bar away from the

ends, according to the *Saint-Venant's principle* [Timoshenko and Goodier, 1970]. By "of the order of the thickness," we usually mean a distance of about two or three times the thickness of the bar. There are some thin-walled structures, such as monocoque cylinders under concentrated loads and many sheet-and-stringer geometries, for which the effect of loading is not restricted to the region near the point of application of the load [Hoff, 1945].

It is not easy to make a completely general and concise statement of Saint-Venant's principle, because one would need to state estimates of the magnitude of the deviation between the actual and approximate stresses and displacements in the vicinity of the regions near the boundary [Boresi and Chong, 1987]. A general proof of Saint-Venant's principle has been presented [Goodier, 1937]. From the standpoint of engineering design, we may summarize Saint-Venant's principle as follows. Two different but statically equivalent force systems (systems having the same resultant force and resultant moment) that act over a small part of the surface of a body produce approximately the same stress distribution and displacements in regions sufficiently far removed from the region where the forces act.

Based on Saint-Venant's principle, we may be assured that the stresses in the bar at distances of about two or three times the thickness of the bar may be determined from eq. (2-83) for any symmetrical temperature distribution. In this case, the resultant force on the ends of the bar is P_e, and the resultant moment is zero.

Let us consider the case in which the bar is unconstrained at the ends or $P_e = 0$. Using eqs. (2-80) and (2-81) for this case, we find the overall strain as

$$\varepsilon_0 = \frac{F_T}{EA} = \alpha \Delta T_m$$

The stress distribution is found from eq. (2-83):

$$\sigma = -\alpha E \left[\Delta T(y) - \Delta T_m \right]$$

We observe that, although there are no *external* constraints, *internal* constraints do exist, because the outer surface and inner regions try to expand (or contract) at different rates, yet the bar must remain in one piece. This means that, even though there are no external forces on the bar, thermal stresses are developed due to internal constraints.

Example 2-6 A rectangular bar having a thickness h and length L has both ends rigidly fixed, as shown in Figure 2-11. Because of energy dissipation within the bar, the following symmetrical temperature distribution is produced:

$$T = T_s + (T_c - T_s) \left[1 - \left(\frac{2y}{h} \right)^2 \right]$$

The temperature change from the stress-free condition T_0 is

$$\Delta T = T - T_0 = \Delta T_s + (T_c - T_s) \left[1 - \left(\frac{2y}{h} \right)^2 \right]$$

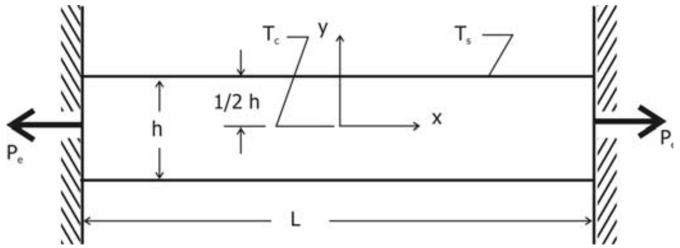

Figure 2-11. Sketch for Example 2-6.

The quantity T_s is the surface temperature of the bar, T_c is the center temperature of the bar, and ΔT_s is $(T_s - T_0)$. The coordinate y is measured from the center of the bar. Determine the stress distribution in the bar and the resultant force at the ends of the bar.

If the width of the bar is constant, the mean temperature change may be evaluated as follows:

$$\Delta T_m = \frac{1}{h}\int_{-h/2}^{h/2} \Delta T(y)\, dy$$

$$\Delta T_m = \frac{1}{h}\int_{-h/2}^{h/2} \left[\Delta T_s + (T_c - T_s)\left(1 - \frac{4y^2}{h^2}\right)\right] dy$$

$$\Delta T_m = \Delta T_s + (T_c - T_s)\left[\frac{y}{h} - \frac{4y^3}{3h^3}\right]_{-h/2}^{+h/2} = \Delta T_s + \frac{2}{3}(T_c - T_s)$$

Using eq. (2-83) for zero overall strain ($\varepsilon_0 = 0$), we obtain the following expression for the stress in the bar:

$$\sigma = -\alpha E \Delta T(y) = -\alpha E \left\{\Delta T_s + (T_c - T_s)\left[1 - \left(\frac{2y}{h}\right)^2\right]\right\}$$

The expression for the stress distribution may be written in dimensionless form:

$$\frac{\sigma}{\alpha E\,(T_c - T_s)} = -\left(\frac{\Delta T_s}{T_c - T_s}\right) - \left[1 - \left(\frac{2y}{h}\right)^2\right]$$

A plot of the stress distribution is shown in Figure 2-12.

The maximum stress occurs at the upper or lower surface of the bar, $y = \pm h/2$, for

$$\left(\frac{\Delta T_s}{T_c - T_s}\right) < -\frac{1}{2}$$

$$\frac{\sigma_{\max}}{\alpha E\,(T_c - T_s)} = -\frac{\Delta T_s}{T_c - T_s}$$

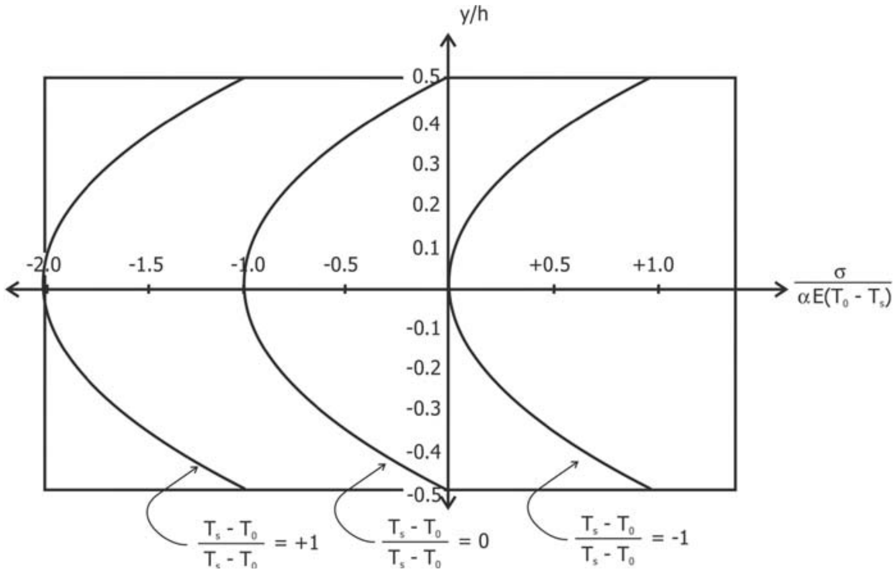

Figure 2-12. Dimensionless stress distribution for Example 2-6, for the temperature ratio, $(T_s - T_0)/(T_c - T_s) = +1$, 0, and -1.

On the other hand, the maximum stress occurs at the center of the bar, $y = 0$, for

$$\left(\frac{\Delta T_s}{T_c - T_s}\right) \geq -\frac{1}{2}$$

$$\frac{\sigma_{\max}}{\alpha E\,(T_c - T_s)} = -\frac{\Delta T_s}{T_c - T_s} - 1 = -\frac{T_c - T_0}{T_c - T_s}$$

The end reaction P_e for the bar may be determined from eq. (2-81), with $\varepsilon_0 = 0$:

$$P_e = -F_T = -\alpha E A \Delta T_m = -\alpha E A \left[\Delta T_s + \tfrac{2}{3}\,(T_c - T_s)\right]$$

The end reaction is compressive (negative) for $\Delta T_s > -\frac{2}{3}(T_c - T_s)$, and the end reaction is tensile (positive) for $\Delta T_s < -\frac{2}{3}(T_c - T_s)$.

Suppose the bar material is Inconel, with a thermal expansion coefficient of 13.0×10^{-6} K^{-1} (7.22×10^{-6} °F^{-1}) and Young's modulus of 215 GPa (31.2×10^6 psi). The thickness of the bar is 30 mm (1.18 in.), the width of the bar is 40 mm (1.57 in.), and the length is 450 mm (17.72 in.). The bar is stress-free at a temperature $T_0 = 20$°C (68°F). The surface temperature of the bar is 30°C (86°F), and the center temperature is 60°C (140°F). The temperature distribution is that given at the beginning of this example. Determine the maximum stress and the end reaction force.

Let us first calculate the temperature ratio:

$$\frac{\Delta T_s}{T_c - T_s} = \frac{30^\circ - 20^\circ}{60^\circ - 30^\circ} = +\frac{1}{3} > -\frac{1}{2}$$

The maximum stress occurs at the center of the bar ($y = 0$):

$$\frac{\sigma_{\max}}{\alpha E\,(T_c - T_s)} = -\frac{1}{3} - 1 = -\frac{4}{3}$$

$$\sigma_{\max} = \left(-\tfrac{4}{3}\right)\left(13.0 \times 10^{-6}\right)\left(215 \times 10^{9}\right)(60^\circ - 30^\circ)$$
$$= -111.8\,\text{MPa (16,200 psi)}$$

The reactive force at the ends of the bar is

$$P_e = -\left(13.0 \times 10^{-6}\right)\left(215 \times 10^{9}\right)(0.030)\,(0.040)\left[10^\circ + \tfrac{2}{3}(60^\circ - 30^\circ)\right]$$
$$P_e = -100.6 \times 10^{3}\ N = -100.6\ \text{kN} \quad (-22{,}600\ \text{lb}_\text{f})$$

Let us calculate the maximum stress for the case in which the bar has no force on the ends. The mean temperature change is

$$\Delta T_m = \left(30^\circ - 20^\circ\right) + \tfrac{2}{3}\left(60^\circ - 30^\circ\right) = 30^\circ\text{C}\left(54^\circ\ \text{F}\right)$$

The stress at any point in the bar is given by

$$\sigma = -\alpha E\left\{\Delta T_s + (T_c - T_s)\left[1 - \left(\frac{2y}{h}\right)^2\right] - \Delta T_m\right\}$$

or

$$\sigma = -\alpha E\left\{(T_c - T_s)\left[1 - \left(\frac{2y}{h}\right)^2\right] - \frac{2}{3}(T_c - T_s)\right\}$$
$$= -\alpha E\,(T_c - T_s)\left[\frac{1}{3} - \left(\frac{2y}{h}\right)^2\right]$$

The maximum stress occurs at the surface of the bar, $y = \pm h/2$:

$$\sigma_{\max} = +\tfrac{2}{3}\alpha E\,(T_c - T_s)$$
$$\sigma_{\max} = \left(\tfrac{2}{3}\right)\left(13.0 \times 10^{-6}\right)\left(215 \times 10^{9}\right)\left(60^\circ - 30^\circ\right) = 55.0\,\text{MPa} \quad (8110\,\text{psi})$$

The stress at the center of the bar, $y = 0$, is

$$\sigma\,(y = 0) = -\tfrac{1}{3}\alpha E\,(T_c - T_s) = -\left(\tfrac{1}{3}\right)\left(13.0 \times 10^{-6}\right)\left(215 \times 10^{9}\right)\left(30^\circ\right)$$
$$\sigma\,(y = 0) = -27.95\,\text{MPa}\ \ (4050\,\text{psi})$$

The stress at the surface of the bar is tensile, and the stress at the center is compressive.

2.6 HISTORICAL NOTE

Barré de Saint-Venant (1797–1886), see Figure 2-13, demonstrated outstanding mathematical abilities at an early age [Timoshenko, 1953]. He entered the École Polytechnique when he was 16 years old and became first in his class. During his second year, the students of the École Polytechnique were mobilized into the French army, and Saint-Venant expressed his conscientious objection to the conflict. Because of this incident, Saint-Venant was not allowed to continue his studies at the university.

After about eight years of work as an assistant in the powder industry, Saint-Venant entered the École des Ponts et Chaussées, completed his university course work, and graduated first in his class in 1825. After graduation, he worked with an engineering group on river channels and did theoretical work in mechanics and fluid dynamics in his spare time. In 1837, Saint-Venant was asked to lecture on strength of materials at the École des Ponts et Chaussées, where he presented some of his developments in the theory of elasticity. He believed that engineering work should involve both experimental and theoretical or analytical components.

In a basic discussion of bending of beams, Saint-Venant stated the principle that we call *Saint-Venant's principle* today. He mentioned that the analytical solutions he obtained for bending of beams would be exact only if the support

Figure 2-13. Barré de Saint-Venant. [From *History of Strength of Materials*, S. P. Timoshenko (1953). Used by permission of Dover Publications, Inc., New York, NY]

forces at the ends of the beam were distributed in a specific manner. He stated, however, that the solution would be sufficiently accurate for any other support force distribution with the same resultant force and resultant moment as the one prescribed in the analytical solution. Saint-Venant carried out some experiments with rubber bars to demonstrate that substantial difference in the deformation is produced only in the vicinity of the applied forces.

Although Saint-Venant published no books on the theory of elasticity, he wrote several papers (mémoires) and edited books by Navier and Clebsch, a German scientist and engineer. In these two books, Saint-Venant added extensive notes and appendices to the original material.

PROBLEMS

2-1. A rod having a diameter of 16 mm (0.630 in.) and a length of 450 mm (17.72 in.) is rigidly attached to supports at each end. The rod is constructed of C1020 carbon steel, with $E = 205$ GPa (29.7×10^6 psi) and $\alpha = 11.9 \times 10^{-6}\ K^{-1}$ ($6.6 \times 10^{-6}\ {}^\circ F^{-1}$. The rod is initially stress-free at a temperature of 25°C (77°F). If the bar is uniformly cooled to −5°C (27°F), determine the resulting stress in the bar and the force on the ends of the bar.

2-2. Suppose the rod given in Problem 2-1 has an initial tensile stress of 50 MPa (7250 psi) when the rod is at 25°C (77°F). Determine the temperature at which the stress in the rod would be zero.

2-3. A tie rod has a diameter of 12.62 mm (0.497 in.; cross-sectional area, 1.250 cm^2 or 0.1938 in^2) and an effective length of 1.250 m (4.10 ft). The rod material is 4340 alloy steel, with a Young's modulus of 189 GPa (27.4×10^6 psi) and a thermal expansion coefficient of $11.8 \times 10^{-6}\ K^{-1}$ ($6.56 \times 10^{-6}\ {}^\circ F^{-1}$). One end of the rod has a connector with a gap of 0.12 mm (0.0047 in.) when the system is at 25°C (77°F), the stress-free condition. The other end of the rod is fixed. If the rod is cooled to a uniform temperature of −15°C (+5°F), determine the stress in the rod. Determine the gap width required for a stress of 35 MPa (5080 psi) in the rod at −15°C.

2-4. A structural member has a length of 828 mm (32.6 in.) and a cross-sectional area of 12.0 cm^2 (1.86 in^2). The member is made of 6061-T6 aluminum, with Young's modulus, 69.0 GPa (10.0×10^6 psi), and thermal expansion coefficient, $23.4 \times 10^{-6}\ K^{-1}$ ($13.0 \times 10^{-6}\ {}^\circ F^{-1}$). One end of the member is rigidly fixed, and the other end is supported by a spring. The system is initially stress-free at a temperature of 20°C (68°F). Determine the required spring constant, if the thermal stress in the member is to be limited to 12 MPa (1740 psi) when the member is heated to a uniform temperature of 120°C (248°F).

2-5. A vacuum-jacketed cryogenic transfer line consists of two concentric tubes, connected at each end. The inner tube is aluminum, with Young's

modulus, 70 GPa (10.2×10^6 psi) and thermal expansion coefficient, 22×10^{-6} K^{-1} (12.2×10^{-6} °F^{-1}). The cross-sectional area of the inner tube is 3.86 cm^2 (0.598 in^2). The outer tube is stainless steel, with Young's modulus, 200 GPa (29.0×10^6 psi) and thermal expansion coefficient, 16×10^{-6} K^{-1} (8.89×10^{-6} °F^{-1}). The cross sectional area of the outer tube is 10.66 cm^2 (1.652 in^2). The length of both tubes is 8.00 m (26.25 ft). The system is stress-free when the temperature is 25°C (77°F). The outer tube is heated to +45°C (113°F), and the inner tube is cooled to −55°C (−67°F). Determine the resulting thermal stress in the inner tube and in the outer tube.

2-6. Two concentric pipes, each having a length of 4.50 m (14.76 ft), are rigidly attached to each other at each end. The outer pipe is copper, with the following properties: $E = 112$ GPa (16.2×10^6 psi), $\alpha = 18 \times 10^{-6}$ K^{-1} (10×10^{-6} °F^{-1}), $A = 84$ cm^2 (13.02 in^2, and $S_y = 140$ MPa (20,300 psi). The inner pipe is aluminum, with the following properties: properties: $E =$ 70 GPa (10.2×10^6 psi), $\alpha = 22.5 \times 10^{-6}$K^{-1} ($12.5 \times 10^{-6}$ °F^{-1}), $A =$ 48 cm^2 (7.44 in^2), and $S_y = 215$ MPa (31,200 psi). The system is stress-free at a temperature of 25°C (77°F). If the copper pipe is maintained at 25°C, determine the temperature to which the aluminum pipe must be heated to achieve yielding in one of the pipes. In which pipe will yielding first occur, if the aluminum pipe is heated?

2-7. The two concentric pipes shown in Figure 2-14 are rigidly fixed at one end and are rigidly attached to each other at the other end. The shorter pipe is copper, having the following properties: $E = 110$ GPa (16.0×10^6 psi), $\alpha = 18 \times 10^{-6}$ K^{-1} (10×10^{-6} °F^{-1}), $A = 10$ cm^2 (1.55 in^2), and length, 600 mm (23.6 in.). The longer pipe is aluminum, having the following properties: following properties: $E = 72.9$ GPa (10.6×10^6 psi), $\alpha =$ 23×10^{-6} K^{-1} (12.8×10^{-6} °F^{-1}), $A = 17$ cm^2 (2.64 in^2), and length, 900 mm (35.4 in.). The system is stress-free at a temperature of 20°C (68°F). The copper pipe is cooled to 5°C (39°F). Determine the temperature to which the aluminum pipe must be heated to produce a stress of 140 MPa (20,300 psi) in the copper pipe.

2-8. A carbon steel pipe is constrained by six stainless steel tie rods, as shown in Figure 2-15. The pipe is 80 mm nominal ($3\frac{1}{2}$-in. nom.), with the following properties: Young's modulus, 205 GPa (29.7×10^6 psi); thermal expansion coefficient, 11.9×10^{-6} K^{-1} (6.61×10^{-6} °F^{-1}); cross-sectional area, 17.65 cm^2 (2.74 in^2). The tie rods are stainless steel 14-mm (0.551-in.) diameter rods, with the following properties: Young's modulus, 193 GPa (28.0×10^6 psi); thermal expansion coefficient, 16.0×10^{-6} K^{-1} (8.9×10^{-6} °F^{-1}); cross-sectional area, 1.54 cm^2 (0.239 in^2) each rod, or 9.24 cm^2 (1.432 in^2) total for six rods. The length of the pipe and the rods is 1.50 m (4.92 ft). The system is stress-free at 120°C (248°F). Determine the stress in the pipe and in the rods when

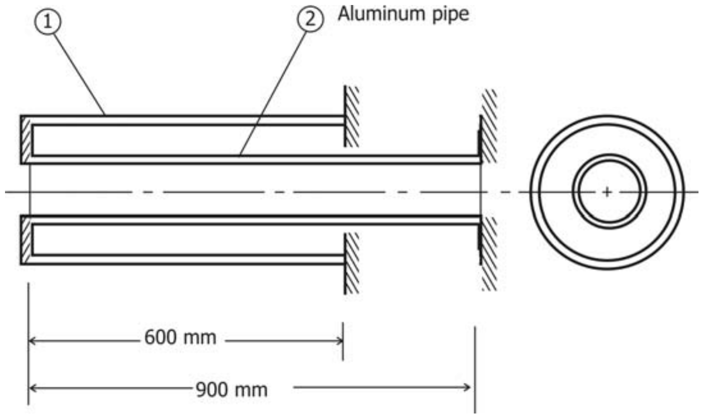

Figure 2-14. Sketch for Problem 2-7.

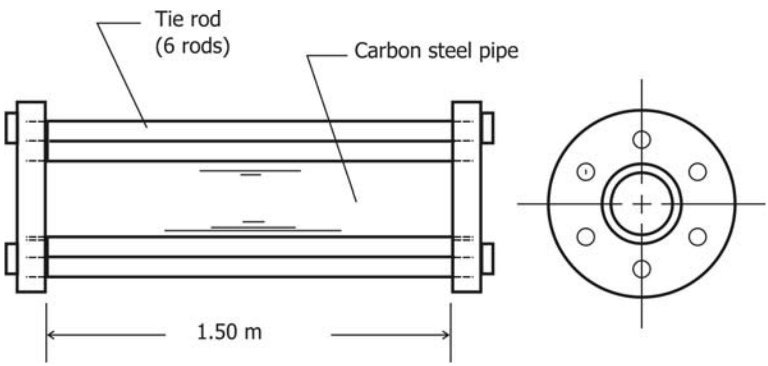

Figure 2-15. Sketch for Problem 2-8.

both are at 25°C (77°F). If the rods are heated to 60°C (140°F) and the pipe is heated to 90°C (194°F), determine the stress in the pipe and in the rods.

2-9. A double-pipe heat exchanger has a carbon steel outer tube, with the following properties: $E = 200\,\text{GPa}$ $(29.0 \times 10^6\,\text{psi})$, $\alpha = 11.5 \times 10^{-6}\,\text{K}^{-1}$ $(6.39 \times 10^{-6}\ {}^\circ\text{F}^{-1})$, and $A = 7.00\,\text{cm}^2$ $(1.085\,\text{in}^2)$. The inner tube is aluminum with the following properties: $E = 69\,\text{GPa}$ $(10.0 \times 10^6\,\text{psi})$, $\alpha = 23.4 \times 10^{-6}\,\text{K}^{-1}$ $(13.0 \times 10^{-6}\ {}^\circ\text{F}^{-1})$, and $A = 3.50\,\text{cm}^2$ $(0.543\,\text{in}^2)$. Both tubes have the same length, 2.50 m (8.20 ft). The heat exchanger is stress-free at a temperature of 25°C (77°F). Under operating conditions, the outer tube is warmed to 45°C (113°F), and the inner tube is heated to 125°C (257°F). Determine the stress in each tube and the net change in length of the heat exchanger tubes.

2-10. An aluminum sleeve having an inside diameter of 15 mm (0.591 in.) and an outside diameter of 25 mm (0.984 in.) is slipped over a 15-mm

(0.591-in.) diameter carbon steel bolt having a length of 250 mm (9.84 in.), as shown in Figure 2-16. The assembly is held in place by a nut that is turned just snug. For aluminum, $E = 69$ GPa (10×10^6 psi) and $\alpha = 23.4^a \times 10^{-6}$ K^{-1} ($13.0^a \times 10^{-6}$ °F^{-1}). For steel, $E =$ 200 GPa (29.0×10^6 psi) and $\alpha = 11.0 \times 10^{-6}$ K^{-1} (6.11×10^{-6} °F^{-1}). The assembly is stress-free at a temperature of 20°C (68°F). The assembly is heated uniformly to a final temperature, and the stress in the compressive sleeve is 80 MPa (11,600 psi). Determine (a) the value of the final temperature, (b) the stress in the bolt at the final condition, and (c) the change in length of the assembly.

2-11. Suppose the inner pipe in Problem 2-9 has a sliding seal, such that the inner pipe may elongate freely through a distance of 1.00 mm (0.0394 in.). Determine the stress in each pipe for this condition. What is the change in length for each pipe?

2-12. If the nut in Problem 2-10 is not tightened, but there is a gap of 0.13 mm (0.0051 in.) initially, determine the final temperature required to induce a compressive stress of 80 MPa (11,600 psi) in the sleeve. What is the total change in length of the assembly for this condition?

2-13. A support rod for a cryogenic fluid storage vessel consists of a 19.05-mm (0.750-in.) diameter and 1.524-m (60.00-in.) long 304 stainless steel rod, which is rigidly fixed at both ends. The rod is subjected to an axial temperature distribution given by

$$T(x) = T_c + (T_h - T_c)(x/L)$$

where T_c is the cold-end temperature, T_h is the warm-end temperature, and L is the length of the rod. The average properties of the 304 stainless steel are $E = 200$ GPa (29.0×10^6 psi), $\alpha = 13.8$°C^{-1} (7.67°F^{-1}), $S_y =$ 1000 MPa (145,000 psi). The hot-end temperature is +20°C (68°F) and the cold-end temperature is −180°C (−292°F) for the support rod. The support rod is stress-free when it is at a uniform temperature of +20°C. Determine the thermal stress in the rod.

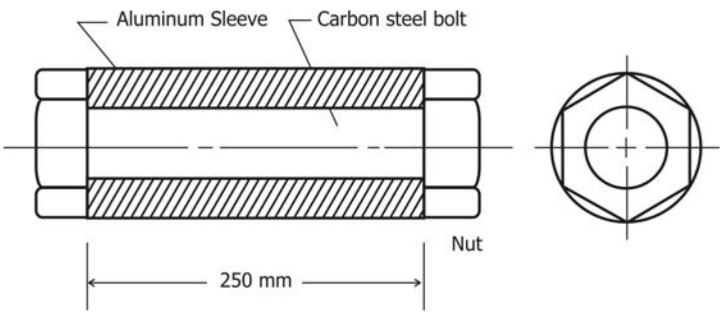

Figure 2-16. Sketch for Problem 2-10.

2-14. In a plate-fin heat exchanger, the rectangular fin (as shown in Figure 2-17) is rigidly fixed at both ends and is subjected to an axial temperature distribution as follows.

$$T(x) = T_0 + (T_f - T_0)\left[1 - \frac{\cosh(mx)}{\cosh(mL)}\right]$$

where $T_f = 170^\circ\mathrm{C}\,(338^\circ\mathrm{F})$ = temperature of the fluid around the fin, $T_0 = 20^\circ\mathrm{C}\,(68^\circ\mathrm{F})$ = stress-free temperature, $m = 70\,\mathrm{m}^{-1}$ ($21.3\mathrm{ft}^{-1}$) = fin parameter, $L = 25\,\mathrm{mm}$ (0.984 in.) = half-length of the fin, and x = distance from the midpoint of the fin. The fin is constructed of copper/10% nickel alloy, with $E = 124\,\mathrm{GPa}$ (18×10^6 psi), $\alpha = 16.2 \times 10^{-6}\,{}^\circ\mathrm{C}^{-1}$($9 \times 10^{-6}\,{}^\circ\mathrm{F}^{-1}$). Show that the mean temperature change is given by

$$\Delta T_m = (T_f - T_0)\left[1 - \frac{\tanh(mL)}{mL}\right]$$

Determine the numerical value of the thermal stress in the fin.

2-15. A bus-bar with a rectangular cross section of height $h = 75\,\mathrm{mm}$ (2.953 in.) has the following temperature distribution across the cross section, as a result of electrical energy dissipation:

$$T(y) = T_0 + (T_c - T_0)\cos(\pi y/h)$$

where $T_c = 75^\circ\mathrm{C}$ ($167^\circ\mathrm{F}$) = center temperature, $T_0 = 25^\circ\mathrm{C}$ ($77^\circ\mathrm{F}$) = stress-free temperature and also the surface temperature for the bar, and y = distance measured from the center of the bar. The bar is constructed of 2024 aluminum, with $E = 73.4\,\mathrm{GPa}$ (10.65×10^6 psi), $\alpha = 22.5 \times 10^{-6\circ}\mathrm{C}^{-1}$ ($12.5 \times 10^{-6}\,{}^\circ\mathrm{F}^{-1}$). Determine the stress in the bar at the center ($y = 0$) and at the surface ($y = 1/2h$) (a) if the bar is not restrained at the ends, and (b) if the rod is rigidly fixed at the ends.

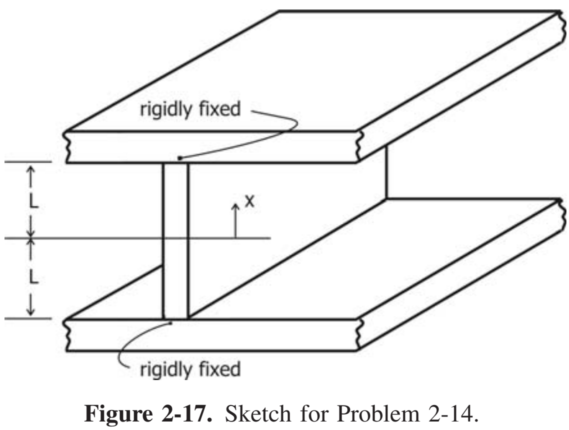

Figure 2-17. Sketch for Problem 2-14.

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3

THERMAL BENDING

3.1 LIMITS ON THE ANALYSIS

In this chapter, we examine the relationships between stress and strain caused by the presence of a temperature variation across the cross section of a beam. The analysis is based on the *strength-of-materials approach*, which involves the following restrictions:

(a) All cross-section planes remain plane after bending.
(b) The beam is relatively narrow so that lateral contractions or expansions (Poisson ratio effects) are negligible.
(c) The longitudinal axis of the beam is straight initially.
(d) The external forces pass through the centroid axis of the cross section, so that no torsion forces are present.
(e) The stresses are below the elastic limit, and the normal bending stresses are directly proportional to the distance from the centroid axis of the beam cross section.

To be precisely correct, the only situation for which the cross sections of a beam would remain plane after bending is the case in which a simple moment is applied to the ends of the beam. Any external transverse forces will produce shearing stresses in the beam, and the shear stresses are responsible for distortion of the cross section when the beam is loaded [Timoshenko and Goodier, 1970]. The importance of the shear stress effect depends on the loading condition; however, in general, the error between the strength-of-materials approach and a more

exact theory-of-elasticity approach is less than 5 percent if the depth of the beam h is less than about $L/10$, where L is the length of the beam.

If the beam is very wide, the effect of lateral contraction or expansion on the stress is important. For a beam width b that is less than about $h/20\mu$, where μ is Poisson's ratio, the effect of lateral contraction is less than about 5 percent, if $b/h < 1$ [Burgreen, 1971]. For Poisson's ratio between 0.25 and 0.33, the beam width should be less than $0.15h$ to $0.20h$ for Poisson's ratio effects to be negligible, where h is the depth of the beam.

If the longitudinal axis of the beam is not straight initially, the stress distribution will not be symmetrical around the neutral axis. In addition, the *neutral axis* (the position at which the fibers experience no longitudinal displacement for bending) and the centroid axis of the beam cross section will not be the same. The bending stress in a curved beam is not directly proportional to the distance from the neutral axis or the centroid axis [Seely and Smith, 1952]. If the depth of a beam h is less than about $0.05R$, where R is the radius of curvature of the beam, the stress calculated by the straight-beam approach is generally less than about 5 percent different from the stress determined with effects of beam curvature considered.

3.2 STRESS RELATIONSHIPS

Let us consider the case of thermal stresses developed in a beam in which the temperature varies in the transverse direction across the cross section only or $T = T(y)$. Assuming that bending of the beam may occur, let us express the total strain as the sum of two components:

(a) A uniform extensional strain ε_0, which is the strain along the centroid axis
(b) A bending strain ε_b, which varies linearly across the cross section in the y-direction

The strain may be written according to eq. (2-5), for temperature-dependent properties, or by eq. (2-6), for constant material properties. Let us work with the case for which the material properties may be considered as constants:

$$\varepsilon = \frac{\sigma}{E} + \alpha \Delta T = \varepsilon_0 + \varepsilon_b \tag{3-1}$$

To evaluate the bending strain component, let us consider the beam shown in Figure 3-1. For small angles of rotation, the tangent of the angle ψ is approximately equal to the angle itself, expressed in radian units:

$$\tan \psi \approx \psi = \frac{\varepsilon_b L}{y} \tag{3-2}$$

From the geometry of the beam deflection, we may also write the rotation angle in terms of the beam length L and the radius of curvature R of the beam after bending:

$$\frac{L}{2R} = \tan \frac{\psi}{2} \approx \frac{\psi}{2} \tag{3-3}$$

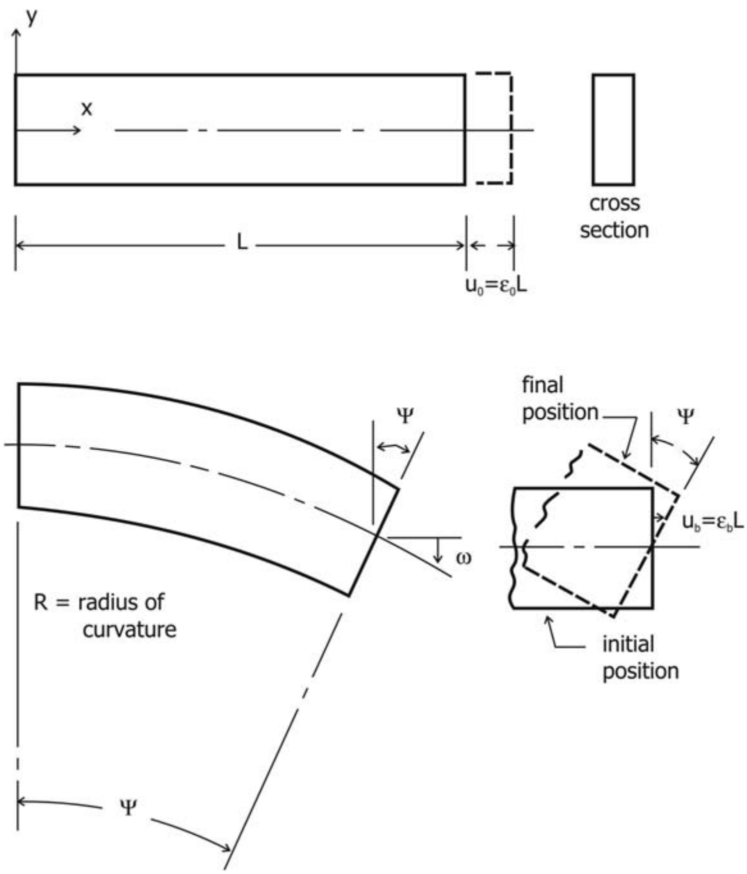

Figure 3-1. Deflection of a beam in bending.

By equating the two expressions for the angle ψ from eqs. (3-2) and (3-3), we obtain the following expression for the bending strain component:

$$\varepsilon_b = \frac{y}{R} \tag{3-4}$$

The stress may be written by combining eqs. (3-1) and (3-4):

$$\sigma = E\varepsilon_0 + \frac{Ey}{R} - \alpha E \Delta T \tag{3-5}$$

If we make a force balance between the end of the beam, at which the applied axial force is P_e, and any cross section of the beam, and use eq. (3-5), we obtain

$$P_e = \int \sigma \, dA = EA\varepsilon_0 + 0 - \alpha E \int \Delta T \, dA \tag{3-6}$$

The second term on the right-hand side is zero, because the origin is located at the centroid of the cross section. The last term may be defined as the "thermal force," according to eq. (2-78). For constant material properties,

$$F_T = \alpha E \int \Delta T \, dA \tag{3-7}$$

We may solve for the strain at the centroid axis by combining eqs. (3-6) and (3-7):

$$\varepsilon_0 = \frac{P_e + F_T}{EA} \tag{3-8}$$

The mechanical bending moment at any point along the length of the beam may be found, using eq. (3-5):

$$M = \int \sigma y \, dA = 0 + \frac{E}{R} \int y^2 \, dA - \alpha E \int \Delta T y \, dA \tag{3-9}$$

The first term on the right-hand side is zero, because the origin is located at the centroid of the cross section. The last term may be called the "thermal moment," and written as

$$M_T = \alpha E \int \Delta T y \, dA \tag{3-10}$$

If the material properties vary with temperature, the thermal moment may be defined as

$$M_T = \int \alpha E \Delta T y \, dA \tag{3-11}$$

The area moment of inertia of the cross section about the z-axis of the beam ($I = I_z$) is defined by

$$I = \int y^2 \, dA \tag{3-12}$$

If we make the substitutions from eqs. (3-10) and (3-12) into eq. (3-9), we may solve for the radius of curvature of the beam or its reciprocal

$$\frac{1}{R} = \frac{M + M_T}{EI} \tag{3-13}$$

By substituting the expression for the strain at the centroid axis ε_0 from eq. (3-8) and the radius of curvature R from eq. (3-13) into the stress relationship, eq. (3-5), we obtain the general expression for determining the stress in a beam subject to mechanical and thermal loads:

$$\sigma = \frac{P_e + F_T}{A} + \left(\frac{M + M_T}{I} \right) y - \alpha E \Delta T \tag{3-14}$$

The sign convention for the mechanical moment is that a positive bending moment is one that would produce a tensile stress or an extension in the $+y$ portion of the cross section. This sign convention is necessary to assure that the mechanical moment M and the thermal moment M_T are defined in a compatible manner.

We may observe that the maximum stress with thermal bending does not necessarily occur at the outermost fiber, as is the case for an isothermal beam-bending problem. We examine some examples of the application of the general relationship, eq. (3-14), in the following sections.

If the temperature varies in both the transverse and spanwise directions, $T = T(y,\ z)$, or the beam cross section is not symmetrical, as shown in Figure 3-2, then the stress in the beam must be determined from the following expression:

$$\sigma = \frac{P_e + F_T}{A} + \frac{My + K_y y + K_z z}{I_z} - \alpha E \Delta T \tag{3-15}$$

The quantities K_y and K_z are defined as follows:

$$K_y = \frac{I_y M_T - I_{yz} M_{Tz}}{I_y - (I_{yz}^2/I_z)} \tag{3-16}$$

$$K_z = \frac{I_z M_{Tz} - I_{yz} M_T}{I_y - (I_{yz}^2/I_z)} \tag{3-17}$$

The spanwise thermal moment is defined by

$$M_{Tz} = \alpha E \int \Delta T z \, dA \tag{3-18}$$

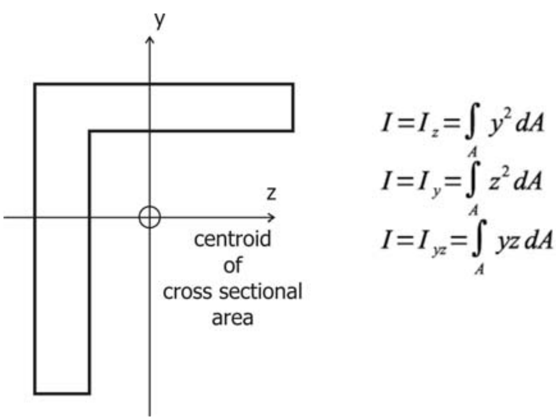

Figure 3-2. Nonsymmetrical cross section for beam bending.

The area moment of inertia I_y and the product of inertia I_{yz} are defined by

$$I_y = \int z^2 \, dA \tag{3-19}$$

$$I_{yz} = \int yz \, dA \tag{3-20}$$

3.3 DISPLACEMENT RELATIONS

Let us consider the beam element in bending, as shown in Figure 3-3. The bending strain may be determined from the general strain definition:

$$\varepsilon_b = \frac{\text{change in length}}{\text{initial length}} = \frac{-y \, d\omega}{ds} \approx \frac{-y \, d\omega}{dx} \tag{3-21}$$

The negative sign is introduced because the rotation of the beam ω is defined by

$$\omega = \frac{dv}{dx} \tag{3-22}$$

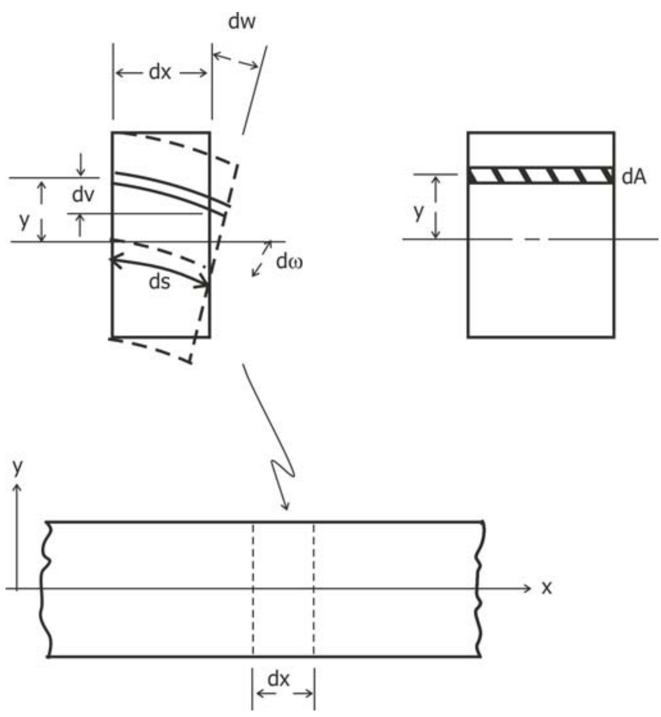

Figure 3-3. Differential element of a beam in bending.

The lateral displacement v is taken as positive when the displacement is in the positive y-direction. The approximation $ds \approx dx$ restricts the analysis to the case in which the lateral displacement v is small in comparison with the depth of the beam.

Using eq. (3-16) in eq. (3-15), we obtain the following expression for the bending strain in terms of the transverse displacement:

$$\varepsilon_b = -y\frac{d^2v}{dx^2} \tag{3-23}$$

By comparing eqs. (3-4) and (3-23), we find the following expression for the radius of curvature of the beam:

$$\frac{1}{R} = -\frac{d^2v}{dx^2} \tag{3-24}$$

In addition, by using the radius of curvature from eq. (3-13), we find the following important relationship:

$$\frac{d^2v}{dx^2} = -\frac{M + M_T}{EI} \tag{3-25}$$

If we express the mechanical bending moment and thermal moment as a function of the position x along the length of the beam, we may integrate both sides of eq. (3-25) to obtain the expression for the transverse displacement v of the beam.

3.4 GENERAL THERMAL BENDING RELATIONS

Let us consider the differential element of the beam, as shown in Figure 3-4. The quantity q is the applied transverse load per unit length {N/m or lb_f/ft}. This quantity may vary with longitudinal position x and is generally a known quantity in the design of the beam. The transverse load is considered to be positive if the direction of the force is in the positive y-direction.

The quantity V is the *total shear force* {units: N or lb_f} across a cross section of the beam. If we make a force balance in the y-direction for the beam element, we obtain

$$\Sigma F = 0 = -V + (V + dV) + q\,dx$$

$$q = -\frac{dV}{dx} \tag{3-26}$$

If we sum the mechanical bending moments for the beam element around an axis at the left end of the element, we obtain

$$\Sigma M = 0 = -M + (M + dM) - (V + dV)\,dx - \tfrac{1}{2}q\,(dx)^2$$

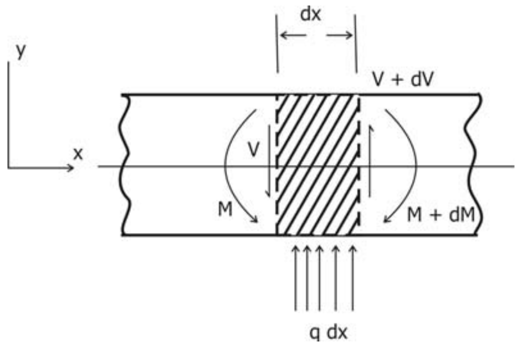

Figure 3-4. Loading for a beam element in bending.

If we neglect the higher-order differentials, $(dV\,dx)$ and $(dx)^2$, we obtain the relationship between the transverse shear force and the mechanical bending moment:

$$V = \frac{dM}{dx} \tag{3-27}$$

Equation (3-26) may be used to relate the mechanical bending moment and the applied transverse load:

$$q = -\frac{dV}{dx} = -\frac{d^2M}{dx^2} \tag{3-28}$$

We may solve for the mechanical moment from eq. (3-25) in terms of the transverse displacement:

$$M = -EI\frac{d^2v}{dx^2} - M_T \tag{3-29}$$

Using this expression in the transverse shear expression, eq. (3-27), we obtain

$$V = \frac{dM}{dx} = -EI\frac{d^3v}{dx^3} - \frac{dM_T}{dx} \tag{3-30}$$

Finally, we may relate the transverse load q to the transverse displacement v by combining eqs. (3-28) and (3-30):

$$q = -\frac{dV}{dx} = EI\frac{d^4v}{dx^4} + \frac{d^2M_T}{dx^2} \tag{3-31}$$

or

$$\frac{d^4v}{dx^4} = \frac{1}{EI}\left(q - \frac{d^2M_T}{dx^2}\right) \tag{3-32}$$

3.5 SHEAR STRESSES

Shear stresses will occur in beams under thermal loads, unless the axial normal stress is constant or varies linearly with axial distance along the beam. If the temperature change is a function of y alone, $\Delta T = \Delta T(y)$, and is not a function of the axial coordinate x, the thermal loads will produce no shear stresses in statically determinant beams.

Let us consider the beam element shown in Figure 3-5. The width of the beam cross section z_b is a function of the coordinate y. Only the case of a beam with a symmetrical cross section will be considered in this section. In general, the temperature change will be assumed to be a function of both the x- and y-coordinates, $\Delta T = \Delta T(x, y)$. The thermal force F_T and thermal moment M_T involve integration of the temperature change across the cross section, so these quantities are functions of the axial coordinate only.

Let us make a force balance for the shaded portion of the beam. The result is

$$\tau z_b \, dx + \int_y^{y_b} \left(\frac{\partial \sigma}{\partial x} dx \right) dA = 0 \tag{3-33}$$

The quantity y_b is the value of the coordinate y at the top of the beam cross section, as illustrated in Figure 3-5. The expression for the shear stress may be written in the following form:

$$\tau = -\frac{1}{z_b} \int_y^{y_b} \left(\frac{\partial \sigma}{\partial x} \right) dA = -\frac{1}{z_b} \frac{\partial}{\partial x} \left(\int_y^{y_b} \sigma \, dA \right) \tag{3-34}$$

The direct stress may be found from eq. (3-14):

$$\sigma = \frac{P_e + F_T}{A} + \left(\frac{M + M_T}{I} \right) y - \alpha E \Delta T \tag{3-35}$$

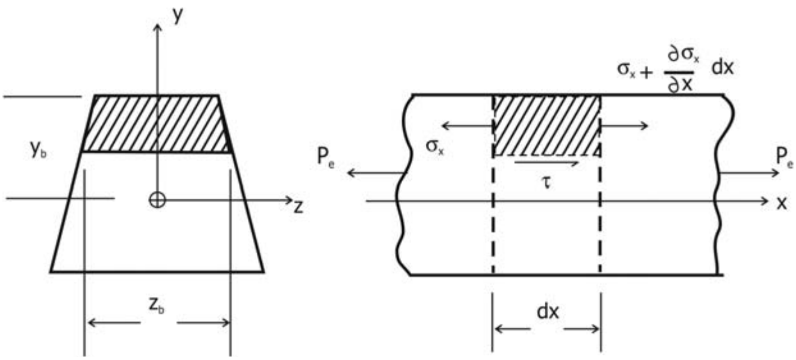

Figure 3-5. Shear stresses in a beam.

Making this substitution into eq. (3-34), we obtain

$$\tau = -\frac{1}{z_b}\frac{\partial}{\partial x}\left[\frac{P_e + F_T}{A}\int_y^{y_b} dA + \frac{M + M_T}{I}\int_y^{y_b} y\,dA - \alpha E\int_y^{y_b} \Delta T\,dA\right] \tag{3-36}$$

We note that, with the absence of traction (shear) forces on the surface of the beam, the external axial force P_e is constant.

Let us define the following quantities:

$$\Gamma_y = \int_y^{y_b} y\,dA \tag{3-37}$$

$$A_y = \int_y^{y_b} dA \tag{3-38}$$

$$A_y \Delta T_y = \int_y^{y_b} \Delta T\,dA \tag{3-39}$$

If we make these substitutions into eq. (3-36), we obtain

$$\tau = -\frac{1}{z_b}\frac{\partial}{\partial x}\left[\frac{A_y F_T}{A} + \frac{(M + M_T)\Gamma_y}{I} - \alpha E A_y \Delta T_y\right] \tag{3-40}$$

Carrying out the differentiation, we obtain

$$\tau = -\frac{1}{z_b}\left[\frac{A_y}{A}\frac{dF_T}{dx} + \frac{\Gamma_y}{I}\left(\frac{dM}{dx} + \frac{dM_T}{dx}\right) - \alpha E A_y \frac{d(\Delta T_y)}{dx}\right] \tag{3-41}$$

To further simplify the expression for the shear stress in the beam, let us define the *thermal shear force* V_T as

$$V_T = \frac{dM_T}{dx} \tag{3-42}$$

The mechanical shear force is related to the mechanical moment through eq. (3-27). If we make this substitution into eq. (3-42), we obtain the final expression for the shear stress in the beam.

$$\tau = -\frac{\Gamma_y}{z_b r_g^2}\left(\frac{V}{A} + \frac{V_T}{A}\right) - \frac{A_y}{z_b A}\frac{dF_T}{dx} + \frac{\alpha E A_y}{z_b}\frac{d(\Delta T_y)}{dx} \tag{3-43}$$

The quantity r_g is the *radius of gyration* for the cross-sectional area, defined by

$$r_g^2 = \frac{I}{A} \tag{3-44}$$

3.6 BEAM BENDING EXAMPLES

In this section, we consider some examples of the application of the general thermal bending relationships for beams. In actual beams, the stresses in the immediate vicinity of the ends may be two dimensional and depart somewhat from the strength-of-materials approach adopted in this chapter. With a factor of safety included, the results are generally satisfactory for design purposes.

3.6.1 Cantilever Beam

Let us consider the cantilever beam of length L with rectangular cross section, as shown in Figure 3-6. The beam has the following temperature distribution:

$$\Delta T = \frac{1}{4}\Delta T_0 \left(1 + \frac{2y}{h}\right)^2 \left(\frac{x}{L}\right)^2 \tag{3-45}$$

In this problem, the external mechanical loads P_e and M are zero.

3.6.1.1 Direct Thermal Stresses. We will use eq. (3-14) to evaluate the distribution of the direct thermal stresses. The thermal force term is evaluated from

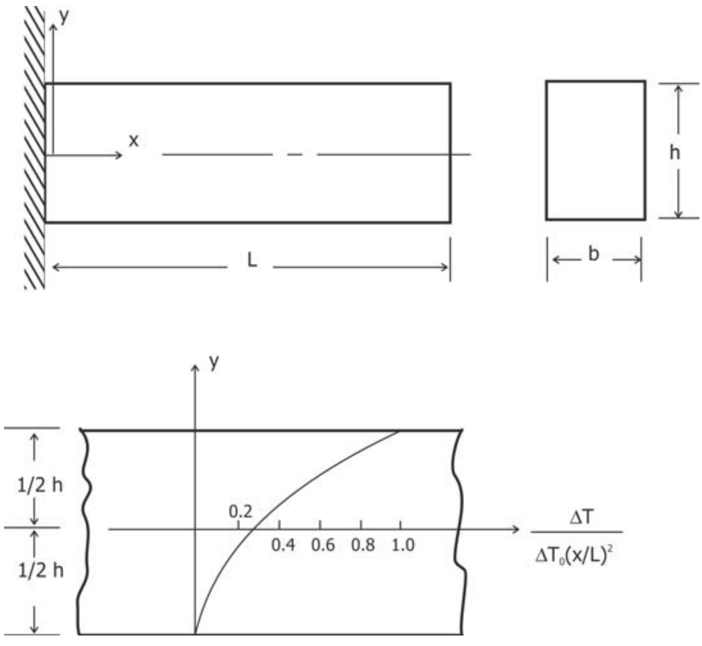

Figure 3-6. Cantilever beam of rectangular cross section.

its definition, eq. (3-7), where the differential area is $dA = b\,dy$:

$$F_T = \alpha E \int_{-h/2}^{h/2} \Delta T\, dA = \frac{1}{4} b\alpha E \Delta T_0 (x/L)^2 \int_{-h/2}^{h/2} \left(1 + \frac{2y}{h}\right)^2 dy \tag{3-46}$$

$$F_T = \tfrac{1}{3}\alpha E \Delta T_0 bh(x/L)^2 \tag{3-47}$$

The thermal moment is evaluated from eq. (3-10):

$$M_T = \alpha E \int_{-h/2}^{h/2} \Delta T y\, dA = \frac{1}{4} b\alpha E \Delta T_0 (x/L)^2 \int_{-h/2}^{h/2} \left(1 + \frac{2y}{h}\right)^2 y\, dy \tag{3-48}$$

$$M_T = \tfrac{1}{12} bh^2 \alpha E \Delta T_0 (x/L)^2 \tag{3-49}$$

The area moment of inertia for the cross section is $I = bh^3/12$, so the thermal moment expression, eq. (3-49), may be written in the following alternate form:

$$M_T = \frac{\alpha EI\, \Delta T_0}{h} \left(\frac{x}{L}\right)^2 \tag{3-50}$$

Making the substitutions from eqs. (3-45), (3-47), and (3-50) into the general thermal stress relationship, eq. (3-14), we obtain the following stress distribution:

$$\sigma = \frac{1}{3}\alpha E \Delta T_0 \left(\frac{x}{L}\right)^2 + \frac{\alpha E \Delta T_0 y}{h} \left(\frac{x}{L}\right)^2 - \frac{1}{4}\alpha E \Delta T_0 \left(1 + \frac{2y}{h}\right)^2 \left(\frac{x}{L}\right)^2 \tag{3-51}$$

This expression may be simplified, as follows:

$$\sigma = -\frac{\alpha E \Delta T_0}{12} \left(\frac{x}{L}\right)^2 \left[3\left(\frac{2y}{h}\right)^2 - 1\right] \tag{3-52}$$

The stress distribution is shown in Figure 3-7.

The maximum direct stress occurs at the upper or lower surface of the beam, $y = \pm h/2$.

$$\sigma_{\max} = -\tfrac{1}{6}\alpha E \Delta T_0 (x/L)^2 \tag{3-53}$$

3.6.1.2 Displacement. In this example, both the mechanical and thermal moments are known; therefore, we may use the displacement–moment relationship, eq. (3-25), to determine the transverse displacement of the beam:

$$\frac{d^2 v}{dx^2} = -\frac{M + M_T}{EI} = -\frac{\alpha EI\, \Delta T_0}{IEh} \left(\frac{x}{L}\right)^2 = -\frac{\alpha \Delta T_0}{h} \left(\frac{x}{L}\right)^2 \tag{3-54}$$

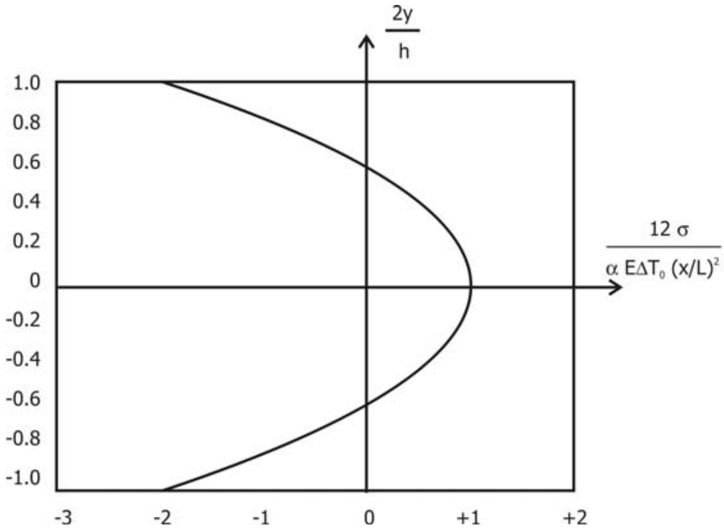

Figure 3-7. Bending stress distribution for cantilever beam example.

The rotation or slope of the beam at any point may be found by integrating eq. (3-54) one time:

$$\frac{dv}{dx} = \omega = -\frac{\alpha \Delta T_0 L}{3h}\left(\frac{x}{L}\right)^3 + C_1$$

At the fixed end of the beam ($x = 0$), the slope is zero, so $C_1 = 0$.

The transverse displacement may be found by integrating again:

$$v = -\frac{\alpha \Delta T_0 L^2}{12h}\left(\frac{x}{L}\right)^4 + C_2$$

At the fixed end of the beam ($x = 0$), the transverse displacement is also zero, so $C_2 = 0$. The expression for the transverse displacement of the beam is

$$v = -\frac{\alpha \Delta T_0 L^2}{12h}\left(\frac{x}{L}\right)^4 \tag{3-55}$$

The negative sign in eq. (3-55) means that the displacement is in the negative y-direction, if the temperature change ΔT_0 is positive. The maximum transverse displacement occurs at the free end ($x = L$) of the beam:

$$v_{\max} = -\frac{\alpha \Delta T_0 L^2}{12h} \tag{3-56}$$

3.6.1.3 Shear stress distribution. The shear stress distribution may be evaluated from eq. (3-43). Let us first determine the quantities defined in

eqs. (3-37)–(3-39). The width of the beam $z_b = b$ is constant, and the distance from the centroid axis to the top of the beam is $y_b = \frac{1}{2}h$.

$$A_y = \int_y^{h/2} b\,dy = \frac{1}{2}bh\left[1 - \left(\frac{2y}{h}\right)\right] \tag{3-57}$$

$$\Gamma_y = \int_y^{h/2} y\,dA = \frac{1}{8}bh^2\left[1 - \left(\frac{2y}{h}\right)^2\right] \tag{3-58}$$

$$A_y\Delta T_y = \int_y^{h/2} \Delta T\,dA = \frac{1}{4}b\Delta T_0(x/L)^2 \int_y^{h/2} \left[1 + \left(\frac{2y}{h}\right)\right]^2 dy$$

$$A_y\Delta T_y = \frac{bh\Delta T_0}{24}\left(\frac{x}{L}\right)^2\left[8 - \left(1 + \frac{2y}{h}\right)^3\right] \tag{3-59}$$

The radius of gyration for the rectangular cross section is found from eq. (3-44):

$$r_g^2 = \frac{I}{A} = \frac{bh^3/12}{bh} = \frac{h^2}{12} \tag{3-60}$$

The change in the thermal force per unit length of the beam may be calculated, where the thermal force expression is given by eq. (3-47):

$$\frac{dF_T}{dx} = \frac{2\alpha EA\Delta T_0}{3L}\left(\frac{x}{L}\right) \tag{3-61}$$

Using the thermal moment expression from eq. (3-49), we may determine the thermal shear force from its definition, eq. (3-42):

$$V_T = \frac{dM_T}{dx} = \frac{2\alpha\, EI\, \Delta T_0}{hL}\left(\frac{x}{L}\right) \tag{3-62}$$

Let us make these substitutions into eq. (3-43) to evaluate the shear stress distribution across the beam cross section:

$$\tau = -\frac{\alpha Eh\Delta T_0}{4L}\left[1 - \left(\frac{2y}{h}\right)^2\right]\left(\frac{x}{L}\right) - \frac{\alpha Eh\Delta T_0}{3L}\left[1 - \left(\frac{2y}{h}\right)\right]\left(\frac{x}{L}\right) + \frac{\alpha Eh\Delta T_0}{12L}\left[8 - \left(1 + \frac{2y}{h}\right)^3\right]\left(\frac{x}{L}\right)$$

This expression may be simplified, as follows.

$$\tau = -\frac{\alpha Eh\Delta T_0}{12L}\left(\frac{x}{L}\right)\left\{3\left[1 - \left(\frac{2y}{h}\right)^2\right] + 4\left(1 - \frac{2y}{h}\right) - \left[8 - \left(1 + \frac{2y}{h}\right)^3\right]\right\} \tag{3-63}$$

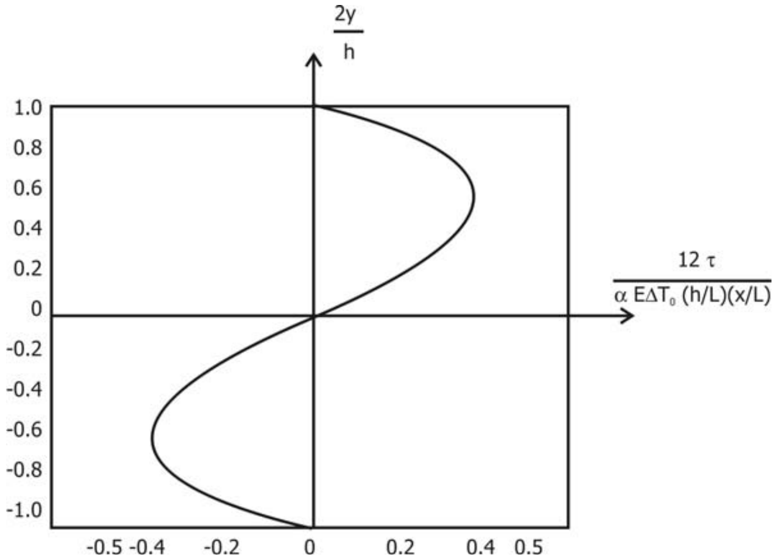

Figure 3-8. Shear stress distribution for cantilever beam example.

or

$$\tau = \frac{\alpha E \Delta T_0 (h/L)}{12} \left(\frac{x}{L}\right) \left(\frac{2y}{h}\right) \left[1 - \left(\frac{2y}{h}\right)^2\right] \tag{3-64}$$

This stress distribution is shown in Figure 3-8. We note that the shear stress is zero at the top and bottom surface of the beam, $y = \pm h/2$, and at the center of the beam, $y = 0$. The shear stress reaches a maximum value at $y = \pm h/[2(3)^{1/2}] = 0.577(h/2)$. The value for the shear stress at this point is

$$\tau_{\max} = 0.032075 \alpha E \Delta T_0 (h/L)(x/L) \tag{3-65}$$

We note from eq. (3-52) that the direct stress σ is zero at $2y/h = 1/3^{1/2}$. Because of the small magnitude of the factor (*h*/*L*), the maximum shear stress is generally much smaller than the maximum direct stress for this example.

Example 3-1 A beam constructed of C1020 steel ($E = 205\,\text{GPa} = 29.7 \times 10^6\,\text{psi}$); $\alpha = 11.9 \times 10^{-6}\,\text{K}^{-1} = 6.61 \times 10^{-6\circ}\text{F}^{-1}$) has a temperature distribution given by eq. (3-45). The beam has a depth $h = 150\,\text{mm}$ (5.91 in.), a width $b = 50\,\text{mm}$ (1.97 in.), and a length $L = 2.50\,\text{m}$ (8.202 ft). The beam is stress-free at a uniform temperature $T_0 = 20^\circ\text{C}$ (68°F). If the lower surface of the beam is maintained at 20°C, determine the temperature for the top surface of the beam in order that the direct stress in the beam not exceed ±80 MPa (11,600 psi). Also, determine the maximum transverse deflection for the beam and the maximum shear stress for this condition.

The temperature difference ΔT_0 may be determined from eq. (3-53):

$$\Delta T_0 = \frac{6\sigma_{\max}}{\alpha E} = \frac{(6)(\pm 80)(10^6)}{(11.9 \times 10^{-6})(205 \times 10^9)} = \pm 197 \text{ K} \quad (\pm 386^\circ\text{F})$$

The temperature at the top surface of the beam is

$$T_1 = 20 \pm 197 = 217^\circ\text{C } (422^\circ\text{F}) \text{ or } -177^\circ\text{C} \quad (-286^\circ\text{F})$$

The maximum transverse deflection for the beam (at the free end, $x = L$) may be calculated from eq. (3-56):

$$v_{\max} = -\frac{(11.9 \times 10^{-6})(\pm 197)(2.50)^2}{(12)(0.150)}$$
$$= \mp 8.14 \times 10^{-3} \text{ m} = 8.14 \text{ mm} \quad (0.320 \text{ in.})$$

The maximum shear stress for the beam is found from eq. (3-65):

$$\tau_{\max} = (0.032075)(11.9 \times 10^{-6})(205 \times 10^9)(197)(0.150/2.50)$$
$$\tau_{\max} = 0.925 \times 10^6 \text{Pa} = 0.925 \text{ MPa} \quad (134 \text{ psi})$$

We note that the ratio $(\tau_{\max}/\tau_{\max}) = 0.012$ in this case. Also, the direct stress σ is zero at the point at which the shear stress τ is a maximum $(2y/h = 1/3^{1/2} = 0.577)$, and the shear stress is zero at the point where the direct stress is a maximum $(2y/h = \pm 1)$.

3.6.2 Simply-supported Beam

Let us consider the simply-supported beam with a uniformly distributed mechanical load $-q_0$, as shown in Figure 3-9. The beam has a rectangular cross section, with a depth h and a width b. The temperature distribution across the beam cross section is

$$\Delta T = \Delta T_0 \sin(\pi y/h) \tag{3-66}$$

In this problem, the external axial load P_e is zero; however, the uniform load does impose a mechanical moment on the beam.

3.6.2.1 Direct thermal stresses. The thermal force term F_T is zero, in this case, because the temperature distribution is an odd function of the coordinate y. This may be demonstrated by using the definition of the thermal force, eq. (3-7), where the differential area is $dA = b\,dy$:

$$F_T = \alpha E \Delta T_0 b \int_{-h/2}^{h/2} \sin(\pi y/h) = \alpha E \Delta T_0 (bh/\pi) \left[\cos(\pi y/h)\right]_{-h/2}^{h/2} = 0$$

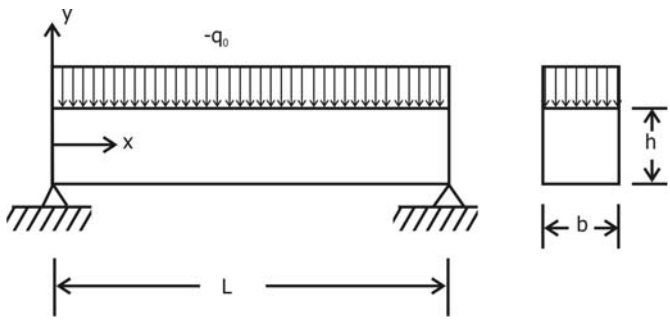

Figure 3-9. Simply-supported beam with a uniform mechanical load.

The reaction force at the supports for the beam is $F = \frac{1}{2}q_0 L$, so the total mechanical moment may be written as

$$\sum M(x) = 0 = M + Fx - (q_0 x)(x/2)$$

$$M = -\frac{1}{2}q_0 Lx + \frac{1}{2}q_0 x^2 = -\frac{1}{2}q_0 L^2\left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2\right] \tag{3-67}$$

In summing moments about the z-axis at any point x, a clockwise moment is considered as a positive moment, because it would produce a tensile stress in the $+y$ portion of the cross section.

The thermal moment may be found from eq. (3-10):

$$M_T = \alpha E \Delta T_0 b \int_{-h/2}^{h/2} \sin(\pi y/h) y\, dy$$

$$M_T = \alpha E \Delta T_0 (bh^2/\pi^2)\left[\sin(\pi y/h) - (\pi y/h)\cos(\pi y/h)\right]_{-h/2}^{h/2}$$

$$M_T = \frac{2\alpha E \Delta T_0 bh^2}{\pi^2} = \frac{24\alpha EI\Delta T_0}{\pi^2} \tag{3-68}$$

We have substituted the area moment of inertia, $I = bh^3/12$, in eq. (3-68).

The direct stress may be evaluated from eq. (3-14):

$$\sigma = -\frac{q_0 L^2}{2I}\left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2\right] y + \frac{24\alpha E \Delta T_0 y}{\pi^2 h} - \alpha E \Delta T_0 \sin(\pi y/h)$$

This expression may be simplified, as follows:

$$\sigma = -\frac{q_0 L^2 h}{4I}\left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2\right]\left(\frac{2y}{h}\right) + \alpha E \Delta T_0\left[\frac{24y}{\pi^2 h} - \sin(\pi y/h)\right] \tag{3-69}$$

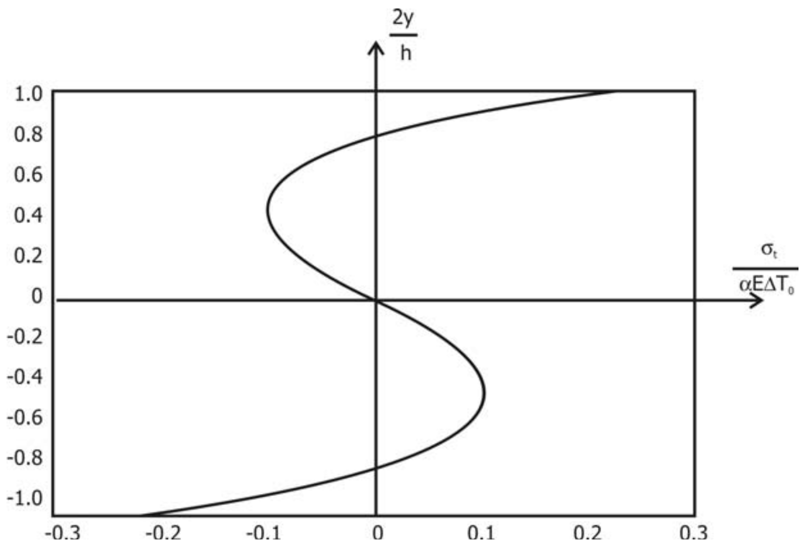

Figure 3-10. Bending stress distribution for the simply-supported beam example.

The first term in eq. (3-69) represents the stress component resulting from the applied mechanical load, and the second term represents the stress component resulting from the thermal load. A plot of the stress distribution associated with the thermal load is shown in Figure 3-10.

At either end of the beam ($x = 0$ or $x = L$), the stress component resulting from the mechanical load is zero. For this case, the maximum stress occurs at the surface of the beam ($y = \pm h/2$):

$$\sigma(x = 0;\, y = \pm h/2) = \pm\left(\frac{12}{\pi^2} - 1\right)\alpha E \Delta T_0 \tag{3-70}$$

The stress component resulting from the mechanical load has a maximum value at the center of the beam span ($x = L/2$). For this location, the stress distribution is given by the following, from eq. (3-69):

$$\sigma(x = L/2) = -\frac{3q_0L^2}{4bh^2}\left(\frac{2y}{h}\right) + \alpha E \Delta T_0\left[\frac{12}{\pi^2}\left(\frac{2y}{h}\right) - \sin\left(\frac{\pi 2y}{2h}\right)\right] \tag{3-71}$$

The location of the maximum stress at the center of the beam depends on the relative size of the mechanical and thermal loads. The maximum stress will occur either at the surface of the beam ($y = h/2$) or at the location given by

$$\cos\left(\frac{\pi 2y}{2h}\right) = \frac{24}{\pi^3}\left[1 - \frac{\pi^2 q_0 L^2}{16bh^2\alpha E \Delta T_0}\right] \tag{3-72}$$

The stress at the surface of the beam at the center of the span is found from eq. (3-71):

$$\sigma\left(x = \frac{1}{2}L;\, y = \frac{1}{2}h\right) = -\frac{3q_0L^2}{4bh^2} + \left(\frac{12}{\pi^2} - 1\right)\alpha E\Delta T_0 \tag{3-73}$$

It may possible to make the stress at the surface of the beam at the center of the span equal to zero if both the mechanical load q_0 and the temperature change ΔT_0 have the same algebraic sign. If we set the stress in eq. (3-73) equal to zero, we obtain the following condition:

$$\frac{L}{h} = \left[\frac{4b\alpha E\Delta T_0}{3q_0}\left(\frac{12}{\pi^2} - 1\right)\right]^{1/2} \tag{3-74}$$

The ratio $(L/h > 5)$ if the strength-of-materials analysis is to be valid.

The stress is not identically zero for the condition given by eq. (3-74). If we make the substitution from eq. (3-74) into eq. (3-73), we obtain the stress distribution at the beam center for this condition:

$$\sigma^*(x = L/2) = -\alpha E\Delta T_0\left[\sin\left(\frac{\pi 2y}{2h}\right) - \left(\frac{2y}{h}\right)\right] \tag{3-75}$$

The maximum value of the stress given by eq. (3-75) occurs at $2y/h = \pm 0.56607$ and has the following value:

$$\sigma^*_{\max}(x = L/2) = \mp 0.2105\alpha E\Delta T_0 \tag{3-76}$$

3.6.2.2 Displacement. The transverse displacement of the beam in this example may be determined from eq. (3-25):

$$\frac{d^2v}{dx^2} = -\frac{M + M_T}{EI} = \frac{q_0L^2}{2EI}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^2\right] - \frac{24\alpha\Delta T_0}{\pi^2 h} \tag{3-77}$$

The rotation or slope of the beam at any point is found by integrating eq. (3-77):

$$\frac{dv}{dx} = \omega = \frac{q_0L^3}{6EI}\left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right] - \frac{24\alpha\Delta T_0 L}{\pi^2 h}\left(\frac{x}{L}\right) + C_1 \tag{3-78}$$

The transverse displacement may be found by integrating again:

$$v = \frac{q_0L^4}{12EI}\left[2\left(\frac{x}{L}\right)^3 - \left(\frac{x}{L}\right)^4\right] - \frac{12\alpha\Delta T_0 L^2}{\pi^2 h}\left(\frac{x}{L}\right)^2 + C_1x + C_2 \tag{3-79}$$

The displacement at the left end $(x = 0)$ of the beam is zero; therefore, $C_2 = 0$ in eq. (3-79). The displacement at the right end $(x = L)$ of the beam is also zero:

$$C_1 = -\frac{q_0L^3}{12EI} + \frac{12\alpha\Delta T_0 L}{\pi^2 h} \tag{3-80}$$

After making the substitutions for the constants of integration into eq. (3-79), we obtain the following expression for the transverse displacement of the beam:

$$v = -\frac{q_0 L^4}{12EI}\left[1 - 2\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right]\left(\frac{x}{L}\right) + \frac{12\alpha\Delta T_0 L^2}{\pi^2 h}\left[1 - \left(\frac{x}{L}\right)\right]\left(\frac{x}{L}\right) \tag{3-81}$$

The expression for the transverse displacement may be written in a slightly different form by introducing the expression for the area moment of inertia for the rectangular cross section, $I = bh^3/12$:

$$v = -\frac{q_0 L^4}{bh^3 E}\left[1 - 2\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right]\left(\frac{x}{L}\right) + \frac{12\alpha\Delta T_0}{\pi^2 h}\left[1 - \left(\frac{x}{L}\right)\right]\left(\frac{x}{L}\right) \tag{3-81}$$

The rotation of the beam may be found from eq. (3-78) after substituting for the constant of integration from eq. (3-80):

$$\omega = \frac{dv}{dx} = -\frac{q_0 L^3}{bh^3 E}\left[1 - 6\left(\frac{x}{L}\right)^2 + 4\left(\frac{x}{L}\right)^3\right] + \frac{12\alpha\Delta T_0 L}{\pi^2 h}\left[1 - 2\left(\frac{x}{L}\right)\right] \tag{3-82}$$

The rotation (slope) of the beam is zero at the center of the beam ($x = L/2$); therefore, the maximum transverse displacement also occurs at this location:

$$v_{\max} = v\left(\frac{1}{2}L\right) = -\frac{5q_0 L^4}{16bh^3 E} + \frac{3\alpha\Delta T_0 L^2}{\pi^2 h} \tag{3-83}$$

If the mechanical load q_0 and the temperature change ΔT_0 have the same algebraic sign, it may be possible to make the transverse displacement at the center of the span equal to zero. If we set the displacement equal to zero in eq. (3-83), we obtain the following condition:

$$\frac{L}{h} = \left(\frac{48b\alpha E\Delta T_0}{5\pi^2 q_0}\right)^{1/2} \tag{3-84}$$

This condition is not quite the same as that given by eq. (3-74). The ratio L/h must be > 5 if the strength-of-materials analysis is to be valid.

The displacement is not identically zero for the condition given by eq. (3-84). If we make the substitution from eq. (3-84) into the displacement relationship, eq. (3-81), we obtain the distribution for the transverse displacement under this condition:

$$v^* = \frac{12\alpha\Delta T_0 L^2}{5\pi^2 h}\left[1 - 4\left(\frac{x}{L}\right) + 4\left(\frac{x}{L}\right)^2\right]\left[1 - \left(\frac{x}{L}\right)\right]\left(\frac{x}{L}\right) \tag{3-85}$$

For this condition, the maximum transverse displacement occurs at the locations $x = 0.1464L$ and $x = 0.8536L$, and has the following value:

$$v^*_{\max} = \frac{3\alpha \Delta T_0 L^2}{20\pi^2 h} \tag{3-86}$$

Example 3-2 A beam is constructed of 6061-T6 aluminum ($\alpha = 23.4 \times 10^{-6}\text{K}^{-1} = 13.0 \times 10^{-6\circ}\text{F}^{-1}$; $E = 69.0\,\text{GPa} = 10.0 \times 10^6\,\text{psi}$; $S_y = 275\,\text{MPa} = 40{,}000\,\text{psi}$) with a length between supports of 2.250 m (7.38 ft). The beam is simply supported at each end. The cross section of the beam is rectangular, with the width equal to $\frac{1}{3}$ of the height. There is a uniformly distributed mechanical load directed downward of 1.55 kN/m (106.2 lb_f/ft = 8.85 lb_f/in.). The temperature distribution across the depth of the beam is given by eq. (3-66), with $\Delta T_0 = 120^\circ\text{C}$ (216°F). If the depth of the beam cross section is selected such that the stress at the top and bottom surfaces of the beam is zero at the center of the span of the beam, determine the width and height of the beam. Also, determine the transverse deflection at the center of the span of the beam.

The dimensions of the beam may be determined from eq. (3-74):

$$\frac{L}{h} = \left[\frac{(4)\left(\frac{1}{3}\right)(h)(23.4 \times 10^{-6})(69.0 \times 10^9)(120)}{(3)(1550)}\left(\frac{12}{\pi^2} - 1\right)\right]^{1/2}$$

$$\frac{L}{h^{3/2}} = [(55{,}556)\,(0.21585)]^{1/2} = 109.51$$

The required depth of the beam is

$$h = \left(\frac{2.250}{109.51}\right)^{2/3} = 0.0750\,\text{m} = 75.0\,\text{mm} \quad (2.953\,\text{in.})$$

The width of the beam is

$$w = \tfrac{1}{3}h = 25.0\,\text{mm} \quad (0.984\,\text{in.})$$

We note that $(L/h) = (2.250/0.075) = 30 > 5$, so the calculated beam dimensions are satisfactory.

The maximum stress in the beam may be found from eq. (3-76):

$$\sigma^*_{\max}(x = L/2) = \mp(0.2105)(23.4 \times 10^{-6})(69.0 \times 10^9)(120)$$

$$\sigma^*_{\max} = \mp 40.78 \times 10^6\,\text{Pa} = \mp 40.78\,\text{MPa} \quad (5920\,\text{psi})$$

This stress is less than the yield strength for the material ($S_y = 275\,\text{MPa}$).

The transverse displacement of the beam at the center of the span is found from eq. (3-83):

$$v(L/2) = -\frac{(5)(1550)(2.25)^4}{(16)(0.025)(0.075)^3(69.0 \times 10^9)} + \frac{(3)(23.4 \times 10^{-6})(120)(2.25)^2}{(\pi^2)(0.075)}$$

$$v(L/2) = -0.0171 + 0.0576 = 0.0405\,\text{m} = 40.5\,\text{mm} \quad (1.60\,\text{in.})$$

We note that, in this case, the beam deflects upward, because the displacement due to thermal effects (the second term) more than offsets the displacement due to the distributed mechanical load (the first term).

3.6.3 Statically Indeterminate Beam

Let us consider the beam shown in Figure 3-11, which is simply supported at the center and at each end. The beam has a rectangular cross section, and no external mechanical loads, other than those of the supports, are applied. The beam has the following temperature distribution across the cross section:

$$\Delta T = \frac{1}{2}\Delta T_0 \left[1 + \left(\frac{2y}{h}\right)\right] \tag{3-87}$$

The beam is statically indeterminate because the external forces due to the supports cannot be determined by force and moment balances alone. From a force and moment balance, we can determine that the end loads are equal and the center load is twice that of the end force. The value of the support force F_1 must be determined from information about the displacement of the beam at the supports. The displacement at each support is taken as zero, in this problem, because the beam is considered to be simply supported.

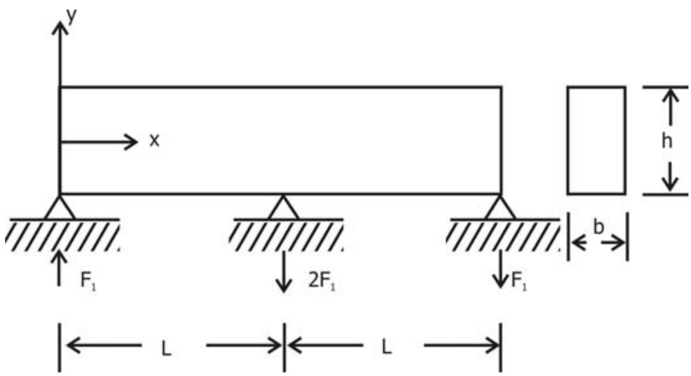

Figure 3-11. Simply-supported beam with three supports.

3.6.3.1 Direct thermal stresses. The thermal force may be evaluated from its definition, eq. (3-7):

$$F_T = \frac{1}{2}\alpha E \Delta T_0 b \int_{-h/2}^{h/2} \left[1 + \left(\frac{2y}{h}\right)\right] dy$$

$$F_T = \frac{1}{2}\alpha E \Delta T_0 b \left[y + \frac{y^2}{h}\right]_{-h/2}^{h/2} = \frac{1}{2}\alpha E \Delta T_0 bh = \frac{1}{2}\alpha E \Delta T_0 A \tag{3-88}$$

The quantity $A = bh$ is the cross-sectional area of the beam.

The thermal moment may be evaluated from eq. (3-10):

$$M_T = \frac{1}{2}\alpha E \Delta T_0 b \int_{-h/2}^{h/2} \left[1 + \left(\frac{2y}{h}\right)\right] y\, dy$$

$$M_T = \frac{1}{2}\alpha E \Delta T_0 b \left[\frac{1}{2}y^2 + \frac{2}{3}\frac{y^3}{h}\right]_{-h/2}^{h/2} = \frac{\alpha E \Delta T_0 bh^2}{12} \tag{3-89}$$

The area moment of inertia for the rectangular beam cross section is $I = bh^3/12$, so the thermal moment may be written in the following form:

$$M_T = \frac{\alpha EI \Delta T_0}{h} \tag{3-90}$$

The mechanical moment may be written in terms of the moment produced by the support forces.

$$\text{For } 0 \le x \le L: \quad M = -F_1 x \tag{3-91a}$$

$$\text{For } L \le x \le 2L: \quad M = -F_1 x + 2F_1(x - L) = -F_1(2L - x) \tag{3-91b}$$

The stress distribution within the beam may be evaluated from eq. (3-14):

$$\sigma = \frac{F_T}{A} + \frac{(M + M_T)\, y}{I} - \alpha E \Delta T \tag{3-92}$$

Let us examine the following terms:

$$\frac{F_T}{A} + \frac{M_T y}{I} - \alpha E \Delta T = \frac{1}{2}\alpha E \Delta T_0 + \frac{\alpha E \Delta T_0}{2}\left(\frac{2y}{h}\right) - \frac{1}{2}\alpha E \Delta T_0 \left[1 + \left(\frac{2y}{h}\right)\right] = 0$$

Therefore, the stress at any location is given by the following:

$$\text{For } 0 \le x \le L: \quad \sigma = -\frac{F_1 x y}{I} \tag{3-93a}$$

$$\text{For } L \le x \le 2L: \quad \sigma = -\frac{F_1 (2L - x) y}{I} \tag{3-93b}$$

The support force is not known, and we cannot evaluate this quantity until we have an expression for the transverse displacement of the beam.

3.6.3.2 Displacement. The transverse displacement of the beam may be determined from eq. (3-25). For the left section of the beam, $0 \le x \le L$, we have

$$\frac{d^2 v}{dx^2} = -\frac{M + M_T}{EI} = \frac{F_1 x}{EI} - \frac{\alpha \Delta T_0}{h} \tag{3-94}$$

Integrating once, we obtain the general expression for the slope for the left section:

$$\omega = \frac{dv}{dx} = \frac{F_1 x^2}{2EI} - \frac{\alpha \Delta T_0 x}{h} + C_1 \tag{3-95}$$

Integrating again, we obtain the general expression for the transverse displacement for the left section:

$$v = \frac{F_1 x^3}{6EI} - \frac{\alpha \Delta T_0 x^2}{2h} + C_1 x + C_2 \tag{3-96}$$

At the left support, $v(0) = 0$; therefore, $C_2 = 0$. At the center support, $v(L) = 0$, and we find the following value for the constant of integration C_1:

$$C_1 = -\frac{F_1 L^2}{6EI} + \frac{\alpha \Delta T_0 L}{2h} \tag{3-97}$$

Making the substitutions for the constants of integration into eq. (3-96), we obtain the expression for the transverse displacement and rotation for the left portion of the beam, $0 \le x \le L$:

$$v = -\frac{F_1 L^3}{6EI}\left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^3\right] + \frac{\alpha \Delta T_0 L^2}{2h}\left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2\right] \tag{3-98}$$

$$\omega = -\frac{F_1 L^2}{6EI}\left[1 - 3\left(\frac{x}{L}\right)^2\right] + \frac{\alpha \Delta T_0 L}{2h}\left[1 - 2\left(\frac{x}{L}\right)\right] \tag{3-99}$$

For the right section of the beam, $L < x \le 2L$, we have the following expression to solve for the displacement:

$$\frac{d^2 v}{dx^2} = \frac{F_1 (2L - x)}{EI} - \frac{\alpha \Delta T_0}{h} \tag{3-100}$$

The general expressions for the slope and displacement are found by integration:

$$\omega = \frac{dv}{dx} = \frac{F_1}{2EI}\left(4Lx - x^2\right) - \frac{\alpha \Delta T_0 x}{h} + C_3 \tag{3-101}$$

$$v = \frac{F_1}{6EI}\left(6Lx^2 - x^3\right) - \frac{\alpha \Delta T_0 x^2}{2h} + C_3 x + C_4 \tag{3-102}$$

The constants of integration may be found from the conditions that the displacement is zero at each support point, $v(L) = v(2L) = 0$. Using these conditions, we find the following values for the constants of integration:

$$C_3 = -\frac{11 F_1 L^2}{6EI} + \frac{3\alpha \Delta T_0 L}{2h} \tag{3-103}$$

$$C_4 = \frac{F_1 L^3}{EI} - \frac{\alpha \Delta T_0 L^2}{h} \tag{3-104}$$

If we substitute the expressions for the constants of integration into eqs. (3-101) and (3-102), we obtain the following expressions for the slope and displacement for the right portion of the beam, $L < x \le 2L$:

$$v = \frac{F_1 L^3}{6EI}\left[6 - 11\left(\frac{x}{L}\right) + 6\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3\right] - \frac{\alpha \Delta T_0 L^2}{2h}\left[2 - 3\left(\frac{x}{L}\right) + \left(\frac{x}{L}\right)^2\right] \tag{3-105}$$

$$\omega = -\frac{F_1 L^2}{6EI}\left[11 - 12\left(\frac{x}{L}\right) + 3\left(\frac{x}{L}\right)^2\right] + \frac{\alpha \Delta T_0 L}{2h}\left[3 - 2\left(\frac{x}{L}\right)\right] \tag{3-106}$$

3.6.3.3 Support reaction. The expression for the support reaction F_1 may be found from the condition that the slope or rotation under the middle load (at $x = L$) is continuous. If we equate the rotation given by eq. (3-99) at $x = L$ to that given by eq. (3-106) at $x = L$, we obtain

$$\frac{F_1 L^2}{3EI} - \frac{\alpha \Delta T_0 L}{2h} = -\frac{F_1 L^2}{3EI} + \frac{\alpha \Delta T_0 L}{2h} \tag{3-107}$$

If we solve for the reaction force, we obtain

$$F_1 = \frac{3\alpha EI \Delta T_0}{2hL} \tag{3-108}$$

Because of the symmetry of the loading, the slope or rotation at the center of the beam ($x = L$) should be zero. If we make the substitution from eq. (3-108) into either eq. (3-99) or (3-106) at $x = L$, we do obtain the result that $\omega = 0$ at this point.

3.6.3.4 Maximum stress and deflection. The stress distribution may be written by combining eqs. (3-93) and (3-108).

$$\text{For } 0 \le x \le L: \quad \sigma = -\frac{3\alpha E \Delta T_0}{4}\left(\frac{x}{L}\right)\left(\frac{2y}{h}\right) \tag{3-109a}$$

$$\text{For } L \le x \le 2L: \quad \sigma = -\frac{3\alpha E \Delta T_0}{4}\left(2 - \frac{x}{L}\right)\left(\frac{2y}{h}\right) \tag{3-109b}$$

The maximum stress occurs at the center of the span ($x = L$) and at the upper and lower surfaces ($y = \pm h/2$):

$$\sigma_{\max} = \mp\tfrac{3}{4}\alpha E \Delta T_0 \tag{3-110}$$

We note from eq. (3-110) that the dimensions (length, width, or depth) do not affect the maximum stress, in this case. The designer cannot control the stresses by selection of the dimensions of the beam, for this case.

If we substitute the expression for the reaction force F_1 from eq. (3-110) into eqs. (3-98) and (3-105), we obtain the following expressions for the transverse displacement and rotation of the beam, in terms of the temperature distribution.

$$\text{For } 0 \le x \le L: \quad v = \frac{\alpha \Delta T_0 L^2}{4h}\left[\left(\frac{x}{L}\right) - 2\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right] \tag{3-111a}$$

$$\text{For } L \le x \le 2L: \quad v = \frac{\alpha \Delta T_0 L^2}{4h}\left[2 - 5\left(\frac{x}{L}\right) + 4\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3\right] \tag{3-111b}$$

$$\text{For } 0 \le x \le L: \quad \omega = \frac{\alpha \Delta T_0 L}{4h}\left[1 - 4\left(\frac{x}{L}\right) + 3\left(\frac{x}{L}\right)^2\right] \tag{3-112a}$$

$$\text{For } L \le x \le 2L: \quad \omega = \frac{\alpha \Delta T_0 L}{4h}\left[-5 + 8\left(\frac{x}{L}\right) - 3\left(\frac{x}{L}\right)^2\right] \tag{3-112b}$$

From eq. (3-112a), we see that the slope is zero at $(x/L) = 1$ and $(x/L) = \frac{1}{3}$. Since the displacement is zero at $(x/L) = 1$, we conclude that the maximum displacement occurs at $(x/L) = \frac{1}{3}$. Making this substitution into eq. (3-111a), we find the following expression for the maximum transverse displacement:

$$v_{\max} = v\left(\frac{1}{3}L\right) = \frac{\alpha \Delta T_0 L^2}{27h} \tag{3-113}$$

The same displacement occurs for the right portion of the beam at $(x/L) = \frac{5}{3}$.

3.6.3.5 Partial constraint of the beam. One of the basic observations is that thermal stresses arise due to constraints; therefore, if we can control the degree of constraint for the system, we may control the stress level. If the beam were not supported in the center, it would be free to deflect as a result of the temperature change. Because the temperature change (in this example) is linear across the cross section, and the bending stress is a linear function of the coordinate y, zero thermal stresses would result if the center support were removed.

If we set $F_1 = 0$ in eq. (3-94) and integrate, we obtain the general expressions for the slope and displacement for this case:

$$\omega^o = -\frac{\alpha \Delta T_0 x}{h} + C_5 \tag{3-114}$$

$$v^o = -\frac{\alpha \Delta T_0 x^2}{2h} + C_5 x + C_6 \tag{3-115}$$

We may evaluate the constants of integration from the conditions that at $x = 0$, the displacement is zero, and at $x = L$, the slope is zero. The result for the transverse displacement for the case of $F_1 = 0$ is

$$v^o = \frac{\alpha \Delta T_0 L^2}{2h}\left[2 - \left(\frac{x}{L}\right)\right]\left(\frac{x}{L}\right) \tag{3-116}$$

The displacement at the center of the beam is given by

$$v^o(L) = \frac{\alpha \Delta T_0 L^2}{2h} = \delta_{\mathrm{cr}} \tag{3-117}$$

Let us consider the case in which the displacement at the center support is δ, instead of being equal to zero. It is assumed that $\delta \le \delta_{\mathrm{cr}}$; otherwise, the support force would be zero. Let us define the coefficient of constraint ξ_c by

$$\xi_c \equiv \frac{\delta_{\mathrm{cr}} - \delta}{\delta_{\mathrm{cr}}} = 1 - \frac{\delta}{\delta_{\mathrm{cr}}} \qquad \text{where } 0 \le \xi_c \le 1 \tag{3-118}$$

A value of $\xi_c = 0$ implies no constraint ($F_1 = 0$), and a value of $\xi_c = 1$ implies complete constraint ($v = 0$) at the center support, $x = L$. The displacement at the support may be written in terms of the coefficient of constraint by using eq. (3-117):

$$\delta = (1 - \xi_c)\,\delta_{\mathrm{cr}} = (1 - \xi_c)\,\frac{\alpha \Delta T_0 L^2}{2h} \tag{3-119}$$

If we use the condition $v(L) = \delta$ in eq. (3-96), we obtain the following expression for the transverse displacement for the left portion of the beam, $0 \le x \le L$:

$$v^* = -\frac{F_1 L^3}{6EI}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^3\right] + \frac{\alpha \Delta T_0 L^2}{2h}\left[(2 - \xi_c)\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2\right] \tag{3-120}$$

As a check, we note that eq. (3-120) reduces to eq. (3-98) for $\xi_c = 1$, and for $\xi_c = 0$, the expression reduces to eq. (3-116) for zero constraint. If we use the condition $v(L) = \delta$ in eq. (3-102) for the right portion of the beam, we may obtain the expression for the transverse displacement for $L < x \le 2L$:

$$v^* = \frac{F_1 L^3}{6EI}\left[6 - 11\left(\frac{x}{L}\right) + 6\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3\right]$$

$$- \frac{\alpha \Delta T_0 L^2}{2h}\left[2\xi_c - (2+\xi_c)\left(\frac{x}{L}\right) + \left(\frac{x}{L}\right)^2\right] \qquad (3\text{-}121)$$

Finally, if we equate the slope for the left and right portions of the beam at the center support ($x = L$), we obtain the following expression for the support reaction F_1:

$$F_1^* = \frac{3\alpha EI\,\Delta T_0 \xi_c}{2hL} \qquad (3\text{-}122)$$

The maximum stress for partial constraint of the beam may be found by combining eqs. (3-93) and (3-122).

$$\sigma_{\max} = \mp \frac{3}{4}\alpha E \Delta T_0 \xi_c \qquad (3\text{-}123)$$

The designer has the ability to control the thermal stress level through selection of the degree of constraint at the center support of the beam. It may not be practical to have the beam totally unconstrained at the center; however, it is often practical to introduce partial constraint, $\delta = 0$ at the center of the span.

For the partially constrained beam, if the coefficient of constraint $\xi_c \le \frac{2}{3}$, the maximum transverse displacement occurs at the center of the span and has the value equal to the gap width, δ. For the case for $\frac{2}{3} \le \xi_c \le 1$, we may find the maximum displacement, as follows. Let us substitute the support reaction F_1 from eq. (3-123) into the displacement expression, eq. (3-120) for $0 \le x \le L$:

$$v^* = \frac{\alpha \Delta T_0 L^2}{4h}\left[(4 - 3\xi_c)\left(\frac{x}{L}\right) - 2\left(\frac{x}{L}\right)^2 + \xi_c\left(\frac{x}{L}\right)^3\right] \qquad (3\text{-}124)$$

If we take the derivative of the displacement given by eq. (3-124) and set the result equal to zero, we obtain the following quadratic equation for the location of the maximum transverse displacement for the case of partial constraint of the beam for $\frac{2}{3} \le \xi_c \le 1$:

$$\left(\frac{x}{L}\right)^2 - \frac{4}{3\xi_c}\left(\frac{x}{L}\right) + \frac{4 - 3\xi_c}{3\xi_c} = 0 \qquad (3\text{-}125)$$

The solution for the position of maximum transverse displacement is found from eq. (3-125):

$$\frac{x}{L} = \frac{2}{3\xi_c}\left[1 - \sqrt{1 - 3\xi_c\left(1 - \tfrac{3}{4}\xi_c\right)}\right] \quad \text{For} \quad \tfrac{2}{3} \le \xi_c \le 1 \qquad (3\text{-}126)$$

Example 3-3 A beam constructed of 304 stainless steel has a loading as given in Fig. 3-11. The properties of 304 stainless steel are $\alpha = 16 \times 10^{-6}\text{K}^{-1}$ ($8.89 \times 10^{-6\circ}\text{F}^{-1}$), $E = 193\,\text{GPa}$ (28.0×10^6 psi), and $S_y = 232\,\text{MPa}$ (33,650 psi). The length of the beam between supports is 4.25 m (13.94 ft), the depth of the beam is 600 mm (23.62 in.), and the width of the beam is 100 mm (3.94 in.). The maximum temperature change for the beam is $\Delta T_0 = 120^\circ\text{C}$ (216°F). Determine the gap width required to limit the stress in the beam to 125 MPa (18,130 psi).

The maximum stress for the partially constrained beam is given by eq. (3-123):

$$\sigma_{\max} = 125 \times 10^6 = \tfrac{3}{4}\left(16 \times 10^{-6}\right)\left(193 \times 10^9\right)\left(120^\circ\right)\xi_c$$

The constraint coefficient is

$$\xi_c = \frac{125}{277.9} = 0.450 = 1 - \frac{\delta}{\delta_{\text{cr}}}$$

The critical gap width is found from eq. (3-117):

$$\delta_{\text{cr}} = \frac{\alpha \Delta T_0 L^2}{2h} = \frac{\left(16 \times 10^{-6}\right)(120^o)(4.25)^2}{(2)(0.600)} = 0.0289\,\text{m} = 28.9\,\text{mm}$$

The required gap width is found from the definition of the constraint coefficient:

$$\delta = (1 - 0.450)(28.9) = 15.9\,\text{mm} \quad (0.626\,\text{in.})$$

Since the constraint coefficient in this example is less than $\frac{2}{3}$, the maximum transverse deflection of the beam occurs at the center support and is equal to $15.9\,\text{mm} = \delta$.

3.6.4 Beam on an Elastic Foundation

In the previous examples, the mechanical bending moment distribution could be determined in terms of the external forces. If the beam is supported on an elastic foundation, however, the external support force is a function of the displacement of the beam, which is not known at the beginning of the solution. In this case, we will need to use the general expression for the displacement, eq. (3-32). The mechanical bending moment may then be evaluated from the general relations given in Section 3-4.

Let us consider a beam supported along its length by a continuous elastic foundation, as shown in Figure 3-12. The elastic foundation will be considered to be linearly elastic, for which the reaction force between the foundation and the beam is directly proportional to the displacement of the beam. The reaction force per unit length may be written as

$$q_f = -kv \tag{3-127}$$

The negative sign is included, because a negative deflection v produces a reaction force in the positive y-direction. The quantity k {units: (N/m)/m or N/m^2 = Pa} is called the *foundation modulus* [Timoshenko and Langer, 1932].

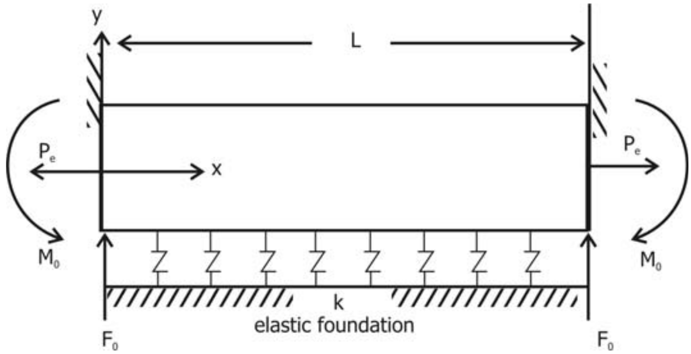

Figure 3-12. Beam on an elastic foundation.

The analysis of the problem of beam bending on an elastic foundation with no thermal loading was developed by E. Winkler [Winkler, 1867]. The initial application for the solution was in determining the stress in railway tracks supported on a crosstie and ballast foundation [Timoshenko, 1956].

3.6.4.1 General Equation Solution. The general expression for the transverse displacement of the beam is eq. (3-32):

$$\frac{d^4 v}{dx^4} = \frac{1}{EI}\left(q - \frac{d^2 M_T}{dx^2}\right) \tag{3-128}$$

Suppose the only mechanical force along the length of the beam is the foundation reaction, given by eq. (3-127), or $q = q_f$. For this problem, let us consider a beam with a rectangular cross section of depth h and width b. Suppose the temperature distribution is given by

$$\Delta T(x, y) = \frac{1}{2}\Delta T_0 \left[1 + \left(\frac{2y}{h}\right)\right] \sin(\pi x/L) \tag{3-129}$$

The thermal moment may be determined:

$$M_T = \alpha E \int \Delta T y \, dA = \frac{\alpha EI \, \Delta T_0}{h} \sin(\pi x/L) \tag{3-130}$$

The thermal shear force may be determined from its definition:

$$V_T = \frac{dM_T}{dx} = \frac{\pi \alpha EI \, \Delta T_0}{hL} \cos(\pi x/L) \tag{3-131}$$

The second derivative of the thermal moment is

$$\frac{d^2 M_T}{dx^2} = -\frac{\pi^2 \alpha EI \, \Delta T_0}{hL^2} \sin(\pi x/L) \tag{3-132}$$

The thermal force may also be determined:

$$F_T = \alpha E \int \Delta T \, dA = \tfrac{1}{4}\alpha E A \Delta T_0 \sin(\pi x/L) \tag{3-133}$$

If we make the substitutions for the mechanical and thermal loading into eq. (3-128), we obtain the following differential equation to solve for the displacement:

$$\frac{d^4 v}{dx^4} = -\frac{kv}{EI} + \frac{\pi^2 \alpha \Delta T_0}{hL^2} \sin(\pi x/L) \tag{3-134}$$

For convenience, let us define the following quantities:

$$\beta^4 = \frac{k}{4EI} \tag{3-135}$$

$$\gamma_P = \frac{\pi^2 \alpha \Delta T_0}{hL^2} \tag{3-136}$$

We may now write eq. (3-134) as

$$\frac{d^4 v}{dx^4} + 4\beta^4 v = \gamma_P \sin(\pi x/L) \tag{3-137}$$

The general solution of eq. (3-137) may be written as the sum of the solution of the homogeneous equation, obtained by setting the right side of the complete equation equal to zero, and a particular solution. The solution of the homogeneous equation is independent of the thermal loading; however, the particular solution depends on the specific expression for the thermal moment. For this example, the general solution is

$$\begin{aligned} v(x) = {} & e^{-\beta x}(C_1 \sin\beta x + C_2 \cos\beta x) \\ & + e^{+\beta x}(C_3 \sin\beta x + C_4 \cos\beta x) + \delta_P \sin(\pi x/L) \end{aligned} \tag{3-138}$$

The quantity δ_P is defined as follows:

$$\delta_P = \frac{\gamma_P}{\frac{\pi^4}{L^4} + 4\beta^4} = \frac{\dfrac{\alpha \Delta T_0 L^2}{\pi^2 h}}{1 + 4\left(\frac{\beta L}{\pi}\right)^4} \tag{3-139}$$

The constants of integration, $C_1, \ldots, C_4$, must be determined from the boundary conditions for the beam.

3.6.4.2 Long-beam solution. If the beam is very long, it would appear that the conditions at the ends of the beam would not greatly influence the stress and deflection in the center region. For an extreme case, think of a beam with a length

of 1.20 km (about $\frac{3}{4}$ mile). What one does at the end of the beam supported on an elastic foundation would have no discernable effect $\frac{3}{8}$ of a mile away. For practical purposes, a "long" beam may be considered as one for which $\beta L < 5$.

For the long beam, the term involving $e^{+\beta x}$ would be quite large, unless we require that $C_3 = C_4 = 0$. Thus, the general solution, eq. (3-139), reduces to the following expression for a long beam:

$$v(x) = e^{-\beta x}\,(C_1 \sin\beta x + C_2 \cos\beta x) + \delta_P \sin(\pi x/L) \tag{3-140}$$

The rotation of the long beam may be determined, as follows:

$$\omega(x) = \frac{dv}{dx} = \beta e^{-\beta x}\,[C_1(\cos\beta x - \sin\beta x) - C_2(\cos\beta x + \sin\beta x)] + (\pi\delta_P/L)\cos(\pi x/L) \tag{3-141}$$

The deflection is zero at the end of the beam:

$$v(0) = 0 = C_1$$

Similarly, for a clamped-end beam, the rotation is also zero at the end:

$$\omega(0) = 0 = \beta C_1 + (\pi\delta_P/L)$$

or

$$C_1 = -\frac{\pi\delta_P}{\beta L}$$

Using these results for the constants of integration, we may determine the expressions for the deflection and rotation for this example:

$$v(x) = \delta_P\left[-\frac{\pi}{\beta L}e^{-\beta x}\sin\beta x + \sin(\pi x/L)\right] \tag{3-142}$$

$$\omega(x) = \frac{\pi\delta_P}{L}\left[e^{-\beta x}(\sin\beta x - \cos\beta x) + \cos(\pi x/L)\right] \tag{3-143}$$

We will need the second and third derivatives of the displacement to evaluate the bending moment and shear force terms:

$$\frac{d^2v}{dx^2} = \frac{\pi\delta_P}{L^2}\left[2\beta L e^{-\beta x}\cos\beta x - \pi\sin(\pi x/L)\right] \tag{3-144}$$

$$\frac{d^3v}{dx^3} = -\frac{\pi\delta_P}{L^3}\left[2\beta^2L^2e^{-\beta x}(\cos\beta x + \sin\beta x) + \pi^2\cos(\pi x/L)\right] \tag{3-145}$$

The mechanical bending moment may be determined from eqs. (3-29) and (3-144):

$$M(x) = -\frac{\pi \delta_P EI}{L^2}\left[2\beta L e^{-\beta x}\cos\beta x - \pi \sin(\pi x/L)\right] - \frac{\alpha EI\,\Delta T_0}{h}\sin(\pi x/L) \tag{3-146}$$

The bending moment at the clamped end of the beam may now be evaluated:

$$M_0 = M(0) = -\frac{2\pi \delta_P EI\,\beta L}{L^2} \tag{3-147}$$

The end moment may be written in terms of the temperature difference by using eq. (3-139) for the term δ_P.

$$M_0 = -\frac{\dfrac{4\beta L\alpha EI\,\Delta T_0}{\pi h}}{1 + 4(\beta L/\pi)^4} \tag{3-148}$$

The shear force at any point along the beam may be found from eqs. (3-30) and (3-145):

$$V(x) = +\frac{\pi \delta_P EI}{L^3}\left[2\beta^2 L^2 e^{-\beta x}(\cos\beta x + \sin\beta x) + \pi^2\cos(\pi x/L)\right] - V_T \tag{3-149}$$

The thermal shear force is given by eq. (3-131) for this example. The vertical reaction force at the end of the beam may be found from

$$F_0 = -V(0) = -\frac{\pi(\beta L)^2\,\delta_P EI}{L^3}\left[2 + \left(\frac{\pi}{\beta L}\right)^2\right] + \frac{\pi\alpha EI\,\Delta T_0}{hL} \tag{3-150}$$

The vertical reaction force may be written in terms of the temperature difference by using eq. (3-139):

$$F_0 = -\frac{\alpha EI\,\Delta T_0}{\pi h L}\left\{\frac{(\beta L)^2\left[2 + (\pi/\beta L)^2\right]}{1 + 4(\beta L/\pi)^4} - \pi^2\right\} \tag{3-151}$$

In this problem, the end is considered to be clamped, so there will be an axial reaction force P_e at the ends of the beam. The axial reaction force may be determined from the condition that the change in length of the beam is zero:

$$\Delta L = 0 = \int_0^L \varepsilon_0\,dx = \frac{1}{EA}\int_0^L (P_e + F_T)\,dx \tag{3-152}$$

The expression for the thermal force is given by eq. (3-133) in this case:

$$0 = \frac{P_e L}{EA} + \frac{1}{4}\alpha\Delta T_0\int_0^L \sin(\pi x/L)\,dx = \frac{P_e L}{EA} + \frac{\alpha\Delta T_0 L}{2\pi} \tag{3-153}$$

The axial reaction force is given by

$$P_e = -\frac{\alpha EA\Delta T_0}{2\pi} \tag{3-154}$$

We now have enough information to evaluate the thermal stress from eq. (3-14):

$$\sigma(x, y) = \frac{P_e + F_T}{A} + \frac{(M + M_T)\, y}{I} - \alpha E\Delta T(x, y) \tag{3-155}$$

From eqs. (3-129), (3-133), and (3-154), we may evaluate the following set of terms:

$$\Sigma = \frac{P_e + F_T}{A} - \alpha E\Delta T = -\frac{\alpha E\Delta T_0}{2\pi}\left\{1 + \pi\left[\frac{1}{2} + \left(\frac{2y}{h}\right)\right]\sin(\pi x/L)\right\} \tag{3-156}$$

Example 3-4 A beam is constructed of 4340 alloy steel with a loading as shown in Fig. 3-12. The properties of the steel are $\alpha = 11.2 \times 10^{-6}\text{K}^{-1}$ ($6.22 \times 10^{-6\circ}\text{F}^{-1}$); $E = 214\,\text{GPa}$ (31.0×10^6 psi); $S_y = 934\,\text{MPa}$ (135,500 psi). The dimensions of the beam, which has a rectangular cross section, are height, 150 mm (5.91 in.); width, 65 mm (2.56 in.); and length, 6.711 m (22.02 ft). The temperature distribution is given by eq. (3-129), with a maximum temperature change of $\Delta T_o = 80^\circ\text{C}$ (144°F). The foundation modulus for the support is $k = 10\,\text{MPa}$ (1450 psi), and both ends of the beam are rigidly clamped. Determine the maximum transverse deflection and maximum stress for the beam.

First, let us calculate the reciprocal relaxation length parameter β from eq. (3-135). The area moment of inertia for the beam cross section is

$$I = \frac{bh^3}{12} = \frac{(0.065)(0.150)^3}{(12)} = 1828 \times 10^{-8}\,\text{m}^4 = 1828\,\text{cm}^4 \quad (43.92\,\text{in}^4)$$

The reciprocal relaxation length parameter may now be calculated:

$$\beta^4 = \frac{k}{4EI} = \frac{10 \times 10^6}{(4)(214 \times 10^9)(1828 \times 10^{-8})} = 0.6939\,\text{m}^{-4}$$

$$\beta = (0.6939)^{1/4} = 0.8941\,\text{m}^{-1}$$

The dimensionless parameter βL is

$$\beta L = (0.8941)(6.711) = 6.00$$

The beam dimensions correspond to a "long" beam. Note that βL must be in *radian* units when used in a trigonometric term.

Let us now determine the deflection parameter from its definition in eq. (3-139):

$$\delta_P = \frac{\left[\dfrac{(11.2 \times 10^{-6})(80^0)(6.711)^2}{(\pi^2)(0.150)}\right]}{1 + \frac{(4)(6.00)^4}{\pi^4}} = \frac{0.02726}{54.22} = 0.503 \times 10^{-3}\,\text{m} = 0.503\,\text{mm}$$

The maximum deflection occurs at the center of the span ($x = \frac{1}{2}L$). Using eq. (3-142), we may evaluate the maximum deflection of the beam.

$$v_{\max} = v\left(\tfrac{1}{2}L\right) = \delta_P\left[1 - (\pi/\beta L)\,e^{-\beta L/2}\sin(\beta L/2)\right]$$

$$v_{\max} = (0.503)\left[1 - (\pi/6.00)\,e^{-3.00}\sin(3.00)\right]$$

$$v_{\max} = (0.503)\,(1 - 0.00368) = 0.501\,\text{mm} \quad (0.0197\,\text{in.})$$

We note that, at the center of the span, the term involving the end condition is only about 0.4 percent of the term involving the thermal loading. This behavior is what we would anticipate from the "long"-beam solution.

The bending moment at the clamped ends of the beam may be calculated from eq. (3-147) or eq. (3-148):

$$M_0 = M(0) = -\frac{(2\pi)(0.503 \times 10^{-3})(214 \times 10^9)(1828 \times 10^{-8})(6.00)}{(6.711)^2}$$

$$M_0 = -1647\,\text{N-m} = -1.647\,\text{kN-m} \quad (1215\,\text{in-lb}_\text{f})$$

The bending moment at any location may be found from eq. (3-146):

$$M(x) = M_0\left[e^{-\beta x}\cos\beta x - (\pi/2\beta L)\sin(\pi x/L)\right] - \frac{\alpha EI\,\Delta T_0}{h}\sin(\pi x/L)$$

Let us calculate the numerical value for the coefficient in the last term:

$$\frac{\alpha EI\,\Delta T_0}{h} = \frac{(11.2 \times 10^{-6})(214 \times 10^9)(1828 \times 10^{-8})(80^0)}{(0.150)}$$
$$= 23.368 \times 10^3\,\text{N-m}$$

This term is the thermal moment at the center of the beam ($x = \frac{1}{2}L$), according to eq. (3-130). The mechanical bending moment at the center of the span is

$$M\left(\tfrac{1}{2}L\right) = (-1647)\left(e^{-3}\cos 3 - \pi/12\right) - (23.37 \times 10^3)(1)$$

$$M\left(\tfrac{1}{2}L\right) = +512 - 23{,}368 = -22{,}856\,\text{N-m} = -22.86\,\text{kN-N} \quad (-16{,}860\,\text{in-lb}_\text{f})$$

The last set of terms needed to determine the stress in the beam is the quantity given by eq. (3-156). Let us evaluate this term at the upper surface of the beam, $y = \frac{1}{2}h$:

$$\Sigma = -\left[\frac{(11.2 \times 10^{-6})(214 \times 10^{9})(80^{0})}{2\pi}\right]\left[1 + \pi\left(\frac{1}{2} + 1\right)\sin(\pi x/L)\right]$$

$$\Sigma = -\left(30.517 \times 10^{6}\right)\left[1 + \tfrac{3}{2}\pi \sin(\pi x/L)\right]$$

At the center of the span, this quantity has the following value at the upper surface ($y = +\frac{1}{2}h$):

$$\Sigma\left(\tfrac{1}{2}L\right) = -\left(30.517 \times 10^{6}\right)\left(1 + \tfrac{3}{2}\pi\right) = -174.325 \times 10^{6}\,\text{Pa} = -174.325\,\text{MPa}$$

At the lower surface ($y = -\frac{1}{2}h$), we find the following value:

$$\Sigma\left(\tfrac{1}{2}L\right) = -\left(30.517 \times 10^{6}\right)\left(1 - \tfrac{3}{2}\pi\right) = +113.291 \times 10^{6}\,\text{Pa} = 113.291\,\text{MPa}$$

We have calculated all quantities needed to determine the stress in the beam:

$$\sigma\,(x, y) = \Sigma(x, y) + \frac{(M + M_T)\,y}{I}$$

At the center of the span ($x = \frac{1}{2}L$) and at the upper surface ($y = \frac{1}{2}h$), we find the following value for the stress:

$$\sigma = \left(-174.325 \times 10^{6}\right) + \frac{(-22.856 + 23.368)\left(10^{3}\right)(0.075)}{\left(1828 \times 10^{-8}\right)}$$

$$\sigma = -174.325 \times 10^{6} + 2.101 \times 10^{6}$$

$$= -172.224 \times 10^{6}\,\text{Pa} = -172.22\,\text{MPa} \quad (-24{,}980\,\text{psi})$$

Similarly, at the center of the span and at the lower surface ($y = -\frac{1}{2}h$), we find the following stress:

$$\sigma = +113.291 \times 10^{6} - 2.101 \times 10^{6} = +111.190 \times 10^{6}\,\text{Pa}$$

$$= 111.19\,\text{MPa} \quad (16{,}130\,\text{psi})$$

At the clamped end of the beam, the thermal force and thermal moment are zero and the mechanical moment is equal to M_0. The stress at the end of the beam and at the upper surface is

$$\sigma = -30.517 \times 10^{6} + \frac{(-1647)\,(0.075)}{\left(1828 \times 10^{-8}\right)} = (-30.517 - 6.757)\left(10^{6}\right)$$

$$\sigma = -37.274 \times 10^{6}\,\text{Pa} = -37.27\,\text{MPa} \quad (-5410\,\text{psi})$$

From this series of calculations, we find that the maximum stress occurs at the center of the span and at the upper surface of the beam. This stress is compressive:

$$\sigma_{\max} = \sigma\left(\tfrac{1}{2}L, \tfrac{1}{2}h\right) = -172.22\,\text{MPa} \quad (24{,}980\,\text{psi})$$

3.6.4.3 Short-beam solution. If the dimensions of the beam are such that $\beta L < 5$, we find that the solution outlined in the previous section is not valid for this case. For mathematical convenience, we may take the general solution for the displacement in the following alternate form:

$$\begin{aligned} v(x) &= \sinh\beta x\left(C_1^* \sin\beta x + C_2^* \cos\beta x\right) \\ &\quad + \cosh\beta x\left(C_3^* \sin\beta x + C_4^* \cos\beta x\right) + v_P(x) \end{aligned} \tag{3-157}$$

The quantity $v_P(x)$ is the particular solution of eq. (3-128). This quantity depends on the thermal moment. For the temperature distribution given by eq. (3-129), the particular solution is

$$v_P(x) = \delta_P \sin(\pi x/L) \tag{3-158}$$

The rotation may be found from the displacement expression:

$$\begin{aligned} \omega = \frac{dv}{dx} &= \beta\cosh\beta x\left[\left(C_1^* - C_4^*\right)\sin\beta x + \left(C_2^* + C_3^*\right)\cos\beta x\right] \\ &\quad + \beta\sinh\beta x\left[\left(C_1^* + C_4^*\right)\cos\beta x + \left(C_3^* - C_2^*\right)\sin\beta x\right] + \frac{dv_P}{dx} \end{aligned} \tag{3-159}$$

For the temperature distribution given by eq. (3-129), the last term is

$$\frac{dv_P}{dx} = \left(\frac{\pi\delta_P}{L}\right)\cos(\pi x/L) \tag{3-160}$$

Let us consider the case for which the temperature distribution is given by eq. (3-129) and both ends of the beam are clamped. From the condition that the displacement is zero at the left end of the beam ($x = 0$), we find that $C_4^* = 0$. From the condition that the rotation is also zero at the left end of the beam, we find that $\left(C_2^* + C_3^*\right) = -\left(\pi\delta_P/\beta L\right)$. The other two conditions are that the displacement and rotation are also zero at the right end of the beam ($x = L$). Let us define the following quantities:

$$f_1 \equiv \sinh\beta L\ \sin\beta L \tag{3-161a}$$

$$f_2 \equiv \sinh\beta L\ \cos\beta L \tag{3-161b}$$

$$f_3 \equiv \cosh\beta L\ \sin\beta L \tag{3-161c}$$

$$f_4 \equiv \cosh\beta L\ \cos\beta L \tag{3-161d}$$

These terms may be used to express the two additional equations to determine the remaining constants of integration:

$$v(L) = 0 = C_1^* f_1 + C_2^* f_2 + C_3^* f_3 + \delta_P \tag{3-162a}$$

$$\omega(L) = 0 = \beta\left[C_1^*\left(f_2 + f_3\right) + \left(C_2^* + C_3^*\right) f_4 + \left(C_3^* - C_2^*\right) f_1 + \left(\pi\delta_P/\beta L\right)\right] \tag{3-162b}$$

If we substitute $C_3^* = -\left(\pi\delta_P/\beta L\right) - C_2^*$, we obtain the two expressions, which, if solved simultaneously, will yield the values for the constants of integration:

$$C_1^* f_1 + C_2^*\left(f_2 - f_3\right) + \left(1 - \frac{\pi f_3}{\beta L}\right)\delta_P = 0 \tag{3-163a}$$

$$C_1^*\left(f_2 + f_3\right) - 2C_2^* f_1 + \left(\frac{\pi\delta_P}{\beta L}\right)\left(1 - f_1 - f_4\right) = 0 \tag{3-163b}$$

The second derivative is related to the bending moment:

$$\frac{d^2v}{dx^2} = -\frac{M + M_T}{EI} = 2\beta^2\left[\cosh\beta x\left(C_1^*\cos\beta x - C_2^*\sin\beta x\right) + C_3^*\sinh\beta x\cos\beta x\right]$$
$$-\left(\frac{\pi^2\delta_P}{L^2}\right)\sin\left(\pi x/L\right) \tag{3-164}$$

The bending moment at the clamped end of the beam ($x = 0$) may be evaluated, for this case:

$$M_0 = M(0) = -2\beta^2 EI\, C_1^* \tag{3-165}$$

The third derivative may be used to express the shear force distribution:

$$\frac{d^3v}{dx^3} = -\frac{V + V_T}{EI} = -2\beta^3\left\{\cosh\beta x\left[C_1^*\sin\beta x + \left(C_2^* - C_3^*\right)\cos\beta x\right]\right\}$$
$$-2\beta^3\left(C_2^* + C_3^*\right)\sinh\beta x\sin\beta x - \left(\frac{\pi^3\delta_P}{L^3}\right)\cos\left(\pi x/L\right) \tag{3-166}$$

The vertical reaction force may be written as

$$F_0 = -V(0) = -2\beta^3 EI\left(C_2^* - C_3^*\right) - \left(\pi^3 EI\,\delta_P/L^3\right) + V_T(0) \tag{3-167}$$

If we make the substitutions for the constant C_3^* and the thermal shear force Q_{T} from eq. (3-131), the vertical reaction force may be written as

$$F_0 = -\left(\frac{\pi EI}{L^3}\right)\left[\frac{4\left(\beta L\right)^3 C_2^*}{\pi} + \left(2\beta^2L^2 + \pi^2\right)\delta_P\right] + \frac{\pi\alpha EI\,\Delta T_0}{hL} \tag{3-168}$$

3.7 THERMAL BOWING OF PIPES

When cryogenic liquids, such as liquid oxygen and liquid nitrogen, flow inside a pipeline at partial fill levels, severe thermal stresses and significant deflections may result [Flieder et al., 1961]. This problem may arise in partially filled steam lines, but it is especially troublesome in cryogenic pipelines. For these systems, the pipe material, such as 304 stainless steel, has a low thermal conductivity, so that significant temperature gradients may develop around the circumference of the pipe exposed to the vapor phase. In addition, the vapor tends to become thermally stratified as it is warmed by the heat transfer from the ambient temperature surroundings. The pipe temperature gradients tend to be sustained by this condition.

3.7.1 Thermal Analysis

In the examples considered previously in this chapter, the temperature distribution was given. We would like to illustrate how a reasonable temperature distribution in the pipe wall may be obtained for this problem, however.

Let us consider the situation shown in Figure 3-13. A horizontal pipe is partially filled with a liquid to a level ζD_m, where ζ is the liquid fill level parameter and ζD_m is the mean diameter of the pipe. Note that the mean pipe diameter is related to the pipe OD and ID by $D_m = D_0 - t = D_i + t$. The quantity t is the pipe wall thickness. The fill angle ϕ_0 and the liquid fill level parameter ζ are related:

$$\zeta D_m = \tfrac{1}{2} D_m (1 + \cos \phi_0) \qquad (3\text{-}169)$$

If we solve for the liquid fill parameter ζ, we obtain

$$\zeta = \tfrac{1}{2} (1 + \cos \phi_0) \qquad (3\text{-}170)$$

Similarly, if we solve for the fill angle ϕ_0, we obtain the following relationship.

$$\phi_0 = \cos^{-1} (2\zeta - 1) \qquad (3\text{-}171)$$

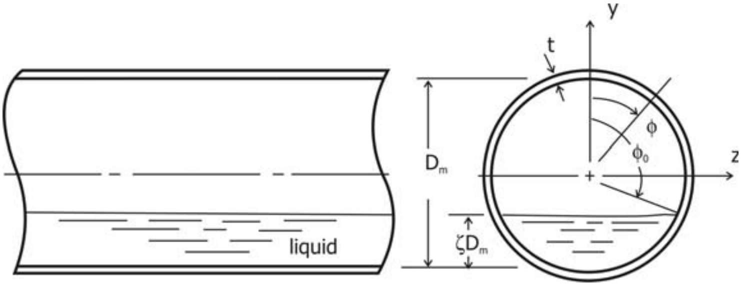

Figure 3-13. Horizontal pipe partially filled with a liquid.

The rate equation for conduction heat transfer is the *Fourier rate equation*, which is examined in more detail in Chapter 5. The heat transfer rate $\dot{Q}$ is related to the temperature gradient as follows:

$$\dot{Q} = -k_t A \frac{dT}{dx} \tag{3-172}$$

The quantity k_t is the thermal conductivity of the material {units: W/m-K or Btu/hr-ft-^0F}, A is the area through which the energy is conducted, and x is the coordinate perpendicular to the area.

If we apply the *conservation of energy principle* for steady-state conditions to the differential pipe wall element shown in Figure 3-14, we obtain

$$\dot{Q}_{\text{in}} + q_a\left[\tfrac{1}{2}\left(D_m + t\right) d\phi L\right] - \dot{Q}_{\text{in}} - \frac{d}{d\phi}\left(-\frac{2k_t t L}{D_m}\frac{dT}{d\phi}\right) d\phi = 0 \tag{3-173}$$

The quantity q_a is the heat transfer rate from the ambient surroundings per unit outside surface area of the pipe. If the thermal conductivity can be treated as constant, the differential equation for the temperature distribution, eq. (3-173), may be written as

$$\frac{d^2T}{d\phi^2} + \frac{q_a\left(D_m + t\right) D_m}{4k_t t} = 0 \tag{3-174}$$

Let us make the following definitions. The second term in eq. (3-174) is constant, so we may define θ_q as

$$\theta_q = \frac{q_a\left(D_m + t\right) D_m}{8k_t t} \tag{3-175}$$

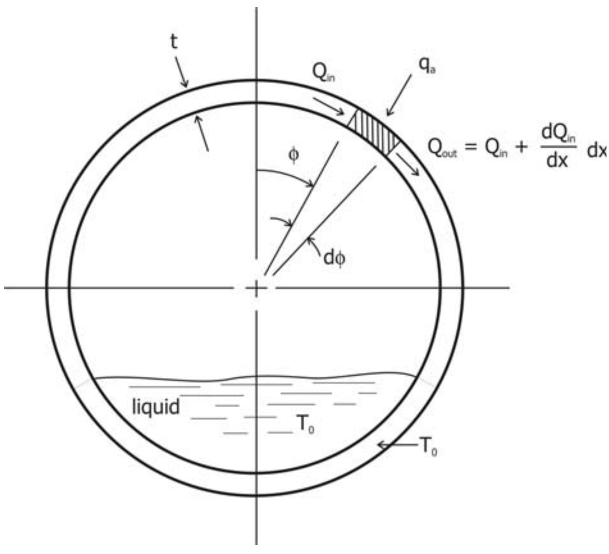

Figure 3-14. Differential element of the pipe wall for thermal analysis.

The temperature of the pipe exposed to the liquid will be approximately constant and equal to the liquid temperature T_0, because the liquid has a much higher thermal conductance than the vapor phase. Let us define the temperature difference parameter as

$$\Delta T = T - T_0 \tag{3-176}$$

With these terms, we may write the governing differential equation as

$$\frac{d^2 \Delta T}{d\phi^2} + 2\theta_q = 0 \tag{3-177}$$

We obtain the following result after integrating eq. (3-177) once:

$$\frac{d\Delta T}{d\phi} + 2\theta_q \phi = C_1 \tag{3-178}$$

Because the temperature distribution must be symmetrical with respect to the vertical axis, the temperature gradient is zero at the top of the pipe, $\phi = 0$; therefore, $C_1 = 0$. If we integrate eq. (3-178), we obtain the following result:

$$\Delta T + \theta_q \phi^2 = C_2 \tag{3-179}$$

At the liquid level, $\phi = \phi_0$, the pipe wall temperature is approximately equal to the liquid temperature, $T = T_0$, or $\Delta T = 0$; therefore, $C_2 = \theta_q \phi_0^2$.

The final expression for the temperature distribution may be written as

$$\Delta T = T - T_0 = \begin{cases} \theta_q \left(\phi_0^2 - \phi^2\right) & \text{for} \quad 0 \le \phi \le \phi_0 \\ 0 & \text{for} \quad \phi_0 < \phi \le \pi \end{cases} \tag{3-180}$$

The angle coordinate ϕ is related to the vertical coordinate y as follows:

$$y = \tfrac{1}{2} D_m \cos\phi \tag{3-181}$$

The temperature distribution expression is a good fit to experimental measurements on 8-in. and 12-in. (200 mm and 300 mm) nominal diameter lines half-filled with liquid oxygen (fill angle equal to 90°) for the upper portion of the pipe, between the top of the line and a location angle of approximately 75° [Arthur D. Little, 1959].

We note that the maximum temperature difference ΔT_0 occurs at the top of the pipe ($\phi = 0$):

$$\Delta T_0 = T_{\max} - T_0 = \theta_q \phi_0^2 = \frac{q_a (D_m + t) D_m \phi_0^2}{8 k_t t} \tag{3-182}$$

We observe that, for a given heat flux from ambient q_a, the maximum temperature difference will be increased if the thermal conductivity k_t or the pipe wall thickness t is decreased. The maximum temperature will be increased if the pipe diameter D_m is increased.

3.7.2 Thermal Stress Analysis

To concentrate on temperature effects, let us consider the case of a pipe of length L that is simply supported at each end, with no external mechanical loading. The stress at any point in the pipe wall may be written from eq. (3-14), with P_e and M zero:

$$\sigma = \frac{F_T}{A} + \frac{M_T y}{I} - \alpha E \Delta T \tag{3-183}$$

The thermal force parameter may be evaluated from eq. (3-7):

$$F_T = \alpha E \int \Delta T \, dA = 2\alpha E \int_0^{\phi_0} \theta_q \left(\phi_0^2 - \phi^2\right) \left(\tfrac{1}{2} D_m t\right) d\phi \tag{3-184}$$

The factor 2 is introduced, because the area and temperature distribution between 0 and $-\phi_0$ is the same as that between 0 and $+\phi_0$. The temperature difference ΔT is zero for the portion of the pipe exposed to the liquid, $\phi_0 < \phi \le \pi$. We obtain the following result after carrying out the integration:

$$F_T = \alpha E \theta_q D_m t \left[\left(\phi_0^2 \phi - \tfrac{1}{3}\phi^3\right)\right]_0^{\phi_0} = \tfrac{2}{3}\alpha E \theta_q D_m t \phi_0^3 \tag{3-185}$$

The pipe wall cross-sectional area is $A = \pi D_m t$, so the first term in the stress expression, eq. (3-183), may be written as

$$\frac{F_T}{A} = \frac{2\alpha E \theta_q \phi_0^3}{3\pi} = \frac{2\alpha E \Delta T_0 \phi_0}{3\pi} = \frac{\alpha E q_a \left(D_m + t\right) D_m \phi_0^3}{12\pi k_t t} \tag{3-186}$$

Next, let us determine the thermal moment:

$$M_T = \alpha E \int y \Delta T \, dA = 2\alpha E \int_0^{\phi_0} \left(\frac{1}{2} D_m \cos\phi\right) \theta_q \left(\phi_0^2 - \phi^2\right) \left(\frac{1}{2} D_m t\right) d\phi$$

$$M_T = \tfrac{1}{2}\alpha E D_m^2 t \theta_q \int_0^{\phi_0} \cos\phi \left(\phi_0^2 - \phi^2\right) d\phi$$

$$M_T = \tfrac{1}{2}\alpha E D_m^2 t \theta_q \left[\phi_0^2 \sin\phi - 2\phi\cos\phi - \left(\phi^2 - 2\right)\sin\phi\right]\Big|_0^{\phi_0}$$

The final expression for the thermal moment is

$$M_T = \alpha E D_m^2 t \theta_q \left(\sin\phi_0 - \phi_0 \cos\phi_0\right) \tag{3-187}$$

The area moment of inertia for the pipe wall cross section may be written as

$$I = \frac{1}{8}\pi \left(D_m^2 + t^2\right) D_m t = \frac{1}{8}\pi D_m^3 t \left[1 + \left(\frac{t}{D_m}\right)^2\right] \tag{3-188}$$

The term in brackets in eq. (3-188) is usually approximately equal to unity for the thin-walled pipes used in cryogenic systems. For example, for a 100-mm nominal (4-in. nominal) SCH 10 pipe (wall thickness, 3.1 mm or 0.120 in.), the mean pipe diameter is 111.3 mm (4.380 in.). The numerical value for the term in brackets is as follows, for this example:

$$1 + \left(\frac{t}{D_m}\right)^2 = 1 + \left(\frac{3.1}{111.3}\right)^2 = 1.00075$$

Thus, we may approximate the moment of inertia for thin-walled pipes ($t/D_m \leq 0.1$) by the following expression, with an error of 1 percent or less.

$$I = \tfrac{1}{8}\pi D_m^3 t \tag{3-189}$$

If we use the moment of inertia from eq. (3-189) in eq. (3-187), the thermal moment may be written in the following form:

$$M_T = \frac{8\alpha EI\theta_q}{\pi D_m}(\sin\phi_0 - \phi_0\cos\phi_0) \tag{3-190}$$

The thermal moment expression may also be written in terms of the fill level parameter ζ by using eq. (3-171):

$$\cos\phi_0 = 2\zeta - 1 \tag{3-191}$$

$$\sin\phi_0 = \sqrt{1 - \cos^2\phi_0} = 2\sqrt{\zeta\,(1-\zeta)} \tag{3-192}$$

The thermal moment expression is

$$M_T = \frac{8\alpha EI\theta_q}{\pi D_m}\left[2\sqrt{\zeta\,(1-\zeta)} - (2\zeta - 1)\cos^{-1}(2\zeta - 1)\right] \tag{3-193}$$

If we make the substitutions from eq. (3-186) for the thermal force and eq. (3-190) for the thermal moment into eq. (3-183), the following expressions for the thermal stress distribution may be obtained.

(a) For the portion of the pipe exposed to the vapor, $0 \leq \phi \leq \phi_0$:

$$\frac{\sigma}{\alpha E\theta_q} = -\left(\phi_0^2 - \phi^2 - \frac{2\phi_0^3}{3\pi}\right) + \frac{4}{\pi}(\sin\phi_0 - \phi_0\cos\phi_0)\cos\phi \tag{3-194a}$$

(b) For the portion of the pipe exposed to the liquid, $\phi_0 < \phi \leq \pi$:

$$\frac{\sigma}{\alpha E\theta_q} = \frac{2\phi_0^3}{3\pi} + \frac{4}{\pi}(\sin\phi_0 - \phi_0\cos\phi_0)\cos\phi \tag{3-194b}$$

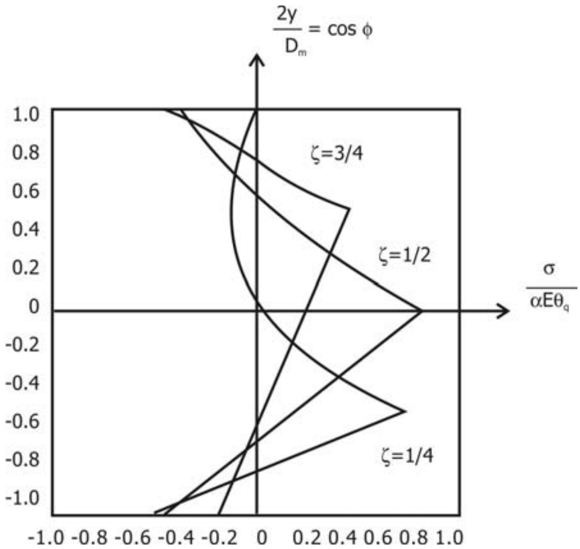

Figure 3-15. Thermal stress distribution for partially filled pipes.

The stress distribution for a few cases is shown in Figure 3-15.

The maximum stress occurs at the fill level, $\phi = \phi_0$:

$$\frac{\sigma_{\max}}{\alpha E\theta_q} = \frac{2\phi_0^3}{3\pi} + \frac{4}{\pi}(\sin\phi_0 - \phi_0\cos\phi_0)\cos\phi_0 \tag{3-195}$$

The maximum stress may also be expressed in terms of the fill level parameter ζ:

$$\begin{aligned}\frac{\sigma_{\max}}{\alpha E\theta_q} &= \frac{2}{3\pi}\left[\cos^{-1}(2\zeta - 1)\right]^3 \\ &\quad + \frac{4}{\pi}\left[2\sqrt{\zeta(1-\zeta)} - (2\zeta - 1)\cos^{-1}(2\zeta - 1)\right](2\zeta - 1)\end{aligned} \tag{3-196}$$

The maximum stress is shown in Figure 3-16 as a function of the fill level parameter ζ.

We observe that the stress is more severe when the pipe is filled with a small quantity of liquid (ζ small).

3.7.3 Deflection Analysis

To illustrate the pipe bowing phenomenon, let us consider a horizontal pipe with no external loading. The pipe has a length L and may be considered to be simply supported at each end. For this problem, let us take the origin of our coordinate system at the center of the pipe span. The temperature distribution in the pipe wall is given by eq. (3-180), and the thermal moment, which is constant, is given by eq. (3-187) or eq. (3-193).

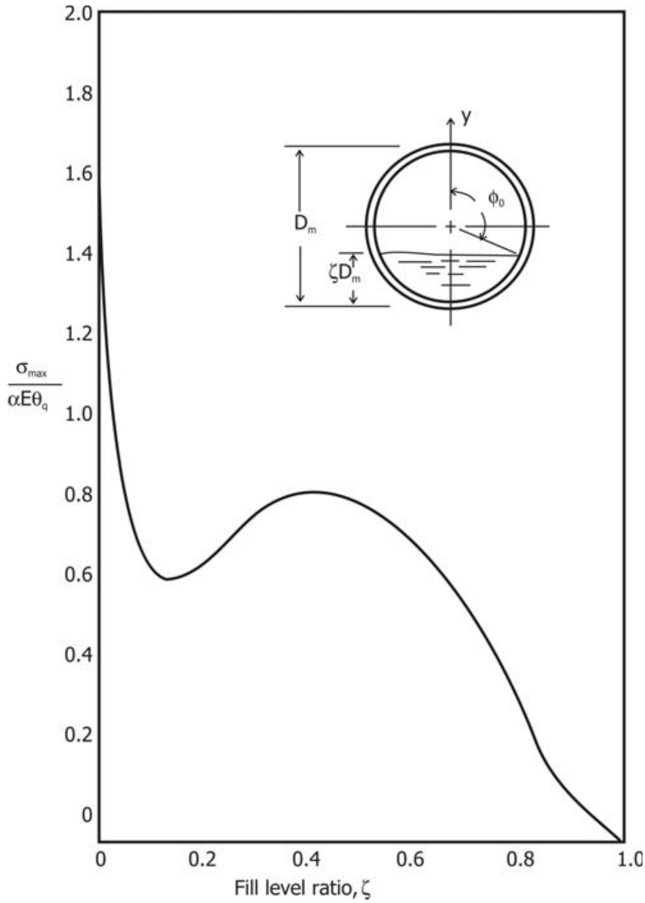

Figure 3-16. Maximum stress as a function of the fill level ζ for a partially filled pipe.

The governing expression for the transverse deflection for this problem is

$$\frac{d^2v}{dx^2} = -\frac{M + M_T}{EI} = -\frac{M_T}{EI} = \text{constant} \tag{3-197}$$

If we integrate twice, we obtain

$$v(x) = -\frac{M_T x^2}{2EI} + C_1 x + C_2 \tag{3-198}$$

The constants of integration may be evaluated from the condition that $v = 0$ at either end of the pipe ($x = \pm L/2$):

$$C_1 = 0 \quad \text{and} \quad C_2 = \frac{M_T L^2}{8EI}$$

Making these substitutions into eq. (3-198), we obtain the expression for the transverse displacement of the pipe at any point:

$$v = \frac{M_T L^2}{8EI}\left[1 - \left(\frac{2x}{L}\right)^2\right] \tag{3-199}$$

The maximum deflection occurs at the center of the pipe run ($x = 0$):

$$v_{\max} = v\,(0) = \frac{M_T L^2}{8EI} = \frac{\alpha\theta_q L^2}{\pi D}\,(\sin\phi_0 - \phi_0\cos\phi_0) \tag{3-200}$$

The maximum deflection may also be written in the following alternate form in terms of the fill level parameter ζ:

$$\frac{v_{\max} D_m}{\alpha\theta_q L^2} = \frac{1}{\pi}\left[2\sqrt{\zeta\,(1-\zeta)} - (2\zeta - 1)\cos^{-1}(2\zeta - 1)\right] \tag{3-201}$$

Example 3-5 A cryogenic pipeline is consists of a 304 stainless steel 200 mm nominal (8 in. nominal) Schedule 5 pipe having a length of 12.5 m (41.01 ft). The outside diameter of the pipe is 219.1 mm (8.625 in.), and the pipe wall thickness is 2.8 mm (0.109 in.). The pipe is half-full of liquid nitrogen ($\zeta = 0.500$) at a temperature of –195°C (–319°F). The properties of the pipe material are: Young's modulus, 193 GPa (28.0×10^6 psi); thermal expansion coefficient, $16 \times 10^{-6}\mathrm{K}^{-1}$ ($8.89 \times 10^{-6}{}^\circ\mathrm{F}^{-1}$); thermal conductivity, 12.3 W/m-K (7.11 Btu/hr-ft-°F). The heat flux based on the pipe outside surface area is 471.2 W/m^2 (149.4 Btu/hr-ft^2). Determine the maximum stress and deflection for the pipe, if the pipe is simply supported at both ends and there are no mechanical loads.

The line is half-full, so the liquid fill angle is

$$\phi_0 = \cos^{-1}(2\zeta - 1) = \cos^{-1}\left[(2)\left(\tfrac{1}{2}\right) - 1\right] = 90^\circ = \tfrac{1}{2}\pi \text{ rad}$$

The mean diameter of the pipe is $D_m = 219.1 - 2.8 = 216.3$ mm (8.516 in.). The thermal parameter may be evaluated from eq. (3-175):

$$\theta_q = \frac{(471.2)\,(216.3 + 2.8)\left(10^{-3}\right)(0.2163)}{(8)\,(12.3)\,(0.0028)} = 81.05 \text{ K}$$

The maximum temperature difference in the pipe wall is

$$\Delta T_0 = \theta_q\phi_0^2 = (81.05)\left(\tfrac{1}{2}\pi\right)^2 = 200.0\,\mathrm{K}\ (360^\circ\mathrm{F})$$

The maximum stress may be evaluated from eq. (3-195):

$$\frac{\sigma_{\max}}{\alpha E\theta_q} = \frac{(2)\left(\frac{1}{2}\pi\right)^3}{(3\pi)} + \frac{4}{\pi}\left[\sin\left(\frac{1}{2}\pi\right) - \left(\frac{1}{2}\pi\right)\cos\left(\frac{1}{2}\pi\right)\right]\cos\left(\frac{1}{2}\pi\right) = 0.8225$$

Then

$$\sigma_{\max} = (0.8225)\left(16.0 \times 10^{-6}\right)\left(193 \times 10^{9}\right)(81.05) = 205.9 \times 10^{6}\,\text{Pa}$$

$$\sigma_{\max} = 205.9\,\text{MPa}\ (29{,}860\,\text{psi})$$

The maximum deflection for the pipe may be found using eq. (3-200):

$$\frac{v_{\max} D_m}{\alpha \theta_q L^2} = \frac{1}{\pi}\left[\sin\left(\frac{1}{2}\pi\right) - \left(\frac{1}{2}\pi\right)\cos\left(\frac{1}{2}\pi\right)\right] = \frac{1}{\pi}$$

Solving for the maximum deflection, we obtain the following value:

$$v_{\max} = \frac{\left(16.0 \times 10^{-6}\right)(81.05)(12.5)^2}{(\pi)(0.2163)} = 0.298\,\text{m} = 298\,\text{mm}\,(11.74\,\text{in.})$$

The stress in this example is not excessive, since the yield strength for 304 stainless steel is 232 MPa (33,600 psi), which is 1.13 times the maximum stress. If the designer wants to reduce the thermal stress, one approach is to apply additional insulation to the pipe and reduce the heat flux to the pipe surface. Suppose we would like to reduce the maximum stress to $\sigma_{\max} = 125\,\text{MPa}\ (18{,}100\,\text{psi})$. The required value of the thermal parameter may be found as follows:

$$\theta_q = \frac{125 \times 10^{6}}{(0.8225)\left(16.0 \times 10^{-6}\right)\left(193 \times 10^{9}\right)} = 49.2\ \text{K}$$

Using eq. (3-175), the definition of the thermal parameter, we may find the required heat flux:

$$q_a = \frac{(8)(12.3)(0.0028)(49.2)}{(0.2191)(0.2163)} = 286\,\text{W/m}^2 \quad \left(90.7\,\text{Btu/hr-ft}^2\right)$$

If the heat flux is reduced to 286 W/m^2, the maximum deflection will be reduced to a value of (298 mm)(49.2/81.05) = 181 mm (7.12 in.).

One approach that could be used to reduce the maximum deflection of the pipe would be to anchor the pipe at the center, similar to the support system shown in Figure 3-11 and discussed in Section 3.6.3. The disadvantage to this approach is that one must contend with large anchor support reactions.

By combining eqs. (3-90) and (3-113), we may write the end support force in terms of the thermal moment:

$$F_1 = \frac{3M_T}{L}$$

The force at the center of the space is $2F_1$. For this problem, the expression for the thermal moment, eq. (3-190), reduces to

$$M_T = \frac{8\alpha E I \theta_q}{\pi D_m}\left[\sin\left(\frac{1}{2}\pi\right) - \left(\frac{1}{2}\pi\right)\cos\left(\frac{1}{2}\pi\right)\right] = \frac{8\alpha E I \theta_q}{\pi D_m}$$

The area moment of inertia for the pipe is given by eq. (3-189):

$$I = \frac{\pi}{8} D_m^3 t = \left(\frac{\pi}{8}\right)(0.2163)^3(0.0028)$$
$$= 1113 \times 10^{-8}\,\text{m}^4 = 1113\,\text{cm}^4 \quad \left(26.73\,\text{in}^4\right)$$

The value for the thermal moment is

$$M_T = \frac{(8)\left(16.0 \times 10^{-6}\right)\left(193 \times 10^9\right)\left(1113 \times 10^{-8}\right)(81.05)}{(\pi)(0.2163)}$$
$$M_T = 32.790 \times 10^3\,\text{N-m} = 32.79\,\text{kN-m}$$

We may now evaluate the support reactions for the pipe anchored (simply-supported) at each end and at the center of the span. The reaction force at either end of the pipe run is

$$F_1 = \frac{(3)(32{,}790)}{(12.5)} = 7870\,\text{N} = 7.78\,\text{kN} \quad (1770\,\text{lb}_\text{f})$$

The center reaction force is twice that at either end:

$$2F_1 = 15{,}740\,\text{N} = 15.74\,\text{kN} \quad (3540\,\text{lb}_\text{f})$$

The maximum deflection for the constrained pipe may be found in terms of the thermal moment by combining eqs. (3-90) and (3-108):

$$v_{\max} = \frac{M_T L^2}{108EI}$$

The maximum deflection for the anchored pipe is

$$v_{\max} = \frac{(32{,}790)(12.5)^2}{(108)\left(193 \times 10^9\right)\left(1113 \times 10^{-8}\right)} = 0.0221\,\text{m} = 22.1\,\text{mm} \quad (0.870\,\text{in.})$$

The stress distribution in the anchored pipe will involve an additional mechanical bending moment, which is zero at the ends of the pipe and equal to $-F_1 L/2 = -\frac{3}{2} M_T$ at the center of the span. At the ends of the span, the stress distribution is the same as determined previously for the unanchored pipe, because the mechanical bending moment is zero at the ends.

On the other hand, the anchor force produces the following stress distribution at the center of the span.

(a) Above the liquid level, $0 \le \phi \le \phi_0$:

$$\frac{\sigma}{\alpha E \theta_q} = -\left(\phi_0^2 - \phi^2 - \frac{2\phi_0^3}{3\pi}\right) - \frac{2}{\pi}(\sin\phi_0 - \phi_0 \cos\phi_0)\cos\phi$$

(b) Below the liquid level, $\phi_0 < \phi \leq \pi$:

$$\frac{\sigma}{\alpha E \theta_q} = \frac{2\phi_0^3}{3\pi} - \frac{2}{\pi}(\sin\phi_0 - \phi_0\cos\phi_0)\cos\phi$$

For the case considered in this problem of a half-filled pipe ($\phi_0 = \pi/2$), these expressions reduce to the following.

For $0 \leq \phi \leq \pi/2$:

$$\frac{\sigma}{\alpha E \theta_q} = -\left(\frac{\pi^2}{6} - \phi^2\right) - \frac{2}{\pi}\cos\phi$$

For $\pi/2 < \phi \leq \pi$:

$$\frac{\sigma}{\alpha E \theta_q} = \frac{\pi^2}{12} - \frac{2}{\pi}\cos\phi$$

The numerical values for the stress function at the top, liquid level, and bottom of the pipe are

$$\frac{\sigma}{\alpha E \theta_p} = \begin{cases} -\dfrac{\pi^2}{6} - \dfrac{2}{\pi} = -2.2816 & (\text{for } \phi = 0) \\ \dfrac{\pi^2}{12} = +0.8225 & (\text{for } \phi = \pi/2) \\ \dfrac{\pi^2}{12} + \dfrac{2}{\pi} = +1.4591 & (\text{for } \phi = \pi) \end{cases}$$

Thus, the maximum stress for the anchored case occurs at the center of the span and at the top of the pipe:

$$\sigma_{\max} = -(2.2816)\left(16.0 \times 10^{-6}\right)\left(193 \times 10^9\right)(81.05)$$

$$= 571 \times 10^6\,\text{Pa} = 571\,\text{MPa}\ (82{,}800\,\text{psi})$$

This value of stress *is* excessive, because the ultimate strength of 304 stainless steel is 516 MPa.

We conclude that constraint of the pipe, by introducing a simply supported point at the center of the span, would result in unacceptable stress levels, even though the deflection could be significantly reduced. We could use partial constraint (allow some motion at the center support) to reduce the stress in the pipe for the case of a center support; however, the deflection would be larger than that determined for the case of complete constraint. Increasing the thermal insulation or operating the pipeline completely filled (if possible) would probably be more practical solutions to the pipe bowing problem.

3.8 HISTORICAL NOTE

Beam-bending analysis has a long and interesting history. The first written record of an attempt to analyze bending of a beam was presented by Galilei Galileo in 1638 [Galileo, 1933]. His book *Two New Sciences* was the first published book on strength of materials.

Galileo noted that the "strength" of a bar in tension (which he also called the "absolute resistance to fracture") was proportional to the cross-sectional area of the bar. This means that Galileo's term "strength" actually referred to the *tensile force* required to break the bar, instead of the ultimate tensile strength that we consider today.

Galileo then investigated the "strength" of the bar if it were used as a cantilever beam. He incorrectly assumed that the "resistance" was uniformly distributed across the cross section of the beam, as shown in Figure 3-17. Galileo also assumed that the beam would tend to rotate about point B at the lower surface at the fixed end, as shown in Figure 3-17. Although he did not use the term *neutral axis*, his discussion indicated that he was assuming that the neutral axis for the cross section was at the lowest point of the cross section.

According to Galileo's assumption, the force required to cause yielding of the cantilever beam (or rupture for a "fully plastic" material) would be

$$F_{\text{yield}} = \frac{3M_0}{2L} = \frac{3S_y bh^2}{2L} \tag{3-202}$$

The actual failure load [Hodge, 1959] for a cantilever beam is

$$F_{\text{yield}} = \frac{M_0}{L} = \frac{S_y bh^2}{2L} \tag{3-203}$$

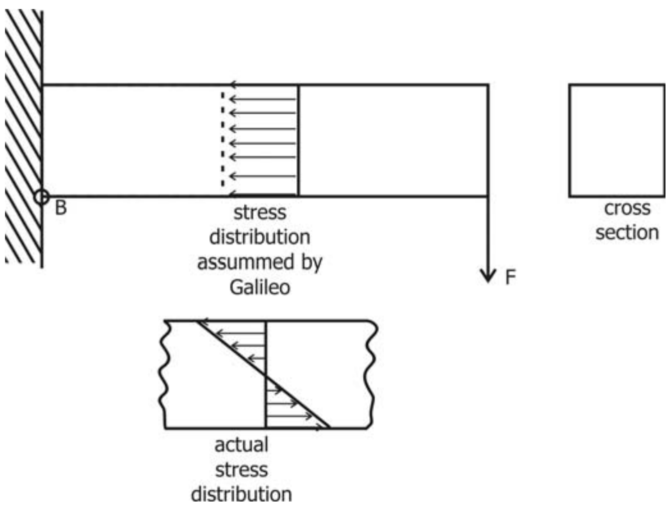

Figure 3-17. Cantilever beam according to Galileo.

Galileo's analysis yields a failure load that is three times larger than the actual failure load based on data from a simple tensile test.

E. Mariotte (1620–1684) conducted experiments with wooden and glass beams, and he observed that Galileo's theory predicted breaking loads that were larger than the loads found experimentally. Mariotte developed his own theory for determining the breaking load for cantilever beams, in which the stress distribution was assumed to be linear (zero at the lower fiber and maximum at the upper fiber), but the neutral axis was at the lowest point of the cross section, the same as Galileo's assumption. Mariotte obtained an expression for the critical force at the end of the beam:

$$F_{\text{yield}} = \frac{S_y b h^2}{L} \tag{3-204}$$

This expression was in error by "only" a factor of 2, instead of a factor of 3.

Around 1694, Jacob Bernoulli (the uncle of the renowned Daniel Bernoulli) calculated the deflection curve for a cantilever beam; however, he obtained an erroneous solution, because he also assumed that the neutral axis was at the lowest point of the cross section of the beam.

In 1713, Parent, who worked as an assistant with Des Billettes at the French Academy of Sciences, published a memoir in which he concluded that the stress distribution across the cross section of a cantilever beam with a load at the end was linear, but the bending stress was zero at the centroid of the cross-sectional area [Parent, 1713]. Thus, Parent assumed that the neutral axis for bending was also the centroid axis for the cross section. Because Parent's work was not published by the prestigious French Academy, it remained largely unnoticed by the scientific community for several years.

In 1773, C. A. Coulomb published a paper in which he analyzed the bending of a cantilever beam using the correct stress distribution and location of the neutral axis. He did not reference Parent's work, so it is apparent that Coulomb was unaware of Parent's prior work. Coulomb obtained the correct expression, eq. (3-203), for the force required to initiate yielding in the cantilever beam.

In 1826, M. Navier published a book on strength of materials in which he assumed that plane cross sections remained plane during bending of a beam and that the neutral axis passed through the centroid axis of the cross section [Burstall, 1965]. Navier utilized the following relationship for determining the deflection of beams:

$$\frac{d^2v}{dx^2} = -\frac{M}{EI} \tag{3-205}$$

Leonard Euler had developed this expression previously using calculus of variations around 1744, when he published a book on calculus of variations and elastic curves.

Around 1837, Saint-Venant developed a technique, which we now call the area–moment method, for calculating the deflection of a beam without integrating eq. (3-205). During the latter half of the 1880s, beam-bending theory and strength

of materials principles were extensively used in design of railway bridges and other bridges.

PROBLEMS

3-1. A beam has a symmetrical cross section, as shown in Figure 3-18. The beam is subjected to a temperature change given by the following relationship.

(a) For the upper flange, $\Delta T = \Delta T_0 = \text{constant}$

(b) For the web,

$$\Delta T\,(y) = \Delta T_0 \left\{ \frac{\cosh 2 - \cosh \left[1 - \left(\frac{2y}{h} \right) \right]}{(\cosh 2) - 1} \right\}$$

(c) For the lower flange, $\Delta T = 0$

The height of the web is denoted by h, and the thickness of the web is denoted by b. The thickness (height) for each flange is denoted by h_1, and the flange width is denoted by b_1. Determine the expressions for the thermal force F_T and the thermal moment M_T. Determine the numerical values for the thermal force and thermal moment if the beam is constructed of structural steel ($E = 200\,\text{GPa} = 29.0 \times 10^6\,\text{psi}$; $\alpha = 12.0 \times 10^{-6}\text{K}^{-1} = 6.67 \times 10^{-6\circ}\text{F}^{-1}$) with the following dimensions: $h = 116\,\text{mm}$ (4.567 in.), $b = 12\,\text{mm}$ (0.472 in.), $h_1 = 18\,\text{mm}$ (0.709 in.), and $b_1 = 35\,\text{mm}$ (1.378 in.) and $\Delta T_0 = 75^\circ\text{C}$ (135°F).

3-2. A cantilever beam having a rectangular cross section is subjected to a uniformly distributed load q_0, as shown in Figure 3-19. The temperature

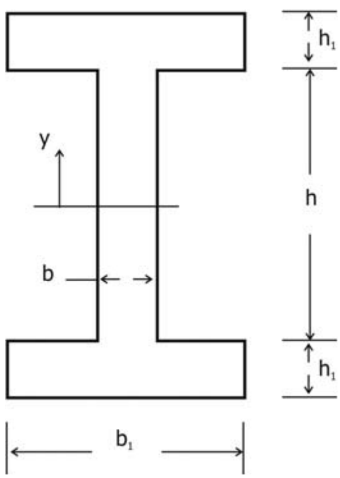

Figure 3-18. Beam cross section for Problem 3-1.

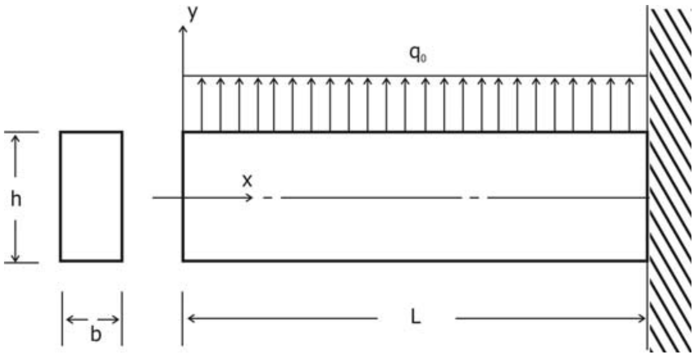

Figure 3-19. Beam loading for Problem 3-2.

distribution across the cross section is given by

$$\Delta T\,(x, y) = \frac{\Delta T_0 x^2 y}{L^2 h}$$

The quantity h is the depth of the beam, and L is the length of the beam. The coordinate y is measured from the centroid axis, and the coordinate x is measured from the free end of the beam. Determine (a) the expressions for the stress distribution and maximum stress for the beam, (b) the expressions for the transverse displacement and maximum deflection for the beam, and (c) the temperature difference parameter ΔT_0 that would result in zero deflection for the beam, if the beam is loaded with a uniform load $q_0 = -2.048$ kN/m (-1510 lb$_f$/ft), directed downward (in the negative y-direction). The dimensions of the beam are depth $h = 40$ mm (1.575 in.), width $b = 24$ mm (0.945 in.), and length $L = 550$ mm (21.65 in.). The beam is constructed of 70/30 brass, with a thermal expansion coefficient of $11 \times 10^{-6}\text{K}^{-1}$ ($6.1 \times 10^{-6\circ}\text{F}^{-1}$) and a Young's modulus of 110 GPa (15.95×10^6 psi). What is the maximum stress in the beam for this condition?

3-3. A cantilever beam having a rectangular cross section with a depth h, as shown in Figure 3-20, is subjected to the following temperature distribution:

$$\Delta T\,(x, y) = \Delta T_0\,(x/L) \sin\,(\pi y/h)$$

The beam is constructed of Class 30 gray cast iron ($E = 100\,\text{GPa} = 14.5 \times 10^6$ psi; $\alpha = 10 \times 10^{-6}\text{K}^{-1} = 5.56 \times 10^{-6\circ}\text{F}^{-1}$). The beam has a depth $h = 100$ mm (3.937 in.), a width $b = 75$ mm (2.953 in.), and a length $L = 1.250$ m (49.21 in.). The maximum temperature difference parameter $\Delta T_0 = 100^\circ$C (180°F). Determine: (a) the expression for the stress distribution in the beam and the maximum stress, (b) the expression for the transverse displacement of the beam and the maximum deflection, (c) the

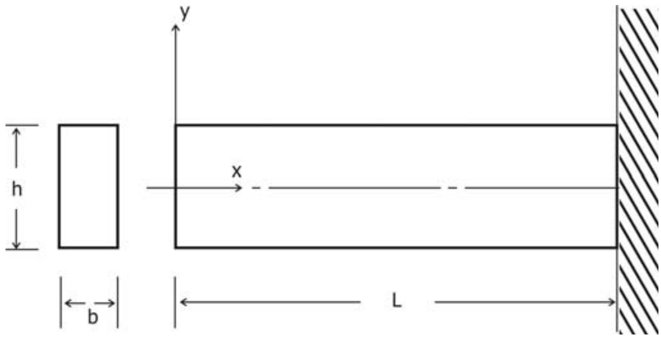

Figure 3-20. Beam for Problems 3-3 and 3-4.

numerical value for the maximum stress, and (d) the numerical value for the maximum deflection.

3-4. A cantilever beam having a rectangular cross section with a depth h, as shown in Figure 3-20, is subjected to the following temperature distribution:

$$\Delta T\ (x, y) = \Delta T_0\ (2y/h)^3 \cos\ (\pi x/2L)$$

The beam is constructed of stainless steel ($E = 200\,\text{GPa} = 29.0 \times 10^6$ psi; $\alpha = 16 \times 10^{-6}\text{K}^{-1} = 8.89 \times 10^{-6\circ}\text{F}^{-1}$). The beam has a depth $h = 90\,\text{mm}$ (3.543 in.), a width $b = 16.5\,\text{mm}$ (0.650 in.), and a length $L = 500\,\text{mm}$ (41.69 in.). The maximum deflection of the beam is 2.25 mm (0.0886 in.). Determine: (a) the required value of the maximum temperature change parameter ΔT_0, and (b) the maximum stress in the beam.

3-5. A simply-supported beam, as shown in Figure 3-21, has a rectangular cross section, with a depth $h = 105\,\text{mm}$ (4.134 in.), a width $b = 80\,\text{mm}$ (3.150 in.), and a length $L = 2.45\,\text{m}$ (8.038 ft). The beam is subjected to the following temperature distribution:

$$\Delta T\ (x, y) = \Delta T_0\ (2x/L)^2 \sin\ (\pi y/h)$$

The coordinate x is measured from the center of the span, and the coordinate y is measured from the centroid axis. The maximum temperature change parameter is $\Delta T_0 = 65^\circ\text{C}$ (117°F). The beam is constructed of 2024-T6 aluminum, for which the Young's modulus is 73.4 GPa (10.6×10^6 psi) and the thermal expansion coefficient is $22.5 \times 10^{-6}\text{K}^{-1}$ ($12.5 \times 10^{-6\circ}\text{F}^{-1}$). Determine: (a) the expression for the stress distribution and the numerical value for the maximum stress for the beam, and (b) the expression for the deflection of the beam as a function of the coordinate x and the numerical value for the maximum deflection.

3-6. A cantilever beam having a rectangular cross section, as shown in Figure 3-22 has one end (at $x = L$) rigidly fixed, and the other end

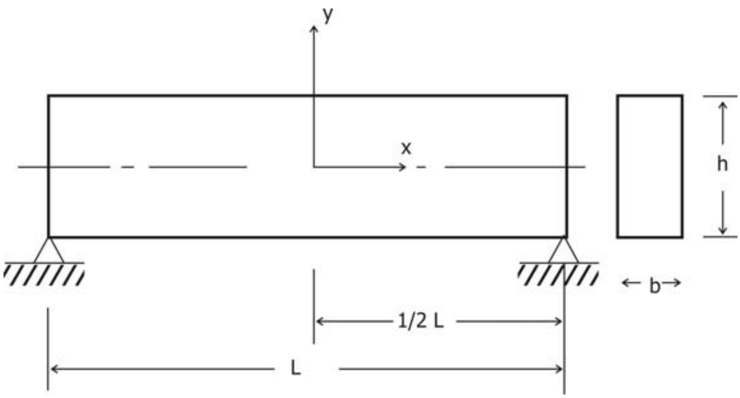

Figure 3-21. Beam for Problem 3-5.

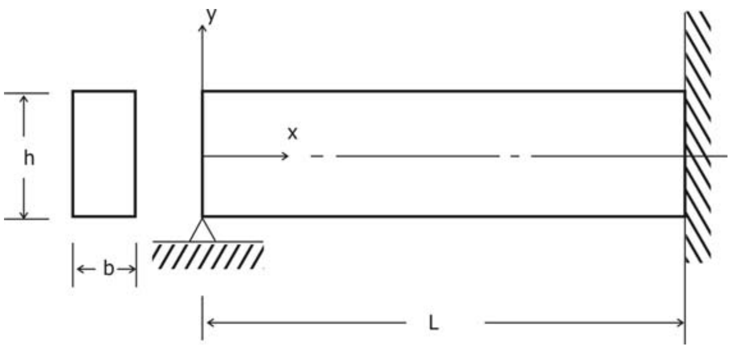

Figure 3-22. Beam geometry for Problem 3-6.

(at $x = 0$) is simply supported. The depth of the beam is denoted by h and the width is denoted by b. The beam is subjected to the following temperature distribution:

$$\Delta T\ (y) = \Delta T_1 \exp\left[\left(\frac{2y}{h}\right) - 1\right]$$

The quantity ΔT_1 is the temperature change at the top surface ($y = +h/2$) of the beam.

Determine (a) the expression for the support reaction force at the simply supported end of the beam, (b) the expression for the transverse displacement of the beam as a function of the coordinate x, and (c) the expression for the maximum stress in the beam. Suppose the beam is constructed of steel, for which $E = 200\,\text{GPa}$ (29×10^6 psi) and $\alpha = 11 \times 10^{-6}\text{K}^{-1}$ ($6.11 \times 10^{-6\circ}\text{F}^{-1}$). The beam dimensions are depth, 100 mm (3.937 in.); width, 72 mm (2.835 in.); and length, 2.00 m (6.562 ft). The maximum temperature change parameter $\Delta T_1 = 60^\circ\text{C}$

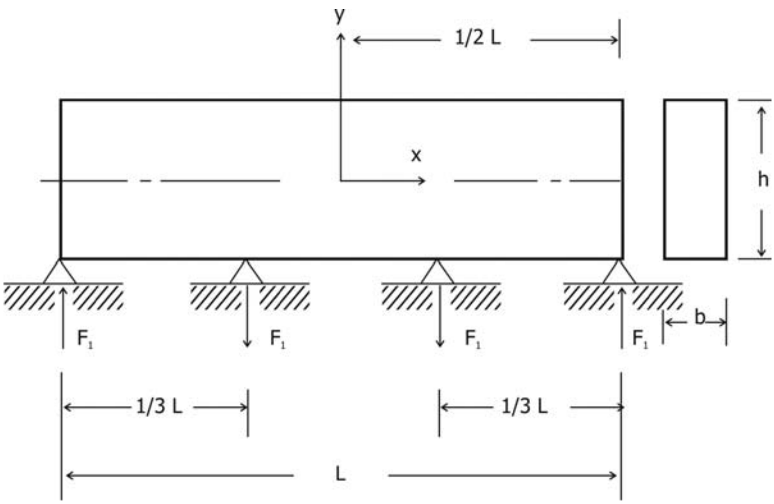

Figure 3-23. Beam support for Problem 3-7.

(108°F). Determine the numerical value of the maximum stress. Note that

$$\int e^u (u + 1)\, du = ue^u$$

3-7. A beam with a rectangular cross section is supported by four equally spaced simple supports, as shown in Figure 3-23. The beam has no mechanical loading, and the temperature distribution across the cross section of the beam is given by

$$\Delta T(y) = \begin{cases} \Delta T_0 & \text{for} \quad 0 \le y \le h/2 \\ 0 & \text{for} \quad -h/2 \le y < 0 \end{cases}$$

Determine (a) the expression for the stress distribution in the beam, and (b) the expression for the maximum stress. If the beam is constructed of 4130 steel, with Young's modulus, 214 GPa (31×10^6 psi) and thermal expansion coefficient, $11.2 \times 10^{-6}\text{K}^{-1}$ ($6.2 \times 10^{-6\circ}\text{F}^{-1}$), determine the numerical value of the maximum stress. The temperature difference parameter is $\Delta T_0 = 75^\circ\text{C}$ (135°F).

3-8. A cantilever beam with a circular cross section of diameter D, as shown in Figure 3-24, is buried in the ground, so that is has an elastic support with a foundation modulus k. The beam is subjected to a temperature distribution as follows:

$$\Delta T(x, y) = \begin{cases} \Delta T_0\,(x/L)^2\,(2y/D)^2 & \text{for} \quad 0 \le y \le D/2 \\ -\Delta T_0\,(x/L)^2\,(2y/d)^2 & \text{for} \quad -D/2 \le y < 0 \end{cases}$$

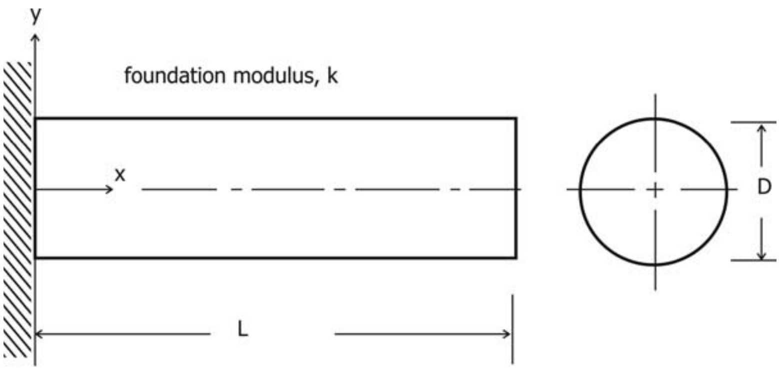

Figure 3-24. Elastically-supported beam for Problem 3-8.

The coordinate x is measured from the clamped end of the beam. The beam may be treated as a "long" beam. (a) Show that the thermal moment is given by

$$M_T = \frac{64\alpha EI\,\Delta T_0}{15\pi D}\left(\frac{x}{L}\right)^2$$

(b) Determine the expressions for the stress distribution and the transverse deflection of the beam. (c) Suppose the beam is constructed of a structural steel with a Young's modulus of 200 GPa (29.0×10^6 psi) and thermal expansion coefficient of $12 \times 10^{-6}\mathrm{K}^{-1}$ ($6.67 \times 10^{-6}{}^\circ\mathrm{F}^{-1}$). The beam has a length of 12.0 m (39.37 ft) and a diameter of 115 mm (4.528 in.). The temperature difference parameter is $\Delta T_0 = 75^\circ\mathrm{C}$ ($135^\circ\mathrm{F}$), and the foundation modulus is 10.52 MPa (1526 psi). Show that the "long"-beam solution is valid for this case. Determine the numerical value of the maximum stress and maximum transverse deflection for the beam.

3-9. The temperature distribution in the wall of a partially filled pipe is given by

$$\Delta T(y) = T(y) - T_0 = \begin{cases} \Delta T_0 \cos^2\phi & \text{for} \quad 0 \le \phi \le \frac{1}{2}\pi \\ 0 & \text{for} \quad \frac{1}{2}\pi < \phi \le \pi \end{cases}$$

The angle ϕ is measured from the top of the pipe, as shown in Figure 3-13. The temperature difference parameter $\Delta T_0 = T_1 - T_0$, where T_1 is the temperature at the top of the pipe cross section. The pipe has a mean diameter D_m, a wall thickness t, and a length L. The pipe is simply supported at each end. Determine: (a) the equation for the stress distribution and maximum stress in the pipe wall, and (b) the equation for the transverse deflection of the pipe and the maximum deflection. (c) Suppose the pipe is constructed of 304 stainless steel, with Young's modulus of 193 GPa (28.0×10^6 psi) and thermal expansion coefficient of $16.0 \times 10^{-6}\mathrm{K}^{-1}$ ($8.89 \times 10^{-6}{}^\circ\mathrm{F}^{-1}$). The dimensions of the pipe are mean diameter 319.28 mm (12.570 in.),

wall thickness 4.57 mm (0.180 in.), and length 12.19 m (40.0 ft). The temperature of the lower portion of the pipe is $T_0 = -180^\circ$C (-292°F), and the temperature at the top of the pipe is $T_1 = +20^\circ$C ($+68^\circ$F). Determine the numerical value for the maximum stress and maximum deflection for the pipe.

3-10. Suppose the pipe given in Problem 3-9 is simply supported at the center and at both ends, as shown in Figure 3-11. Determine (a) the expression for the end reaction force, F_1, (b) the numerical value for the end reaction force, and (c) the numerical value for the maximum stress in the pipe.

3-11. A section of a pipeline is clamped at each end, but free to move axially, as shown in Figure 3-25. The transverse deflection and rotation at each end is zero, but the axial displacement is not zero. The pipe is subjected to the following temperature distribution:

$$\Delta T(y) = T(y) - T_0 = \begin{cases} \Delta T_0 \cos\phi & \text{for} \quad 0 \le \phi \le \frac{1}{2}\pi \\ 0 & \text{for} \quad \frac{1}{2}\pi < \phi \le \pi \end{cases}$$

The location angle ϕ is measured from the top of the pipe. Determine (a) the expression for the end reaction moment M_0, (b) the transverse displacement, and (c) the expression for the stress distribution. (d) Determine the numerical value for the maximum stress in the pipe, if the pipe is constructed of stainless steel with Young's modulus of 193 GPa (28.0×10^6 psi) and thermal expansion coefficient of $16.2 \times 10^{-6}\text{K}^{-1}$ ($9.0 \times 10^{-6\circ}\text{F}^{-1}$). The mean pipe diameter is 161.2 mm (6.345 in.), the pipe wall thickness is 7.1 mm (0.280 in.), and the pipe length is 9.15 m (30.0 ft). The temperature difference parameter ΔT_0 is 200°C (360°F)

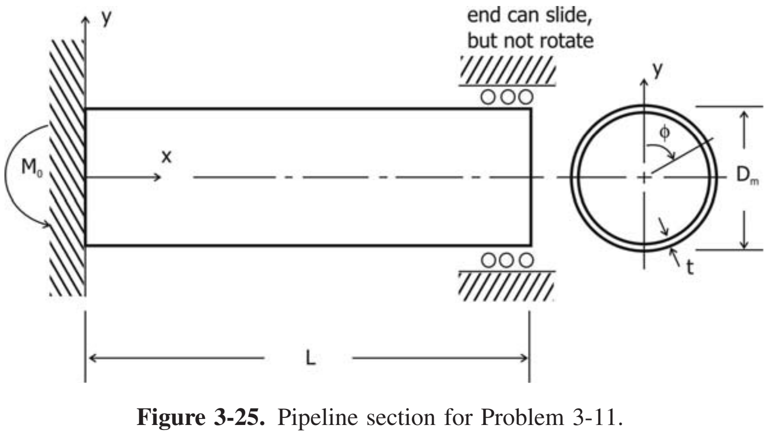

Figure 3-25. Pipeline section for Problem 3-11.

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4

THERMAL STRESSES IN TRUSSES AND FRAMES

A static structure composed of two or more members connected such that the members support only axial tensile or compressive loads is called a *truss*. Roof supports and bridges are two examples of trusses in engineering structures. The members may be connected at the ends by large bolts or pins, or by welding or riveting the ends to a common plate, called a gusset plate. If the ends are welded or riveted, the members may be treated as pin-connected members if the centerlines of the members intersect at a common point at each connection or joint.

A static structure composed of two or more members in which the members support both axial loading and bending loading is called a *frame*. Pipe U-bends and automobile suspension structures are two examples of frames.

In this chapter, we analyze the trusses and frames using the "strength-of-materials" approach, as discussed in Chapter 3.

4.1 ELASTIC ENERGY METHOD

The analysis of statically determinant isothermal trusses and frames is often accomplished by a consideration of forces alone. An alternate method of analysis, which is advantageous for cases involving thermal effects, is the *elastic energy method* [Van den Broek, 1942]. The elastic energy method applies for materials that do not exhibit hysteresis or permanent deformation under the applied loads. The elastic energy method may also be applied in analysis of indeterminate trusses and frames.

There are three general approaches for the elastic energy method: (a) *Castigliano's method*, or the strain energy method; (b) *Engesser's method*, or the

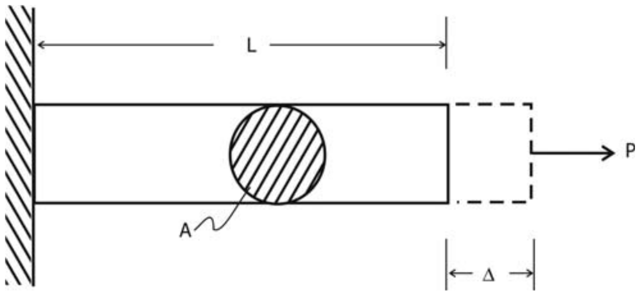

Figure 4-1. Bar subjected to an axial force and temperature change ΔT.

complementary energy method; and (c) the *Maxwell-Mohr method*, or the unit-load method.

We will first develop the complementary energy method for axial forces or for simple elongation with application to trusses. Let us consider the bar shown in Figure 4-1. We will denote the external applied force by P and the internal force by F. Note that, in this simple case, $F = P$. The displacement at the point of application of the external force will be denoted by Δ and the elongation of the member by e. Again, in this simple case, we note that $e = \Delta$. The ratio of force to mechanical elongation is called the *spring constant* k_{sp}:

$$k_{\text{sp}} = \frac{F}{e_m} = \frac{\sigma A}{\varepsilon_m L} = \frac{AE}{L} \tag{4-1}$$

If the bar is first subjected to a temperature change ΔT, then the external force is applied, we obtain a force–displacement curve as shown in Figure 4-2. The work done by the external force on the member is given by

$$W = \int P d\Delta \tag{4-2}$$

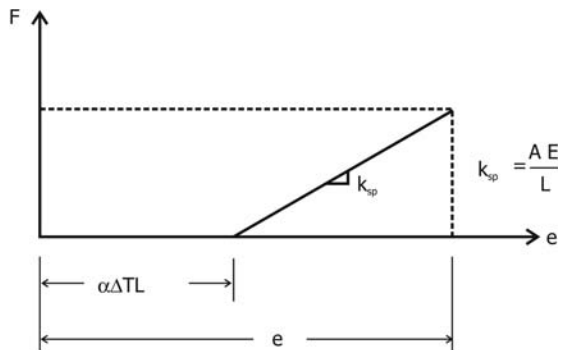

Figure 4-2. Force–displacement curve for a bar with axial loading, as shown in Figure 4-1.

The *complementary work* is defined as

$$W_c = \int \Delta\, dP = P\Delta - W \tag{4-3}$$

The *strain energy* or the mechanical energy stored in the bar is given as

$$U_s = \int F de = \int k_{\mathrm{sp}} e de = \frac{1}{2} k_{\mathrm{sp}} e_m^2 = \frac{F^2}{2k_{\mathrm{sp}}} \tag{4-4}$$

We may define the *complementary strain energy* as follows:

$$U_c = \int e dF = Fe - U_s = F\alpha \Delta T L + \frac{F^2}{2k_{\mathrm{sp}}} \tag{4-5}$$

If there are no dissipative effects present (hysteresis, etc.), the thermal energy added (heat transfer) becomes internal energy associated with the temperature change of the member, and the work energy is added to the member as strain energy. According to the *conservation of energy principle*, we have the following result:

$$W = U_s \tag{4-6}$$

Using eqs. (4-3) and (4-5) for the work and strain energy, we find

$$P\Delta - W_c = Fe - U_c \tag{4-7}$$

We note that $P\Delta = Fe$; therefore, the complementary work and complementary strain energy are equal:

$$W_c = U_c = \int \Delta\, dP \tag{4-8}$$

If we differentiate both sides of eq. (4-8) with respect to the applied load P, we obtain

$$\Delta = \frac{dU_c}{dP} \tag{4-9}$$

To apply the complementary energy method, we must express the internal force F as a function of the external force P, or

$$F = F(P) \tag{4-10}$$

This procedure is illustrated in Example 4-1.

For a system composed of several linear elastic members, the total complementary energy is equal to the sum of the complementary energy for each member:

$$U_c = \sum_j F_j \alpha_j \Delta T_j L_j + \sum_j \frac{F_j^2}{2k_j} \tag{4-11}$$

The temperature change ΔT_j is the average temperature change for the jth member. If P_x and P_y are the externally applied forced in the x- and y-directions, respectively, the displacements of the system at the point where the external

forces are applied and in the direction of the applied forces may be determined by an expression similar to eq. (4-9):

$$\Delta_x = \frac{\partial U_c}{\partial P_x} \tag{4-12a}$$

$$\Delta_y = \frac{\partial U_c}{\partial P_y} \tag{4-12b}$$

The complementary energy method is easily applied for manual calculations; however, the unit-load method or finite element method is better suited for digital computer solutions. We have examined the complementary energy method first, because the unit-load method is a modification of the complementary energy method.

Example 4-1 The pin-connected truss shown in Figure 4-3 is loaded by a vertical force of 5 kN (1124 lb_f). The cross-sectional area for each member is $A_1 = 60\ mm^2$ (0.0930 in^2) and $A_2 = 300\ mm^2$ (0.465 in^2). The length of member 2 is $L_2 = 1.200$ m (47.2 in.), and the angle between the two members is 30°. The material properties are Young's modulus, 200 GPa (29.0×10^6 psi) and thermal expansion coefficient, $11 \times 10^{-6}\ K^{-1}$ ($6.11 \times 10^{-6}\ {}^\circ F^{-1}$). The temperature change for member 1 is $\Delta T_1 = 60°C$ (108°F) and the temperature change of member 2 is zero. Determine the vertical and horizontal deflections of the joint at which the load is applied.

The spring constants for each member may be determined:

$$k_1 = \frac{A_1 E_1}{L_1} = \frac{(60 \times 10^{-6})(200 \times 10^9)}{(1.200)\cos 30^\circ} = 11.55 \times 10^6\ \text{N/m} = 11.55\ \text{MN/m}$$

$$k_2 = \frac{(300 \times 10^{-6})(200 \times 10^9)}{(1.200)} = 50.0 \times 10^6\ \text{N/m} = 50.0\ \text{MN/m}$$

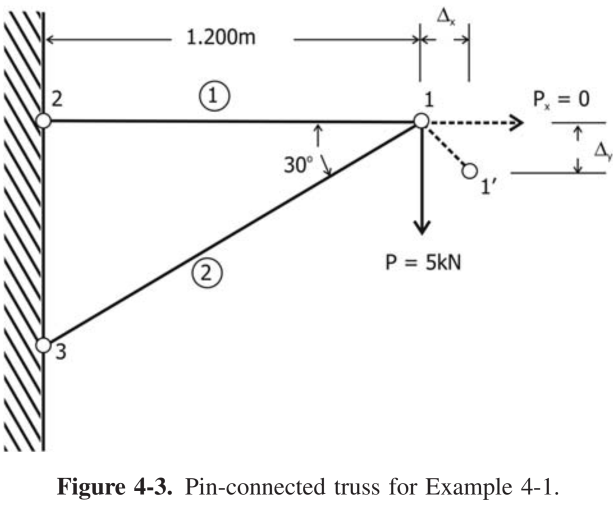

Figure 4-3. Pin-connected truss for Example 4-1.

Let us calculate the values of the following thermal parameters:

$$\alpha_1 \Delta T_1 L_1 = \left(11 \times 10^{-6}\right)(60)(1.039) = 0.6857 \times 10^{-3}\ \text{m}$$
$$= 0.6857\ \text{mm}\ \ (0.0270\ \text{in.})$$
$$\alpha_2 \Delta T_2 L_2 = 0$$

Because there is no external force acting in the horizontal direction, we must add a virtual horizontal force P_x and then set the force equal to zero after the differentiation. With this in mind, we can write the following expressions for the internal forces in each member:

$$F_1 = P_y \cot 30^\circ + P_x = \sqrt{3}P_y + P_x$$
$$F_2 = P_y \csc 30^\circ = 2P_y$$

The complementary energy for the two members may be written

$$U_c = \left(\sqrt{3}P_y + P_x\right)\alpha_1 \Delta T_1 L_1 + 0 + \frac{\left(\sqrt{3}P_y + P_x\right)^2}{2k_1} + \frac{\left(2P_y\right)^2}{2k_2}$$

The deflections may now be found from eq. (4-12). The horizontal deflection is

$$\Delta_x = \frac{\partial U_c}{\partial P_x} = \alpha_1 \Delta T_1 L_1 + \frac{\sqrt{3}P_y + P_x}{k_1}$$

$$\Delta_x = 0.6857 \times 10^{-3} + \frac{\sqrt{3}\,(5000) + 0}{11.55 \times 10^6} = 1.436 \times 10^{-3}\ \text{m}$$
$$= 1.436\ \text{mm}\ \ (0.0565\ \text{in.})$$

The vertical deflection is found as follows:

$$\Delta_y = \frac{\partial U_c}{\partial P_y} = \sqrt{3}\alpha_1 \Delta T_1 L_1 + \frac{\sqrt{3}\left(\sqrt{3}P_y + P_x\right)}{k_1} + \frac{4P_y}{k_2}$$

$$\Delta_y = \left(\sqrt{3}\right)\left(0.6857 \times 10^{-3}\right) + \frac{\left(\sqrt{3}\right)\left(5000\sqrt{3} + 0\right)}{11.55 \times 10^6} + \frac{(4)(5000)}{50.0 \times 10^6}$$

$$\Delta_y = 2.886 \times 10^{-3}\ \text{m} = 2.886\ \text{mm}\ \ (0.1136\ \text{in.})$$

The algebraic sign for each of the deflections is positive; therefore, the direction of the deflections Δ_x and Δ_y is in the same direction as the applied forces P_x and P_y, respectively.

4.2 UNIT-LOAD METHOD

The unit-load method and the complementary energy method for linear elastic materials are similar; however, the mathematical manipulations for the unit-load method are less extensive than for the complementary energy method. In addition, the unit-load method is well suited for digital computer applications.

4.2.1 Development of the Unit-load Method

Let us examine the unit-load method for trusses. By combining eqs. (4-9) and (4-11), we obtain the following expression for the deflection in the horizontal direction:

$$\Delta_x = \frac{\partial U_c}{\partial P_x} = \sum_j \frac{\partial F_j}{\partial P_x}\alpha_j \Delta T_j L_j + \sum_j \frac{F_j}{k_j}\frac{\partial F_j}{\partial P_x} \tag{4-13}$$

For a linear elastic material, the internal forces F_j are linear functions of the applied loads; therefore, the partial derivatives are constants. For example, we found the following expression for the force in member 1 in Example 4-1:

$$F_1 = \sqrt{3}P_y + P_x \quad \text{so} \quad \frac{\partial F_1}{\partial P_x} = 1 \tag{4-14}$$

We note that, if we apply a unit load, $f_x = 1$, in the horizontal direction, the resulting force in member 1 is $f_1 = 1$. In general, we conclude that the internal forces produced by a unit load in the horizontal direction are given by

$$f_j = \frac{\partial F_j}{\partial P_x} \tag{4-15}$$

We obtain a similar result for a unit load applied in the vertical direction.

If we make the substitution from eq. (4-15) into eq. (4-13), we obtain the important result for application of the unit load method:

$$\Delta_x = \sum_j f_j \alpha_j \Delta T_j L_j + \sum_j \frac{F_j f_j}{k_j} \tag{4-16}$$

where

F_j = actual force in the jth member
f_i = "dummy" force in the jth member due to the application of a unit load $f_x = 1$ in the x-direction at the point at which we want to determine the deflection in the x-direction
$k_j = A_j E_j / L_j$ = spring constant for the jth member

Although the quantity f_j is called a *dummy force*, it is actually dimensionless, as indicated by its definition in eq. (4-15). The dummy force is the rate of change of the internal force per unit change of the applied external force.

4.2.2 Determination of the Forces in a Truss

For simple systems, the solution for the forces in each member may be found "by hand"; however, for trusses with a large number of members, it becomes more effective to use methods amendable to digital computer use. The method described in this section for finding the forces in each member is sometimes called the *method of joints* [Meriam, 1975].

Let us consider the joint or connection between members 1 and 2, with applied external (known) forces P_x and P_y, as shown in Figure 4-4. If we set the sum of forces in the horizontal and vertical directions equal to zero, we obtain the following equations, which if solved simultaneously would yield values for the forces F_1 and F_2 in members 1 and 2, respectively:

$$F_1 \cos\theta_1 + F_2 \cos\theta_2 + P_x = 0 \tag{4-17a}$$

$$F_1 \sin\theta_1 + F_2 \sin\theta_2 + P_y = 0 \tag{4-17b}$$

These two expressions may be written in matrix notation:

$$\begin{bmatrix} \cos\theta_1 & \cos\theta_2 \\ \sin\theta_1 & \sin\theta_2 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} + \begin{Bmatrix} P_x \\ P_y \end{Bmatrix} = 0 \tag{4-18}$$

Equation (4-18) may also be written in a more compact form, if we define the matrix **B** and the matrix **F** as follows:

$$\mathbf{B} = \begin{bmatrix} \cos\theta_1 & \cos\theta_2 \\ \sin\theta_1 & \sin\theta_2 \end{bmatrix} \tag{4-19}$$

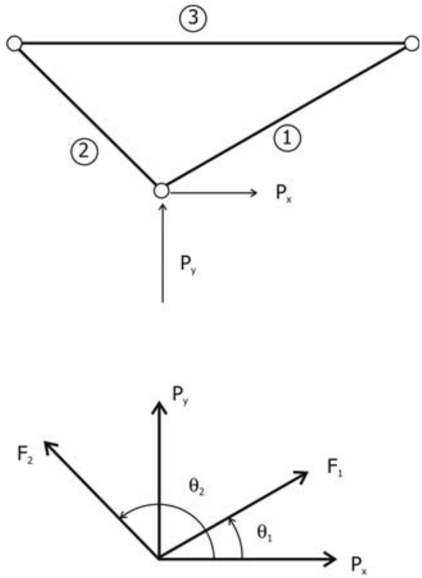

Figure 4-4. Forces at the joint between members 1 and 2 in Figure 4-3.

$$\mathbf{F} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{4-20}$$

The matrix equation resulting from the equilibrium force balance at the joint is

$$\mathbf{BF} + \mathbf{P} = 0 \tag{4-21}$$

We may solve for the unknown forces in the two members by multiplying both sides of eq. (4-21) by the inverse of matrix **B**:

$$\mathbf{B}^{-1}\mathbf{B}\,\mathbf{F} = \mathbf{F} = -\mathbf{B}^{-1}\mathbf{P} \tag{4-22}$$

The inverse of the 2×2 matrix **B** is

$$\mathbf{B}^{-1} = \frac{1}{\det \mathbf{B}} \begin{bmatrix} \sin\theta_2 & -\cos\theta_2 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \tag{4-23}$$

The determinant for matrix **B** is

$$\det \mathbf{B} = \begin{vmatrix} \cos\theta_1 & \cos\theta_2 \\ \sin\theta_1 & \sin\theta_2 \end{vmatrix} = \cos\theta_1 \sin\theta_2 - \sin\theta_1 \cos\theta_2 \tag{4-24}$$

The values for the direction angles for the jth member may be found from the coordinates of each end of the member, as indicated in Figure 4-5.

$$\cos\theta_j = \frac{(x_2 - x_1)_j}{L_j} \tag{4-25}$$

$$\sin\theta_j = \frac{(y_2 - y_1)_j}{L_j} \tag{4-26}$$

The length of each member may also be determined from the coordinates of each end of the member:

$$L_j = \sqrt{(x_2 - x_1)_j^2 + (y_2 - y_1)_j^2} \tag{4-27}$$

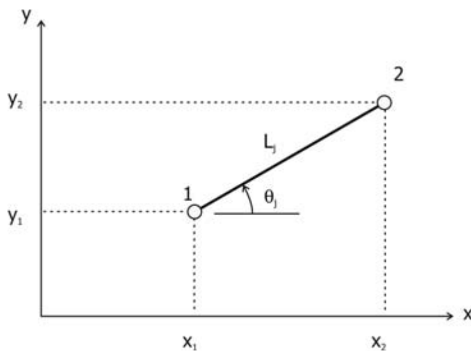

Figure 4-5. Direction angles for the jth member in a truss.

The coordinates (x_1, y_1) are the coordinates at the end of the member at the point at which the loads P are applied, and (x_2, y_2) are the coordinates of the other end of the member.

Example 4-2 Let us work the problem given in Example 4-1 using the unit-load method. The coordinates for the ends of each member are given in Table 4-1.

The length of each member may be found from eq. (4-27):

$$L_1 = 1.0392\text{ m}$$

$$L_2 = \sqrt{(1.0392)^2 + (0.600)^2} = 1.200\text{ m}$$

The direction angles may be determined from eqs. (4-25) and (4-26):

$$\cos\theta_1 = 1 \quad \text{and} \quad \sin\theta_1 = 0$$

$$\cos\theta_2 = \frac{-1.0392}{1.200} = -0.866 \quad \text{and} \quad \sin\theta_2 = \frac{-0.600}{1.200} = -0.500$$

The direction cosine matrix $\mathbf{B}$ is

$$\mathbf{B} = \begin{bmatrix} -1.000 & -0.866 \\ 0 & -0.500 \end{bmatrix}$$

The determinant of matrix $\mathbf{B}$ is found from eq. (4-24):

$$\det\mathbf{B} = (-1.000)\,(-0.500) - (-0.866)\,(0) = +0.500$$

Finally, the inverse of matrix $\mathbf{B}$ is found from eq. (4-23):

$$\mathbf{B}^{-1} = \frac{1}{0.500}\begin{bmatrix} -0.500 & +0.866 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1.00 & 1.732 \\ 0 & -2 \end{bmatrix}$$

The internal force matrix $\mathbf{F}$ may be found from eq. (4-22):

$$\mathbf{F} = -\begin{bmatrix} -1.00 & 1.732 \\ 0 & -2 \end{bmatrix}\begin{Bmatrix} 0 \\ -5000 \end{Bmatrix} = \begin{Bmatrix} +8{,}660 \\ -10{,}000 \end{Bmatrix}\text{N} = \begin{Bmatrix} +8.66 \\ -10.00 \end{Bmatrix}\text{kN}$$

TABLE 4-1. Data for Example 4-2

Member	End Coordinates	
1	$(x_1, y_1)_1 = (0, 0)$	$(x_2, y_2)_1 = (-1.0392, 0)$
2	$(x_1, y_1)_2 = (0, 0)$	$(x_2, y_2)_2 = (-1.0392, -0.600)$

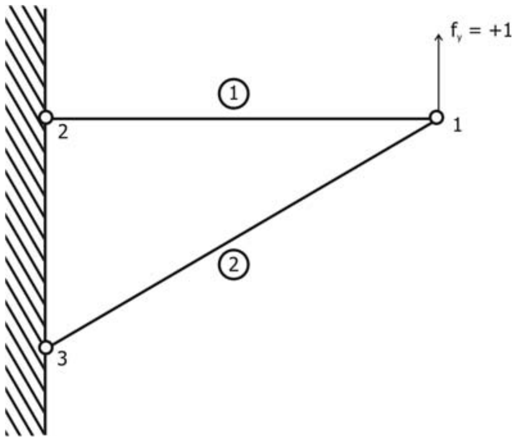

Figure 4-6. Application of the unit loads in the vertical direction for the truss shown in Figure 4-3.

We may also evaluate the dummy forces due to the application of a unit load $f_y = +1$ in the y-direction at the joint between the two members, as shown in Figure 4-6.

$$\mathbf{f}_y = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}_y = -\mathbf{B}^{-1}\mathbf{f}_{yu} = -\begin{bmatrix} -1.00 & 1.732 \\ 0 & -2 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -1.732 \\ +2.000 \end{Bmatrix}$$

We may calculate the terms in eq. (4-16) to determine the vertical deflection:

$$\sum_j f_j \alpha_j \Delta T_j L_j = (-1.732)\left(11 \times 10^{-6}\right)\left(60^\circ\right)(1.0392) + 0 = -1.1877 \times 10^{-3}\ \text{m}$$

$$\sum_j \frac{F_j f_j}{k_j} = \frac{(8660)\,(-1.732)}{11.55 \times 10^6} + \frac{(-10{,}000)\,(+2.000)}{50.0 \times 10^6} = -1.6987 \times 10^{-3}\ \text{m}$$

The deflection in the vertical direction is found from eq. (4-16):

$$\Delta_y = (-1.1877 - 1.6987)\left(10^{-3}\right) = -2.886 \times 10^{-3}\ \text{m} = -2.886\ \text{mm}\ (0.1136\ \text{in.})$$

This is the same result that we obtained using the complementary energy method. The calculations may be arranged in tabular form, as shown in Table 4-2. The use of the tabular format is quite effective for more complicated structures. The negative sign on the deflection means that the actual deflection is in the direction opposite to that of the unit load f_y, i.e., the vertical deflection is in the downward direction.

Similarly, the dummy forces for a unit load applied in the x-direction, as shown in Figure 4-7, are

$$\mathbf{f}_x = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}_x = -\mathbf{B}^{-1}\mathbf{f}_{xu} = -\begin{bmatrix} -1.00 & 1.732 \\ 0 & -2 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

TABLE 4-2. Solution for Example 4-2

Member j	F_j, N	f_j	k_j, N/m	$F_j f_j/k_j$, m	$\alpha_j \Delta T_j L_j$, m	$f_j \alpha_j \Delta T_j L_j$, m	$\sigma_j = F_j/A_j$, MPa
Vertical deflection							
1	+8660	1.000	11.55×10^6	0.7498×10^{-3}	0.6857×10^{-3}	0.6857×10^{-3}	144.3
2	−10,000	0	50.00×10^6	0	0	0	−33.3
Total				0.7498×10^{-3}		0.6857×10^{-3}	
Horizontal deflection							
1	+8660	−1.732	11.55×10^6	-1.2987×10^{-3}	0.6857×10^{-3}	-1.1877×10^{-3}	144.3
2	−10,000	+2.000	50.00×10^6	-0.4000×10^{-3}	0	0	−33.3
Total				-1.6987×10^{-3}		-1.1877×10^{-3}	

$\Delta_x = (0.7498 + 0.6857)(10^{-3}) = 1.4355 \times 10^{-3}$ m $= 1.436$ mm (0.0565 in.)
$\Delta_y = (-1.6987 - 1.1877)(10^{-3}) = -2.8864 \times 10^{-3}$ m $= 2.886$ mm (0.1136 in.)

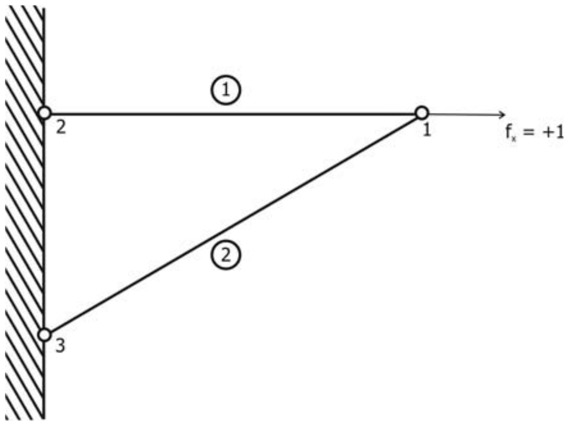

Figure 4-7. Application of the unit loads in the horizontal direction for the truss shown in Figure 4-3.

The horizontal deflection may be found from eq. (4-16):

$$\Delta_x = \sum_j \frac{F_j f_j}{k_j} + \sum_j f_j \alpha_j \Delta T_j L_j = \frac{(8660)\,(1)}{11.55 \times 10^6} + (1)\left(0.6857 \times 10^{-3}\right)$$

$$\Delta_x = (0.7498 + 0.6857)\left(10^{-3}\right) = 1.4355 \times 10^{-3}\ \text{m} = 1.436\,\text{mm}\ \ (0.0565\ \text{in.})$$

The stress in each member may be calculated from the known internal forces and cross-sectional areas. For example, the stress in member 1 is

$$\sigma_1 = \frac{F_1}{A_1} = \frac{8660}{60 \times 10^{-6}} = +144.3 \times 10^6\,\text{Pa} = 144.3\,\text{MPa}\ \ (20{,}900\,\text{psi})$$

The stress for each member is also given in Table 4-2.

4.3 TRUSSES WITH EXTERNAL CONSTRAINTS

In this section, we examine the analysis for trusses that are made statically indeterminate by having support reactions that cannot be found by direct application of the force equilibrium equations alone to the entire structure.

Let us consider the truss shown in Figure 4-8. The vertical support reaction P_y is zero, because there are no external loads applied in the vertical direction. The horizontal reaction for is unknown; however, we do know that the deflection at the support point is zero. This fact will be used to determine the reaction force through an analysis technique similar to that used in Section 4.2.

Example 4-3 All members of the truss shown in Figure 4-8 are constructed of structural steel with $E = 200\,\text{GPa}$ $(29.0 \times 10^6\ \text{psi})$ and $\alpha = 11 \times 10^{-6}\,\text{K}^{-1}$ $(6.11 \times 10^{-6}\ {}^\circ\text{F}^{-1})$. The cross-sectional area and temperature changes for each

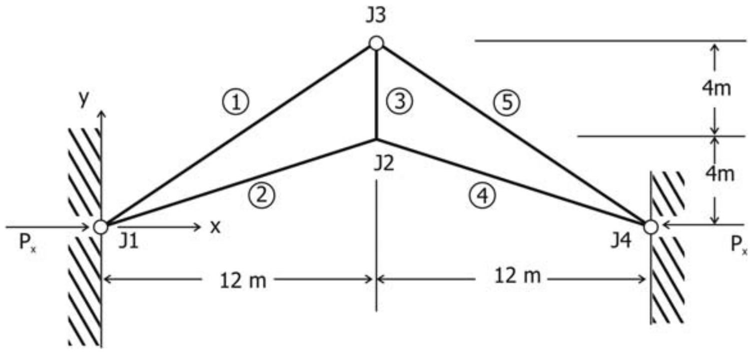

Figure 4-8. Pin-connected truss for Example 4-3.

TABLE 4-3. Data for Example 4-3

	End Coordinates, m						
Member j	$(x_1, y_1)_j$	$(x_2, y_2)_j$	L_j, m	A_j, cm^2	k_j, MN/m	ΔT_j, °C	$\alpha_j \Delta T_j L_j$, mm
1	(0, 0)	(12, 8)	14.422	160	221.88	10°	1.586
2	(0, 0)	(12, 4)	12.649	100	158.12	30°	4.174
3	(12, 4)	(12, 8)	4.000	60	300.00	20°	0.880
4	(12, 4)	(24, 0)	12.649	100	158.12	30°	4.174
5	(12, 8)	(24, 0)	14.422	160	221.88	10°	1.586

member are given in Table 4-3. The coordinates for the ends of each member are also given in Table 4-3. Determine the stress in each member.

The length of each member may be determined from eq. (4-27). For member 1 (or member 5), we find the following length:

$$L_1 = \sqrt{(12-0)^2 + (8-0)^2} = 14.422 \text{ m} \quad (47.32 \text{ ft})$$

The lengths for the other members are shown in Table 4-3.

The direction angles are determined from eqs. (4-25) and (4-26). For member 1 at the support point, we find the following values.

$$\cos\theta_1 = \frac{24-12}{14.422} = 0.83205$$

$$\sin\theta_1 = \frac{8-0}{14.422} = 0.55470$$

The direction angle is 33.7°. The direction cosine matrix for the members at the pin at the left support (joint J1 and members 1 and 2) is

$$\mathbf{B}^{(\mathrm{J1})} = \begin{bmatrix} 0.83205 & 0.94868 \\ 0.55470 & 0.31623 \end{bmatrix}$$

The determinant of the direction matrix for joint J1 is

$$\det \mathbf{B}^{(\mathrm{J1})} = \begin{vmatrix} 0.83205 & 0.94868 \\ 0.55470 & 0.31623 \end{vmatrix} = -0.26311$$

The inverse matrix for joint J1 is

$$\mathbf{B}^{-1} = \frac{1}{-0.26311}\begin{bmatrix} 0.31623 & -0.94868 \\ -0.55470 & 0.83205 \end{bmatrix} = \begin{bmatrix} -1.20188 & 3.60559 \\ 2.10821 & -3.16232 \end{bmatrix}$$

The force matrix for joint J1 is

$$\mathbf{F}^{(\mathrm{J1})} = -\begin{bmatrix} -1.20188 & 3.60559 \\ 2.10821 & -3.16232 \end{bmatrix}\begin{Bmatrix} P_x \\ 0 \end{Bmatrix} = \begin{Bmatrix} +1.20188P_x \\ -2.10821P_x \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

If we use F_2 as a known force at joint J2, we may solve for the forces in members 3 and 4. The force in member 5 (and a check on the force in member 3) may be found by using the force F_1 as a known force at joint J3. The resulting values for the forces in each member are shown in Table 2-4.

The "dummy" forces are given in Table 4-4 for the other members.

The horizontal deflection at the support point (which is $\Delta_x = 0$) may be used to find the horizontal reaction force P_x:

$$\Delta_x = 0 = -14.9603 \times 10^{-3} + 75.1619 \times 10^{-9} P_x$$

$$P_x = 199.04 \times 10^3\ N = 199.04\,\mathrm{kN}\ \ (44{,}740\,\mathrm{lb_f})$$

The positive sign for the support reaction P_x means that the load is in the same direction as that used in calculating the forces F_1 and F_2.

The required horizontal gap distance at the support may be found to limit the stresses to a prescribed value. For example, suppose we want to limit the stress in member 3 to 20 MPa (2900 psi) compressive. The required force in member 3

TABLE 4-4. Results for Example 4-3

Member j	Force, F_j	"Dummy" Force, f_j	$F_j f_j/k_j$, m	$f_j \alpha_j \Delta T_j L_j$, m	Stress, $\sigma_j = F_j/A_j$, MPa
1	$+1.2019P_x$	$+1.2019$	$6.5100 \times 10^{-9} P_x$	$+1.9061 \times 10^{-3}$	+14.95
2	$-2.1082P_x$	-2.1082	$28.1080 \times 10^{-9} P_x$	-8.7996×10^{-3}	−41.96
3	$-1.3333P_x$	-1.3333	$5.9259 \times 10^{-9} P_x$	-1.1733×10^{-3}	−44.23
4	$-2.1082P_x$	-2.1082	$28.1080 \times 10^{-9} P_x$	-8.7996×10^{-3}	−41.96
5	$+1.2019P_x$	$+1.2019$	$6.5100 \times 10^{-9} P_x$	$+1.9061 \times 10^{-3}$	+14.95
Total			$75.1619 \times 10^{-9} P_x$	-14.9603×10^{-3}	

$$P_x = \frac{14.9603 \times 10^{-3}}{75.1619 \times 10^{-9}} = 0.19904 \times 10^6\ \mathrm{N} = 199.04\ \mathrm{kN}\ \ (44{,}700\ \mathrm{lb_f})$$

to achieve this stress level is

$$F_3 = \sigma_3 A_3 = \left(-20 \times 10^6\right)\left(60 \times 10^{-6}\right) = -120 \times 10^3 \text{ Pa} = -120\,\text{kN}$$

The required horizontal reaction force may be determined:

$$F_3 = -1.333 P_x = -120{,}000\,\text{N}$$

$$P_x = 90{,}000\,\text{N} = 90\,\text{kN} \;\; (20{,}230\,\text{lb}_\text{f})$$

The gap distance may be determined from eq. (4-16):

$$\delta = -14.9603 \times 10^{-3} + \left(75.1619 \times 10^{-9}\right)\left(90 \times 10^3\right)$$

$$\delta = -8.20 \times 10^{-3} \text{ m} = -8.20\,\text{mm} \;\; (-0.323 \text{ in.})$$

The minus sign denotes that the gap is in the direction opposite to that of the reaction force P_x, or to the left from joint J1.

4.4 TRUSSES WITH INTERNAL CONSTRAINTS

In a statically determinant truss, changes in temperature of the members do not result in thermal stresses, because the members are not constrained and are free to deflect as the members expand or contract. This fact may be seen from the truss shown in Figure 4-3. If the external force P_y were equal to zero, the forces F_1 and F_2 in each member would be zero. The deflection at the end would not be zero, if member 1 experiences a temperature change; however, this would not affect the stresses in each member.

On the other hand, if the truss has more members than are necessary to prevent collapse of the truss, then the extra members are *redundant members* and the truss is statically indeterminate. In this case, thermal stresses may develop, because the truss is constrained internally.

4.4.1 Trusses with Thermal Loads Only

Let us develop the unit load method for trusses with redundant members in two steps. First, we will consider the case in which only temperature changes of the members are involved and no mechanical forces are applied. Then, we will consider the case in which both thermal and mechanical loads are applied.

Let us consider the truss shown in Figure 4-9. The truss is statically indeterminate because there is one redundant member or one more member than is required to keep the structure in static equilibrium. Let us remove the nth member and replace the member with the force F_R in the nth member acting at joints A and B. We may determine the deflection Δ_{AB} between the points A and B by applying a unit load $f_R = 1$ at these points, as shown in Figure 4-10. The forces in the other members of the truss may be found from the following:

$$F_j = f_j F_R \tag{4-28}$$

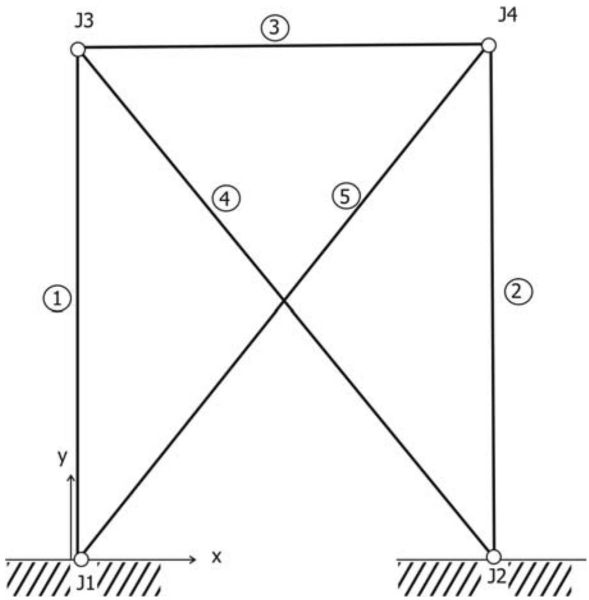

Figure 4-9. Pin-connected truss with a redundant member.

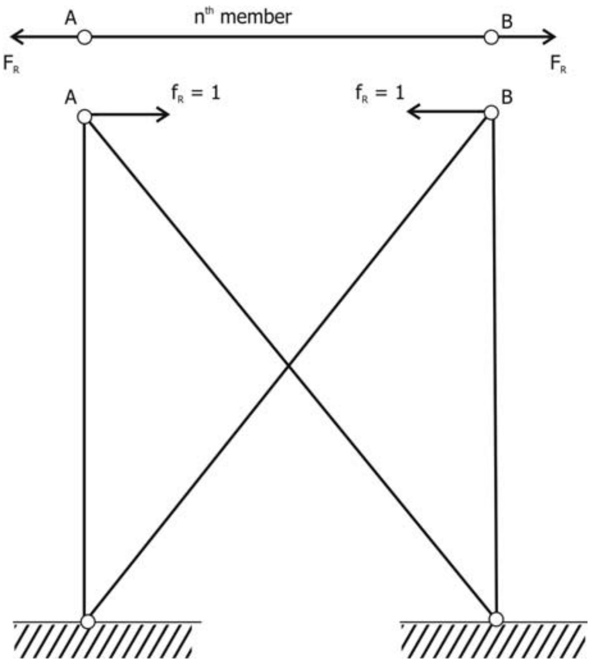

Figure 4-10. Application of a unit load for the truss shown in Figure 4-9.

The deflection between points A and B may be found from eq. (4-16):

$$\Delta_{AB} = \sum_{j \neq n} f_j \alpha_j \Delta T_j L_j + \sum_{j \neq n} \frac{F_j f_j}{k_j} = \sum_{j \neq n} f_j \alpha_j \Delta T_j L_j + \sum_{j \neq n} \frac{F_R f_j^2}{k_j} \qquad (4\text{-}29)$$

The force in the redundant member F_R is a constant, as far as the summation is concerned in eq. (4-29):

$$\Delta_{AB} = \sum_{j \neq n} f_j \alpha_j \Delta T_j L_j + F_R \sum_{j \neq n} \frac{f_j^2}{k_j} \qquad (4\text{-}30)$$

The deflection of the redundant bar between points A and B may be written as

$$\Delta_n = \alpha_n \Delta T_n L_n + \frac{F_R}{k_n} = f_R \alpha_n \Delta T_n L_n + \frac{F_R f_R^2}{k_n} \qquad (4\text{-}31)$$

The last expression may be written, because $f_n = f_R = 1$.

The redundant bar and the remainder of the truss remain in contact after the thermal expansion takes place; therefore, we may write

$$\Delta_{AB} + \Delta_n = 0 \qquad (4\text{-}32)$$

If we make the substitutions from eqs. (4-30) and (4-31) into eq. (4-32), we obtain

$$\sum_{j \neq n} f_j \alpha_j \Delta T_j L_j + f_R \alpha_n \Delta T_n L_n + F_R \sum_{j \neq n} \frac{f_j^2}{k_j} + F_R \frac{f_R^2}{k_n} = 0 \qquad (4\text{-}33)$$

Or, we may write eq. (4-33) in the following form:

$$\sum_{\text{all } j} f_j \alpha_j \Delta T_j L_j + F_R \sum_{\text{all } j} \frac{f_j^2}{k_j} = 0 \qquad (4\text{-}34)$$

The force in the redundant member may be determined from the following expression:

$$F_R = -\frac{\displaystyle\sum_j f_j \alpha_j \Delta T_j L_j}{\displaystyle\sum_j \frac{f_j^2}{k_j}} \qquad (4\text{-}35)$$

If the calculated value of the force F_R is positive, the force is tensile; if F_R is negative, the force is compressive. The forces in the other members may be determined from the "dummy" loads f_j and the force in the redundant member F_R by using eq. (4-28).

Example 4-4 The coordinates for the joints in the truss shown in Figure 4-9 are given in Table 4-5. The truss is constructed of steel with $E = 200\,\text{GPa}$ (29×10^6 psi) and $\alpha = 11 \times 10^{-6}\ \text{K}^{-1}$ ($6.11 \times 10^{-6}\ {}^\circ\text{F}^{-1}$). Member 3 has a

TABLE 4-5. Data for Example 4-4

	End Coordinates, m						
Member j	$(x_1, y_1)_j$	$(x_2, y_2)_j$	L_j, m	A_j, cm^2	k_j, MN/m	ΔT_j, °C	$\alpha_j \Delta T_j L_j$, mm
1	(0, 0)	(0, 1.20)	1.20	10	166.67	0°	0
2	(1.60, 0)	(1.60, 1.20)	1.20	10	166.67	0°	0
3	(0, 1.20)	(1.60, 1.20)	1.60	15	187.50	80°	1.408
4	(0, 0)	(1.60, 1.20)	2.00	20	200.00	0°	0
5	(0, 1.20)	(1.60, 0)	2.00	20	200.00	0°	0

temperature change $\Delta T_3 = +80°\text{C}$ (144°F), and the other members have zero temperature change. Determine the stress in each member.

Let us remove member 3 and apply a unit load at joints J3 and J4 ($f_3 = f_R = 1$). There will be a corresponding unit reaction at joints J1 and J2. The direction angles at joint J1 are

$$\cos\theta_1 = 0 \quad \text{and} \quad \sin\theta_1 = 1$$
$$\cos\theta_4 = \frac{1.60}{2.00} = 0.800 \quad \text{and} \quad \sin\theta_4 = \frac{1.20}{2.00} = 0.600$$

The direction cosine matrix at joint J1 is

$$\mathbf{B}^{(\mathrm{J1})} = \begin{bmatrix} 0 & 0.800 \\ 1 & 0.600 \end{bmatrix}$$

The determinant for the direction cosine matrix is found as follows:

$$\det \mathbf{B} = 0 - 0.800 = -0.800$$

The inverse matrix may be found from eq. (4-23):

$$\mathbf{B}^{-1} = \frac{1}{-0.800}\begin{bmatrix} 0.600 & -0.800 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -0.750 & +1 \\ +1.250 & 0 \end{bmatrix}$$

The "dummy" forces in members 1 and 4 may now be determined:

$$\mathbf{f}^{(\mathrm{J1})} = \begin{Bmatrix} f_1 \\ f_4 \end{Bmatrix} = -\begin{bmatrix} -0.750 & 1 \\ +1.250 & 0 \end{bmatrix}\begin{Bmatrix} +1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} +0.750 \\ -1.250 \end{Bmatrix}$$

The other calculations are summarized in Table 4-6. The force in the redundant member 3 may be found from eq. (4-35):

$$F_R = F_3 = -\frac{\sum_j f_j \alpha_j \Delta T_j L_j}{\sum_j \dfrac{f_j^2}{k_j}} = -\frac{1.408 \times 10^{-3}}{27.7083 \times 10^{-9}} = -50.815 \times 10^3 \text{ N}$$
$$= -50.815 \text{ kN}$$

TABLE 4-6. Results for Example 4-4

Member j	"Dummy" Force, f_j	f_j^2/k_j, m/N	$f_j \alpha_j \Delta T_j L_j$, m	$F_j = f_j F_R$, kN	Stress, $\sigma_j = F_j/A_j$, MPa
1	+0.750	3.375×10^{-9}	0	−38.111	−38.11
2	+0.750	3.375×10^{-9}	0	−38.111	−38.11
3	+1.000	5.333×10^{-9}	1.408×10^{-3}	−50.815	−33.88
4	−1.250	7.8125×10^{-9}	0	+63.519	+31.76
5	−1.250	7.8125×10^{-9}	0	+63.519	+38.11
Total		27.7083×10^{-9}	1.408×10^{-3}		

$$F_R = F_3 = -\frac{1.408 \times 10^{-3}}{27.7083 \times 10^{-9}} = -50.815 \times 10^3 \text{ N} = -50.815 \text{ kN}$$

The forces in the other members may be found from eq. (4-28). For example, for member 1, the force is

$$F_1 = f_1 F_R = (0.750)\,(-50.815) = -38.11 \text{ kN} \;\; (-8567 \text{ lb}_\text{f})$$

The other forces are given in Table 4-6.

The stress in the members may be calculated from the forces and cross-sectional areas. For example, for member 1, we find the following stress:

$$\sigma_1 = \frac{F_1}{A_1} = \frac{-38.111 \times 10^3}{10 \times 10^{-4}} = -38.11 \times 10^6 \text{ Pa} = -38.11 \text{ MPa} \;\; (-5530 \text{ psi})$$

The stress in the other members is also given in Table 4-6.

4.4.2 Trusses with Both Thermal and Mechanical Loads

Let us consider the truss shown in Figure 4-11, on which both external mechanical loads (P_x) and thermal loads are applied. The truss is statically indeterminate, so we cannot determine the forces in the members by a force balance alone. Let us remove the nth member and replace the member with the force F_R acting between joints A and B. We may determine the deflection Δ_{AB} between these points by applying a unit load $f_R = 1$ at these joints. The "dummy" loads in the other members f_j may be determined. Let us denote the forces in the members of the statically determinate truss, with member n removed and the external loads applied, by F_{Pj}. These forces may be determined from force balances applied at each joint, because the modified truss is statically determinate. Finally, the forces in each member may be determined from the following:

$$F_j = F_{Pj} + f_j F_R \tag{4-36}$$

The deflection between points A and B may be written, using eq. (4-16). We note that F_R is a constant, as far as the summation is concerned:

$$\Delta_{AB} = \sum_{j \neq n} f_j \alpha_j \Delta T_j L_j + \sum_{j \neq n} \frac{\left(F_{Pj} + f_j F_R\right) f_j}{k_j} \tag{4-37}$$

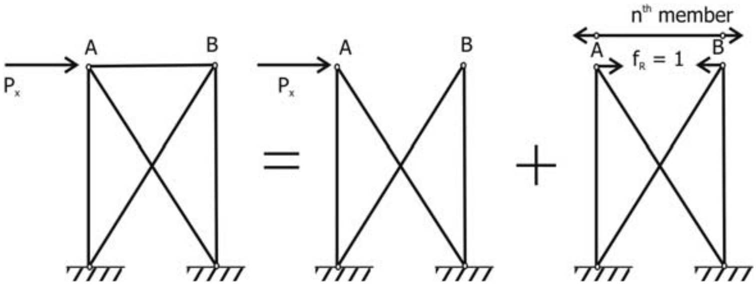

Figure 4-11. Superposition applied to the truss shown in Figure 4-9.

The deflection of the redundant bar between points A and B may be written as

$$\Delta_n = \alpha_n \Delta T_n L_n + \frac{F_R}{k_n} = f_R \alpha_n \Delta T_n L_n + \frac{(F_{Pn} + f_n F_R)\, f_R^2}{k_n} \tag{4-38}$$

The last expression is valid because $f_R = f_n = 1$ and $F_{Pn} = 0$.

The redundant bar and the remainder of the truss remain in contact after the thermal expansion and external loads are applied, so eq. (4-32) is valid in this case. If we add the deflections from eqs. (4-37) and (4-38), we obtain

$$\sum_{\text{all } j} f_j \alpha_j \Delta T_j L_j + \sum_{\text{all } j} \frac{F_{Pj} f_j}{k_j} + F_R \sum_{\text{all } j} \frac{f_j^2}{k_j} = 0 \tag{4-39}$$

The force in the redundant member may be determined from the following expression:

$$F_R = -\frac{\displaystyle\sum_j f_j \alpha_j \Delta T_j L_j + \sum_j \frac{F_{Pj} f_j}{k_j}}{\displaystyle\sum_j \frac{f_j^2}{k_j}} \tag{4-40}$$

After the force in the redundant member has been determined, the forces in the other members of the truss may be found from eq. (4-36).

Example 4-5 In the truss shown in Figure 4-12 has a vertical load applied at joint J3 of 400 kN (89,920 lb_f). The coordinates of the joints and cross-sectional areas of the members are given in Table 4-7. The truss material has a Young's modulus of 200 GPa (29×10^6 psi) and thermal expansion coefficient of $11.0 \times 10^{-6}\ \text{K}^{-1}$ ($6.11 \times 10^{-6}\ {}^\circ\text{F}^{-1}$). Member 5 experiences a temperature change of $+50^\circ$C (90°F), and all other members experience no temperature change. Determine the stress in each member.

Let us chose member 5 as the redundant member. We may apply a unit tensile load at joints J1 and J3, as shown in Figure 4-13. The resulting "dummy" loads may be determined.

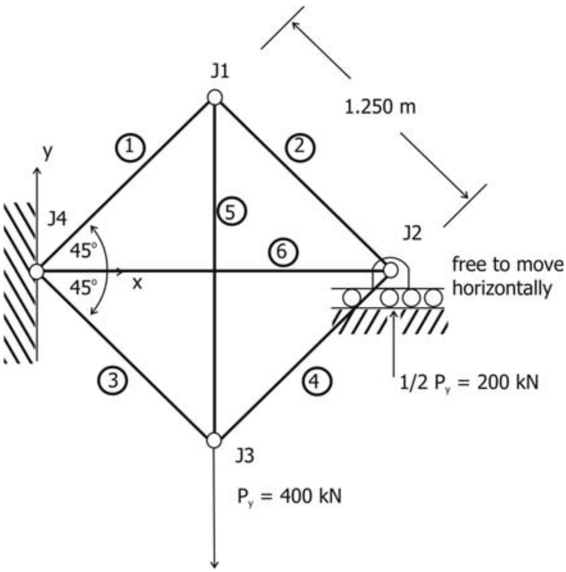

Figure 4-12. Pin-connected truss for Example 4-5.

For example, the direction angles for members 1 and 2 at joint J1 are

$$\mathbf{B}^{(J1)} = \begin{bmatrix} \cos\theta_1 & \cos\theta_2 \\ \sin\theta_1 & \sin\theta_2 \end{bmatrix} = \begin{bmatrix} -0.7071 & +0.7071 \\ -0.7071 & -0.7071 \end{bmatrix}$$

The determinant of the direction matrix is found from eq. (4-24):

$$\det \mathbf{B}^{(J1)} = (-0.7071)\,(-0.7071) - (0.7071)\,(-0.7071) = +1$$

The inverse of the direction matrix is found from eq. (4-23):

$$\mathbf{B}^{-1} = \begin{bmatrix} -0.7071 & -0.7071 \\ +0.7071 & -0.7071 \end{bmatrix}$$

The "dummy" loads for members 1 and 2 are found as follows:

$$\mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = -\mathbf{B}^{-1} \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} = \begin{Bmatrix} -0.7071 \\ -0.7071 \end{Bmatrix}$$

The values for the "dummy" loads in the other members are given in Table 4-8:

$$\mathbf{B}^{(\mathrm{J1})} = \begin{bmatrix} \cos\theta_1 & \cos\theta_2 \\ \sin\theta_1 & \sin\theta_2 \end{bmatrix} = \begin{bmatrix} -0.7071 & +0.7071 \\ -0.7071 & -0.7071 \end{bmatrix}$$

The determinant of the direction matrix is found from eq. (4-24):

$$\det \mathbf{B}^{(\mathrm{J1})} = (-0.7071)\,(-0.7071) - (0.7071)\,(-0.7071) = +1$$

TABLE 4-7. Data for Example 4-5

	End Coordinates, m						
Member j	$(x_1, y_1)_j$	$(x_2, y_2)_j$	L_j, m	A_j, cm^2	k_j, MN/m	ΔT_j, °C	$\alpha_j \Delta T_j L_j$, mm
1	(0, 0)	(1.7678, 1.7678)	2.500	125	1000	0	0
2	(1.7678, 1.7678)	(3.5355, 0)	2.500	125	1000	0	0
3	(0, 0)	(1.7678, −1.7678)	2.500	75	600	0	0
4	(1.7678, −1.7678)	(3.5355, 0)	2.500	75	600	0	0
5	(1.7678, −1.7678)	(1.7678, 1.7678)	3.5355	100	565.7	50	1.9445×10^{-3}
6	(0, 0)	(3.5355, 0)	3.5355	150	848.5	0	0

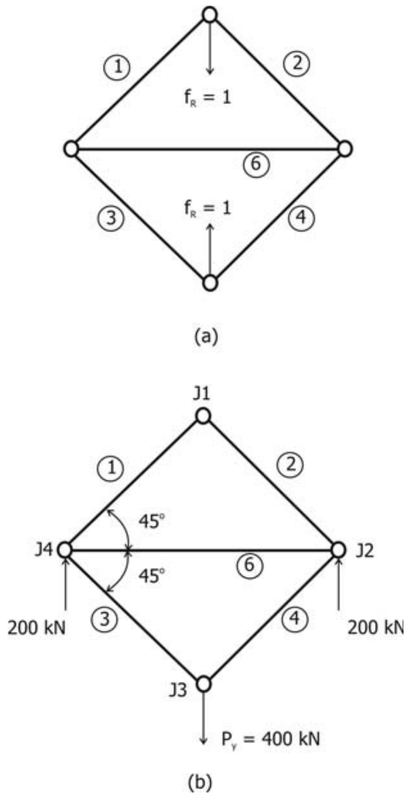

Figure 4-13. Application of a unit load for the truss shown in Figure 4-12.

The inverse of the direction matrix is found from eq. (4-23):

$$\mathbf{B}^{-1} = \begin{bmatrix} -0.7071 & -0.7071 \\ +0.7071 & -0.7071 \end{bmatrix}$$

The "dummy" loads for members 1 and 2 are found as follows:

$$\mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = -\mathbf{B}^{-1} \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} = \begin{Bmatrix} -0.7071 \\ -0.7071 \end{Bmatrix}$$

The values for the "dummy" loads in the other members are given in Table 4-8.

Next, let us determine the values for the loads in the truss with the member 5 removed, but with the external mechanical load $P_y = 400\,\text{kN}$ applied, as shown in Figure 4-13. The direction cosine matrix for joint J3 is

$$\mathbf{B}^{(\text{J3})} = \begin{bmatrix} \cos\theta_3 & \cos\theta_4 \\ \sin\theta_3 & \sin\theta_4 \end{bmatrix} = \begin{bmatrix} -0.7071 & 0.7071 \\ +0.7071 & 0.7071 \end{bmatrix}$$

The determinant of the direction cosine matrix is

$$\det \mathbf{B}^{(\text{J3})} = (-0.7071)\,(0.7071) - (0.7071)\,(0.7071) = -1.00$$

TABLE 4-8. Results for Example 4-5

Member j	"Dummy" Force, f_j	$f_j{}^2/k_j$, m/N	$f_j \alpha_j \Delta T_j L_j$, mm	F_{Pj}, kN	$F_{Pj} f_j/k_j$, mm	F_j, kN	$\sigma_j = F_j/A_j$, MPa
1	−0.7071	0.5000×10^{-9}	0	0	0	+131.3	+10.50
2	−0.7071	0.5000×10^{-9}	0	0	0	+131.3	+10.50
3	−0.7071	0.8333×10^{-9}	0	+282.8	−0.3333	+414.1	+55.21
4	−0.7071	0.8333×10^{-9}	0	+282.8	−0.3333	+414.1	+55.21
5	+1.000	1.7677×10^{-9}	1.9445	0	0	−185.7	−18.57
6	+1.000	1.1786×10^{-9}	0	−200.0	−0.2357	−385.7	−25.71
Total		5.6129×10^{-9}	1.9445		−0.9024		

The inverse of the direction cosine matrix is found from eq. (4-23):

$$\mathbf{B}^{-1} = \frac{1}{\det \mathbf{B}} \begin{bmatrix} \sin\theta_4 & -\cos\theta_4 \\ -\sin\theta_3 & \cos\theta_3 \end{bmatrix} = \begin{bmatrix} +0.7071 & -0.7071 \\ -0.7071 & -0.7071 \end{bmatrix}$$

The forces in members 3 and 4 may be determined:

$$\mathbf{F}^{(J3)} = \begin{Bmatrix} F_{P3} \\ F_{P4} \end{Bmatrix} = \mathbf{B}^{-1} \begin{Bmatrix} 0 \\ -400 \end{Bmatrix} = \begin{Bmatrix} +282.8 \\ +282.8 \end{Bmatrix} \text{kN}$$

The forces in the other members are given in Table 4-8.

The force in the redundant member, member 5 in this case, may be found from eq. (4-40):

$$F_R = F_5 = -\frac{1.9445 \times 10^{-3} - 0.9024 \times 10^{-3}}{5.6129 \times 10^{-9}} = -185.66 \times 10^3\ \text{N} = -185.66\ \text{kN}$$

The forces in the other members may be found from eq. (4-36). For example, the force in member 3 is

$$F_3 = F_{P3} + f_3 F_R = +282.8 + (-0.7071)\,(-185.66) = +414.1\ \text{kN}\ \ (93{,}100\ \text{lb}_\text{f})$$

The forces in the other members are given in Table 4-8, along with the stresses in each member.

Trusses with two or more redundant members are not utilized extensively. The general method described in this section for analysis of trusses with one redundant member may be extended to the case of more than one redundant member, however. The detailed analysis may be found in the reference by Van den Broek [1942, pp. 26–30].

4.5 THE FINITE ELEMENT METHOD

The *finite element method* of analysis has developed into a powerful tool for analysis of structural systems, thermal systems, aerodynamic and hydraulic systems, and electrical and magnetic systems. In overly simplified terms, the finite element method involves developing and solving simultaneous equations relating the forces and displacements (or the equivalent parameters in other systems) in a mechanical system that has been broken down into many connected finite elements. Because of the large number of equations that are usually handled, only the mathematically masochistic person would attempt to solve a finite element problem "by hand." Several large-scale digital computer programs, such as ANSYS and NASTRAN, are available to carry out the mathematical manipulations of finite element analysis.

The basic concepts of the finite element analysis had their origin in analysis of aircraft structures. Hrenikoff [1941] discussed a method of the approximate solution of elasticity problems that he called the "framework method." Courant

[1943] has been recognized as the first person to develop the finite element method. The name "finite element method" was first applied to the technique in 1960 by Clough [1960]. Zienkiewicz and Chung wrote the first book on the finite element method in 1967 [Zienkiewicz and Chung, 1967]. There are several textbooks available on the finite element method today [Moaveni, 1999].

4.5.1 Finite Element Formulation

Let us consider the truss member shown in Figure 4-14. The displacements and forces may be expressed in terms of a coordinate system attached to the member (local coordinates) or in terms of an overall coordinate system (global coordinates). The stress in the member may be written as

$$\sigma = \frac{F}{A} = E\varepsilon - \alpha E \Delta T \tag{4-41}$$

The axial displacement of the bar is related to the strain by $e = \varepsilon L$. The thermal force is defined by eq. (3-7):

$$F_T = \alpha E A \Delta T \tag{4-42}$$

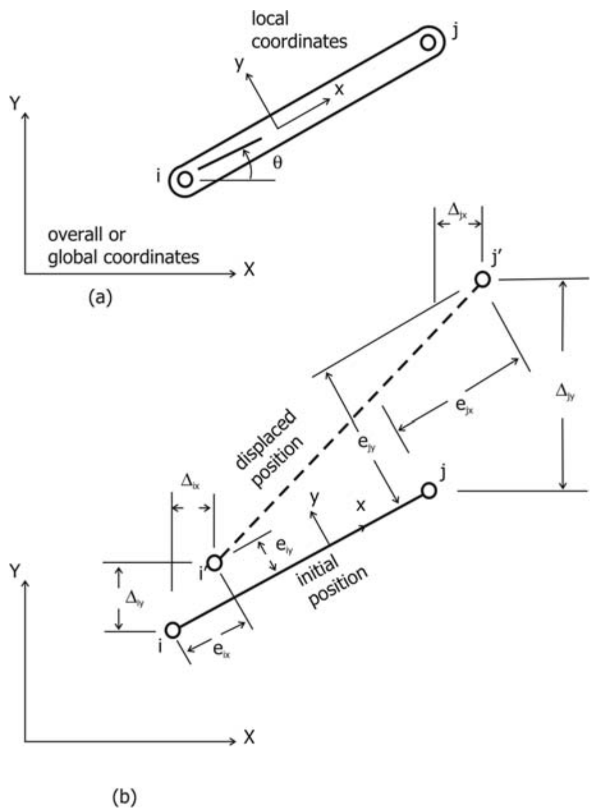

Figure 4-14. Truss member subjected to a tensile or compressive load only.

If we make these substitutions into eq. (4-41), we obtain the following expression for the force in the member:

$$F + F_T = \frac{EA}{L}e = k_{sp}e \tag{4-43}$$

The quantity $k_{sp} = EA/L$ is the spring constant for the member or *element*.

The displacements of each end of a member or *element* (i, j) in terms of the local coordinates may be written as displacements of each joint or *node* (Ji, Jj) in the overall or global coordinates:

$$\Delta_{iX} = e_{ix}\cos\theta - e_{iy}\sin\theta \tag{4-44a}$$

$$\Delta_{iY} = e_{ix}\sin\theta + e_{iy}\cos\theta \tag{4-44b}$$

$$\Delta_{jX} = e_{jx}\cos\theta - e_{jy}\sin\theta \tag{4-44c}$$

$$\Delta_{jY} = e_{jx}\sin\theta + e_{jy}\cos\theta \tag{4-44d}$$

The angle θ is the angle between the overall X-axis and the member, as shown in Figure 4-14. These four equations may be written in compact matrix form:

$$\mathbf{D}^{(e)} = \mathbf{T}^{(e)}\mathbf{u}^{(e)} \tag{4-45}$$

where

$$\mathbf{D}^{(e)} = \begin{Bmatrix} \Delta_{iX} \\ \Delta_{iY} \\ \Delta_{jX} \\ \Delta_{jY} \end{Bmatrix} \quad \text{and} \quad \mathbf{u}^{(e)} = \begin{Bmatrix} e_{ix} \\ e_{iy} \\ e_{jx} \\ e_{jy} \end{Bmatrix} \tag{4-46}$$

$\mathbf{D}^{(e)}$ and $\mathbf{u}^{(e)}$ represent the displacements of joints or nodes i and j with respect to the global coordinates X and Y and the local coordinates x and y, respectively.

The transformation matrix is

$$\mathbf{T}^{(e)} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \lambda & -\mu & 0 & 0 \\ \mu & \lambda & 0 & 0 \\ 0 & 0 & \lambda & -\mu \\ 0 & 0 & \mu & \lambda \end{bmatrix} \tag{4-47}$$

We have used $\lambda = \cos\theta$ and $\mu = \sin\theta$.

The forces at each end of the member may also be written as follows, in terms of the local and global coordinates:

$$F_{iX} = F_{ix}\cos\theta - F_{iy}\sin\theta \tag{4-48a}$$

$$F_{iY} = F_{ix}\sin\theta + F_{iy}\cos\theta \tag{4-48b}$$

$$F_{jX} = F_{jx}\cos\theta - F_{jy}\sin\theta \tag{4-48c}$$

$$F_{jY} = F_{jx}\sin\theta + F_{jy}\cos\theta \tag{4-48d}$$

We note that, because the force in the element is applied along the axis of the element, the forces $F_{iy} = F_{jy} = 0$. The force expression, eq. (4-48), may also be written in matrix form:

$$\mathbf{F}^{(e)} = \mathbf{T}^{(e)}\mathbf{f}^{(e)} \tag{4-49}$$

where

$$\mathbf{F}^{(e)} = \begin{Bmatrix} F_{iX} \\ F_{iY} \\ F_{jX} \\ F_{jY} \end{Bmatrix} \quad \text{and} \quad \mathbf{f}^{(e)} = \begin{Bmatrix} F_{ix} \\ F_{iy} \\ F_{jx} \\ F_{jy} \end{Bmatrix} = \begin{Bmatrix} F_{ix} \\ 0 \\ F_{jx} \\ 0 \end{Bmatrix} \tag{4-50}$$

The forces and displacements of the ends of each element are related through the following set of equations:

$$\begin{Bmatrix} F_{ix} \\ F_{iy} \\ F_{jx} \\ F_{jy} \end{Bmatrix} + F_T \begin{Bmatrix} -1 \\ 0 \\ +1 \\ 0 \end{Bmatrix} = \begin{bmatrix} k_{\text{sp}} & 0 & -k_{\text{sp}} & 0 \\ 0 & 0 & 0 & 0 \\ -k_{\text{sp}} & 0 & k_{\text{sp}} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} e_{ix} \\ e_{iy} \\ e_{jx} \\ e_{jy} \end{Bmatrix} \tag{4-51}$$

or

$$\mathbf{f}^{(e)} + F_T\mathbf{S} = \mathbf{k}^{(e)}\mathbf{u}^{(e)} \tag{4-52}$$

The matrix $\mathbf{S}$ is defined by

$$\mathbf{S} = \begin{Bmatrix} -1 \\ 0 \\ +1 \\ 0 \end{Bmatrix} \tag{4-53}$$

If we solve for $\mathbf{f}$ from eq. (4-49) and solve for $\mathbf{u}$ from eq. (4-45), then make the substitutions in eq. (4-52), we obtain

$$\mathbf{T}^{-1}\mathbf{F}^{(e)} + F_T\mathbf{S} = \mathbf{k}^{(e)}\mathbf{T}^{-1}\mathbf{D}^{(e)} \tag{4-54}$$

If we multiply both sides of eq. (4-54) by the transformation matrix $\mathbf{T}$, we obtain

$$\mathbf{T}\mathbf{T}^{-1}\mathbf{F}^{(e)} + F_T\mathbf{T}\mathbf{S} = \mathbf{F}^{(e)} + \mathbf{F}_T^{(e)} = \mathbf{T}\mathbf{k}^{(e)}\mathbf{T}^{-1}\mathbf{D}^{(e)} \tag{4-55}$$

The final form for eq. (4-55) is

$$\mathbf{F}^{(e)} + \mathbf{F}_T^{(e)} = \mathbf{K}^{(e)}\mathbf{D}^{(e)} \tag{4-56}$$

The thermal force matrix relative to the global coordinates for an element may be shown to be

$$\mathbf{F}_T^{(e)} = F_T\mathbf{T}\mathbf{S} = \begin{Bmatrix} -F_T\lambda \\ -F_T\mu \\ +F_T\lambda \\ +F_T\mu \end{Bmatrix} \tag{4-57}$$

The spring constant matrix or *stiffness matrix* for any element relative to the global coordinates is given by

$$\mathbf{K}^{(e)} = \mathbf{T}\mathbf{k}^{(e)}\mathbf{T}^{-1} = k_{\mathrm{sp},e}\begin{bmatrix} \lambda^2 & \lambda\mu & -\lambda^2 & -\lambda\mu \\ \lambda\mu & \mu^2 & -\lambda\mu & -\mu \\ -\lambda^2 & -\lambda\mu & \lambda^2 & \lambda\mu \\ -\lambda\mu & -\mu^2 & \lambda\mu & \mu^2 \end{bmatrix} \tag{4-58}$$

The overall or *global* stiffness matrix $\mathbf{K}^{(G)}$ for the system may be obtained by combining or *assembling* the elemental stiffness matrices. This procedure is illustrated in the following example.

4.5.2 Finite Element Analysis

In any finite element analysis, there are three general phases: (a) *preprocessing phase*, which involves "setting up" the problem to be analyzed, (b) *solution phase*, which involves solving the set of simultaneous equations for the displacements of the various elements, and (c) *postprocessing phase*, which involves calculating the stresses and reaction forces from the displacement solution and displaying the final results.

The *preprocessing phase* involves five general steps. First, the system must be defined in terms of nodes and elements, or the system is *discretized*. In the case of a truss, the nodes are the joints and the elements are the members of the truss. The element may be defined in terms of the coordinates of each joint of the truss and the direction (which is the ith joint and which is the jth joint) and the designation (number) for the member. It is important at this stage to designate or number the members and joints in such a manner that the joint numbers for an element are as near as possible to the element number. For example, the member (element) between joints (nodes) 6 and 7 should be numbered 6 or 7, if possible. This procedure results in a more effective numerical solution.

The second general step in the preprocessing phase is to develop a solution that approximates the behavior of an element. In the case of a truss, this step involves determining the spring constant and thermal loading for each member. For a distributed system, such as a flat plate, the displacement must be approximated by a polynomial (usually). The coefficients of the polynomial are determined by energy methods or other approximation techniques to "best fit" the actual displacement for the element.

The third general step in the preprocessing phase is to develop the relationships between stress, strain, and temperature for the elements. For the case of a truss, this relationship is given by eq. (4-56).

The fourth step in the preprocessing phase is to combine or assemble the relationships for the individual elements into one set of overall equations for the entire system. This step is illustrated in the following example.

The fifth step in the preprocessing phase involves application of the boundary conditions, because some of the displacements of the nodes may actually be known quantities. For example, the displacement of the node or joint at a support for the truss is usually a known quantity.

The *solution phase* involves solving the system of algebraic equations simultaneously for the displacements of the nodes of the system. Because of the large number of equations involved, the solution phase is usually carried out using a numerical analysis technique [Bathe and Wilson, 1976].

The *postprocessing phase* involves calculation of the reaction forces and the stress in each member. The reaction forces $\mathbf{F}_R$ for the truss may be calculated from the displacements as follows:

$$\mathbf{F}_R = \mathbf{K}^{(G)}\mathbf{D}^{(G)} - \mathbf{P} - \mathbf{F}_T^{(G)} \tag{4-59}$$

The matrix $\mathbf{P}$ is the matrix of the external forces:

$$\mathbf{P} = \begin{Bmatrix} P_{X1} \\ P_{Y1} \\ P_{X2} \\ P_{Y2} \\ \text{etc.} \end{Bmatrix} \tag{4-60}$$

The force in each member may be written from eq. (4-52):

$$F_{ix} = k_{\text{sp}}\left(e_{ix} - e_{jx}\right) + F_T \tag{4-61a}$$

$$F_{jx} = k_{\text{sp}}\left(e_{jx} - e_{ix}\right) + F_T \tag{4-61b}$$

We note that the sum of F_{ix} and F_{jx} is zero. The two expressions may be written in matrix form:

$$\mathbf{f}^{(e)} = \mathbf{k}^{(e)}\mathbf{u}^{(e)} - F_T\mathbf{S} \tag{4-62}$$

The stress for each member may be calculated from the force, $\sigma = F_x/A$, where A is the cross-sectional area of the member.

Example 4-6 Let us work the problem given in Example 4-1, using the finite element approach. The definition of the elements (members) and nodes (joints) is shown in Table 4-9.

The orientation angles for each member may be found from the joint coordinates and eqs. (4-25) and (4-27). For example, for member 1, the length is found from eq. (4-27):

$$L^{(1)} = \sqrt{\left(X_j - X_i\right)^2 + \left(Y_j - Y_i\right)^2} = \sqrt{(1.0392 - 0)^2 + (0.600 - 0.600)^2}$$

$$L^{(1)} = 1.0392\ \text{m} \quad (40.91\ \text{in.})$$

Similarly, $L^{(2)} = 1.200$ m (47.24 in.), for member 2. The angles may be determined from eq. (4-25). For example, for member 1,

$$\cos\theta^{(1)} = \frac{\left(X_j - X_i\right)^{(1)}}{L^{(1)}} = \frac{1.0392 - 0}{1.0392} = 1.000 \quad \text{or} \quad \theta^{(1)} = 0^\circ$$

Similarly, $\cos\theta^{(2)} = 0.8660$ and $\theta^{(2)} = 30^\circ$.

TABLE 4-9. Data for Example 4-6

Element	Node i	Node j	Angle θ	k, MN/m
(1)	1	2	0°	11.55
(2)	3	2	30°	500

Joint Global Coordinates

Joint (node)	X, m	Y, m
J1	0	0.600
J2	1.0392	0.600
J3	0	0

The spring constants for the individual members have been calculated in Example 4-1, and are listed in Table 4-9. The stiffness matrix for each of the elements may be found from eq. (4-58). For example, for member 1,

$$\lambda_1 = \cos\theta^{(1)} = 1.000 \quad \text{and} \quad \mu_1 = \sin\theta^{(1)} = 0$$

Similarly, for member 2,

$$\lambda_2 = \cos\left(30^\circ\right) = 0.8660 \quad \text{and} \quad \mu_2 = \sin\left(30^\circ\right) = 0.5000$$

Making these substitutions into eq. (4-58), the stiffness matrix for element 1 may be found:

$$\mathbf{K}^{(1)} = \left(11.55 \times 10^6\right) \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{N/m}$$

Similarly, the stiffness matrix for element 2 may be calculated:

$$\mathbf{K}^{(2)} = \left(50.00 \times 10^6\right) \begin{bmatrix} 0.750 & 0.433 & -0.750 & -0.433 \\ 0.433 & 0.250 & -0.433 & -0.250 \\ -0.750 & -0.433 & 0.750 & 0.433 \\ -0.433 & -0.250 & 0.433 & 0.250 \end{bmatrix} \text{N/m}$$

The thermal force matrix for the elements may be evaluated from eq. (4-57). For member 1, we have

$$F_{T1} = \alpha E A_1 \Delta T_1 = \left(11 \times 10^{-6}\right)\left(200 \times 10^9\right)\left(60 \times 10^{-6}\right)\left(60^\circ\right) = 7920\,\text{N}$$

Also,

$$F_{T2} = 0$$

The thermal force matrix for element 1 is

$$\mathbf{F}_T^{(1)} = \begin{Bmatrix} -7920 \\ 0 \\ +7920 \\ 0 \end{Bmatrix} \text{N}$$

For element 2, we obtain the following result:

$$\mathbf{F}_T^{(2)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The next step involves combining or assembling the individual stiffness matrices to form the overall or global stiffness matrix. To understand this process, we may first look at a simple example in which we have two sets of equations for three unknowns, x, y, and z:

$$\begin{cases} 3x + 2y = 7 \\ 2x + y = 4 \end{cases} \quad \text{and} \quad \begin{cases} 3y + 2z = 12 \\ y + 2z = 8 \end{cases}$$

These two sets of equations may be written in matrix form:

$$\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 7 \\ 4 \end{Bmatrix} \quad \text{or} \quad \begin{bmatrix} 3 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 7 \\ 4 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} y \\ z \end{Bmatrix} = \begin{Bmatrix} 12 \\ 8 \end{Bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 12 \\ 8 \end{Bmatrix}$$

The last two expressions in each case may be added together to obtain

$$\begin{bmatrix} 3 & 2 & 0 \\ 2 & (1+3) & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 7 \\ (4+12) \\ 8 \end{Bmatrix}$$

The 3×3 matrix on the left side of the above expression may be considered the "assembled" matrix from the previous two matrix relations.

Let us follow the same procedure to assemble the global stiffness matrix from the two matrices for the elements. The matrices are as follows, in "augmented" form:

$$\mathbf{K}^{(1)} = (10^6) \begin{bmatrix} 11.55 & 0 & -11.55 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -11.55 & 0 & 11.55 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{K}^{(2)} = (10^6) \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 37.50 & 21.65 & -37.50 & -21.65 \\ 0 & 0 & 21.65 & 12.50 & -21.65 & -12.50 \\ 0 & 0 & -37.50 & -21.65 & 37.50 & 21.65 \\ 0 & 0 & -21.65 & -12.50 & 21.65 & 12.50 \end{bmatrix}$$

If we add these element matrices, we obtain the overall or global stiffness matrix for this problem:

$$\mathbf{K}^{(G)} = \left(10^6\right)\begin{bmatrix} 11.55 & 0 & -11.55 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -11.55 & 0 & (11.55+37.50) & (21.65+0) & -37.50 & -21.65 \\ 0 & 0 & (21.65+0) & (12.50+0) & -21.65 & -12.50 \\ 0 & 0 & -37.50 & -21.65 & 37.50 & 21.65 \\ 0 & 0 & -21.65 & -12.50 & 21.65 & 12.50 \end{bmatrix} \begin{vmatrix} \Delta_{1X} \\ \Delta_{1Y} \\ \Delta_{2X} \\ \Delta_{2Y} \\ \Delta_{3X} \\ \Delta_{3Y} \end{vmatrix}$$

For a truss with three joints, the global stiffness matrix is a $(2)(3) = 6 \times 6$ matrix. If we had a problem involving a truss with 100 joints, the global stiffness matrix would be a 200×200 matrix. It is obvious that, in this case, we would graciously allow the computer program to assemble the global stiffness matrix.

The final step in the preprocessing phase ("getting ready" to solve the problem) is to apply the boundary conditions or to "condition" the global stiffness matrix. Returning to the example of simultaneous equation solution discussed previously, the final form for the equations is

$$\begin{bmatrix} 3 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 7 \\ 16 \\ 8 \end{Bmatrix}$$

or

$$\begin{cases} 3x + 2y = 7 \\ 2x + 4y + 2z = 16 \\ y + 2z = 8 \end{cases}$$

Suppose we have given (a "condition" for the problem) that $x = 1$. We only need the last two equations to solve the problem, so the first equation (first row in the matrix) may be discarded. We may transfer the "known" quantities to the right side of the equation to yield

$$\begin{cases} 4y + 2z = 16 - (2)(1) = 14 \\ y + 2z = 8 - (0)(1) = 8 \end{cases}$$

or

$$\begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} y \\ z \end{Bmatrix} = \begin{Bmatrix} 14 \\ 8 \end{Bmatrix}$$

This last result corresponds to the "conditioned" matrix in the finite element solution.

For this problem, we have the following boundary conditions:

Joint J1: $\Delta_{1X} = 0$ and $\Delta_{1Y} = 0$

Joint J3: $\Delta_{3X} = 0$ and $\Delta_{3Y} = 0$

In the global stiffness matrix, we may discard the first, second, fifth, and sixth rows. Because the known quantities (displacements) are zero in this problem, the

conditioned global stiffness matrix reduces to

$$\mathbf{K}^{(G)} = \left(10^6\right)\begin{bmatrix} 49.05 & 21.65 \\ 21.65 & 12.50 \end{bmatrix} \begin{vmatrix} \Delta_{2X} \\ \Delta_{2Y} \end{vmatrix}$$

In a similar manner, we may find the global thermal force matrix:

$$\mathbf{F}_T^{(G)} = \begin{Bmatrix} -7920 \\ 0 \\ (7920+0) \\ (0+0) \\ 0 \\ 0 \end{Bmatrix}$$

The conditioned thermal force matrix is obtained by deleting the first, second, fifth, and sixth rows, corresponding to the known displacement values. The resulting conditioned global thermal force matrix is

$$\mathbf{F}_T^{(G)} = \begin{Bmatrix} 7920 \\ 0 \end{Bmatrix} \text{N}$$

The reduced global force matrix reduces to

$$\mathbf{F}^{(G)} = \begin{Bmatrix} F_{jX}^{(1)} + F_{jX}^{(2)} \\ F_{jY}^{(1)} + F_{jY}^{(2)} \end{Bmatrix} = \begin{Bmatrix} P_X \\ P_Y \end{Bmatrix} = \begin{Bmatrix} 0 \\ P_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ -5000 \end{Bmatrix} \text{N}$$

The final expression that must be solved for the displacements is

$$\mathbf{F}^{(G)} + \mathbf{F}_T^{(G)} = \mathbf{K}^{(G)}\mathbf{D}^{(G)}$$

The matrix expression may be written out as

$$\begin{Bmatrix} 0 \\ -5000 \end{Bmatrix} + \begin{Bmatrix} 7920 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 7920 \\ -5000 \end{Bmatrix} = \left(10^6\right)\begin{bmatrix} 49.05 & 21.65 \\ 21.65 & 12.50 \end{bmatrix}\begin{Bmatrix} \Delta_{2X} \\ \Delta_{2Y} \end{Bmatrix}$$

For large matrices (greater than about 10×10), the matrix inversion technique (multiplying the right side by the inverse matrix) is usually not as computationally effective as other numerical techniques, such as the Gauss elimination method or the Gauss-Seidel iteration method [Pearson, 1986]. The solution of the equilibrium equations for the truss is

$$\mathbf{D}^{(G)} = \begin{Bmatrix} \Delta_{2X} \\ \Delta_{2Y} \end{Bmatrix} = \begin{Bmatrix} 1.436 \times 10^{-3} \\ -2.886 \times 10^{-3} \end{Bmatrix} \text{ m} = \begin{Bmatrix} 1.436 \\ -2.886 \end{Bmatrix} \text{ mm}$$

The final part of the problem is the postprocessing phase or “displaying” the results of the solution. The reaction forces may be found from eq. (4-59):

$$\mathbf{F}_R = \begin{Bmatrix} F_{R1X} \\ F_{R1Y} \\ F_{R2X} \\ F_{R2Y} \\ F_{R3X} \\ F_{R3Y} \end{Bmatrix} = \left(10^3\right)\begin{bmatrix} 11.55 & 0 & -11.55 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -11.55 & 0 & 49.05 & 21.65 & -37.50 & -21.65 \\ 0 & 0 & 21.65 & 12.50 & -21.65 & -12.50 \\ 0 & 0 & -37.50 & -21.65 & 37.50 & 21.65 \\ 0 & 0 & -21.65 & -12.50 & 21.65 & 12.50 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 1.436 \\ -2.886 \\ 0 \\ 0 \end{Bmatrix}$$

$$- \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -5000 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -7920 \\ 0 \\ 7920 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

If we carry out the operations, we obtain the following result for the reaction forces:

$$\mathbf{F}_R = \begin{Bmatrix} -8660 \\ 0 \\ 0 \\ 0 \\ +8660 \\ +5000 \end{Bmatrix} \text{N}$$

The forces in the individual members may be found from eq. (4-52). We may evaluate the element displacement from eq. (4-45):

$$\mathbf{T}^{-1}\mathbf{D}^{(e)} = \mathbf{T}^{-1}\mathbf{T}^{(e)}\mathbf{u}^{(e)} = \mathbf{u}^{(e)}$$

The inverse of the transformation matrix $\mathbf{T}$ is

$$\mathbf{T}^{-1} = \begin{bmatrix} \lambda & \mu & 0 & 0 \\ -\mu & \lambda & 0 & 0 \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & -\mu & \lambda \end{bmatrix}$$

The quantities $\lambda = \cos\theta$ and $\mu = \sin\theta$ for the element.

The local displacement vector for element (1) may be written as

$$\mathbf{u}^{(1)} = \begin{Bmatrix} e_{1x} \\ e_{1y} \\ e_{2x} \\ e_{2y} \end{Bmatrix} = \mathbf{T}^{-1}\mathbf{D}^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 1.436 \\ -2.886 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1.436 \\ -2.886 \end{Bmatrix} \text{mm}$$

For member 2, we obtain

$$\mathbf{u}^{(2)} = \begin{Bmatrix} e_{2x} \\ e_{2y} \\ e_{3x} \\ e_{3y} \end{Bmatrix} = \mathbf{T}^{-1}\mathbf{D}^{(2)} = \begin{bmatrix} 0.866 & 0.500 & 0 & 0 \\ -0.500 & 0.866 & 0 & 0 \\ 0 & 0 & 0.866 & 0.500 \\ 0 & 0 & -0.500 & 0.866 \end{bmatrix} \begin{Bmatrix} 1.436 \\ -2.886 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} -0.200 \\ -3.217 \\ 0 \\ 0 \end{Bmatrix} \text{ mm}$$

The force in member 1 may now be calculated:

$$F_{2x}^{(1)} = k_1 (e_{2x} - e_{1x}) - F_T^{(1)}$$

$$F_{2x}^{(1)} = \left(11.55 \times 10^6\right)\left(1.436 \times 10^{-3} - 0\right) - 7920 = 8660 \text{ N} = 8.66 \text{ kN}$$

The stress for member 1 is

$$\sigma_1 = \frac{F_{2x}^{(1)}}{A_1} = \frac{8660}{60 \times 10^{-6}} = 144.3 \times 10^6 \text{ Pa} = 144.3 \text{ MPa}$$

We find the following values for member 2:

$$F_{2x}^{(2)} = k_2 (e_{2x} - e_{3x}) - F_T^{(2)}$$

$$F_{2x}^{(2)} = \left(50 \times 10^6\right)\left(-0.200 \times 10^{-3} - 0\right) - 0 = -10{,}000 \text{ N} = -10.0 \text{ kN}$$

$$\sigma_2 = \frac{-10{,}000}{300 \times 10^{-6}} = -33.33 \times 10^6 \text{ Pa} = -33.33 \text{ MPa}$$

4.6 ELASTIC ENERGY IN BENDING

In the previous sections, we considered the analysis of *trusses*, in which the members supported axial (tensile or compressive) loads only. Let us now examine the analysis of thermal stresses in *frames*, in which both axial loading and bending loading may be present. Although there are several techniques available for analysis of frames, we will use the unit-load method, developed from elastic energy theory, in this section.

The stress–strain curve for a linear elastic member with thermal strain is shown in Figure 4-15. The complementary energy per unit volume may be written as follows:

$$\frac{dU_c}{d\text{Vol}} = \sigma\left(\alpha\Delta T + \frac{\sigma}{E}\right) - \frac{\sigma^2}{2E} = \sigma\alpha\Delta T + \frac{\sigma^2}{2E} \tag{4-63}$$

The volume element may be written as

$$d\text{Vol} = dA\, ds \tag{4-64}$$

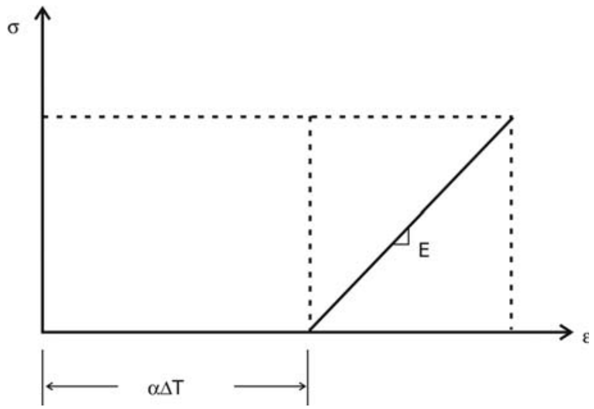

Figure 4-15. Stress–strain curve for a linear elastic element with thermal strain.

The quantity A is the cross-sectional area, and s is the distance measured along the centroid axis of the element. The complementary energy per unit volume must be used, because the stress and strain vary across the cross section when bending is present.

The stress at any point is given by eq. (3-14):

$$\sigma = \frac{F + F_T}{A} + \frac{(M + M_T)\, y}{I} - \alpha E \Delta T \tag{4-65}$$

If we make the substitution from eq. (4-65) for the stress into eq. (4-63) and integrate, we obtain the expression for the total complementary energy:

$$\begin{aligned} U_c = &\iint\limits_{s,A} \alpha \Delta T \left[\frac{F + F_T}{A} + \frac{(M + M_T)\, y}{I} - \alpha E \Delta T \right] dA\, ds \\ &+ \iint\limits_{s,A} \frac{1}{2E} \left[\frac{F + F_T}{A} + \frac{(M + M_T)\, y}{I} - \alpha E \Delta T \right]^2 dA\, ds \end{aligned} \tag{4-66}$$

The transverse deflection of the beam may be found by taking the derivative of the complementary energy with respect to an external load P applied at the point at which we wish to determine the deflection. Note that the thermal force and thermal moment are not functions of the applied load. The result is

$$\begin{aligned} \Delta_y = \frac{dU_c}{dP} = &\iint\limits_{s,A} \alpha \Delta T \left(\frac{1}{A} \frac{dF}{dP} + \frac{y}{I} \frac{dM}{dP} \right) dA\, ds \\ &+ \iint\limits_{s,A} \frac{1}{E} \left[\frac{F + F_T}{A} + \frac{(M + M_T)\, y}{I} - \alpha E \Delta T \right] \left[\frac{1}{A} \frac{dF}{dP} + \frac{y}{I} \frac{dM}{dP} \right] dA\, ds \end{aligned} \tag{4-67}$$

Because we are considering a linear elastic material, the internal forces F and moments M in the beam are directly proportional to the applied load P. This means that the derivatives are constant at any cross section. We may define the following dummy loads f and dummy moments m that would result if a unit load ($load = 1$) were applied at the point at which the deflection is desired.

$$f \equiv \frac{dF}{dP} \quad \text{and} \quad m \equiv \frac{dM}{dP} \tag{4-68}$$

Let us consider the first integral on the right side of eq. (4-67):

$$\iint\limits_{s,A} \alpha \Delta T \left(\frac{f}{A} + \frac{my}{I} \right) dA\, ds = \int\limits_{s} \left(\frac{f}{EA} \int\limits_{A} \alpha E \Delta T\, dA + \frac{m}{EI} \int\limits_{A} \alpha E \Delta T y\, dA \right) ds$$

or

$$\iint\limits_{s,A} \alpha \Delta T \left(\frac{f}{A} + \frac{my}{I} \right) dA\, ds = \int \left(\frac{f F_T}{EA} \right) ds + \int \left(\frac{m M_T}{EI} \right) ds \tag{4-69}$$

Next, let us examine the first part of the second integral on the right side of eq. (4-67):

$$\iint\limits_{s,A} \frac{f}{EA} \left[\frac{F + F_T}{A} - \alpha E \Delta T + \frac{(M + M_T)\, y}{I} \right] dA\, ds$$

$$= \int\limits_{s} \frac{f}{EA} \left[\frac{F + F_T}{A} \int\limits_{A} dA - \int\limits_{A} \alpha E \Delta T dA + \frac{M + M_T}{I} \int\limits_{A} y\, dA \right] ds$$

The second term on the right side in brackets is the thermal force F_T. Note also that $\int y\, dA = 0$ because the origin for the y-axis is on the centroid axis. The integral reduces to

$$\iint\limits_{s,A} \frac{f}{EA} [\cdots]\, dA\, ds = \int \frac{fF}{EA} ds \tag{4-70}$$

Finally, let us examine the other part of the second integral on the right side of eq. (4-67):

$$\iint\limits_{s,A} \frac{m}{EI} \left[\frac{F + F_T}{A} + \frac{(M + M_T)\, y}{I} - \alpha E \Delta T \right] y\, dA\, ds$$

$$= \int\limits_{s} \frac{m}{EI} \left[\frac{F + F_T}{A} \int\limits_{A} y\, dA + \frac{(M + M_T)}{I} \int\limits_{A} y^2 dA - \int\limits_{A} \alpha E \Delta T\, y\, dA \right] ds$$

The last term in brackets is the thermal moment M_T. We note that $\int y^2 dA = I$, the area moment of inertia for the cross section. The integral reduces to

$$\iint_{s,A} \frac{m}{EI}[\cdots]\, dA\, ds = \int \frac{mM}{EI} ds \tag{4-71}$$

If we make the substitutions for the integrals from eqs. (4-69), (4-70), and (4-71) into eq. (4-67), we obtain the final expression for determining the transverse deflection for an element with bending present:

$$\Delta_y = \int \frac{(M + M_T)\, m\, ds}{EI} + \int \frac{(F + F_T)\, f\, ds}{EA} \tag{4-72}$$

The rotation or slope of the beam may be found by taking the derivative of the complementary energy with respect to an external moment M_e applied at the point at which we wish to determine the rotation. We may define the unit moment m_r as follows:

$$m_r \equiv \frac{dM}{dM_e} \tag{4-73}$$

With this term, the rotation may be written as

$$\omega = \frac{dU_c}{dM_e} = \iint_{s,A} \alpha \Delta T \frac{y}{I} \frac{dM}{dM_e} dA\, ds + \iint_{s,A} \frac{1}{E}\left[\frac{F + F_T}{A} - \alpha E \Delta T + \frac{(M + M_T)\, y}{I}\right]\left(\frac{y}{I}\right) \frac{dM}{dM_e} dA\, ds \tag{4-74}$$

If we carry out the integrations over the cross-sectional area of the member, the only nonzero terms are

$$\omega = \int \frac{m_r}{EI}\left[\int_A \frac{\alpha E \Delta T y\, dA}{EI} - \int_A \frac{\alpha E \Delta T y\, dA}{EI} + \frac{M + M_T}{I} \int_A y^2 dA\right] ds \tag{4-75}$$

The integral in the last term is the area moment of inertia, $I = \int y^2 dA$. The final expression for the rotation is

$$\omega = \int \frac{(M + M_T)\, m_r ds}{EI} \tag{4-76}$$

The application of these general relationships may be best demonstrated by an example.

Example 4-7 Let us solve the problem presented in Section 3.6.1 by the elastic energy method. A cantilever beam shown in Figure 4-16 has a rectangular cross section, with a height h, a width b, and a length L. The beam has no mechanical

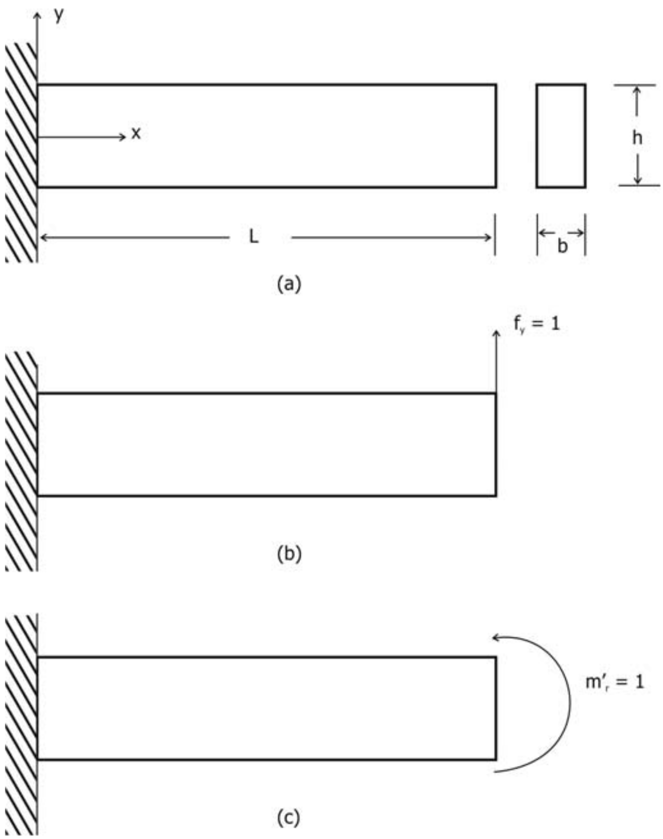

Figure 4-16. Cantilever beam in Example 4-7.

loads, and the temperature distribution in the beam is given by

$$\Delta T = \frac{1}{4}\Delta T_0 \left(1 + \frac{2y}{h}\right)^2 \left(\frac{x}{L}\right)^2$$

The thermal force, given by eq. (3-47), is

$$F_T = \tfrac{1}{3}\alpha E \Delta T_0 bh\,(x/L)^2$$

The thermal moment is given by eq. (3-50):

$$M_T = \frac{\alpha E I \Delta T_0}{h}\left(\frac{x}{L}\right)^2$$

Let us determine the transverse displacement and the rotation for the beam at the end of the beam ($x = L$).

The mechanical loads are zero, so $F = M = 0$. The differential distance along the centroid axis is $ds = dx$. The dummy moment due to the application of the unit force at the end of the beam may be written as

$$m = -(L - x)$$

The algebraic sign is negative, because the bending moment caused by the applied unit load would produce a *compressive* stress (or compressive strain) in the upper part (the $+y$ half) of the beam. Note that the unit load causes no axial forces in the beam, so $f = 0$ for the beam. Similarly, the bending moment produced by the application of a unit moment at the end of the beam is

$$m_r = -1$$

The applied unit moment shown in Figure 4-16 would cause a compressive stress in the $+y$ part of the beam; therefore, the moment is negative.

For this problem, eq. (4-72) reduces to

$$\Delta_y = \int_0^L \frac{(0 + M_T)\, m\, dx}{EI} + 0 = -\frac{\alpha \Delta T_0}{h} \int_0^L \left(\frac{x}{L}\right)^2 (L - x)\, dx$$

Carrying out the integration, we obtain the following expression for the deflection:

$$\Delta_y = -\frac{\alpha \Delta T_0 L^2}{h} \left[\frac{1}{3}\left(\frac{x}{L}\right)^3 - \frac{1}{4}\left(\frac{x}{L}\right)^4\right]_0^L = -\frac{\alpha \Delta T_0 L^2}{12h}$$

The rotation of the beam at the end ($x = L$) may be found from eq. (4-76), which reduces to

$$\omega = \int_0^L \frac{0 + M_T}{EI} (-1)\, dx = -\frac{\alpha \Delta T}{h} \int_0^L \left(\frac{x}{L}\right)^2 dx$$

If we carry out the integration, we obtain the following result for the rotation at the end of the beam. The *negative* sign means that the rotation is *downward*.

$$\omega = -\frac{\alpha \Delta T_0 L}{3h}$$

These results are the same as those obtained in Section 3.6.1. The use of the elastic energy method requires no solution of differential equations, however.

4.7 PIPE THERMAL EXPANSION LOOPS

We have noted that thermal stresses arise as a result of constraints on the system. It follows that thermal stresses would usually be decreased if the system could be made more flexible. A system supporting a load in bending is usually much more flexible than the same system with a direct or extensional loading.

The spring constant for a rod in tension or compression is

$$k_{\text{ext}} = \frac{EA}{L} \tag{4-77}$$

If the system is a cantilever beam clamped at one end and loaded with a transverse load at the other end, the spring constant for this bending situation is

$$k_{\text{bend}} = \frac{3EI}{L^3} = \frac{P}{\Delta_y} \tag{4-78}$$

The quantity A is the cross-sectional area, I is the area moment of inertia, and L is the length of the member. The ratio of the two spring constants depends on the dimensions of the member only:

$$\frac{k_{\text{ext}}}{k_{\text{bend}}} = \frac{AL^2}{3I} \tag{4-79}$$

For a pipe, the area moment of inertia is given by eq. (3-189), and the cross-sectional area is $A = \pi D_m t$, where D_m is the mean diameter of the pipe and t is the pipe wall thickness. The stiffness ratio of a pipe may be found from eq. (4-79):

$$\left(\frac{k_{\text{ext}}}{k_{\text{bend}}}\right)_{\text{pipe}} = \frac{8L^2}{3D_m^2} \tag{4-80}$$

If the length-to-diameter ratio is greater than 10, the stiffness ratio will be as follows:

$$\left(\frac{k_{\text{ext}}}{k_{\text{bend}}}\right)_{\text{pipe}} > \left(\frac{8}{3}\right)(10)^2 = 267 = \frac{1}{0.00375}$$

The pipe is 267 times stiffer in tension than in bending or, conversely, the pipe is 267 times more flexible in bending than in tension. This concept is used to reduce thermal stresses in piping systems through the use of pipe expansion loops.

There are many different forms of pipe expansion loops that have been used to reduce thermal stresses in piping systems. Let us consider the simple expansion shown in Figure 4-17. The pipe is anchored at each end, and the distance between the anchors is denoted by L. The width of the loop is W and the height of the loop is H. The distance a is related to W and L by $a = \frac{1}{2}(L - W)$. The temperature change is uniform across the pipe cross section, $\Delta T = \Delta T_0 =$ constant.

The quantities needed for eq. (4-72) are as follows. Because the temperature is uniform across the cross section, the thermal moment is zero, $M_T = 0$, and the thermal force is given by

$$F_T = \alpha EA \Delta T_0 \tag{4-81}$$

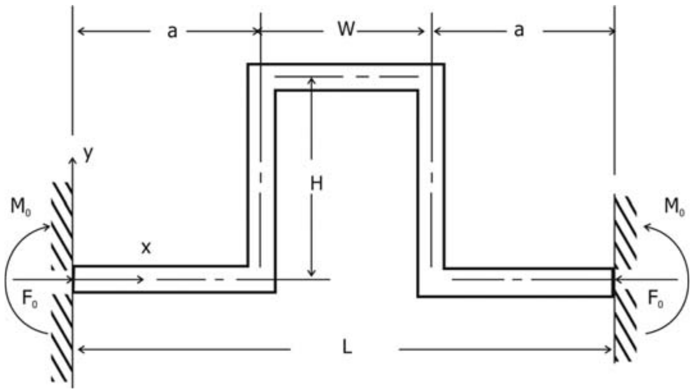

Figure 4-17. Pipe expansion loop.

The mechanical moments and forces at each of the five sections of the loop are

$$M = \begin{cases} M_0 \\ M_0 - F_0 y \\ M_0 - F_0 H \\ M_0 - F_0 y \\ M_0 \end{cases} \quad F = \begin{cases} -F_0 \\ 0 \\ -F_0 \\ 0 \\ -F_0 \end{cases} \quad ds = \begin{cases} dx \\ dy \\ dx \\ -dy \\ dx \end{cases} \quad \begin{cases} 0 \le x \le a \\ 0 \le y \le H \\ a \le x \le (a+W) \\ H \ge y \ge 0 \\ (L-a) \le x \le L \end{cases}$$

Similarly, the dummy moments and loads due to the application of a unit force and unit moment at the support point, as shown in Figure 4-18, are

$$m = \begin{cases} 0 \\ -y \\ -H \\ -y \\ 0 \end{cases} \quad f = \begin{cases} -1 \\ 0 \\ -1 \\ 0 \\ -1 \end{cases} \quad m_r = \begin{cases} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{cases} \quad \begin{cases} 0 \le x \le a \\ 0 \le y \le H \\ a \le x \le (a+W) \\ H \ge y \ge 0 \\ (L-a) \le x \le L \end{cases}$$

4.7.1 Support Reactions

Let us evaluate the integrals in eq. (4-72). First, the integral involving the direct and thermal forces is

$$\int (F + F_T)\, f\, ds = \int_0^a (-F_0 + F_T)(-1)\, dx + 0$$

$$+ \int_a^{a+W} (-F_0 + F_T)(-1)\, dx + 0 + \int_{L-a}^{L} (-F_0 + F_T)(-1)\, dx$$

$$\int (F + F_T)\, f\, ds = \left(F_0 - F_T\right)(2a + W) = (F_0 - F_T)\, L \tag{4-82}$$

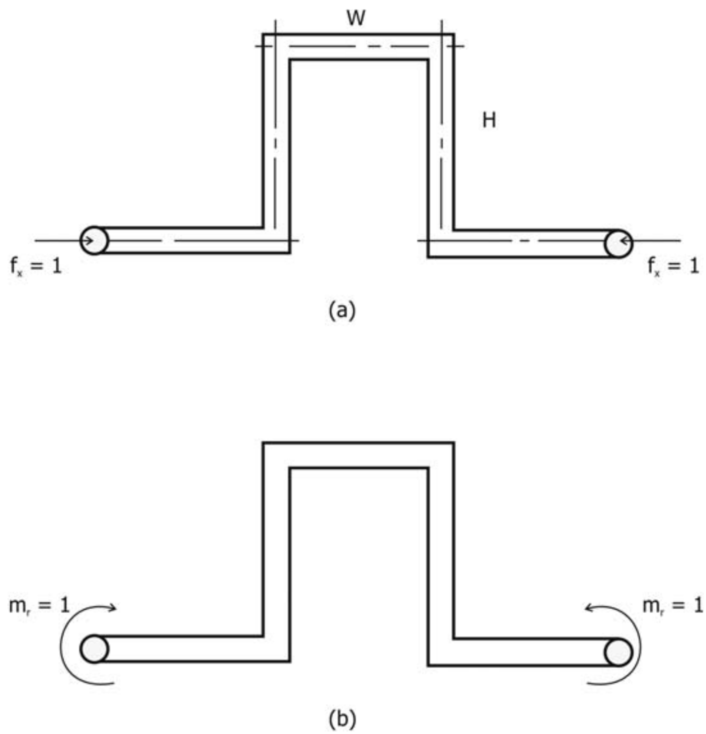

Figure 4-18. Reactions at the fixed ends (anchors) for the expansion loop shown in Figure 4-17.

Next, the integral involving the moments may be evaluated:

$$\int (M + M_T)\, m ds = 0 + \int_0^H (M_0 - F_0 y)\,(-y)\,dy + \int_a^{a+W} (M_0 - F_0 H)(-H)\,dx$$
$$+ \int_H^0 (M_0 - F_0 y)\,(-y)\,(-dy) + 0$$

$$\int (M + M_T)\, m ds = -M_0\left(H^2 + HW\right) + F_0\left(H^2 W + \tfrac{2}{3}H^3\right) \tag{4-83}$$

The displacement in the x-direction at the fixed support is zero; therefore, eq. (4-72) may be written as

$$\Delta_x = 0 = \frac{(F_0 - F_T)\,L}{EA} - \frac{M_0\left(H^2 + WH\right)}{EI} + \frac{F_0\left(H^2 W + \frac{2}{3}H^3\right)}{EI} \tag{4-84}$$

This expression may be simplified by substituting for the thermal force term:

$$\alpha EI \Delta T_0 L = -M_0\left(H^2 + WH\right) + F_0\left(H^2 W + \frac{2}{3}H^3 + \frac{LI}{A}\right) \tag{4-85}$$

We may evaluate the integrals in eq. (4-76) as follows:

$$\begin{aligned}\int (M + M_T)\, m_r ds = &\int_0^a (M_0 + 0)\,(1)\, dx + \int_0^H (M_0 - F_0 y + 0)\,(1)\, dy \\ &+ \int_a^{a+W} (M_0 - F_0 H + 0)\, dx + \int_H^0 (M_0 - F_0 y)\,(-dy) \\ &+ \int_{L-a}^{L} (M_0 + 0)\, dx\end{aligned}$$

Making this substitution into eq. (4-76) and noting that the rotation at the fixed end is also zero, we obtain

$$EI\omega = 0 = M_0\,(L + 2H) - F_0\left(H^2 + WH\right) \tag{4-86}$$

Equations (4-85) and (4-86) may be solved simultaneously to obtain the expressions for the support bending moment and the support force reaction:

$$F_0 = (\alpha EI \Delta T_0 L)\left[\frac{3(2H + L)}{H^2\left(2HL + H^2 + 3WL - 3W^2\right) + 3\left(\dfrac{LI}{A}\right)(2H + L)}\right] \tag{4-87}$$

$$M_0 = (\alpha EI \Delta T_0 L)\left[\frac{3(H + W)H}{H^2\left(2HL + H^2 + 3WL - 3W^2\right) + 3\left(\dfrac{LI}{A}\right)(2H + L)}\right] \tag{4-88}$$

For a pipe, we may use the area moment of inertia expression, eq. (3-189), to simplify the ratio in the denominator of eqs. (4-87) and (4-88):

$$\frac{LI}{A} = \frac{L\left(\frac{1}{8}\pi D_m^3 t\right)}{\pi D_m t} = \frac{1}{8} D_m^2 L \tag{4-89}$$

The expressions for the support reactions, eqs. (4-87) and (4-88), may be written in dimensionless form, as follows. Let us introduce the following length ratios:

$$\eta \equiv \frac{H}{L} \quad \text{and} \quad \xi \equiv \frac{W}{L} \tag{4-90}$$

If we substitute these parameters into eqs. (4-87) and (4-88), we obtain the following dimensionless expressions:

$$\frac{F_0 L^2}{\alpha EI \Delta T_0} = \frac{3\,(1 + 2\eta)}{\eta^2\left(2\eta + \eta^2 + 3\xi - 3\xi^2\right) + 3\left(\dfrac{D_m^2}{8L^2}\right)(1 + 2\eta)} = K_1 \tag{4-91}$$

$$\frac{M_0 L}{\alpha EI \Delta T_0} = \frac{3\eta\,(\eta + \xi)}{\eta^2\left(2\eta + \eta^2 + 3\xi - 3\xi^2\right) + 3\left(D_m^2/8L^2\right)(1 + 2\eta)} = K_2 \tag{4-92}$$

4.7.2 Stresses

The maximum stress in the pipe bend would occur either in the pipe section connected to the support ($0 \le x = 0$, $y = 0$) or in the horizontal portion of the bend ($a < x < a + W$, $y = H$). Generally, the first case applies when the width W of the pipe bend is large (or for $\xi = W/L > \frac{1}{2}$), and the second case applies when the width of the pipe bend is small (or for $\xi < \frac{1}{2}$).

The stress may be determined from the support reaction expressions and eq. (3-14). In this example, the thermal moment is zero, $M_T = 0$, and the thermal force is constant, $F_T = \alpha EA\Delta T_0$. For this case, the expression for the maximum stress reduces to the following. Note that the maximum bending stress occurs on the outer surface of the pipe and the outside diameter of the pipe is related to the mean diameter by $D_0 = D_m + t$.

$$\sigma = \frac{F}{A} \pm \frac{M(D_m + t)}{2I} \tag{4-93}$$

The maximum stress in the pipe bend may be written in dimensionless form.

For $\xi \ge \frac{1}{2}$:

$$\sigma = -\frac{F_0}{A} - \frac{M_0\,(D_m + t)}{2I}$$

$$\mathrm{Y} \equiv \left(\frac{\sigma}{\alpha E\Delta T_0}\right)\left[\frac{2L}{D_m\,(1 + t/D_m)}\right] = -K_2 - K_1\left[\frac{D_m}{4L\,(1 + t/D_m)}\right] \tag{4-94}$$

For $\xi < \frac{1}{2}$:

$$\sigma = -\frac{F_0}{A} - \frac{(M_0 - F_0 H)\,(D_m + t)}{2I}$$

$$Y = \left(\frac{\sigma}{\alpha E\Delta T_0}\right)\left[\frac{2L}{D_m\,(1 + t/D_m)}\right] = -K_2 - K_1\left[\frac{D_m}{4L\,(1 + t/D_m)} - \eta\right] \tag{4-95}$$

We may check the stress expressions for two obvious limiting cases, extremely large values for H and extremely small values of H. First, for H very large or for $\eta \to \infty$, we observe from eqs. (4-91) and (4-92) that $K_1 = K_2 = 0$. Therefore, if the height of the U-bend is very large, the stress in the U-bend approaches zero, as would be expected.

Second, for H very small or for $\eta \to 0$, we find the following values for K_1 and K_2:

$$K_1 \to \frac{8L^2}{D_m^2} \quad \text{and} \quad K_2 \to 0$$

If we make these substitutions into eq. (4-94), the dimensionless stress function reduces to the following for $\eta = 0$:

$$Y = \left(\frac{\sigma}{\alpha E\Delta T_0}\right)\left[\frac{2L}{D_m\,(1 + t/D_m)}\right] = 0 - \frac{2L}{D_m\,(1 + t/D_m)}$$

Therefore, the stress in the system for which the height of the U-bend is zero (or actually, if no U-bend is present) reduces to

$$\sigma = -\alpha E \Delta T_0$$

This is the same expression that would be obtained for a straight run of pipe with no U-bend.

The ratio of the pipe wall thickness to the mean pipe diameter (t/D_m) is usually small. For example, for Schedule 40 pipe (standard pipe), we find that $0.03 < t/D_m < 0.11$ for nominal pipe diameters from 1 in. (25 mm) to 12 in. (300 mm). The term in brackets in eq. (4-94) is also small unless the pipe run is short. For example, if the ratio of the distance between anchors to the pipe outside diameter $\{L/[D_m(1 + t/D_m)]\}$ is greater than about 25, the bracketed term in eq. (4-94) has a value less than 1 percent.

$$\left[\frac{D_m}{4L\left(1 + \dfrac{t}{D_m}\right)}\right] \le \frac{1/25}{4} = 0.010$$

The denominators of the factor K_1 and K_2 are equal, so we may examine the numerator terms to determine the value of $\xi = W/L$ that results in the smallest maximum stress. If we neglect the term involving diameter-to-length ratio, the maximum stress in the U-bend may be written from eqs. (4-94) and (4-95).

For $\xi \ge \frac{1}{2}$:

$$Y = -K_2 = -\frac{3\eta(\eta + \xi)}{\text{Denom}}$$

For $\xi \le \frac{1}{2}$:

$$Y = -K_2 + \eta K_1 = -\frac{3\eta^2 + 3\eta\xi - 3\eta - 6\eta^2}{\text{Denom}} = -\frac{3\eta(\xi - 1 - \eta)}{\text{Denom}}$$

We note from the plot of the numerator terms given in Figure 4-19 that the magnitude of the numerator increases as ξ increased for the first case, and the magnitude $3\eta(1 + \eta - \xi)$ of the numerator decreases as ξ is increased for the second case.

If we equate the magnitude of the numerator terms, we obtain

$$3\eta^2 + 3\eta\xi = \left|3\eta\xi - 3\eta - 3\eta^2\right| = 3\eta^2 - 3\eta\xi + 3\eta$$

or

$$6\xi = 3 \quad \text{and} \quad \xi = \tfrac{1}{2}$$

From the preceding discussion, we conclude that the best choice for the width of the U-bend is $\xi = \frac{1}{2}$ or $W = \frac{1}{2}L$. For this value of the ratio ξ, the dimensionless

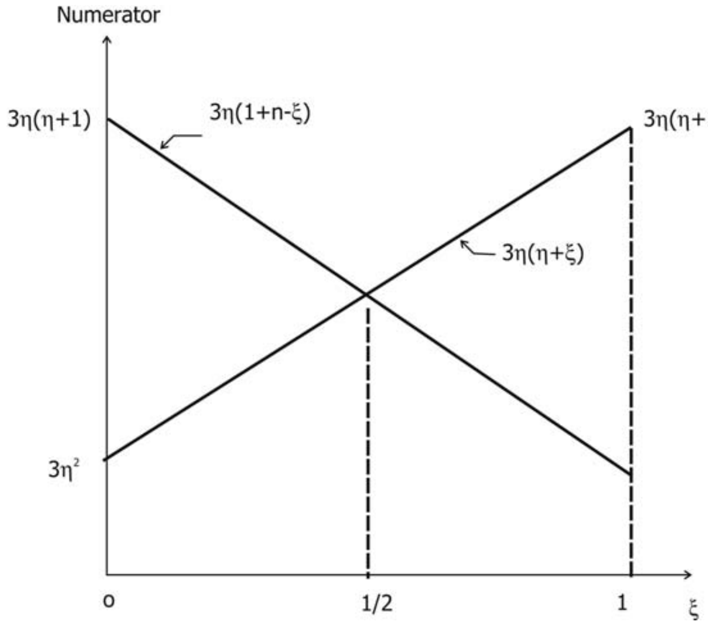

Figure 4-19. Plot of the numerator terms in the dimensionless stress parameter Y for $\xi = W/L = \frac{1}{2}$ for the expansion loop shown in Figure 4-17.

parameters in eqs. (4-91) and (4-92) reduce to the following expressions:

$$K_1 = \frac{12}{\eta^2(3+2\eta) + \left(\dfrac{3D_m^2}{2L^2}\right)} \quad \text{for } \xi = \tfrac{1}{2} \tag{4-96}$$

$$K_2 = \frac{6\eta}{\eta^2(3+2\eta) + \left(\dfrac{3D_m^2}{2L^2}\right)} \quad \text{for } \xi = \tfrac{1}{2} \tag{4-97}$$

The dimensionless stress function given by eq. (4-94) reduces to

$$Y = \left(\frac{\sigma}{\alpha E \Delta T_0}\right)\left[\frac{2L}{D_m\left(1+\dfrac{t}{D_m}\right)}\right] = -\frac{6\left[\eta + \dfrac{D_m}{2L\left(1+\dfrac{t}{D_m}\right)}\right]}{\eta^2(3+2\eta) + \left(\dfrac{3D_m^2}{2L^2}\right)} \quad \text{for } \xi = \tfrac{1}{2} \tag{4-98}$$

The data obtained from eq. (4-98) for $(L/D_m) = 26.5$ and $(t/D_m) = 0.060$ (and for $L/D_0 = 25$) are tabulated in Table 4-10. For $\eta = H/L > 0.35$, the effect of these ratios is less than 5 percent on the value of the dimensionless stress function. If the second terms in the numerator and denominator of eq. (4-98) are

TABLE 4-10. Dimensionless stress Y as a function of dimensionless pipe loop depth $\eta = H/L$ for a pipe loop dimensionless width $\xi = W/L = \frac{1}{2}$. The tabulated values correspond to a length-to-diameter ratio $L/D_m = 26.5$ and a wall thickness ratio $t/D_m = 0.060$ in eq. (4-98). Use logarithmic interpolation with the tabular data.

$\eta = H/L$	Y	$\eta = H/L$	Y
0.15	13.180	0.9	1.4156
0.20	9.460	1.0	1.2208
0.25	7.274	1.1	1.0656
0.30	5.847	1.2	0.9394
0.35	4.846	1.3	0.8353
0.40	4.109	1.4	0.7482
0.45	3.544	1.5	0.6745
0.50	3.100	1.6	0.6115
0.55	2.742	1.8	0.5100
0.60	2.448	2.0	0.4324
0.65	2.203	2.5	0.3021
0.70	1.996	3.0	0.2235
0.75	1.818	4.0	0.1370
0.80	1.6655	5.0	0.0926

$$\frac{\ln \eta - \ln \eta_1}{\ln Y - \ln Y_1} = \frac{\ln \eta_2 - \ln \eta_1}{\ln Y_2 - \ln Y_1} \quad \text{or} \quad \frac{\eta}{\eta_1} = \left(\frac{Y}{Y_1}\right)^{\frac{\eta_2/\eta_1}{Y_2/Y_1}}$$

neglected, the stress expression reduces to

$$Y \approx -\frac{6}{\eta\,(3 + 2\eta)} \tag{4-99}$$

For design purposes, the following relationship (obtained from a curve-fit technique) may be used to determine the loop depth ratio, for the range $0.2 \leq \eta \leq 2.0$.

$$\ln \eta = 0.1400 - 0.6945 \ln Y - 0.0374\,(\ln Y)^2 \tag{4-100}$$

The design principles for pipe expansion loops may be illustrated by an example.

Example 4-8 A pipe loop, shown in Figure 4-17, is anchored at a distance of 12 m (39.37 ft) between anchors. The pipe is 150 mm nominal (6 in. nominal) Schedule 40 pipe, with an outside diameter of 168.3 mm (6.625 in.) and a wall thickness of 7.1 mm (0.280 in.). The pipe material is 304 stainless steel with $E = 193$ GPa (28.0×10^6 psi), and $\alpha = 16.0 \times 10^{-6}\,\text{K}^{-1}$ ($8.89 \times 10^{-6}\,{}^\circ\text{F}^{-1}$). The thermal stress in the pipe must be limited to 18 MPa (2610 psi). The pipe carries liquid oxygen, so the pipe wall will be cooled from ambient temperature (300 K or 80°F), the stress-free condition, to an operating temperature of 90 K (−298°F). Determine the dimensions of the pipe loop.

The temperature change for the pipe is $\Delta T_0 = 90 - 300 = -210$ K (−378°F). Let us first check whether a pipe expansion loop is needed in this case. The thermal stress for a straight run of pipe anchored at each end is found as follows:

$$\sigma = \alpha E \Delta T_0 = \left(16.0 \times 10^{-6}\right)\left(193 \times 10^9\right)(210) = 648.5 \times 10^6\ \text{Pa} = 648.5\ \text{MPa}$$

Since this value greatly exceeds the allowable stress of 18 MPa, an expansion loop is required.

The mean diameter of the pipe is

$$D_m = D_0 - t = 168.3 - 7.1 = 161.2\ \text{mm} \quad (6.345\ \text{in.})$$

Generally, the optimum width for the pipe loop is one-half of the distance between anchors:

$$W = \tfrac{1}{2}L = \tfrac{1}{2}(12) = 6.00\ \text{m} \quad (19.685\ \text{ft})$$

The dimensionless stress function may be calculated from its definition:

$$Y \equiv \left(\frac{\sigma}{\alpha E \Delta T_0}\right)\left[\frac{2L}{D_m\left(1+\dfrac{t}{D_m}\right)}\right] = \frac{18\times 10^6}{648.5\times 10^6}\frac{(2)(12.0)}{(0.1612)\left(1+\dfrac{7.1}{161.2}\right)}$$

$$= 3.9582$$

If we use this value in eq. (4-98), we obtain the following value for the dimensionless height of the loop:

$$\eta = 0.4042 = \frac{H}{L}$$

The required height of the expansion loop is

$$H = (0.4042)(12.0) = 4.850\ \text{m} \quad (15.91\ \text{ft})$$

The anchor reactions may be evaluated for the expansion loop. The diameter-to-length term in the denominator of eqs. (4-96) and (4-97) has the following value:

$$\left(\frac{3D_m^2}{2L^2}\right) = \frac{(3)(0.1612)^2}{(2)(12.0)^2} = 0.2707\times 10^{-3}$$

The factors given by eqs. (4-96) and (4-97) are

$$K_1 = \frac{(12)}{(0.4042)^2\,[3+(2)(0.4042)] + 0.0002707} = \frac{12}{0.62248} = 19.278$$

and

$$K_2 = \frac{(6)(0.4042)}{0.62248} = 3.896$$

The area moment of inertia for the pipe is

$$I = \frac{\pi}{8}D_m^3 t = \left(\frac{\pi}{8}\right)(0.1612)^3\,(0.0071) = 1170\times 10^{-8}\ \text{m}^4 = 1170\ \text{cm}^4$$

The axial force at the anchor may be calculated from eq. (4-91):

$$F_0 = \frac{\left(648.5\times 10^6\right)\left(1170\times 10^{-8}\right)(19.278)}{(12.0)^2} = 1016\ \text{N} \ \ (228\ \text{lb}_\text{f})$$

The moment at the anchor may be calculated from eq. (4-92):

$$M_0 = \frac{\left(648.5 \times 10^6\right)\left(1170 \times 10^{-8}\right)(3.896)}{(12.0)} = 2463 \text{ N-m} \ \ (1817 \text{ lb}_\text{f}\text{-ft})$$

4.7.3 Effect of Pipe Fittings (Elbows)

Sharp right-angled joints are generally not used in piping systems, because of the larger fluid pressure drop occurring in the sharp bend. Instead, the straight runs of pipe are jointed by pipe fittings called *elbows*.

When an elbow is subjected to bending, it has been shown both experimentally and analytically that the circular cross section of the elbow becomes oval [von Kármán, 1911] The result of the ovalization is that the element becomes more flexible and the local stress is higher than the overall bending stress value given by curved beam analysis [Den Hartog, 1952] The detailed analysis for this condition is complex; however, flexibility factors or ratios and stress intensification factors are used for design purposes to reduce the complexity. The *flexibility factor* is the ratio of the actual flexibility to the flexibility with ovalization not considered. The *stress intensification factor* is the ratio of the maximum local stress to the stress that would result for a straight run of pipe in bending.

The ASME Piping Code [ASME, 1999] recommends the following expression for the flexibility factor K_fl:

$$K_\text{fl} = \begin{cases} \dfrac{1.65}{\lambda_p} & \text{for} \quad \lambda_p \le 1.65 \\ 1 & \text{for} \quad \lambda_p > 1.65 \end{cases} \tag{4-101}$$

The factor λ_p is defined as

$$\lambda_p = \frac{4tR_m}{D_m^2} \tag{4-102}$$

The quantity t is the wall thickness of the elbow, R_m is the mean radius of the elbow bend, and D_m is the mean diameter of the elbow cross section, as illustrated in Figure 4-20.

For the case of pipe expansion loops, the effect of the increased flexibility is that the elbow acts as if the distance from the face of the elbow to the centerline of the other face R_m is increased by a length ΔL, when compared with the square-corner deflection analysis. The increase in length or virtual length for a 90° elbow is given by the following expression [Kellogg, 1956]:

$$\Delta L = R_m \left(\frac{\pi}{4} K_\text{fl} - 1\right) \tag{4-103}$$

This additional length is added to the straight pipe run on either side of the corner (intersection of the pipe run centerlines), as shown in Figure 4-21.

The stress intensity factor recommended by the ASME Piping Code is

$$K_\text{si} = \begin{cases} \dfrac{0.90}{\lambda_p^{2/3}} & \text{for} \quad \lambda_p \le 0.85 \\ 1 & \text{for} \quad \lambda_p < 0.85 \end{cases} \tag{4-104}$$

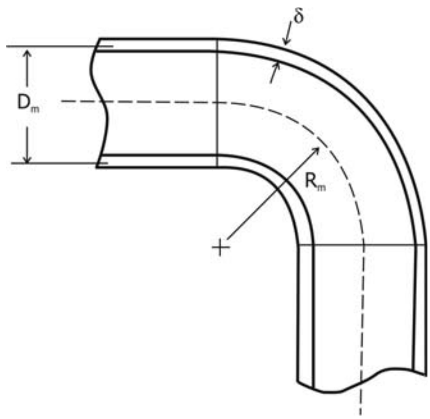

Figure 4-20. Dimensions for a pipe elbow.

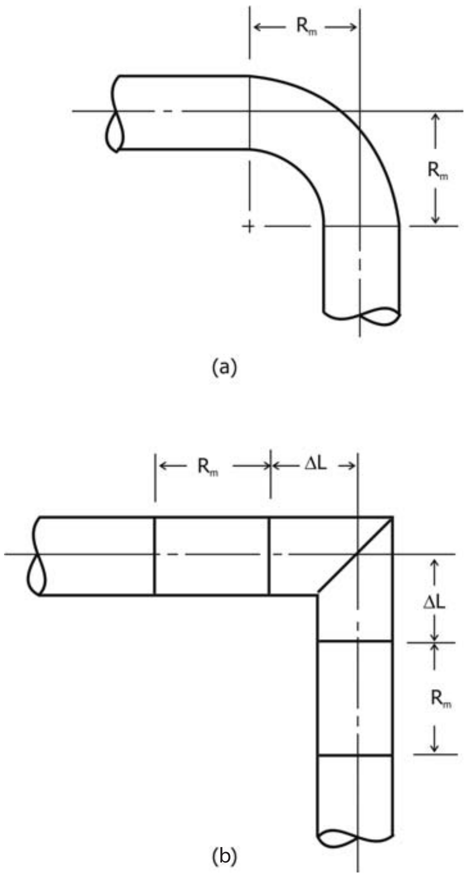

Figure 4-21. Additional virtual length to account for elbow flexibility.

This multiplier is applied to the bending stress only. The flexibility of the pipe fitting has a minor effect on direct stresses produced by axial forces.

The maximum stress for a pipe expansion loop, considering the stress in the elbows, may be written as follows, which is a modification of eqs. (4-94) and (4-95).

For $\xi = W/L \geq \frac{1}{2}$:

$$Y = -K_2 K_{si} - K_1 \left[\frac{D_m}{4L\left(1 + \dfrac{t}{D_m}\right)} \right] \tag{4-105}$$

For $\xi < \frac{1}{2}$:

$$Y = -K_2 K_{si} - K_1 \left[\frac{D_m}{4L\left(1 + \dfrac{t}{D_m}\right)} - \eta K_{si} \right] \tag{4-106}$$

For the special case of $\xi = \frac{1}{2}$, eq. (4-98) may be modified as follows:

$$Y = -\frac{6\left[\eta K_{si} + \dfrac{D_m}{2L\left(1 + \dfrac{t}{D_m}\right)}\right]}{\eta^2 (3 + 2\eta) + \left(\dfrac{3D_m^2}{2L^2}\right)} \quad \text{for} \quad \xi = \tfrac{1}{2} \tag{4-107}$$

The stress intensity factor is associated with the local stresses in the pipe fittings, so the support reactions M_0 and F_0 are not affected by the increased stress in the elbows. On the other hand, the increased flexibility of the elbows does affect the support reactions. The support reactions may be calculated from eqs. (4-91) and (4-92), using the equivalent lengths for each member.

Example 4-9 Let us work Example 4-8, considering the flexibility of the pipe elbows. The four elbows are long-radius 90° elbows with an elbow radius of 228.6 mm (9.00 in.) and the same mean diameter and wall thickness as the straight run of pipe.

Let us first calculate the equivalent dimensions for the pipe runs in the expansion loop, with elbow flexibility considered. The dimensionless parameter defined by eq. (4-102) is

$$\lambda_p = \frac{(4)\,(7.1)\,(228.6)}{(161.2)^2} = 0.250$$

The flexibility factor is given by eq. (4-101):

$$K_{fl} = \frac{1.65}{0.250} = 6.60$$

The "virtual" length for each end of the elbows may be found from eq. (4-103):

$$\Delta L = (0.2286)\left[\frac{\pi}{4}(6.60) - 1\right] = 0.956\ \text{m}$$

The equivalent lengths for each run are

$$W_e = W + 2\Delta L = 6.00 + (2)(0.956) = 7.912\ \text{m} \quad \text{(2 elbows in } W\text{-run)}$$
$$L_e = L + 4\Delta L = 12.00 + (4)(0.956) = 15.824\ \text{m}$$

The dimensionless ratio ξ is

$$\xi = \frac{W_e}{L_e} = \frac{7.912}{15.824} = 0.500$$

The stress intensification factor may be calculated from eq. (4-104):

$$K_{\text{si}} = \frac{0.90}{(0.250)^{2/3}} = 2.268$$

The following ratios may be calculated, using the equivalent length L_e:

$$\frac{D_m}{2L_e\left(1 + \dfrac{t}{D_m}\right)} = \frac{0.1612}{(2)(15.824)\left(1 + \dfrac{7.1}{161.2}\right)} = 4.878 \times 10^{-3}$$
$$\frac{3D_m^2}{2L_e^2} = \frac{(3)\,(0.1612)^2}{(2)(15.824)^2} = 0.1557 \times 10^{-3}$$
$$\frac{2L_e}{D_m\left(1 + \dfrac{t}{D_e}\right)} = \frac{(2)(15.824)}{(0.1612)\left(1 + \dfrac{7.1}{161.2}\right)} = 188.03$$

The stress ratio is

$$Y = \frac{\left(18 \times 10^6\right)(188.03)}{648.5 \times 10^6} = 5.219 = \frac{6\left(2.268\eta + 4.878 \times 10^{-3}\right)}{\eta^2\,(3 + 2\eta) + 0.1557 \times 10^{-3}}$$

The dimensionless ratio η may be found:

$$\eta = 0.6177 = \frac{H_e}{L_e}$$

The equivalent height for the expansion loop, including flexibility of the elbows, is

$$H_e = (0.6177)(15.824) = 9.774\ \text{m}$$

The actual height for the expansion loop may be determined, for the case of two elbows in the pipe run:

$$H = 9.774 - (2)(0.956) = 7.862\ \text{m} \quad (25.79\ \text{ft})$$

In Example 4-8, we found a height of 4.850 m, neglecting the effects of the elbows. Considering the stress concentration and flexibility of the elbows, the expansion loop height was $(7.862/4.950) = 1.588$ times the length for square

corners. The stress concentration factor for the elbows is 2.268, so the flexibility of the elbows does have the effect of reducing the stress.

4.8 PIPE BENDS

In some types of shell-and-tube heat exchangers, the heat exchanger tubes are anchored in a tube sheet, as shown in Figure 4-22. The tubes have a straight length L, and the mean radius of the pipe bend is R. The system is considered to be rigidly anchored at the ends of the straight sections, where the support bending moment is M_0 and the transverse force is F_0. The entire system is subjected to a uniform temperature change ΔT_0.

The bending moment and axial force at any location may be determined:

$$M = \begin{cases} M_0 - F_0 x & \text{for} \quad 0 \leq x \leq L \\ (M_0 - F_0 L) - F_0 R \sin\phi & \text{for} \quad 0 \leq \phi \leq \pi \\ \mathrm{M}_0 - F_0 x & \text{for} \quad 0 \leq x \leq L \end{cases}$$

$$F = \begin{cases} 0 & \text{for} \quad 0 \leq x \leq L \\ -F_0 \sin\phi & \text{for} \quad 0 \leq \phi \leq \pi \\ 0 & \text{for} \quad 0 \leq x \leq L \end{cases}$$

The coordinate for each section is

$$ds = \begin{cases} dx & \text{for} \quad 0 \leq x \leq L \\ R d\phi & \text{for} \quad 0 \leq \phi \leq \pi \\ -dx & \text{for} \quad 0 \leq x \leq L \end{cases}$$

If we apply a unit load $f_y = 1$ at the supports points, as shown in Figure 4-23, the resulting dummy moments and axial forces are

$$m = \begin{cases} -x \\ -L - R\sin\phi \\ -x \end{cases} \qquad f = \begin{cases} 0 & \text{for} \quad 0 \leq x \leq L \\ -\sin\phi & \text{for} \quad 0 \leq \phi \leq \pi \\ 0 & \text{for} \quad 0 \leq x \leq L \end{cases}$$

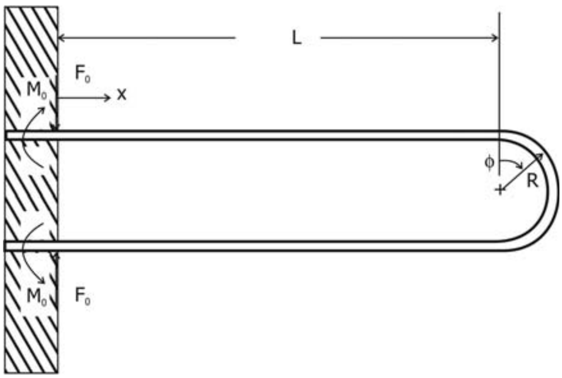

Figure 4-22. Pipe bends, such as those in a U-tube shell-and-tube heat exchanger.

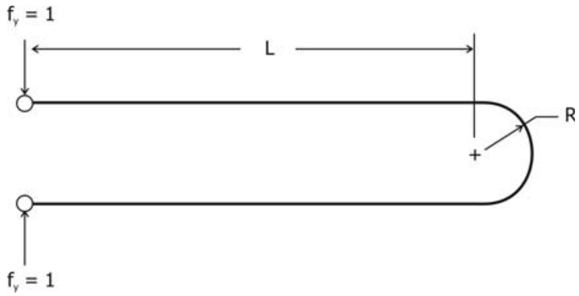

Figure 4-23. Application of a unit load for the pipe bend shown in Figure 4-22.

For the given thermal loading, we find that the thermal moment $M_T = 0$ and the thermal force $F_T = \alpha EA\Delta T_0$. These quantities may be used to evaluate the integrals in eq. (4-72). The integral involving the axial force and thermal force may be evaluated:

$$\begin{aligned}\int (F + F_T) f ds &= 0 + \int_0^{\pi} (-F_0 \sin\phi + F_T)(-\sin\phi) R d\phi + 0 \\ &= \tfrac{1}{2}\pi F_0 R - 2\alpha EA\Delta T_0 R\end{aligned}$$

The integral involving the moments may be evaluated also:

$$\begin{aligned}\int (M + M_T) m ds = &\int_0^L (M_0 - F_0 x)(-x)\, dx \\ &+ \int_0^{\pi} (M_0 - F_0 L - F_0 R \sin\phi)(-L - R\sin\phi) R d\phi \\ &+ \int_L^0 (M_0 - F_0 x)(-x)(-dx)\end{aligned}$$

After evaluation of the integrals, we find

$$\int (M + M_T) m ds = -M_0\left(L^2 + \pi LR + 2R^2\right) + F_0\left(\tfrac{2}{3}L^3 + \pi L^2 R + 4LR^2 + \tfrac{1}{2}\pi R^3\right)$$

The transverse displacement at the support point is zero, so eq. (4-72) may be written in the following form:

$$\begin{aligned}EI\Delta_y = 0 = &-M_0\left(L^2 + \pi LR + 2R^2\right) + F_0\left(\tfrac{2}{3}L^3 + \pi L^2 R + 4LR^2 + \tfrac{1}{2}\pi R^3\right) \\ &+ \frac{\pi IRF_0}{2A} - 2\alpha EI\Delta T_0 R \qquad (4\text{-}108)\end{aligned}$$

The rotation at the support point is also zero. If we apply a unit moment at the support point, as shown in Figure 4-24, then the dummy moment in each

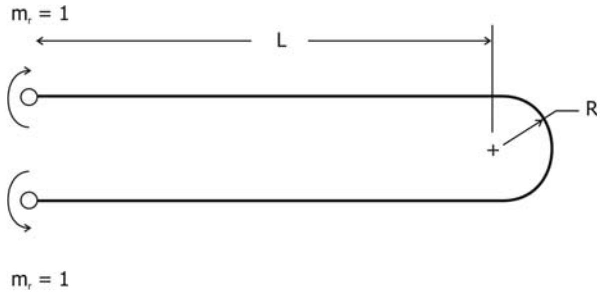

Figure 4-24. Application of a unit moment for the pipe bend shown in Figure 4-22.

section is $m_r = 1$. If we make this substitution in eq. (4-76), we obtain

$$EI\omega = 0 = \int (M + M_T)\, m_r ds = \int M ds$$

If we make the substitutions for the bending moment, we obtain

$$0 = \int_0^L (M_0 - F_0 x)\, dx + \int_0^\pi (M_0 - F_0 L - F_0 R \sin\phi) R d\phi + \int_L^0 (M_0 - F_0 x)(-dx)$$

After evaluating the integrals, we obtain the following relationship:

$$0 = M_0\,(2L + \pi R) - F_0 \left(L^2 + \pi LR + 2R^2\right)$$

The bending moment and transverse force at the support point are related through the following expression:

$$M_0 = F_0 \left(\frac{L^2 + \pi LR + 2R^2}{2L + \pi R}\right) \tag{4-109}$$

We may combine eqs. (4-108) and (4-109) to obtain the expressions for the support force and bending moment. If we introduce the following dimensionless parameter; $\eta = L/R$, we obtain

$$F_0 = K_3 \frac{\alpha EI \Delta T_0}{R^2} \tag{4-110}$$

where

$$K_3 = \frac{6\,(2\eta + \pi)}{\eta^4 + 2\pi\eta^3 + 12\eta^2 + 3\pi\eta + \left(\frac{3}{2}\pi - 12\right) + \frac{3}{2}\pi\,(2\eta + \pi)\left(I/AR^2\right)} \tag{4-111}$$

The term involving the area moment of inertia may be simplified by using eq. (3-189):

$$\frac{I}{AR^2} = \frac{\frac{1}{8}\pi D_m^3 t}{\pi D_n t R^2} = \frac{D_m^2}{8R^2} \tag{4-112}$$

By using eq. (4-110) for the force in eq. (4-109), we may solve for the bending moment at the support point:

$$M_0 = K_4 \frac{\alpha EI \Delta T_0}{R} \tag{4-113}$$

where

$$K_4 = \frac{6\left(\eta^2 + \pi\eta + 2\right)}{\eta^4 + 2\pi\eta^3 + 12\eta^2 + 3\pi\eta + \left(\frac{3}{2}\pi^2 - 12\right) + \frac{3}{2}\pi\,(2\eta + \pi)\left(\dfrac{I}{AR^2}\right)} \tag{4-114}$$

The maximum thermal stress generally occurs at the support point:

$$\sigma = \pm\frac{M_0 D_0}{2I} = \pm\frac{K_4 \alpha E \Delta T_0 D_m \left(1 + \dfrac{t}{D_m}\right)}{2R} \tag{4-115}$$

The dimensionless stress at the support point may be written as

$$Y \equiv \frac{\sigma}{\alpha E \Delta T_0}\left[\frac{2R}{D_m\left(1 + \dfrac{t}{D_m}\right)}\right] = K_4 \tag{4-116}$$

This function is plotted in Figure 4-25.

The stress at the top of the pipe bend ($\phi = \frac{1}{2}\pi$) is as follows, where the stress concentration effects of "ovaling" are not included:

$$\sigma = \pm\frac{[M_0 - F_0\,(L + R)]\,D_0}{2I} - \frac{F_0}{A} \tag{4-117}$$

The stress at this point may be written in dimensionless form as

$$Y = K_4 - K_3\left[1 + \eta + \frac{D_m}{4R\left(1 + \dfrac{t}{D_m}\right)}\right] \tag{4-118}$$

If the stress intensity factor from the ASME Piping Code is included with the bending terms, the dimensionless stress in the pipe bend is

$$Y = K_4 K_{\text{si}} - K_3\left[(1 + \eta)\,K_{\text{si}} + \frac{D_m}{4R\left(1 + \dfrac{t}{D_m}\right)}\right] \tag{4-119}$$

The stress intensity factor K_{si} is given by eq. (4-104).

The largest stress occurs for the condition in which the pipe bend is least flexible. This case corresponds to case with a straight pipe length of zero or for $\eta = L/R = 0$. The dimensionless stress for this condition may be found from eq. (4-116):

$$Y = K_4\,(\eta = 0) = \frac{1}{\left(\dfrac{\pi^2}{8} - 1\right) + \left(\dfrac{\pi D_m}{8R}\right)^2} \tag{4-120}$$

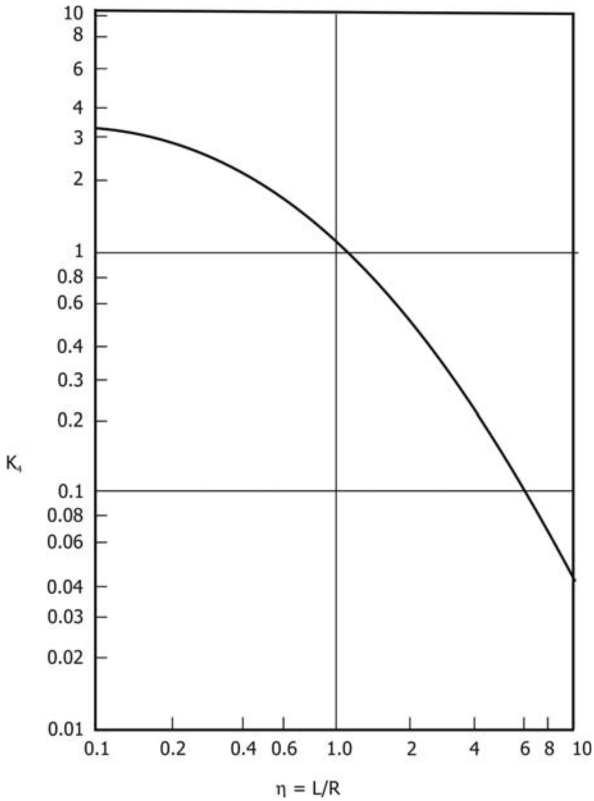

Figure 4-25. Plot of the maximum stress for the pipe bend shown in Figure 4-22.

The stress is smallest when the pipe bend flexibility is largest, which corresponds to the case of a very long, straight pipe length or for η very large. The dimensionless stress for this condition is

$$Y = K_4\,(\eta > 100) \approx \frac{6}{\eta^2} \tag{4-121}$$

The difference between the stress value from eq. (4-116) and from eq. (4-121) is less than 3 percent for $\eta > 100$.

Example 4-10 A heat exchanger U-tube is constructed of Cu/10% Ni with Young's modulus, 124 GPa (18×10^6 psi) and thermal expansion coefficient, $\alpha = 16.2 \times 10^{-6}\ \mathrm{K}^{-1}$ ($9 \times 10^{-6}\ {}^\circ\mathrm{F}^{-1}$). The tubing is 1 in. OD (25.4 mm OD) 16 BWG tube, with a wall thickness of 1.65 mm (0.065 in.) and a mean diameter of 23.75 mm (22.1 in.). The radius of the tube bend is 38.1 mm (1.500 in.) and the length of the straight pipe runs is 900 mm (35.43 in.). The thermal stress in the pipe bend is to be limited to 2.0 MPa (290 psi). Determine the maximum allowable temperature change for the pipe bend.

The geometry ratio η is

$$\eta = \frac{L}{R} = \frac{900}{38.1} = 23.62$$

The term involving the area moment of inertia in eq. (4-114) may be calculated:

$$\frac{I}{AR^2} = \frac{D_m^2}{8R^2} = \frac{(23.75)^2}{(8)\,(38.1)^2} = 0.04857$$

Next, let us calculate the numerical value for the denominator of eq. (4-114):

$$(23.62)^4 + (2\pi)(23.62)^3 + (12)(23.62)^2 + (3\pi)(23.62) + \left(\tfrac{3}{2}\pi^2 - 12\right) \\ + \left(\tfrac{3}{2}\pi\right)(47.24 + \pi)(0.04857) = 401.115 \times 10^3$$

Making this substitution into eq. (4-114), we find the following value for the dimensionless parameter K_4:

$$K_4 = \frac{(6)\left(23.62^2 + 23.62\pi + 2\right)}{401.115 \times 10^3} = 0.009487 = Y$$

The maximum allowable temperature change may be determined from eq. (4-116):

$$Y = 0.009487 = \frac{\left(2.0 \times 10^6\right)(2)(38.1)}{\left(16.2 \times 10^{-6}\right)\left(124 \times 10^9\right)(\Delta T_0)\,(23.75)\left(1 + \dfrac{1.65}{23.75}\right)} = \frac{2.987}{\Delta T_0}$$

The maximum allowable temperature change is

$$\Delta T_0 = \frac{2.987}{0.009487} = 315 \text{ K} \quad \left(567^\circ\text{F}\right)$$

Let us check the stress at the top of the pipe bend ($\phi = \frac{1}{2}\pi$). The value of the parameter K_3 may be determined from eq. (4-111):

$$K_3 = \frac{(6)\,[(2)\,(23.62) + \pi]}{401.115 \times 10^3} = 0.7537 \times 10^{-3}$$

The value of the dimensionless diameter–radius ratio is

$$\frac{D_m}{4R\left(1 + \dfrac{t}{D_m}\right)} = \frac{23.75}{(4)\,(38.1)\left(1 + \dfrac{1.65}{23.75}\right)} = 0.1457$$

The stress at the top of the pipe bend, excluding the effects of stress concentration due to ovaling, may be determined from eq. (4-118):

$$Y = 0.009487 - \left(0.7537 \times 10^{-3}\right)(1 + 23.62 + 0.1457) = -0.009180$$

The value of the stress, excluding the effects of stress concentration, is

$$\sigma = (-0.009180)\left(16.2 \times 10^{-6}\right)\left(124 \times 10^{9}\right)(315)\frac{25.4}{(2)(38.1)}$$
$$\sigma = -1.936 \times 10^{6}\,\text{Pa} = -1.936\,\text{MPa} \;\; (-281\,\text{psi})$$

This value is less than the maximum allowable stress (2.00 MPa); however, stress concentration in the pipe bend will result in a higher stress.

Let us examine the effect of the stress concentration. The dimensionless parameter from eq. (4-102) may be evaluated:

$$\lambda_p = \frac{4tR}{D_m^2} = \frac{(4)(1.65)(38.1)}{(23.75)^2} = 0.4458 < 0.85$$

The stress intensity factor may be found using eq. (4-104):

$$K_{\text{si}} = \frac{0.90}{(0.4458)^{2/3}} = 1.542$$

The stress at the top of the pipe bend, including the effects of stress concentration due to ovaling, may be determined from eq. (4-119):

$$Y = (0.009487)(1.542) - \left(0.7537 \times 10^{-3}\right)[(1 + 23.62)(1.542) + 0.1457]$$
$$= -0.01409$$

The value of the stress for a temperature change of 315 K is

$$\sigma = (-0.01409)\left(16.2 \times 10^{-6}\right)\left(124 \times 10^{9}\right)(315)\left(\tfrac{1}{3}\right) = -2.973 \times 10^{6}\,\text{Pa}$$
$$= -2.973\,\text{MPa}$$

This value of stress is larger than the allowable stress; therefore, the maximum temperature change is governed by the stress in the bend, when stress concentration is considered:

$$\Delta T_0 = (315)\left(\frac{2.00}{2.973}\right) = 212\,\text{K} \quad \left(382^\circ\text{F}\right)$$

4.9 ELASTIC ENERGY IN TORSION

In general for a homogeneous, isotropic material, temperature changes do not directly result in torsion stresses. However, for some structures, thermal bending moments in one portion of the structure may produce a torsion or twisting in other portions. In this section, we consider the elastic energy method as it applies for torsion loading.

4.9.1 Shear Stress–Strain Relations

The shearing strain in torsion is defined as the change in the angle of a line drawn parallel to the axis of the unstrained material when a torque M_t is applied,

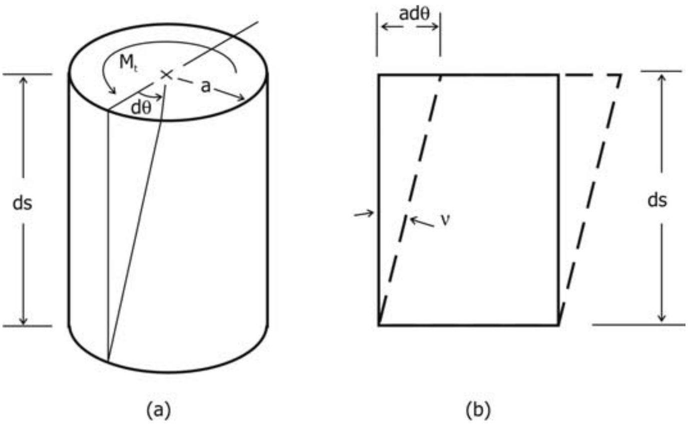

Figure 4-26. Strain for a rod subjected to torsional loading.

as shown in Figure 4-26. The shear strain γ at the outer fiber for the element is given by

$$\gamma \approx \tan\gamma = \frac{a\,d\theta}{ds} \tag{4-122}$$

The shear strain γ is a linear function of the radial coordinate r, so the strain at any point for a circular cross section is given by

$$\gamma = \frac{r\,d\theta}{ds} \tag{4-123}$$

For a linear elastic material, the shear stress and shear strain are related through the shear modulus G, as given in eq. (1-17):

$$\tau = G\gamma = Gr\frac{d\theta}{ds} \tag{4-124}$$

The torque M_t acting on a differential cross-sectional area dA located at a distance r from the center of the cross section may be written as

$$dM_t = r\tau dA = Gr^2 dA\frac{d\theta}{ds} \tag{4-125}$$

If we integrate the torque expression given by eq. (4-125) over the entire cross section, we obtain the expression for the total torque for an element with a circular cross section.

$$M_t = G\frac{d\theta}{ds}\int_A r^2 dA = GJ\frac{d\theta}{ds} \tag{4-126}$$

The quantity J is the *polar moment of inertia*, defined as

$$J \equiv \int_A r^2 dA = \int_A \left(x^2 + y^2\right)dA = I_y + I_x \tag{4-127}$$

If the element cross section is not circular, the determination of the torque–twist relationship is more complicated [Timoshenko and Goodier, 1970]. For example, for a bar with a rectangular cross section, $b \times h$, the effective polar moment of inertia may be written as follows, where $h \geq b$:

$$\frac{J_{\text{eff}}}{J} = \frac{4b^2 f(h/b)}{b^2 + h^2} \tag{4-128}$$

The polar moment of inertia for the rectangular cross section is

$$J = \frac{bh}{12}\left(b^2 + h^2\right) \tag{4-129}$$

The function of the aspect ratio for the cross section is

$$f(h/b) = 1 - \frac{192}{\pi^5}\frac{b}{h} \sum_{j=0,1,\ldots}^{\infty} \frac{1}{(2j+1)^5} \tanh\left[\frac{(2j+1)\,\pi h}{2b}\right] \tag{4-130}$$

The series in eq. (4-130) generally converges quite rapidly; in fact, only the first term is required for an accuracy of at least 0.5 percent.

$$f(h/b) \approx 1 - \frac{192}{\pi^5}\frac{b}{h} \tanh\left(\frac{\pi h}{2b}\right) \tag{4-131}$$

We may substitute the angle of twist per unit length, $d\theta/ds$, from eq. (4-126) into eq. (4-124) to obtain the relationship between the shear stress and torque for an element with a circular cross section:

$$\tau = \frac{M_t r}{J} = G\gamma \tag{4-132}$$

The maximum shear stress occurs at the outermost fiber, $r = \frac{1}{2}D$.

For an element with a rectangular cross section, the stress–torque relationship is as follows [Timoshenko and Goodier, 1970]:

$$\tau_{\max} = \frac{f(h/b)}{3 f_1(h/b)} \tag{4-133}$$

The function f_1 is given by

$$f_1(h/b) = 1 - \frac{8}{\pi^2} \sum_{j=0,1,\ldots}^{\infty} \frac{1}{(2j+1)^2 \cosh\left[\dfrac{(2j+1)\,\pi h}{2b}\right]} \tag{4-134}$$

For an accuracy of at least 0.5 percent, the series in eq. (4-134) may be represented by the first term:

$$f_1\,(h/b) \approx 1 - \frac{8}{\pi^2 \cosh(\pi h/2b)} \tag{4-135}$$

For a thin rectangular element (h/b very large), the second term approaches zero, and f_1 is approximately equal to unity.

4.9.2 Elastic Energy Relations for Torsion

The strain energy per unit volume for a material with a circular cross section in shear is

$$\frac{dU_s}{d\text{Vol}} = \int \tau d\gamma = G \int \gamma d\gamma = \frac{1}{2} G\gamma^2 = \frac{\tau^2}{2G} \tag{4-136}$$

We may use eq. (4-132) for the shear stress in the circular shaft.

$$\frac{dU_s}{d\text{Vol}} = \frac{M_t^2 r^2}{2GJ^2} \tag{4-137}$$

The total strain energy may be found by integrating eq. (4-137) over the volume of the element:

$$U_s = \iint \frac{M_t^2 r^2}{2GJ^2} dA\, ds = \int \frac{M_t^2}{2GJ^2} \left(\int r^2 dA \right) ds = \int \frac{M_t^2 ds}{2GJ} \tag{4-138}$$

The quantity s is the distance measured along the axis of the circular cross section.

There is no "torsional thermal expansion coefficient" for a homogeneous isotropic material, so the strain energy and the complementary strain energy in torsion are equal:

$$U_s = U_c = \int \frac{M_t^2 ds}{2GJ} \qquad \text{for torsion} \tag{4-139}$$

The linear deflection Δ at the point of application of an external load P may be found from eq. (4-9):

$$\Delta = \frac{\partial U_c}{\partial P} = \int \frac{M_t ds}{GJ} \left(\frac{\partial M_t}{\partial P} \right) \tag{4-140}$$

The derivative may be interpreted as the torque per unit applied load (*dummy torque*) resulting from the application of a unit load ($f = 1$) at the point at which we wish to determine the deflection:

$$m_t = \frac{\partial M_t}{\partial P} \tag{4-141}$$

If we make the substitution from eq. (4-141) into eq. (4-140), we obtain the following expression for the torsional contribution to the deflection:

$$\Delta = \int \frac{M_t m_t ds}{GJ} \tag{4-142}$$

In the solution for the deflection, we must include the effects of bending and extensional forces, in addition to the torsion contributions.

In a similar manner, we may show that the rotation at a particular point due to the application of a torque on an element may be found by application of a unit torque at the point:

$$\theta = \int \frac{M_t m_{t,r} ds}{GJ} \tag{4-143}$$

The quantity $m_{t,r}$ is the dummy torque at any point resulting from the application of a unit torque at the point at which we wish to determine the rotation.

4.9.3 Application

Let us consider the pipe bend shown in Figure 4-27. Suppose all three legs of the pipe bend are subject to the following temperature distribution:

$$\Delta T = \left(\frac{2y}{D_m}\right)\Delta T_0 \quad \text{or,} \quad \Delta T = \cos\phi \Delta T_0 \tag{4-144}$$

The quantity D_m is the mean diameter of the pipe, and the pipe has a wall thickness t. The angle ϕ is measured from the top of the pipe cross section, as shown in Figure 4-27.

Because the center run (length L) will bend under the action of the thermal moment, a torque M_{to} will be imposed on the other two legs of the pipe bend.

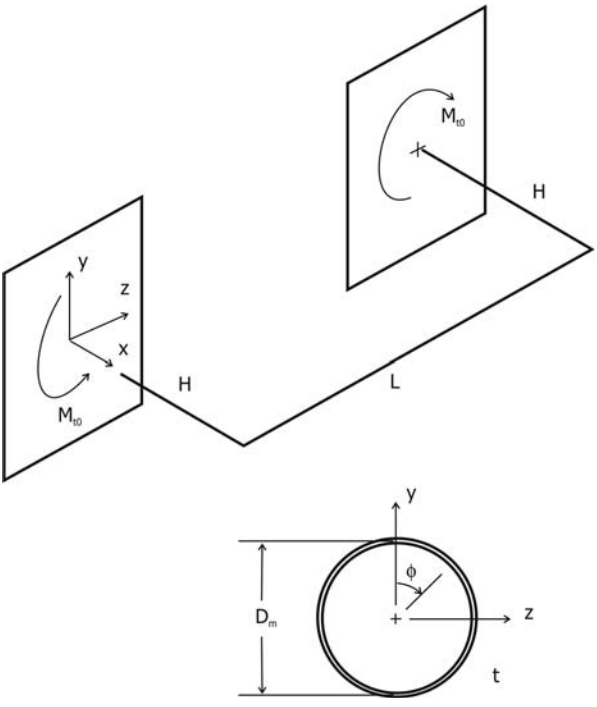

Figure 4-27. Pipe bend with out-of-plane thermal loading.

Thus, there will be an external balancing torque at the support points; however, there will be no support moments or support forces.

The thermal force N_T is zero, because the temperature distribution is symmetrical. The thermal moment may be evaluated as follows:

$$M_T = \alpha E \int \Delta T y dA = 2\alpha E \Delta T_0 \int_0^{\pi} \cos^2\phi \left(\tfrac{1}{2}D_m\right)\left(\tfrac{1}{2}D_m t d\phi\right) \tag{4-145}$$

If we carry out the integration, we obtain the following expression for the constant thermal moment present in all three portions of the pipe bend:

$$M_T = \tfrac{1}{4}\pi\alpha E \Delta T_0 D_m^2 t \tag{4-146}$$

If we substitute the expression for the area moment of inertia of a pipe from eq. (3-189), we obtain an alternate expression for the thermal moment:

$$M_T = \frac{2\alpha E I \Delta T_0}{D_m} \tag{4-147}$$

The actual torques, moments, and forces in the pipe bend may be listed as follows:

$$M_t = \begin{cases} M_{to} \\ 0 \\ M_{to} \end{cases} \quad M = \begin{cases} 0 \\ M_{to} \\ 0 \end{cases} \quad F = \begin{cases} 0 \\ 0 \\ 0 \end{cases} \quad ds = \begin{cases} dx & \text{for} \quad 0 \le x \le H \\ dz & \text{for} \quad 0 \le z \le L \\ -dx & \text{for} \quad H \ge x \ge 0 \end{cases} \tag{4-148}$$

If we apply a unit torque $m_t = 1$ at the support point in the same direction as the support torque M_{to}, we obtain the following dummy torques and moments:

$$m_{t,r} = \begin{cases} 1 \\ 0 \\ 1 \end{cases} \qquad m_r = \begin{cases} 0 & \text{for} \quad 0 \le x \le H \\ 1 & \text{for} \quad 0 \le z \le L \\ 0 & \text{for} \quad 0 \le x \le H \end{cases} \tag{4-149}$$

The rotation at the support point is zero, so we may evaluate the rotation from the unit load expressions:

$$\omega = \int \frac{M_t m_{t,r} ds}{GJ} + \int \frac{(M + M_T)\, m_r ds}{EI} \tag{4-150}$$

or

$$\omega = 0 = \frac{2M_{to}}{GJ}\int_0^H dx + \frac{M_{to} + M_T}{EI}\int_0^L dz = \frac{2M_{to}H}{GJ} + \frac{(M_{to} + M_T)\, L}{EI} \tag{4-151}$$

We may solve for the torque at the supports to obtain

$$M_{to} = -\frac{M_T L}{\dfrac{2HEI}{GJ} + L} = -\frac{M_T}{1 + \dfrac{2EIH}{GJL}} \tag{4-152}$$

For a circular pipe, the polar moment of inertia is equal to twice the area moment of inertia, $J = 2I$. For a homogeneous isotropic material, the shear

modulus is related to Young's modulus and Poisson's ratio by eq. (1-18). If we make these substitutions into eq. (4-152), we obtain

$$M_{to} = -\frac{M_T}{1 + 2\,(1 + \mu)\,(H/L)} = \frac{2\alpha EI\Delta T_0/D_m}{1 + 2\,(1 + \mu)(H/L)} \tag{4-153}$$

The stress in the support legs (length H) is shear stress only. The maximum shear stress in these elements is determined from eq. (4-132) applied at the outside diameter, $D_0 = D_m + t$:

$$\tau_{max} = \frac{M_{to}\,(D_m + t)}{2J} = -\frac{\frac{1}{2}\alpha E\Delta T_0\left(1 + \frac{t}{D_m}\right)}{1 + 2\,(1 + \mu)\,(H/L)} \tag{4-154}$$

The stress in the center leg (length L) is bending stress only. The stress distribution in this element may be determined from eq. (3-14):

$$\sigma = 0 + \frac{(M_{to} + M_T)\,y}{I} - \alpha E\Delta T_0\left(\frac{2y}{D_m}\right) \tag{4-155}$$

Using the relation between the support torque and the thermal moment given by eq. (4-152), we obtain

$$\sigma = \frac{M_T D_m\,(2y/D_m)}{2I}\left[1 - \frac{1}{1 + 2\,(1 + \mu)\,(H/L)}\right] - \alpha E\Delta T_0\left(\frac{2y}{D_m}\right) \tag{4-156}$$

We may simplify this expression by substituting for the thermal moment from eq. (4-147). The final expression for the bending stress at any point in the central element is

$$\sigma = -\frac{\alpha E\Delta T_0}{1 + 2\,(1 + \mu)\,(H/L)}\left(\frac{2y}{D_m}\right) \tag{4-157}$$

The maximum bending stress occurs at the outermost fiber, $y = \frac{1}{2}\,(D_m + t)$.

$$\sigma_{max} = -\frac{\alpha E\Delta T_0\left(1 + \frac{t}{D_m}\right)}{1 + 2\,(1 + \mu)\,(H/L)} \tag{4-158}$$

If the center leg of the pipe bend is very short ($L \to 0$), the thermal stress is zero, because the torque supplied by the center member is no longer present. On the other hand, if the legs at the support points are very short ($H \to 0$), the thermal stress distribution approaches that which would be obtained for a beam fixed at both ends and subjected to a constant thermal bending moment M_T:

$$\sigma_{max}\,(H = 0) = -\alpha E\Delta T_0\left(1 + \frac{t}{D_m}\right) \tag{4-159}$$

Example 4-11 A pipe loop, as shown in Figure 4-27, has a distance of 6 m (19.7 ft) between anchors. The pipe is 150 mm nominal (6 in. nominal) Schedule 40 pipe, with an outside diameter of 168.3 mm (6.625 in.) and a wall thickness of 7.1 mm (0.280 in.). The pipe material is 304 stainless steel, with $E = 193$ GPa

(28.0×10^6 psi), $\mu = 0.305$, and $\alpha = 16.0 \times 10^{-6}\,\mathrm{K}^{-1}$ ($8.89 \times 10^{-6}\,{}^\circ\mathrm{F}^{-1}$). The shear stress due to thermal effects must be limited to 40 MPa (5800 psi). The pipe is stress-free at a temperature of 300 K (26.8°C or 80°F). The pipe is subjected to a linear temperature change, with the upper portion of the pipe heated to 405 K (131.8°C or 269°F) and the lower portion is cooled to 195 K (−78.2°C or −109°F). Determine the dimension of the leg of the pipe bend attached at the anchor.

The mean pipe diameter is

$$D_m = D_0 - t = 168.3 - 7.1 = 161.2\ \mathrm{mm} \quad (6.345\ \mathrm{in.})$$

Let us determine the value of the dimensionless stress ratio:

$$\frac{2\tau_{\max}}{\alpha E \Delta T_0 \left(1 + \dfrac{t}{D_m}\right)} = \frac{(2)\left(40 \times 10^6\right)}{\left(16 \times 10^{-6}\right)\left(193 \times 10^9\right)(405 - 300)\left(1 + \dfrac{7.1}{161.2}\right)}$$
$$= 0.2363$$

The ratio of the lengths of the bend legs may be found from eq. (4-154):

$$\frac{H}{L} = \frac{1}{(2)(1 + 0.305)}\left(\frac{1}{0.2363} - 1\right) = 1.238$$

The dimension of the leg of the pipe bend attached at the anchor is as follows.

$$H = (6.00)(1.238) = 7.429\ \mathrm{m} \quad (24.37\ \mathrm{ft})$$

By comparing eqs. (4-154) and (4-158), we note that the magnitude of the maximum bending stress is twice the maximum shear stress.

4.10 HISTORICAL NOTE

Wooden bridges and roofs were the first constructions to utilize trusses [Timoshenko, 1983]. Although the Romans used the arch extensively, they also used wooden trusses in bridges such as the bridge over the Danube River built under the orders of Emperor Trajan. European architects had a renewed interest in wooden truss construction during the 18th century for bridges having a span length of 30 to 60 m (100 to 200 ft).

When railroad construction began in the 19th century, there was a need for bridges that would support heavier loads and would last longer than was possible with wooden bridges. Cast iron girders and arches were used in many of the early railroad bridges. The first all-metal trusses were built in the United States in 1840 by the engineer S. Whipple. He was also the first person to publish a book [Whipple, 1847] describing engineering analysis of trusses.

Several investigators used strain–energy relationships in analyzing trusses and other mechanical systems in the early portion of the 19th century. Carlo Alberto Castigliano (Figure 4-28) presented his famous theorem, along with some examples of the application of the strain energy method, in his thesis at the Turin

Figure 4-28. Carlo Alberto Castigliano.

Polytechnic Institute in 1873. Castigliano applied the strain energy method in the analysis of statically indeterminate trusses or trusses with redundant members.

The complementary energy method was developed as a generalization of Castigliano's theorem by F. Engesser around 1889 [Engesser, 1889]. The displacements in a system must be a linear function of the external loads for Castigliano's theorem to be valid. Engesser demonstrated that the complementary energy method applied for both linear and nonlinear systems.

Around 1864 James Clerk Maxwell developed the unit load method for determining the deflection of trusses. His presentation of the method, as applied to analysis of indeterminate trusses, did not include illustrations and was in a somewhat abstract form, so engineers involved in practical design did not use the method initially. Ten years later, Otto Mohr rediscovered Maxwell's theorem and published a paper describing several applications in structural analysis. Because Mohr developed the unit load method without knowledge of Maxwell's prior work, and because the method began to be used by engineers to solve practical design problems only after Mohr's publication, the unit load method is usually called the *Maxwell-Mohr method*.

PROBLEMS

4-1. The truss shown in Figure 4-29 is constructed of cast iron (Young's modulus, 96.5 GPa or 14.0×10^6 psi; thermal expansion coefficient, $10.8 \times 10^{-6}\ \mathrm{K}^{-1}$ or $6.0 \times 10^{-6}\ {}^\circ\mathrm{F}^{-1}$). All joints are pin-connected. The structure is stress-free at a temperature of 20°C (68°F). The dimensions and temperature of each member are given in the table. Determine (a) the vertical deflection of the structure at the point of application of the 10-kN (2248-$\mathrm{lb_f}$) load, and (b) the stress in each member.

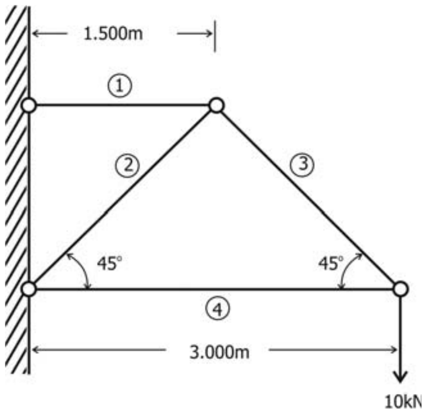

Figure 4-29. Figure for Problem 4-1.

Table for Problem 4-1

Member	Area, cm^2	Length, m	Temperature, °C
1	6.50	1.500	20°
2	4.50	2.121	30°
3	4.50	2.121	50°
4	3.00	3.000	60°

4-2. The Warren truss shown in Figure 4-30 is constructed of structural steel (Young's modulus, 200 GPa or 29×10^6 psi; thermal expansion coefficient, $12 \times 10^{-6}\ K^{-1}$ or $6.67 \times 10^{-6}\ {}^\circ F^{-1}$). All joints are pin-connected, and the length of each member is 5.00 m (18.04 ft). A force of 125 kN (28,100 lb_f or 14.05 tons) is applied at the center of the truss. The structure is stress-free at a temperature of 25°C (77°F). The dimensions and temperature of each member are given in the table. One support of the truss is fixed, and the other support is free to move. Determine: (a) the vertical deflection of the structure at the point of application of the 125 kN (2248 lb_f) load, and (b) the stress in each member.

Table for Problem 4-2

Member	Area, cm^2	Temperature, °C
1	15.0	45°
2	8.0	30°
3	15.0	45°
4	15.0	70°
5	15.0	45°
6	8.0	30°
7	15.0	45°

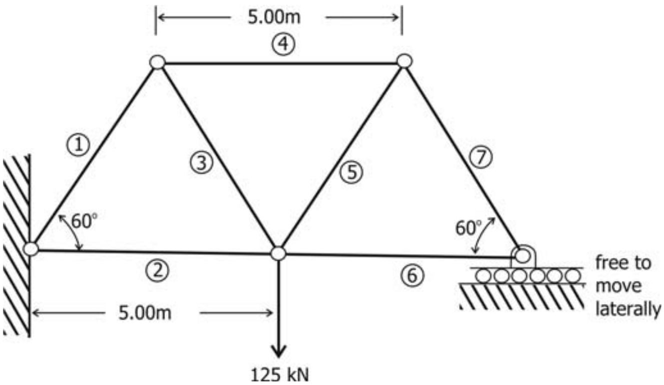

Figure 4-30. Figure for Problem 4-2.

4-3. The truss shown in Figure 4-31 is constructed of carbon steel (Young's modulus, 200 GPa or 29×10^6 psi; thermal expansion coefficient, 11×10^{-6} K^{-1} or 6.1×10^{-6} $°F^{-1}$). All joints are pin-connected, and both support points are fixed. The structure is stress-free at a temperature of 25°C (77°F). The dimensions and temperature of each member are given in the table. The applied load at the center of the structure is 300 kN (67,400 lb_f or 33.7 tons). Determine (a) the horizontal reaction force at the supports, and (b) the stress in each member.

Table for Problem 4-3

Member	Area, cm^2	Length, m	Temperature, °C
1	35	2.475	25°
2	25	3.500	35°
3	35	2.475	25°
4	50	3.500	50°
5	35	2.475	25°
6	25	3.500	35°
7	35	2.475	25°

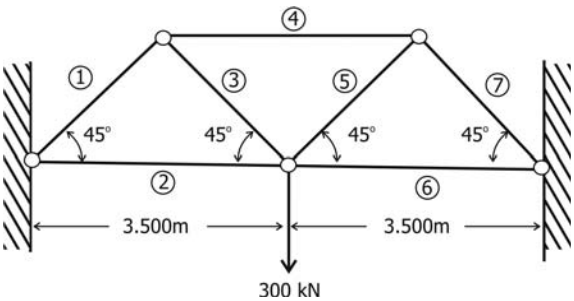

Figure 4-31. Figure for Problem 4-3.

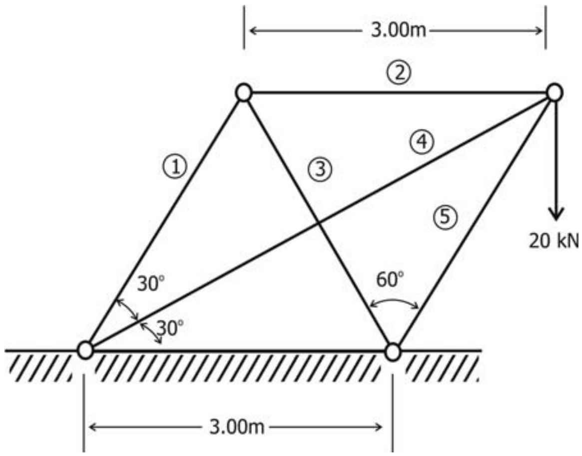

Figure 4-32. Figure for Problem 4-4.

4-4. The pin-connected truss shown in Figure 4-32 is constructed of 6061-T6 aluminum (Young's modulus, 69 GPa or 10×10^6 psi; thermal expansion coefficient, 23.4×10^{-6} K^{-1} or 13.0×10^{-6} $°F^{-1}$). Both support points are fixed. The structure is stress-free at a temperature of 25°C (77°F). The dimensions and temperature of each member are given in the table. The applied load is 20 kN (4496 lb_f or 2.248 tons). Determine the stress in each member.

Table for Problem 4-4

Member	Area, cm^2	Length, m	Temperature, °C
1	6.0	3.000	25°
2	6.0	3.000	35°
3	6.0	3.000	35°
4	12.0	5.196	30°
5	12.0	3.000	50°

4-5. In the pin-connected truss shown in Figure 4-33, member 6 is subjected to a temperature change from 20°C (68°F) to 55°C (131°F), while all other members remain at 20°C. The truss is constructed of an alloy steel, with Young's modulus of 200 GPa (29×10^6 psi) and thermal expansion coefficient of 12.0×10^{-6} K^{-1} (6.67×10^{-1} $°F^{-1}$). One support for the truss is fixed, and the other is free. The dimensions of the members are given in the table. Determine the stress in each member.

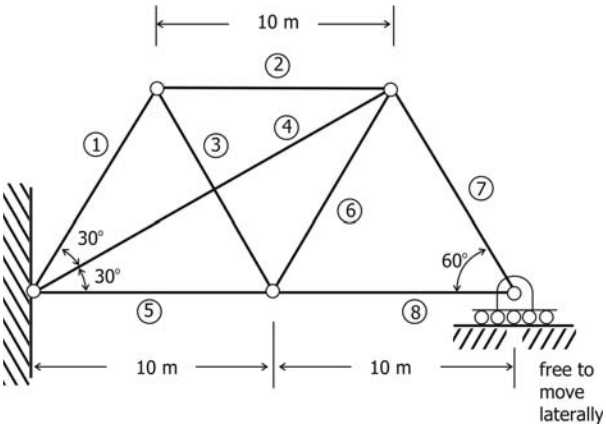

Figure 4-33. Figure for Problem 4-5.

Table for Problem 4-5

Member	Area, cm^2	Length, m
1	20	10
2	20	10
3	25	10
4	15	17.321
5	20	10
6	25	10
7	20	10
8	20	10

4-6. The pin-connected truss shown in Figure 4-34 is constructed of an alloy steel, with Young's modulus of 200 GPa (29 × 10^6 psi) and thermal expansion coefficient of 12.0 × 10^{-6} K^{-1} (6.67 × 10^{-1} $°F^{-1}$). Both supports for the truss are fixed. Member 4 is heated to 75°C (167°F) while all other members remain at 25°C (77°F). The dimensions of each member are given in the table. Determine the stress in each member.

Table for Problem 4-6

Member	Area, cm^2	Length, m
1	1.00	3.00
2	1.75	5.196
3	1.00	6.00
4	2.00	6.00
5	1.00	6.00
6	1.75	5.196
7	1.00	3.00

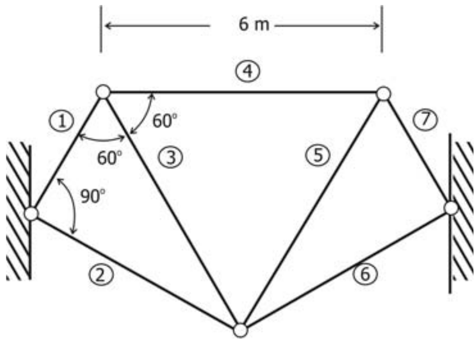

Figure 4-34. Figure for Problem 4-6.

4-7. Member 5 in the pin-connected truss shown in Figure 4-35 is heated from 25°C (77°F) to 105°C (221°F) while all other members remain at 25°C. Both supports for the truss are fixed, and the system is stress-free at 25°C. The truss material has a Young's modulus of 200 GPa (29.0×10^6 psi) and a thermal expansion coefficient of $16 \times 10^{-6}\ K^{-1}$ ($8.89 \times 10^{-6}\ {}^{\circ}F^{-1}$). The dimensions of each member are given in the table. Determine the stress in each member.

Table for Problem 4-7

Member	Area, cm^2	Length, m
1	3.00	5.00
2	3.00	5.00
3	3.00	5.00
4	3.00	5.00
5	3.00	7.071

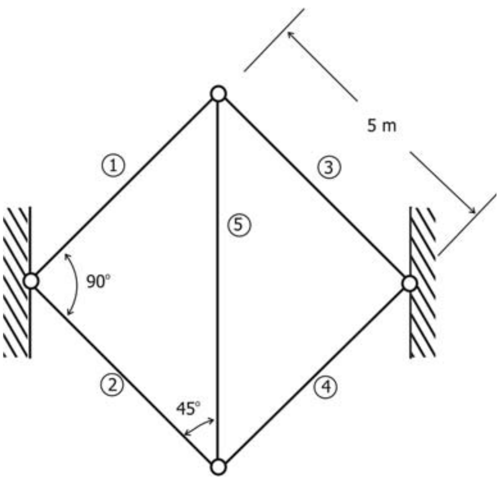

Figure 4-35. Figure for Problem 4-7.

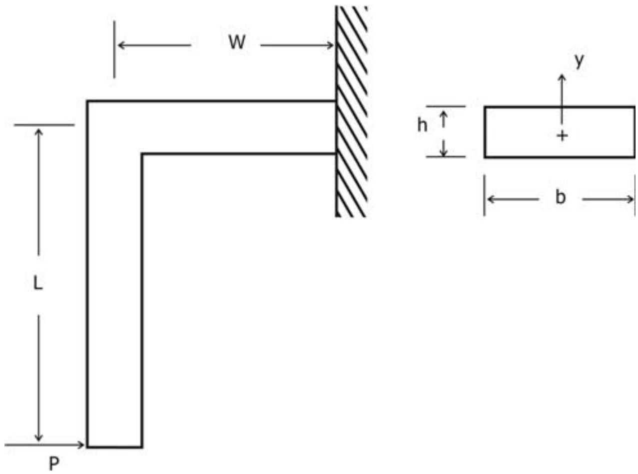

Figure 4-36. Figure for Problem 4-8.

4-8. The system shown in Figure 4-36 has a rectangular cross section of height h and width b. The leg having a length W has the following temperature distribution:

$$\Delta T = T - T_0 = (T_1 - T_0)\,(2y/h) = \Delta T_1\,(2y/h)$$

The leg having a length L has a constant temperature distribution, $T = T_0$, where T_0 is the temperature at which the system is stress-free. Determine the expression for the horizontal deflection at the point of application of the mechanical load P. Determine the numerical value of the horizontal deflection for the following condition. The cross section dimensions are $h = 60$ mm (2.36 in.) and $b = 360$ mm (14.17 in.). The dimensions of the system are $W = 2.00$ m (6.56 ft) and $L = 3$ m (9.84 ft). The applied load is $P = 3$ kN (674 lb_f), and the temperature change across the cross section is $\Delta T_1 = 40^\circ$C (72°F). The material is C1020 carbon steel.

4-9. The pipe ring shown in Figure 4-37 is constructed of a thin-walled tube having a mean diameter D_m and a wall thickness t. The ring material is

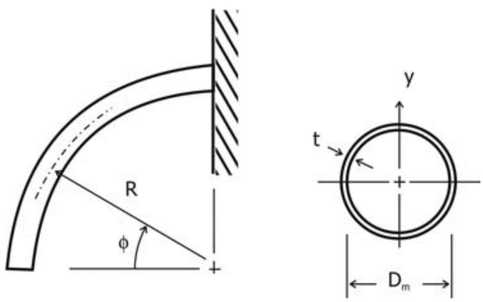

Figure 4-37. Figure for Problem 4-9.

304 stainless steel, with $E = 193$ GPa (28.0×10^6 psi) and $\alpha = 16.0 \times 10^{-6}\ \mathrm{K}^{-1}$ ($8.89 \times 10^{-6}\ {}^\circ\mathrm{F}^{-1}$). The mean bend radius of the ring is R. The ring is subjected to a temperature distribution as follows:

$$\Delta T = T - T_0 = (T_1 - T_0)\,(2y/D_m) = \Delta T_1\,(2y/D_m)$$

Determine the expression for the horizontal deflection of the free end of the ring. Determine the numerical value for the horizontal deflection of the free end of the ring and the maximum stress in the ring for the following values of the dimensions. $R = 1.500$ m (4.921 ft); $D_m = 150$ mm (5.906 in.); and $t = 5$ mm (0.197 in.). The temperature change parameter for the ring is $\Delta T_1 = 25^\circ$C (45°F). The applied mechanical load $P = 5$ kN (1124 $\mathrm{lb_f}$)

4-10. The frame shown in Figure 4-38 has a vertical length H and a horizontal length W. The cross section of the frame is rectangular, with dimensions b (width) and h (thickness). Only the horizontal portion is subjected to a temperature change given by

$$\Delta T = T - T_0 = (T_1 - T_0)\,(2y/h) = \Delta T_1\,(2y/h)$$

The vertical legs remain at a temperature T_0, the initial stress-free temperature. The support points for the frame are pin-connected, i.e., there is no bending moment at the supports. The material is C1020 steel, with $E =$ 205 GPa (29.7×10^6 psi) and $\alpha = 11.9 \times 10^{-1}\ \mathrm{K}^{-1}$ ($6.61 \times 10^{-6}\ {}^\circ\mathrm{F}^{-1}$). Determine the expressions for the horizontal reaction force and the

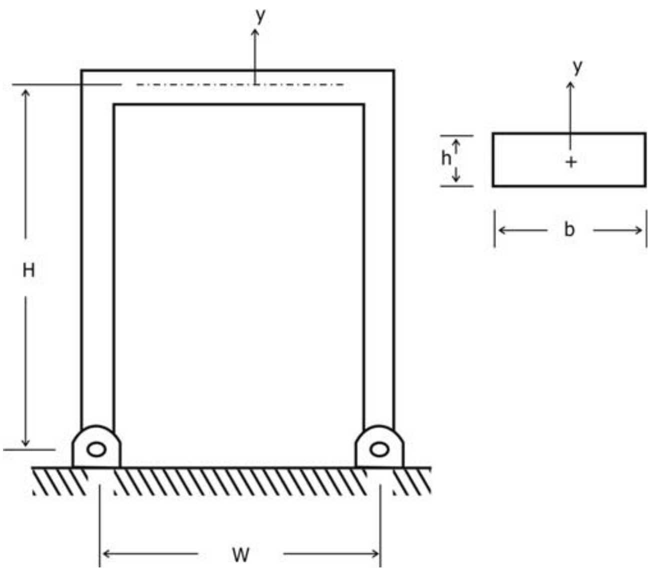

Figure 4-38. Figure for Problem 4-10.

maximum stress in the frame. Determine the numerical values for the horizontal reaction force and the maximum stress in the frame for the following dimensions: $H = 3.60$ m (11.81 ft), $W = 2.40$ m (7.87 ft), $b = 150$ mm (5.91 in.), and $h = 50$ mm (1.97 in.). The temperature change parameter for the frame is $\Delta T_1 = 50^\circ$C (90°F).

4-11. A pipe loop, shown in Figure 4-17, is constructed of C1020 steel, with Young's modulus, 200 GPa (29.0×10^6 psi), and thermal expansion coefficient, $12.0 \times 10^{-6}\ \text{K}^{-1}$ ($6.67 \times 10^{-6}\ {}^\circ\text{F}^{-1}$). The pipe is 80 mm nominal (3 in. nominal) SCH 40 pipe with an OD of 88.9 mm (3.500 in.) and a wall thickness of 5.5 mm (0.216 in.). The distance between the anchors is 6.00 m (19.69 ft) and the width of the loop is $W = 3.00$ m (9.84 ft). The pipe is subjected to a uniform temperature change of 125°C (225°F). Neglecting the effects of elbow flexibility and stress concentration in the elbows, determine the height of the pipe loop H to limit the thermal stress in the pipe to 10.0 MPa (1450 psi).

4-12. Rework Problem 4-11 by considering the elbow flexibility and stress concentration in the elbows. The mean radius of each of the four elbows in the pipe loop is 114.3 mm (4.500 in.).

4-13. A pipe bend shown in Figure 4-22 is constructed of 70/30 brass, with Young's modulus 110 GPa (16.0×10^6 psi) and thermal expansion coefficient $11 \times 10^{-6}\ \text{K}^{-1}$ ($6.1 \times 10^{-6}\ {}^\circ\text{F}^{-1}$). The pipe is 50 mm nominal (2 in. nominal) with an OD of 60.325 mm (2.375 in.) and a wall thickness of 3.975 mm (0.1565 in.). The mean radius of the bend is 482.6 mm (19.0 in.). The pipe bend is subjected to a uniform temperature change of 150°C (270°F). Neglecting the stress concentration in the bend due to ovaling of the pipe, determine the required length L of the straight run of pipe to limit the thermal stress in the pipe bend to 15 MPa (2175 psi).

4-14. Rework Problem 4-13 by including the stress concentration in the bend due to ovaling of the pipe.

4-15. The right-angle pipe bend shown in Figure 4-39 has legs of equal length L, and both ends of the bend are rigidly fixed. The entire pipe bend is subject to a temperature distribution as follows:

$$\Delta T = \Delta T_0 \left(\frac{2y}{D_m} \right)$$

The thermal moment corresponding to this temperature distribution is constant:

$$M_T = \frac{2\alpha E I \Delta T_0}{D_m}$$

The quantity D_m is the mean diameter of the pipe. (a) Determine the maximum bending stress σ_b and the torsional stress τ in the pipe.

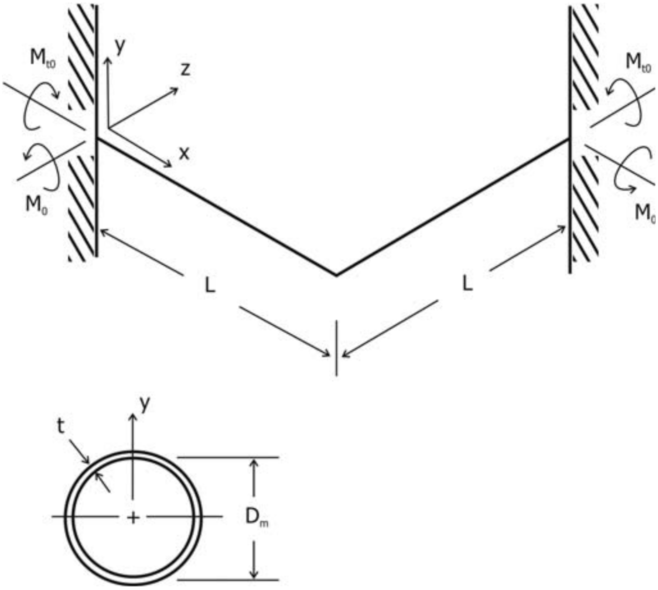

Figure 4-39. Figure for Problem 4-15.

(b) Show that the *von Mises stress*,

$$\sigma_{\mathrm{vM}} = \left(\sigma_b^2 + 3\tau^2\right)^{1/2}$$

is given by the following expression at the outermost fiber of the pipe:

$$\sigma_{\mathrm{vM}} = \frac{\sqrt{7}\alpha E \Delta T_0}{2\,(2+\mu)}\left(1 + \frac{t}{D_m}\right)$$

(c) If the pipe is constructed of 304 stainless steel with Young's modulus, 192 GPa (27.8×10^6 psi), thermal expansion coefficient, $16 \times 10^{-6}\ \mathrm{K}^{-1}$ ($8.89 \times 10^{-6}\ {}^\circ\mathrm{F}^{-1}$), and Poisson's ratio, 0.305, determine the von Mises stress at the outermost fiber of the pipe. The pipe is 200-mm nominal (8-in. nominal) with an OD of 219.1 mm (8.625 in.) and a wall thickness of 3.76 mm (0.148 in.). The length of the legs in the pipe bend is $L = 2.50$ m (8.20 ft). The temperature change parameter is $\Delta T_0 =$ 75 K (135°F). Do not consider stress concentration and ovaling effects for the pipe elbow (i.e., consider a "square" corner).

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5

BASIC EQUATIONS OF THERMOELASTICITY

5.1 INTRODUCTION

In the preceding chapters, we had used the so-called “strength of materials” approach in calculating thermal stresses and deformations. The restrictions on the strength of materials analysis are listed in Section 3.1. When plates are subjected to mechanical and thermal loading, the effect of lateral strain (Poisson’s ratio effects) may be significant, since neglecting these effects result in errors of 10 percent or more in the calculated stress. In addition, the analysis was essentially one dimensional. Even in the case of bending stresses, the variation of the displacement in the direction perpendicular to the neutral axis was taken as a linear variation in the strength of materials approach.

In this chapter, we will examine the fundamental equations for thermoelasticity. It is important to understand the restrictions on the basic equations. An extremely complicated analysis is meaningless if the basic equations used in the analysis are not applicable to the problem. Also, it is helpful in understanding the analysis if the physical meaning for the mathematical terms is understood.

The basic thermoelasticity equations include (a) definitions of strain in terms of the displacements, (b) strain compatibility relations, (c) stress equilibrium relations (force balances), (d) stress–strain relations (generalized Hooke’s law), and (e) the temperature field equations. Material property relations are also required in the solution of thermoelasticity problems, in general.

In the discussion of the stress–strain relationships and temperature field equations in this chapter, we will consider only materials that may be considered homogeneous (the same material throughout) and isotropic (properties the same

in all directions). For design purposes, many engineering materials may be treated as homogeneous and isotropic with little error. Materials such as wood, layered composites, and reinforced concrete are exceptions to this statement.

5.2 STRAIN RELATIONSHIPS

The relationships between the displacement of parts of a material and the resulting strain on the material depend only on the geometry or coordinate system used. The forces causing the strain do not enter into the strain expressions, and no material properties appear. Let us look first at the definitions of strain, and then consider the relationships involving the strain components.

5.2.1 Strain Components

Let us consider the element in Cartesian coordinates shown in Figure 5-1. Let us denote the components of the displacement of a particular point (x, y, z) as u (x-displacement), v (y-displacement), and w (z-displacement). In general, the displacement of a point a small distance from the point in question will be somewhat different.

The *extensional strain* may be defined as the change in length per unit original length. For example, the extensional strain in the x-direction is

$$\varepsilon_x \equiv \frac{\text{(change in length)}}{\text{(original length)}} = \frac{\left(u + \dfrac{\partial u}{\partial x}dx\right) - u}{dx} = \frac{\partial u}{\partial x} \tag{5-1}$$

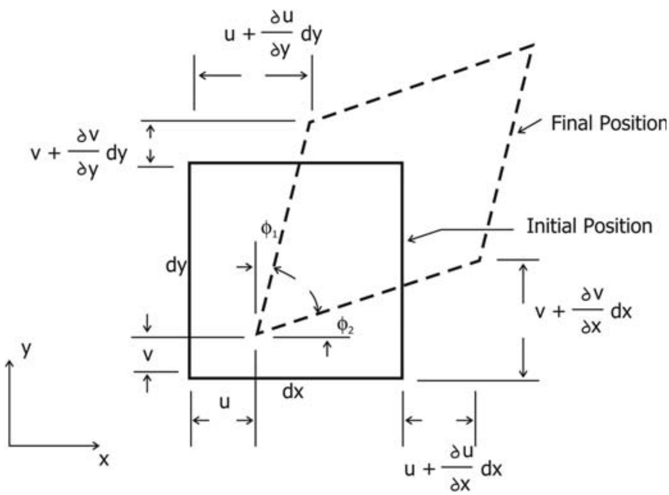


Figure 5-1. Differential element used to determine the strains in Cartesian coordinates.

Similarly, the extensional strains in the other two directions may be found from the following relations:

$$\varepsilon_y = \frac{\partial v}{\partial y} \tag{5-2}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \tag{5-3}$$

The *shear strains* or *distortion strains* are defined as the change in the angle between two intersecting planes of the element. Because the strains are small, the tangent of the angle is practically equal to the angle itself. It may be observed that tan $\phi = \phi$, with less than 1 percent error, when $\phi \leq 0.17\,\mathrm{rad} = 10^\circ$. The shearing strain between the edge of the element in the x-direction and the side of the element in the y-direction is found as follows:

$$\gamma_{xy} = \frac{\left(u + \dfrac{\partial u}{\partial y}dy\right) - u}{dy} + \frac{\left(v + \dfrac{\partial v}{\partial x}dx\right) - v}{dx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{5-4}$$

The other shear or distortion strains may be found in a similar manner:

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{5-5}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \tag{5-6}$$

It is apparent from the definition of the shear strains that the following is true:

$$\gamma_{xy} = \gamma_{yx} \quad \text{and} \quad \gamma_{xz} = \gamma_{zx} \quad \text{and} \quad \gamma_{yz} = \gamma_{zy} \tag{5-7}$$

Thus, there are only three *different* shear strains.

The strain quantities that we have defined are sometimes called *engineering strains*, because these strains correspond to those used in the strength-of-materials analysis. An alternate definition for the strains may be developed as follows.

Let us use an index notation for the coordinates and the displacement in the corresponding coordinate direction:

$$x_1 = x \quad \text{and} \quad x_2 = y \quad \text{and} \quad x_3 = z \tag{5-8}$$

$$u_1 = u \quad \text{and} \quad u_2 = v \quad \text{and} \quad u_3 = w \tag{5-9}$$

With this index notation, the *mathematical strains* may be defined:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (i, j = 1, 2, 3) \tag{5-10}$$

From this definition, we observe that the mathematical strains and engineering strains are related as follows:

$$\varepsilon_x = \varepsilon_{11} \quad \text{and} \quad \varepsilon_y = \varepsilon_{22} \quad \text{and} \quad \varepsilon_z = \varepsilon_{33} \tag{5-11a}$$

$$\gamma_{xy} = 2\varepsilon_{12} \quad \text{and} \quad \gamma_{xz} = 2\varepsilon_{13} \quad \text{and} \quad \gamma_{yz} = 2\varepsilon_{23} \tag{5-11b}$$

The strain components may also be expressed in cylindrical coordinates (r, θ, z). The direct or extensional strains in the r-, θ-, and z-directions are

$$\varepsilon_r = \frac{\partial u}{\partial r} \quad \text{and} \quad \varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial r} \quad \text{and} \quad \varepsilon_z = \frac{\partial w}{\partial z} \tag{5-12}$$

The displacements u, v, and w are displacements in the r-, θ-, and z-directions, respectively. The shearing strains in cylindrical coordinates are

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \quad \text{and} \quad \gamma_{\theta \mathrm{z}} = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta} \quad \text{and} \quad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \tag{5-13}$$

The strain components may also be expressed in spherical coordinates (r, ϕ, θ), where ϕ is the *azimuth angle* (measured from the z-axis), and θ is the *polar angle* (measured in the equatorial plane from the x-axis). Let us denote u_{r} as the displacement in the radial direction, u_ϕ as the displacement in the azimuth direction perpendicular to the radial line, and u_θ as the displacement in the polar direction. These terms are illustrated in Figure 5-2.

$$\varepsilon_r = \frac{\partial u_r}{\partial r}; \quad \varepsilon_\phi = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi}; \quad \varepsilon_\theta = \frac{u_r}{r} + \frac{u_\phi}{r \tan\phi} + \frac{1}{r \sin\phi}\frac{\partial u_\theta}{\partial \theta} \tag{5-14}$$

$$\gamma_{r\phi} = \frac{1}{r}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}; \quad \gamma_{r\theta} = \frac{1}{r\sin\phi}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r};$$

$$\gamma_{\phi\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \phi} - \frac{u_\theta}{r\tan\phi} + \frac{1}{r\sin\phi}\frac{\partial u_\phi}{\partial \theta} \tag{5-15}$$

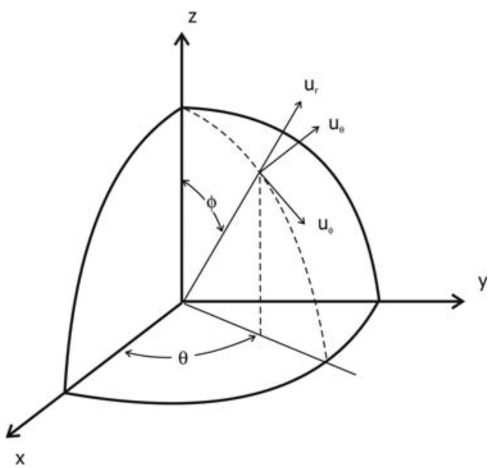


Figure 5-2. Illustration of the angles for spherical coordinates.

5.2.2 Strain Compatibility Relations

We observe that there are three components of displacement, and there are six component of strain, defined in terms of the displacements. If we solve a problem for the displacement components, the strain components may be calculated uniquely. However, if we solve for the strain components, the displacement components cannot be determined, in general, unless we have additional relationships between the strain components. These relationships are called the *strain compatibility conditions*.

Let us differentiate the direct or extensional strain expressions, eqs. (5-1) and (5-2):

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^3 u}{\partial y^2 \partial x} \qquad \text{and} \qquad \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^3 v}{\partial x^2 \partial y} \tag{5-16}$$

Next, let us take the mixed derivative of the shear strain, eq. (5-4):

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^3 u}{\partial y^2 \partial x} + \frac{\partial^3 v}{\partial x^2 \partial y} \tag{5-17}$$

By comparing eqs. (5-16) and (5-17), we obtain the first strain compatibility condition:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{5-18a}$$

Two other strain compatibility conditions may be obtained in a similar manner:

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \tag{5-18b}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{xz}}{\partial x \partial z} \tag{5-18c}$$

Three additional relationships may be found by taking the mixed partial derivatives:

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{yz}}{\partial x}\right) \tag{5-19a}$$

$$2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{xz}}{\partial y}\right) \tag{5-19b}$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) \tag{5-19c}$$

The six compatibility conditions (for small displacements) may be written in compact form in terms of the mathematical strains using the index notation:

$$\frac{\partial^2 \varepsilon_{ij}}{\partial x_m \partial x_n} + \frac{\partial^2 \varepsilon_{mn}}{\partial x_i \partial x_j} = \frac{\partial^2 \varepsilon_{im}}{\partial x_j \partial x_n} + \frac{\partial^2 \varepsilon_{jn}}{\partial x_i \partial x_m} \qquad (i, j, m, n = 1, 2, 3) \tag{5-20}$$

These equations are called the *Saint-Venant strain compatibility conditions*. The six components of strain must satisfy these equations if the solution for the components of displacement is to be physically realizable.

Note that for two-dimensional cases, in which the strain in the z-direction is constant (or zero) and the strain variation in the z-direction is zero, the only nonzero strain compatibility relation is eq. (5-18a).

5.2.3 Large Strains

If the displacements are not negligibly small, then the strain components must be defined as the ratio of the difference in the squares of the original and final element length, divided by twice the square of the original element length. For example, for the extensional strain in the x-direction, we have the following distances, where ds is the final length of the element and dx is the original element length:

$$(ds)^2 = \left(dx + \frac{\partial u}{\partial x}dx\right)^2 + \left(\frac{\partial v}{\partial x}dx\right)^2 + \left(\frac{\partial w}{\partial x}dx\right)^2 \tag{5-21}$$

$$(ds)^2 = \left[\left(1 + \frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right](dx)^2 \tag{5-22}$$

The strain in the x-direction for large strains is related to the displacement components as follows:

$$\varepsilon_x = \frac{(ds)^2 - (dx)^2}{2\,(dx)^2} = \frac{\partial u}{\partial x} + \tfrac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \tag{5-23}$$

For large deformations, the mathematical strains may be written in index notation:

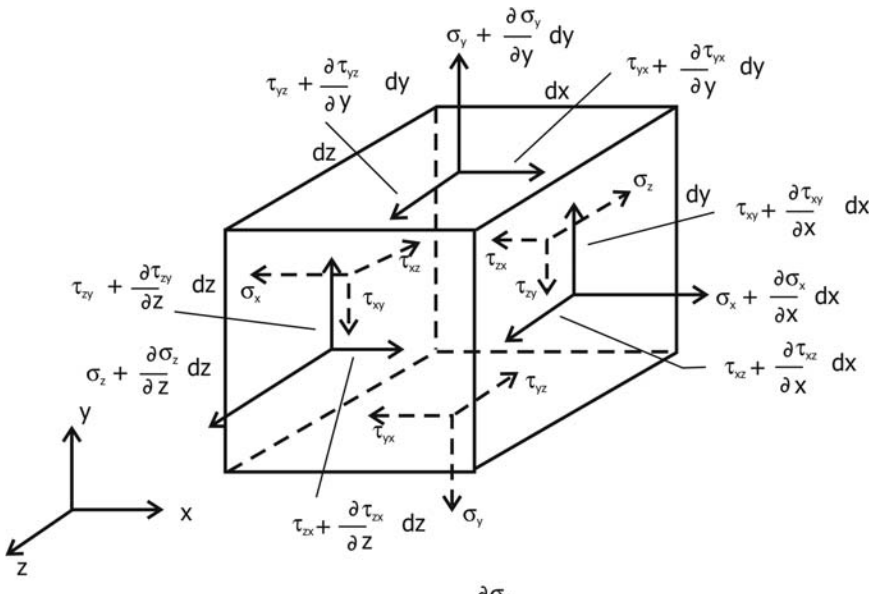
$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\sum_{m=1}^{3}\left(\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right) \tag{5-24}$$

In this text, we will consider only the situation in which the strains are small (on the order of 10^{-2} or smaller). The effects of large strain are discussed in advanced texts on theory of elasticity [Boresi and Chong, 1987]. This small-strain limitation is not severe for metals. For example, the strain at the yield point for some materials is as follows. For C1020 steel, $S_y/E = (324 \times 10^6)/(205 \times 10^9) = 0.0016 < 0.01$; for 304 stainless steel, $S_y/E = 0.0012 < 0.01$; and for 6061-T6 aluminum, $S_y/E = 0.0040 < 0.01$. The strains for these metals will be small up to the material yield point, where the material begins to permanently deform. Large strains can be achieved in materials such as molded rubber, however.

5.3 STRESS RELATIONSHIPS

The relationships between the stresses acting on a material involve application of Newton's second law of motion. For dynamic problems (vibration, wave transmission, etc.), the stress relationships involve the material density and the displacements. For static problems, the stress relationships do not involve material properties or displacements, except possibly when "body forces" are included.

The stresses and forces acting on a differential element in Cartesian coordinates are shown in Figure 5-3. In general, there are two types of forces that act on an element of material: (a) *surface forces*, which result from the direct or normal stresses in the x-, y-, and z-directions (σ_x, σ_y, and σ_z) and the shear stresses (τ_{xy}, τ_{xz}, and τ_{yz}), and (b) *body forces* (forces per unit volume, X, Y, and Z), which may arise from gravitational forces (weight of the material), magnetic forces, inertial forces (centrifugal effects), etc.



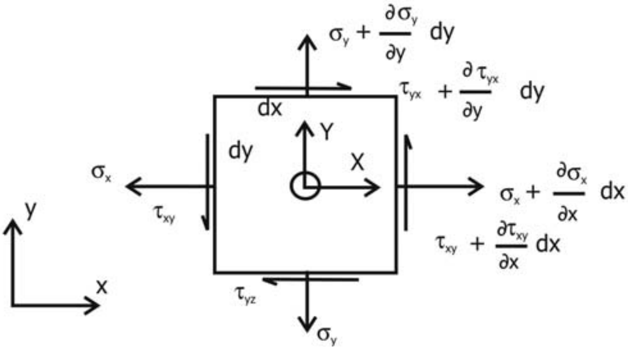


Figure 5-3. Stresses acting on a differential element in Cartesian coordinates.

The convention for the algebraic signs for the stress components is that the stress is considered to be *positive* if the direction of the stress component is in the *positive* coordinate direction. Therefore, if the stress σ_x is directed in the positive x-direction, it is considered to be a positive stress.

The significance of the subscripts for the shear stress components is that the first subscript (x, for example, in τ_{xy}) states that the stress acts on a plane perpendicular to the specified coordinate axis (perpendicular to the x-axis for τ_{xy}, for example). The second subscript (y, for example, in τ_{xy}) states that the shear stress acts in the specified coordinate direction (in the y-direction for τ_{xy}, for example). We may make moment balances around axes along the upper edges of the differential element, as shown in Figure 5-4, and show that the actual order of the subscripts is not critical, because the following results:

$$\tau_{xy} = \tau_{yx} \quad \text{and} \quad \tau_{xz} = \tau_{zx} \quad \text{and} \quad \tau_{yz} = \tau_{zy} \tag{5-25}$$

Because the specific relationship for the body forces depends on the type of body force, let us denote the body force per unit volume {*units:* N/m^3 or lb$_f$/in^3) by X, Y, and Z in the x-, y-, and z-directions, respectively. The specific form for the body force must be introduced for the particular problem being solved.

The velocity of the element in the x-direction, for example, may be expressed as the rate of change of the x-displacement with time t:

$$\overline{v}_x = \frac{\partial u}{\partial t} \tag{5-26}$$

Similarly, the acceleration of the element in the x-direction, for example, may be expressed as the rate of change of the x-velocity component with respect to time t:

$$\overline{a}_x = \frac{\partial \overline{v}_x}{\partial t} = \frac{\partial^2 u}{\partial t^2} \tag{5-27}$$

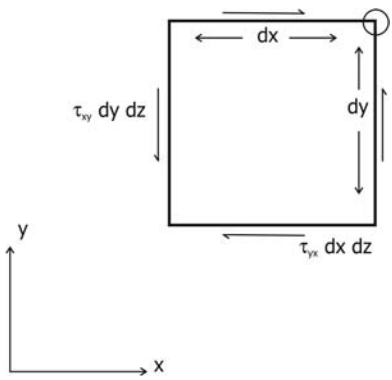


Figure 5-4. Illustration of the equality of τ_{xy} and τ_{yx}. Summing moments about the upper corner, $\tau_{xy}(dy\,dz)\,dy = \tau_{yz}(dx\,dz)\,dy$.

If we apply Newton's second law of motion, $\Sigma F = m\overline{a}$, in the x-direction, we obtain

$$\frac{\partial \sigma_x}{\partial x} dx\,(dy\,dz) + \frac{\partial \sigma_y}{\partial y} dy\,(dx\,dz) + \frac{\partial \sigma_z}{\partial z} dz\,(dx\,dy) + X\,(dx\,dy\,dz) = \rho(dx\,dy\,dz)\frac{\partial^2 u}{\partial t^2}$$

The stress relationship for the x-direction may be written as

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = \rho \frac{\partial^2 u}{\partial t^2} \tag{5-28a}$$

The other two stress relationships may be found by applying Newton's law for the y- and z-directions, respectively.

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y = \rho \frac{\partial^2 v}{\partial t^2} \tag{5-28b}$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = \rho \frac{\partial^2 w}{\partial t^2} \tag{5-28c}$$

For steady-state or static conditions, the right side of eqs. (5-28a), (5-28b), and (5-28c) is zero.

The stress relations may be written in compact form, using the index notation. Let us define the following quantities:

$$\sigma_x = \sigma_{11} \quad \text{and} \quad \sigma_y = \sigma_{22} \quad \text{and} \quad \sigma_z = \sigma_{33} \tag{5-29a}$$

$$\tau_{xy} = \sigma_{12} \quad \text{and} \quad \tau_{xz} = \sigma_{13} \quad \text{and} \quad \tau_{yz} = \sigma_{23} \tag{5-29b}$$

$$X = F_{b,1} \quad \text{and} \quad Y = F_{b,2} \quad \text{and} \quad Z = F_{b,3} \tag{5-29c}$$

If we use these definitions, along with those given in eqs. (5-8) and (5-9), we may write the stress relations in the following compact form:

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ji}}{\partial x_j} + F_{b,i} = \rho \frac{\partial^2 u_i}{\partial t^2} \qquad (i = 1, 2, 3) \tag{5-30}$$

The stress relations in cylindrical coordinates may be written as follows:

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + R = \rho \frac{\partial^2 u}{\partial t^2} \tag{5-31a}$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial z} + \frac{2\tau_{r\theta}}{r} + \Theta = \rho \frac{\partial^2 v}{\partial t^2} \tag{5-31b}$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + Z = \rho \frac{\partial^2 w}{\partial t^2} \tag{5-31c}$$

The quantities R, Θ, and Z are the body forces per unit volume in the r-, θ-, and z-directions, respectively.

The stress relations may be written in spherical coordinates as follows. The angle ϕ is the azimuth angle, measured from the z-axis, and the angle θ is the polar angle, measured from the x-axis in the x-y plane.

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\phi r}}{\partial \phi} + \frac{1}{r \sin\phi}\frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{2\sigma_r - \sigma_\phi - \sigma_\theta + \tau_{\phi r}}{r} + F_{b,r} = \rho \frac{\partial^2 u_r}{\partial t^2} \tag{5-32a}$$

$$\frac{\partial \tau_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\phi}{\partial \phi} + \frac{1}{r \sin\phi}\frac{\partial \tau_{\theta\phi}}{\partial \theta} + \frac{\left(\sigma_\phi - \sigma_\theta\right)\cot\phi + 3\tau_{r\phi}}{r} + F_{b,\phi} = \rho \frac{\partial^2 u_\phi}{\partial t^2} \tag{5-32b}$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\phi\theta}}{\partial \phi} + \frac{1}{r \sin\phi}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{3\tau_{r\theta} + 2\tau_{\phi\theta}\cot\phi}{r} + F_{b,\theta} = \rho \frac{\partial^2 u_\theta}{\partial t^2} \tag{5-32c}$$

5.4 STRESS–STRAIN RELATIONS

For a material that is homogeneous and isotropic, we may develop the relations between the stress in the material and the strains present. These constitutive equations are a generalization of the one-dimensional Hooke's law given by eq. (2-6). We note that the strain compatibility relations do not directly involve stress, and the stress equilibrium equations (for steady-state) do not directly involve strain or displacement. The stress–strain equations that we examine in this section relate the stress and strain through the material properties.

The mechanical strain in the x-direction caused by a stress σ_x is given by the expression, σ_x/E, where E is Young's modulus. In addition, if stresses σ_y and σ_z are applied in the y- and z-directions, respectively, strains, $-\mu\ \sigma_y/E$ and $-\mu\ \sigma_z/E$, will be induced in the x-direction as a result of the Poisson ratio effect. The quantity μ is Poisson's ratio, and the negative sign is used, because most materials contract in the direction perpendicular to an applied tensile stress. Finally, if the temperature T changes from the stress-free temperature T_0 by an amount $\Delta T = (T - T_0)$, a thermal stain, $\alpha\Delta T$, will result. If we add these four strain components, we obtain the stress–strain relationship for the x-direction.

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \mu\left(\sigma_y + \sigma_z\right)\right] + \alpha\Delta T \tag{5-33a}$$

By similar reasoning, we may obtain the stress–strain relations for the other two coordinate directions:

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \mu\left(\sigma_x + \sigma_z\right)\right] + \alpha\Delta T \tag{5-33b}$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \mu\left(\sigma_x + \sigma_y\right)\right] + \alpha\Delta T \tag{5-33c}$$

For a homogeneous isotropic material, shear stresses (τ_{xy}, for example) do not produce strains in the third coordinate direction (the z-direction, for example). This means that there is no Poisson's ratio effect for shear stresses in a homogeneous isotropic material. Similarly, temperature changes generally produce expansions or contractions, and do not produce distortion strains. The shear strains are related to the shear stresses by eq. (1-17):

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2(1+\mu)\,\tau_{xy}}{E} \tag{5-34a}$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G} = \frac{2(1+\mu)\,\tau_{xz}}{E} \tag{5-34b}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} = \frac{2(1+\mu)\,\tau_{yz}}{E} \tag{5-34c}$$

The shear modulus G is related to Young's modulus E through eq. (1-18) for a homogeneous isotropic material:

$$G = \frac{E}{2(1+\mu)} \tag{5-35}$$

The six stress–strain relations may be written in compact form using the index notation and the mathematical strain terms:

$$\varepsilon_{ij} = \frac{1}{E}\left[(1+\mu)\,\sigma_{ij} - \mu\,(\sigma_{11}+\sigma_{22}+\sigma_{33})\,\delta_{ij}\right] + \alpha\,\Delta T\,\delta_{ij} \qquad (i, j = 1, 2, 3) \tag{5-36}$$

The quantity δ_{ij} is called the *Kroneker delta* and is defined by

$$\delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases} \tag{5-37}$$

The quantity $s_m = (\sigma_{11}+\sigma_{22}+\sigma_{33})/3 = (\sigma_x+\sigma_y+\sigma_z)/3$ is called the *mean applied stress*. This term has the physical interpretation of a pressure or stress that is independent of direction.

We may solve the stress–strain relations in terms of stress as follows:

$$\sigma_x = (\lambda_L + 2G)\,\varepsilon_x + \lambda_L\left(\varepsilon_y + \varepsilon_z\right) - \beta_t B\Delta T \tag{5-38a}$$

$$\sigma_y = (\lambda_L + 2G)\,\varepsilon_y + \lambda_L\,(\varepsilon_x + \varepsilon_z) - \beta_t B\Delta T \tag{5-38b}$$

$$\sigma_z = (\lambda_L + 2G)\,\varepsilon_z + \lambda_L\left(\varepsilon_x + \varepsilon_y\right) - \beta_t B\Delta T \tag{5-38c}$$

$$\tau_{xy} = G\gamma_{xy} \quad \text{and} \quad \tau_{xz} = G\gamma_{xz} \quad \text{and} \quad \tau_{yz} = G\gamma_{yz} \tag{5-39}$$

The quantity λ_L is the *Lamè elastic constant* given by eq. (1-22):

$$\lambda_L = \frac{\mu E}{(1+\mu)(1-2\mu)} \tag{5-40}$$

The quantity β_t is the volumetric thermal expansion coefficient given by eq. (1-6):

$$\beta_t = 3\alpha \tag{5-41}$$

The quantity B is the volume modulus of elasticity given by eq. (1-20):

$$B = \frac{E}{3\,(1-2\mu)} \tag{5-42}$$

We note that the coefficient in the first term in eq. (5-38) can be written in the following form:

$$\lambda_L + 2G = \frac{(1-\mu)E}{(1+\mu)\,(1-2\mu)} = \frac{3\,(1-\mu)\,B}{1+\mu} \tag{5-43}$$

The eqs. (5-38) and (5-39) may be written in compact form using the mathematical strain and the index notation:

$$\sigma_{ij} = 2G\varepsilon_{ij} + [\lambda_L\,(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) - \beta_t B\Delta T]\,\delta_{ij} \qquad (i,j=1,2,3) \tag{5-44}$$

The quantity $e_v = (\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) = (\varepsilon_x+\varepsilon_y+\varepsilon_z)$ is called the *volumetric dilatation*. This quantity has the physical interpretation of the change in volume of the element per unit volume.

If we add eqs. (5-33a), (5-33b), and (5-33c), we obtain

$$\varepsilon_x+\varepsilon_y+\varepsilon_z = \frac{1-2\mu}{E}\left(\sigma_x+\sigma_y+\sigma_z\right) + 3\alpha\Delta T \tag{5-45}$$

The mean stress and volumetric dilatation may be related as follows:

$$e_v = \frac{s_m}{B} + \beta_t\Delta T \tag{5-46}$$

5.5 TEMPERATURE FIELD EQUATION

To calculate the thermal stress distribution within a material, we must first determine the temperature distribution in the material. For steady-state conditions (temperature and strain are not changing with time), the determination of the temperature distribution may be carried out independent of the strain distribution. On the other hand, for fairly rapid temperature and strain changes, the governing equation for the temperature distribution involves strain also. In other words, the temperature and strain equations are *coupled* [Boley and Weiner, 1960].

Let us consider the element shown in Figure 5-5. The governing rate equation for conduction heat transfer rate is the *Fourier rate equation*:

$$\dot{Q} = -k_t A\frac{\partial T}{\partial x} \tag{5-47}$$

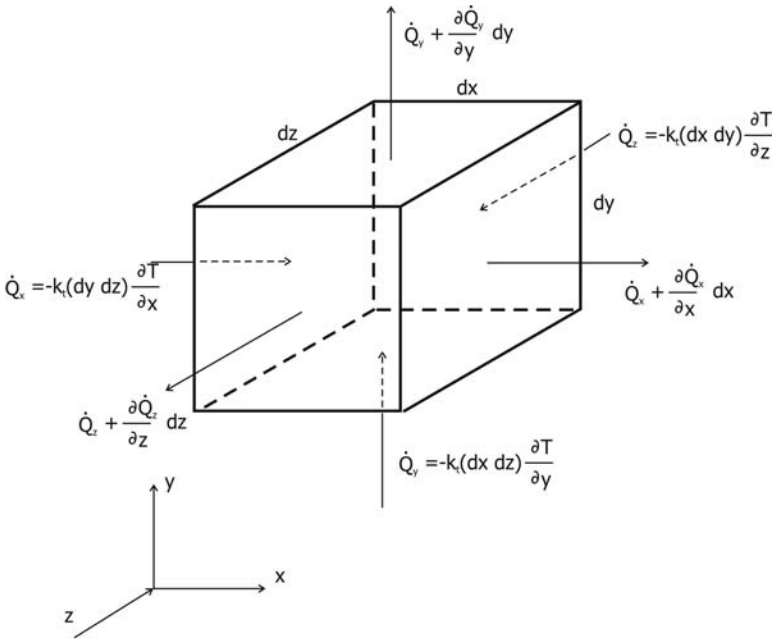


Figure 5-5. Heat conduction for a differential element in Cartesian coordinates.

The quantity k_t is the thermal conductivity for the material, A is the area through which heat is conducted, and $\partial T/\partial x$ is the temperature gradient in direction in which heat is conducted. The thermal conductivity may be a function of temperature, as is the case for heat transfer in cryogenic systems. The negative sign appears in the rate equation because heat is conducted in the direction of decreasing temperature with distance.

The conduction heat transfer rate *into* the element in the x-direction may be written as

$$\dot{Q}_{x,\text{in}} = -k_t\,(dy\,dz)\,\frac{\partial T}{\partial x} \tag{5-48}$$

The conduction heat transfer rate *from* the element in the x-direction may be written as

$$\dot{Q}_{x,\text{out}} = \dot{Q}_{x,\text{in}} + \frac{\partial \dot{Q}_x}{\partial x}dx = \dot{Q}_{x,\text{in}} - \frac{\partial}{\partial x}\left(k_t\frac{\partial T}{\partial x}\right) dx\ dy\ dz \tag{5-49}$$

The *net* heat conduction rate in the x-direction is

$$\dot{Q}_{x,\text{net}} = \dot{Q}_{x,\text{in}} - \dot{Q}_{x,\text{out}} = +\frac{\partial}{\partial x}\left(k_t\frac{\partial T}{\partial x}\right) dx\ dy\ dz \tag{5-50}$$

Similar expressions may be written for the net heat conduction rate in the other two directions. The net heat conducted into the element may be written as

the sum of these three contributions:

$$\dot{Q}_{\text{net}} = \left[\frac{\partial}{\partial x}\left(k_t\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_t\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_t\frac{\partial T}{\partial z}\right)\right] dx\ dy\ dz \tag{5-51}$$

The mechanical work transfer rate or volume-change work may be written as

$$\dot{W}_{\text{vc}} = -\left(\sigma_x\frac{\partial \varepsilon_x}{\partial t} + \sigma_y\frac{\partial \varepsilon_y}{\partial t} + \sigma_z\frac{\partial \varepsilon_z}{\partial t}\right) dx\ dy\ dz \tag{5-52}$$

The negative sign is included because work done on the element is considered negative work in the thermodynamic sense. The other work rates, such as electrical work (I^2R or *Joule heating*), magnetic work, etc. or the "heat generation" or "heat source" terms may be represented by the energy dissipation per unit volume q_g:

$$\dot{W}_{\text{other}} = -\dot{q}_g\, dx\, dy\, dz \tag{5-53}$$

The total work transfer rate is given by the sum of the terms from eqs. (5-52) and (5-53).

The rate of change of energy stored in the element involves both thermal energy and strain energy [Noda et al., 2003]:

$$\begin{aligned}\frac{\partial U}{\partial t} &= \rho c_\varepsilon \frac{\partial T}{\partial t} dx\, dy\, dz + \left(\sigma_x\frac{\partial \varepsilon}{\partial t} + \sigma_y\frac{\partial \varepsilon_y}{\partial t} + \sigma_z\frac{\partial \varepsilon_z}{\partial t}\right) dx\, dy\, dz \\ &\quad + \beta_t T B\left(\frac{\partial \varepsilon_x}{\partial t} + \frac{\partial \varepsilon_y}{\partial t} + \frac{\partial \varepsilon_z}{\partial t}\right) dx\, dy\, dz\end{aligned} \tag{5-54}$$

The quantity c_ε is the specific heat at constant strain, or the specific heat at constant volume, for all practical purposes. We note that the coefficient in the last term may be written as follows:

$$\beta_t B = \frac{\alpha E}{1 - 2\mu} \tag{5-55}$$

In addition, the temperature in the last term of eq. (5-54) is the *absolute temperature*, which is generally much larger than the temperature change. We may then use the stress-free temperature (absolute temperature) T_0 as an approximation to the temperature T.

The final temperature field equation may be obtained by making these substitutions into the conservation of energy principle or first law of thermodynamics:

$$\dot{Q}_{\text{net}} - \dot{W}_{\text{net}} = \frac{\partial U}{\partial t} \tag{5-56}$$

The result is

$$\begin{aligned}\frac{\partial}{\partial x}\left(k_t\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_t\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_t\frac{\partial T}{\partial z}\right) + \dot{q}_g &= \rho c_\varepsilon \frac{\partial T}{\partial t} \\ + \frac{\alpha E T_0}{1 - 2\mu}\left(\frac{\partial \varepsilon_x}{\partial t} + \frac{\partial \varepsilon_y}{\partial t} + \frac{\partial \varepsilon_z}{\partial t}\right)&\end{aligned} \tag{5-57}$$

The second term on the right side of eq. (5-57) represents a coupling of the strain and temperature relations. The coefficient of the term may be written in terms of the *Grüneisen constant* γ_G, which is defined by the following [Yates, 1972]:

$$\gamma_G = \frac{\beta_t B}{\rho c_v} = \frac{\alpha E}{\rho c_v (1 - 2\mu)} \tag{5-58}$$

Values for the Grüneisen constant for various materials are given in Table 5-1. The right side of eq. (5-57) may be written in the following form, where we have introduced the volumetric dilatation $e_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$:

$$\rho c_v \frac{\partial T}{\partial t} + \rho c_v \gamma_G T_0 \frac{\partial e_v}{\partial t} \tag{5-59}$$

We note from Table 5-1 that the Grüneisen constant γ_G is on the order of unity or between about 1 and 2.4 for most metals (Invar is an exception). From this observation, we conclude that coupling effects will be negligible when the strain rate is much smaller than the rate of temperature change:

$$\frac{\partial e_v}{\partial t} \ll \frac{1}{T_0} \frac{\partial T}{\partial t} \tag{5-60}$$

If the strain were totally thermal strain, the volumetric dilatation would be

$$e_v = \beta_t (T - T_0) \quad \text{or} \quad \frac{\partial e_v}{\partial t} = \beta_t \frac{\partial T}{\partial t} \tag{5-61}$$

Thus, for coupling to be negligible in this case, we have the following condition:

$$\beta_t T_0 = 3\alpha T_0 \ll 1 \tag{5-62}$$

For most metals (except Invar), the thermal expansion coefficient α is in the range between $10 \times 10^{-6}\ \mathrm{K}^{-1}$ and $20 \times 10^{-6}\ \mathrm{K}^{-1}$, as noted from Appendix B. If the stress-free temperature is near ambient temperature (300 K or 540°R), we find the following numerical value:

$$3\alpha T_0 = (3)\,(10 \text{ to } 20)\left(10^{-6}\right)(300) = 0.009 \text{ to } 0.018$$

TABLE 5-1. Values of the Grüneisen Constant γ_G for Various Materials

Material	γ_G	Material	γ_G
2024-T3 aluminum	2.12	nickel	1.88
brass	1.18	nylon	0.66
bronz	1.53	Pyrex glass	0.21
copper (pure)	1.96	silver	2.40
copper/10% Ni	1.95	C1020 carbon steel	1.54
gold	2.40	9% Ni steel	1.63
Invar	0.27	304 stainless steel	1.67
lead	2.73	titanium	1.30

The coupling effect in this case is on the order of 10^{-2} or 1 percent and can be neglected for many engineering applications.

If the thermal conductivity may be treated as constant, then the temperature field equation, eq. (5-57), reduces to the following form for Cartesian coordinates:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}_g}{k_t} = \frac{1}{\kappa}\frac{\partial T}{\partial t} + \frac{\gamma_G T_0}{\kappa}\frac{\partial e_v}{\partial t} \tag{5-63}$$

For steady-state conditions, the right side of eq. (5-63) is equal to zero. The temperature field equation in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}_g}{k_t} = \frac{1}{\kappa}\frac{\partial T}{\partial t} + \frac{\gamma_G T_0}{\kappa}\frac{\partial e_v}{\partial t} \tag{5-64}$$

where $e_v = \varepsilon_r + \varepsilon_\theta + \varepsilon_z$

For spherical coordinates, the temperature field equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 T}{\partial \theta^2} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial T}{\partial \phi}\right) + \frac{\dot{q}_g}{k_t} = \frac{1}{\kappa}\frac{\partial T}{\partial t} + \frac{\gamma_G T_0}{\kappa}\frac{\partial e_v}{\partial t} \tag{5-65}$$

where $e_v = \varepsilon_r + \varepsilon_\theta + \varepsilon_\phi$

The temperature field equation may be written in a more compact general form by introducing the *Laplacian operator*. For Cartesian coordinates, the Laplacian operator ∇^2 may be written as follows:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{5-66}$$

In terms of the Laplacian operator, the temperature field equation may be written

$$\nabla^2 T = \frac{1}{\kappa}\frac{\partial T}{\partial t} + \frac{\gamma_G T_0}{\kappa}\frac{\partial e_v}{\partial t} \tag{5-67}$$

5.6 REDUCTION OF THE GOVERNING EQUATIONS

If the coupling effect is negligible in the temperature field equation, eq. (5-67), the equation may be solved to obtain the temperature distribution independent of the stress problem. This result means that the temperature distribution becomes a *known* quantity in the stress–strain relations.

The number of basic equations is sufficient to solve for the stresses or strains. We have the following relationships:

6 strain–displacement equations, eqs. (5-1)–(5-6)

3 stress equilibrium or Newton's second law equations, eqs. (5-28a, b, c)

6 stress–strain equations, eqs. (5-33a)–(5-34c)

Thus, we have a total of 15 governing equations relating the 15 unknown quantities:

6 strain components, ε_x, ε_y, ε_z, γ_{xy}, γ_{xz}, and γ_{yz}
3 displacement components, u, v, and w
6 stress components, σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, and τ_{yz}

Although the thermal stress problem is defined, the actual solution for the three-dimensional thermal stresses is far from simple, in general. The remaining chapters of this text are devoted to demonstrating techniques for solution of two-dimensional thermal stress problems arising in a variety of practical situations.

In some applications, the boundary conditions are such that the stresses are known at the boundaries. For these cases, we may reduce the set of equations to six relationships involving stress and temperature change by eliminating the strains in the strain compatibility relations, eqs. (5-18a)–(5-19c), by using the stress equilibrium relations, eqs. (5-28a, b, c), and the stress–strain relations, (5-33a)–(5-34c).

The six stress compatibility relations, or the *Beltrami-Michell thermoelasticity equations*, for Cartesian coordinates and for steady-state conditions are

$$(1+\mu)\nabla^2\sigma_x + \frac{\partial^2}{\partial x^2}\left(\sigma_x+\sigma_y+\sigma_z\right) + \alpha E\left[\frac{1+\mu}{1-\mu}\nabla^2(\Delta T) + \frac{\partial^2(\Delta T)}{\partial x^2}\right] + (1+\mu)\left[\frac{\mu}{1-\mu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right) + 2\frac{\partial X}{\partial x}\right] = 0 \tag{5-68a}$$

$$(1+\mu)\nabla^2\sigma_y + \frac{\partial^2}{\partial y^2}\left(\sigma_x+\sigma_y+\sigma_z\right) + \alpha E\left[\frac{1+\mu}{1-\mu}\nabla^2(\Delta T) + \frac{\partial^2(\Delta T)}{\partial y^2}\right] + (1+\mu)\left[\frac{\mu}{1-\mu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right) + 2\frac{\partial Y}{\partial y}\right] = 0 \tag{5-68b}$$

$$(1+\mu)\nabla^2\sigma_z + \frac{\partial^2}{\partial z^2}\left(\sigma_x+\sigma_y+\sigma_z\right) + \alpha E\left[\frac{1+\mu}{1-\mu}\nabla^2(\Delta T) + \frac{\partial^2(\Delta T)}{\partial z^2}\right] + (1+\mu)\left[\frac{\mu}{1-\mu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right) + 2\frac{\partial Z}{\partial z}\right] = 0 \tag{5-68c}$$

The relationships involving the shear stresses are

$$(1+\mu)\nabla^2\tau_{xy} + \frac{\partial^2}{\partial x\partial y}\left(\sigma_x+\sigma_y+\sigma_z\right) + \alpha E\frac{\partial^2(\Delta T)}{\partial x\partial y} + (1+\mu)\left(\frac{\partial X}{\partial y}+\frac{\partial Y}{\partial x}\right) = 0 \tag{5-69a}$$

$$(1+\mu)\nabla^2\tau_{yz} + \frac{\partial^2}{\partial y\partial z}\left(\sigma_x+\sigma_y+\sigma_z\right) + \alpha E\frac{\partial^2(\Delta T)}{\partial y\partial z} + (1+\mu)\left(\frac{\partial Y}{\partial z}+\frac{\partial Z}{\partial y}\right) = 0 \tag{5-69b}$$

$$(1+\mu)\,\nabla^2\tau_{zx}+\frac{\partial^2}{\partial z\partial x}\left(\sigma_x+\sigma_y+\sigma_z\right)+\alpha E\frac{\partial^2(\Delta T)}{\partial z\partial x}+(1+\mu)\left(\frac{\partial Z}{\partial x}+\frac{\partial X}{\partial z}\right)=0 \tag{5-69c}$$

The six stress compatibility relations may be written in compact form using index notation:

$$(1+\mu)\,\nabla^2\sigma_{ij}+3\frac{\partial^2 s_m}{\partial x_i\partial x_j}+\alpha E\left[\frac{1+\mu}{1-\mu}\nabla^2(\Delta T)\,\delta_{ij}+\frac{\partial^2(\Delta T)}{\partial x_i x_j}\right]$$
$$+(1+\mu)\left(\frac{\mu}{1-\mu}\sum_k\frac{\partial F_{b,k}}{\partial x_k}\delta_{ij}+\frac{\partial F_{b,i}}{\partial x_j}+\frac{\partial F_{b,j}}{\partial x_i}\right)=0 \tag{5-70}$$

The quantity $s_m=\frac{1}{3}\sum\limits_k\sigma_{kk}$, δ_{ij} is the Kronecker delta, and $\Delta T=T-T_0$.

In other applications, the boundary conditions are such that the displacements or strains are known at the boundaries. For these cases, we may reduce the set of equations to three relationships involving displacement and temperature change by eliminating the stresses in the stress equilibrium relations, eqs. (5-28a, b, c), by using the strain–displacement relations, eqs. (5-1) through (5-6) and the stress–strain relations, (5-33a) through (5-34c).

The three displacement relations, or the Navier thermoelasticity equations, for Cartesian coordinates and for steady-state conditions are

$$G\nabla^2 u+(\lambda_L+G)\,\frac{\partial e_v}{\partial x}-\beta_t B\frac{\partial(\Delta T)}{\partial x}+X=0 \tag{5-71a}$$

$$G\nabla^2 v+(\lambda_L+G)\,\frac{\partial e_v}{\partial y}-\beta_t B\frac{\partial(\Delta T)}{\partial y}+Y=0 \tag{5-71b}$$

$$G\nabla^2 w+(\lambda_L+G)\,\frac{\partial e_v}{\partial z}-\beta_t B\frac{\partial(\Delta T)}{\partial z}+Z=0 \tag{5-71c}$$

The quantity G is the shear modulus, λ_L is the Lamé constant, β_t is the volumetric coefficient of thermal expansion, B is the bulk modulus, and e_v is the dilatation:

$$e_v=\varepsilon_x+\varepsilon_y+\varepsilon_z=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} \tag{5-72}$$

These three relations may be written in compact form using the index notation:

$$G\nabla^2 u_i+(\lambda_L+G)\sum_j\frac{\partial^2 u_j}{\partial x_i\partial x_j}-\beta_t B\frac{\partial(\Delta T)}{\partial x_i}+F_{b,i}=0 \tag{5-73}$$

The Navier equations may be written in cylindrical coordinates:

$$G\left(\nabla^2 u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right)+(\lambda_L+G)\,\frac{\partial e_v}{\partial r}-\beta_t B\frac{\partial(\Delta T)}{\partial r}+F_r=0 \tag{5-74a}$$

$$G\left(\nabla^2 u_\theta + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) + (\lambda_L + G)\frac{1}{r}\frac{\partial e_v}{\partial \theta} - \beta_t B \frac{1}{r}\frac{\partial (\Delta T)}{\partial \theta} + F_\theta = 0 \qquad \text{(5-74b)}$$

$$G\nabla^2 u_z + (\lambda_L + G)\frac{\partial e_v}{\partial z} - \beta_t B \frac{\partial (\Delta T)}{\partial z} + F_z = 0 \qquad \text{(5-74c)}$$

The volumetric dilatation e_v in cylindrical coordinates is

$$e_v = \varepsilon_r + \varepsilon_\theta + \varepsilon_z = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \qquad \text{(5-75)}$$

The Laplacian operator in cylindrical coordinates may be written in the following form:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \qquad \text{(5-76)}$$

5.7 HISTORICAL NOTE

The trio Navier, Beltrami and Michell made significant contributions in establishing the foundation of thermal stress analysis; however, they made important contributions in other areas of technology as well. Claud Louis Marie Henri Navier is given credit for developing the first satisfactory theory of plate bending in a paper presented in 1820 and published in 1823 [Timoshenko, 1983]. His expression for the lateral displacement of a plate w did not include thermal effects, however:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D} \qquad \text{(5-77)}$$

The quantity p is the lateral load or pressure on the plate, and D is the flexural rigidity:

$$D = \frac{Eh^3}{12\left(1 - \mu^2\right)} \qquad \text{(5-78)}$$

The quantity h is the thickness of the plate. Navier used one elastic constant ($\lambda_L = G$) in his development of the plate-bending equations, so his result corresponded to a material having a Poisson's ratio $\mu = \frac{1}{4}$. Navier used a double trigonometric series in solving the bending problem for rectangular plates.

Navier also developed fundamental relationships in fluid mechanics, called the *Navier-Stokes equations* in a paper presented in 1822 [Schlichting, 1968]. These equations are basically Newton's second law of motion applied to a differential fluid element, similar to the stress relations for a solid material, eqs. (5-28a, b, and c). The difference between the two sets of equations is that the shear stress in a "Newtonian" fluid is proportional to the strain *rate* in the fluid instead of being proportional to the strain, as is the case for an elastic solid. There is only one fluid property parameter in the fluid mechanics equations, the fluid *viscosity*,

Figure 5-6. Eugenio Beltrami.

which plays a mathematical role similar to the shear modulus in the equations for a solid. The thermal effects in a fluid result in buoyancy forces or natural convection characteristics.

Eugenio Beltrami (Figure 5-6) was born in Cremona in Lombardy, which was a part of the Austrian Empire at that time (it is now a part of Italy). He studied mathematics at the University of Pavia, but he was forced to leave after voicing his political opinions. He worked for a railroad company until he was appointed professor at the University of Bologna in 1862.

In 1885, Beltrami proposed the failure theory bearing his name, also called the *maximum total strain energy theory*. This theory asserts that the amount of total strain energy per unit volume for a material at failure under combined loading should be the same as the total strain energy per unit volume for a material in the simple uniaxial tensile test:

$$U_s = \tfrac{1}{2}\left(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3\right) = \tfrac{1}{2}S_y\varepsilon_y \tag{5-79}$$

The quantities σ_1, σ_2, and σ_3 are the three principal stresses, and ε_1, ε_2, and ε_3 are the corresponding principal strains. If we make the substitutions from eqs. (5-33a, b, and c) and do not include the thermal term, eq. (5-79) may be written in the following form:

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3\right) = S_y^2 \tag{5-80}$$

According to the maximum total strain energy theory, a material would fail under complex loading when eq. (5-80) is satisfied. For a material subjected to

hydrostatic loading ($\sigma_1 = \sigma_2 = \sigma_3 = p/3$), it has been observed experimentally that a large amount of strain energy can be stored without yielding or fracturing the material. Thus, the maximum total strain energy failure theory does not agree with experimental failure data. For this reason, the Beltrami failure theory is not used by designers, but the theory is of historical significance because it contributed to the development of the widely used distortion energy theory or *Von Mises-Hencky* failure theory.

John Henry Michell was born in Australia and conducted his graduate studies at Cambridge University, where he received his M.A. in 1887. After teaching at Cambridge for several years, he returned to Australia and was appointed lecturer in mathematics at the University of Melbourne. In addition to the development of the stress compatibility relations, Michell solved a variety of problems in the area of theory of elasticity, including torsion of a circular shaft of variable diameter, stress distributions in a semi-infinite plate, and stresses and deflections of plates.

PROBLEMS

5-1. In a particular problem, the strain components in a material were determined as follows:

$$\varepsilon_x = C_1 \cos(\pi x/a)\sin(\pi y/b)$$

$$\varepsilon_y = C_2 \cos(\pi x/a)\sin(\pi y/b)$$

$$\gamma_{xy} = (C_1 + C_2)\sin(\pi x/a)\cos(\pi y/b)$$

$$\text{where} \qquad C_1 = 0.002$$

$$a = 250 \text{ mm}$$

$$b = 200 \text{ mm}$$

All other strain components are zero. Using the strain compatibility relationships, determine the numerical value for the constant C_2.

5-2. Stress functions are used in the solution of some two-dimensional thermoelastic problems. Let us consider a stress function $\Phi(x, y)$ defined by the following relations:

$$\sigma_x = \frac{\partial^2\Psi}{\partial y^2} \qquad \sigma_y = \frac{\partial^2\Psi}{\partial x^2} \quad \text{and} \qquad \tau_{xy} = -\frac{\partial^2\Psi}{\partial x\partial y}$$

The other stresses (σ_z, τ_{xz}, and τ_{yz}) and the body forces (X, Y, and Z) are zero. (a) Using the stress equilibrium relations (5-28a) and (5-28b) for static conditions, show that the stress function definition satisfies the stress equilibrium equations identically, i.e., the equations are satisfied no matter what form the stress function Ψ takes. (b) Using the strain compatibility

relation (5-18a) and the stress–strain relations (5-33a), (5-33b), and (5-34a), show that the stress function must satisfy the following differential equation:

$$\frac{\partial^4\Psi}{\partial x^4}+2\frac{\partial^4\Psi}{\partial x^2\partial y^2}+\frac{\partial^4\Psi}{\partial y^4}+\alpha E\left[\frac{\partial^2(\Delta T)}{\partial x^2}+\frac{\partial^2(\Delta T)}{\partial y^2}\right]=0$$

or

$$\nabla^2\left(\nabla^2\Psi+\alpha E\Delta T\right)=0$$

5-3. Suppose the body forces X and Y in a two-dimensional problem are not zero, but are related to a potential function $V(x,y)$ by the following relations:

$$X=-\frac{\partial V}{\partial x}\quad\text{and}\quad Y=-\frac{\partial V}{\partial y}\quad\text{with } Z=0$$

Let us consider the stress function $\Psi(x,y)$ defined by the following relations:

$$\sigma_x=\frac{\partial^2\Psi}{\partial y^2}+V\qquad\sigma_y=\frac{\partial^2\Psi}{\partial x^2}+V\qquad\tau_{xy}=-\frac{\partial^2\Psi}{\partial x\partial y}$$

The other stresses (σ_z, τ_{xz} and τ_{yz}) are zero. (a) Using the stress equilibrium relations (5-28a) and (5-28b) for static conditions, show that the stress function definition satisfies the stress equilibrium equations identically, i.e., the equations are satisfied no matter what form the stress function Ψ takes. (b) Using the strain compatibility relation (5-18a) and the stress–strain relations (5-33a), (5-33b), and (5-34a), show that the stress function must satisfy the following differential equation:

$$\frac{\partial^4\Psi}{\partial x^4}+2\frac{\partial^4\Psi}{\partial x^2\partial y^2}+\frac{\partial^4\Psi}{\partial y^4}+\alpha E\left[\frac{\partial^2(\Delta T)}{\partial x^2}+\frac{\partial^2(\Delta T)}{\partial y^2}\right]+(1-\mu)\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)=0$$

or

$$\nabla^2\left[\nabla^2\Psi+\alpha E\Delta T+(1-\mu)V\right]=0$$

5-4. (a) For a particular situation, the direct stress σ_z in the z-direction is identically zero. Using eqs. (5-33a) and (5-33b), determine the corresponding expressions for the direct stresses σ_x and σ_y in terms of the strains ε_x and ε_y. This condition is called the *plane stress* condition. (b) Suppose that the direct mechanical strain in the z-direction is identically zero and the stress σ_z is not zero, but is given by:

$$\sigma_z=\mu\left(\sigma_x+\sigma_y\right)-\alpha E,\Delta T$$

For this condition, determine the relationships between the direct stresses and the direct strains. This condition is called the *plane strain* condition.

5-5. The stress–strain relationships for plane stress in Cartesian coordinates may be written as follows:

$$\sigma_x = \frac{E}{1-\mu^2}\left(\varepsilon_x + \mu\varepsilon_y\right) - \frac{\alpha E \Delta T}{1-\mu}$$
$$\sigma_y = \frac{E}{1-\mu^2}\left(\varepsilon_y + \mu\varepsilon_x\right) - \frac{\alpha E \Delta T}{1-\mu}$$
$$\tau_{xy} = \frac{E}{2\left(1+\mu\right)}\gamma_{xy} \qquad \sigma_z = \tau_{xz} = \tau_{yz} = 0$$

Using the stress equilibrium relations for steady-state and zero body forces, eqs. (5-28a) and (5-28b), and the strain–displacement relations, eqs. (5-1), (5-2), and (5-4), develop the following relationships for the displacements:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \left(\frac{1+\mu}{1-\mu}\right)\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} - 2\alpha\frac{\partial\left(\Delta T\right)}{\partial x}\right] = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \left(\frac{1+\mu}{1-\mu}\right)\left[\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} - 2\alpha\frac{\partial\left(\Delta T\right)}{\partial y}\right] = 0$$

5-6. Displacement potential functions are used in the solution of some two-dimensional thermal stress problems. Let us define the displacement potential function $\Phi(x,y)$ as follows:

$$u = \frac{\partial\Phi}{\partial x} \quad \text{and} \quad v = \frac{\partial\Phi}{\partial y}$$

Using these definitions in the two equations obtained in Problem 5-5, we find

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \left(\frac{1+\mu}{1-\mu}\right)\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} - 2\alpha\frac{\partial\left(\Delta T\right)}{\partial x}\right] = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \left(\frac{1+\mu}{1-\mu}\right)\left[\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} - 2\alpha\frac{\partial\left(\Delta T\right)}{\partial y}\right] = 0$$

Differentiate the first equation with respect to x and the second with respect to y. Add the resulting two relationships to obtain the governing equation for the displacement potential for two-dimensional thermal stress problems with zero body forces:

$$\frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4} - (1+\mu)\alpha\left[\frac{\partial^2\left(\Delta T\right)}{\partial x^2} + \frac{\partial^2\left(\Delta T\right)}{\partial y^2}\right] = 0$$

or

$$\nabla^2\left(\nabla^2\Phi - (1+\mu)\alpha\Delta T\right) = 0$$

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6

PLANE STRESS

6.1 INTRODUCTION

In this chapter, we examine some thermal stress problems that can be solved by the application of the *plane stress* conditions. The general thermoelastic equations are usually so mathematically complicated that an analytical solution can be obtained for only a few idealized situations. Many practical solutions may be obtained by making certain approximations, however. The plane stress condition is one of the approximations that may be used for design and analysis of plates without bending.

A *plate* is an element in which the thickness (the dimension in the z-direction, for example) is much smaller than the element dimensions in the other two coordinate directions. For practical purposes, the plate thickness should be less than about 10 percent of the smaller value of the length or width of the plate (the dimensions in the x- and y-directions, for example), or the radius for a circular plate.

The restrictions on the accurate use of the plane stress approach include the following.

(a) The element must be flat. The case for curved plates or shells under membrane loading is considered in a subsequent chapter.
(b) There are no lateral loads on the plate surface. This restriction assures that the mechanical loads cause no bending of the plate. Plate bending problems are also considered in a subsequent chapter.
(c) The temperature within the plate is not a function of the thickness coordinate. A less restrictive condition is that the temperature is an even function

(z^2, cos z, etc.) of the thickness coordinate. Both of these conditions result in no thermal bending of the plate.

(d) The stresses are below the elastic limit.

The mathematical conditions for plane stress are

$$\sigma_x = \sigma_x(x, y) \quad \text{and} \quad \sigma_y = \sigma_y(x, y) \quad \text{and} \quad \tau_{xy} = \tau_{xy}(x, y)$$
$$\sigma_z = \tau_{xz} = \tau_{yz} = 0 \tag{6-1}$$

In other words, the stress condition is two dimensional, and there are no stresses acting perpendicular to the surface of the plate.

6.2 STRESS RESULTANTS

If the temperature distribution is uniform across the cross section, the radial and circumferential stress distribution will also be uniform at distances from the edge on the order of the plate thickness. For a temperature distribution that is an even function of the thickness coordinate z (such that no bending occurs), the stress distribution will not be uniform; however, the variation of the stress across the cross section will generally be small compared with the stress variation in the other coordinate directions. For these cases, we may use the average stress across the thickness of the plate.

The radial and circumferential *stress resultants* are defined as follows:

$$N_r = \int_{-h/2}^{h/2} \sigma_r \, dz \tag{6-2}$$

$$N_\theta = \int_{-h/2}^{h/2} \sigma_\theta \, dz \tag{6-3}$$

The quantity h is the thickness of the plate. The origin for the z-axis is taken at the center of the plate thickness. The units for the stress resultants are {N/m} or {lb_f/in.}. The *average* radial and circumferential stresses may be found from the stress resultants:

$$\overline{\sigma_r} = \frac{N_r}{h} \quad \text{and} \quad \overline{\sigma_\theta} = \frac{N_\theta}{h} \tag{6-4}$$

In addition, let us define the *thermal stress resultant* N_T as

$$N_T = \alpha E \int_{-h/2}^{h/2} \Delta T \, dz \tag{6-5}$$

For the conditions of plane stress ($\sigma_z = 0$, etc.), the expressions for the radial and circumferential strains may be written as

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \mu\sigma_\theta) + \alpha\Delta T \tag{6-6a}$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \mu\sigma_r) + \alpha\Delta T \tag{6-6b}$$

If we solve for the radial and circumferential stresses from eqs. (6-6a) and (6-6b), we obtain the following expressions for the case of plane stress:

$$\sigma_r = \frac{E}{1-\mu^2}(\varepsilon_r + \mu\varepsilon_\theta) - \frac{\alpha E\Delta T}{1-\mu} \tag{6-7a}$$

$$\sigma_\theta = \frac{E}{1-\mu^2}(\varepsilon_\theta + \mu\varepsilon_r) - \frac{\alpha E\Delta T}{1-\mu} \tag{6-7b}$$

If we multiply both sides of eqs. (6-7a) and (6-7b) by dz and integrate across the plate thickness, we obtain the following expressions for the stress resultants:

$$N_r = \frac{Eh}{1-\mu^2}(\overline{\varepsilon_r} + \mu\overline{\varepsilon_\theta}) - \frac{N_T}{1-\mu} \tag{6-8a}$$

$$N_\theta = \frac{Eh}{1-\mu^2}(\overline{\varepsilon_\theta} + \mu\overline{\varepsilon_r}) - \frac{N_T}{1-\mu} \tag{6-8b}$$

The overbar denotes the average value of the strain across the plate thickness.

If we solve for the average strain terms in eq. (6-8a), for example, we obtain

$$\overline{\varepsilon_r} + \mu\overline{\varepsilon_\theta} = \frac{\left(1-\mu^2\right)N_r + (1+\mu)N_T}{Eh} \tag{6-9}$$

For the conditions of plane stress, the radial and circumferential strains are approximately uniform across the plate thickness at any selected location, so we may substitute the strain relation into eq. (6-7) to obtain the expression for the stress:

$$\sigma_r = \frac{N_r}{h} + \frac{N_T}{(1-\mu)h} - \frac{\alpha E\Delta T}{1-\mu} \tag{6-10}$$

The average temperature change across the plate thickness is found from its definition:

$$\Delta T_m = \frac{1}{h}\int_{-h/2}^{h/2} \Delta T\,dz \tag{6-11}$$

By combining eqs. (6-5) and (6-11), we obtain the relation between the thermal stress resultant and the average temperature change across the cross section:

$$N_T = \alpha Eh\Delta T_m \tag{6-12}$$

Using the relationship from eq. (6-12) into eq. (6-10), we obtain

$$\sigma_r = \frac{N_r}{h} + \frac{\alpha E}{1-\mu}(\Delta T_m - \Delta T) \tag{6-13a}$$

A similar expression may be obtained for the circumferential stress:

$$\sigma_\theta = \frac{N_\theta}{h} + \frac{\alpha E}{1-\mu}(\Delta T_m - \Delta T) \tag{6-13b}$$

Generally, the variation of ΔT across the thickness is much smaller than the variation in the in-plane directions. The first term on the right side of eqs. (6-13) is equal to the stress averaged across the thickness. The second term represents the stress variation due to nonuniform temperature distributions in the thickness direction. If the temperature is uniform across the thickness of the plate, the second term is zero.

6.3 CIRCULAR PLATE WITH A HOT SPOT

6.3.1 Problem Statement

As an example of the application of the thermal stress analysis, let us consider the case of a circular plate having a radius a with a hot spot of radius b at the center of the plate, as shown in Figure 6-1. Let us consider the case in which the temperature distribution and the stress distribution are one dimensional, i.e., functions of the radial coordinate only, and not a function of the circumferential coordinate. Also, let us consider the case in which there are no body forces acting on the plate.

Because there is no variation of the stresses in the θ direction, it would seem intuitively obvious that the shear stress $\tau_{r\theta}$ would be zero for this problem. This fact may be demonstrated, as follows. The circumferential stress σ_θ is not a function of the coordinate θ (the loading is one dimensional), the shear stress $\tau_{r\theta}$ is zero (plane stress condition), and the body force is zero; therefore, the stress equilibrium equation in the circumferential direction, eq. (5-31b), reduces to

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} = 0 \tag{6-14}$$

The general solution of this problem is

$$\tau_{r\theta} = \frac{C}{r^2} \tag{6-15}$$

We know that the shear stress must be finite, so C must be equal to zero for the shear stress to be finite at the center of the disc. For $C = 0$, we see that $\tau_{r\theta} = 0$.

The strain–displacement relations for this problem may be obtained from eq. (5-12), noting that the radial displacement is $u = u(r)$ and the circumferential displacement $v = 0$:

$$\varepsilon_r = \frac{du}{dr} \tag{6-16a}$$

$$\varepsilon_\theta = \frac{u}{r} \tag{6-16b}$$

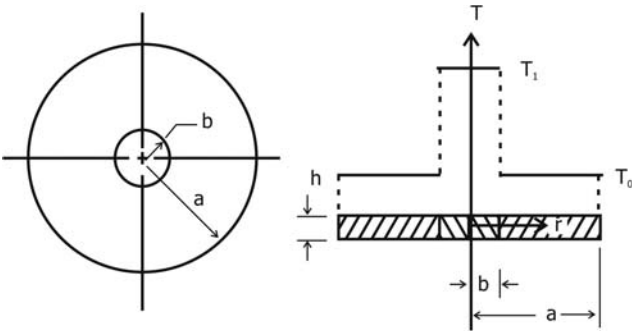

Figure 6-1. A circular plate with a circular hot spot at the center of the plate.

These two relations may be combined to yield

$$\varepsilon_r = \frac{d\,(r\varepsilon_\theta)}{dr} \tag{6-17}$$

For steady-state operation and zero body forces, the stress equilibrium relation in the radial direction, eq. (5-31a), reduces to

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{6-18}$$

Let us consider the case for which the temperature is uniform across the thickness of the plate. We may introduce the stress resultants into eq. (6-18) to obtain

$$\frac{d\,(rN_r)}{dr} - N_\theta = 0 \tag{6-19}$$

The stress–strain relations for this problem are given by eqs. (6-8a) and (6-8b).

The problem may be solved by using two approaches: (a) the *displacement formulation*, in which all quantities are related to the radial displacement u, and (b) the *stress formulation*, in which all quantities are related to the radial stress resultant N_r or the radial stress σ_r. The displacement formulation is generally more convenient to use for problems in which the boundary conditions involve only displacements or strains. On the other hand, if the boundary conditions involve only stresses (or forces), then the stress formulation is usually more convenient to use in solving the thermal stress problem. For problems in which the boundary conditions involve both displacement and stresses (the so-called *mixed problem*), then either formulation results in about the same mathematical complexity for the solution.

Let us solve this problem using each of the approaches, and subject to the temperature distribution given by the following relationship:

$$\Delta T = \begin{cases} \Delta T_1 & \text{for} \quad 0 \le r < b \\ 0 & \text{for} \quad b < r \le a \end{cases} \tag{6-20}$$

The thermal stress resultant for this temperature distribution is

$$N_T = \alpha E \int_{-h/2}^{h/2} \Delta T \, dz = \begin{cases} \alpha E h \Delta T_1 & \text{for} \quad 0 \le r < b \\ 0 & \text{for} \quad b < r \le a \end{cases} \tag{6-21}$$

6.3.2 Displacement formulation

We may start with the stress–strain equations, eqs. (6-18a) and (6-18b), and substitute the strain components in terms of the radial displacement given by eqs. (6-16a) and (6-16b):

$$N_r = \frac{Eh}{1-\mu^2}\left(\frac{du}{dr} + \mu\frac{u}{r}\right) - \frac{N_T}{1-\mu} \tag{6-22a}$$

$$N_\theta = \frac{Eh}{1-\mu^2}\left(\frac{u}{r} + \mu\frac{du}{dr}\right) - \frac{N_T}{1-\mu} \tag{6-22b}$$

The stress–strain relations may then be substituted into the radial stress equilibrium relation, eq. (6-19), to obtain

$$r\frac{d}{dr}\left[\frac{1}{r}\frac{d\,(ru)}{dr}\right] = \frac{(1+\mu)\,r}{Eh}\frac{dN_T}{dr} \tag{6-23}$$

The radial displacement may be found by integrating eq. (6-23). If we integrate once, we obtain

$$\frac{1}{r}\frac{d\,(ru)}{dr} = \frac{(1+\mu)\,N_T}{Eh} + 2C_1 \tag{6-24}$$

Separating variables and integrating eq. (6-24), we obtain the general expression for the radial displacement:

$$u = \frac{1+\mu}{Ehr}\int N_T r\,dr + C_1 r + \frac{C_2}{r} \tag{6-25}$$

The constants C_1 and C_2 are to be determined from the boundary conditions for a specific problem.

The general expression for the stress resultants may be found by substituting the displacement from eq. (6-25) into the stress–strain relations, eqs. (6-22a and 22b):

$$N_r = -\frac{1}{r^2}\int N_T r\,dr + \frac{EhC_1}{1-\mu} - \frac{EhC_2}{(1+\mu)\,r^2} \tag{6-26a}$$

$$N_\theta = \frac{1}{r^2}\int N_T r\,dr - N_T + \frac{EhC_1}{1-\mu} + \frac{EhC_2}{(1+\mu)\,r^2} \tag{6-26b}$$

Now, let us consider the specific problem of determining the thermal stresses for the temperature distribution given by eq. (6-20). First, for the region inside the hot spot, $0 \leq r < b$, we find

$$\int N_T r\, dr = \tfrac{1}{2} Eh\alpha \Delta T_1 r^2 \tag{6-27}$$

In order that the displacement be finite at the center of the hot spot ($r = 0$), we must have the constant $C_2 = 0$ in eqn. (6-25).

The expressions for the radial displacement and the stress resultants may be found from these results and eqs. (6-25), (6-26a), and (6-26b).

$$u = \tfrac{1}{2}(1+\mu)\,\alpha \Delta T_1 r + C_1 r \tag{6-28a}$$

$$N_r = -\frac{1}{2} Eh\alpha\Delta T_1 + \frac{EhC_1}{1-\mu} \tag{6-28b}$$

$$N_\theta = -\frac{1}{2} Eh\alpha\Delta T_1 + \frac{EhC_1}{1-\mu} \tag{6-28c}$$

For the region outside the hot spot ($b < r \leq a$), the thermal stress resultant is zero. The constants of integration will be different in this region, and we may write the expressions for the radial displacement and stress resultants as

$$u = C_3 r + \frac{C_4}{r} \tag{6-29a}$$

$$N_r = \frac{EhC_3}{1-\mu} - \frac{EhC_4}{(1+\mu)\,r^2} \tag{6-29b}$$

$$N_\theta = \frac{EhC_3}{1-\mu} + \frac{EhC_4}{(1+\mu)\,r^2} \tag{6-29c}$$

Let us consider the case for which there are no external forces acting on the outer edge of the plate. For this condition, $N_r = 0$ at $r = a$. Using this condition in eq. (6-29b), we obtain

$$\frac{C_4}{1+\mu} = \frac{a^2 C_3}{1-\mu} \tag{6-30}$$

The radial displacement and stress resultants outside the hot spot ($b < r \leq a$) may be written in terms of the constant of integration C_3:

$$u = C_3 r\left[1 + \left(\frac{1+\mu}{1-\mu}\right)\left(\frac{a}{r}\right)^2\right] \tag{6-31a}$$

$$N_r = \frac{EhC_3}{1-\mu}\left[1 - \left(\frac{a}{r}\right)^2\right] \tag{6-31b}$$

$$N_\theta = \frac{EhC_3}{1-\mu}\left[1 + \left(\frac{a}{r}\right)^2\right] \tag{6-31c}$$

The expressions for the constants C_1 and C_3 may be found from the conditions at the edge of the hot spot. At this point, the radial stress resultant (radial force per unit circumferential length) is continuous.

$$N_r\,(r = b-) = N_r\,(r = b+) \tag{6-32}$$

The notation $b-$ means that the equation for the region *inside* the hot spot is used at the interface, and $b+$ means that the equation for the region *outside* the hot spot is used. The radial displacement is also continuous across the edge of the hot spot.

$$u\,(r = b-) = u\,(r = b+) \tag{6-33}$$

If we make the substitutions from eqs. (6-32) and (6-33), we obtain the following expressions for the constants of integration:

$$C_1 = \tfrac{1}{4}\alpha\Delta T_1\,(1-\mu)^2\,(b/a)^2 \tag{6-34a}$$

$$C_3 = \frac{\frac{1}{2}\alpha\Delta T_1 Eh}{\left(\dfrac{a}{b}\right)^2 - 1}\left[1 - \frac{1}{2}\,(1-\mu)\left(\frac{b}{a}\right)^2\right] \tag{6-34b}$$

These expressions may be used to determine the final relations for the radial displacement and stress resultants. For the region inside the hot spot ($0 \le r < b$);

$$u = \tfrac{1}{2}\alpha\Delta T_1\left[(1+\mu) + \tfrac{1}{2}\,(1-\mu)^2\,(b/a)^2\right]r \tag{6-35a}$$

$$N_r = N_\theta = -\tfrac{1}{2}Eh\alpha\Delta T_1\left[1 - \tfrac{1}{2}\,(1-\mu)\,(b/a)^2\right] \tag{6-35b}$$

For the region outside the hot spot ($b < r \le a$);

$$u = \frac{\frac{1}{2}\alpha\Delta T_1\,(1-\mu)}{\left(\dfrac{a}{b}\right)^2 - 1}\left[1 - \frac{1}{2}\,(1-\mu)\left(\frac{b}{a}\right)^2\right]\left[1 + \left(\frac{1+\mu}{1-\mu}\right)\left(\frac{a}{r}\right)^2\right]r \tag{6-36a}$$

$$N_r = -\frac{\frac{1}{2}\alpha\Delta T_1 Eh}{\left(\dfrac{a}{b}\right)^2 - 1}\left[1 - \frac{1}{2}\,(1-\mu)\left(\frac{b}{a}\right)^2\right]\left[\left(\frac{a}{r}\right)^2 - 1\right] \tag{6-36b}$$

$$N_\theta = \frac{\frac{1}{2}\alpha\Delta T_1 Eh}{\left(\dfrac{a}{b}\right)^2 - 1}\left[1 - \frac{1}{2}\,(1-\mu)\left(\frac{b}{a}\right)^2\right]\left[\left(\frac{a}{r}\right)^2 + 1\right] \tag{6-36c}$$

We note that, because the temperature is uniform through the thickness of the plate in this problem, the radial and circumferential stress from eqs. (6-13a and 13b) are given by

$$\sigma_r = \frac{N_r}{h} \quad \text{and} \quad \sigma_\theta = \frac{N_\theta}{h} \tag{6-37}$$

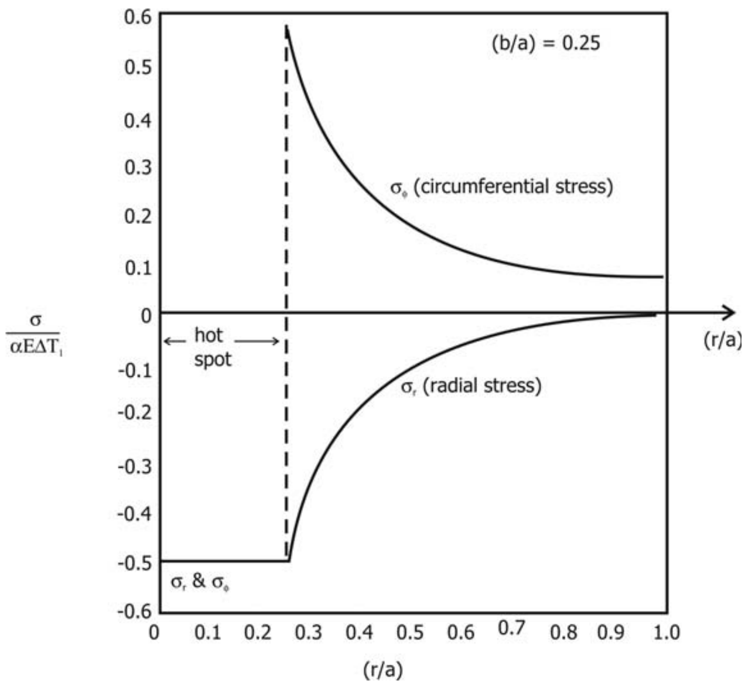

Figure 6-2. Plot of the radial and circumferential stresses for a plate with a circular hot spot at the center. The ratio of the hot-spot diameter to the plate diameter is 0.250.

A plot of the radial and circumferential stresses for the special case of $(b/a) = 0.25$ is shown in Figure 6-2.

For the case in which the hot-spot diameter is very small compared to the disc diameter ($b/a \to 0$), the stress and displacement relations, eqs. (6-35) and (6-36), reduce

$$(0 \le r < b): \quad u = \tfrac{1}{2}(1+\mu)\,\alpha\Delta T_1 b\,(r/b) \tag{6-38a}$$

$$\sigma_r = \sigma_\theta = -\tfrac{1}{2}\alpha E\Delta T_1 \tag{6-38b}$$

$$(b < r \le a): \quad u = \tfrac{1}{2}(1+\mu)\,\alpha\Delta T_1 b\,(b/r) \tag{6-39a}$$

$$\sigma_r = -\sigma_\theta = -\tfrac{1}{2}\alpha E\Delta T_1\,(b/r)^2 \tag{6-39b}$$

In general, the maximum stress is the circumferential stress immediately outside the hot spot ($r = b+$):

$$(\sigma_\theta)_{\max} = \tfrac{1}{2}\alpha E\Delta T_1\left[1 - \frac{1}{2}(1-\mu)\left(\frac{b}{a}\right)^2\right]\left[\frac{1+\left(\dfrac{b}{a}\right)^2}{1-\left(\dfrac{b}{a}\right)^2}\right] \tag{6-40}$$

If the temperature change ΔT_1 is positive (the spot is "heated"), then the maximum circumferential stress is positive or tensile. High tensile stress in the

circumferential direction could present a problem for materials such as concrete or some plastics, which are weak in tension or have a relatively low ultimate tensile strength.

6.3.3 Stress Formulation

The problem of stresses around a hot spot is a problem with "mixed" boundary conditions (boundary conditions involving both stress and displacement), so neither the displacement formulation nor the stress formulation presents any special advantages for this problem. However, let us develop the stress formulation for the hot-spot problem to illustrate the technique.

We may start with the stress–strain equations written as follows:

$$\varepsilon_r = \frac{du}{dr} = \frac{1}{Eh}\,(N_r - \mu N_\theta + N_T) \tag{6-41a}$$

$$\varepsilon_\theta = \frac{u}{r} = \frac{1}{Eh}\,(N_\theta - \mu N_r + N_T) \tag{6-41b}$$

The stress–strain relations may be substituted into the strain compatibility relation for plane stress, eq. (6-17):

$$\varepsilon_r = \frac{d\,(r\varepsilon_\theta)}{dr} \tag{6-42}$$

The result is

$$N_r - \mu\frac{d\,(rN_r)}{dr} - \frac{d\,(rN_\theta)}{dr} + \mu\frac{dN_\theta}{dr} = \frac{d\,(rN_T)}{dr} - N_T \tag{6-43}$$

The circumferential stress resultant may be eliminated by using the stress–equilibrium relationship, eq. (6-19). The result is

$$N_r - \frac{d}{dr}\left[r\frac{d\,(rN_r)}{dr}\right] = r\frac{dN_T}{dr} \tag{6-44}$$

This relationship may be simplified as follows:

$$r\frac{d}{dr}\left[\frac{1}{r}\frac{d\left(r^2N_r\right)}{dr}\right] = -r\frac{dN_T}{dr} \tag{6-45}$$

If we integrate eq. (6-45) once, we obtain

$$\frac{1}{r}\frac{d\left(r^2N_r\right)}{dr} = -N_T + 2C_1 \tag{6-46}$$

Finally, integration of eq. (6-46) yields the general solution for the radial stress resultant:

$$N_r = -\frac{1}{r^2}\int N_T r\,dr + C_1 + \frac{C_2}{r^2} \tag{6-47}$$

The expression for the circumferential stress resultant may be found by combining eqs. (6-47) and (6-19):

$$N_\theta = \frac{d\,(rN_r)}{dr} = \frac{1}{r^2}\int N_T r\,dr - N_T + C_1 - \frac{C_2}{r^2} \tag{6-48}$$

Finally, the general expression for the radial displacement may be found from the stress–strain relation for the circumferential direction, eq. (6-41b):

$$u = \frac{(1+\mu)\,r}{Eh}\left[\frac{1}{r^2}\int N_T r\,dr + \left(\frac{1-\mu}{1+\mu}\right)C_1 - \frac{C_2}{r^2}\right] \tag{6-49}$$

For the case of the hot spot in a circular plate with a temperature distribution given by eq. (6-29), the integral in eq. (6-47) is given by eq. (6-27) for the region inside the hot spot, $0 \le r < b$. Using this result in eq. (6-47), we obtain the following expression for the radial stress resultant within the hot spot:

$$N_r = C_1 + \frac{C_2}{r^2} - \frac{1}{2}Eh\alpha\Delta T_1 = C_1 - \frac{1}{2}Eh\alpha\Delta T_1 \tag{6-50a}$$

We have set $C_2 = 0$ because the stress at the center of the hot spot $(r = 0)$ must be finite.

The circumferential stress resultant and radial displacement expressions for the region inside the hot spot are

$$N_\theta = C_1 - \tfrac{1}{2}Eh\alpha\Delta T_1 \tag{6-50b}$$

$$u = \frac{r}{Eh}\left[(1-\mu)\,C_1 + \frac{1}{2}\,(1+\mu)\,Eh\alpha\Delta T_1\right] \tag{6-50c}$$

The temperature change and thermal stress resultant are zero outside the hot spot. For the stress formulation, the stress resultants and radial displacement reduce to the following expressions for $(b < r \le a)$:

$$N_r = C_3 + \frac{C_4}{r^2} \tag{6-51a}$$

$$N_\theta = C_3 - \frac{C_4}{r^2} \tag{6-51b}$$

$$u = \frac{r}{Eh}\left[(1-\mu)\,C_3 - (1+\mu)\,\frac{C_4}{r^2}\right] \tag{6-51c}$$

The radial stress resultant is zero at the outer edge of the plate $(r = a)$, because there is no applied radial force. We find the following from eq. (6-51a):

$$C_3 = -\frac{C_4}{a^2} \tag{6-52}$$

If we apply the conditions at the interface between the heated and unheated portions of the disc, $N_r(b-) = N_r(b+)$ and $u(b-) = u(b+)$, we find the following values for the other constants:

$$C_1 = \tfrac{1}{4}\alpha E\Delta T_1 h\,(1-\mu)\,(b/a)^2 \tag{6-53}$$

$$C_4 = -\frac{1}{2}\alpha E \Delta T_1 h \frac{a^2\left[1 - \frac{1}{2}(1-\mu)(b/a)^2\right]}{(a/b)^2 - 1} \tag{6-54}$$

If we substitute the expressions for the constants, eqs. (6-52), (6-53), and (6-54), into the expressions for the stress resultants and radial displacements, we obtain the same expressions, eqs. (6-35) and (6-36), that were obtained using the displacement formulation.

Example 6-1 A circular disc having a diameter of 70 mm (2.755 in.) is to be heated over a circular area at the center of the disc. The heated area has a diameter of 14 mm (0.551 in.). The plate thickness is 4.1 mm (0.162 in.). The disc material is 6061-T6 aluminum with the following properties: $\alpha = 23.4 \times 10^{-6}\,\mathrm{K}^{-1}$ ($13.0 \times 10^{-6}\,{}^\circ\mathrm{F}^{-1}$); $E = 69.0\,\mathrm{GPa}$ (10.0×10^6 psi); $\mu = 0.30$. Determine the maximum temperature to which the hot spot may be heated such that the maximum circumferential stress is limited to 35 MPa (5080 psi) if the temperature of the unheated portion of the plate is maintained at 24°C (75.2°F).

The diameter ratio for this problem is

$$b/a = 14/70 = 0.200$$

Using eq. (6-40) for the maximum circumferential stress, we may calculate the dimensionless ratio:

$$\frac{(\sigma_\theta)_{\max}}{\frac{1}{2}\alpha E \Delta T_1} = \left[1 - \frac{1}{2}(1-0.30)(0.200)^2\right]\left[\frac{1+(0.200)^2}{1-(0.200)^2}\right] = (0.986)(1.083) = 1.068$$

The maximum allowable temperature rise may now be determined:

$$\Delta T_1 = \frac{(2)\left(35 \times 10^6\right)}{\left(23.4 \times 10^{-6}\right)\left(69.0 \times 10^9\right)(1.068)} = 40.6^\circ\mathrm{C} \qquad \left(105.1^\circ\mathrm{F}\right)$$

The maximum allowable temperature for the hot spot is

$$(T_1)_{\max} = T_0 + \Delta T_1 = 24^\circ + 40.6^\circ = 64.6^\circ\mathrm{C} \qquad \left(148.3^\circ\mathrm{F}\right)$$

The radial displacement at the outer edge of the plate ($r = a$) may be found from eq. (6-36a):

$$\frac{u}{\frac{1}{2}\alpha \Delta T_1 a} = \frac{(1-0.300)}{(5.00)^2 - 1}\left[1 - \frac{1}{2}(1-0.300)(0.200)^2\right]\left[1 + \frac{1+0.300}{1-0.300}\right] = 0.08217$$

$$u = (0.08217)\left(\tfrac{1}{2}\right)\left(23.4 \times 10^{-6}\right)\left(40.6^\circ\right)(0.035) = 1.366 \times 10^{-6}\,\mathrm{m}$$

$$u = 1.366\,\mathrm{mm} \qquad \left(0.538 \times 10^{-3}\,\mathrm{in.}\right)$$

6.3.4 Temperature Distribution Outside the Hot Spot

In the previous sections, the temperature distribution was given as a somewhat physically artificial function, since discontinuous temperature distributions are not usually encountered in practice. In this section, we develop a more reasonable temperature distribution for the portion of the plate outside the hot spot.

Let us consider the case in which the plate is exposed to air on both the top and the bottom of the plate. If the hot spot of radius b is maintained at a temperature T_1, heat will be conducted from the hot spot into the region around the hot spot. This energy will eventually be transferred by convection from the plate surface to the surrounding air at a temperature T_0, which is also considered as the "stress-free temperature" for the plate. The temperature at the edge of the hot spot is T_1, and the outer edge of the plate (at $r = a$) is considered to be insulated or have negligible heat transfer out the edge.

Let us consider the energy transfers for the differential element of the plate shown in Figure 6-3. The heat transfer rate into the element by conduction is given by the Fourier rate equation for conduction:

$$\dot{Q}_{\text{in}} = -k_t\,(2\pi r h)\,\frac{dT}{dr} \tag{6-55}$$

The quantity k_t is the thermal conductivity of the plate material, and h is the plate thickness.

The heat conducted out of the element may be written as follows:

$$\dot{Q}_{\text{out}} = \dot{Q}_{\text{in}} + d\dot{Q}_{\text{in}} = \dot{Q}_{in} - 2\pi h k_t \frac{d}{dr}\left(r\frac{dT}{dr}\right)dr \tag{6-56}$$

Finally, the heat transfer from the upper and lower surface of the plate by convection may be expressed in terms of the convection rate equation, where h_C is the convective heat transfer coefficient (*units*: W/m^2-°C or Btu/hr-ft^2-°F):

$$d\dot{Q}_{\text{conv}} = h_C(2)\,(2\pi r\,dr)\,(T - T_0) = 4\pi h_C r \Delta T_1 dr \tag{6-57}$$

Correlations for the convective heat transfer coefficient may be found in many heat transfer texts [Holman, 1997].

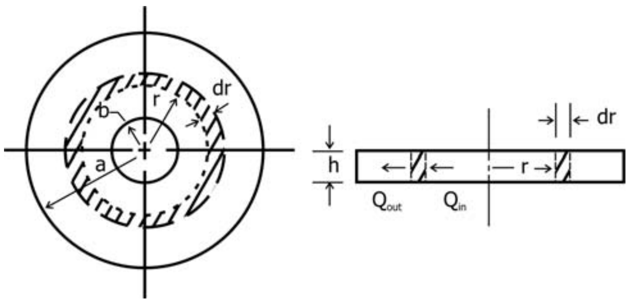

Figure 6-3. Plate differential element used to develop the temperature distribution around a circular hot spot in the plate.

If we use the first law of thermodynamics (*conservation of energy principle*),

$$\dot{Q}_{\text{in}} = \dot{Q}_{\text{out}} + d\dot{Q}_{\text{conv}} \tag{6-58}$$

we obtain the following differential equation for the temperature distribution in the portion of the plate outside the hot spot:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) - \left(\frac{2h_C}{k_t h}\right)(T - T_0) = 0 \tag{6-59}$$

We may simplify eq. (6-59) by introducing the temperature difference $\Delta T = T - T_0$ and the parameter M, defined by

$$M^2 = \frac{2h_C}{k_t h} \tag{6-60}$$

With these quantities, eq. (6-59) may be written as follows:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Delta T}{dr}\right) - M^2\Delta T = 0 \tag{6-61}$$

or

$$\frac{d^2\Delta T}{dr^2} + \frac{1}{r}\frac{d\Delta T}{dr} - M^2\Delta T = 0 \tag{6-62}$$

The equation for the temperature distribution is one form of the modified Bessel equation, as discussed in Appendix D. By comparison with the general equation, eq. (D-44), we see that the solution of eq. (6-62) may be written as follows:

$$\Delta T = C_1 I_0\,(Mr) + C_2 K_0\,(Mr) \tag{6-63}$$

The quantities $I_0(Mr)$ and $K_0(Mr)$ are the modified Bessel functions of the first and second kind of order 0.

At the outer edge of the plate ($r = a$), the heat transfer is negligible, because of the very small plate thickness or small conduction heat transfer area. If we use the Fourier rate equation for conduction, we obtain the following expression for the edge boundary condition:

$$\left(\frac{d\Delta T}{dr}\right)_{r=a} = 0 \tag{6-64}$$

From eq. (D-52), the values for the derivatives of the modified Bessel functions are

$$\frac{dI_0\,(Mr)}{dr} = MI_1\,(Mr) \quad \text{and} \quad \frac{dK_0\,(Mr)}{dr} = -MK_1\,(Mr) \tag{6-65}$$

If we make these substitutions into eq. (6-64), we find

$$C_1 MI_1\,(Ma) - C_2 MK_1\,(Ma) = 0 \tag{6-66}$$

The temperature at the edge of the hot spot ($r = b$) is T_1 or the temperature difference is $\Delta T(b) = \Delta T_1$. Making this substitution into eq. (6-63), we obtain a second relationship for the constants of integration:

$$\Delta T_1 = C_1 I_0 (Mb) + C_2 K_0 (Mb) \tag{6-67}$$

If we solve for the constants C_1 and C_2 from eqs. (6-66) and (6-67) and substitute into eq. (6-63), we obtain the following expression for the temperature distribution outside the hot spot:

$$(b \le r \le a) \qquad \frac{\Delta T\,(r)}{\Delta T_1} = \frac{K_1\,(Ma)\,I_0\,(Mr) + I_1\,(Ma)\,K_0\,(Mr)}{K_1\,(Ma)\,I_0\,(Mb) + I_1\,(Ma)\,K_0\,(Mb)} \tag{6-68}$$

The temperature distribution is plotted in Figure 6-4.

For the case of a large plate (a large, or actually for Ma large), the modified Bessel function $I_1(Ma)$ becomes very large (see the asymptotic expression, eq. (D-55)). This behavior requires that the constant $C_1 = 0$ for the case of $a \to \infty$, as required by the boundary condition from eq. (6-66). Using this result in

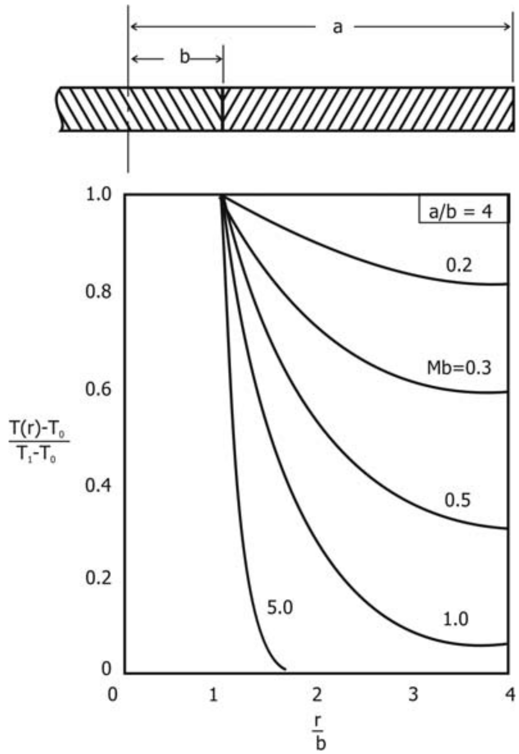

Figure 6-4. Temperature distribution outside a hot spot of radius b in a circular plate of radius a.

eq. (6-67) to evaluate the other constant C_2, we find the following result for the temperature distribution for a very large plate:

$$\frac{\Delta T\,(r)}{\Delta T_1} = \frac{K_0\,(Mr)}{K_0\,(Mb)} \tag{6-69}$$

The limiting expression, eq. (6-69), is a good approximation (within 1 or 2 K agreement) to the more complicated temperature expression, eq. (6-68), when the parameter $Ma > 5$. For example, with $Ma = 5$ and $Mb = 1.25$, we find the following values for the modified Bessel functions from Appendix D:

$$I_0(5) = 27.240 \qquad K_0(5) = 0.0036911$$

$$I_1(5) = 24.336 \qquad K_1(5) = 0.0040446$$

$$I_0(1.25) = 1.4305 \qquad K_0(1.25) = 0.29774$$

Using these values for a temperature change $\Delta T_1 = 100^\circ\mathrm{C}$ (180°F) in eq. (6-68), we obtain the following temperature change at the edge of the plate ($r = a$):

$$\Delta T = (0.02756)(100^\circ) = 2.76^\circ\mathrm{C}$$

Similarly, if we use eq. (6-69) for the same conditions, we obtain the following temperature difference:

$$\Delta T = (0.01240)(100^\circ) = 1.24^\circ\mathrm{C}$$

There is a difference of only 1.5°C (2.7°F) between the values obtained from the two equations.

It would be of interest to determine the size of the plate that met the condition $Ma = 5$. For a steel plate ($k_t = 54$ W/m-K $= 31.2$ Btu/hr-ft-°F) having a thickness of 1.6 mm (0.063 in.), we may find the required outer radius for this condition. For forced convection heat transfer to a gas, such as air, the convective heat transfer coefficient is on the order of $h_C = 200$ W/m^2-K (35.2Btu/hr-ft^2-°F). Using eq. (6-60), the value of the parameter M is

$$M = \left[\frac{(2)\,(200)}{(54)\,(0.0016)}\right]^{1/2} = 68.0\ \mathrm{m}^{-1}$$

The radius of the disc may be determined:

$$a = \frac{5}{68.0} = 0.0735\ \mathrm{m} = 73.5\ \mathrm{mm} \qquad (2.89\ \mathrm{in.})$$

This is definitely a size that could be encountered in practice.

6.3.5 Stress Distribution for a Large Plate with a Hot Spot

Let us use the temperature distribution from Section 6.3.4 to determine the stress distribution in a large plate with a hot spot of radius b maintained at a temperature T_1. The temperature distribution for this case is

$$\Delta T\,(r) = \begin{cases} \Delta T_1 & \text{for} \quad 0 \le r \le b \\ \Delta T_1 \dfrac{K_0\,(Mr)}{K_0\,(Mb)} & \text{for} \quad b < r \end{cases} \tag{6-70}$$

The thermal stress resultant for this temperature distribution is

$$N_T = \begin{cases} \alpha E h \Delta T_1 & \text{for} \quad 0 \le r \le b \\ \alpha E h \Delta T_1 \dfrac{K_0\,(Mr)}{K_0\,(Mb)} & \text{for} \quad b < r \end{cases} \tag{6-71}$$

The temperature distribution and stress resultant for the region inside the hot spot is the same as that given in eqs. (6-20) and (6-21), so the general expressions for the radial displacement and stresses are also given by eq. (6-28):

$$u = \left[\tfrac{1}{2}\,(1+\mu)\,\alpha \Delta T_1 + C_1\right] r \tag{6-72a}$$

$$N_r = N_\theta = -\frac{1}{2} E h \left(\alpha \Delta T_1 - \frac{2C_1}{1-\mu}\right) \tag{6-72b}$$

For the region outside the hot spot ($r > b$), we may evaluate the integral required in eqs. (6-25) and (6-26) for the radial displacement and stress resultants:

$$\int N_T r\,dr = \frac{\alpha E h \Delta T_1}{K_0\,(Mb)} \int K_0\,(Mr)\,r\,dr = -\frac{\alpha E h \Delta T_1 K_1\,(Mr)\,r}{M K_0\,(Mb)} \tag{6-73}$$

Equation (D-50) with $n = 1$ was used to evaluate the integral of the modified Bessel function of the second kind.

According to eq. (6-25), in order that the radial displacement remain finite as $r \to \infty$, we must take $C_3 = 0$. The general displacement and stress expressions for the region outside the hot spot ($r > b$) are

$$u = -\frac{(1+\mu)\,\alpha \Delta T_1 K_1\,(Mr)}{M K_0\,(Mb)} + \frac{C_4}{r} \tag{6-74a}$$

$$N_r = \frac{\alpha E h \Delta T_1 K_1\,(Mr)}{M r K_0\,(Mb)} - \frac{E h C_4}{(1+\mu)\,r^2} \tag{6-74b}$$

$$N_\theta = -\frac{\alpha E h \Delta T_1}{K_0\,(Mb)} \left[\frac{K_1\,(Mr)}{Mr} + K_0\,(Mr)\right] + \frac{E h C_4}{(1+\mu)\,r^2} \tag{6-74c}$$

The constants of integration may be determined from the conditions at the edge of the hot spot that the displacement is continuous $u(b-) = u(b+)$ and that

the radial stress resultant is continuous $N_r(b-) = N_r(b+)$. The resulting values for the constants are

$$C_1 = 0 \quad \text{and} \quad C_4 = \frac{1}{2}(1+\mu)\,\alpha\Delta T_1 b^2 \left[1 + \frac{2K_1(Mb)}{Mb\,K_0(Mb)}\right] \tag{6-75}$$

If we make these substitutions into eqs. (6-72) and (6-74), we obtain the following expressions for the radial displacement and the stress distribution:

For $0 \le r \le b$:

$$\frac{2u}{(1+\mu)\,\alpha\Delta T_1 b} = \frac{r}{b} \tag{6-76a}$$

$$\frac{2\sigma_r}{\alpha E\Delta T_1} = \frac{2\sigma_\theta}{\alpha E\Delta T_1} = -1 \tag{6-76b}$$

For $b \le r$:

$$\frac{2u}{(1+\mu)\,\alpha\Delta T_1 b} = \left[1 + \frac{2K_1(Mb) - 2(r/b)K_1(Mr)}{MbK_0(Mb)}\right]\left(\frac{b}{r}\right) \tag{6-77a}$$

$$\frac{2\sigma_r}{\alpha E\Delta T_1} = -\left(\frac{b}{r}\right)^2 - \left(\frac{2b}{r}\right)\frac{[(b/r)K_1(Mb) - K_1(Mr)]}{MbK_0(Mb)} \tag{6-77b}$$

$$\frac{2\sigma_\theta}{\alpha E\Delta T_1} = \left(\frac{b}{r}\right)^2 + \left(\frac{2b}{r}\right)\frac{[(b/r)K_1(Mb) - K_1(Mr)]}{MbK_0(Mb)} - \frac{2K_0(Mr)}{K_0(Mb)} \tag{6-77c}$$

Example 6-2 A 304 stainless steel disk having a diameter of 300 mm (11.81 in.) and thickness of 4.5 mm (0.177 in.) is heated at the center over a diameter of 25 mm (0.984 in.) to a temperature of 125°C (257°F). The surface of the disk outside the hot spot is exposed to air at 25°C (77°F), with a convective heat transfer coefficient of 360 W/m^2-°C (63.4 Btu/hr-ft^2-°F) on both sides of the plate. Mechanical properties of 304 stainless steel are as follows: thermal expansion coefficient, 16.6×10^{-6} °C^{-1} (9.22×10^{-6} °F^{-1}); Young's modulus, 190 GPa (27.6×10^6 psi); thermal conductivity, 16 W/m-°C (9.24 Btu/hr-ft-°F). Determine the temperature and thermal stresses at a radial distance of 25 mm (0.984 in.) from the center of the disk.

The fin parameter M may be found by using eq. (6-60):

$$M = \sqrt{\frac{2h_C}{k_t h}} = \sqrt{\frac{(2)(360)}{(16)(0.0045)}} = 100 \text{ m}^{-1}$$

Then,

$$Ma = (100)(0.150) = 15.0 \quad \text{and} \quad Ma > 5$$
$$Mb = (100)(0.0125) = 1.250$$
$$Mr = (100)(0.025) = 2.50$$

The required modified Bessel functions of the second kind may be found in Table D-2:

$$K_0(Mb) = K_0(1.25) = 0.29774 \quad K_1(Mb) = K_1(1.25) = 0.40252$$

$$K_0(Mr) = K_0(2.50) = 0.06235 \quad K_1(Mr) = K_0(2.50) = 0.07389$$

The temperature at $r = 25$ mm (outside the hot spot) is found from eq. (6-70):

$$\Delta T(r) = T(r) - T_0 = \Delta T_1 \frac{K_0(Mr)}{K_0(Mb)} = (125^\circ - 25^\circ)\frac{0.06235}{0.29774} = (100^\circ)(0.20941)$$

$$T(r) = 25^\circ + 20.9^\circ = 45.9^\circ\text{C} \qquad (114.6^\circ\text{F})$$

The radial stress at a radial distance of 25 mm from the disk center is found from eq. (6-77b):

$$\frac{2\sigma_r}{\alpha E \Delta T_1} = -\left(\frac{12.5}{25}\right)^2 - \left(\frac{25}{25}\right)\frac{[(0.500)(0.40252) - 0.07389]}{(1.250)(0.29774)} = -0.250 - 0.3422$$

$$\frac{2\sigma_r}{\alpha E \Delta T_1} = -0.5922$$

$$\sigma_r = \tfrac{1}{2}(16.6 \times 10^{-6})(190 \times 10^9)(125^\circ - 25^\circ)(-0.5922)$$

$$\sigma_r = (157.7 \times 10^6)(-0.5922) = -93.39 \times 10^6 \text{ Pa}$$
$$= -93.39 \text{ MPa} \quad (-13{,}545 \text{ psi})$$

The circumferential stress at a radial distance of 25 mm from the disk center is found from eq. (6-77c):

$$\frac{2\sigma_\theta}{\alpha E \Delta T_1} = +(0.500)^2 + 0.3422 - \frac{(2)(0.06235)}{0.29774} = 0.5922 - 0.4188 = +0.1734$$

$$\sigma_\theta = (157.7 \times 10^6)(+0.1734) = +27.34 \times 10^6 \text{ Pa} = +27.34 \text{ MPa (3965 psi)}$$

The yield strength of 304 stainless steel is $S_y = 210$ MPa (30,460 psi), so the imposed thermal stresses are within satisfactory limits.

6.4 TWO-DIMENSIONAL PROBLEMS

As one would anticipate, two-dimensional thermal stress problems are much more difficult to analyze (model) than one-dimensional ones. In fact, many practical two-dimensional problems can be solved only by numerical techniques, such as the finite element methods. In the following material, we consider two-dimensional problems for the case of plane stress, as discussed in Section 6.1.

6.4.1 Stress–Strain Relationships

For the condition of plane stress ($\sigma_z = \gamma_{zx} = \gamma_{zy} = 0$), the strain components in Cartesian coordinates may be written as follows:

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \mu\sigma_y) + \alpha\Delta T \tag{6-78a}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \mu\sigma_x) + \alpha\Delta T \tag{6-78b}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \tag{6-78c}$$

Let us define the following stress resultants (force per unit length):

$$N_x = \int_{-h/2}^{+h/2} \sigma_x \, dz \tag{6-79a}$$

$$N_y = \int_{-h/2}^{+h/2} \sigma_y \, dz \tag{6-79b}$$

$$N_{xy} = \int_{-h/2}^{+h/2} \tau_{xy} \, dz \tag{6-79c}$$

$$N_T = \int_{-h/2}^{+h/2} \alpha E \Delta T \, dz = \alpha E h \Delta T_m \tag{6-79d}$$

where ΔT_m is the mean temperature change across the thickness h of the plate. The stress resultants are illustrated in Figure 6-5.

If we integrate eqs. (6-78) across the thickness of the plate, the stress–strain relations may be written in terms of the stress resultants:

$$\overline{\varepsilon}_x = \frac{1}{Eh}(N_x - \mu N_y + N_T) \tag{6-80a}$$

$$\overline{\varepsilon}_y = \frac{1}{Eh}(N_y - \mu N_x + N_T) \tag{6-80b}$$

$$\overline{\gamma}_{xy} = \frac{1}{Gh} N_{xy} = \frac{2(1+\mu)}{Eh} N_{xy} \tag{6-80c}$$

The strains (averaged across the plate thickness) are related to the displacements u and v (the x- and y-components of the displacement, respectively) as follows:

$$\overline{\varepsilon}_x = \frac{\partial u}{\partial x} \tag{6-81a}$$

$$\overline{\varepsilon}_y = \frac{\partial v}{\partial x} \tag{6-81b}$$

$$\overline{\gamma}_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{6-81c}$$

If we solve for the stress resultants from eqs. (6-80), the following relationships are obtained for plane stress:

$$N_x = \frac{Eh}{1-\mu^2}(\overline{\varepsilon}_x + \mu\overline{\varepsilon}_y) - \frac{1+\mu}{1-\mu^2}N_T = K(\overline{\varepsilon}_x + \mu\overline{\varepsilon}_y) - \frac{N_T}{1-\mu} \tag{6-82a}$$

$$N_y = \frac{Eh}{1-\mu^2}(\overline{\varepsilon}_y + \mu\overline{\varepsilon}_x) - \frac{1+\mu}{1-\mu^2}N_T = K(\overline{\varepsilon}_y + \mu\overline{\varepsilon}_x) - \frac{N_T}{1-\mu} \tag{6-82b}$$

$$N_{xy} = Gh\overline{\gamma}_{xy} = \frac{Eh}{2(1+\mu)}\overline{\gamma}_{xy} = \frac{1}{2}(1-\mu)K\overline{\gamma}_{xy} \tag{6-82c}$$

The quantity K is called the *extensional rigidity*.

$$K = \frac{Eh}{1-\mu^2} \tag{6-83}$$

6.4.2 Stress Equilibrium Relations

If we sum forces in the x-direction, as shown in Figure 6-5, the following is obtained:

$$-N_x\,dy + \left(N_x + \frac{\partial N_x}{\partial x}\,dx\right)dy - N_{xy}\,dx + \left(N_{xy} + \frac{\partial N_{xy}}{\partial y}\,dy\right)dx = 0 \tag{6-84}$$

The stress–equilibrium relationship, eq. (6-84), may be simplified as follows:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \tag{6-85a}$$

A similar expression is obtained from a force balance in the y-direction:

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \tag{6-85b}$$

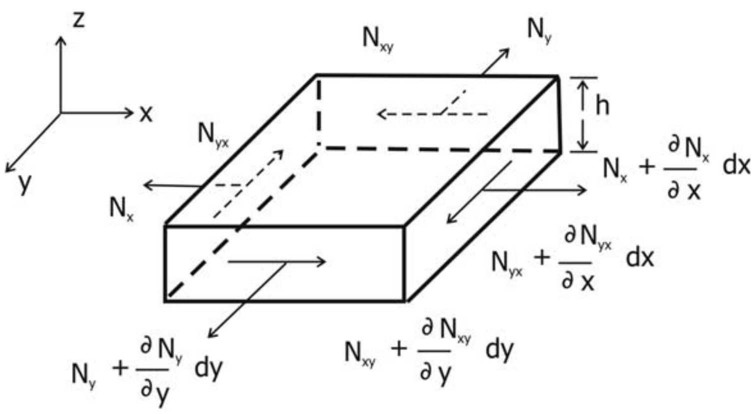

Figure 6-5. Stress resultants in two-dimensional Cartesian coordinates.

6.4.3 Displacement Formulation

There are a total of 8 unknowns (u, v, $\overline{\varepsilon_x}, \overline{e_y}, \overline{\gamma_{xy}}$, N_x, N_y, and N_{xy}) and there are 8 equations relating these unknowns, eqs. (6-80a, 6-80b, 6-80c, 6-82a, 6-82b, 6-82c, 6-85a, and 6-85b); therefore, the problem is defined.

If we substitute the expressions for the strain from eqs. (6-81) into the stress–strain relations, eqs. (6-82), the following is obtained:

$$N_x = K\left(\frac{\partial u}{\partial x} + \mu\frac{\partial v}{\partial y}\right) - \frac{N_T}{1-\mu} \tag{6-86a}$$

$$N_y = K\left(\frac{\partial v}{\partial y} + \mu\frac{\partial u}{\partial x}\right) - \frac{N_T}{1-\mu} \tag{6-86b}$$

$$N_{xy} = \frac{1}{2}(1-\mu)K\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{6-86c}$$

If we substitute the stress resultant expressions, eqs. (6-86), into the stress–equilibrium relations, eqs. (6-85), the following relationship are obtained:

$$K\left(\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 v}{\partial x \partial y}\right) - \frac{1}{1-\mu}\frac{\partial N_T}{\partial x} + \frac{1}{2}(1-\mu)K\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) = 0 \tag{6-87a}$$

$$K\left(\frac{\partial^2 v}{\partial y^2} + \mu\frac{\partial^2 u}{\partial x \partial y}\right) - \frac{1}{1-\mu}\frac{\partial N_T}{\partial y} + \frac{1}{2}(1-\mu)K\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y}\right) = 0 \tag{6-87b}$$

Let us introduce the *Goodier displacement function* (thermoelastic potential function) Φ, defined by the following [Goodier, 1937; Timoshenko and Goodier, 1970]:

$$u = \frac{\partial \Phi}{\partial x} \tag{6-88a}$$

$$v = \frac{\partial \Phi}{\partial y} \tag{6-88b}$$

If we make the substitutions from eqs. (6-88) into eqs. (6-87), the following expressions are obtained:

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) = \frac{1}{(1-\mu)K}\frac{\partial N_T}{\partial x} \tag{6-89a}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) = \frac{1}{(1-\mu)K}\frac{\partial N_T}{\partial y} \tag{6-89b}$$

The *Laplacian operator* in two-dimensional Cartesian coordinates is given by

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \tag{6-90}$$

If we differentiate eq. (6-89a) with respect to x and differentiate eq. (6-89b) with respect to y, and add the resulting expressions, the following relationship is obtained:

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right)=\frac{1}{(1-\mu)K}\left(\frac{\partial^2 N_T}{\partial x^2}+\frac{\partial^2 N_T}{\partial y^2}\right) \tag{6-91}$$

This expression may be written in compact form in terms of the Laplacian operator:

$$\nabla^2(\nabla^2\Phi)=\frac{1}{(1-\mu)K}\nabla^2 N_T \tag{6-92}$$

Equation (6-92) may be written in alternate form by introducing the biharmonic operator, defined by

$$\nabla^4\Phi=\nabla^2(\nabla^2\Phi) \tag{6-93}$$

Using eq. (6-83) for the extensional rigidity K, eq. (6-92) may be written as

$$\nabla^4\Phi-\left(\frac{1+\mu}{Eh}\right)\nabla^2 N_T=0 \tag{6-94}$$

For steady-state two-dimensional temperature distributions with no "heat generation," the mean temperature obeys the temperature field equation, given by

$$\nabla^2\theta_m=0 \quad \text{or} \quad \nabla^2 N_T=0 \tag{6-95}$$

For steady-state thermal stress problems, the governing equation from the displacement formulation is

$$\nabla^4\Phi=0 \tag{6-96}$$

The solution for the stress distribution for two-dimensional thermal stress problems (plane stress) is deceptively simple. First, eq. (6-96) is solved for the thermoelastic potential Φ, including the boundary conditions for the problem. Second, the displacements are found from eqs. (6-88). Third, the stress resultants (and, hence, the average stresses) may be found by using eqs. (6-86). That first step is often the *really* difficult step, however, particularly for systems with irregular shape or nonuniform boundary conditions.

6.4.4 Stress Formulation

An alternate formulation for the thermal stress problem for plane stress conditions may be developed in terms of the *Airy stress function* [Airy, 1862; Noda et al., 2003]. Beginning with the first strain compatibility relationship, eq. (5-18a),

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\,\partial y} \tag{6-97}$$

we may substitute for the strains in terms of the stress resultants from eqs. (6-80) to obtain

$$\frac{\partial^2 N_x}{\partial y^2} + \frac{\partial^2 N_y}{\partial x^2} - \mu\left(\frac{\partial^2 N_x}{\partial x^2} + \frac{\partial^2 N_y}{\partial y^2}\right) + \left(\frac{\partial^2 N_T}{\partial x^2} + \frac{\partial^2 N_T}{\partial y^2}\right) = 2(1+\mu)\frac{\partial^2 N_{xy}}{\partial x\,\partial y} \tag{6-98}$$

For steady-state plane stress conditions ($\sigma_z = \tau_{xz} = \tau_{yz} = 0$) with no body forces, the stress equilibrium relations, eqs. (5-28), reduce to the following, in terms of the stress resultants:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \tag{6-99a}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \tag{6-99b}$$

If we differentiate both sides of eq. (6-99a) with respect to x and differentiate both sides of eq. (6-99b) with respect to y, the following relationships are obtained:

$$\frac{\partial^2 N_x}{\partial x^2} = -\frac{\partial^2 N_{xy}}{\partial x \partial y} \tag{6-100a}$$

$$\frac{\partial^2 N_y}{\partial y^2} = -\frac{\partial^2 N_{xy}}{\partial x\,\partial y} = +\frac{\partial^2 N_x}{\partial x^2} \tag{6-100b}$$

If we make the substitutions from eqs. (6-100) into eq. (6-98), the following result is obtained:

$$\frac{\partial^2 N_x}{\partial y^2} + \frac{\partial^2 N_y}{\partial x^2} + \left(\frac{\partial^2 N_T}{\partial x^2} + \frac{\partial^2 N_T}{\partial y^2}\right) = -\frac{\partial^2 N_x}{\partial x^2} - \frac{\partial^2 N_y}{\partial y^2} \tag{6-101}$$

This expression may be written more compactly using the Laplacian operator:

$$\nabla^2 N_x + \nabla^2 N_y + \nabla^2 N_T = \nabla^2(N_x + N_y + N_T) = 0 \tag{6-102}$$

The Airy stress function Ψ is defined as follows:

$$N_x = \frac{\partial^2 \Psi}{\partial y^2} \tag{6-103a}$$

$$N_y = \frac{\partial^2 \Psi}{\partial x^2} \tag{6-103b}$$

$$N_{xy} = -\frac{\partial^2 \Psi}{\partial x\,\partial y} \tag{6-103c}$$

It is noted that the Airy stress function identically satisfies the stress equilibrium relations, eqs. (6-99). If we substitute the expressions from eqs. (6-103) into the governing equation, eq. (6-102), the following is obtained:

$$\nabla^2\left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + N_T\right) = 0 \tag{6-104}$$

Introducing the Laplacian operator, we may write eq. (6-104) as

$$\nabla^2(\nabla^2\Psi + N_T) = 0 \tag{6-105}$$

Using the biharmonic operator, the following is obtained:

$$\nabla^4\Psi + \nabla^2 N_T = 0 \tag{6-106}$$

For steady-state two-dimensional problems with no "heat generation," the second term in eq. (6-106), $\nabla^2 N_T = 0$, and the expression reduces to

$$\nabla^4\Psi = 0 \tag{6-107}$$

6.4.5 Governing Equations in Cylindrical Coordinates

The governing equations (strain–displacement relations, force balance, stress–strain relations, etc.) may be expressed in two-dimensional cylindrical coordinates by making a coordinate transformation or by using "first principles." The two-dimensional (plane stress) strain–displacement relations in cylindrical coordinates are as follows.

Radial strain: $$\varepsilon_r = \frac{\partial u}{\partial r} \tag{6-108a}$$

Circumferential strain : $$\varepsilon_\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} \tag{6-108b}$$

Shear strain: $$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \tag{6-108c}$$

The quantities u and v are the displacements in the radial and circumferential directions, respectively.

The stress–strain relationships are given by eqs. (6-8). The shear stress resultant $N_{r\theta}$ is related to the shear strain by

$$N_{r\theta} = Gh\gamma_{r\theta} = \frac{Eh}{2(1+\mu)}\gamma_{r\theta} = \frac{1}{2}(1-\mu)K\gamma_{r\theta} \tag{6-109}$$

The stress equilibrium relations (radial and circumferential force balances) may be found from Figure 6-6. For the radial force balance, we have

$$\left(N_r + \frac{\partial N_r}{\partial r}dr\right)(r+dr)d\theta - N_r r\,d\theta + \left(N_{r\theta} + \frac{\partial N_{r\theta}}{\partial \theta}\,d\theta\right)\cos\left(\frac{1}{2}d\theta\right)dr$$
$$-N_{r\theta}\cos\frac{1}{2}d\theta\,dr - \left(N_\theta + \frac{\partial N_\theta}{\partial \theta}d\theta\right)\sin\left(\frac{1}{2}d\theta\right)dr - N_\theta\sin\left(\frac{1}{2}d\theta\right)dr = 0 \tag{6-110}$$

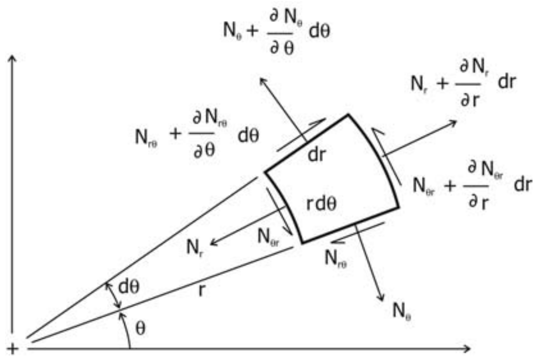

Figure 6-6. Stress resultants for two-dimensional cylindrical coordinates.

If we note that (for very small angles), $\cos\left(\frac{1}{2}d\theta\right) \approx 1$ and $\sin\left(\frac{1}{2}d\theta\right) \approx \frac{1}{2}d\theta$, eq. (6-110) may be simplified:

$$N_r\, dr\, d\theta + \frac{\partial N_r}{\partial r} r\, dr\, d\theta + \frac{\partial N_r}{\partial r} dr\, dr\, d\theta + \frac{\partial N_{r\theta}}{\partial \theta} dr\, d\theta - \frac{\partial N_\theta}{\partial \theta} dr\, d\theta\, d\theta - N_\theta dr\, d\theta = 0 \tag{6-111}$$

If we divide each term in eq. (6-111) by $(dr\, d\theta)$, we obtain

$$\frac{\partial N_r}{\partial r} + \frac{1}{r}\frac{\partial N_{r\theta}}{\partial \theta} + \frac{(N_r - N_\theta)}{r} + \frac{\partial N_r}{\partial r}\frac{dr}{r} - \frac{\partial N_\theta}{\partial \theta}\frac{d\theta}{r} = 0 \tag{6-112}$$

Because the differentials dr are $d\theta$ and infinitesimally small, the last two terms in eq. (6-112) are negligibly small, and the radial force balance equation may be written as follows:

$$\frac{\partial N_r}{\partial r} + \frac{1}{r}\frac{\partial N_{r\theta}}{\partial \theta} + \frac{N_r - N_\theta}{r} = 0 \tag{6-113a}$$

By making a force balance in the circumferential direction for the element shown in Figure 6-6 and simplifying in the same manner as for the radial force balance, the following expression is obtained:

$$\frac{1}{r}\frac{\partial N_\theta}{\partial \theta} + \frac{\partial N_{r\theta}}{\partial r} + \frac{2N_{r\theta}}{r} = 0 \tag{6-113b}$$

The final expressions from the displacement formulation, eq. (6-96), and the stress formulation, eq. (6-107), are valid in cylindrical coordinates. The Laplacian operator in cylindrical coordinates may be written as

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} \tag{6-114}$$

The Goodier displacement function Φ in cylindrical coordinates is

$$u = \frac{\partial \Phi}{\partial r} \tag{6-115a}$$

$$v = \frac{1}{r}\frac{\partial \Phi}{\partial \theta} \tag{6-115b}$$

Similarly, the Airy stress function Ψ in cylindrical coordinates is

$$N_r = \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} \tag{6-116a}$$

$$N_\theta = \frac{\partial^2 \Psi}{\partial r^2} \tag{6-116b}$$

$$N_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi}{\partial \theta}\right) \tag{6-116c}$$

6.5 PLATE WITH A CIRCULAR HOLE

As an example of two-dimensional thermal stress problems, let us consider the case of a large plate containing a hole of radius b, as shown in Figure 6-7. At distances far from the hole, there is a uniform heat flux, such that the temperature distribution far from the hole is linear with a constant temperature gradient, g_T. The temperature at any x-coordinate along the line $y = 0$ is $T_0 =$ constant.

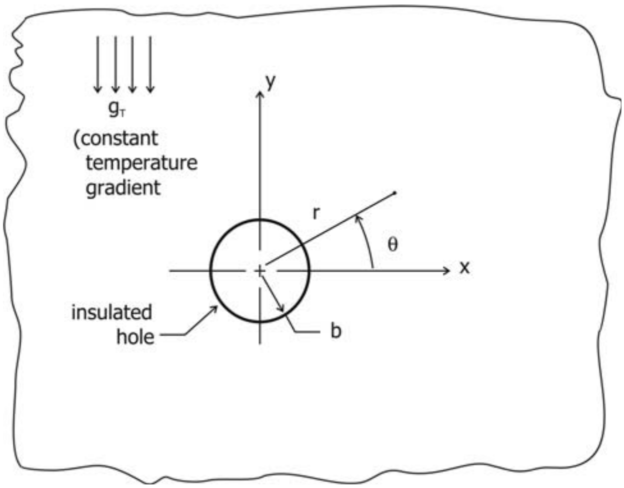

Figure 6-7. Large plate with a circular hole and a linear temperature distribution at large distances from the hole.

6.5.1 Temperature Distribution

The general temperature field equation in cylindrical coordinates is given by eq. (5-64). For two-dimensional steady-state heat transfer with no heat generation, eq. (5-64) reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0 \tag{6-117}$$

The linear temperature distribution introduces no thermal stresses, so we may introduce the following temperature variable:

$$\Delta T = T(x, y) - T_0 - g_T y = T(r, \theta) - T_0 - g_T r \sin\theta \tag{6-118}$$

By direct substitution, we may show that the following applies:

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{\partial}{\partial r}\left(r\frac{\partial \Delta T}{\partial r}\right) + g_T \sin\theta \tag{6-119}$$

$$\frac{\partial^2 T}{\partial \theta^2} = \frac{\partial^2 \Delta T}{\partial \theta^2} - g_T r \sin\theta \tag{6-120}$$

If we make the substitutions from eqs. (6-119) and (6-120) into the temperature field equation, eq. (6-117), we obtain:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Delta T}{\partial r}\right) + \frac{\partial^2 \Delta T}{\partial \theta^2} = 0 \tag{6-121}$$

We have the following boundary conditions on the temperature distribution.

(a) At distances far from the hole (as $r \to \infty$), the temperature distribution is linear:

$$T(\infty, y) = T_0 + g_T y = T(\infty, \theta) = T_0 + g_T r \sin\theta$$

Therefore,

$$\Delta T(\infty, \theta) = 0$$

(b) At the surface of the hole ($r = b$), the heat transfer rate is zero. From the Fourier rate equation, this condition means that the following applies:

$$\left(\frac{\partial T}{\partial r}\right)_{r=b} = 0 = \left(\frac{\partial \Delta T}{\partial r}\right)_{r=b} + g_T \sin\theta$$

(c) At the condition $y = 0$ or $\theta = 0$, the temperature is constant,

$$T(r,\ 0) = T_0 \quad \text{or} \quad \Delta T(r,\ 0) = 0$$

To solve eq. (6-121) for the temperature variable, let us try a product solution of the following form:

$$\Delta T(r, \theta) = R(r)\Theta(\theta) \tag{6-122}$$

where $R(r)$ is a function of the coordinate r alone, and $\Theta(\theta)$ is a function of the coordinate θ alone. If we make the substitution from eq. (6-122) directly into eq. (6-121) and divide through by the product $R\Theta$, we obtain

$$\frac{r}{R}\frac{d(rR')}{dr} = -\frac{\Theta''}{\Theta} = \text{constant} = -n^2 \tag{6-123}$$

The two parts of eq. (6-123) must be equal to a constant, because a constant is the only function that can "depend on" both r and θ at the same time. The constant must be negative in order that the solution be a trigonometric function to make the temperature at $\theta = 0$ be the same as that at $\theta = 2\pi$.

We may separate variables in the temperature field equation, eq. (6-121), to obtain the following ordinary differential equations from eq. (6-123):

$$r\frac{d(rR')}{dr} - n^2 R = 0 \tag{6-124a}$$

$$\Theta'' + n^2\Theta = 0 \tag{6-124b}$$

The solution of eq. (6-124b) is

$$\Theta(\theta) = C_1 \sin(n\theta) + C_2 \cos(n\theta) \tag{6-125}$$

From the boundary condition, $\Delta T(r, 0) = 0$ for all r, or $\Theta(0) = 0$, we must have $C_2 = 0$.

Equation (6-124a) may be written in the following form:

$$r^2 R'' + rR' - n^2 R = 0 \tag{6-126}$$

The solution of eq. (6-126) is

$$R(r) = C_3 r^{-n} + C_4 r^n \tag{6-127}$$

From the boundary condition that $\Delta T(\infty, \theta) = 0$ or $R(\infty) = 0$, we must have $C_4 = 0$.

Setting the constants C_2 and C_4 equal to zero, the following expression is obtained for the temperature function:

$$\Delta T(r, \theta) = R(r)\Theta(\theta) = Cr^{-n} \sin(n\theta) \tag{6-128}$$

where $C = C_1 C_3$. From the zero-heat transfer boundary condition at the surface of the hole, we may evaluate the following:

$$\left(\frac{\partial \Delta T}{\partial r}\right)_{r=b} = -g_T \sin\theta = -Cnr^{-n-1} \sin n\theta\ ]_{r=b} = -Cnb^{-n-1} \sin n\theta$$

By comparing the sin terms, we see that $n = 1$. Using $n = 1$, we must have the following:

$$g_T = Cb^{-2} \quad \text{or} \quad C = g_T b^2$$

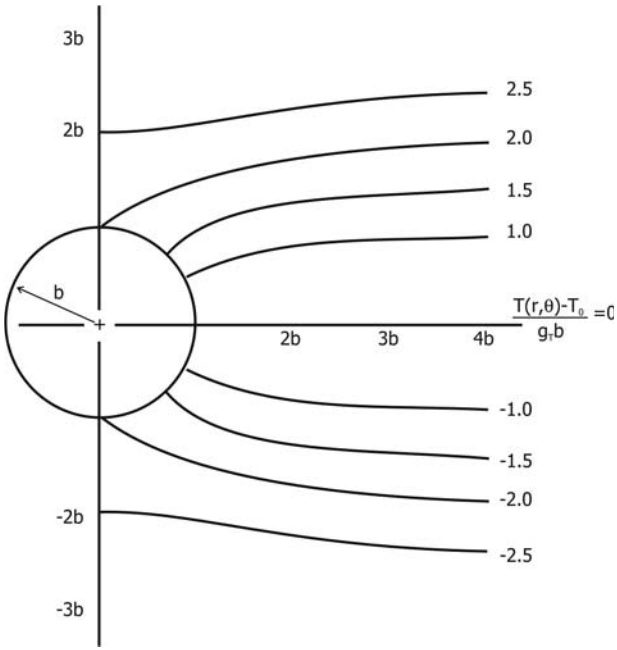

Figure 6-8. Temperature distribution in a large plate with a circular hole.

Making the substitutions for the constants into eq. (6-128), the final form for the temperature distribution is obtained:

$$\Delta T(r,\theta) = \left(\frac{g_T b^2}{r}\right)\sin\theta \tag{6-129}$$

The temperature distribution is found from eq. (6-118):

$$T(r,\theta) = T_0 + g_T b\left(\frac{r}{b} + \frac{b}{r}\right)\sin\theta \tag{6-130}$$

Isotherms for this temperature distribution are illustrated in Figure 6-8.

The thermal stress resultant is

$$N_T = \alpha E h \Delta T = \left(\frac{\alpha E h g_T b^2}{r}\right)\sin\theta \tag{6-131}$$

6.5.2 Evaluation of the Stress Function

At distances far from the hole, the stresses are known (all are zero), so let us use the stress formulation in the solution of this problem. Equation (6-105) may be written in the following form:

$$\nabla^2\Gamma(r,\theta) = 0 \tag{6-132a}$$

$$\Gamma(r,\theta) = \nabla^2\Psi(r,\theta) + N_T \tag{6-132b}$$

Eqn. (6-132a) may be written in the expanded form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Gamma}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Gamma}{\partial\theta^2}=0 \tag{6-133}$$

Let us try a solution of the following form:

$$\Gamma(r,\theta)=\rho(r)\sin\theta \tag{6-134}$$

If we make the substitution from eq. (6-134) into eq. (6-133), the following total differential equation is obtained:

$$r\frac{d}{dr}\left(r\frac{d\rho}{dr}\right)-\rho=0 \tag{6-135}$$

If we make the change of variable, $r=e^t$ or $t=\ln\, r$, eq. (6-133) may be written in the following form:

$$\frac{d^2\rho}{dr^2}-\rho=0 \tag{6-136}$$

The solution of eq. (6-136) is

$$\rho(r)=C_5e^t+C_6e^{-t}=C_5r+\frac{C_6}{r} \tag{6-137}$$

The complete solution for the function $\Gamma(r,\theta)$ is

$$\Gamma(r,\theta)=\left(C_5r+\frac{C_6}{r}\right)\sin\theta \tag{6-138}$$

where C_5 and C_6 are constants of integration to be determined from the boundary conditions on the stresses.

If we make the substitutions from eqs. (6-131) and (6-138) into eqn. (6-132b) and expand the Laplacian operator, we get

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}+\frac{\alpha Ehg_Tb^2}{r}\sin\theta=\left(C_5r+\frac{C_6}{r}\right)\sin\theta \tag{6-139}$$

The final solution will involve another $(r^{-1}\sin\theta)$ function, so we may set $C_6=0$, with no loss of generality.

Suppose we again try a solution of the following form:

$$\Psi(r,\theta)=f(r)\sin\theta \tag{6-140}$$

If we make the substitution from eq. (6-140) into eq. (6-139), the following total differential equation is obtained:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)-\frac{f}{r^2}+\frac{B}{r}=C_5 \tag{6-141}$$

where, for "shorthand," we have defined the constant B as

$$B \equiv \alpha E h g_T b^2 \tag{6-142}$$

If we again make the change of variable, $r = e^t$ or $t = \ln\ r$, eq. (6-141) may be written in the following form:

$$\frac{d^2 f}{dt^2} - f = C_5 e^{3t} - B e^t \tag{6-143}$$

The solution for eq. (6-143) may be obtained in two parts: f_{hom}, the solution of the homogeneous equation (with the right side of eq. (6-143) equal to zero), and f_{part}, a particular solution of eq. (6-143), which involves exponential functions, because the right side of eq. (6-143) involves exponential functions. These two parts are

$$f_{\text{hom}} = C_7 e^t + C_8 e^{-t} \tag{6-144a}$$

$$f_{\text{part}} = \tfrac{1}{8} C_5 e^{3t} - \tfrac{1}{2} B(2 + t) e^t \tag{6-144b}$$

The complete solution for eq. (6-143) is

$$f(t) = f_{\text{hom}} + f_{\text{part}} = C_7 e^t + C_8 e^{-t} + \tfrac{1}{8} C_5 e^{3t} - \tfrac{1}{2} B t e^t \tag{6-145}$$

In terms of the radial coordinate r, the function $f(r)$ is

$$f(r) = C_7 r + \frac{C_8}{r} + \frac{1}{8} C_5 r^3 - \tfrac{1}{2} B r \ln r \tag{6-146}$$

Finally, the Airy stress function may be written

$$\Psi(r, \theta) = \left(C_7 r + \frac{C_8}{r} + \frac{1}{8} C_5 r^3 - \frac{1}{2} \alpha E h g_T b^2 r \ln r \right) \sin\theta \tag{6-147}$$

6.5.3 Evaluation of the Thermal Stresses

The shear stress resultant may be evaluated from eq. (6-116c).

$$N_{r\theta} = \left(\frac{2C_8}{r^3} - \tfrac{1}{4} C_5 r + \frac{\alpha E h g_T b^2}{2r} \right) \cos\theta \tag{6-148}$$

At distances far from the hole (for $r \to \infty$), the shear stress is zero; therefore, we have that $C_5 = 0$. At the edge of the hole (at $r = b$), there are no applied external forces, so the shear stress is also zero at this point.

$$N_{r\theta}(b, \theta) = 0 = \left(\frac{2C_8}{b^3} + \frac{1}{2} \alpha E h g_T b \right) \cos\theta \tag{6-149}$$

The constant C_8 is

$$C_8 = -\tfrac{1}{4}\alpha E h g_T b^4 \tag{6-150}$$

The final expression for the shear stress resultant is found by substituting the value of C_8 from eq. (6-150) into eq. (6-148):

$$N_{r\theta} = \frac{1}{2}\alpha E h g_T b\left(\frac{b}{r} - \frac{b^3}{r^3}\right)\cos\theta \tag{6-151}$$

The shear stress is

$$\tau_{r\theta} = \frac{N_{r\theta}}{h} = \tfrac{1}{2}\alpha E g_T b\left(\frac{b}{r} - \frac{b^3}{r^3}\right)\cos\theta \tag{6-152}$$

At this point in the solution, the Airy stress function may be written as follows, using the value of C_8 from eq. (6-150):

$$\Psi(r,\theta) = \left[C_7 r - \tfrac{1}{4}\alpha E h g_T b^3\left(\frac{b}{r} + \frac{2r}{b}\ln r\right)\right]\sin\theta \tag{6-153}$$

The radial stress resultant may be evaluated from eqs. (6-116a) and (6-153):

$$N_r = -\tfrac{1}{2}\alpha E h g_T b\left(\frac{b}{r} - \frac{b^3}{r^3}\right)\sin\theta \tag{6-154}$$

The radial stress is

$$\sigma_r = \frac{N_r}{h} = -\tfrac{1}{2}\alpha E g_T b\left(\frac{b}{r} - \frac{b^3}{r^3}\right)\sin\theta \tag{6-155}$$

The circumferential stress resultant is found from eqs. (6-116b) and (6-153):

$$N_\theta = -\tfrac{1}{2}\alpha E h g_T b\left(\frac{b}{r} + \frac{b^3}{r^3}\right)\sin\theta \tag{6-156}$$

The circumferential stress is

$$\sigma_\theta = \frac{N_\theta}{h} = -\frac{1}{2}\alpha E g_T b\left(\frac{b}{r} + \frac{b^3}{r^3}\right)\sin\theta \tag{6-157}$$

A plot of the radial and circumferential stress distribution at $\theta = \frac{1}{2}\pi = 90^\circ$ is shown in Figure 6-9. Note that the shear stress at $\theta = 90^\circ$ is zero; however, the shear stress at $\theta = 0^\circ$ is the negative of the radial stress at $\theta = 90^\circ$.

The maximum circumferential stress occurs at the surface of the hole (at $r = b$) and at the angular position $\theta = \pm\frac{1}{2}\pi$.

$$(\sigma_\theta)_{\max} = \pm\alpha E g_T b \tag{6-158}$$

The point at which the maximum radial stress occurs may be found from

$$\frac{\partial\sigma_r}{\partial r} = 0 = -\frac{1}{2}\alpha E g_T b\left(-\frac{b}{r^2} + \frac{3b^3}{r^4}\right)\sin\theta$$

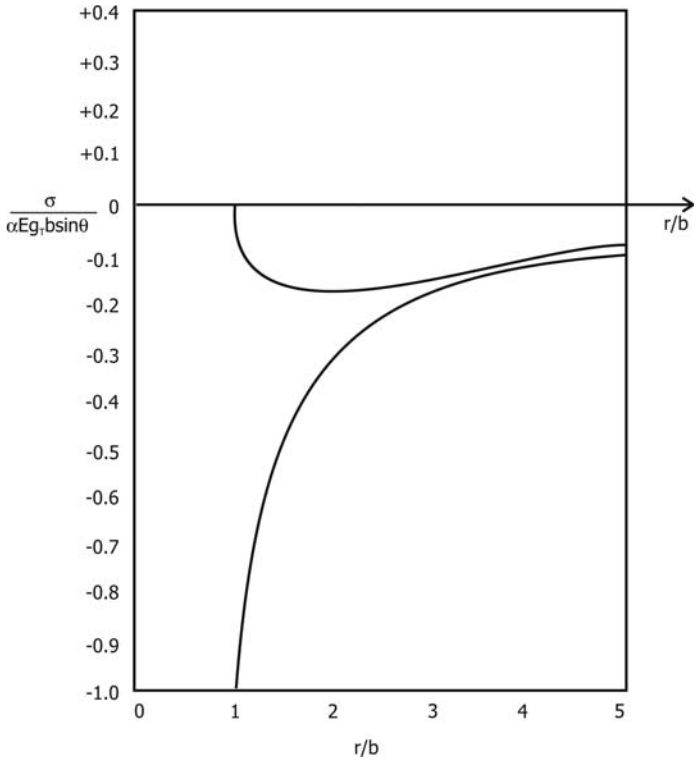

Figure 6-9. Radial and circumferential stress distribution in a large plate with a circular hole and a linear temperature distribution at large distances from the hole.

or

$$r_{\max} = \sqrt{3}b \quad \text{and} \quad \theta_{\max} = \pm\tfrac{1}{2}\pi$$

The maximum radial stress occurs at $r/b = \sqrt{3}$ and at $\theta = \pm\frac{1}{2}\pi$ and has the following value:

$$(\sigma_r)_{\max} = \pm\frac{\alpha E g_T b}{3\sqrt{3}} = \pm 0.19245\alpha E g_T b \tag{6-159}$$

Similarly, the maximum shear stress occurs at $r/b = \sqrt{3}$ and $\theta = 0$ or π and has the following value:

$$(\tau_{r\theta})_{\max} = \pm\frac{\alpha E g_T b}{3\sqrt{3}} \tag{6-160}$$

We might be interested to know the "practical" size of a plate for which the preceding solution for an "infinite" plate would apply. If we accept an error of $\pm 1\%$ of the maximum stress at the edge of the actual plate (where the stress is zero), we may write the following from eq. (6-157):

$$\frac{(\sigma_\theta)_{\theta=\pi/2}}{(\sigma_\theta)_{\max}} = 0.01 = \frac{1}{2}\left(\frac{b}{r} + \frac{b^3}{r^3}\right)$$

Solving for the radial coordinate, we obtain

$$\frac{r}{b} = \frac{1}{0.019992} \approx 50$$

Thus, if no dimension (length or width) of the actual plate is less than 50 times the hole diameter, the preceding stress expressions are applicable within $\pm 1\%$ error.

Example 6-3 A 304 stainless steel square plate has edge dimensions of 1.060 m (41.73 in.), and the plate has a circular hole in the center of the plate. The thickness of the plate is 9.5 mm (0.375 in.). One edge of the plate is maintained at 20°C (68°F) and the opposite edge is maintained at 830°C (1526°F), while the other two edges are adiabatic (no heat transfer). The top and bottom surfaces of the plate are thermally insulated. Mechanical properties of 304 stainless steel include: $\alpha = 16.6 \times 10^{-6}\ {}^\circ\mathrm{C}^{-1}$ ($9.22 \times 10^{-6}\ {}^\circ\mathrm{F}^{-1}$); $E = 190\,\mathrm{GPa}$ (27.6×10^6 psi); $k_t = 16$ W/m-°C (9.24 Btu/hr-ft-°F). The maximum stress in the plate is to be limited to 25 MPa (3626 psi). Determine the maximum allowable diameter of the hole such that this stress is not exceeded.

The temperature gradient is as follows:

$$g_T = -\frac{T_2 - T_1}{L} = -\frac{830^\circ - 20^\circ}{1.060} = -764.2^\circ\mathrm{C/m} \quad (-34.94^\circ\mathrm{F/in.})$$

The maximum circumferential stress is the largest of the maximum stresses. The maximum stress is tensile at the upper side of the hole ($\theta = +\frac{1}{2}\pi$) and compressive at the lower side of the hole ($\theta = -\frac{1}{2}\pi$). Using eqn. (6-158), we may determine the maximum hole radius b:

$$b = \frac{(\sigma_\theta)_{\max}}{\alpha E |g_T|} = \frac{25 \times 10^6}{(16.6 \times 10^{-6})(190 \times 10^9)(764.2)} = 0.01037\,\mathrm{m} = 10.37\,\mathrm{mm}$$

The maximum allowable hole diameter is

$$D_h = 2b = (2)(10.37) = 20.74\,\mathrm{mm} \quad (0.817\,\mathrm{in.})$$

We may check the applicability of eq. (6-158) for this case:

$$\frac{L}{D_h} = \frac{1.060}{0.02074} = 51.1 > 50 \quad (\text{solution } \textit{is} \text{ valid})$$

The midplate temperature T_0 may be found as follows:

$$T_0 = \tfrac{1}{2}(T_1 + T_2) = \tfrac{1}{2}(20^\circ + 830^\circ) = 425^\circ\mathrm{C} \quad (797^\circ\mathrm{F})$$

The temperature at the edge of the hole ($r = b$) is found from eq. (6-130):

$$T(b, \theta) = 425^\circ + (-764.2)(0.01037)(1 + 1)\sin\theta = 425^\circ - 15.8^\circ \sin\theta$$

At the upper side of the hole ($\theta = 90^\circ$), the temperature is

$$T(b, \tfrac{1}{2}\pi) = 425^\circ - 15.8^\circ = 409.2^\circ\text{C} \quad (768.6^\circ\text{F})$$

Similarly, at the lower side of the hole ($\theta = -90^\circ$), the temperature is

$$T(b, -\tfrac{1}{2}\pi) = 425^\circ + 15.8^\circ = 440.8^\circ\text{C} \quad (825.4^\circ\text{F})$$

6.6 HISTORICAL NOTE

Sir George Biddell Airy (Figure 6-10) was appointed as professor of mathematics at Cambridge University in 1826 [Timoshenko, 1983]. Although some mathematicians of his time looked on the subject of mathematics as being a "pure science," Airy was always interested in the application of mathematics in the solution of practical engineering problems. In a letter to George B. Stokes in 1868, he mentioned that some mathematical subjects were "pursued in the closet, without the effort of looking into the scientific world to see what is wanted there."

In 1828, Airy was elected to the position of professor of astronomy at Cambridge University. In 1835, he was appointed to the position of Astronomer Royal and moved to Greenwich. He continued his interest in applying mathematics to the solution of practical problems. In 1840, Airy drew up specifications for the tower clock at Westminster. His specifications required that the clock should be accurate to within one second a day, which was considered by the clockmakers

Figure 6-10. Sir George Biddell Airy.

of the day to be an almost impossible accuracy to achieve. However, based on these specifications, the clockwork of Big Ben was constructed and the clock has run accurately for more than a hundred years.

It was in a paper presented to the Royal Society in 1862 that Airy introduced the stress function that bears his name today as a part of the two-dimensional solution for stresses in bending of rectangular beams. In this paper, Airy expressed the stress function in the form of a polynomial, and he determined the values of the coefficients in the polynomial such that the boundary conditions were satisfied.

J. N. Goodier was professor of engineering mechanics and head of the department at Cornell University for almost a decade before he accepted an appointment as professor of engineering mechanics at Stanford University in 1947. He introduced and utilized his *thermoelastic potential* (displacement function) in many thermal stress problems in his technical publications while he was at Cornell. While he was at Stanford, he co-authored a classic book, *Theory of Elasticity*, with Steven P. Timoshenko.

PROBLEMS

6-1. In a circular plate of radius b and thickness h, the temperature distribution is given by

$$\Delta T = T - T_0 = \Delta T_1 \left(1 - \frac{r}{b}\right)$$

where ΔT_1 is the temperature difference at the center of the plate. The temperature is uniform across the plate thickness at any location r. The outer edge of the plate (at $r = b$) is rigidly fixed, such that the radial displacement u is zero at the outer edge. Using the displacement formulation,

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d(ru)}{dr}\right] = \frac{(1+\mu)}{Eh}\frac{dN_T}{dr}$$

determine the equations for the radial displacement u, the radial stress σ_r, and the circumferential stress σ_θ as a function of the radial coordinate r. For a plate constructed of C1020 carbon steel with a plate radius $b = 250$ mm (9.843 in.) and a center temperature difference $\Delta T_1 = 80^\circ$C (144°F), determine the location and numerical value of the maximum radial displacement, maximum radial stress, and maximum circumferential stress.

6-2. Suppose that the plate given in Problem 6-1 has the outer edge free, instead of being rigidly fixed, such that the radial stress σ_r is zero at the outer edge, $r = b$. Using the stress formulation,

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d(r^2 N_r)}{dr}\right] + \frac{dN_T}{dr} = 0$$

determine the equations for the radial displacement u, the radial stress σ_r, and the circumferential stress σ_θ as a function of the radial coordinate r. For a plate constructed of C1020 carbon steel with a plate radius $b = 250\,\text{mm}$ (9.843 in.) and a center temperature difference $\Delta T_1 = 80^\circ\text{C}$ (144°F), determine the location and numerical value of the maximum radial displacement, maximum radial stress, and maximum circumferential stress.

6-3. A circular plate of thickness h has a radius b and is subjected to the following temperature distribution:

$$\Delta T = T(r) - T_0 = (T_1 - T_0)\cos\left(\frac{\pi r}{2b}\right)$$

The temperature is uniform through the thickness of the plate. The outer edge of the plate has no force applied, so that the radial stress resultant is zero at this point, $N_r(r = b) = 0$. Determine (a) the expressions for the radial and circumferential stresses as a function of the radial coordinate r, (b) the expression for the radial displacement u as a function of r. (c) If the plate is constructed of 304 stainless steel, with $E = 193$ GPa (28.0×10^6 psi), $\alpha = 16.0 \times 10^{-6}\ ^\circ\text{C}^{-1}$ ($8.9 \times 10^{-6}\ ^\circ\text{F}^{-1}$), and $\mu = 0.305$, determine the numerical values for the maximum stress $\sigma_{\max}$ and the radial displacement at the outer edge ($r = b$) of the plate. The plate radius is 150 mm (5.91 in.), the stress-free temperature $T_0 = 25^\circ\text{C}$ (77°F), and the temperature at the center of the plate $T_1 = 100^\circ\text{C}$ (212°F).

6-4. A circular plate of thickness h and outer radius b has a hole of radius a in the center, as shown in Figure 6-11. An electrical current is passed through the plate such that there is an energy dissipation rate per unit volume of q_{gen}. The edge surface of the hole at $r = a$ is thermally insulated $(dT/dr)_{r=a} = 0$, and the outer edge of the plate at $r = b$ is maintained at the temperature T_0. The temperature field equation for this case is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{q_{\text{gen}}}{k_t} = 0$$

Determine (a) the expression for the temperature distribution, $T(r)$. (b) If the plate material is 600 Inconel, with thermal conductivity $k_t = 15$ W/m-°C (8.67 Btu/hr-ft-°F), a plate thickness of 25 mm (0.984 in.), an outer radius of 120 mm (4.724 in.), an inner radius of 60 mm (2.362 in.), and an energy dissipation rate per unit volume of $q_{\text{gen}} = 250\,\text{kW/m}^3$ (24,160 Btu/hr-ft^3), determine the numerical value of the plate temperature at the surface of the hole ($r = a$) if $\Delta T_0 = 25^\circ\text{C}$.

6-5. Suppose the temperature distribution for the plate given in Problem 6-4 is given by

$$\Delta T = T(r) - T_0 = \left(\frac{q_{\text{gen}} b^2}{4k_t}\right)\left[1 - \left(\frac{r}{b}\right)^2 + 2\left(\frac{a}{b}\right)^2 \ln\left(\frac{r}{b}\right)\right]$$

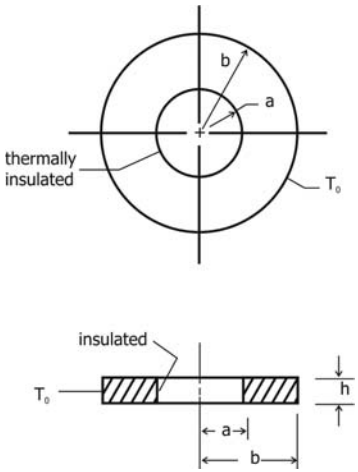

Figure 6-11. Sketch for Problem 6-4 and Problem 6-6.

The temperature is uniform across the thickness of the plate. The radial forces at both the inner edge and outer edge of the plate are zero, such that $N_r(a) = N_r(b) = 0$. Determine (a) the expressions for the radial stress σ_r and the circumferential stress σ_θ as a function of the radial coordinate r. (b) If the plate is constructed of 600 Inconel (properties given in Appendix B), with a plate thickness of 25 mm (0.984 in.), an outer radius of 120 mm (4.724 in.), an inner radius of 60 mm (2.362 in.), and an energy dissipation rate per unit volume of $q_{\text{gen}} = 250\,\text{kW/m}^3$ (24,160 Btu/hr-ft^3), determine the numerical value of the maximum stress in the plate. if $\Delta T_0 = 25^\circ$C.

6-6. A circular plate of thickness h and radius b, with a hole of radius a in the center, as shown in Figure 6-11, has the following temperature distribution imposed on the plate:

$$\Delta T = T - T_0 = \Delta T_1 \left(\frac{r}{b}\right)^2 \qquad \text{for } a \le r \le b$$

where ΔT_1 is the temperature difference $(T_1 - T_0)$ at the edge of the plate $(r = b)$. The temperature is uniform across the plate thickness at any location r. Both the inner edge and the outer edge of the plate have no external loads applied, such that the radial stress resultants $N_r(r = a)$ and $N_r(r = b)$ are both zero. Using the stress formulation, determine (a) the equations for the radial stress σ_r, the circumferential stress σ_θ, and the radial displacement u as a function of the radial coordinate r; (b) the location and numerical value of the maximum radial stress, maximum circumferential stress, and maximum radial displacement, if the plate is constructed of titanium alloy. The temperature at the inner edge is $T(r = a) = T_0 = 25^\circ$C (77°F) and the temperature at the outer edge is $T(r = b) = T_1 = 125^\circ$C (257°F). The inner radius of the plate is 25 mm

(0.984 in.), and the outer radius of the plate is 125 mm (4.921 in.). Properties of the titanium alloy include $E = 117.2\,\text{GPa}$ (17.0×10^6 psi), $\alpha = 10.3 \times 10^{-6}\,\text{K}^{-1}$ ($5.7 \times 10^{-6}\,{}^\circ\text{F}^{-1}$), and $\mu = 0.330$.

6-7. A circular disk of Teflon with Young's modulus 0.40 GPa (58,000 psi), thermal expansion coefficient $100 \times 10^{-6}\,{}^\circ\text{C}^{-1}$($55.6 \times 10^{-6}\,{}^\circ\text{F}^{-1}$), and Poisson's ratio 0.40, has an outer diameter of 60 mm (2.362 in.). Suddenly, an intense heat source is applied to a circular spot of diameter 20 mm (0.787 in.) at the center of the disk, such that the temperature within the hot spot is suddenly raised from 25°C (77°F) to a temperature T_1, while the portion of the disk outside the hot spot remains essentially at 25°C. If the maximum stress in the disk is to be limited to 2.50 MPa (363 psi), determine the maximum allowable temperature T_1 in the hot spot.

6-8. A very large plate having a thickness of $h = 6$ mm (0.236 in.) has a rigid inclusion of radius $b = 25$ mm (0.984 in.) at its center (at $r = 0$). The temperature distribution in the plate is given by

$$\Delta T = T - T_0 = (T_1 - T_0)\frac{K_0(Mr)}{K_0(Mb)}$$

where T_0 = air temperature around the plate (30°C or 86°F), and T_1 = temperature of the inclusion (120°C or 248°F). The parameter M is defined by

$$M = \sqrt{\frac{2h_c}{k_t h}}$$

where h_c = convective heat transfer coefficient at the plate surfaces (240 W/m^2-°C or 42.3 Btu/hr-ft^2-°F), and k_t = thermal conductivity of the plate material (50 W/m-°C or 28.9 Btu/hr-ft-°F). The plate material is C1020 carbon steel, for which the thermal expansion coefficient is $11.0 \times 10^{-6}\,\text{K}^{-1}$ ($6.1 \times 10^{-6}\,{}^\circ\text{F}^{-1}$), Young's modulus is 200 GPa (29.0×10^6 psi), and Poisson's ratio is 0.280. Determine: (a) the equations for the radial and circumferential stress distribution in the plate, assuming that the displacement of the plate is zero at the edge of the inclusion (at $r = b$), (b) the numerical value of the maximum stress, and (c) the numerical values of the radial stress and circumferential stress at a distance of $r = 50$ mm (1.969 in.) from the center of the inclusion.

6-9. A rectangular plate having a dimension of $2a$ parallel to the x-axis and a dimension $2b$ parallel to the y-axis, as shown in Figure 6-12, has all four edges fixed, such that the strain is zero at the edges. The plate is subjected to a uniform temperature change, $\Delta T = \Delta T_1$ = constant. (a) Show that the Airy stress function

$$\Psi(x, y) = \tfrac{1}{2}C(y^2 + x^2)$$

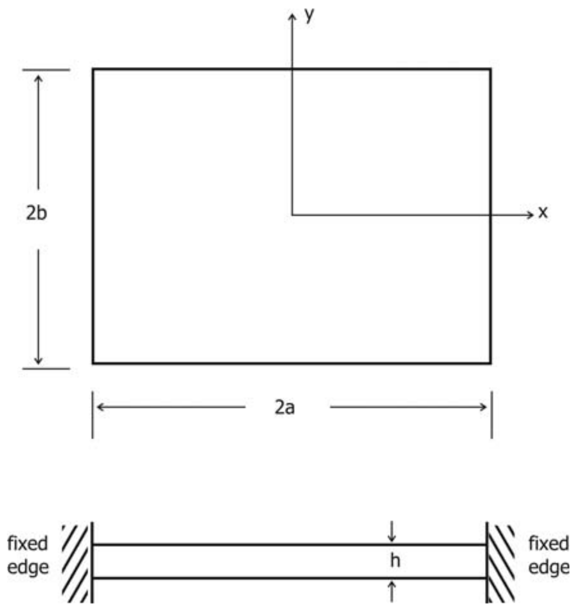

Figure 6-12. Sketch for Problem 6-9.

satisfies the governing equation, eq. (6-105). (b) Determine the expressions for the direct stresses σ_x and σ_y in terms of the temperature change ΔT_1, subject to the condition that the strains are zero at the edges of the plate. *Suggestion:* Use eq. (6-78) to evaluate the constant C in the stress function. (c) If the plate is constructed of C1020 carbon steel (annealed) with $E = 205$ GPa (29.7×10^6 psi), $\alpha = 11.9 \times 10^{-6}\ {}^\circ\text{C}^{-1}$ ($6.6 \times 10^{-6}\ {}^\circ\text{F}^{-1}$), and $\mu = 0.28$, and the plate dimensions are $(2a) = 200$ mm (7.874 in.), $(2b) = 300$ mm (11.81 in.), and thickness $h = 20$ mm (0.787 in.), determine the numerical values for the direct stresses, σ_x and σ_y. The temperature change for the plate is $\Delta T_1 = T_1 - T_0 = 40^\circ\text{C}$ (72°F).

6-10. A very large plate having a thickness h has a rigid thermally-insulating inclusion of radius b at its center, as shown in Figure 6-13. There is a uniform heat transfer rate through the plate, such that the temperature gradient $(dT/dy) = g_T$ is constant for large distances from the inclusion. The temperature distribution in the plate in polar coordinates is given by

$$T(r,\theta) = T_0 + g_T b\left(\frac{r}{b} + \frac{b}{r}\right)\sin\theta$$

and the thermal stress resultant is given by

$$N_T = \alpha E h \Delta T = \left(\frac{\alpha E h g_T b^2}{r}\right)\sin\theta$$

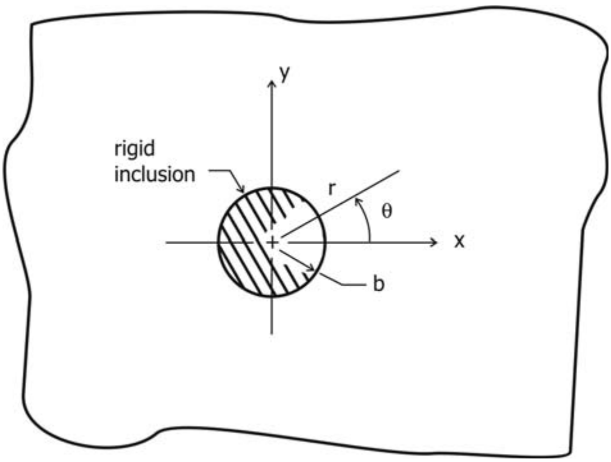

Figure 6-13. Sketch for Problem 6-10.

(a) Using the displacement formulation,

$$\nabla^2 \left(\nabla^2 \Phi - \frac{1+\mu}{Eh} N_T \right) = 0$$

show that the following Goodier displacement function (thermoelastic potential) satisfies the governing equation, eq. (6-94):

$$\Phi(r,\theta) = \left[C_1 r^3 + C_2 r + \frac{C_3}{r} + \frac{1}{2}(1+\mu)\alpha g_T b^2 r \ln\left(\frac{r}{b}\right) \right] \sin\theta$$

The boundary conditions for this problem include the following: (1) the radial displacement u at the surface of the inclusion ($r = b$) is zero, (2) the radial strain ε_r at distances far from the inclusion ($r \to \infty$) is zero, and (3) the radial strain ε_r at the surface of the inclusion ($r = b$) is zero. (b) Determine the equations for the radial stress σ_{r} and circumferential stress σ_θ as a function of r and θ. (c) If the plate material is titanium ($E = 117.2$ GPa $=$ 17.0×10^6 psi, $\alpha = 10.3 \times 10^{-6}\mathrm{K}^{-1} = 5.7 \times 10^{-6}\ {}^\circ\mathrm{F}^{-1}$, and $\mu = 0.33$), the inclusion radius is 6.50 mm (0.256 in.), the temperature gradient is 12,000°C/m (549°F/in.), and the temperature $T_0 = 25$°C (77°F), determine the numerical values for the maximum radial stress and maximum circumferential stress.

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7

BENDING THERMAL STRESSES IN PLATES

7.1 INTRODUCTION

In this chapter, we examine some thermal stress problems that involve bending of flat plates. Certain approximations, similar to those associated with the strength-of-materials approach to beam bending, are usually introduced to solve the problem of determining the stresses and strains for plate bending.

The model that we will consider was developed by Kirchhoff, Navier, and Levy [Kirchhoff, 1876] and expanded by Love [1944], so it is called the *Kirchhoff-Love hypothesis* [Noda et al., 2003]. The restrictions on the use of the Kirchhoff-Love plate-bending approach include the following.

(a) The deflection of the midplane of the plate must be small. From a practical standpoint, this means that the deflection normal to the surface of the plate w should be less than about $\frac{1}{10}$ of the thickness h of the plate. This factor also means that the slope ω of the deflected surface is very small and the square of the slope is negligible compared with unity.

(b) Plane sections normal to the midplane surface before bending of the plate remain plane and normal to the deflected midplane surface after bending. This part of the model means that the vertical shear strains γ_{xz} and γ_{yz} are negligible. The deflection of the plate is associated mainly with bending strains.

(c) Normal stresses and normal strains in the direction normal to the plate surface are negligible compared with the bending stresses and strains.

This component of the model becomes inaccurate in the vicinity of any highly concentrated loads on the plate surface, however.

(d) The membrane stresses resulting from the deflection of the plate in the z-direction are negligible, which means that the midplane remains essentially unstrained after the plate is subjected to bending.

(e) The temperature distribution across the thickness of the plate is an *odd function* of the thickness coordinate (z, z^3, sin z, etc.) such that the mean temperature across any cross section at any location (x, y) is equal to the temperature at the midplane at that same location.

The mathematical conditions for the Kirchhoff-Love hypothesis for pure plate bending include

$$\sigma_z = \varepsilon_z = \gamma_{xz} = \gamma_{yz} \approx 0 \tag{7-1}$$

7.2 GOVERNING RELATIONS FOR BENDING OF RECTANGULAR PLATES

7.2.1 Bending and Transverse Shear Stress Resultants

The stresses that are distributed over any cross section of the plate produce bending moments, twisting moments, and vertical shear forces. These moments and forces per unit length are called *stress resultants*.

The *bending* stress resultants (bending moment per unit length) in Cartesian coordinates are defined as follows.

$$M_x = \int_{-h/2}^{+h/2} \sigma_x z\, dz \tag{7-2a}$$

$$M_y = \int_{-h/2}^{+h/2} \sigma_y z\, dz \tag{7-2b}$$

The *torsional* (twisting) stress resultant (torque per unit length) is defined in a similar manner:

$$M_{xy} = \int_{-h/2}^{+h/2} \tau_{xy} z\, dz = M_{yx} \tag{7-3}$$

The units for the bending and torsional stress resultants are {N–m/m = N} in the SI system and {in.–lb_f/in. = lb_f} in the conventional system. We note that the bending and torsional stress resultants are positive if the direct and shear stresses are positive.

The transverse shear stress resultants are defined as follows:

$$Q_x = \int_{-h/2}^{+h/2} \tau_{xz}\, dz \tag{7-4a}$$

$$Q_y = \int_{-h/2}^{+h/2} \tau_{yz}\, dz \tag{7-4b}$$

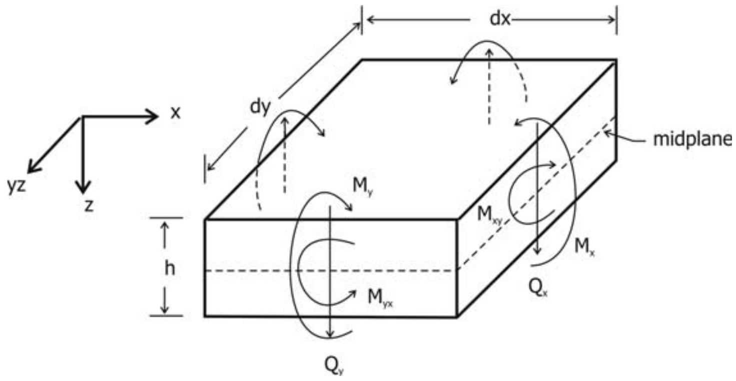

Figure 7-1. Illustration of the stress resultants on a plane element.

The units for the shear stress resultants are {N/m} in the SI system and {lb_f/in.} for the conventional system. The shear stress resultants are positive if they are directed in the positive z-direction. These stress resultants are illustrated in Figure 7-1.

In analogy with the bending stress resultants, let us define the thermal moment M_T as

$$M_T = \int_{-h/2}^{+h/2} E\alpha \Delta T z\, dz \tag{7-5}$$

where the quantity $\Delta T = T - T_0$, is the temperature change from the stress-free condition. If the material properties may be treated as constants, eq. (7-5) reduces to

$$M_T = E\alpha \int_{-h/2}^{+h/2} \Delta T z\, dz \tag{7-6}$$

Because the direct strains are linear function of the transverse coordinate (the z-coordinate) the direct stresses are also linear functions of z. As we will demonstrate in a subsequent section, the stresses may be related to the stress resultants as follows:

$$\sigma_x = \frac{12M_x}{h^3} z \tag{7-7a}$$

$$\sigma_y = \frac{12M_y}{h^3} z \tag{7-7b}$$

$$\tau_{xy} = \frac{12M_{xy}}{h^3} z \tag{7-7c}$$

The maximum stress occurs at the top or bottom surface of the plate, at $z = \pm\frac{1}{2}h$.

$$(\sigma_x)_{\max} = \pm\frac{6M_x}{h^2} \tag{7-8a}$$

$$(\sigma_y)_{\max} = \pm\frac{6M_y}{h^2} \tag{7-8b}$$

$$(\tau_{xy})_{\max} = \pm\frac{6M_{xy}}{h^2} \tag{7-8c}$$

The average transverse shear stresses are related to the shear stress resultants as follows:

$$\overline{\tau}_{xz} = \overline{\tau}_{zx} = \frac{Q_x}{h} \tag{7-9a}$$

$$\overline{\tau}_{yz} = \overline{\tau}_{zy} = \frac{Q_y}{h} \tag{7-9b}$$

It has been shown [Timoshenko, 1959] that the shearing stresses τ_{xz} and τ_{yz} are distributed across the plate thickness in a parabolic manner (proportional to z^2), such that the maximum value of these stresses is given by

$$(\tau_{xz})_{\max} = \frac{3Q_x}{2h} \tag{7-10a}$$

$$(\tau_{yz})_{\max} = \frac{3Q_y}{2h} \tag{7-10b}$$

7.2.2 Stress–Strain Relations for Thermal Bending

The angle of rotation in the x-direction (or simply, the *rotation* ω_x) of a plate element subject to bending is illustrated in Figure 7-2. The rotation is related to the transverse displacement of the midplane w as follows for small deflections:

$$\omega_x \approx \tan\omega_x = \frac{\left(w + \dfrac{\partial w}{\partial x}\,dx\right) - w}{dx} = \frac{\partial w}{\partial x} \tag{7-11a}$$

Similarly, the rotation in the y-direction is given by

$$\omega_y = \frac{\partial w}{\partial y} \tag{7-11b}$$

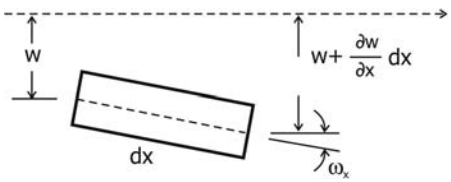

Figure 7-2. Rotation of a plate element in the x-direction.

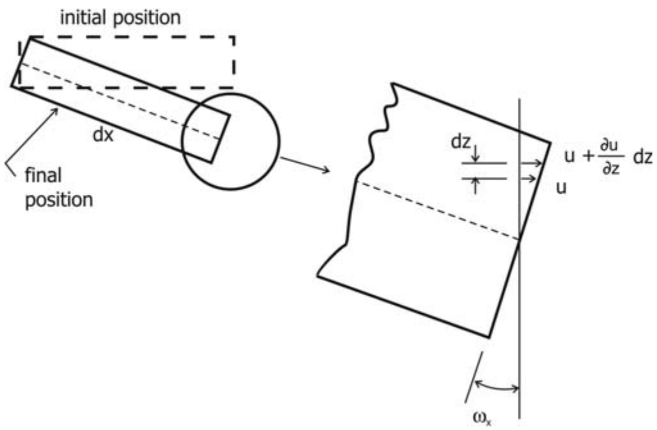

Figure 7-3. Rotation at the end of a differential element in bending.

The rotation is also related to the displacements in the direction parallel to the midplane, as illustrated in Figure 7-3. The rotation in the x-direction may be written in the following form:

$$\omega_x \approx \tan\omega_x = -\frac{\left(u + \dfrac{\partial u}{\partial z}\,dz\right) - u}{dz} = -\frac{\partial u}{\partial z} \tag{7-12a}$$

Similarly, the rotation in the y-direction is also given by

$$\omega_y = -\frac{\partial v}{\partial z} \tag{7-12b}$$

As a consequence of the Kirchhoff-Love hypothesis that planes perpendicular to the midplane remain plane after bending of the plate element, the displacement in the x-direction u is a linear function of the coordinate z at any location. Using this condition with eqs. (7-11), we may write the expressions for the in-plane displacements u and v as

$$u = -z\omega_x = -z\left(\frac{\partial w}{\partial x}\right) \tag{7-13a}$$

$$v = -z\omega_y = -z\left(\frac{\partial w}{\partial y}\right) \tag{7-13b}$$

In this case, only displacements due to bending are considered, because we are considering the case of no in-plane (membrane) forces acting on the plate—only bending forces.

The strains in the x- and y-directions may be found from the basic definition of extensional strain, eqs. (5-1) and (5-2). In the Kirchhoff-Love model, the

transverse strain ε_z is negligible:

$$\varepsilon_x = \frac{\partial u}{\partial x} = -z\frac{\partial \omega_x}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \tag{7-14a}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = -z\frac{\partial \omega_y}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} \tag{7-14b}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -z\left(\frac{\partial \omega_x}{\partial y} + \frac{\partial \omega_y}{\partial x}\right) = -2z\frac{\partial^2 w}{\partial x \partial y} \tag{7-14c}$$

The curvature of the midplane surface is the reciprocal of the radius of curvature and is defined as the change in the rotation per unit distance. Because the deflection of the midplane surface is considered to be small, the square of the rotation terms is negligible compared with unity. With this restriction, the curvatures may be written as follows:

$$\kappa_x = \frac{1}{r_x} = \frac{\partial \omega_x}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial^2 w}{\partial x^2} \tag{7-15a}$$

$$\kappa_y = \frac{1}{r_y} = \frac{\partial \omega_y}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial^2 w}{\partial y^2} \tag{7-15b}$$

$$\kappa_{xy} = \frac{1}{r_{xy}} = \frac{\partial \omega_x}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial^2 w}{\partial x \partial y} \tag{7-15c}$$

The quantity κ_{xy} is also called the *twist* of the midplane surface.

Because the transverse stress σ_z is negligible in the Kirchhoff-Love model, the stress–strain relations given by eqs. (6-78) apply:

$$\sigma_x = \frac{E}{1-\mu^2}(\varepsilon_x + \mu\varepsilon_y) - \frac{\alpha E \Delta T}{1-\mu} \tag{7-16a}$$

$$\sigma_y = \frac{E}{1-\mu^2}(\varepsilon_y + \mu\varepsilon_x) - \frac{\alpha E \Delta T}{1-\mu} \tag{7-16b}$$

$$\tau_{xy} = \frac{E}{2(1+\mu)}\gamma_{xy} \tag{7-16c}$$

Using the definition of the bending and twisting stress resultants, eqs. (7-2) and (7-3), with the strain relations, eqs. (7-14) and the stress relations, eqs. (7-16), we may write the following for the bending stress resultant in the x-direction:

$$\begin{aligned} M_x &= \int_{-h/2}^{+h/2} \sigma_x z\, dz \\ &= \int_{-h/2}^{+h/2} \frac{E}{1-\mu^2}\left(-\frac{\partial^2 w}{\partial x^2} - \mu\frac{\partial^2 w}{\partial y^2}\right) z^2\, dz - \int_{-h/2}^{+h/2} \frac{E\alpha\Delta T}{1-\mu} z\, dz \end{aligned}$$

Carrying out the integrations, we obtain the following for constant material properties:

$$M_x = \frac{Eh^3}{12(1-\mu^2)}\left(-\frac{\partial^2 w}{\partial x^2} - \mu\frac{\partial^2 w}{\partial y^2}\right) - \frac{E\alpha}{1-\mu}\int_{-h/2}^{+h/2} \Delta T z\, dz \tag{7-17}$$

The coefficient of the first term on the right side of eq. (7-17) is called the *flexural rigidity* of a plate:

$$D = \frac{Eh^3}{12(1-\mu^2)} \tag{7-18}$$

The flexural rigidity in plate bending plays the same role as the factor EI in beam bending. The units for the flexural rigidity are {Pa-m or N/m} in the SI system and {lb_f/in.} in the conventional system.

The second term is related to the thermal moment given by eq. (7-6). Introducing these factors, the following expression may be written for the bending stress resultant in the x-direction:

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right) - \frac{M_T}{1-\mu} \tag{7-19a}$$

The relations for the other stress resultants may be written in a similar form:

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right) - \frac{M_T}{1-\mu} \tag{7-19b}$$

$$M_{xy} = -(1-\mu)D\frac{\partial^2 w}{\partial x \partial y} \tag{7-19c}$$

7.2.3 Equilibrium Relationships for Plate Bending

For the plate element shown in Figure 7-4, we may sum forces in the vertical direction:

$$-Q_x\, dy + \left(Q_x + \frac{\partial Q_x}{\partial x}\, dx\right) dy - Q_y\, dx + \left(Q_y + \frac{\partial Q_y}{\partial y}\, dy\right) dx + p\, dx\, dy = 0$$

or

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p = 0 \tag{7-20}$$

Summing moments about an axis along the left face of the element, parallel to the y-coordinate axis, we obtain

$$M_x\, dy - \left(M_x + \frac{\partial M_x}{\partial x}\, dx\right) dy + M_{yx}\, dx - \left(M_{yx} + \frac{\partial M_{yx}}{\partial y}\, dy\right)$$
$$+ \left(Q_x + \frac{\partial Q_x}{\partial x}\, dx\right) dy\, dx + p\, dx\, dy\left(\frac{1}{2}\, dx\right) = 0$$

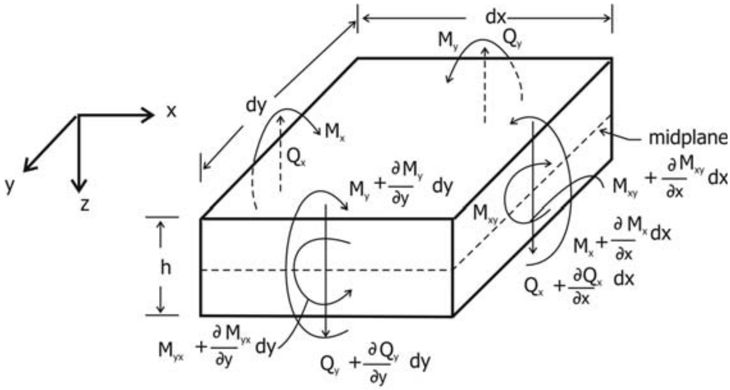

Figure 7-4. Stress resultants applied to a plate element.

If we neglect the terms containing higher-order differentials $(dx)^2 dy$ and simplify, the following result is obtained from the moment balance:

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} = Q_x \tag{7-21a}$$

A similar expression may be obtained by summing moments about an axis along the back face of the element, parallel to the x-coordinate axis:

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = Q_y \tag{7-21b}$$

From a moment balance along an axis parallel to the z-coordinate axis, we find that the twisting moments are related:

$$M_{xy} = M_{yx} \tag{7-21c}$$

The shear stress resultants may be related to the transverse deflection of the plate by using eqs. (7-19) for the moment stress resultants and making this substitution into the force–balance expression, eqs. (7-21). The result is

$$Q_x = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - \frac{1}{1-\mu}\frac{\partial M_T}{\partial x} \tag{7-22a}$$

$$Q_y = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - \frac{1}{1-\mu}\frac{\partial M_T}{\partial y} \tag{7-22b}$$

7.2.4 Governing Differential Equation for Plate Bending

The equations for thermal bending of flat plates may be reduced to a single differential equation in terms of the transverse deflection of the plate. If we

substitute the transverse shear resultant expressions, eqs. (7-21a) and (7-21b), into the vertical equilibrium relation, eq. (7-20), we obtain

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + p = 0 \tag{7-23}$$

The stress resultants may be eliminated from eq. (7-23) by using eqs. (7-19):

$$-D\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right) - 2(1-\mu)D\frac{\partial^4 w}{\partial x^2 \partial y^2} - D\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right)$$
$$-\frac{1}{(1-\mu)}\left(\frac{\partial^2 M_T}{\partial x^2} + \frac{\partial^2 M_T}{\partial y^2}\right) + p = 0 \tag{7-24}$$

Equation (7-24) may be simplified:

$$-D\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\right]$$
$$-\frac{1}{1-\mu}\left(\frac{\partial^2 M_T}{\partial x^2} + \frac{\partial^2 M_T}{\partial y^2}\right) + p = 0 \tag{7-25}$$

For a two-dimensional Cartesian coordinate system, the Laplacian operator may be written as

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \tag{7-26}$$

In terms of the Laplacian operator, eq. (7-25) may be written as

$$D\left[\frac{\partial^2}{\partial x^2}(\nabla^2 w) + \frac{\partial^2}{\partial y^2}(\nabla^2 w)\right] = p - \frac{\nabla^2 M_T}{1-\mu} \tag{7-27}$$

The expression may be further compacted by introducing the biharmonic operator,

$$\nabla^4 w = \nabla^2(\nabla^2 w) \tag{7-28}$$

The final result is

$$\nabla^4 w = \frac{p}{D} - \frac{\nabla^2 M_T}{(1-\mu)D} \tag{7-29}$$

In general, the applied loading p is known and the thermal moment M_T is also known or may be found from information about the temperature distribution across the thickness of the plate. With p and M_T as known functions of x and y for the right side, eq. (7-29) may be solved (in principle) for the transverse deflection w of the plate, subject to the applicable boundary conditions for the situation. After the deflection has been determined, all of the other quantities may be evaluated. For example, the strains may be evaluated from eqs. (7-14); the

stresses may be evaluated from eqs. (7-16) or (7-7); the stress resultants may be evaluated from eqs. (7-19); and the transverse shear resultants may be evaluated from eqs. (7-22).

An alternate solution method involves a two-step procedure. If we add eqs. (7-19a) and (7-19b) for the stress resultants, we get

$$\begin{aligned} M_x + M_y &= -D(1+\mu)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - \frac{2M_T}{1-\mu} \\ &= -D(1+\mu)\nabla^2 w - \frac{2(1+\mu)M_T}{1-\mu^2} \end{aligned}$$

Let us define the moment function *M(x,y)* as follows:

$$M = \frac{M_x + M_y}{1+\mu} = -D\nabla^2 w - \frac{2M_T}{1-\mu^2} \tag{7-30}$$

Equation (7-30) may be written in an alternate form:

$$\nabla^2 w = -\frac{M}{D} - \frac{2M_T}{(1-\mu^2)D} \tag{7-31}$$

If we make the substitution from eq. (7-31) into eq. (7-29), the following is obtained:

$$\nabla^2 M = -p - \frac{1}{1+\mu}\nabla^2 M_T \tag{7-32a}$$

or

$$\nabla^2\left(M + \frac{M_T}{1+\mu}\right) = -p \tag{7-32b}$$

In the alternate two-step solution approach, the moment function *M(x, y)* is determined from eq. (7-32) with known mechanical loading *p(x, y)* and thermal loading $M_T(x, y)$. This solution is then used in eq. (7-31), with M and M_T known, to solve for the lateral deflection *w(x, y)*. After the deflection has been determined, all of the other quantities may be evaluated, as described previously. Some examples of plate-bending solutions are given by Timoshenko [1959] and Ugural [1999] for isothermal conditions.

7.3 BOUNDARY CONDITIONS FOR PLATE BENDING

There are many different conditions that could be imposed along the edges of a plate; however, there are at least four commonly encountered boundary conditions, as illustrated in Figure 7-5.

(a) *Clamped or built-in edge*. This condition is encountered, for example, when the edge of the plate is welded to a stiff support structure. In this case, the support structure prevents the edge of the plate from deflecting or

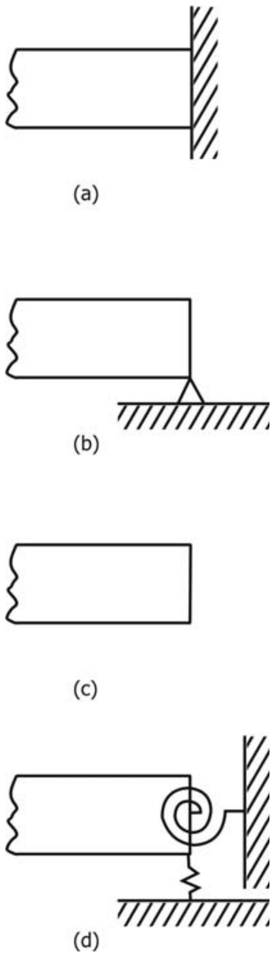

Figure 7-5. Illustration of plate boundary conditions: (a) clamped edge, (b) simply-supported edge, (c) free edge, and (d) elastically supported edge.

rotating. If the clamped edge occurs at $x = a$, the two applicable boundary conditions are

$$w(x = a) = 0 \quad \text{and} \quad \omega_x(x = a) = \left(\frac{\partial w}{\partial x}\right)_{x=a} = 0 \tag{7-33}$$

(b) *Simply-supported edge*. In this case, the plate is supported at the edge, but the rotation of the edge is not prevented. Instead, the moment resultant is zero along the edge, if there are no applied external edge moments. If the clamped edge occurs at $x = a$, the two applicable boundary conditions are

$$w(x = a) = 0 \quad \text{and} \quad M_x(x = a) = 0 \tag{7-34}$$

Equation (7-18a) may be used to relate the stress resultant and the transverse deflection w at the plate edge.

(c) *Free edge*. In this case, there are no applied external bending moments and zero transverse forces. At first look, it would seem that we could set the moment stress resultant, the twisting stress resultant, and the transverse shear resultant all equal to zero at the edge (no applied forces or moments); however, this approach involves one too many restrictions or conditions.

It was shown by Kirchhoff [1850] that the correct boundary condition for a free edge involved only two conditions: the stress resultant $M_x(x = a) = 0$ and an equivalent shear resultant $V_x(x = a) = 0$. The equivalent shear resultant is illustrated in Figure 7-6.

For two adjacent differential elements on the edge, a twisting moment $[M_{xy}\ dy]$ acts on one element and the other element is subjected to a twisting moment of $\left[M_{xy} + (\partial M_{xy}/\partial y)\,dy\right] dy$. These twisting moments may be replaced by a statically equivalent force couple, and at the interface between the two elements, there is a net force of $\left[(\partial M_{xy}/\partial y)\,dy\right]$ acting in the z-direction. The effective transverse shear force per unit length is the sum of the transverse shear resultant and the net force produced by the twisting moment:

$$V_x = Q_x + \frac{\partial M_{xy}}{\partial y} = -\frac{\partial}{\partial x}\left[D\nabla^2 w + (1-\mu)D\frac{\partial^2 w}{\partial y^2} + \frac{M_T}{1-\mu}\right] \quad (7\text{-}35)$$

The correct boundary conditions for a free edge, say along $x = a$, is

$$M_x(x = a) = 0 \quad \text{and} \quad V_x(x = a) = 0 \quad (7\text{-}36)$$

(d) *Corner conditions*. It has been observed that a uniform loading on a simply-supported plate tends to cause the corners to deflect either upward or downward. As a result, concentrated forces are required to maintain zero

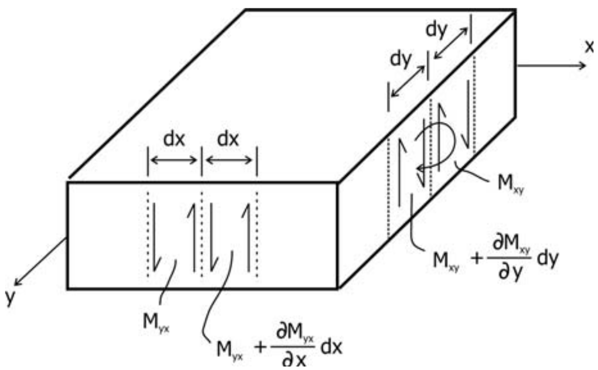

Figure 7-6. Illustration of Kirchhoff's condition for a free edge.

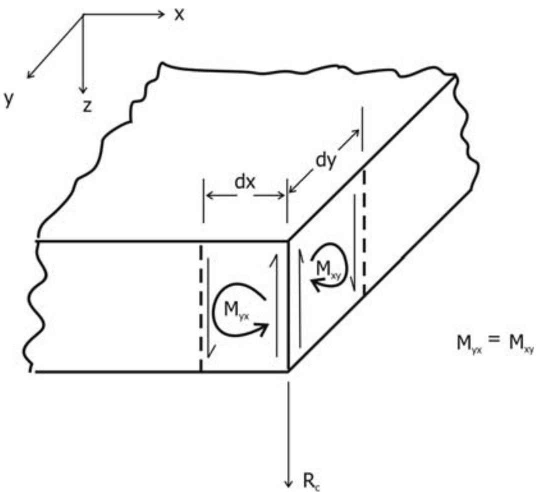

Figure 7-7. Illustration of corner concentrated forces for a rectangular plate.

deflection of the corners, as shown in Figure 7-7. Note that corner forces are not required for plate edges that are fixed or free.

By replacing the twisting moments at the corner with an equivalent set of force couples, it may be observed from Figure 7-7 that the corner resultant force R_c is given by the following for a rectangular plate of dimensions a (in the x-direction) and b (in the y-direction):

$$R_c = 2(M_{xy})_{x=a;y=b} = -2(1-\mu)D\left(\frac{\partial^2 w}{\partial x \partial y}\right)_{x=a;y=b} \tag{7-37}$$

(e) *Elastically supported edge*. In this case, the edge is restrained from bending and rotating by an elastic support structure. For *linear* elastic restraints, the edge force is proportional to the deflection at the edge, and the edge bending moment is proportional to the rotation at the edge. These conditions may be expressed mathematically as follows:

$$M_x = k_\theta \omega_x = k_\theta \frac{\partial w}{\partial x} \quad \text{or} \quad M_y = k_\theta \omega_y = k_\theta \frac{\partial w}{\partial y} \tag{7-38a}$$

$$V_x = Q_x + \frac{\partial M_{xy}}{\partial y} = -k_{\text{sp}} w \quad \text{or} \quad V_y = Q_y + \frac{\partial M_{xy}}{\partial x} = -k_{\text{sp}} w \tag{7-38b}$$

The quantity k_θ is the rotational "spring constant" or rotational elastic constant for the support material, with units of {N/rad} or {lb_f /radian}, and the quantity k_{sp} is the linear elastic constant (spring constant) for the restraining material, with units of {N/m} or {lb_f/in.}.

7.4 BENDING OF SIMPLY-SUPPORTED RECTANGULAR PLATES

In this section, we consider the thermal bending of a simply-supported rectangular plate of dimensions $a \times b$, with the a-dimension parallel to the x-axis, and thickness h, as shown in Figure 7-8. For this example, let us consider the case in which there is no mechanical loading, or $p = 0$. Let us consider the case of one-dimensional conduction heat transfer through the thickness of the plate. The plate center temperature is T_0 (the stress-free temperature), and the upper surface of the plate has a uniform temperature of T_1. The temperature change function is given by

$$\Delta T = T - T_0 = \left(\frac{2z}{h}\right)\Delta T_1 \tag{7-39}$$

where $\Delta T_1 = T_1 - T_0$.

The thermal moment is given by eq. (7-6) and is constant for this problem.

$$M_T = \alpha E \int_{-h/2}^{+h/2} \Delta T\, z\, dz = \alpha E \int_{-h/2}^{+h/2} \Delta T_1 \left(\frac{2z}{h}\right) z\, dz$$

$$M_T = \alpha E \Delta T_1 \left[\frac{2z^2}{3h}\right]_{-h/2}^{+h/2} = \frac{1}{6}\alpha E \Delta T_1 h^2 = M_{T,0} = \text{constant} \tag{7-40}$$

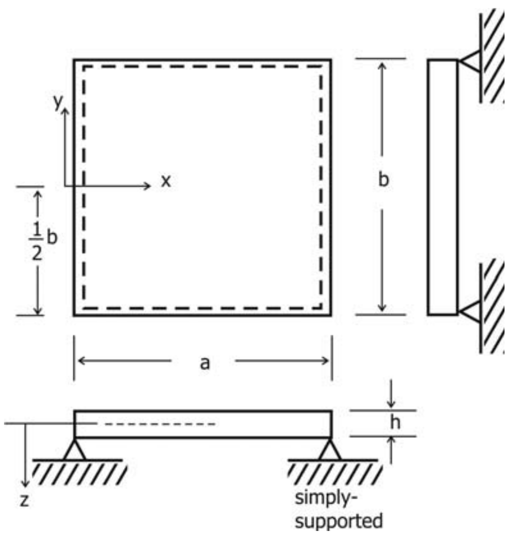

Figure 7-8. Simply-supported rectangular plate.

7.4.1 Evaluation of the Moment Function *M*

The expression for the moment function *M(x, y)*, given by eq. (7-32), is as follows for this problem for the mechanical load $p = 0$:

$$\frac{\partial^2}{\partial x^2}\left(M + \frac{M_{T,0}}{1+\mu}\right) + \frac{\partial^2}{\partial y^2}\left(M + \frac{M_{T,0}}{1+\mu}\right) = 0 \tag{7-41}$$

Equation (7-41) may be solved by a separation-of-variables technique to yield the following solution:

$$M(x, y) + \frac{M_{T.0}}{1+\mu} = [C_1 \cos(\lambda x) + C_2 \sin(\lambda x)][C_3 \cosh(\lambda y) + C_4 \sinh(\lambda y)] \tag{7-42}$$

where λ is a constant. For the simply-supported plate, the boundary conditions are that the moment function, given by eq. (7-30), is equal to $[-M_{T,0}/(1+\mu)]$ along the edges of the plate. Along the left edge ($x = 0$), we have

$$M(0, y) + \frac{M_{T,0}}{1+\mu} = C_1[C_3 \cosh(\lambda y) + C_4 \sinh(\lambda y)] = 0$$

Therefore, $C_1 = 0$. Along the upper and lower edges ($y = \pm\frac{1}{2}b$), the following relations are obtained:

$$M\left(x, +\frac{1}{2}b\right) + \frac{M_{T,0}}{1+\mu} = C_2 \sin(\lambda x)\left[C_3 \cosh\left(\frac{1}{2}\lambda b\right) + C_4 \sinh\left(\frac{1}{2}\lambda b\right)\right] = 0$$

$$M\left(x, -\frac{1}{2}b\right) + \frac{M_{T,0}}{1+\mu} = C_2 \sin(\lambda x)\left[C_3 \cosh\left(\frac{1}{2}\lambda b\right) - C_4 \sinh\left(\frac{1}{2}\lambda b\right)\right] = 0$$

One solution is that $C_3 = C_4 = 0$; therefore, the moment function $M(x, y) = -M_{T,0}/(1+\mu)$ for this problem.

7.4.2 Evaluation of the Transverse Deflection *w*

With the result that the moment function is identically zero, eq. (7-30) reduces to the following expression:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{M_{T,0}}{(1-\mu)D} = -\frac{\alpha E \Delta T_1 h^2}{6(1-\mu)D} \tag{7-43}$$

By using eq. (7-18) for the flexural rigidity D, the right side of eq. (7-43) may be written as

$$\frac{\alpha E \Delta T_1 h^2}{6(1-\mu)D} = \frac{2(1+\mu)\alpha \Delta T_1}{h} \tag{7-44}$$

As a solution for the transverse deflection *w(x, y)* from eq. (7-43), let us try a function of the form

$$w(x, y) = \sum_{m=1,3,\ldots}^{\infty} Y_m(y) \sin(m\pi x/a) \tag{7-45}$$

This function identically satisfies the boundary conditions, $w(0) = w(a) = 0$. We chose a as the smaller dimension of the plate ($a \leq b$), for reasons that will be discussed later.

If we make the substitution from eq. (7-45) directly into the governing equation for the transverse displacement, eq. (7-43), the following is obtained:

$$\sum_{m=1,3,\ldots}^{\infty} \left[\left(-\frac{m^2\pi^2}{a^2}\right) Y_m + \frac{d^2 Y_m}{dy^2}\right] \sin\left(\frac{m\pi x}{a}\right) = -\frac{M_{T,0}}{(1-\mu)D} \tag{7-46}$$

We need some technique for eliminating the sine term to complete the solution. In this problem, the thermal moment is constant; however, in other cases, the thermal moment may be a function of the coordinate x, depending on the temperature distribution.

Suppose we consider the following:

$$M_T(x) = M_{T,0} f(x) \tag{7-47}$$

where $M_{T,0}$ is a constant. According to Fourier series principles, we may expand the function *f(x)* as a sine series on the range between 0 and a:

$$f(x) = \sum_{m=1}^{\infty} \beta_m \sin(m\pi x a) \tag{7-48}$$

According to Fourier series analysis, the constants β_m may be evaluated from

$$\int_0^a f(x) \sin(m\pi x/a)\, dx = \beta_m \int_0^a \sin^2(m\pi x/a)\, dx \tag{7-49}$$

The integral on the right side of eq. (7-49) has a value of (*a/2*); therefore, the general expression for the coefficient β_m is

$$\beta_m = \left(\frac{2}{a}\right) \int_0^a f(x) \sin(m\pi x/a)\, dx \tag{7-50}$$

In the problem that we are considering, the function $f(x) = 1$. Making this substitution into eq. (7-50), the following result is obtained:

$$\beta_m = \left(\frac{2}{m\pi}\right) [1 - \cos(m\pi)] = \frac{4}{m\pi} \quad \text{for } m = 1, 3, 5, \ldots \tag{7-51}$$

The constant bending moment may be written as

$$M_T = \left(\frac{4M_{T,0}}{\pi}\right) \sum_{m=1,3,\ldots}^{\infty} \left(\frac{1}{m}\right) \sin\left(m\pi x/a\right) \tag{7-52}$$

When we make the substitution from eq. (7-52) for the thermal moment into eq. (7-46), the sine terms cancel out term-by-term, and the following total differential equation is obtained for the functions $Y_m(x)$:

$$\frac{d^2 Y_m}{dy^2} - \left(\frac{m\pi}{a}\right)^2 Y_m = -\frac{4M_{T,0}}{\pi m(1-\mu)D} \tag{7-53}$$

The right side of eq. (7-53) is a constant, so the general solution is

$$Y_m(y) = C_1 \cosh\left(m\pi y/a\right) + C_2 \sinh\left(m\pi y/a\right) + \left(\frac{a}{m\pi}\right)^2 \left(\frac{4}{m\pi}\right) \frac{2(1+\mu)\alpha\Delta T_1}{h} \tag{7-54}$$

The lateral displacement along the edges $y = \pm\frac{1}{2}b$ is zero for the simply-supported plate; therefore, $Y_m(\pm\frac{1}{2}b) = 0$.

$$0 = C_1 \cosh\left(m\pi b/2a\right) + C_2 \sinh\left(m\pi b/2a\right) + \left(\frac{2}{m\pi}\right)^3 \frac{a^2(1+\mu)\alpha\Delta T_1}{h} \tag{7-55a}$$

$$0 = C_1 \cosh\left(m\pi b/2a\right) - C_2 \sinh\left(m\pi b/2a\right) + \left(\frac{2}{m\pi}\right)^3 \frac{a^2(1+\mu)\alpha\Delta T_1}{h} \tag{7-55b}$$

From these boundary conditions, we see that $C_2 = 0$. Solving for the other constant, C_1, we obtain

$$C_1 = -\left(\frac{2}{m\pi}\right)^3 \frac{a^2(1+\mu)\alpha\Delta T_1}{h\cosh\left(m\pi b/2a\right)} \tag{7-56}$$

The solution for the function $Y_m(y)$ may be obtained by substituting the expressions for the constants of integration into eq. (7-54):

$$Y_m(y) = \left(\frac{2}{m\pi}\right)^3 \frac{a^2(1+\mu)\alpha\Delta T_1}{h} \left[1 - \frac{\cosh\left(m\pi y/a\right)}{\cosh\left(m\pi b/2a\right)}\right] \tag{7-57}$$

The final expression for the transverse displacement of the plate is obtained by substituting eq. (7-57) into eq. (7-45):

$$w(x,y) = \frac{8(1+\mu)\alpha\Delta T_1 a^2}{\pi^3 h} \sum_{m=1,3,\ldots}^{\infty} \frac{\sin\left(m\pi x/a\right)}{m^3} \left[1 - \frac{\cosh\left(m\pi y/a\right)}{\cosh\left(m\pi b/2a\right)}\right] \tag{7-58}$$

At this point, we may understand why we selected dimension a as the smaller dimension. If $b/a \geq 1$, the $\cosh(m\pi b/2a)$ term in the denominator of eq. (7-58) is relatively large. For example, if $b/a = 1, \cosh(\pi/2) = 2.509$; and if $b/a = 4, \cosh(2\pi) = 267.7$. Except for the regions near the edges $y = \pm\frac{1}{2}b$, the second term in brackets in eq. (7-58) will be negligible if $a/b > 4$. The remaining term represents a rapidly converging series:

$$\sum_{m=1,3,\ldots}^{\infty} \frac{\sin(m\pi x/a)}{m^3} = \left(\frac{1}{8}\pi^3\right)[(x/a) - (x/a)^2] \tag{7-59}$$

For example, for $x/a = \frac{1}{2}$, the right side of eq. (7-59) is 0.9689, and the first 3 terms for the series yield a value of 0.9710, which differs by about 0.2% from the exact value (the right-side value). This is a characteristic that is advantageous for numerical and design calculations.

The maximum deflection occurs at the center of the plate, $x = \frac{1}{2}a$, $y = 0$:

$$w_{\max} = \frac{8(1+\mu)\alpha\Delta T_1 a^2}{\pi^3 h} \sum_{m=1,3,\ldots}^{\infty} \frac{\sin(m\pi/2)}{m^3}\left[1 - \frac{1}{\cosh(m\pi b/2a)}\right] \tag{7-60}$$

Equation (7-59) may be used to simplify the series in eq. (7-60):

$$w_{\max} = \frac{(1+\mu)\alpha\Delta T_1 a^2}{4\pi h}\left[1 - \frac{32}{\pi^2}\sum_{m=1.3,\ldots}^{\infty} \frac{(-1)^{(m-1)/2}}{m^3\cosh(m\pi b/2a)}\right] \tag{7-61}$$

If the plate dimensions are such that $b/a \geq 4$, the second term in brackets is less than about 0.01, which is may be considered negligible compared with unity.

7.4.3 Evaluation of the Stresses

The stress resultants may be evaluated using the transverse deflection from eq. (7-58) in eqs. (7-19). The results are as follows:

$$M_x = -\frac{2\alpha E\Delta T_1 h^2}{3\pi} \sum_{m=1,3,\ldots}^{\infty} \frac{\sin(m\pi x/a)}{m}\frac{\cosh(m\pi y/a)}{\cosh(m\pi b/2a)} \tag{7-62a}$$

$$M_y = -\frac{1}{6}\alpha E\Delta T_1 h^2\left[1 - \frac{4}{\pi}\sum_{m=1,3,\ldots}^{\infty} \frac{\sin(m\pi x/a)}{m}\frac{\cosh(m\pi y/a)}{\cosh(m\pi b/2a)}\right] \tag{7-62b}$$

$$M_{xy} = -\frac{2\alpha E\Delta T_1 h^2}{3\pi} \sum_{m=1,3,\ldots}^{\infty} \frac{\cos(m\pi x/a)}{m}\frac{\sinh(m\pi y/a)}{\cosh(m\pi b/2a)} \tag{7-62c}$$

The stresses at the top and bottom surfaces of the plate are found from eqs. (7-8):

$$\sigma_x = \pm\frac{6M_x}{h^2} = \mp\frac{4}{\pi}\alpha E \Delta T_1 \sum_{m=1,3,\ldots}^{\infty} \frac{\sin(m\pi x/a)}{m}\frac{\cosh(m\pi y/a)}{\cosh(m\pi b/2a)} \tag{7-63a}$$

$$\sigma_y = \pm\frac{6M_y}{h^2} = \mp\alpha E \Delta T_1 \left[1 - \frac{4}{\pi}\sum_{m=1,3,\ldots}^{\infty} \frac{\sin(m\pi x/a)}{m}\frac{\cosh(m\pi y/a)}{\cosh(m\pi b/2a)}\right] \tag{7-63b}$$

The maximum bending stresses occur along the edges of the plate. The expressions for these maximum stresses are relatively simple:

$$(\sigma_x)_{\max} = (\sigma_x)_{y=\pm b/2} = \frac{4}{\pi}\alpha E \Delta T_1 \sum_{m=1,3,\ldots}^{\infty} \frac{\sin(m\pi x/a)}{m} = \alpha E \Delta T_1 \tag{7-64a}$$

$$(\sigma_y)_{\max} = (\sigma_y)_{x=0;x=a} = \alpha E \Delta T_1 \tag{7-64b}$$

Example 7-1 A wooden door, having dimensions of 0.900 m by 2.00 m by 45 mm thick (35.43 in. × 78.74 in. × 1.772 in. thick), absorbs solar radiation at a rate of 75 W/m^2 (23.78 Btu/hr-ft^2) on the outer surface. The inner surface of the door is maintained at 25°C (77°F). The properties of the wood are as follows: $\alpha = 3.6 \times 10^{-5}{}^\circ\text{C}^{-1}$ $(2.0 \times 10^{-5}{}^\circ\text{F}^{-1})$, $E = 12.5$ GPa $(1.81 \times 10^6$ psi$)$, $\mu = 0.15$, $k_t = 0.1688$ W/m-°C (0.0975 Btu/hr-ft-°F), and $S_y = 8.7$ MPa (1260 psi). The door may be treated as a simply-supported plate. Determine the maximum lateral deflection of the door and the maximum thermal stress in the door.

The Fourier rate equation for conduction is

$$q = \frac{k_t(2\Delta T_1)}{h}$$

The temperature difference across the thickness of the door due to the solar radiation heat transfer is

$$2\Delta T_1 = \frac{(75)(0.045)}{0.1688} = 20.0^\circ\text{C} \quad \text{or} \quad \Delta T_1 = 10.0^\circ\text{C}$$

The temperature of the outer surface of the door is $(25^\circ + 20^\circ) = 45^\circ$C (113°F), and the midplane temperature of the door is $T_0 = 25^\circ + 10^\circ = 35^\circ$C (95°F).

Let us first calculate the coefficient factor in eq. (7-61). Note that the smaller dimension of the door is $a = 0.900$ m:

$$\frac{(1+\mu)\alpha \Delta T_1 a^2}{4\pi h} = \frac{(1.15)(3.6 \times 10^{-5})(10^\circ)(0.900)^2}{(4\pi)(0.045)}$$

$$= 0.593 \times 10^{-3}\ \text{m} = 0.593\ \text{mm}$$

Next, let us calculate the term in the series,

$$\frac{\pi b}{2a} = \frac{\pi(2.000)}{(2)(0.900)} = 3.49066$$

The numerical value of the series in eq. (7-61) is

$$\sum_{m=1,3,\ldots}^{\infty} \frac{(-1)^{(m-1)/2}}{m^3 \cosh(m\pi b/2a)} = \frac{1}{\cosh(3.49066)} - \frac{1}{(27)\cosh(10.47198)} + \cdots$$

$$= 0.0609050 - 0.0000021 + \cdots = 0.0609029\ldots$$

We note that, in this example, the magnitude of the first term in the series is about 29,000 times that of the second term; therefore, the use of only the first term in the infinite series would yield very good accuracy.

The value of the maximum deflection may now be calculated from eq. (7-61):

$$w_{\max} = (0.593)\left[1 - \left(\frac{32}{\pi^2}\right)(0.06090)\right] = 0.476\,\text{mm} \quad (0.0187\,\text{in.})$$

The value of the maximum stress in the door is found from eq. (7-64):

$$(\sigma_x)_{\max} = (\sigma_y)_{\max} = (3.6 \times 10^{-5})(12.5 \times 10^9)(10^\circ) = 4.50 \times 10^6\,\text{Pa}$$

$$= 4.50\,\text{MPa}\ (653\,\text{psi})$$

We note that the factor of safety for the door is

$$f_s = \frac{S_y}{\sigma_{\max}} = \frac{8.70}{4.50} = 1.93$$

If the temperature difference across the thickness of the door were to be increased from 20 to 40°C (outer surface temperature of 65°C or 149°F), the thermal stress would exceed the elastic limit for the wood.

7.5 RECTANGULAR PLATES WITH TWO-DIMENSIONAL TEMPERATURE DISTRIBUTIONS

Let us consider the same plate-bending problem as in the previous section (a simply-supported rectangular plate), shown in Figure 7-9, except that the temperature of the midplane depends on both the x- and y-coordinate. Suppose the temperature distribution is given by

$$\Delta T(x, y, z) = T(x, y, z) - T_0 = \left(\frac{2z}{h}\right)\Delta T_1 f(x, y) \tag{7-65}$$

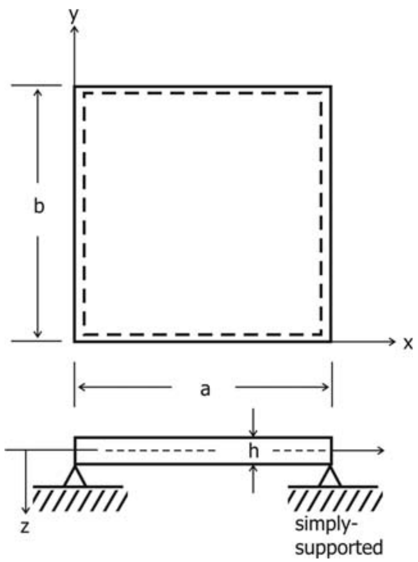

Figure 7-9. Simply-supported rectangular plate with a thermal moment that depends on both the x- and y-coordinates.

The expression for the thermal moment M_T is the same as that given by eq. (7-40), except that the thermal moment is no longer a constant:

$$M_T(x, y) = \tfrac{1}{6}\alpha E \Delta T_1 h^2 f(x, y) = M_{T,0} f(x, y) \tag{7-66}$$

For the simply-supported edge condition, we find that the solution for eq. (7-41) is the same as in the previous problem:

$$M + \frac{M_T}{1+\mu} = 0 \tag{7-67}$$

where M is the moment function given by eq. (7-30). Equation (7-30) may be written as follows:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{2(1+\mu)\alpha \Delta T_1 f(x, y)}{h} \tag{7-68}$$

The simply-supported boundary conditions are that the transverse deflection is zero along the edges of the plate, or

$$w(0, y) = w(a, y) = w(x, 0) = w(x, b) = 0 \tag{7-69}$$

Because of the additional complexity of the thermal moment in this problem, we must expand the thermal moment as a double-sum Fourier series, as follows.

This solution approach was first suggested by M. Navier, who presented the solution for mechanical loading in a paper to the French Academy of Science in 1820 and published the paper in 1823 [Navier, 1823].

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \beta_{mn} \sin(m\pi x/a) \sin(n\pi y/b) \tag{7-70}$$

From Fourier series principles, the coefficients β_{mn} may be determined from the following:

$$\beta_{mn} = \frac{4}{ab}\int_0^a \int_0^b f(x, y) \sin(m\pi x/a) \sin(n\pi y/b)\, dx\, dy \tag{7-71}$$

The lateral displacement may also be expanded in a double-sum Fourier series, which satisfies the boundary conditions, eq. (7-69), identically:

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \sin(m\pi x/a) \sin(n\pi y/b) \tag{7-72}$$

This expression for w, eq. (7-72), and the expression for the thermal moment function $f(x, y)$, eq. (7-70), may be substituted into the governing equation for zero mechanical loading:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{M_{T,0} f(x, y)}{(1-\mu)D} \tag{7-73}$$

The sine terms cancel out term by term, and the following expression is obtained:

$$B_{mn}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] = \frac{\beta_{mn} M_{T,0}}{(1-\mu)D} \tag{7-74}$$

The series coefficients in the expansion for the lateral loading are

$$B_{mn} = \frac{M_{T,0}\beta_{mn} a^2}{\pi^2(1-\mu)D[m^2 + n^2(a/b)^2]} = \frac{2\beta_{mn}(1+\mu)\alpha\Delta T_1 a^2}{\pi^2 h[m^2 + n^2(a/b)^2]} \tag{7-75}$$

Example 7-2 Let us determine the expression for the transverse deflection of a simply-supported rectangular plate of dimensions $a \times b$ and thickness h, as shown in Figure 7-9, with the following temperature distribution:

$$T - T_0 = \Delta T = \frac{4\Delta T_1}{(a^2 + b^2)}[x(a-x) + y(b-y)]\left(\frac{2z}{h}\right) = \Delta T_1 f(x, y)(2z/h)$$

where ΔT_1 is a constant. This is a first approximation to the temperature distribution in a plate with constant energy dissipation,

$$q_{\text{gen}} = \frac{16k_t \Delta T_1}{(a^2 + b^2)}$$

The complete solution for the temperature distribution in a plate at $z = \frac{1}{2}h$ with constant energy dissipation rate [Jacob, 1949] is

$$\begin{aligned}\frac{4k_t(T - T_0)}{q_{\text{gen}}} &= x(a - x) + y(b - y)\\ &- \frac{8}{\pi^3} \sum_{n=1,3,\ldots}^{\infty} \frac{(-1)^{(n-1)/2}}{n^3} \left\{ \frac{b^2 \cosh[(n\pi/b)(x - \frac{1}{2}a)]}{\cosh(n\pi a/2b)} \cos[(n\pi/b)\left(y - \frac{1}{2}b\right)\right\}\\ &- \frac{8}{\pi^3} \sum_{n=1,3,\ldots}^{\infty} \frac{(-1)^{(n-1)/2}}{n^3} \left\{ \frac{a^2 \cosh[(n\pi/a)(y - \frac{1}{2}b)]}{\cosh(n\pi b/2a)} \cos[(n\pi/a)\left(x - \frac{1}{2}a\right)\right\}\end{aligned}$$

The thermal moment from eq. (7-6) is

$$M_T = \tfrac{1}{6}\alpha E \Delta T_1 f(x, y)$$

where

$$f(x, y) = \left(\frac{4}{a^2 + b^2}\right)[x(a - x) + y(b - y)]$$

The first integral in eq. (7-71) is

$$\int_0^a f(x, y) \sin(m\pi x/a)\, dx = \frac{4a[1 - (-1)^m]}{m\pi(a^2 + b^2)} \left[\left(\frac{2a^2}{m^2\pi^2}\right) + y(b - y)\right]$$

Carrying out the second integration in eq. (7-71), the following is obtained for the Fourier coefficient:

$$\beta_{m,n} = \frac{128}{mn\pi^4(a^2 + b^2)} \left(\frac{a^2}{m^2} + \frac{b^2}{n^2}\right) \qquad m = 1, 3, \ldots; n = 1, 3, \ldots$$

If we make this substitution into eq. (7-75), we obtain the final solution for the displacement function coefficients:

$$B_{m,n} = \frac{256(1 + \mu)\alpha \Delta T_1 a^2}{m^3 n^3 \pi^4 h \left[1 + \left(\frac{a}{b}\right)^2\right]} \qquad m = 1, 3, \ldots; n = 1, 3, \ldots$$

The lateral displacement of the plate in this problem is

$$w(x, y) = \frac{256(1+\mu)\alpha \Delta T_1 a^2}{\pi^4 h \left[1 + \left(\frac{a}{b}\right)^2\right]} \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{m^3 n^3} \sin(m\pi x/a)\sin(n\pi y/b)$$

The maximum deflection occurs at the center of the plate ($x = \frac{1}{2}a$; $y = \frac{1}{2}b$). In this case, the infinite series may be evaluated [Gradshteyn and Ryzhik, 1965] as follows:

$$\sum_{m=1,3,\ldots}^{\infty} \frac{\sin(m\pi/2)}{m^3} = \sum_{n=1,3,\ldots}^{\infty} \frac{\sin(n\pi/2)}{n^3} = \sum_{m=1,3,\ldots}^{\infty} \frac{(-1)^{(m-1)/2}}{m^3} = \frac{\pi^3}{32}$$

The expression for the maximum deflection of the plate for this problem is

$$w_{\max} = \frac{\pi^2(1+\mu)\alpha \Delta T_1 a^2}{4h\left[1 + \left(\frac{a}{b}\right)^2\right]}$$

For example, suppose the plate is constructed of nichrome ($\alpha = 10.8 \times 10^{-6\circ}\mathrm{C}^{-1} = 6.0 \times 10^{-6\circ}\mathrm{F}^{-1}$; $\mu = 0.30$), with dimensions $a = 150\,\mathrm{mm}$ (5.91 in.), $b = 225\,\mathrm{mm}$ (8.86 in.), and $h = 10\,\mathrm{mm}$ (0.394 in.), and the temperature difference factor $\Delta T_1 = 25^\circ\mathrm{C}$ (45°F). The corresponding deflection at the center of the plate is

$$w_{\max} = \frac{\pi^2(1+0.30)(10.8\times 10^{-6})(25^\circ)(0.150)^2}{(4)(0.010)\left[1 + \left(\frac{150}{225}\right)^2\right]}$$
$$= 0.00135\,\mathrm{m} = 1.35\,\mathrm{mm}\ (0.53\,\mathrm{in.})$$

7.6 AXISYMMETRIC BENDING OF CIRCULAR PLATES

In this section, we consider the thermal bending of a circular plate in which the loading is *axisymmetric*, which means that the thermal and mechanical loads are symmetric about the z-axis (perpendicular to the plate surface). Because of the symmetry, the twisting moments and the transverse shear resultant in the θ-direction are zero, and the stress resultants and transverse shear resultants are functions of the radial coordinate r only.

7.6.1 Bending and Transverse Shear Stress Resultants

The stress resultants in cylindrical coordinates are similar in form to those in Cartesian coordinates, given by eqs. (7-2) and (7-4):

$$M_r = \int_{-h/2}^{+h/2} \sigma_r z\, dz \tag{7-76a}$$

$$M_\theta = \int_{-h/2}^{+h/2} \sigma_\theta z \, dz \tag{7-76b}$$

$$Q_r = \int_{-h/2}^{+h/2} \tau_{rz} \, dz \tag{7-76c}$$

For pure bending of the circular plate, the stresses are related to the stress resultants by the following:

$$\sigma_r = \frac{12M_r}{h^3} z \tag{7-77a}$$

$$\sigma_\theta = \frac{12M_\theta}{h^3} z \tag{7-77b}$$

The maximum stress occurs at the top or bottom surface of the plate (at $z = \pm\frac{1}{2}h$) and is given by

$$(\sigma_r)_{\max} = \pm\frac{6M_r}{h^2} \tag{7-78a}$$

$$(\sigma_\theta)_{\max} = \pm\frac{6M_\theta}{h^2} \tag{7-78b}$$

The maximum value of the transverse shear stress is given by

$$(\tau_{rz})_{\max} = \frac{3Q_r}{2h} \tag{7-79}$$

7.6.2 Stress–Strain Relations for Axisymmetric Loading

For axisymmetric loading of circular plates, there is no rotation in the θ-direction. The radial rotation is related to the transverse deflection is a manner similar to that given by eq. (7-11) for rectangular plates:

$$\omega_r = \frac{dw}{dr} \tag{7-80}$$

Using the Kirchhoff-Love hypothesis for bending, the radial displacement is linearly related to the rotation of the element.

$$u = -z\omega_r = -z\left(\frac{dw}{dr}\right) \tag{7-81}$$

The radial and circumferential strains are given by eq. (5-12) with all changes with respect to θ equal to zero. The strains are illustrated in Figure 7-10.

$$\varepsilon_r = \frac{\left(dr + \dfrac{du}{dr}dr\right) - dr}{dr} = \frac{du}{dr} \tag{7-82a}$$

$$\varepsilon_\theta = \frac{(r+u)d\theta - r d\theta}{r d\theta} = \frac{u}{r} \tag{7-82b}$$

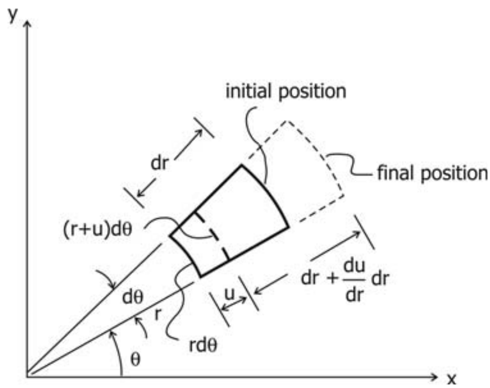

Figure 7-10. Illustration of strains for axisymmetric loading of a circular plate.

Using eq. (7-81), the strains may also be related to the transverse displacement,

$$\varepsilon_r = -z\frac{d^2w}{dr^2} \tag{7-83a}$$

$$\varepsilon_\theta = -z\frac{1}{r}\frac{dw}{dr} \tag{7-83b}$$

The radial and circumferential stresses are related to the strains by Hooke's law:

$$\sigma_r = \frac{E}{1-\mu^2}(\varepsilon_r + \mu\varepsilon_\theta) - \frac{\alpha E \Delta T}{1-\mu} \tag{7-84a}$$

$$\sigma_\theta = \frac{E}{1-\mu^2}(\varepsilon_\theta + \mu\varepsilon_r) - \frac{\alpha E \Delta T}{1-\mu} \tag{7-84b}$$

The bending stress resultants may be found by integrating the moment across the thickness of the plate:

$$M_r = \int_{-h/2}^{+h/2} \sigma_r z\, dz = \frac{Eh^3}{12(1-\mu^2)}\left(-\frac{d^2w}{dr^2} - \frac{\mu}{r}\frac{dw}{dr}\right) - \frac{E\alpha}{1-\mu}\int_{-h/2}^{+h/2} \Delta T z\, dz \tag{7-85}$$

Introducing the flexural rigidity from eq. (7-18) and the thermal moment from eq. (7-6), we may write the radial stress resultant as

$$M_r = -D\left(\frac{d^2w}{dr^2} + \frac{\mu}{r}\frac{dw}{dr}\right) - \frac{M_T}{1-\mu} \tag{7-86a}$$

The circumferential stress resultant may be found in a similar manner:

$$M_\theta = -D\left(\frac{1}{r}\frac{dw}{dr} + \mu\frac{d^2w}{dr^2}\right) - \frac{M_T}{1-\mu} \tag{7-86b}$$

7.6.3 Equilibrium Relationships for Axisymmetric Loading of a Circular Plate

For the plate element shown in Figure 7-11, we may make a force balance for the z-direction:

$$-Q_r r d\theta + (Q_r + dQ_r)(r + dr)d\theta + pr\,dr\,d\theta = 0$$

If we simplify this expression and neglect the term with the higher-order differentials ($dQ_r\,dr\,d\theta$), the following equilibrium relationship is obtained:

$$\frac{d(rQ_r)}{dr} + rp = 0 \tag{7-87}$$

By summing moments in the radial direction, the following is obtained:

$$-M_r r\,d\theta + (M_r + dM_r)(r + dr)\,d\theta - 2(M_\theta\,dr)\left(\frac{1}{2}d\theta\right) - (Q_r r\,d\theta)\,dr = 0$$

We may neglect the term with the higher-order differential ($M_r\,dr\,d\theta$) and simplify the moment balance expression to obtain

$$Q_r = \frac{dM_r}{dr} + \frac{(M_r - M_\theta)}{r} \tag{7-88}$$

The transverse shear resultant may also be written in terms of the transverse deflection of the plate by using eq. (7-86) in eq. (7-88):

$$Q_r = -D\frac{d}{dr}\left(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) - \frac{1}{1-\mu}\frac{dM_T}{dr} \tag{7-89}$$

Equation (7-89) may be written in an alternate form:

$$Q_r = -D\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right] - \frac{1}{1-\mu}\frac{dM_T}{dr} \tag{7-90}$$

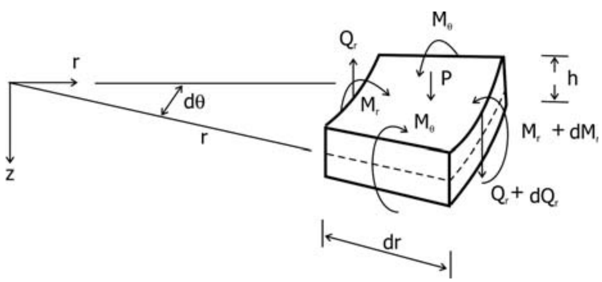

Figure 7-11. Forces and moments acting on a symmetrically loaded plate element.

7.6.4 Governing Differential Equation for Axisymmetric Loading

The equations for thermal bending of an axisymmetrical loaded circular plate may be reduced to a single relationship in terms of the transverse deflection of the plate. If we substitute the expression for the transverse shear resultant from eq. (7-90) into the vertical force balance expression, eq. (7-87), the following is obtained:

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\} = \frac{p}{D} - \frac{1}{(1-\mu)D}\frac{1}{r}\frac{d}{dr}\left(r\frac{dM_T}{dr}\right) \tag{7-91}$$

The Laplacian operator in one-dimensional polar coordinates may be written as follows from eq. (5-76):

$$\nabla^2 w = \frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) \tag{7-92}$$

The left side of eq. (7-91) is the biharmonic operator applied to the deflection w:

$$\nabla^4 w = \nabla^2(\nabla^2 w) = \frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\} \tag{7-93}$$

The governing equation for axisymmetric bending of a circular plate, eq. (7-91), may be written in terms of the biharmonic operator:

$$\nabla^4 w = \frac{p}{D} - \frac{1}{(1-\mu)D}\nabla^2 M_T \tag{7-94}$$

7.6.5 Boundary Conditions for Axisymmetric Bending

The boundary conditions for axisymmetric bending of circular plates are similar in concept to those for bending of rectangular plates. For a circular plate of radius b, some typical conditions at the edge of the plate are as follows:

(a) *Built-in or fixed edge*. The transverse deflection and rotation are zero.

$$w(r=b) = 0 \quad \text{and} \quad \omega_r(r=b) = \left(\frac{dw}{dr}\right)_{r=b} = 0 \tag{7-95}$$

(b) *Simply-supported edge.* In this case, the transverse deflection and radial bending stress resultant are zero at the edge.

$$w(r=b) = 0 \tag{7-96a}$$

$$M_r(r=b) = 0 \quad \text{or} \quad \left(\frac{d^2w}{dr^2} + \frac{\mu}{r}\frac{dw}{dr}\right)_{r=b} = -\frac{(M_T)_{r=b}}{(1-\mu)D} \tag{7-96b}$$

(c) *Free edge*. For a free edge, the radial bending stress resultant and the net edge force are zero.

$$M_r(r=b)=0 \quad \text{or} \quad \left(\frac{d^2w}{dr^2}+\frac{\mu}{r}\frac{dw}{dr}\right)_{r=b}=-\frac{(M_T)_{r=b}}{(1-\mu)D} \tag{7-97a}$$

$$V_r(r=b)=0 \quad \text{or} \quad \left\{\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\}_{r=b}=-\frac{1}{(1-\mu)D}\left(\frac{dM_T}{dr}\right)_{r=b} \tag{7-97b}$$

(d) *Elastically supported edge*. For linearly elastic edge support, the edge bending moment is proportional to the rotation at the edge. Because of symmetry, we may measure the deflection from the final position of the edge or take the edge deflection as zero.

$$w(r=b)=0 \tag{7-98a}$$

$$M_r(r=b)=k_\theta\omega_r=k_\theta\left(\frac{dw}{dr}\right)_{r=b} \quad \text{or}$$

$$\left(\frac{d^2w}{dr^2}+\frac{\mu}{r}\frac{dw}{dr}\right)_{r=b}=-\frac{(M_T)_{r=b}}{(1-\mu)D}-\frac{k_\theta}{D}\left(\frac{dw}{dr}\right)_{r=b} \tag{7-98b}$$

where k_θ is the rotational spring constant or rotational elastic constant for the support material.

7.7 AXISYMMETRIC THERMAL BENDING EXAMPLES

7.7.1 General Solution

The solution of the homogeneous equation $\nabla^4 w_h=0$ may be found by successive integration. Integrating once, the following is obtained:

$$r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw_h}{dr}\right)\right]=4C_1 \tag{7-99}$$

Note that the extra factor 4 is used to avoid fractions in the final solution. Integrating eq. (7-99), the following is obtained:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dw_h}{dr}\right)=4C_1\ln r+4(C_2+C_1) \tag{7-100}$$

Integration of eq. (7-100) yields the homogeneous solution for the radial rotation:

$$\frac{dw_h}{dr}=2C_1 r\ln r+(2C_2+C_1)r+\frac{C_3}{r} \tag{7-101}$$

Integration of eq. (7-101) yields the homogeneous solution for the transverse deflection:

$$w_h = C_1 r^2 \ln r + C_2 r^2 + C_3 \ln r + C_4 \tag{7-102}$$

7.7.2 Circular Plate with a Built-in Edge

Let us consider the case of a circular plate, such as a bolted flange, in which the edge can be treated as a built-in edge, as shown in Figure 7-12(a). Suppose the plate is subjected to the following temperature distribution:

$$\Delta T = T - T_0 = \frac{1}{2}\left(\frac{2z}{h} + 1\right)\Delta T_1 \tag{7-103}$$

For this temperature distribution, the top surface ($z = -\frac{1}{2}h$) is maintained at T_0 and the lower surface ($z = +\frac{1}{2}h$) is maintained at T_1.

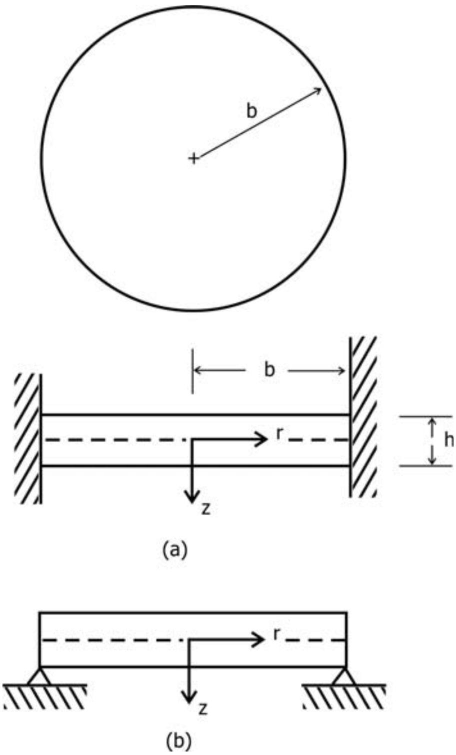

Figure 7-12. Thermal bending of a circular plate with (a) a built-in edge and (b) a simply-supported edge.

The thermal moment may be determined from its definition, eq. (7-6):

$$M_T = \frac{1}{2}\alpha E \Delta T_1 \int_{-h/2}^{+h/2} \left(\frac{2z}{h} + 1\right) z\, dz = \frac{\alpha E \Delta T_1 h^2}{12} = \text{constant} \qquad (7\text{-}104)$$

Because the thermal moment is constant, the Laplacian of the thermal moment is zero, $\nabla^2 M_T = 0$. For the case of no applied mechanical loads $p = 0$, the governing equation, eq. (7-94) reduces to $\nabla^4 w = 0$, which has the solution given by eq. (7-102).

For the solution to be physically realistic, the transverse deflection must be finite at the center of the plate (at $r = 0$). This condition requires that the term involving ln r drop out of the solution, or that the constant $C_3 = 0$.

Using eq. (7-101) with $C_3 = 0$, we obtain

$$\omega_r = \frac{dw_h}{dr} = 2C_1 r \ln r + (2C_2 + C_1) r \qquad (7\text{-}105)$$

$$\frac{d\omega_r}{dr} = \frac{d^2 w}{dr^2} = 2C_1 \ln r + 2(C_2 + C_1) \qquad (7\text{-}106)$$

Substituting the expressions from eqs. (7-105) and (7-106) into eq. (7-86a) for the radial stress resultant, we get

$$M_r = -D\,[(2+\mu)C_1 \ln r + 2(1+\mu)C_2 + (2+\mu)C_1] - \frac{M_T}{1-\mu} \qquad (7\text{-}107)$$

The radial stress resultant must also be finite at the center of the plate (at $r = 0$) for the solution to be physically realistic, This condition requires that the term involving ln r drop out of the solution, or that the constant $C_1 = 0$.

The expressions for the radial rotation and the transverse deflection may be written as follows, using $C_1 = C_3 = 0$:

$$\omega_r = \frac{dw}{dr} = 2C_2 r \qquad (7\text{-}108a)$$

$$w = C_2 r^2 + C_4 \qquad (7\text{-}108b)$$

The plate is built-in or clamped at the edge (at $r = b$), so that both the rotation and the deflection are zero at the edge. Using eq. (7-108a),

$$(w_r)_{r=b} = 0 = 2C_2 b \quad \text{or} \quad C_2 = 0$$

Using eq. (7-108b),

$$(w)_{r=b} = 0 = C_4 \quad \text{or} \quad C_4 = 0$$

For this problem, the solution for the transverse deflection is $w = 0$, or the plate remains flat under the action of the thermal moment. The radial and circumferential stress resultants are given by eq. (7-86):

$$M_r = M_\theta = -\frac{M_T}{(1-\mu)} = -\frac{\alpha E \Delta T_1 h^2}{12(1-\mu)} \tag{7-109}$$

The maximum bending stress occurs at the top and bottom surfaces ($z = \pm\frac{1}{2}h$) and is given by eq. (7-78):

$$(\sigma_r)_{\max} = (\sigma_\theta)_{\max} = \pm\frac{\alpha E \Delta T_1}{2(1-\mu)} \tag{7-110}$$

If the temperature change ΔT_1 is *positive*, the thermal moment is *negative*; therefore, there will be a compressive (*negative*) stress at the bottom surface ($z = +\frac{1}{2}h$) of the plate and a tensile (positive) stress at the top surface ($z = -\frac{1}{2}h$) of the plate.

7.7.3 Circular Plate with a Simply-Supported Edge

Let us consider the case of a circular plate, such as a manway cover, in which the edge can be treated as a simply-supported edge, as shown in Figure 7-12(b). Suppose the plate is subjected to the linear temperature distribution given by eq. (7-103), so that the thermal moment is constant.

In this problem, the transverse deflection and the radial stress resultant must be finite at the center of the plate; therefore, $C_1 = C_3 = 0$, as was the case for the plate with a built-in edge. The solution for the deflection is given by eq. (7-108b). For the simply-supported plate, the radial stress resultant is zero at the edge of the plate ($r = b$). Using eq. (7-86a), the following is obtained:

$$(M_r)_{r=b} = -2D(1+\mu)C_2 - \frac{M_T}{1-\mu} = 0 \tag{7-111}$$

The constant C_2 is

$$C_2 = -\frac{M_T}{2(1-\mu^2)D} = -\frac{6M_T}{Eh^3} = -\frac{\alpha \Delta T_1}{2h} \tag{7-112}$$

For the simply-supported plate, the deflection at the edge of the plate is also zero:

$$(w)_{r=b} = 0 = C_2 b^2 + C_4$$

The constant C_4 is

$$C_4 = -C_2 b^2 = +\frac{\alpha \Delta T_1 b^2}{2h} \tag{7-113}$$

Using these values for C_2 and C_4, the expression for the transverse deflection for a simply-supported circular plate may be written as

$$w = \frac{\alpha \Delta T_1 b^2}{2h}\left[1 - \left(\frac{r}{b}\right)^2\right] \tag{7-114}$$

The maximum deflection occurs at the center of the plate, at which point the rotation ω_r is zero:

$$w_{\text{max}} = \frac{\alpha \Delta T_1 b^2}{2h} \tag{7-115}$$

If the temperature change ΔT_1 is *positive*, the deflection will be *downward* (in the $+z$-direction).

From eq. (7-111), the radial stress resultant M_r is constant; therefore, it is also zero for the entire plate, and the radial and circumferential stresses are also zero for the simply-supported plate.

Example 7-3 Determine the maximum temperature change such that the deflection for a simply-supported circular plate is limited to 1.00 mm (0.039 in.), if the plate has a diameter of 600 mm (23.62 in.) and a thickness of 24 mm (0.945 in.). The plate is constructed of C1020 carbon steel, for which $\alpha = 11.9 \times 10^{-6\circ}\text{C}^{-1}$ ($6.6 \times 10^{-6\circ}\text{F}^{-1}$), $E = 205$ GPa (29.7×10^6 psi), and $\mu = 0.28$.

Using eq. (7-115) for the simply-supported plate, the following maximum temperature change is obtained:

$$\Delta T_1 = \frac{2h w_{\text{max}}}{\alpha b^2} = \frac{(2)(0.024)(0.001)}{(11.9 \times 10^{-6})(0.300)^2} = 44.8^\circ\text{C} \quad (80.6^\circ\text{F})$$

Suppose the edge of the plate in this problem is built-in and the temperature change is 44.8°C. What would the stress in the plate be for this condition? Using eq. (7-110) for the plate with a built-in edge, the maximum bending stress is

$$(\sigma_r)_{\text{max}} = (\sigma_\theta)_{\text{max}} = \pm\frac{\alpha E \Delta T_1}{2(1-\mu)} = \pm\frac{(11.9 \times 10^{-6})(205 \times 10^9)(44.8^\circ)}{(2)(1-0.28)}$$

$$(\sigma)_{\text{max}} = \pm 75.90 \times 10^6 \text{ Pa} = \pm 75.90 \text{ MPa} \quad (11{,}010 \text{ psi})$$

The yield strength for annealed C-1020 steel is 324 MPa, so the factor of safety is

$$f_s = \frac{S_y}{\sigma_{\text{max}}} = \frac{324}{75.90} = 4.3$$

7.7.4 Circular Plate with Variable Thermal Moment

For the case in which the thermal moment is not a constant (or, in fact, when $\nabla^2 M_T \neq 0$), a nonzero particular solution of eq. (7-94) is required. For $p = 0$, we may write eq. (7-94) as

$$\nabla^2 \left[\nabla^2 w + \frac{M_T}{(1-\mu)D} \right] = 0 \tag{7-116}$$

Separating variables and integrating twice, the following is obtained:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr} \right) = 4C_1 \ln r + 4(C_2 + C_1) - \frac{M_T}{(1-\mu)D} \tag{7-117}$$

Integration of eq. (7-117) yields the general solution for the radial rotation of the circular plate:

$$\frac{dw}{dr} = \omega_r = 2C_1 r \ln r + (2C_2 + C_1)r + \frac{C_3}{r} - \frac{1}{(1-\mu)Dr}\int M_T r\,dr \tag{7-118}$$

Integration of eq. (7-118) yields the general solution for the transverse deflection of a circular plate:

$$w = C_1 r^2 \ln r + C_2 r^2 + C_3 \ln r + C_4 - \frac{1}{(1-\mu)D}\int \frac{1}{r}\left(\int M_T r\,dr \right) dr \tag{7-119}$$

If we substitute the general expression for the transverse displacement from eq. (7-119) into eqs. (7-86), the following general expressions for the stress resultants may be obtained:

$$M_r = -D\left\{ [2(1+\mu)\ln r + (3+\mu)]C_1 + 2(1+\mu)C_2 - (1-\mu)\frac{C_3}{r^2} \right\} - \frac{1}{r^2}\int M_T r\,dr \tag{7-120a}$$

$$M_\theta = -D\left\{ [2(1+\mu)\ln r + (1+3\mu)]C_1 + 2(1+\mu)C_2 + (1-\mu)\frac{C_3}{r^2} \right\} - M_T + \frac{1}{r^2}\int M_T r\,dr \tag{7-120b}$$

The general expression for the shear resultant may be obtained from eq. (7-89):

$$Q_r = -\frac{4DC_1}{r} \tag{7-120c}$$

As a specific example, let us consider the case of a circular plate having a radius b and a built-in or clamped edge with the following temperature distribution:

$$\Delta T = T - T_0 = \frac{1}{2}\Delta T_1 \left(\frac{2z}{h} + 1\right)\left[1 - \left(\frac{r}{b}\right)^2\right] \tag{7-121}$$

where ΔT_1 is the temperature change at the center of the plate at the midplane and is a constant. This temperature distribution is an approximation for a plate with a constant energy dissipation rate per unit volume q_{gen}, where $q_{\text{gen}} = 4k_t\Delta T_1/b^2$, and k_t is the thermal conductivity of the plate.

The thermal moment for this case is given by

$$M_T = \frac{\alpha E \Delta T_1 h^2}{12}\left[1 - \left(\frac{r}{b}\right)^2\right] \tag{7-122}$$

Introducing the expression for the flexural rigidity from eq. (7-18), we may write

$$\frac{M_T}{(1-\mu)D} = \frac{(1+\mu)\alpha\Delta T_1}{h}\left[1 - \left(\frac{r}{b}\right)^2\right] \tag{7-123}$$

If we carry out the integrations using eq. (7-123), the last term in the general solution, eq. (7-119), may be written as

$$\frac{1}{(1-\mu)D}\int \frac{1}{r}\left(\int M_T r\,dr\right)dr = \frac{(1+\mu)\alpha\Delta T_1 b^2}{12h}\left[3\left(\frac{r}{b}\right)^2 - \left(\frac{r}{b}\right)^4\right] \tag{7-124}$$

Making this substitution into the general expression for the transverse deflection, eq. (7-119) and combining the r^2 term in the particular solution with the second term (C_2r^2) in the homogeneous solution, the following result is obtained:

$$w = C_1 r^2 \ln r + C_2 r^2 + C_3 \ln r + C_4 + \frac{(1+\mu)\alpha\Delta T_1 b^2}{12h}\left(\frac{r}{b}\right)^4 \tag{7-125}$$

In order that the transverse displacement be finite at the center of the plate, $C_3 = 0$. Similarly, in order that the radial stress resultant be finite at the center of the plate, $C_1 = 0$, as in the previous example. The transverse deflection expression reduces to

$$w = C_2 r^2 + C_4 + \frac{(1+\mu)\alpha\Delta T_1 b^2}{12h}\left(\frac{r}{b}\right)^4 \tag{7-126}$$

The radial rotation is

$$\omega_r = \frac{dw}{dr} = 2C_2 r + \frac{(1+\mu)\alpha\Delta T_1 b}{3h}\left(\frac{r}{b}\right)^3 \tag{7-127}$$

At the fixed edge of the plate, the rotation $\omega_r(r = b) = 0$; therefore, the constant of integration C_2 is

$$C_2 = -\frac{(1+\mu)\alpha\Delta T_1}{6h} \tag{7-128}$$

Also, the transverse deflection $w(r = b) = 0$ at the fixed edge; therefore, the constant of integration C_4 is

$$C_4 = \frac{(1+\mu)\alpha\Delta T_1 b^2}{12h} \tag{7-129}$$

The final expression for the transverse deflection may be found using these values for the constants of integration:

$$w = \frac{(1+\mu)\alpha\Delta T_1 b^2}{12h}\left[\left(\frac{r}{b}\right)^4 - 2\left(\frac{r}{b}\right)^2 + 1\right] = \frac{(1+\mu)\alpha\Delta T_1 b^2}{12h}\left[1 - \left(\frac{r}{b}\right)^2\right]^2 \tag{7-130}$$

The maximum transverse deflection occurs at the center of the plate and is given by

$$w_{\max} = \frac{(1+\mu)\alpha\Delta T_1 b^2}{12h} \tag{7-131}$$

The stress resultants may be found from eqs. (7-86) and (7-126):

$$M_r = -\frac{\alpha E\Delta T_1 h^2}{36(1-\mu)}\left[(2-\mu) + \mu\left(\frac{r}{b}\right)^2\right] \tag{7-132a}$$

$$M_\theta = -\frac{\alpha E\Delta T_1 h^2}{36(1-\mu)}\left[(2-\mu) - (2-3\mu)\left(\frac{r}{b}\right)^2\right] \tag{7-132b}$$

Note that $Q_r = 0$ in this case.

The maximum stress resultant occurs at the fixed edge of the plate ($r = b$):

$$(M_r)_{r=b} = -\frac{\alpha E\Delta T_1 h^2}{18(1-\mu)} \tag{7-133a}$$

$$(M_\theta)_{r=b} = -\frac{\mu\alpha E\Delta T_1 h^2}{18(1-\mu)} \tag{7-133b}$$

The maximum stress is the radial stress at the edge of the plate:

$$(\sigma_r)_{\max} = \pm\frac{6(M_r)_{\max}}{h^2} = \frac{\alpha E\Delta T_1}{3(1-\mu)} \tag{7-134}$$

7.7.5 Annular Circular Plate with Axisymmetric Loading

For an annular circular plate with inner radius a and outer radius b, the solutions for the deflection and stress resultants are much more algebraically complex that those for a plate with no hole, because the ln r terms in eq. (7-119) do not drop out.

There are several possible combinations of edge conditions for the annular plate. Some solutions for the case of a thermal moment that is dependent on the coordinate r are as follows.

(a) *Both edges built-in*, Figure 7-13a.

$$w = \frac{1}{(1-\mu)D}\left\{\varphi_1 \int_a^b \frac{1}{r}\left[\int_a^r M_T(r) r\, dr\right] dr + \varphi_2 \int_a^b M_T(r) r\, dr\right\} - \frac{1}{(1-\mu)D}\int_a^r \frac{1}{r}\left[\int_a^r M_T(r) r\, dr\right] dr \tag{7-135}$$

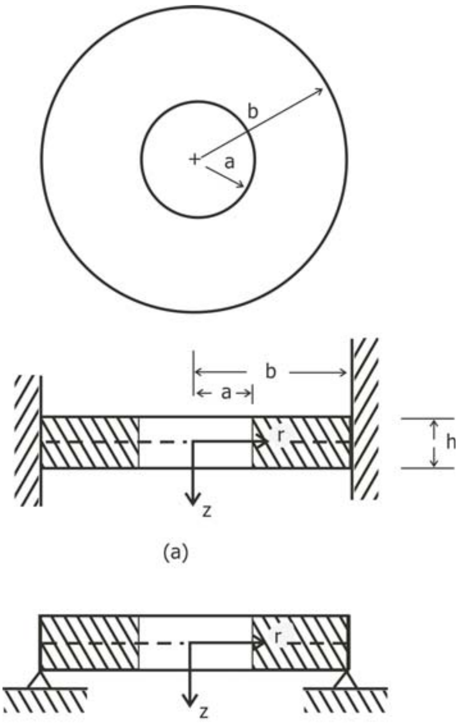

Figure 7-13. Thermal bending of an annular circular plate with (a) both edges built-in and (b) both edges simply-supported.

$$\varphi_1 = \frac{1}{B}\left[-\frac{(b^2-a^2)}{ab}\left(\frac{r^2}{b^2}-\frac{a^2}{b^2}\right) - 2\left(\frac{b}{a}\right)\left(\frac{r^2}{b^2}\right)\ln\left(\frac{b}{r}\right)\right.$$
$$\left. -2\left(\frac{a}{b}\right)\left(\frac{r^2}{b^2}\right)\ln\left(\frac{r}{a}\right) + 2\left(\frac{a}{b}\right)\left(1+2\ln\frac{r}{a}\right)\right] \tag{7-136a}$$

$$\varphi_2 = \frac{1}{B}\left[-2\left(\frac{a}{b}\right)\left(1-\frac{r^2}{b^2}\right)\ln\left(\frac{b}{a}\right)\ln\left(\frac{r}{a}\right)\right.$$
$$\left. + \frac{(b^2-a^2)}{ab}\left(\frac{r^2}{b^2}-\frac{a^2}{b^2}\right)\ln\left(\frac{b}{r}\right)\right] \tag{7-136b}$$

$$B = 4\left(\frac{a}{b}\right)\left(\ln\frac{b}{a}\right)^2 - \left(\frac{b}{a}\right)\left(1-\frac{a^2}{b^2}\right) \tag{7-136c}$$

$$M_r = \frac{4}{(1-\mu)b^2}\varphi_3\int_a^b \frac{1}{r}\left[\int_a^r M_T(r)r\,dr\right]dr - \frac{1}{(1-\mu)b^2}\varphi_4\int_a^b M_T(r)r\,dr$$
$$-\frac{1}{r^2}\int_a^r M_T(r)r\,dr \tag{7-137}$$

$$\varphi_3 = \frac{1}{B}\left\{(1+\mu)\left[\left(\frac{b}{a}\right)\ln\left(\frac{b}{r}\right) + \left(\frac{a}{b}\right)\ln\left(\frac{r}{a}\right)\right]\right.$$
$$\left. +(1-\mu)\left(\frac{a}{b}\right)\left(\frac{b}{r}\right)^2\ln\left(\frac{b}{a}\right) - \left(\frac{b}{a}\right)\left(1-\frac{a^2}{b^2}\right)\right\} \tag{7-138a}$$

$$\varphi_4 = \frac{1}{B}\left\{4(1+\mu)\left(\frac{a}{b}\right)\ln\left(\frac{b}{a}\right)\ln\left(\frac{r}{a}\right) + 2(1+\mu)\left(\frac{b}{a}\right)\left(1-\frac{a^2}{b^2}\right)\ln\left(\frac{b}{r}\right)\right.$$
$$+2\left(\frac{a}{b}\right)\left[3+\mu+(1-\mu)\left(\frac{b}{r}\right)^2\right]\ln\left(\frac{b}{a}\right)$$
$$\left. -\left(\frac{b}{a}\right)\left(1-\frac{a^2}{b^2}\right)\left[3+\mu+(1-\mu)\left(\frac{a}{r}\right)^2\right]\right\} \tag{7-138b}$$

$$M_\theta = \frac{4}{(1-\mu)b^2}\varphi_5\int_a^b \frac{1}{r}\left[\int_a^r M_T(r)r\,dr\right]dr - \frac{1}{(1-\mu)b^2}\varphi_6\int_a^b M_T(r)r\,dr$$
$$-M_T(r) + \frac{1}{r^2}\int_a^r M_T(r)r\,dr \tag{7-139}$$

$$\varphi_5 = \frac{1}{B}\left\{(1+\mu)\left[\left(\frac{b}{a}\right)\ln\left(\frac{b}{r}\right) + \left(\frac{a}{b}\right)\ln\left(\frac{r}{a}\right)\right]\right.$$
$$\left. -(1-\mu)\left(\frac{a}{b}\right)\left(\frac{b}{r}\right)^2\ln\left(\frac{b}{a}\right) - \mu\left(\frac{b}{a}\right)\left(1-\frac{a^2}{b^2}\right)\right\} \tag{7-140a}$$

$$\varphi_6 = \frac{1}{B}\left\{4(1+\mu)\left(\frac{a}{b}\right)\ln\left(\frac{b}{a}\right)\ln\left(\frac{r}{a}\right) + 2(1+\mu)\left(\frac{b}{a}\right)\left(1-\frac{a^2}{b^2}\right)\ln\left(\frac{b}{r}\right)\right.$$
$$+ 2\left(\frac{a}{b}\right)\left[1+3\mu-(1-\mu)\left(\frac{b}{r}\right)^2\right]\ln\left(\frac{b}{a}\right)$$
$$\left. - \left(\frac{b}{a}\right)\left(1-\frac{a^2}{b^2}\right)\left[1+3\mu-(1-\mu)\left(\frac{a}{r}\right)^2\right]\right\} \tag{7-140b}$$

$$Q_r = -\frac{4}{(1-\mu)b^3 B}\left(\frac{b}{r}\right)\left\{2\left(\frac{b}{a}\right)\left(1-\frac{a^2}{b^2}\right)\int_a^b \frac{1}{r}\left[\int_a^r M_T(r)r\,dr\right]dr\right.$$
$$\left. + \left[2\left(\frac{a}{b}\right)\ln\left(\frac{b}{a}\right) - \left(\frac{b}{a}\right)\left(1-\frac{a^2}{b^2}\right)\right]\int_a^b M_T(r)r\,dr\right\} \tag{7-141}$$

(b) *Both edges simply-supported*, Figure 7-13b.

$$w = \frac{1}{(1-\mu)D}\left\{\varphi_7\int_a^b \frac{1}{r}\left[\int_a^r M_T(r)r\,dr\right]dr + \varphi_8\int_a^b M_T(r)r\,dr\right.$$
$$\left. - \int_a^r \frac{1}{r}\left[\int_a^r M_T(r)r\,dr\right]dr\right\} \tag{7-142}$$

$$\varphi_7 = \frac{1}{B_1}\left[4(1+\mu^2)\ln\left(\frac{r}{a}\right)\ln\left(\frac{b}{a}\right) + 2(1-\mu^2)\left(\frac{r^2}{a^2}\ln\frac{b}{r} + \frac{r^2}{b^2}\ln\frac{r}{a} - \ln\frac{b}{a}\right)\right.$$
$$\left. + (3+\mu)(1-\mu)\left(\frac{b}{a}\right)^2\left(1-\frac{a^2}{b^2}\right)\left(1-\frac{r^2}{b^2}\right)\right] \tag{7-143a}$$

$$\varphi_8 = \frac{1}{B_1}\left\{2(1+\mu)\left(1-\frac{r^2}{b^2}\right)\ln\left(\frac{r}{a}\right)\ln\left(\frac{b}{a}\right) - 4\left(1-\frac{r^2}{b^2}\right)\ln\left(\frac{b}{a}\right)\right.$$
$$\left. + \left[3+\mu+(1-\mu)\left(\frac{r}{a}\right)^2\right]\left(1-\frac{a^2}{b^2}\right)\ln\left(\frac{b}{r}\right)\right\} \tag{7-143b}$$

$$B_1 = 4(1+\mu)^2\left(\ln\frac{b}{a}\right)^2 + (1-\mu)(3+\mu)\left(\frac{b^2-a^2}{ab}\right)^2 \tag{7-143c}$$

$$M_r = \frac{4(1+\mu)^2}{b^2}\varphi_9\int_a^b \frac{1}{r}\left[\int_a^r M_T(r)r\,dr\right]dr$$
$$+ \frac{1}{b^2}\varphi_{10}\int_a^b M_T(r)r\,dr - \frac{1}{r^2}\int_a^r M_T(r)r\,dr \tag{7-144}$$

$$\varphi_9 = \frac{1}{B_1}\left[-\left(\frac{b}{a}\right)^2\ln\left(\frac{b}{r}\right) - \ln\left(\frac{r}{a}\right) + \left(\frac{b}{r}\right)^2\ln\left(\frac{b}{a}\right)\right] \tag{7-145a}$$

$$\varphi_{10} = \frac{1}{B_1}\left[4(1+\mu)^2 \ln\left(\frac{r}{a}\right)\ln\left(\frac{b}{a}\right) + (3+\mu)(1-\mu)\left(\frac{b^2}{a^2}-1\right)\left(1-\frac{a^2}{r^2}\right)\right.$$
$$\left. -2(1+\mu^2)\left(\frac{b^2}{a^2}\ln\frac{b}{r} + \ln\frac{r}{a} - \frac{b^2}{r^2}\ln\frac{b}{a}\right)\right] \tag{7-145b}$$

$$M_\theta = -\frac{4}{(1-\mu)b^2}\varphi_{11}\int_a^b \frac{1}{r}\left[\int_a^r M_T(r)r\,dr\right]dr - \frac{1}{b^2}\varphi_{12}\int_a^b M_T(r)r\,dr$$
$$- M_T(r) + \frac{1}{r^2}\int_a^r M_T(r)r\,dr \tag{7-146}$$

$$\varphi_{11} = \frac{1}{B_1}\left\{(1+\mu)\left[\left(\frac{b}{a}\right)^2 \ln\left(\frac{b}{r}\right) + \ln\left(\frac{r}{a}\right) + \left(\frac{b}{r}\right)^2 \ln\left(\frac{b}{a}\right)\right]\right.$$
$$\left. + (1-\mu)\left(\frac{b^2}{a^2}-1\right)\right\} \tag{7-147a}$$

$$\varphi_{12} = \frac{1}{B_1}\left\{4(1+\mu)^2 \ln\left(\frac{r}{a}\right)\ln\left(\frac{b}{a}\right)\right.$$
$$+ (1-\mu)\left(1-\frac{a^2}{b^2}\right)\left[(1+3\mu)\left(\frac{b}{a}\right)^2 + (3+\mu)\left(\frac{b}{r}\right)^2\right]$$
$$\left. -2(1-\mu^2)\left[\left(\frac{b}{a}\right)^2 \ln\left(\frac{b}{r}\right) + \ln\left(\frac{r}{a}\right) + \left(2+\frac{b^2}{r^2}\right)\ln\left(\frac{b}{a}\right)\right]\right\} \tag{7-147b}$$

$$Q_r = \frac{4}{b^3 B_1}\left(\frac{b}{r}\right)\left\{2(1+\mu)\left(\frac{b^2}{a^2}-1\right)\int_a^b \frac{1}{r}\left[\int_a^r M_T(r)r\,dr\right]dr\right.$$
$$\left. + \left[2(1+\mu)\ln\left(\frac{b}{a}\right) + (1-\mu)\left(\frac{b^2}{a^2}-1\right)\right]\int_a^b M_T(r)r\,dr\right\} \tag{7-148}$$

For the special case of constant thermal moment, $M_T = M_{T,0} =$ constant, the integrals in the previous expressions reduce to the following:

$$\int_a^b \frac{1}{r}\left[\int_a^r M_T(r)r\,dr\right]dr = \frac{1}{4}b^2 M_{T,0}\left[\left(1-\frac{a^2}{b^2}\right) - 2\left(\frac{a}{b}\right)^2 \ln\left(\frac{b}{a}\right)\right] \tag{7-149a}$$

$$\int_a^b M_T(r)r\,dr = \frac{1}{2}b^2 M_{T,0}\left(1-\frac{a^2}{b^2}\right) \tag{7-149b}$$

$$\frac{1}{r^2}\int_a^r M_T(r)r\,dr = \frac{1}{2}M_{T,0}\left(1-\frac{a^2}{r^2}\right) \tag{7-149c}$$

$$\int_a^r \frac{1}{r}\left[\int_a^r M_T(r)r\,dr\right]dr = \frac{1}{4}r^2 M_{T,0}\left[\left(1-\frac{a^2}{r^2}\right)-2\left(\frac{a}{r}\right)^2 \ln\left(\frac{r}{a}\right)\right] \tag{7-149d}$$

Example 7-4 An annular circular plate having an OD of 320 mm ($b =$ 160 mm = 6.30 in.) and an ID of 160 mm ($a = 80$ mm = 3.15 in.) is constructed of 9% nickel steel with the following properties: $\alpha = 11.2 \times 10^{-6\circ}\mathrm{C}^{-1}$ ($6.22 \times 10^{-6\circ}\mathrm{F}^{-1}$), $E = 189$ GPa (27.4×10^6 psi), $\mu = 0.286$. The plate thickness is 25 mm (0.984 in.), and the temperature distribution in the plate is given by eq. (7-103), with a temperature difference $\Delta T_1 = 150^\circ$C (270°F). Both the inner and outer edges of the plate are simply-supported. Determine the transverse deflection of the plate at a distance $r = 120$ mm (4.72 in.) from the centerline of the plate.

The deflection is given by eq. (7-142). The components in the deflection expression are functions of two ratios: $b/a = 160/80 = 2.00$ and $r/a = 120/80 = 1.50$. The other ratio depends on these two ratios and is $b/r = (b/a)/(r/a) = 1.333$. The value of B_1 may be found using eq. (7-143c):

$$B_1 = (4)(1.286)^2(\ln 2.00)^2 + (0.714)(3.286)(2.00 - 0.500)^2 = 8.457$$

The parameter φ_7 is found from eq. (7-143a).

$$\varphi_7 = \frac{1}{8.457}\{(4)(1.0818)\ln(1.50)\ln(2.00)$$
$$+ (2)(0.9182)\left[(1.50)^2\ln(1.333) + (0.750)^2\ln(1.50) - \ln(2.00)\right]$$
$$+ (3.286)(0.714)(2.00)^2(1 - 0.500^2)(1 - 0.750^2\ )\}$$

$$\varphi_7 = \frac{(1.2161 + 0.3336 + 3.0794)}{(8.457)} = 0.5474$$

The parameter φ_8 is found from eq. (7-143b):

$$\varphi_8 = \frac{1}{8.457}\left\{(2)(1.286)\left[1 - (0.750)^2\right]\ln(1.50)\ln(2.00) - (4)\left[1 - (0.750)^2\right]\ln(2)\right.$$
$$\left. + [3.286 + (0.714)(1.50)^2](1 - 0.500^2)\ln(1.333\ )\right\}$$

$$\varphi_8 = \frac{(0.3162 - 1.2130 + 1.0556)}{(8.457)} = 0.01878$$

The thermal moment integrals may be evaluated as follows. From eq. (7-149a),

$$\int_a^b \frac{1}{r}\left[\int_a^r M_T r dr\right]dr = \frac{1}{4}b^2 M_{T,0}[(1 - 0.500^2) - (2)(0.500)^2\ln(2)$$
$$= 0.10886 b^2 M_{T,0}$$

From eq. (7-149b),

$$\int_a^b M_T r\,dr = \frac{1}{2}b^2 M_{T,0}(1 - 0.500^2) = 0.3750 b^2 M_{T,0}$$

From eq. (7-149d),

$$\int_a^r \frac{1}{r}\left[\int_a^r M_{T,0} r\,dr\right] dr = \frac{1}{4}r^2 M_{T,0}[1 - 0.6667^2 - (2)(0.6667)^2 \ln(1.500)]$$
$$= 0.04879$$

Using eq. (7-142), we may determine the transverse deflection:

$$(1-\mu)Dw = [(0.5475)(0.10886) + (0.01878)(0.3750)$$
$$- \left(\frac{120}{160}\right)^2 (0.04879)]\, M_{T,0}b^2 = 0.03920 M_{T,0}b^2$$

The constant thermal moment for the temperature distribution for this problem is given by eq. (7-104). The expression for the transverse deflection at $r = 120\,\text{mm}$ is

$$w = 0.03920\frac{\alpha E \Delta T_1 h^2 b^2}{12(1-\mu)D} = \frac{(1+\mu)\alpha \Delta T_1 b^2}{25.51 h}$$

$$w = \frac{(1.286)(11.2 \times 10^{-6})(150^\circ)(0.160)^2}{(25.51)(0.025)} = 0.0867 \times 10^{-3}\ \text{m} = 0.0867\,\text{mm}$$

For a plate with no hole, the transverse deflection at $r = 120\,\text{mm}$ is found from eq. (7-114):

$$w\ (\text{no hole}) = \frac{\alpha \Delta T_1 b^2}{2h}\left[1 - \left(\frac{120}{160}\right)^2\right] = \frac{\alpha \Delta T_1 b^2}{4.571 h}$$

If the plate were solid (no center hole), the transverse deflection at the location $r = 120\,\text{mm}$ would be about $25.52/4.571 = 5.58$ times that for the plate with the hole, or 0.484 mm

7.8 CIRCULAR PLATES WITH A TWO-DIMENSIONAL TEMPERATURE DISTRIBUTION

The solution for thermal bending problems involving a temperature distribution that depends both on the radial coordinate r and the angular coordinate θ follows the same general procedure as that for the case of a rectangular plate discussed in Section 7.5. Suppose the temperature distribution is given by

$$\Delta T(r,\theta,z) = T(r,\theta,z) - T_0 = \left(\frac{2z}{h}\right)\Delta T_1 f(r,\theta) \qquad (7\text{-}150)$$

The expression for the thermal moment may be found from the definition of the thermal moment, eq. (7-5):

$$M_T(r,\theta) = M_{T,0} f(r,\theta) = \tfrac{1}{6}\alpha E \Delta T_1 h^2 f(r,\theta) \tag{7-151}$$

The thermal moment function may be expanded as a complete Fourier series, involving both sine and cosine functions:

$$f(r,\theta) = \beta_0(r) + \sum_{n=1}^{\infty} \left[\beta_{1,n}(r)\cos(n\theta) + \beta_{2,n}(r)\sin(n\theta)\right] \tag{7-152}$$

The Fourier coefficients are given by the following [Arfken, 1966]:

$$\beta_0 = \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)\,d\theta \tag{7-153a}$$

$$\beta_{1,n} = \frac{1}{\pi}\int_0^{2\pi} f(r,\theta)\cos(n\theta)\,d\theta \quad n = 1, 2, 3, \ldots \tag{7-153b}$$

$$\beta_{2,n} = \frac{1}{\pi}\int_0^{2\pi} f(r,\theta)\sin(n\theta)\,d\theta \quad n = 1, 2, 3, \ldots \tag{7-153c}$$

Let us expand the solution for the transverse displacement as a Fourier series also:

$$w(r,\theta) = B_0(r) + \sum_{n=1}^{\infty} \left[B_{1,n}(r)\cos(n\theta) + B_{2,n}(r)\sin(n\theta)\right] \tag{7-154}$$

If we substitute the expressions from eqs. (7-151) and (7-154) into the governing expression eq. (7-94) for the case of no applied mechanical loading ($p = 0$) and equate the coefficients of like cosine and sine terms to zero, the following set of solutions are obtained for the B-functions:

$$B_0(r) = C_1 r^2 \ln r + C_2 r^2 + C_3 \ln r + C_4 - \frac{M_{T,0}}{(1-\mu)D}\int \frac{1}{r}\left[\int \beta_0(r) r\,dr\right]dr \tag{7-155a}$$

$$B_{1,1}(r) = C_{1,1} r \ln r + C_{2,1} r^3 + \frac{C_{3,1}}{r} + C_{4,1} r - \frac{M_{T,0}}{(1-\mu)D}\frac{1}{r}\int r\left[\int \beta_{1,1}(r)\,dr\right]dr \tag{7-155b}$$

$$B_{2,1}(r) = D_{1,1} r \ln r + D_{2,1} r^3 + \frac{D_{3,1}}{r} + D_{4,1} r - \frac{M_{T,0}}{(1-\mu)D}\frac{1}{r}\int r\left[\int \beta_{2,1}(r)\,dr\right]dr \tag{7-155c}$$

$$\begin{aligned} B_{1,n}(r) &= C_{1,n} r^{-n+2} + C_{2,n} r^{n+2} + C_{3,n} r^{-n} + C_{4,n} r^n \\ &\quad - \frac{M_{T,0}}{(1-\mu)D}\frac{1}{2n}\left[r^n \int \beta_{1,n} r^{1-n} dr - r^{-n}\int \beta_{2,n} r^{1+n} dr\right] n = 2, 3, \ldots \end{aligned} \tag{7-155d}$$

$$B_{2,n}(r) = D_{1,n}r^{-n+2} + D_{2,n}r^{n+2} + D_{3,n}r^{-n} + D_{4,n}r^{n} - \frac{M_{T,0}}{(1-\mu)D}\frac{1}{2n}\left[r^n \int \beta_{1,n}r^{1-n}dr - r^{-n}\int \beta_{2,n}r^{1+n}dr\right] n = 2, 3, \ldots \tag{7-155d}$$

If the temperature function is symmetric about the $\theta = 0$ axis $[f(r, +\theta) = f(r, -\theta)]$, the coefficients of the sine terms in eqs. (7-152) and (7-154) are all zero, $\beta_{2,n} = B_{2,n} = 0$. On the other hand, if the temperature function is antisymmetric about the $\theta = 0$ axis $[f(r, +\theta) = -f(r, -\theta)]$, the coefficients of the cosine terms in eqs. (7-152) and (7-154) are all zero, $\beta_0 = \beta_{1,n} = B_0 = B_{1,n} = 0$. If the circular plate has no hole (solid plate), the coefficients of the terms containing $\ln r$ and negative powers of r in eqs. (7-155) are zero, $C_1 = C_3 = C_{1,n} = C_{3,n} = 0$ and$D_{1,n} = D_{3,n} = 0$.

Let us consider the case of a solid circular plate of radius b with a built-in edge having the following temperature distribution:

$$\Delta T(r, \theta, z) = T(r, \theta, z) - T_0 = \left(\frac{2z}{h}\right)\Delta T_1\left(\frac{r}{b}\right)(1 + \cos\theta) \tag{7-156}$$

Or

$$f(r, \theta) = \left(\frac{r}{b}\right)(1 + \cos\theta) \tag{7-157}$$

By comparison with the Fourier expansion, eq. (7-152), we see that

$$\beta_0 = \beta_{1,1} = \frac{r}{b} \quad \text{and} \quad \beta_{2,1} = \beta_{1,n} = \beta_{2,n} = 0 \quad n = 2, 3, \ldots \tag{7-158}$$

Let us determine the values for the integrals in eqs. (7-155a) and (7-155b):

$$\int \frac{1}{r}\left[\int \beta_0(r) r\, dr\right] dr = \int \frac{1}{r}\left[\int \frac{1}{b}r^2 dr\right] dr = \frac{1}{9}\frac{r^3}{b} \tag{7-159a}$$

$$\frac{1}{r}\int r\left[\int \beta_{1,1}(r)\, dr\right] dr = \frac{1}{r}\int r\left[\int \frac{1}{b}r\, dr\right] dr = \frac{1}{8}\frac{r^3}{b} \tag{7-159b}$$

Making these substitutions into the transverse deflection expression, eq. (7-154), the following is obtained:

$$w = C_2 r^2 + C_4 - \frac{M_{T,0}b^2}{(1-\mu)D}\frac{r^3}{9b^3} + \left[C_{2,1}r^3 + C_{4,1}r - \frac{M_{T,0}b^2}{(1-\mu)D}\frac{r^3}{8b^3}\right]\cos\theta \tag{7-160}$$

The expression for the radial rotation is

$$\omega_r = \frac{\partial w}{\partial r} = 2C_2 r - \frac{M_{T,0}b}{(1-\mu)D}\frac{r^2}{3b^2} + \left[3C_{2,1}r^2 + C_{4,1} - \frac{M_{T,0}b}{(1-\mu)D}\frac{3r^2}{8b^2}\right]\cos\theta \tag{7-161}$$

At the built-in edge of the plate ($[r = b]$), the transverse displacement and the rotation are zero. For these conditions to be valid for all values of the angular

coordinate θ, the following result is obtained:

$$C_2 b^2 + C_4 - \frac{M_{T,0} b^2}{9(1-\mu)D} = 0 \tag{7-162a}$$

$$2C_2 b - \frac{M_{T,0} b}{3(1-\mu)D} = 0 \tag{7-162b}$$

$$C_{2,1} b^3 + C_{4,1} b - \frac{M_{T,0} b^2}{8(1-\mu)D} = 0 \tag{7-162c}$$

$$3C_{2,1} b^2 + C_{4,1} - \frac{3M_{T,0} b}{8(1-\mu)D} = 0 \tag{7-162d}$$

Solving for the four constants, we get

$$C_2 = \frac{M_{T,0}}{6(1-\mu)D} \tag{7-163a}$$

$$C_4 = -\frac{M_{T,0} b^2}{18(1-\mu)D} \tag{7-163b}$$

$$C_{2,1} = \frac{M_{T,0}}{8(1-\mu)Db} \tag{7-163c}$$

$$C_{4,1} = 0 \tag{7-163d}$$

For the given temperature distribution,

$$\frac{M_{T,0}}{(1-\mu)D} = \frac{2(1+\mu)\alpha \Delta T_1}{h} \tag{7-164}$$

The final expressions for the transverse deflection and radial rotation are

$$w(r,\theta) = -\frac{(1+\mu)\alpha \Delta T_1 b^2}{9h}\left[1 - 3\left(\frac{r}{b}\right)^2 + 2\left(\frac{r}{b}\right)^3\right] \tag{7-165}$$

$$\omega_r(r,\theta) = \frac{2(1+\mu)\alpha \Delta T_1 b}{3h}\left(\frac{r}{b}\right)\left[1 - \frac{r}{b}\right] \tag{7-166}$$

For nonaxisymmetric thermal loading, the stress resultants are related to the transverse displacements:

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \mu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] - \frac{M_T}{1-\mu} \tag{7-167a}$$

$$M_\theta = -D\left[\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) + \mu\frac{\partial^2 w}{\partial r^2}\right] - \frac{M_T}{1-\mu} \tag{7-167b}$$

$$M_{r\theta} = -(1-\mu)D\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right) \tag{7-167c}$$

The expressions for the shear resultants are

$$Q_r = -D\frac{\partial(\nabla^2 w)}{\partial r} - \frac{1}{1-\mu}\frac{\partial M_T}{\partial r}$$

$$= -D\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right] - \frac{1}{1-\mu}\frac{\partial M_T}{\partial r} \tag{7-168a}$$

$$Q_\theta = -D\frac{1}{r}\frac{\partial(\nabla^2 w)}{\partial \theta} - \frac{1}{(1-\mu)r}\frac{\partial M_T}{\partial \theta}$$

$$= -D\frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right] - \frac{1}{(1-\mu)r}\frac{\partial M_T}{\partial \theta} \tag{7-168b}$$

If we use the expression for the transverse deflection, eq. (7-165), in eqs. (7-167), the following expressions are obtained for the stress resultants in this problem. We observe that the twisting stress resultant $M_{r\theta} = 0$ because the transverse deflection w is not a function of the angular coordinate θ in this example.

$$M_r = -\frac{\alpha E \Delta T_1 h^2}{18(1-\mu)}\left[(1+\mu) + (1-\mu+3\cos\theta)\left(\frac{r}{b}\right)\right] \tag{7-169a}$$

$$M_\theta = -\frac{\alpha E \Delta T_1 h^2}{18(1-\mu)}\left\{(1+\mu) + [2(1-\mu+3\cos\theta]\left(\frac{r}{b}\right)\right\} \tag{7-169b}$$

$$M_{r\theta} = 0 \tag{7-169c}$$

The maximum radial and circumferential stresses occur at the built-in edge ($r = b$):

$$(\sigma_r)_{r=b} = \pm\frac{6(M_r)_{r=b}}{h^2} = \mp\frac{\alpha E \Delta T_1}{3(1-\mu)}(2+3\cos\theta) \tag{7-170a}$$

$$(\sigma_\theta)_{r=b} = \pm\frac{6(M_\theta)_{r=b}}{h^2} = \mp\frac{\alpha E \Delta T_1}{3(1-\mu)}(3-2\mu+3\cos\theta) \tag{7-170b}$$

The point at which the maximum stress occurs is at the built-in edge ($r = b$) and at $\theta = 0^\circ$.

In this problem, the thermal moment and moment stress resultants are linear functions of the radial coordinate, so the transverse shear stress resultants will be functions only of the angular coordinate. If we use the expression for the transverse deflection, eq. (7-165), in eqs. (7-168), the following expressions are obtained for the transverse shear stress resultants in this problem:

$$Q_r = -\frac{\alpha E \Delta T_1 h^2}{6(1-\mu)b}\cos\theta \tag{7-171a}$$

$$Q_\theta = \frac{\alpha E \Delta T_1 h^2}{6(1-\mu)b}\sin\theta \tag{7-171b}$$

The edge reaction force is

$$(V_r)_{r=b} = (Q_r)_{r=b} + \left(\frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta}\right)_{r=b} \tag{7-172}$$

In this problem, the twisting stress resultant $M_{r\theta;}$ is zero, so the edge reaction force per unit length is

$$(V_r)_{r=b} = (Q_r)_{r=b} = -\frac{\alpha E \Delta T_1 h^2}{6(1-\mu)b}\cos\theta \tag{7-173}$$

Note that the net edge force,

$$\int_0^{2\pi} (V_r)_{r=b}\, b d\theta = -\frac{\alpha E \Delta T h^2}{6(1-\mu)}\int_0^{2\pi} \cos\theta d\theta = 0$$

because there are no applied mechanical loads in this problem. If a variable mechanical load per unit surface area $p(r,\theta)$ is applied, the net edge reaction force may be calculated from

$$\int_0^{2\pi} (V_r)_{r=b}\, b d\theta = \int_0^{2\pi}\int_0^{b} p(r,\theta) r\, dr\, d\theta \tag{7-174}$$

7.9 HISTORICAL NOTE

Gustave Robert Kirchhoff (Figure 7-14) was born in Königsberg, East Prussia, in 1824 [Timoshenko, 1983]. He studied physics at Königsberg University where he was a pupil of Franz Ernst Neumann, who encouraged his interest in theory of elasticity. After obtaining his doctorate in 1848, Kirchhoff taught physics at the University of Berlin, and in 1850 he moved to the University of Breslau, where he worked with Bunsen in the area of spectroscopy. Bunsen designed the now-famous Bunsen burner, which was constructed by the university's talented mechanic Peter Desaga in 1854, for use in the new chemistry laboratory. In 1855 Kirchhoff joined the physics faculty at the University of Heidelberg, where he taught and conducted research for the remainder of his career. He continued to collaborate with Bunsen, and they discovered the elements cesium and rubidium in 1861. According to his students, he was an outstanding lecturer and also gave his students excellent laboratory training.

While he was at the University of Breslau, Kirchhoff published a paper on bending of plates in which he presented the first sufficiently accurate treatment of plate bending. He also showed that Poisson's requirement of three edge conditions specifying the bending moment, twisting moment, and shear force at an edge could not be met at the same time, in general, because only two conditions were sufficient, as discussed in Section 7.3.

Kirchhoff's theory of plate bending was based on two hypotheses: (1) planes that are perpendicular to the midplane before bending remain plane (straight) and

Figure 7-14. Gustave Robert Kirchhoff. [From *History of Strength of Materials*, S. P. Timoshenko (1983). Used by permission of Dover Publications, Inc., New York, NY.]

normal to the deflected midplane after bending, and (2) the midplane does not experience stretching when a lateral load is applied to the plate. Using these two conditions, Kirchhoff developed the basic plate-bending differential equation for isothermal conditions:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D} \tag{7-175}$$

Kirchhoff used this general equation to solve the problem of vibration of a circular plate with a free edge, for which Poisson's three edge conditions could not be successfully applied. Kirchhoff attempted to use his solutions for plate vibration with experimental data of E.F.F. Chladni on plate vibrational natural frequencies to determine values for Poisson's ratio. Because the magnitude of Poisson's ratio does not strongly affect the value of the natural frequencies, this technique did not produce accurate values for Poisson's ratio. Later, he used a different experimental technique involving a circular steel cantilever beam loaded in both torsion and bending. A mirror was attached to the end of the bar to measure the angular displacement of that point. From this measurement, Kirchhoff found an experimental value for Poisson's ratio for steel of 0.294.

Kirchhoff made significant contributions in several areas of physics. In addition to his work in spectroscopy and elasticity, Kirchhoff proposed basic principles in thermal radiation regarding emissivity and absorptivity relationships, developed the electrical circuit principles now called *Kirchhoff's laws*, and solved Maxwell's equations in optics to provide a foundation for Huygen's wave propagation principle.

PROBLEMS

7-1. A composite plate is composed of two materials having thermal expansion coefficient α_1, Young's modulus E_1, and thermal conductivity k_{t1} for plate 1 and α_2, E_2, and k_{t2} for plate 2, as shown in Figure 7-15. The thicknesses of the plates are h_1 and h_2, respectively. Heat is conducted through the plate in steady state, such that the temperature distribution in the plate is

$$\Delta T = T(z) - T_0 = \begin{cases} (T_1 - T_0)\left(\dfrac{z}{h_1}\right) = \Delta T_1 \left(\dfrac{z}{h_1}\right) & \text{for } 0 \le z \le h_1 \\ -(T_2 - T_0)\left(\dfrac{z}{h_2}\right) = \left(\dfrac{k_{t1} h_2}{k_{t2} h_1}\right) \Delta T_1 \left(\dfrac{z}{h_2}\right) & \text{for } -h_2 \le z \le 0 \end{cases}$$

Determine the expression for the thermal moment for the composite plate, where

$$M_T = \int \alpha E \Delta T z \, dz = \int_{-h2}^{0} \alpha_2 E_2 \Delta T z \, dz + \int_{0}^{h1} \alpha_1 E_1 \Delta T z \, dz$$

Determine the numerical value of the thermal moment if plate 1 is Teflon (thermal conductivity, $0.26\,\text{W/m-}^\circ\text{C} = 0.150$ Btu/hr-ft-$^\circ$F), and plate 2 is C1020 steel (thermal conductivity, $52\,\text{W/m-}^\circ\text{C} = 30$ Btu/hr-ft-$^\circ$F), with other properties given in Appendix B. The thickness of the steel plate is 12 mm (0.472 in.), and the thickness of the. Teflon plate is 6 mm (0.236 in.). The temperature difference parameter $\Delta T_1 = 50^\circ\text{C}$ (90°F).

7-2. A rectangular flat plate of dimensions $a \times b \times h$, as shown in Figure 7-9, has both a uniform mechanical load $p = p_0$ and a thermal load imposed on it. The thermal moment is given by

$$M_T(x, y) = -\frac{(1-\mu) p_0 a^2}{\pi^2 \left[1 + \left(\dfrac{a}{b}\right)^2\right]} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right)$$

For this thermal moment distribution, determine the expression for the right side of eq. (7-29), the governing equation for thermal bending of a plate.

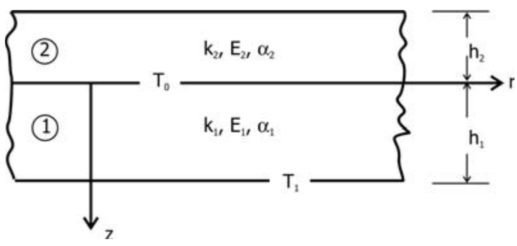

Figure 7-15. Composite plate in Problem 7-1.

7-3. A flat rectangular plate constructed of 304 stainless steel ($\alpha = 16.0 \times 10^{-6\circ}\text{C}^{-1} = 8.9 \times 10^{-6\circ}\text{F}^{-1}$, $E = 193\,\text{GPa} = 28 \times 10^6\,\text{psi}$, $\mu = 0.30$) is simply-supported on all four edges. The plate dimensions are 0.800 m by 1.600 m by 80 mm thick (31.5 in. × 63.0 in. × 3.15 in.) The plate is subjected to the following temperature distribution:

$$\Delta T = T - T_0 = \Delta T_1 \left(\frac{2z}{h}\right)$$

where $\Delta T_1 = T_1 - T_0 = 30^\circ\text{C}$ (54°F). Determine: (a) the numerical values for the stresses σ_x and σ_y at the center of the plate at the plate surface ($z = \frac{1}{2}h$), and (b) the maximum shear stress τ_{xy} at the midpoint of the shorter edge.

7-4. A simply-supported plate constructed of marble ($\alpha = 6.8 \times 10^{-6\circ}\text{C}^{-1} = 3.8 \times 10^{-6\circ}\text{F}^{-1}$, $E = 55.2\,\text{GPa} = 8.0 \times 10^6\,\text{psi}$, $\mu = 0.26$) has dimensions of 1.200 m by 3.600 m by 25 mm thick (47.2 in. × 141.6 in. × 0.984 in.). The plate is subjected to the following temperature distribution:

$$\Delta T = T - T_0 = \begin{cases} -\Delta T_1 \left(\dfrac{2z}{h}\right) & \text{for } 0 < x < \dfrac{1}{2}a \\[2ex] +\Delta T_1 \left(\dfrac{2z}{h}\right) & \text{for } \dfrac{1}{2}a < x < a \end{cases}$$

where a = the shorter dimension and $\Delta T_1 = 50^\circ\text{C}$ (90°F).
Determine (a) the expression for the transverse deflection of the plate (which will involve an infinite series), and (b) the numerical value of the transverse deflection at the center of the plate ($x = \frac{1}{2}a$, $y = 0$). Show that the Fourier coefficients for the series expansion for the thermal moment are as follows:

$$\beta_m = -\frac{2}{m\pi}[1 + \cos(m\pi) - 2\cos(m\pi/2)] = -\frac{8}{m\pi} \quad \text{for } m = 2, 6, 10, \ldots$$

7-5. A circular plate of thickness h and radius b has the following temperature distribution:

$$\Delta T = T - T_0 = \Delta T_1 \left(1 - \frac{r}{b}\right)\left(\frac{2z}{h}\right)$$

where the coordinate z is measured from the midplane and $\Delta T_1 = T_1 - T_0$ is a constant. The edge of the plate is simply-supported. Determine (a) the expression for the transverse deflection *w(r)*, (b) the expressions for the radial stress σ_r and circumferential stress σ_θ at the surface of the plate ($z = \pm\frac{1}{2}h$), (c) the numerical value of the maximum transverse deflection, and (d) the numerical value of the maximum stress. The plate is constructed of 304 stainless steel ($\alpha = 16 \times 10^{-6\circ}\text{C}^{-1} = 8.9 \times 10^{-6\circ}\text{F}^{-1}$, $E = 193\,\text{GPa} = 28 \times 10^6\,\text{psi}$, $\mu = 0.30$). The plate diameter is 450 mm (b = 225 mm = 8.86 in.) and the plate thickness is 20 mm (0.787 in.). The temperature $T_1 = 100^\circ\text{C}$ (212°F) and the stress-free temperature $T_0 = 25^\circ\text{C}$ (77°F).

7-6. A circular plate of thickness h and radius b has the following temperature distribution.

$$\Delta T = T - T_0 = \Delta T_1 \left[1 - \left(\frac{r}{b}\right)^2\right]\left(\frac{2z}{h}\right)$$

where the coordinate z is measured from the midplane and $\Delta T_1 = T_1 - T_0$ is a constant. The edge of the plate is built-in or clamped. Determine (a) the expression for the transverse deflection *w(r)*, (b) the expressions for the radial stress σ_r and circumferential stress σ_θ at the surface of the plate ($z = \pm\frac{1}{2}h$), (c) the numerical value of the maximum transverse deflection, and (d) the numerical value of the maximum stress. The plate is constructed of 6061-T6 aluminum ($\alpha = 23.4 \times 10^{-6\circ}\text{C}^{-1} = 13 \times 10^{-6\circ}\text{F}^{-1}$, $E = 69\,\text{GPa} = 10 \times 10^6\,\text{psi}$, $\mu = 0.30$). The plate diameter is 600 mm ($b = 300\,\text{mm} = 11.81\,\text{in.}$) and the plate thickness is 10 mm (0.394 in.). The temperature $T_1 = 65^\circ\text{C}$ (149°F) and the stress-free temperature $T_0 = 25^\circ\text{C}$ (77°F).

7-7. A circular plate of thickness h and radius b has the following temperature distribution:

$$\Delta T = T - T_0 = \Delta T_1 \left(\frac{2z}{h}\right)$$

where the coordinate z is measured from the midplane and $\Delta T_1 = T_1 - T_0$ is a constant. The edge of the plate is elastically supported, with a rotational spring constant at the edge of k_θ. Determine (a) the expression for the transverse deflection *w(r)*, (b) the expressions for the radial stress σ_r and circumferential stress σ_θ at the surface of the plate ($z = \pm\frac{1}{2}h$), (c) the numerical value of the maximum transverse deflection, and (d) the numerical value of the maximum stress. The plate is constructed of C1020 steel ($\alpha = 12 \times 10^{-6\circ}\text{C}^{-1} = 6.67 \times 10^{-6\circ}\text{F}^{-1}$, $E = 205\,\text{GPa}$, $\mu = 0.28$). The plate diameter is 800 mm ($b = 400\,\text{mm} = 15.75\,\text{in.}$) and the plate thickness is 20 mm (0.787 in.) The temperature $T_1 = 75^\circ\text{C}$ (167°F) and the stress-free temperature $T_0 = 25^\circ\text{C}$ (77°F). The rotational spring constant for the edge support is $k_\theta = 800\,\text{kN/rad}$ (180,000 lb_f/rad).

7-8. An annular circular plate of thickness h, outer radius b, and inner radius a, shown in Figure 7-16. has the following temperature distribution:

$$\Delta T = T - T_0 = \Delta T_1 \left(\frac{2z}{h}\right)$$

where the coordinate z is measured from the midplane and $\Delta T_1 = T_1 - T_0$ is a constant. Both edges of the plate are built-in or clamped. The plate is constructed of bronze ($\alpha = 17.8 \times 10^{-6\circ}\text{C}^{-1} = 9.9 \times 10^{-6\circ}\text{F}^{-1}$, $E = 103\,\text{GPa} = 14.9 \times 10^6\,\text{psi}$, $\mu = 0.30$). The plate outer diameter is 400 mm ($b = 200\,\text{mm} = 7.875\,\text{in.}$), the inner diameter is 160 mm ($a = 80\,\text{mm} = 3.15\,\text{in.}$), and the plate thickness is 8 mm. The temperature $T_1 = 105^\circ\text{C}$ (221°F) and the stress-free temperature $T_0 = 25^\circ\text{C}$ (77°F).

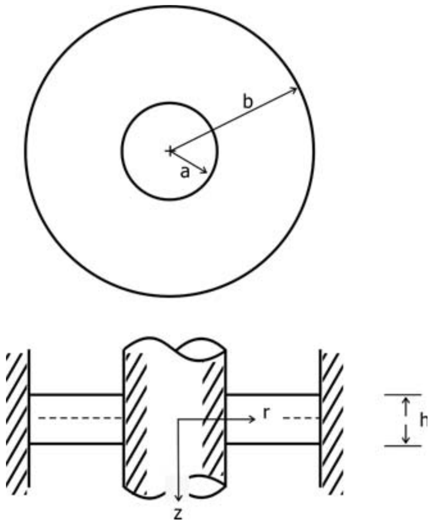

Figure 7-16. Annular circular plate in Problem 7-8.

Determine (a) the numerical value of the transverse deflection at $r = 120\,\text{mm}$ (4.725 in.), and (b) the numerical value of the radial and circumferential stress at the outer edge ($r = 200\,\text{mm} = 7.875\,\text{in.}$) of the plate.

7-9. A circular plate of thickness h and radius b has the following temperature distribution:

$$\Delta T = T - T_0 = \Delta T_1 \left(\frac{2z}{h}\right)\left(1 - \frac{r^2}{b^2}\right)\sin 2\theta$$

where the coordinate z is measured from the midplane and $\Delta T_1 = T_1 - T_0$ is a constant. The edge of the plate is built-in or clamped. Determine (a) the expression for the transverse deflection $w(r,\theta)$, (b) the expressions for the radial stress σ_r and circumferential stress σ_θ at the surface of the plate ($z = \pm\frac{1}{2}h$), (c) the numerical value of the maximum transverse deflection, and (d) the numerical value of the maximum stress. The plate is constructed of Pyrex glass ($\alpha = 3.3 \times 10^{-6}\,{}^\circ\text{C}^{-1} = 1.83 \times 10^{-6}\,{}^\circ\text{F}^{-1}$, $E = 62\,\text{GPa} = 9.0 \times 10^6\,\text{psi}$, $\mu = 0.24$). The plate outer diameter is 250 mm ($b = 125\,\text{mm} = 4.92\,\text{in.}$), and the plate thickness is 6 mm (0.236 in.). The temperature $T_1 = 55^\circ\text{C}$ (131°F), and the stress-free temperature $T_0 = 25^\circ\text{C}$ (77°F).

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8

THERMAL STRESSES IN SHELLS

8.1 INTRODUCTION

A *shell* is a curved structure that can be described by the geometry of its midplane surface and the thickness at any point [Timoshenko and Woinowsky-Krieger, 1959; Ugural, 1999]. In this chapter, we consider thermal stresses in structures generally called *thin shells*, for which the shell thickness h is much smaller than either of the two principal radii of curvature r of the midplane surface. From a practical consideration, a "thin" shell may be taken as a shell for which $(h/r) \leq \frac{1}{10} = 0.10$.

There are several engineering components that may be classed as shells, including pressure vessels, pipes and tubes, shell-and-tube heat exchangers, roof domes, and aircraft fuselages. Two common geometries for engineering applications are either circular cylindrical shells or oval cylinders and spherical shells. These types of shells are illustrated in Figure 8-1. The end closures for cylindrical shells are typically hemispherical heads, semi-elliptical heads, torispherical heads (ASME type), and conical heads. In cases in which the internal pressure is low, a flat end closure may be used, but this type of end closure has a low resistance to bending under internal pressure loading.

Shells of revolution are shells for which the midplane geometry is formed by rotating a plane curve about an axis. Spherical shells (rotating a circle about an axis) and elliptical shells (rotating an ellipse about either the major axis or minor axis) are two examples of shells of revolution. In principle, any plane curve could be used to generate a shell of revolution; however, there are only a few curves that can be used practically (i.e., economically).

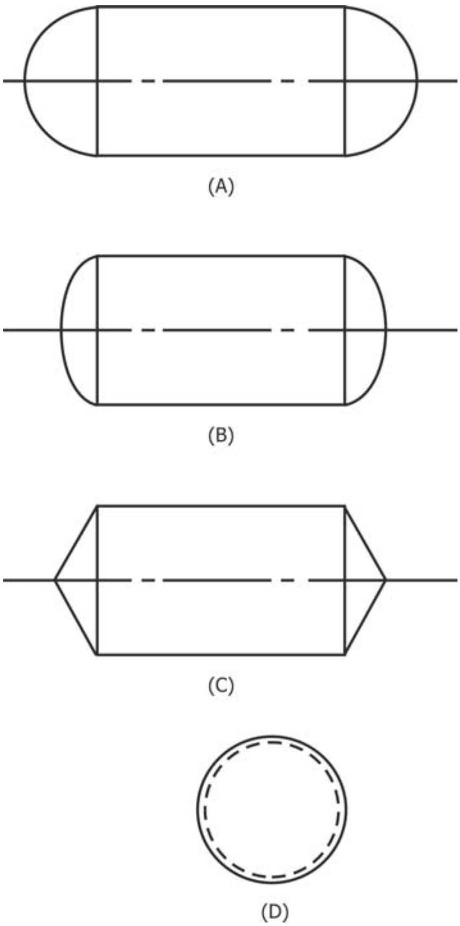


Figure 8-1. Examples of various end closures for cylindrical shells: (A) hemispherical ends, (B) semi-elliptical ends, (C) conical ends. (D) A spherical shell is also one of the commonly used shells for pressure vessels.

There are two broad levels of analysis for shells: (a) membrane analysis, and (b) general analysis, including bending. For membrane analysis, bending and transverse shear forces are not considered. The applied loads on the surface of the membrane are resisted by in-plane (or membrane) forces. The membrane solutions are obtained from static equilibrium considerations. The material properties, such as Young's modulus, do not appear in the membrane solutions for the stresses. For pressure vessels, the membrane stresses are the more important stresses, except near regions of discontinuity or constraint.

On the other hand, the general analysis of shells including bending is a statically indeterminate problem and stress–strain relationships are required to solve the stress problem for this level of analysis. The stresses resulting from bending in shells are most pronounced near discontinuities, such as the junction between

a cylindrical shell and a hemispherical end closure, so these stresses are often called *discontinuity stresses*.

For the general analysis including bending, there are several degrees of complexity, including (from more simple to more complex): (a) cylindrical shells subjected to axisymmetric loading, (b) shells of revolution subject to axisymmetric loading, (c) cylindrical shells subjected to nonsymmetric loading, (d) shells of revolution subject to nonsymmetric loading, and (e) shells of arbitrary shape subjected to general loading conditions. In many cases, the last category can be analyzed only through a numerical analysis.

In this chapter, we consider thermal stresses in cylindrical and spherical shells. Analyses of shells of various shapes with mechanical loading are available in the literature [Flügge, 1960].

8.2 CYLINDRICAL SHELLS WITH AXISYMMETRIC LOADING

8.2.1 Equilibrium Relationships

If both thermal loading and mechanical loading are symmetrical about the axis of the cylinder, the forces and moments acting on a differential element of the shell are as shown in Figure 8-2. The quantity a is the radius of the midplane of the cylindrical shell. Making a radial force balance on the element gives

$$(Q_x + dQ_x)a\ d\theta - Q_x a\ d\theta + p_r a\ d\theta\, dx - 2N_\theta \sin\left(\tfrac{1}{2}d\theta\right) dx = 0 \tag{8-1}$$

If we approximate the sine term with $\sin\left(\frac{1}{2}d\theta\right) \approx \frac{1}{2}d\theta$ for small angles and simplify eq. (8-1), the following radial force balance expression is obtained:

$$\frac{dQ_x}{dx} - \frac{N_\theta}{a} + p_r = 0 \tag{8-2}$$

Making an axial force balance on the element, we obtain

$$(N_x + dN_x)a\ d\theta - N_x a\ d\theta + p_x\ a\ d\theta dx = 0 \tag{8-3}$$

Equation (8-3) may be simplified to obtain the axial force balance expression:

$$\frac{dN_x}{dx} + p_x = 0 \tag{8-4}$$

The axial force resultant may be obtained by integrating eq. (8-4):

$$N_x = -\int p_x dx + C \tag{8-5}$$

If we make a moment balance about an axis through the upper edge of the element, the following is obtained:

$$(M_x + dM_x)a\ d\theta - M_x a\ d\theta - Q_x a\ d\theta dx = 0 \tag{8-6}$$

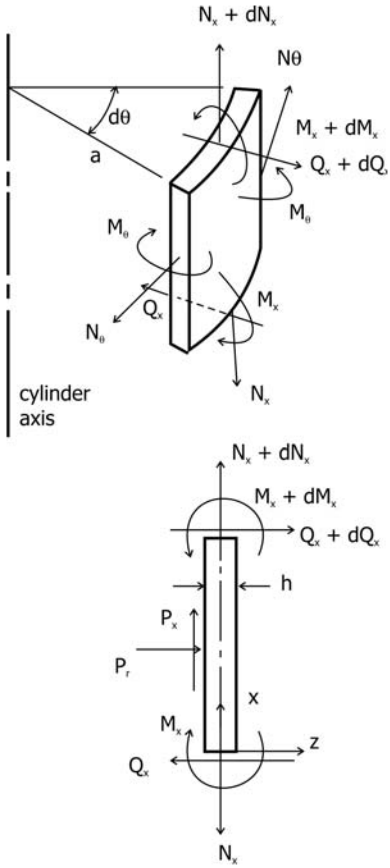


Figure 8-2. Membrane and bending stress resultants acting on a cylindrical shell element with axisymmetric loading.

After simplification, eq. (8-6) reduces to

$$\frac{dM_x}{dx} - Q_x = 0 \tag{8-7}$$

8.2.2 Stress–Strain Relations

Because of symmetry, the circumferential displacement $v = 0$ and the strain–displacement relations for the midplane are

$$\bar{\varepsilon}_x = \frac{(du + dx) - dx}{dx} = \frac{du}{dx} \tag{8-8a}$$

$$\bar{\varepsilon}_\theta = \frac{(a + w)d\theta - ad\theta}{ad\theta} = \frac{w}{a} \tag{8-8b}$$

The stress–strain relations for the midplane are

$$\bar{\varepsilon}_x = \frac{1}{Eh}(N_x - \mu N_\theta) + \frac{N_T}{Eh} \tag{8-9a}$$

$$\bar{\varepsilon}_\theta = \frac{1}{Eh}(N_\theta - \mu N_x) + \frac{N_T}{Eh} \tag{8-9b}$$

The thermal stress resultant N_T is given by

$$N_T = \int_{-h/2}^{+h/2} \alpha E(T - T_0)\,dz = \int_{-h/2}^{+h/2} \alpha E \Delta T\,dz \tag{8-10}$$

If we solve for the stress resultants from eqs. (8-9), the following expressions are obtained:

$$N_x = \frac{Eh}{1-\mu^2}(\bar{\varepsilon}_x + \mu\bar{\varepsilon}_\theta) - \frac{N_T}{1-\mu} = \frac{Eh}{1-\mu^2}\left(\frac{du}{dx} + \mu\frac{w}{a}\right) - \frac{N_T}{1-\mu} \tag{8-11a}$$

$$N_\theta = \frac{Eh}{1-\mu^2}(\bar{\varepsilon}_\theta + \mu\bar{\varepsilon}_x) - \frac{N_T}{1-\mu} = \frac{Ehw}{a} + \mu N_x - N_T \tag{8-11b}$$

The differential equation for the axial displacement u may be obtained from eq. (8-11a):

$$\bar{\varepsilon}_x = \frac{du}{dx} = \frac{(1-\mu^2)N_x}{Eh} - \mu\frac{w}{a} + (1+\mu)N_T \tag{8-12}$$

The solution for the axial displacement may be obtained after the radial displacement w is found.

There is no rotation of the element in the circumferential direction, and the rotation in the axial direction ω_x is given by

$$\omega_x = \frac{dw}{dx} \tag{8-13}$$

The relations for the bending moment resultants are the same as those for plates because there is no change in the radial displacement in the circumferential direction:

$$M_x = -D\frac{d\omega_x}{dx} - \frac{M_T}{1-\mu} = -D\frac{d^2w}{dx^2} - \frac{M_T}{1-\mu} \tag{8-14a}$$

$$M_\theta = -\mu D\frac{d\omega_x}{dx} - \frac{M_T}{1-\mu} = -\mu D\frac{d^2w}{dx^2} - \frac{M_T}{1-\mu} \tag{8-14b}$$

The quantity D is the flexural rigidity for the shell:

$$D = \frac{Eh^3}{1-\mu^2} \tag{8-15}$$

The transverse shear resultant Q_x may be written in terms of the radial displacement w by combining eqs. (8-7) and (8-14a):

$$Q_x = -D\frac{d^3w}{dx^3} - \frac{1}{1-\mu}\frac{dM_T}{dx} \tag{8-16}$$

8.2.3 Displacement Formulation

If we take the derivative of both sides of eq. (8-16) and use the radial force balance expression, eq. (8-2), to eliminate the transverse shear resultant, the governing differential equation is obtained:

$$\frac{d^4w}{dx^4} + \frac{Eh}{Da^2}w = -\frac{\mu N_x}{Da} - \frac{1}{(1-\mu)D}\frac{d^2M_T}{dx^2} + \frac{N_T}{Da} + \frac{p_r}{D} \tag{8-17}$$

Using the definition of the flexural rigidity, eq. (8-15), the coefficient in the second term on the left side of eq. (8-17) may be written as

$$\frac{Eh}{Da^2} = \frac{12(1-\mu^2)}{a^2h^2} \tag{8-18}$$

After the solution for the transverse displacement is obtained, all other quantities may be obtained as follows:

$$M_x = -D\frac{d^2w}{dx^2} - \frac{M_T}{1-\mu} \tag{8-14a}$$

$$M_\theta = -\mu D\frac{d^2w}{dx^2} - \frac{M_T}{1-\mu} \tag{8-14b}$$

$$Q_x = -D\frac{d^3w}{dx^3} - \frac{1}{1-\mu}\frac{dM_T}{dx} \tag{8-16}$$

$$N_x = -\int p_x dx + C \tag{8-5}$$

$$N_\theta = \frac{E\,h\,w}{a} + \mu N_x - N_T \tag{8-11b}$$

The axial and circumferential stresses may be obtained from the solution for the stress resultants. In this case, there are two components of these stresses: (a) the membrane stresses, and (b) the bending stresses. The stresses at the upper or lower surface of the shell are

$$\sigma_x = \frac{N_x}{h} \pm \frac{6M_x}{h^2} \tag{8-19a}$$

$$\sigma_\theta = \frac{N_\theta}{h} \pm \frac{6M_\theta}{h^2} \tag{8-19b}$$

The maximum transverse shear stress occurs at the midplane and is given by the following [Timoshenko, 1959, p. 82; Ugural, 1999, p. 385]:

$$\tau_{xz,\max} = \frac{3Q_x}{2h} \tag{8-19c}$$

8.2.4 Stress Formulation

Equation (8-17) is the result of the *displacement formulation* of the cylindrical shell bending problem. If we eliminate the displacement in the previous relationships and solve for the transverse shear resultant, we may obtain the *stress formulation* of the problem as follows. Using the definition of the rotation, along with eq. (8-8b), we obtain

$$\omega_x = \frac{dw}{dx} = a\frac{d\bar{\varepsilon}_\theta}{dx} \tag{8-20}$$

Using eq. (8-9b) for the circumferential strain, we get the expression for the rotation:

$$\omega_x = \frac{a}{E\,h}\left(\frac{dN_\theta}{dx} - \mu\frac{dN_x}{dx} + \frac{dN_T}{dx}\right) \tag{8-21}$$

The membrane stress resultants may be eliminated using eqs. (8-2) and (8-4):

$$\omega_x = \frac{a^2}{E\,h}\left(\frac{d^2Q_x}{dx^2} + \frac{dp_r}{dx} + \mu\frac{p_x}{a} + \frac{1}{a}\frac{dN_T}{dx}\right) \tag{8-22}$$

If we use eq. (8-14a) with eq. (8-22), we obtain the following expression for the axial moment stress resultant in terms of the transverse shear stress resultant:

$$M_x = -\left(\frac{Da^2}{Eh}\right)\left(\frac{d^3Q_x}{dx^3} + \frac{d^2p_r}{dx^2} + \frac{1}{a}\frac{d^2N_T}{dx^2} + \frac{\mu}{a}\frac{dp_x}{dx}\right) - \frac{M_T}{1-\mu} \tag{8-23}$$

The governing differential equation in the stress formulation for cylindrical shells may be obtained by combining eqs. (8-7) with eq. (8-23):

$$\frac{d^4Q_x}{dx^4} + \left(\frac{Eh}{Da^2}\right)Q_x + \frac{d^3p_r}{dx^3} + \frac{1}{a}\frac{d^3N_T}{dx^3} + \frac{\mu}{a}\frac{d^2p_x}{dx^2} + \frac{Eh}{(1-\mu)Da^2}\frac{dM_T}{dx} = 0 \tag{8-24}$$

As was the case for the displacement formulation, all other quantities may be obtained from the solution for the transverse shear stress resultant:

$$M_x = -\left(\frac{Da^2}{Eh}\right)\left(\frac{d^3Q_x}{dx^3} + \frac{d^2p_r}{dx^2} + \frac{1}{a}\frac{d^2N_T}{dx^2} + \frac{\mu}{a}\frac{dp_x}{dx}\right) - \frac{M_T}{1-\mu} \tag{8-23}$$

$$M_\theta = -\left(\frac{\mu Da^2}{Eh}\right)\left(\frac{d^3Q_x}{dx^3} + \frac{d^2p_r}{dx^2} + \frac{1}{a}\frac{d^2N_T}{dx^2} + \frac{\mu}{a}\frac{dp_x}{dx}\right) - \frac{M_T}{1-\mu} \tag{8-14b}$$

$$N_x = -\int p_x dx + C \tag{8-5}$$

$$N_\theta = a\frac{dQ_x}{dx} + p_r a \tag{8-2}$$

$$\omega_x = \frac{a^2}{Eh}\left(\frac{d^2Q_x}{dx^2} + \frac{dp_r}{dx} + \mu\frac{p_x}{a} + \frac{1}{a}\frac{dN_T}{dx}\right) \tag{8-22}$$

$$w = \left(\frac{a^2}{Eh}\right)\left(\frac{dQ_x}{dx} - \frac{\mu}{a}N_x + \frac{N_T}{a} + p_r\right) \tag{8-11b}$$

Example 8-1 A cylindrical shell of radius a, length L, and thickness h has the following temperature distribution imposed through the thickness of the shell:

$$\Delta T = T - T_0 = \frac{1}{2}\left(\frac{2z}{h}\right)(T_1 - T_0) = \frac{1}{2}\left(\frac{2z}{h}\right)\Delta T_1$$

The thermal moment and thermal stress resultant are as follows.

$$M_T = \tfrac{1}{12}\alpha E \Delta T_1 h^2 \quad \text{and} \quad N_T = 0$$

Both ends of the shell are simply-supported, as shown in Figure 8-3. No mechanical loads are applied to the shell (except at the ends of the shell). Determine the expressions for the transverse displacement of the shell midplane and the stresses if the shell is quite long.

Let us use the displacement formulation in this case. The general expression, eq. (8-17), reduces to the following for this problem:

$$\frac{d^4w}{dx^4} + 4\beta^4 w = 0$$

The quantity β is defined by

$$4\beta^4 \equiv \frac{Eh}{Da^2} \quad \text{or,} \quad \beta = \frac{[3(1-\mu^2)]^{1/4}}{\sqrt{ah}} \tag{8-25}$$

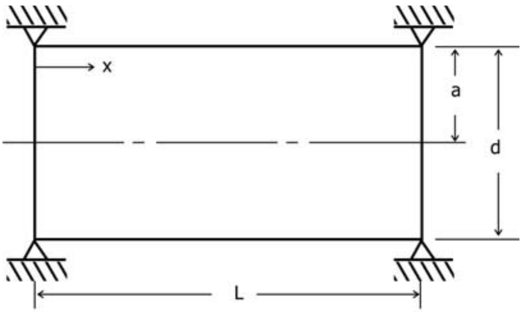
The characteristic values for the governing differential equation are

$$m^4 + 4\beta^4 = 0 \quad \text{or} \quad (m^2 + 2\beta^2)^2 - 4m^2\beta^2 = 0$$

The four solutions for the characteristic values are

$$m^2 + 2\beta^2 = \pm 2m\beta \quad \text{or} \quad m^2 \pm 2m\beta + 2\beta^2 = 0$$

$$m_{1,2,3,4} = \pm\beta \pm \sqrt{-\beta^2} = \pm\beta \pm i\beta = \pm\beta(1 \pm i)$$



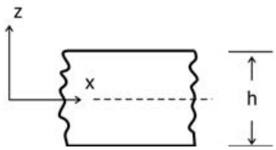


Figure 8-3. Cylindrical vessel with simply-supported edges, Example 8-1.

The general solution is

$$w(x) = C_1 \exp(m_1 x) + C_2 \exp(m_2 x) + C_3 \exp(m_3 x) + C_4 \exp(m_4 x)$$

The solution for the transverse displacement may be written in terms of the trigonometric functions (sine and cosine) and exponential functions:

$$w(x) = e^{-\beta x}(C_1 \cos \beta x + C_2 \sin \beta x) + e^{\beta x}(C_3 \cos \beta x + C_4 \sin \beta x) \quad (8\text{-}26)$$

If the cylinder is very long ($x \to \infty$), the term involving $e^{+\beta x}$ must be zero, if the displacement is to correspond to the physical situation, so we must have $C_3 = C_4 = 0$:

$$w(x) = e^{-\beta x}(C_1 \cos \beta x + C_2 \sin \beta x) \quad (8\text{-}27)$$

The rotation is

$$\omega_x = \frac{dw}{dx} = \beta e^{-\beta x}[(-C_1 + C_2) \cos \beta x - (C_1 + C_2) \sin \beta x] \quad (8\text{-}29)$$

The derivative of the slope is

$$\frac{d\omega_x}{dx} = \frac{d^2 w}{dx^2} = \beta^2 e^{-\beta x}[2C_1 \sin \beta x - 2C_2 \cos \beta x] \quad (8\text{-}30)$$

At the simply-supported edge, the transverse displacement and moment stress resultant are both equal to zero:

$$w(0) = 0 = C_1$$

$$M_r(0) = -D\frac{d^2w}{dx^2} - \frac{M_T}{1-\mu} = 0$$

$$C_2 = \frac{M_T}{2(1-\mu)D\beta^2} = \frac{(1+\mu)\alpha\Delta T_1 h}{2(\beta h)^2}$$

The final expression for the transverse deflection is

$$w(x) = \frac{(1+\mu)\alpha\Delta T_1 h}{2(\beta h)^2} e^{-\beta x}\sin(\beta x) \tag{8-31}$$

The maximum transverse deflection occurs at the point $\left(\beta x = \frac{1}{4}\pi\right)$ and has the following value:

$$w_{\max} = \frac{(1+\mu)\alpha\Delta T_1 h}{2(\beta h)^2} e^{-\pi/4}\sin(\pi/4) = \frac{(1+\mu)\alpha\Delta T_1 h}{6.204(\beta h)^2} \tag{8-32}$$

The rotation is

$$\omega_x(x) = \frac{(1+\mu)\alpha\Delta T_1}{2(\beta h)} e^{-\beta x}(\cos\beta x - \sin\beta x) \tag{8-33}$$

The axial moment stress resultant is

$$M_x(x) = -\frac{\alpha E\Delta T_1 h^2}{12(1-\mu)}\left[1 - e^{-\beta x}\cos(\beta x)\right] \tag{8-34}$$

The radial stress is found from eq. (8-19a) with $N_x = 0$ from eq. (8-4):

$$\sigma_x(x) = \pm\frac{\alpha E\Delta T_1}{2(1-\mu)}\left[1 - e^{-\beta x}\cos(\beta x)\right] \tag{8-35}$$

The maximum axial stress occurs at $\beta x_m = \frac{3}{4}\pi$ and has the following value:

$$(\sigma_x)_{\max} = \pm\frac{\alpha E\Delta T_1}{2(1-\mu)}[1.0670] = \pm\frac{\alpha E\Delta T_1}{1.8744(1-\mu)} \tag{8-36}$$

The axial bending stress far away from the simply-supported edge ($x \to \infty$) is

$$(\sigma_x)_\infty = \pm\frac{\alpha E\Delta T_1}{2(1-\mu)} \tag{8-37}$$

We are now in a position to define what constitutes a very long cylinder. Technically, a "long" cylinder is one for which the effects of bending at the ends of the cylinder have damped out in the middle portion of the cylinder. Suppose we accept a stress deviation of 1 percent from the stress at infinity in the center of the cylinder ($x = \frac{1}{2}L$):

$$\left|\frac{\sigma_x\left(\frac{1}{2}L\right) - (\sigma_x)_{\max}}{(\sigma_x)_{\max}}\right| = e^{-\beta L/2}\cos(\beta L/2) \le e^{-\beta L/2} \le 0.01$$

This condition is met by the following expression:

$$\tfrac{1}{2}\beta L \geq \ln(100) = 4.605$$

The cylinder may be treated as an "infinite cylinder" or a very long cylinder if the following condition is met:

$$\beta L = \frac{[3(1-\mu^2)]^{1/4}}{\sqrt{h/a}}\left(\frac{L}{a}\right) = \frac{[12(1-\mu^2)]^{1/4}}{\sqrt{h/d}}\left(\frac{L}{d}\right) \geq 9.21 \approx 9 \qquad (8\text{-}38)$$

The quantity d is the diameter of the midplane (the vessel *mean diameter*). For the case of Poisson's ratio equal to 0.3, eq. (8-37) may be written as

$$\sqrt{\frac{d}{h}}\left(\frac{L}{d}\right) \geq 5.066 \approx 5 \quad \text{for a long cylinder} \qquad (8\text{-}39)$$

The circumferential membrane stress resultant N_θ is found from eq. (8-11b), with $N_x = N_T = 0$ for this example, and the circumferential bending stress resultant M_θ is found from eq. (8-14b):

$$N_\theta = \frac{Ehw}{a} + \mu N_x + N_T = \frac{(1+\mu)\alpha E \Delta T_1 h^2}{2a(\beta h)^2} e^{-\beta x}\sin(\beta x) \qquad (8\text{-}40)$$

$$M_\theta = \mu M_x = -\frac{\mu\alpha E \Delta T_1 h^2}{12(1-\mu)}\left[1 - e^{-\beta x}\cos(\beta x)\right] \qquad (8\text{-}41)$$

The circumferential stress is found from eq. (8-19b):

$$\sigma_\theta = \frac{N_\theta}{h} \pm \frac{6M_\theta}{h^2} = \frac{\alpha E \Delta T_1}{2(1-\mu)}\left\{e^{-\beta x}\left[\frac{(1-\mu^2)h}{(\beta h)^2 a}\sin(\beta x) \pm \mu\cos(\beta x)\right] \mp \mu\right\} \qquad (8\text{-}42)$$

Using the definition of the β function, eq. (8-25), the following term may be simplified:

$$\frac{(1-\mu^2)h}{(\beta h)^2 a} = \frac{(1-\mu^2)(h/a)}{\sqrt{3(1-\mu^2)}(h/a)} = \sqrt{\tfrac{1}{3}(1-\mu^2)}$$

Using this result, the circumferential stress may be written as

$$\sigma_\theta = \frac{\alpha E \Delta T_1}{2(1-\mu)}\left\{e^{-\beta x}\left[\sqrt{\tfrac{1}{3}(1-\mu^2)}\sin(\beta x) \pm \mu\cos(\beta x)\right] \mp \mu\right\} \qquad (8\text{-}43)$$

The maximum circumferential stress occurs at $(\beta x_m) \approx 2.85$ for Poisson's ratio of $\mu = 0.3$.

The transverse shear resultant may be found from the axial moment balance, eq. (8-7):

$$Q_x = \frac{dM_x}{dx} = -\frac{\alpha E \Delta T_1 h}{12(1-\mu)}(\beta h)e^{-\beta x}(\cos\beta x + \sin\beta x) \qquad (8\text{-}44a)$$

The value of the maximum transverse shear stress is found from eq. (8-19c):

$$\tau_{xz} = \frac{3Q_x}{2h} = -\frac{\alpha E \Delta T_1}{8(1-\mu)}(\beta h)e^{-\beta x}(\cos\beta x + \sin\beta x) \tag{8-44b}$$

To illustrate the magnitude of the quantities that can be obtained, let us determine the deflection and stresses for a cylindrical shell having a diameter $d = 1.80\,\text{m}$ (5.905 ft), with $a = \frac{1}{2}d = 0.90\,\text{m}$, a length $L = 3.60\,\text{m}$ (11.81 ft), and a thickness $h = 18\,\text{mm}$ (0.709 in.). The shell is simply-supported and is constructed of C1020 steel, with $\alpha = 11.9 \times 10^{-6}\,{}^\circ\text{C}^{-1}$ ($6.6 \times 10^{-6}\,{}^\circ\text{F}^{-1}$), $E = 205$ GPa (29.7×10^6 psi), and $\mu = 0.28$. The temperature difference parameter (difference between the upper surface temperature and midplane temperature) is $\Delta T_1 = 35^\circ\text{C}$ (63°F).

First, let us show that this shell falls in the long cylinder category. Using eq. (8-39), the dimensionless parameter βL for the shell is

$$\beta L = \frac{[12(1-0.28^2)]^{1/4}}{\sqrt{0.018/1.80}}\left(\frac{3.60}{1.80}\right) = 36.47 > 9 \qquad \Rightarrow \text{long cylinder}$$

The parameter β is

$$\beta = \frac{36.47}{3.60} = 10.131\,\text{m}^{-1} = \frac{1}{0.0987\,\text{m}}$$

At a distance of about 99 mm (3.9 in.) from the simply-supported edge, the effect of bending at the edge has damped out, for practical purposes. Note that $\beta h = (10.131)(0.018) = 0.1824$.

The maximum deflection of the shell occurs at $\beta x = \frac{1}{4}\pi$, or at

$$x = \frac{\pi}{(4)(10.131)} = 0.0775\text{ m} = 77.4\text{ mm (3.05 in.)}$$

The maximum deflection is determined from eq. (8-32):

$$\begin{aligned} w_{\max} &= \frac{(1+0.28)(11.9\times10^{-6})(35^\circ)(0.018)}{(6.204)(0.1824)^2} \\ &= 0.0465\times10^{-3}\text{ m} = 0.0465\text{ mm} \quad (0.0018\text{ in.}) \end{aligned}$$

The maximum axial bending stress occurs at $\beta x_m = \frac{3}{4}\pi$, or at

$$x = \frac{3\pi}{(4)(10.131)} = 0.233\text{ m} = 233\,\text{mm} \quad (9.15\,\text{in.})$$

The maximum axial bending stress is determined from eq. (8-36):

$$\begin{aligned} (\sigma_x)_{\max} &= \pm\frac{(11.9\times10^{-6})(205\times10^9)(35^\circ)}{(1.8744)(1-0.28)} = 63.3\times10^6\,\text{Pa} \\ &= 63.3\,\text{MPa}\ (9176\,\text{psi}) \end{aligned}$$

Let us calculate the circumferential stress at $(\beta x) = 2.8$ or $x = 276\,\text{mm}$:

$$e^{-\beta x}\left[\sqrt{\tfrac{1}{3}(1-\mu^2)}\sin(\beta x) - \mu\cos(\beta x)\right] + \mu$$
$$= e^{-2.8}\left[\sqrt{\tfrac{1}{3}(1-0.28^2)}\sin(2.8) - (0.28)\cos(2.8)\right] + 0.28 = 0.7295$$

Note that the quantity (βx) must be expressed in *radian* units and not in degrees (even if a radian is dimensionless) for numerical calculations. The maximum circumferential bending stress is determined from eq. (8-43):

$$\sigma_\theta = \frac{(11.9\times10^{-6})(205\times10^9)(35^\circ)}{(2)(1-0.28)}\{0.7295\} = 43.25\times10^6\,\text{Pa}$$
$$= 43.25\,\text{MPa} \quad (6270\,\text{psi})$$

The transverse shear resultant at the simply-supported edge of the shell $(x = 0)$ may be found from eq. (8-7):

$$(Q_x)_{x=0} = -\frac{(11.9\times10^{-6})(205\times10^9)(35^\circ)(0.018)}{(12)(1-0.28)}(0.1824)$$
$$(Q_x)_{x=0} = -29.95\times10^3\,\text{N/m} = 29.95\,\text{kN/m}\ (171\,\text{lb}_\text{f}/\text{in.})$$

The edge support reaction is $V_0 = -(Q_x)x = 0$. The maximum transverse shear stress at the edge of the shell is found from eq. (4-44):

$$\tau_{xz} = \frac{(3)(-29.95\times10^3)}{(3)(0.018)} = -2.49\times10^6\,\text{Pa} = -2.49\,\text{MPa} \quad (-361\,\text{psi})$$

8.3 COOLDOWN OF RING-STIFFENED CYLINDRICAL VESSELS

Let us consider the stress analysis problem involved for the cooldown of a cylindrical vessel that is stiffened by circular rings, as shown in Figure 8-4. This problem is particularly important for cryogenic fluid storage vessels, in which the shell and rings may cool down at different rates and produce serious thermal stresses during the cooldown process.

8.3.1 Thermal Analysis

Using the *lumped-capacity* approach [Holman, 1997], we may write the following energy balance expressions for the shell material (subscript *sh*) and the ring material (subscript *rg*):

$$\rho_\text{sh}V_\text{sh}c_\text{sh}\frac{dT_\text{sh}}{dt} = -h_cA_\text{sh}\left(T_\text{sh} - T_f\right) \tag{8-45a}$$

$$\rho_\text{rg}V_\text{rg}c_\text{rg}\frac{dT_\text{rg}}{dt} = -h_cA_\text{rg}\left(T_\text{rg} - T_f\right) \tag{8-45b}$$

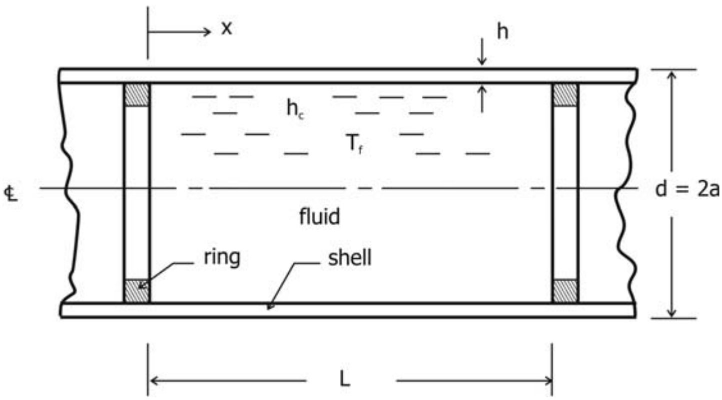


Figure 8-4. Ring-stiffened cylindrical vessel. The ring may be a structural beam rolled to fit the cylindrical vessel. The ring is usually made an integral part of the shell by welding the ring to the shell.

The quantity ρ is the material density, V is the volume of the element, c is the material specific heat, h_c is the convection heat transfer coefficient between the surface and the surrounding fluid, A is the surface area exposed to the fluid, and T_f is the fluid temperature. Initially (at $t = 0$), both the shell and ring are at the same temperature $T(0) = T_0$, the "stress-free temperature."

The solutions of eqs. (8-45), subject to the initial condition, are

$$T_{\mathrm{sh}} - T_f = (T_0 - T_f)\exp(-t/\tau_{\mathrm{sh}}) \tag{8-46a}$$

$$T_{\mathrm{rg}} - T_f = (T_0 - T_f)\exp(-t/\tau_{\mathrm{rg}}) \tag{8-46b}$$

The quantities τ are the *time constants*, which give a measure of how rapidly the components cool down:

$$\tau_{\mathrm{sh}} = \frac{\rho_{\mathrm{sh}} c_{\mathrm{sh}} V_{\mathrm{sh}}}{h_c A_{\mathrm{sh}}} \tag{8-47a}$$

$$\tau_{\mathrm{rg}} = \frac{\rho_{\mathrm{rg}} c_{\mathrm{rg}} V_{\mathrm{rg}}}{h_c A_{\mathrm{rg}}} \tag{8-47b}$$

The system will achieve steady state ($T_{\mathrm{sh}} = T_{\mathrm{rg}} = T_f$) after an elapsed time of approximately $t_{\mathrm{ss}} = 5\tau$.

Initially and at steady state, the shell and ring are at the same temperature, and the thermal stresses will be zero, if the ring and shell are constructed of the same material. At some intermediate time, $t_{\max}$, the temperature difference between the ring and shell ($T_{\mathrm{rg}} - T_{\mathrm{sh}}$) will reach a maximum, and the thermal stresses will be maximum at this time. If we set the derivative of the temperature difference

$$T_{\mathrm{rg}} - T_{\mathrm{sh}} = (T_0 - T_f)[\exp(-t/\tau_{\mathrm{rg}}) - \exp(-t/\tau_{\mathrm{sh}})] \tag{8-48}$$

equal to zero and solve for the time, the following expression is obtained for the time at which the maximum temperature difference occurs:

$$t_{\max} = \tau_{rg} \left[\frac{\ln\left(\dfrac{\tau_{rg}}{\tau_{sh}}\right)}{\left(\dfrac{\tau_{rg}}{\tau_{sh}}\right) - 1} \right] \tag{8-49}$$

8.3.2 Stress Analysis

For a uniform temperature of the shell, the expressions for the thermal membrane stress resultant N_T and thermal bending moment M_T are

$$N_T = \alpha E(T_{sh} - T_0) \int_{-h/2}^{+h/2} dz = \alpha E(T_{sh} - T_0)h \quad \text{(independent of } x\text{)} \tag{8-50a}$$

$$N_T = \alpha E(T_{sh} - T_0) \int_{-h/2}^{+h/2} z\, dz = 0 \tag{8-50b}$$

Let us use the *stress formulation* in this example. The governing equation, eq. (8-24), reduces to

$$\frac{d^4 Q_x}{dx^4} + 4\beta^4 Q_x = 0 \tag{8-51}$$

The quantity β is given by eq. (8-25). The general solution of eq. (8-51) is

$$Q_x = e^{-\beta x}(C_1 \cos\beta x + C_2 \sin\beta x) + e^{\beta x}(C_3 \cos\beta x + C_4 \sin\beta x) \tag{8-52}$$

If the rings are spaced sufficiently far apart or if eq. (8-38) applies, the shell may be treated as a long cylinder, for which the $e^{\beta x}$ term must drop out, or $C_3 = C_4 = 0$. The transverse shear resultant may then be written as follows for a long cylinder:

$$Q_x = e^{-\beta x}(C_1 \cos\beta x + C_2 \sin\beta x) \tag{8-53}$$

The shell rotation as a function of the transverse shear resultant given by eq. (8-22). For no applied mechanical loads ($p_x = p_r = 0$) and a uniform membrane thermal stress resultant, the rotation expression reduces to

$$\omega_x = \frac{a^2}{Eh}\frac{d^2 Q_x}{dx^2} = \frac{2a^2\beta^2}{Eh} e^{-\beta x}\ (C_1 \sin\beta x - C_2 \cos\beta x) \tag{8-54}$$

Because of symmetry, the rotation at the shell–ring junction ($x = 0$) is zero:

$$(\omega_x)_{x=0} = 0 = -\frac{2a^2\beta^2}{Eh} C_2 \quad \text{or} \quad C_2 = 0$$

With this condition, the transverse shear stress resultant may be written as

$$Q_x = C_1 e^{-\beta x} \cos\beta x \tag{8-55}$$

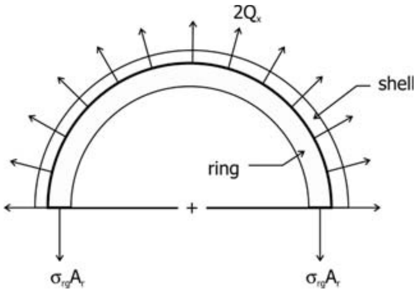


Figure 8-5. Forces acting on the stiffening ring.

Let us consider the loads on the ring caused by the thermal stresses in the shell, as shown in Figure 8-5. If we make a force balance on the half-ring element, the following is obtained:

$$(2a)(2Q_x)_{x=0} = 2\sigma_{\mathrm{rg}} A_r$$

$$(Q_x)_{x=0} = \frac{\sigma_{\mathrm{rg}} A_r}{2a} = C_1 \tag{8-56}$$

The quantity A_r is the cross-sectional area of the ring. Using the result from eq. (8-56) in eq. (8-55), the transverse shear resultant may be written as

$$Q_x = \left(\frac{\sigma_{\mathrm{rg}} A_r}{2a}\right) e^{-\beta x} \cos \beta x \tag{8-57}$$

8.3.3 Stresses in the Ring and Shell

The unknown stress in the ring σ_{rg} may be determined from the fact that the circumferential strain for the ring and for the shell are the same at the shell–ring junction ($x = 0$). The circumferential strain of the shell may be found from eq. (8-9b):

$$(\varepsilon_\theta)_{\mathrm{rg}} = \frac{\sigma_{\mathrm{rg}}}{E} + \alpha(T_{\mathrm{rg}} - T_0) \tag{8-58a}$$

$$(\overline{\varepsilon}_\theta)_{\mathrm{sh}} = \frac{(N_\theta - \mu N_x)_{x=0}}{Eh} + \frac{N_T}{Eh} = \frac{(N_\theta)_{x=0}}{Eh} + \alpha(T_{\mathrm{sh}} - T_0) \tag{8-58b}$$

The circumferential membrane stress resultant is found from eq. (8-2) with no mechanical load applied:

$$N_\theta = a\frac{dQ_x}{dx} = -\frac{1}{2}\sigma_{\mathrm{rg}} A_r \beta e^{-\beta x}(\cos \beta x - \sin \beta x) \tag{8-59}$$

Combining eqs. (8-58) and (8-59), we obtain

$$\frac{\sigma_{\mathrm{rg}}}{E} + \alpha(T_{\mathrm{rg}} - T_0) = -\frac{\sigma_{\mathrm{rg}} A_r \beta}{2Eh} + \alpha(T_{\mathrm{sh}} - T_0) \tag{8-60}$$

The stress in the ring may be written as

$$\sigma_{\mathrm{rg}} = \frac{\alpha E(T_{\mathrm{sh}} - T_{\mathrm{rg}})}{1 + \frac{1}{2}(A_r/h^2)(\beta h)} \tag{8-61}$$

The term in the denominator of eq. (8-61) may be written in the following form:

$$\tfrac{1}{2}(A_r/h^2)(\beta h) = \left[\tfrac{3}{4}(1-\mu^2)\right]^{1/4}(A_r/h^2)\sqrt{h/d} \tag{8-62}$$

The axial bending stress resultant may be obtained from eq. (8-23), with no applied mechanical loads ($p_r = p_x = 0$), constant membrane thermal stress resultant, and zero thermal moment:

$$M_x = -\left(\frac{Da^2}{Eh}\right)\frac{d^3 Q_x}{dx^3} = \left(\frac{Da^2}{Eh}\right)\left(\frac{\sigma_{\mathrm{rg}} A_r}{a}\right)\beta^3 e^{-\beta x}(\sin\beta x - \cos\beta x) \tag{8-63}$$

The coefficient term may be simplified by using eq. (8-25):

$$M_x = \left(\frac{\sigma_{\mathrm{rg}} A_r}{4\beta a}\right) e^{-\beta x}(\sin\beta x - \cos\beta x) \tag{8-64}$$

The axial membrane stress in the shell is zero ($N_x = 0$), so the axial bending stress in the shell may be found from

$$\sigma_x = \pm\frac{6M_x}{h^2} = \pm\left(\frac{3\sigma_{\mathrm{rg}} A_r}{2\beta a h^2}\right) e^{-\beta x}(\sin\beta x - \cos\beta x) \tag{8-65}$$

The coefficient term in eq. (8-65) may be simplified using eq. (8-25):

$$\sigma_x = \pm\left\{\frac{3(A_r/h^2)\sqrt{h/d}}{[12(1-\mu^2)]^{1/4}}\right\}\sigma_{\mathrm{rg}} e^{-\beta x}(\sin\beta x - \cos\beta x) \tag{8-66}$$

The maximum axial stress in the shell occurs at the junction between the shell and the ring ($x = 0$):

$$(\sigma_x)_{\max} = \mp\frac{3(A_r/h^2)\sqrt{h/d}}{[12(1-\mu^2]^{1/4}}\sigma_{\mathrm{rg}} \tag{8-67}$$

The circumferential bending stress resultant is found from eq. (8-14b), with zero mechanical loads ($p_r = p_x = 0$), constant membrane stress resultant, and zero thermal bending stress resultant:

$$M_\theta = -\left(\frac{\mu D a^2}{Eh}\right)\frac{d^3 Q_x}{dx^3} = \mu_x M_x \tag{8-68}$$

The circumferential stress may be found from eq. (8-19b):

$$\sigma_\theta = \frac{N_\theta}{h} \pm \frac{6M_\theta}{h^2} = -\frac{1}{2}(A_r/h^2)(\beta h)\sigma_{\text{rg}} e^{-\beta x}(\cos\beta x - \sin\beta x) \pm \mu\sigma_x \tag{8-69}$$

Using eq. (8-66) for the axial stress σ_x, we may write eq. (8-69)

$$\sigma_\theta = \frac{1}{2}(\beta h)\left[1 \pm \frac{6\mu}{(\beta h)(\beta d)}\right](A_r/h^2)\sigma_{\text{rg}} e^{-\beta x}(\sin\beta x - \cos\beta x) \tag{8-70}$$

The coefficient in eq. (8-70) may be simplified using eq. (8-25):

$$\sigma_\theta = \left[\frac{3}{4}(1-\mu^2)\right]^{1/4}\left[1 \pm \frac{\mu}{\sqrt{\frac{1}{3}(1-\mu^2)}}\right]\left(\frac{A_r}{h^2}\right)\sqrt{\frac{h}{d}}\sigma_{\text{rg}} e^{-\beta x}(\sin\beta x - \cos\beta x) \tag{8-71}$$

The numerical value for the coefficient in eq. (8-71) for $\mu = 0.3$ is

$$\left[\frac{3}{4}(1-\mu^2)\right]^{1/4}\left[1 \pm \frac{\mu}{\sqrt{\frac{1}{3}(1-\mu^2)}}\right] = \begin{cases} 0.9916 \text{ for the } + \text{ sign} \\ 0.8262 \text{ for the } - \text{ sign} \end{cases}$$

The maximum circumferential stress occurs at the shell-ring junction ($x = 0$):

$$(\sigma_\theta)_{\max} = -\left[\frac{3}{4}(1-\mu^2)\right]^{1/4}\left[1 \pm \frac{\mu}{\sqrt{\frac{1}{3}(1-\mu^2)}}\right]\left(\frac{A_r}{h^2}\right)\sqrt{\frac{h}{d}}\sigma_{\text{rg}} \tag{8-72}$$

The ratio of the maximum circumferential to axial stresses is a function of Poisson's ratio only:

$$\frac{(\sigma_\theta)_{\max}}{(\sigma_x)_{\max}} = \mp\left[\sqrt{\tfrac{1}{3}(1-\mu^2)} \pm \mu\right] \tag{8-73}$$

For Poisson's ratio $\mu = 0.3$, this ratio has the following values:

$$\frac{(\sigma_\theta)_{\max}}{(\sigma_x)_{\max}} = \begin{cases} -0.851 \text{ for the upper sign} \\ +0.251 \text{ for the lower sign} \end{cases}$$

The maximum axial stress in the shell is larger than the maximum circumferential stress.

The preceding analysis is based on the condition that the shell can be treated as a "long cylinder." The shell may be treated as a "long cylinder" if the following condition, from eq. (8-38), is valid:

$$\frac{L}{d} \geq \frac{9\sqrt{h/d}}{[12(1-\mu^2)]^{1/4}} \tag{8-74}$$

Example 8-2 A pressure vessel for propane storage is constructed of 304 stainless steel, with $\alpha = 16.0 \times 10^{-6}\ {}^\circ\mathrm{C}^{-1}$ ($8.9 \times 10^{-6}\ {}^\circ\mathrm{F}^{-1}$), $E = 193$ GPa (28.0×10^6 psi), $\mu = 0.305$, density $\rho = 7820\ \mathrm{kg/m^3}$ ($488\ \mathrm{lb_m/ft^3} = 0.283\ \mathrm{lb_m/in^3}$), and specific heat $c = 0.460$ kJ/kg-°C (0.110 Btu/lb$_m$-°F). The vessel is a ring-stiffened cylinder with a spacing between the rings $L = 2.00$ m (78.74 in.), a mean diameter $d = 2.00$ m (78.74 in.), and a shell wall thickness $h = 6$ mm (0.236 in.). The stiffening rings have a cross-sectional area $A_r = 32\ \mathrm{cm^2}$ ($4.96\ \mathrm{in^2}$) and a volume-to-surface area $V_{rg}/A_{rg} = 16$ mm (0.630 in.). The vessel is initially at 25°C (77°F), and liquid propane at $T_f = -45$°C (−49°F) is suddenly introduced into the vessel. The convective heat transfer coefficient between the liquid and the vessel is $h_c = 25\ \mathrm{W/m^2}$-°C (4.40 Btu/hr-ft^2-°F). Determine the maximum stress in the stiffening ring and in the shell.

The volume-to-surface area for the shell is given by

$$\frac{V_{sh}}{A_{sh}} = \frac{\pi d L h}{\pi(d-h)L} = \frac{h}{1-h/d} = \frac{6}{1-6/2000} = 6.018\ \text{mm}\ (0.237\ \text{in.})$$

The time constants for the shell and stiffening ring are found from eqs. (8-47).

$$\text{Shell}: \quad \tau_{sh} = \frac{\rho_{sh} c_{sh}(V_{sh}/A_{sh})}{h_c} = \frac{(7820)(460)(6.018\times 10^{-3})}{(25)} = 865.9\ \text{s} = 14.43\ \text{min.}$$

$$\text{Ring}: \quad \tau_{rg} = \frac{\rho_{rg} c_{rg}(V_{rg}/A_{rg})}{h_c} = \frac{(7820)(460)(16\times 10^{-3})}{(25)} = 2302.2\ \text{s} = 38.37\ \text{min.}$$

We observe that the shell will cool at a faster rate than the ring.

The time at which the maximum temperature difference between the shell and stiffening ring occurs is found from eq. (8-49):

$$t_{max} = (2302.2)\left[\frac{\ln\left(\dfrac{2302.2}{865.9}\right)}{\left(\dfrac{2302.2}{865.9}\right)-1}\right] = (2302.2)(0.5895) = 1357.2\ \text{s} = 22.62\ \text{min.}$$

The maximum temperature difference between the shell and stiffening ring is found from eq. (8-48) using the time $t = t_{max}$:

$$T_{rg} - T_{sh} = (25^\circ + 45^\circ)\left[\exp\left(-\frac{1357.2}{2302.2}\right) - \exp\left(-\frac{1357.2}{865.9}\right)\right] = 24.2^\circ\text{C} \quad (43.6^\circ\text{F})$$

Let us determine if the long cylinder analysis is applicable in this example. Using eq. (8-74), the following is obtained:

$$\frac{9\sqrt{h/d}}{[12(1-\mu^2)]^{1/4}} = \frac{(9)\sqrt{0.006/2.00}}{[(12)(1-0.305^2)]^{1/4}} = 0.895$$

$$\frac{L}{d} = \frac{2.00}{2.00} = 1 > 0.895$$

The long cylinder condition *is* valid for this problem.
The parameter (βh) may be found from eq. (8-25):

$$\beta h = [12(1-\mu^2)]^{1/4}\sqrt{h/d} = [(12)(1-0.305^2)]^{1/4}\sqrt{0.006/2.00} = 0.09948$$

The maximum stress in the stiffening ring is found from eq. (8-61):

$$\sigma_{\mathrm{rg}} = \frac{(16.0\times 10^{-6})(193\times 10^9)(-24.2^\circ)}{1+\left[\dfrac{(32\times 10^{-4})(0.09948)}{(2)(0.006)^2}\right]} = \frac{-74.73\times 10^6}{1+4.421}$$

$$\sigma_{\mathrm{rg}} = -13.78\times 10^6 \text{ Pa} = -13.78 \text{ MPa } (-1999 \text{ psi})$$

The stress in the ring is a *compressive* stress.
The maximum axial bending stress is found from eq. (8-67):

$$(\sigma_r)_{\max} = \mp\frac{(3)(32\times 10^{-4}/0.006^2)\sqrt{(0.006)/(2.00)}}{[(12)(1-0.305^2)]^{1/4}}(-13.78)$$

$$(\sigma_r)_{\max} = \pm(8.041)(13.78) = \pm 110.8 \text{ MPa} \quad (\pm 16{,}070 \text{ psi})$$

The maximum circumferential stress is found from eq. (8-72). For a Poisson's ratio $\mu = 0.305$, we find the following value for the coefficient term:

$$\left[\frac{3}{4}(1-\mu^2)\right]^{1/4}\left[1+\frac{\mu}{\sqrt{\frac{1}{3}(1-\mu^2)}}\right] = 1.412$$

$$(\sigma_\theta)_{\max} = -(1.412)(32\times 10^{-4}/0.006^2)\sqrt{(0.006)/(2.00)}(-13.78)$$

$$(\sigma_\theta)_{\max} = (6.874)(13.78) = 94.7 \text{ MPa} \quad (13{,}740 \text{ psi})$$

8.4 CYLINDRICAL VESSELS WITH AXIAL TEMPERATURE VARIATION

Let us consider another example of thermal stresses in cylindrical shells that involve temperature variation in the axial direction. This situation could arise in a vertical cylindrical vessel, as shown in Figure 8-6, in which the vessel is

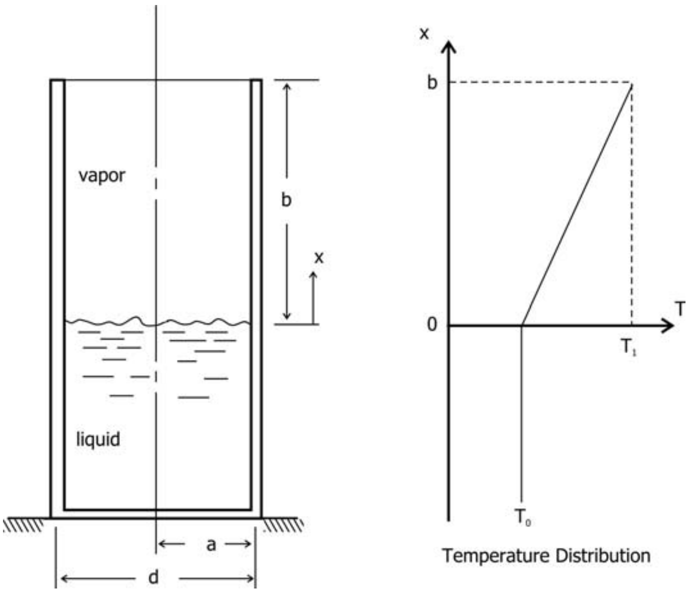


Figure 8-6. Vertical cylindrical vessel partially filled with a liquid. The temperature of the portion of the shell exposed to the liquid is constant, and the temperature of the portion exposed to the vapor varies linearly along the length of the shell.

partially filled with a liquid and the temperature of the vapor above the liquid is not uniform.

For this example, let us consider the case in which the temperature is uniform through the thickness of the shell and varies linearly along the portion of the shell exposed to the vapor:

$$\Delta T(x) = T(x) - T_0 = (T_1 - T_0)(x/b) = \Delta T_1(x/b) \tag{8-75}$$

The temperature of the portion of the shell exposed to the liquid is constant and equal to T_0. For this temperature distribution, the membrane thermal stress resultant and bending stress resultant are

$$N_T = \alpha E \int_{-h/2}^{+h/2} \Delta T_1(x/b)\,dz = \alpha E \Delta T_1 h(x/b) = N_0(x/b) \tag{8-76}$$

The quantity N_0 is

$$N_0 = \alpha E \Delta T_1 h$$

$$M_T = \alpha E \int_{-h/2}^{+h/2} \Delta T_1(x/b) z\,dz = 0 \tag{8-77}$$

Let us use the displacement formulation for this example. For zero applied mechanical loading ($p_r = p_x = 0$), eq. (8-5) yields a value for the axial membrane stress resultant $N_x = 0$. If we make these substitutions into the governing relationship, eq. (8-17), the following is obtained for the portion of the shell exposed to the vapor ($x \geq 0$).

$$\frac{d^4 w}{dx^4} + \frac{Eh}{Da^2} w = \left(\frac{N_0}{Da^2}\right)\left(\frac{x}{b}\right) \tag{8-78}$$

The solution of eq. (8-78) is

$$w(x) = e^{-\beta x}(C_1 \cos \beta x + C_2 \sin \beta x) + e^{\beta x}(C_3 \cos \beta x + C_4 \sin \beta x) + \frac{N_0}{4\beta^4 Da}\left(\frac{x}{b}\right) \tag{8-79}$$

The quantity β is defined by eq. (8-25).

Let us consider the case in which the shell can be treated as a long cylinder, or for which $(\beta b) > 9$. Using eq. (8-25) to simplify the last term on the right side of eq. (8-79), the governing equation for the "long cylinder" case is

$$w(x) = e^{-\beta x}(C_1 \cos \beta x + C_2 \sin \beta x) + \frac{N_0 a}{Eh}\left(\frac{x}{b}\right) \tag{8-80}$$

The radial deflection of the shell at the liquid–vapor interface ($x = 0$) is

$$w(0) = w_0 = C_1 \tag{8-81}$$

The axial rotation of the shell is

$$\omega_x(x) = \frac{dw}{dx} = -\beta e^{-\beta x}[(C_1 - C_2)\cos \beta x + (C_1 + C_2)\sin \beta x] + \frac{N_0 a}{Ehb} \tag{8-82}$$

The axial rotation of the shell at the liquid–vapor interface is

$$\omega_x(0) = -\beta(C_1 - C_2) + \frac{N_0 a}{Ehb} \tag{8-83}$$

Let us evaluate the stress resultants using the deflection solution. The axial bending stress resultant is found from eq. (8-14a) with the thermal moment $M_T = 0$ for this example:

$$M_x = -D\frac{d^2 w}{dx^2} = -2D\beta^2 e^{-\beta x}(C_1 \sin \beta x - C_2 \cos \beta x) \tag{8-84}$$

At the liquid–vapor interface, the axial bending stress resultant is

$$(M_x)_{x=0} = 2D\beta^2 C_2 = M_0 \tag{8-85}$$

The transverse shear stress resultant is found from eq. (8-16) with the thermal moment $M_T = 0$ for this example:

$$Q_x = -D\frac{d^2w}{dx^2} = -2D\beta^3 e^{-\beta x}[(C_1 + C_2)\cos\beta x - (C_1 - C_2)\sin\beta x) \quad (8\text{-}86)$$

At the liquid–vapor interface, the transverse shear stress resultant is

$$(Q_x)_{x=0} = -2D\beta^3(C_1 + C_2) = Q_0 \quad (8\text{-}87)$$

The forces and bending moments at the liquid–vapor interface of the shell are shown in Figure 8-7. For the portion of the shell exposed to the liquid (no thermal stresses) for $x \le 0$, having an applied edge moment M_0 and an applied edge shear $-Q_0$, the displacement and rotation are as follows [Ugural, 1999, p. 412]:

$$w(x) = \frac{e^{+\beta x}}{2D\beta^3}[\beta M_0(\sin\beta x - \cos\beta x) + Q_0\cos\beta x] \quad (8\text{-}88)$$

At the liquid–vapor interface,

$$w(0) = -\frac{1}{2D\beta^3}(\beta M_0 - Q_0) \quad (8\text{-}89a)$$

Similarly, the slope at the interface is given by

$$(\omega_x)_{x=0} = -\frac{1}{2D\beta^2}(2\beta M_0 - Q_0) \quad (8\text{-}89b)$$

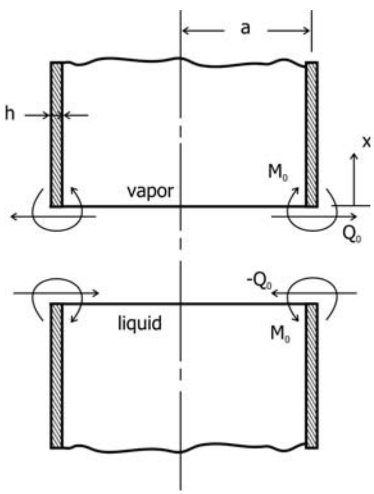


Figure 8-7. Forces and moments in the cylindrical shell at the liquid–vapor interface.

If we combine eqs. (8-81) and (8-88a) by noting that the transverse deflection is the same for the upper and lower portions of the shell at the vapor–liquid interface, then

$$w(0) = C_1 = -\frac{1}{2D\beta^3}[2D\beta^3 C_2 + 2D\beta^3(C_1 + C_2)] = -C_1 - 2C_2 \qquad \text{(8-90a)}$$

or

$$C_1 = -C_2$$

Similarly, we may combine eqs. (8-83) and (8-88a) by noting that the axial rotation is the same for the upper and lower portions of the shell at the liquid–vapor interface:

$$(\omega_x)_{x=0} = -\beta(C_1 - C_2) + \frac{N_0 a}{Eh\beta} = -\frac{1}{2D\beta^2}[4D\beta^3 C_2 - 2D\beta^3(C_1 + C_2)] \qquad \text{(8-90b)}$$

The constants C_1 and C_2 may be found from eqs. (8-90):

$$C_1 = -C_2 = \frac{N_0 a}{4Eh\beta b} = \frac{\alpha \Delta T_1 a}{4\beta b} \qquad \text{(8-91)}$$

The final expressions for the deflection and stress resultants may be obtained by substituting the value for the constant from eq. (8-91) into eqs. (8-80) and (8-84).

$$w(x) = \alpha \Delta T_1 a \left[\left(\frac{x}{b} \right) + \frac{e^{-\beta x}}{4\beta b}(\cos \beta x - \sin \beta x) \right] \qquad \text{(8-92)}$$

$$M_x(x) = -\frac{\alpha \Delta T_1 D a \beta}{2b} e^{-\beta x}(\cos \beta x + \sin \beta x) \qquad \text{(8-93)}$$

The axial bending stress is

$$\sigma_x(x) = \pm \frac{6M_x}{h^2} = \mp \frac{3\alpha \Delta T_1 D a \beta}{bh^2} e^{-\beta x}(\cos \beta x + \sin \beta x) \qquad \text{(8-93)}$$

The maximum axial bending stress occurs at the liquid–vapor interface ($x = 0$):

$$(\sigma_x)_{\max} = \frac{3\alpha \Delta T_1 D \beta a}{bh^2} = \frac{3\alpha E \Delta T_1 d}{2[12(1-\mu^2)]^{3/4} b} \sqrt{\frac{h}{d}} \qquad \text{(8-94)}$$

The axial stress distribution is shown in Figure 8-8.

The circumferential membrane stress resultant is given by eq. (8-11b), with the axial stress resultant $N_x = 0$ in this example:

$$N_\theta = \frac{Ehw}{a} - N_0 \left(\frac{x}{b} \right) + \mu N_x = \frac{\alpha E \Delta T_1 h}{4\beta b} e^{-\beta x}(\cos \beta x - \sin \beta x) \qquad \text{(8-95)}$$

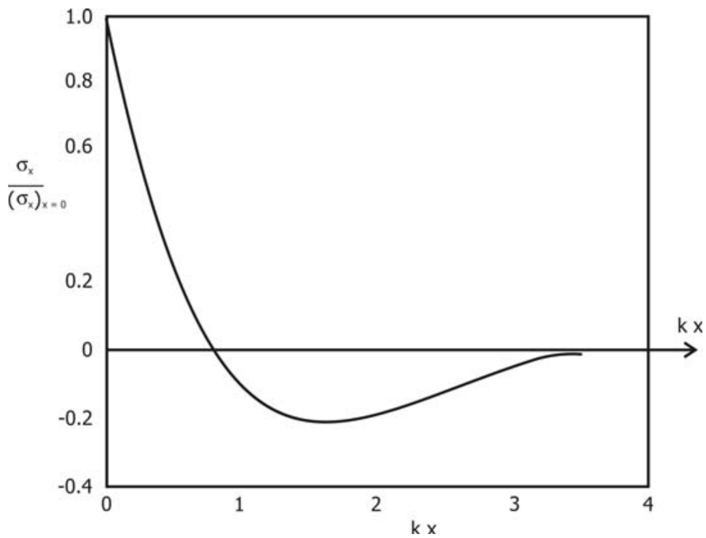


Figure 8-8. Graph of the axial stress distribution for the shell shown in Figure 8-6.

The circumferential bending stress resultant is found from eq. (8-14b), with the thermal moment $M_T = 0$ in this problem:

$$M_\theta = -\mu D \frac{d^2w}{dx^2} - \frac{M_T}{1-\mu} = -\mu D \frac{d^2w}{dx^2} = \mu M_x \tag{8-96}$$

The circumferential stress variation is given by eq. (8-18b):

$$\sigma_\theta(x) = \frac{\alpha E \Delta T_1}{4\beta b} e^{-\beta x} \left[\left(1 \mp \frac{12\mu D a \beta^2}{Eh^2}\right) \cos \beta x - \left(1 \pm \frac{12\mu D a \beta^2}{Eh^2}\right) \sin \beta x \right] \tag{8-97}$$

The following term may be simplified using the definitions of β and the flexural rigidity D:

$$\frac{12\mu D a \beta^2}{Eh^2} = \frac{\mu}{\sqrt{\frac{1}{3}(1-\mu^2)}} \tag{8-98}$$

The maximum circumferential stress occurs at the liquid–vapor interface ($x = 0$):

$$\begin{aligned}(\sigma_\theta)_{\max} &= \frac{\alpha E \Delta T_1}{4\beta b} \left[1 + \frac{\mu}{\sqrt{\frac{1}{3}(1-\mu^2)}}\right] \\ &= \frac{\alpha E \Delta T_1 d}{4[12(1-\mu^2)]^{1/4} b} \sqrt{\frac{h}{d}} \left[1 + \frac{\mu}{\sqrt{\frac{1}{3}(1-\mu^2)}}\right]\end{aligned} \tag{8-99}$$

Example 8-3 A vertical oil storage vessel, as shown in Figure 8-9, is constructed of SA-285 Grade C carbon steel, which has the following properties: $\alpha = 11.5 \times 10^{-6}$ °C−1 (6.4×10^{-6} °F^{-1}), $E = 200$ GPa (29×10^6 psi), $\mu = 0.30$, yield strength $S_y = 207$ MPa (30,000 psi). The vessel outside diameter is $d_0 = 10.00$ m (9.839 ft) and height $H = 10$ m (32.80 ft.). The vessel is designed by the *ASME Code for Unfired Pressure Vessels* for an internal pressure $p_i = 850$ kPa gauge (123.3 psig) and full hydrostatic pressure. The allowable stress (design strength) according to the *Code* is 108 MPa (15,700 psi) and a weld efficiency for spot-examined welds $e_w = 0.850$. The vessel is filled to a level of 4.00 m (13.12 ft) with oil (density, 870 kg/m^3 = 54.0 lb$_m$/ft^3), so the length of the vessel exposed to the vapor $b = 6.00$ m (19.68 ft). The portion of the vessel exposed to the liquid is at a temperature $T_0 = 25$°C (77°F), and the portion exposed to the vapor experiences a linear temperature variation to $T_1 = 75$°C (167°F) at the top of the vessel ($x = b$). Determine the maximum thermal stresses in the shell.

Let us first determine the design shell wall thickness. The hydrostatic pressure at the bottom of the vessel when full of oil is found from the following:

$$p_L = \rho_L g H = (870)(9.806)(10) = 85.3 \times 10^3 \text{ Pa} = 85.3 \text{ kPa} \quad (12.8 \text{ psi})$$

The total design pressure for the vessel is

$$p = p_i + p_L = 850 + 85.3 = 935.3 \text{ kPa gauge} \quad (85.0 \text{ psig})$$

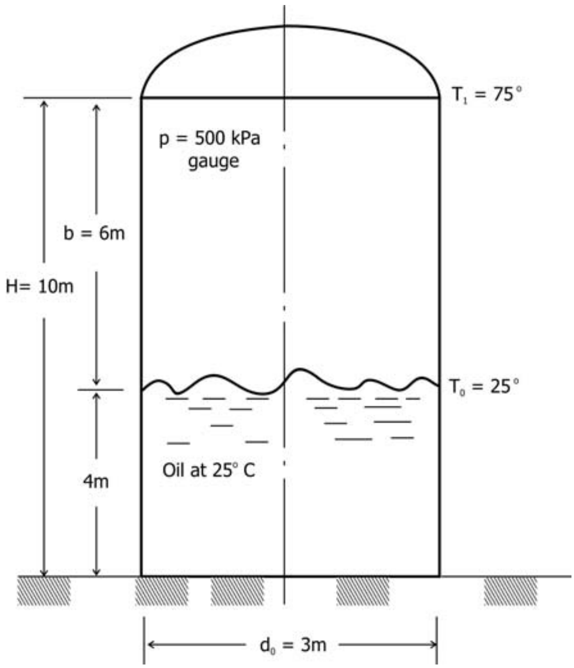


Figure 8-9. Partially filled vertical vessel, Example 8-3.

According to Section VIII, *ASME Code for Unfired Vessels*, the minimum thickness of the shell should be determined from

$$h = \frac{pd_0}{2S_a e_w + 0.8p} = \frac{(935.3)(3.00)}{(2)(108{,}000)(0.85) + (0.8)(935.3)}$$

$$h = 15.22 \times 10^{-3}\ \text{m} = 15.22\ \text{mm} \quad (0.599\ \text{in.})$$

The mill tolerance for metric plate is 0.3 mm under the standard thickness, so the next larger standard thickness (larger than $15.22 + 0.3 = 15.52$ mm) is $h = 16$ mm (0.630 in.).

Let us now verify that the vessel can be treated as a "long cylinder" using eq. (8-39). The mean diameter of the cylinder is $d = d_0 - h = 3.00 - 0.016 = 2.984$ m:

$$\beta b = \frac{[(12)(1 - 0.30^2]^{1/4}}{\sqrt{(0.016)/(2.984)}} \left(\frac{6.00}{2.984}\right) = 49.9 > 9$$

The long cylinder condition is met.

The maximum axial thermal stress is found from eq. (8-94):

$$(\sigma_x)_{\max} = \frac{(3)(11.5 \times 10^{-6})(200 \times 10^9)(75^\circ - 25^\circ)(2.984)}{(2)[(12)(1 - 0.30^2)]^{3/4}(6.00)} \sqrt{\frac{0.016}{2.984}}$$

$$(\sigma_x)_{\max} = 1.046 \times 10^6\ \text{Pa} = 1.046\ \text{MPa} \quad (151.7\ \text{psi})$$

The membrane longitudinal stress due to the internal pressure at the liquid-vapor interface is

$$(\sigma_x)_m = \frac{p_i d}{4h} = \frac{(850)(2.984)}{(4)(0.016)} = 39.630 \times 10^3\ \text{kPa} = 39.630\ \text{MPa} \quad (5748\ \text{psi})$$

The total axial stress at the liquid-vapor interface is

$$\sigma_x = (\sigma_x)_{\max} + (\sigma_x)_m = 1.046 + 39.630 = 40.676\ \text{MPa} \quad (5900\ \text{psi})$$

The maximum circumferential thermal stress is found from eq. (8-99):

$$(\sigma_\theta)_{\max} = \frac{(11.5 \times 10^{-6})(200 \times 10^9)(75^\circ - 25^\circ)(2.984)}{(4)[(12)(1 - 0.30^2)]^{3/4}(6.00)} \times \sqrt{\frac{0.016}{2.984}} \left[1 + \frac{(0.30)}{\sqrt{\frac{1}{3}(1 - 0.30^2}}\right]$$

$$(\sigma_x)_{\max} = 0.269 \times 10^6\ \text{Pa} = 0.269\ \text{MPa} \quad (38.0\ \text{psi})$$

The membrane circumferential stress due to the internal pressure at the liquid–vapor interface is

$$(\sigma_x)_m = \frac{p_i d}{2h} = \frac{(850)(2.984)}{(2)(0.016)} = 79.263 \times 10^3\,\text{kPa} = 79.263\,\text{MPa} \quad (11{,}496\,\text{psi})$$

The total circumferential stress at the liquid–vapor interface is

$$\sigma_x = (\sigma_x)_{\max} + (\sigma_x)_m = 0.269 + 79.263 = 79.532\ \text{MPa} \quad (11{,}535\ \text{psi})$$

Both the axial and membrane stresses at the liquid–vapor interface are less than the effective allowable stress $(S_a e_w) = (108)(0.85) = 91.9\,\text{MPa}$ (13,300 psi). In this example, the thermal stresses are small compared with the membrane stresses produced by the mechanical loading (less than 3% of the membrane stresses).

8.5 SHORT CYLINDERS

For the case in which the shell cannot be treated as a "long cylinder," it is convenient to express the solution for the transverse deflection w in terms of the hyperbolic functions instead of the exponential function; however, the two sets of functions are directly related as follows:

$$\sinh \beta x = \tfrac{1}{2}(e^{\beta x} - e^{-\beta x}) \quad \text{and} \quad \cosh \beta x = \tfrac{1}{2}(e^{\beta x} + e^{-\beta x}) \tag{8-100}$$

The solution of the homogeneous equation (eq. (8-17) with the right side set equal to zero) for the transverse deflection may be written as

$$w_{\text{hom}} = \cosh \beta x (C_1 \cos \beta x + C_2 \sin \beta x) + \sinh \beta x (C_3 \cos \beta x + C_4 \sin \beta x) \tag{8-101}$$

The complete solution is the sum of the solution of the homogeneous equation and a particular solution w_{part} of the complete differential equation, eq. (8-17):

$$w(x) = w_{\text{hom}} + w_{\text{part}} \tag{8-102}$$

The expression for the slope or rotation of the shell in the x-direction is

$$\begin{aligned}\omega_x(x) = \frac{dw}{dx} &= \beta \cosh \beta x[(C_2 + C_3)\cos \beta x + (C_4 - C_1)\sin \beta x] \\ &\quad + \beta \sinh \beta x[(C_1 + C_4)\cos \beta x + (C_2 - C_3)\sin \beta x] + \frac{dw_{\text{part}}}{dx}\end{aligned} \tag{8-103}$$

The second derivative of the transverse displacement (related to the bending stress resultant) is

$$\frac{d^2w}{dx^2} = 2\beta^2[\cosh\beta x(C_4\cos\beta x - C_3\sin\beta x) + \sinh\beta x(C_2\cos\beta x - C_1\sin\beta x)] + \frac{d^2w_{\text{part}}}{dx^2} = -\frac{1}{D}\left(M_x + \frac{M_T}{1-\mu}\right) \tag{8-104a}$$

The third derivative of the transverse displacement (related to the transverse shear stress resultant) is

$$\frac{d^3w}{dx^3} = 2\beta^3\cosh\beta x[(C_2 - C_3)\cos\beta x - (C_1 + C_4)\sin\beta x] + 2\beta^3\sinh\beta x[(C_4 - C_1)\cos\beta x - (C_2 + C_3)\sin\beta x] + \frac{d^3w_{\text{part}}}{dx^3} = -\frac{1}{D}\left(Q_x + \frac{1}{1-\mu}\frac{dM_T}{dx}\right) \tag{8-104b}$$

The four constants of integration may be evaluated from four boundary conditions, usually two conditions at one edge ($x = 0$) and two conditions at the other

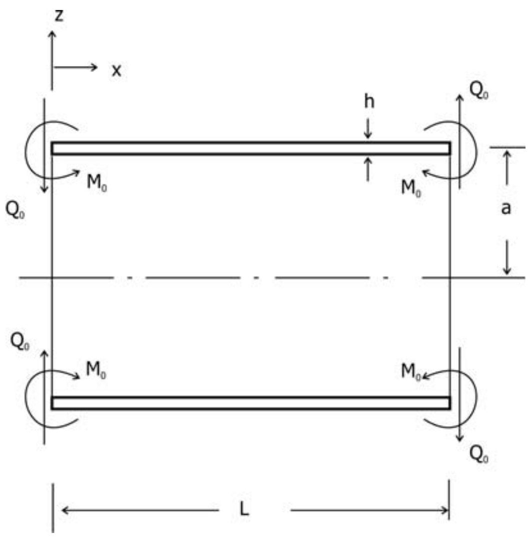


Figure 8-10. Edge-bending moments and transverse shear forces for a short cylinder.

edge ($x = L$). If we consider the case shown in Figure 8-10 and evaluate the constants for known edge bending stress resultant M_0 and transverse shear stress resultant Q_0, the following result is obtained for the transverse displacement and slope at the edges:

$$w(0) = w(L) = -\frac{2\beta a^2}{Eh} \left[\beta\left(M_0 + \frac{M_{T,0}}{1-\mu}\right) f_1(\beta L) + \left(Q_0 + \frac{1}{1-\mu}\frac{dM_{T,0}}{dx}\right) f_2(\beta L)\right] + w_{\text{part}}(0) \tag{8-105a}$$

$$\omega_x(0) = \omega_x(L) = \frac{dw}{dx} = \pm\frac{2\beta^2 a^2}{Eh} \left[2\beta\left(M_0 + \frac{M_{T,0}}{1-\mu}\right) f_3(\beta L) + \left(Q_0 + \frac{1}{1-\mu}\frac{dM_{T,0}}{dx}\right) f_1(\beta L)\right] + \omega_{\text{part}}(0) \tag{8-105b}$$

The quantity $M_{T,0}$ is the thermal moment evaluated at $x = 0$. The functions given in eqs. (8-105) are defined as follows:

$$f_1(\beta L) = \frac{\sinh\beta L - \sin\beta L}{\sinh\beta L + \sin\beta L} \tag{8-106a}$$

$$f_2(\beta L) = \frac{\cosh\beta L + \cos\beta L}{\sinh\beta L + \sin\beta L} \tag{8-106b}$$

$$f_3(\beta L) = \frac{\cosh\beta L - \cos\beta L}{\sinh\beta L + \sin\beta L} \tag{8-106c}$$

The functions are tabulated in Table 8-1 for convenience in numerical calculations. For numerical calculations, the quantity (βL) must be expressed in *radians*. Note that for $(\beta L) = 9$, $f_1(9) = 0.9998 \approx 1$; $f_2(9) = 0.9997 \approx 1$; $f_3(9) = 1.0001 \approx 1$. Equations (8-105) reduce to the case for a "long cylinder" for $(\beta L) \geq 9$.

Example 8-4 To illustrate the effect of shell length, let us consider the case of a shell of mean diameter $d = 2a$, a length L, and a wall thickness h, as shown in Figure 8-11. Both edges of the shell are clamped. The shell is subjected to a uniform temperature change $\Delta T_1 = T_1 - T_0$. Let us determine the edge moment and stresses at the edge for the case of (a) a "long cylinder," and (b) a "short cylinder."

The governing differential equation is eq. (8-17), with $N_x = M_T = p_r = 0$:

$$\frac{d^4 w}{dx^4} + \frac{Eh}{Da^2} w = \frac{N_T}{Da}$$

The thermal membrane stress resultant is evaluated from eq. (8-10):

$$N_T = \alpha E \Delta T_1 h = \text{constant}$$

TABLE 8-1. Values of the Functions for a "Short Cylinder," Eqs. (8-106)

βL	$F_1(\beta L)$	$F_2(\beta L)$	$F_3(\beta L)$
0.0	0.00000	∞	0.00000
0.1	0.00167	10.000	0.05000
0.2	0.00667	5.0003	0.10000
0.3	0.01500	3.3342	0.14999
0.4	0.02666	2.5021	0.19997
0.5	0.04165	2.00416	0.24991
0.6	0.05994	1.67386	0.29978
0.7	0.08153	1.43998	0.34953
0.8	0.10636	1.26701	0.39909
0.9	0.13437	1.13529	0.44837
1.0	0.16549	1.03308	0.49724
1.2	0.23650	0.89003	0.59320
1.5	0.36197	0.77497	0.72972
2.0	0.59909	0.73764	0.92112
2.5	0.81997	0.80184	1.04283
3.0	0.97223	0.89356	1.08356
4.0	1.05705	1.00458	1.05385
5.0	1.02618	1.01706	1.00931

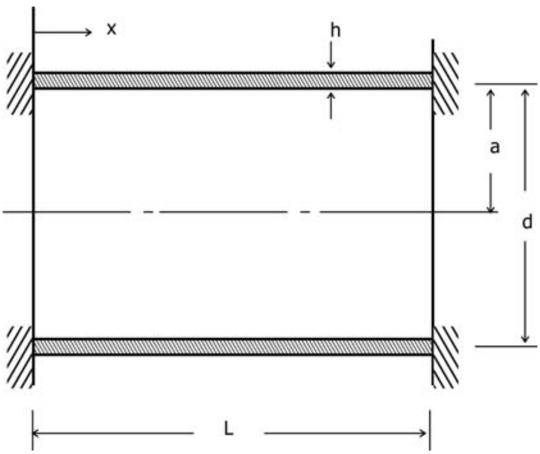


Figure 8-11. Short cylinder with fixed edges, Example 8-4.

(a) *Long cylinder*. For this case, the general expressions for the transverse deflection and axial rotation are

$$w(x) = e^{-\beta x}(C_1 \cos \beta x + C_2 \sin \beta x) + \frac{N_T a}{Eh} \tag{8-107a}$$

$$\omega_x(x) = \frac{dw}{dx} = \beta e^{-\beta x}[(C_2 - C_1) \cos \beta x - (C_1 + C_2) \sin \beta x] \tag{8-107b}$$

Using the conditions that $w(0) = \omega_x(0) = 0$, the following value is obtained for the constants of integration:

$$C_1 = C_2 = -\frac{N_T a}{Eh}$$

The expressions for the transverse deflection and axial rotation for the "long cylinder" case are

$$w(x) = [1 - e^{-\beta x}(\cos\beta x + \sin\beta x)\left(\frac{N_T a}{Eh}\right) \tag{8-108a}$$

$$\omega_x(x) = \frac{dw}{dx} = 2\beta e^{-\beta x}\sin\beta x\left(\frac{N_T a}{Eh}\right) \tag{8-108b}$$

The axial bending stress resultant is found from eq. (8-14a), with $M_T = 0$ for this example:

$$M_x = -D\frac{d^2w}{dx^2} = -\frac{2Da\beta^2 N_T}{Eh}e^{-\beta x}(\cos\beta x - \sin\beta x) \tag{8-109a}$$

$$M_x(0) = M_0 = -\frac{2Da\beta^2 N_T}{Eh} \tag{8-109b}$$

From the definition of the β-parameter, eq. (8-25),

$$\frac{Eh}{Da} = \frac{1}{4\beta^4 a}$$

The edge bending stress resultant may be written as follows for the "long cylinder" case:

$$M_x(0) = M_0 = -\frac{\alpha E\Delta T_1 h}{2\beta^2 a} = -\frac{\alpha E\Delta T_1 h^2}{2\sqrt{3(1-\mu^2)}} \tag{8-110a}$$

The radial bending stress at the edge of the shell is as follows for the "long cylinder" case:

$$(\sigma_x)_{x=0} = \pm\frac{6M_0}{h^2} = \mp\frac{\alpha E\Delta T_1}{\sqrt{\frac{1}{3}(1-\mu^2)}} \tag{8-110b}$$

(b) *Short cylinder*. For this case, the transverse deflection and axial rotation at the edge of the shell are given by eqs. (8-105), with $M_{T,0} = 0$:

$$w(0) = -\frac{2\beta a^2}{Eh}\left[\beta M_0 f_1(\beta L) + Q_0 f_2(\beta L)\right] + \frac{N_T a}{Eh} \tag{8-111a}$$

$$\omega_x(0) = \pm\frac{2\beta^2 a^2}{Eh}\left[2\beta M_0 f_3(\beta L) + Q_0 f_1(\beta L)\right] \tag{8-111b}$$

The axial rotation $\omega_x(0) = 0$ for a clamped edge:

$$Q_0 = -\frac{2\beta M_0 f_3(\beta L)}{f_1(\beta L)}$$

If we make this substitution into eq. (8-11a), the following is obtained for the edge-bending stress resultant for the "short cylinder" case:

$$M_0[2f_2(\beta L)f_3(\beta L) - f_1^2(\beta L)] = -\frac{N_T f_1(\beta L)}{2\beta^2 a} \tag{8-112}$$

By using the definitions of the functions f_1, f_2, and f_3 from eqs. (8-106) and the trigonometric identities,

$$\cos^2(\beta L) = 1 - \sin^2(\beta L) \quad \text{and} \quad \cosh^2(\beta L) = 1 + \sinh^2(\beta L)$$

it can be shown that

$$2f_2(\beta L)f_3(\beta L) - f_1^2(\beta L) = 1$$

The expression for the edge bending stress resultant for the "short cylinder" case may be written as

$$M_0 = -\frac{N_T f_1(\beta L)}{2\beta^2 a} = -\frac{aE\Delta T_1 h^2 f_1(\beta L)}{2\sqrt{3(1-\mu^2)}} \tag{8-113a}$$

The radial bending stress at the edge of the shell is as follows for the "short cylinder" case:

$$(\sigma_x)_{x=0} = \pm\frac{6M_0}{h^2} = \mp\frac{\alpha E\Delta T_1 f_1(\beta L)}{\sqrt{\frac{1}{3}(1-\mu^2)}} = \mp\frac{\alpha E\Delta T_1}{\sqrt{\frac{1}{3}(1-\mu^2)}}\left(\frac{\sinh\beta L - \sin\beta L}{\sinh\beta L + \sin\phi L}\right) \tag{8-113b}$$

If we compare the axial stress at the edge for the short cylinder case, eq. (8-110b), to that for the long cylinder case, eq. (8-113b), we observe that the difference is that the "short cylinder" has the function f_1 in the numerator.

As a numerical example of the effect of the cylinder length, let us consider the case for two shells constructed of SA-285 Grade C steel (properties given in Example 8-3), with a temperature change $\Delta T_1 = 25°\text{C}$ (45°F), mean diameter $d = 2.50$ m (8.20 ft), and shell thickness $h = 12.5$ mm (0.492 in.). One shell has a length of $L_1 = 100$ mm (3.937 in.), and the other shell has a length $L_2 = 1.50$ m (4.921 ft = 59.06 in.)

The right side of eq. (8-74) is

$$\frac{9\sqrt{h/d}}{[12(1-\mu^2)]^{1/4}} = \frac{(9)\sqrt{(0.0125)/(2.50)}}{[(12)(1-0.30^2)]^{1/4}} = 0.350$$

The length-to-diameter ratio for the first shell is $(L_1/d) = (0.100)/(2.50) = 0.040 < 0.350$; therefore, this shell is in the "short cylinder" category. For the

second shell, $(L_2/d) = 0.600 > 0.350$, so this shell is in the "long cylinder" category.

The axial bending stress at the shell edge for the second shell (the "long cylinder") is found from eq. (8-110b):

$$(\sigma_x)_{x=0} = \mp \frac{(11.5 \times 10^{-6})(200 \times 10^9)(25^\circ)}{\sqrt{\frac{1}{3}(1 - 0.30^2)}} = \mp 104.4 \times 10^6 \text{ Pa}$$

$$= 104.4 \text{ MPa} \quad (15{,}140 \text{ psi})$$

The dimensionless parameter βL for the first shell is found from eq. (8-38):

$$\beta L = \frac{[(12)(1 - 0.30^2)]^{1/4}}{\sqrt{(0.0125)/(2.50)}} \left(\frac{0.100}{2.500}\right) = 1.028$$

The function $f_1(\beta L)$ may be found from eq. (8-106a):

$$f_1(1.028) = \frac{\sinh(1.028) - \sin(1.028)}{\sinh(1.028) + \sin(1.028)} = 0.17474$$

The axial bending stress at the shell edge for the first shell (the "short cylinder") is found from eq. (8-113a):

$$(\sigma_x)_{x=0} = (104.4)(0.17474) = 18.24 \text{ MPa} \quad (2646 \text{ psi})$$

The shell length has a significant effect on the stresses and deflections for the shell. This is analogous to the fact that a short beam on an elastic foundation, with a uniform mechanical load, will experience significantly smaller bending stress and deflection than a long beam for the same total load.

8.6 AXISYMMETRIC LOADING OF SPHERICAL SHELLS

Another commonly encountered shell geometry is the *spherical shell*. The coordinate system used to describe the midplane surface is shown in Figure 8-12. If the thermal and mechanical loading on the shell are symmetrical about the polar axis, such that all stresses and bending moments depend on the azimuth angle ϕ only, there will be no in-plane shear forces or twisting stress resultants.

For the case of axisymmetric loading on a spherical shell element, the forces and moments are illustrated in Figure 8-13.

8.6.1 Equilibrium Relationships

If we make a radial force balance for the element shown in Figure 8-13, the following is obtained:

$$\begin{aligned} &- Q_\phi a \sin\phi d\theta + (Q_\phi + dQ_\phi) a \sin(\phi + d\phi) d\theta - (2N_\phi a \sin\phi d\theta)\sin(\tfrac{1}{2}d\phi) \\ &- (2N_\theta a d\phi)\sin(\tfrac{1}{2}d\theta \sin\phi) + p_r a^2 \sin\phi d\phi d\theta = 0 \end{aligned} \tag{8-114}$$

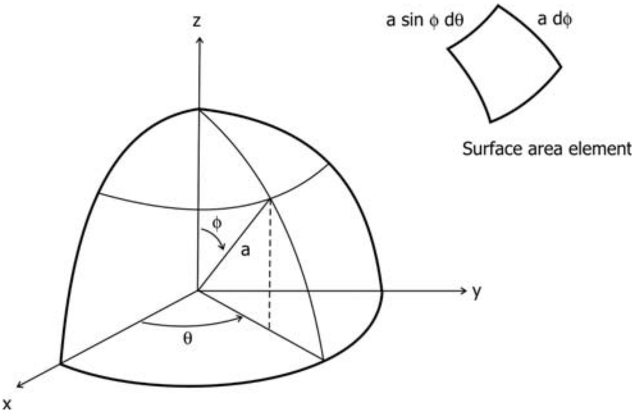


Figure 8-12. Spherical coordinate system. The element dimensions are $(a\ d\phi) \times (a\ \sin\ \phi\ d\theta)$

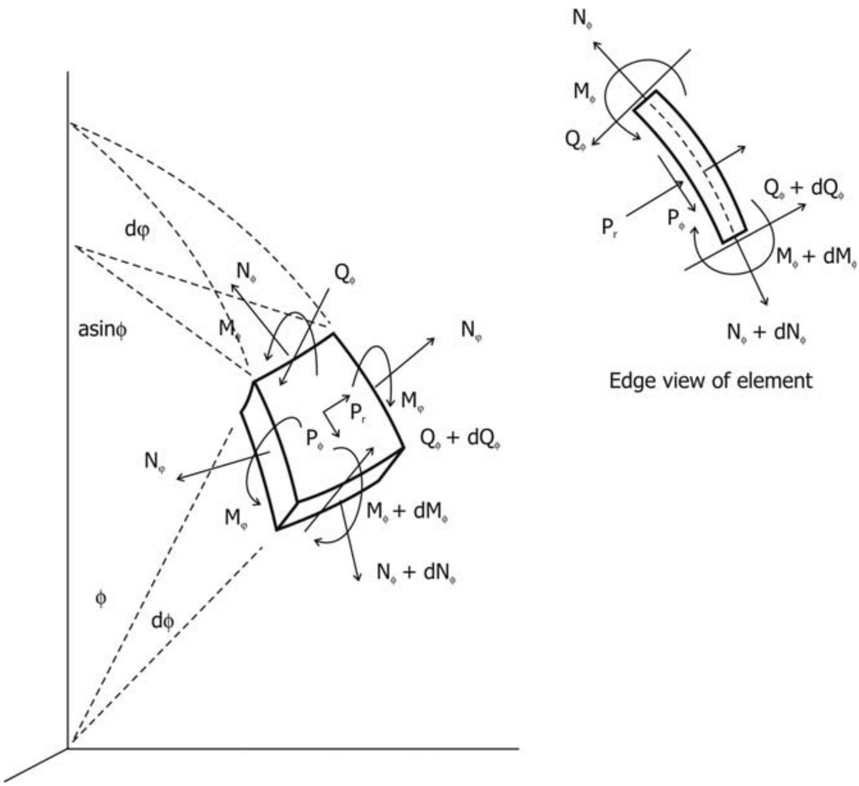


Figure 8-13. Forces and moments acting on a differential element in a spherical shell for axisymmetric loading.

For small angles, the sine of the angle is approximately equal to the angle in radians and the cosine of the small angle is approximately equal to unity. Using the trigonometric identity for the sum on angles, the following result is obtained:

$$\sin(\phi + d\phi) = \sin\phi\cos(d\phi) + \cos\phi\sin(d\phi) = \sin\phi + \cos\phi d\phi \tag{8-115}$$

Making this substitution into eq. (8-114), the following radial equilibrium relationship is obtained:

$$\frac{d}{d\phi}(Q_\phi \sin\phi) - (N_\phi + N_\theta)\sin\phi + p_r a \sin\phi = 0 \tag{8-116}$$

If we make a force balance in the azimuth direction for the element shown in Figure 8-13, the azimuth equilibrium relation is obtained:

$$\frac{d}{d\phi}(N_\phi \sin\phi) - N_\theta \cos\phi + Q_\phi \sin\phi + p_\phi a \sin\phi = 0 \tag{8-117}$$

If we sum moments about the lower edge of the element, the following moment equilibrium relation is obtained:

$$\frac{d}{d\phi}(M_\phi \sin\phi) - M_\theta \cos\phi - aQ_\phi \sin\phi = 0 \tag{8-118a}$$

This expression may be written in alternate form:

$$\frac{dM_\phi}{d\phi} + (M_\phi - M_\theta)\cot\phi - aQ_\phi = 0 \tag{8-118b}$$

8.6.2 Stress–Strain Relationships

Let us consider the element of the midplane shown in Figure 8-14. The strain of the midplane in the azimuth direction is

$$\overline{\varepsilon}_\phi = \frac{(ad\phi + dv)\left(1 + \frac{w}{a}\right) - ad\phi}{ad\phi} = \frac{1}{a}\frac{dv}{d\phi} + \frac{w}{a} = \frac{1}{Eh}(N_\phi - \mu N_\theta + N_T) \tag{8-119}$$

The quantities v and w are the azimuth and radial displacements, respectively.

The strain in the polar direction may also be evaluated from Figure 8-14. The strain of the midplane in the polar direction is

$$\overline{\varepsilon}_\theta = \frac{(a\sin\phi + v\cos\phi + w\sin\phi)d\theta - a\sin\phi d\theta}{a\sin\phi d\theta} = \frac{v\cot\phi + w}{a} \tag{8-120a}$$

$$\overline{\varepsilon}_\theta = \frac{1}{Eh}(N_\theta - \mu N_\phi + N_T) \tag{8-120b}$$

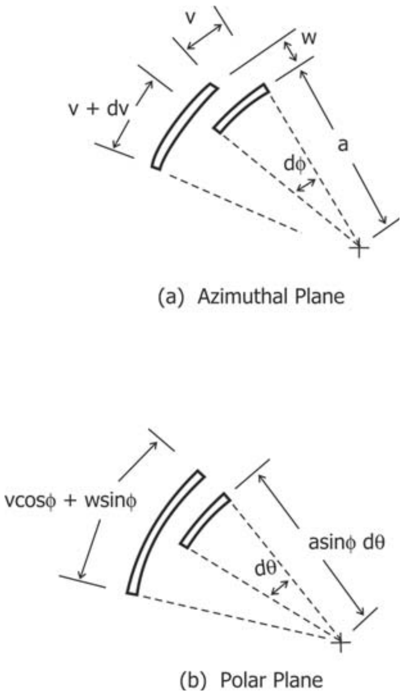


Figure 8-14. Displacements for the midplane element in a spherical shell for axisymmetric loading: (a) azimuth displacements, (b) polar (circumferential) displacements.

If the substitutions of the strain component expressions are made into the membrane stress–strain relations, eqs. (8-11), the following is obtained:

$$N_\phi = K\left[\frac{1}{a}\frac{dv}{d\phi} + \frac{w}{a} + \frac{\mu}{a}(v\cot\phi + w)\right] - \frac{N_T}{1-\mu} \tag{8-121a}$$

$$N_\theta = K\left[\frac{v}{a}\cot\phi + \frac{w}{a} + \frac{\mu}{a}\left(\frac{dv}{d\phi} + w\right)\right] - \frac{N_T}{1-\mu} \tag{8-121a}$$

The quantity K is the *extensional rigidity*:

$$K = \frac{Eh}{1-\mu^2} \tag{8-122}$$

There are two components of the rotation of the top edge of the element in the azimuth direction, as shown in Figure 8-15: a rotation due to the displacement of the element in the polar direction, $(-v/a)$, and an additional rotation due to the displacement in the radial direction, $(1/a)(dw/d\phi)$. The total rotation V may be written as

$$V = \frac{1}{a}\frac{dw}{d\phi} - \frac{v}{a} \tag{8-123}$$

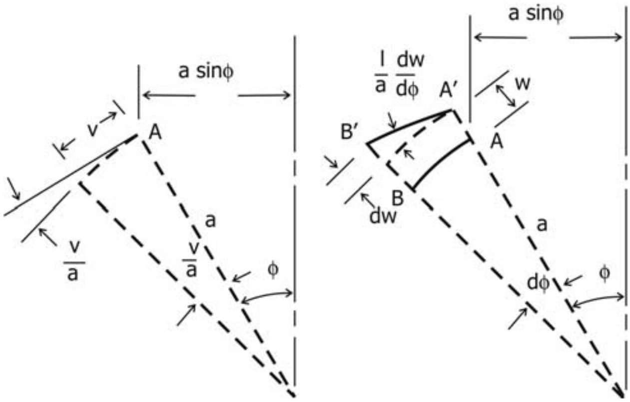


Figure 8-15. Rotation components in the azimuth direction for the element in a spherical shell.

The curvature of the deflected surface (change of rotation) for the two angular coordinate directions is

$$\kappa_\phi = \frac{1}{a}\frac{dV}{d\phi} = \frac{1}{a^2}\frac{d}{d\phi}\left(\frac{dw}{d\phi} - v\right) \tag{8-124a}$$

$$\kappa_\theta = \frac{V\cos\phi}{a\sin\phi} = \frac{1}{a}V\cot\phi = \frac{\cot\phi}{a^2}\left(\frac{dw}{d\phi} - v\right) \tag{8-124b}$$

The bending stress resultants are related to the curvature as follows:

$$M_\phi = -\frac{D}{a}\left(\frac{dV}{d\phi} + \mu V\cot\phi\right) - \frac{M_T}{1-\mu} \tag{8-125a}$$

$$M_\theta = -\frac{D}{a}\left(V\cot\phi + \mu\frac{dV}{d\phi}\right) - \frac{M_T}{1-\mu} \tag{8-125b}$$

8.6.3 Reduction of the Equations for a Sphere

If we subtract eq. (8-120) from eq. (8-119) and utilize eqs. (8-121), the following result is obtained:

$$\frac{dv}{d\phi} - v\cot\phi = \frac{(1+\mu)a}{Eh}(N_\phi - N_\theta) \tag{8-126}$$

If we differentiate both sides of eq. (8-120), we obtain

$$\frac{dv}{d\phi}\cot\phi - \frac{v}{\sin^2\phi} + \frac{dw}{d\phi} = \frac{a}{E\,h}\left(\frac{dN_\theta}{d\phi} - \mu\frac{dN_\phi}{d\phi} + \frac{dN_T}{d\phi}\right) \tag{8-127}$$

Using eq. (8-126) to eliminate the $\left(\frac{dv}{d\phi}\right)$ term, the following is obtained:

$$V = \frac{1}{a}\left(\frac{dw}{d\phi} - v\right) = \frac{1}{E\,h}\left[\frac{dN_\theta}{d\phi} - \mu\frac{dN_\phi}{d\phi} + \frac{dN_T}{d\phi} + (1+\mu)(N_\theta - N_\phi)\cot\phi\right] \tag{8-128}$$

The membrane stress resultants may be written as follows by combining the radial and azimuth equilibrium relations, eqs. (8-116) and (8-117), and integrating. The azimuth membrane stress resultant is

$$N_\phi = Q_\phi \cot\phi + \frac{a}{\sin^2\phi}\int p_r(\phi)\sin\phi\cos\phi\,d\phi - \frac{a}{\sin^2\phi}\int p_\phi(\phi)\sin^2\phi\,d\phi \tag{8-129a}$$

The two integrals are functions of the azimuth angle only:

$$N_\phi = Q_\phi \cot\phi + F(\phi) \tag{8-129b}$$

The polar membrane stress is

$$N_\theta = \frac{dQ_\phi}{d\phi} - F(\phi) + p_r a \tag{8-129c}$$

The following expression is obtained by combining eqs. (8-128) and eqs. (8-129b) and (8-129c):

$$\begin{aligned} EhV = {} & \frac{d^2Q_\phi}{d\phi^2} - (1+\mu)\frac{dF(\phi)}{d\phi} + a\frac{dp_r}{d\phi} + \cot\phi\frac{dQ_\phi}{d\phi} - (\cot^2\phi - \mu)Q_\phi \\ & - 2(1+\mu)F(\phi)\cot\phi + \frac{dN_T}{d\phi} + (1+\mu)ap_r\cot\phi \end{aligned} \tag{8-130}$$

From the definition of the function $F(\phi)$, the derivative may be written as

$$\frac{dF(\phi)}{d\phi} = ap_r - 2F(\phi)\cot\phi \tag{8-131}$$

If we make the substitution from eq. (8-131) into eq. (8-130), we obtain a second-order differential equation relating the transverse shear stress resultant Q_ϕ and the rotation V:

$$\frac{d^2Q_\phi}{d\phi^2} + \cot\phi\frac{dQ_\phi}{d\phi} - (\cot^2\phi - \mu)Q_\phi = EhV - \frac{dN_T}{d\phi} - a\frac{dp_r}{d\phi} \tag{8-132}$$

Similarly, if we make the substitutions for the bending stress resultants from eqs. (8-125) into the moment equilibrium relationship, eq. (8-118), another second-order differential equation relating the transverse shear stress resultant Q_ϕ and the rotation V is obtained:

$$\frac{d^2V}{d\phi^2} + \cot\phi\frac{dV}{d\phi} - (\cot^2 + \mu)V = -\frac{a^2}{D}Q_\phi - \frac{a}{(1-\mu)D}\frac{dM_T}{d\phi} \tag{8-133}$$

Let is define a linear operator L as follows:

$$L = \frac{d^2}{d\phi^2} + \cot\phi \frac{d}{d\phi} - \cot^2\phi \tag{8-134}$$

Using this operator, eqs. (8-132) and (8-133) may be written as

$$LQ_\phi + \mu Q_\phi = EhV - \frac{dN_T}{d\phi} - a\frac{dp_r}{d\phi} \tag{8-135a}$$

$$LV - \mu V = -\frac{a^2}{D}Q_\phi - \frac{a}{(1-\mu)D}\frac{dM_T}{d\phi} \tag{8-135b}$$

If we apply the linear operator again to eq. (8-135a), the following is obtained:

$$L^2Q_\phi + \mu LQ_\phi = Eh\ LV - L\left(\frac{dN_T}{d\phi} + a\frac{dp_r}{d\phi}\right)$$

Using eq. (8-135a) for LV,

$$L^2Q_\phi + \mu LQ_\phi = Eh(\mu V - \frac{a^2}{D}Q_\phi) - \frac{Eha}{(1-\mu)D}\frac{dM_T}{d\phi} - L\left(\frac{dN_T}{d\phi} + a\frac{dp_r}{d\phi}\right) \tag{8-136}$$

Equation (8-135) may be used to eliminate the rotation V from eq. (8-136) to obtain the following expression:

$$L^2Q_\phi + 4\lambda^4Q_\phi = -(L-\mu)\left(\frac{dN_T}{d\phi} + a\frac{dp_r}{d\phi}\right) - \frac{Eha}{(1-\mu)D}\frac{dM_T}{d\phi} \tag{8-137}$$

We may eliminate the transverse shear stress resultant from eqs. (8-135) to obtain a similar expression in terms of the rotation V:

$$L^2V + 4\lambda^4V = \frac{a^2}{D}\left(\frac{dN_T}{d\phi} + a\frac{dp_r}{d\phi}\right) - \frac{a}{(1-\mu)D}\left[(L+\mu)\left(\frac{dM_T}{d\phi}\right)\right] \tag{8-138}$$

The quantity (λ^4) is defined as

$$\lambda^4 = \frac{Eha^2}{4D} - \frac{1}{4}\mu^2 = 3(1-\mu^2)\left(\frac{a}{h}\right)^2 - \frac{1}{4}\mu^2 \tag{8-139}$$

The left side of eqs. (8-137) and (8-138), the homogenous equation, may be factored as follows:

$$(L + 2i\lambda^2)(L - 2i\lambda^2)Q_\phi = 0 \tag{8-140a}$$

$$(L + 2i\lambda^2)(L - 2i\lambda^2)V = 0 \tag{8-140a}$$

In either case, the solution of the homogeneous equation involves the solution of two differential equations:

$$LQ_\phi + 2i\lambda^2 Q_\phi = 0 \quad \text{and} \quad LQ_\phi - 2i\lambda^2 Q_\phi = 0 \tag{8-141}$$

The solutions of these differential equations results in four independent functions, which are the solution of the homogeneous equation corresponding to the governing equation, eq. (8-38). The solutions involve the *hypergeometric functions* [Flügge, 1960, p. 324]; however, the convergence is slow, except for large values of λ or very small values of λ. Calculations have shown [Ekström, 1933] that for $(a/h) = 62.5$, about 18 terms in the hypergeometric series expression are necessary for satisfactory accuracy of the solution. We consider two approximations in the following sections are satisfactory for engineering analysis for shells having a fairly large subtended angle (hemispherical domes, for example) and shells having a relatively small subtended angle (shallow spherical shells or spherical caps).

8.7 APPROXIMATE ANALYSIS OF SPHERICAL SHELLS UNDER AXISYMMETRIC LOADING

8.7.1 Spherical Domes

Let us consider the spherical dome, as shown in Figure 8-16. Detailed calculations [Timoshenko and Woinowsky-Krieger, 1959, p. 548] have shown that for thin shells the derivative of the rotation or transverse shear in eq. (8-132) or (8-133) is much larger than the rotation or transverse shear itself, and the second derivative is large in comparison with the first derivative. For the case in which the angle ϕ is not small, the quantities Q_ϕ and V damp out rapidly from the shell edge, $\phi = \phi_0$. If the Q_ϕ or V terms and the first derivatives of these terms are neglected, eqs. (8-132) and (8-133) reduce to

$$\frac{d^2 Q_\phi}{d\phi^2} = EhV - a\frac{dp_r}{d\phi} \tag{8-142a}$$

$$\frac{d^2 V}{d\phi^2} = -\frac{a^2}{D}Q_\phi - \frac{a}{(1-\mu)D}\frac{dM_T}{d\phi} \tag{8-142b}$$

If we combine eqs. (8-142) to solve for either Q_ϕ or V, the following set of expressions is obtained:

$$\frac{d^4 Q_\phi}{d\phi^4} + \frac{Eha^2}{D}Q_\phi = -a\frac{d^3 p_r}{d\phi^3} - \frac{Eha}{(1-\mu)D}\frac{dM_T}{d\phi} \tag{8-143a}$$

$$\frac{d^4 V}{d\phi^4} + \frac{Eha^2}{D}V = \frac{a^3}{D}\frac{dp_r}{d\phi} - \frac{a}{(1-\mu)D}\frac{d^3 M_T}{d\phi^3} \tag{8-143b}$$

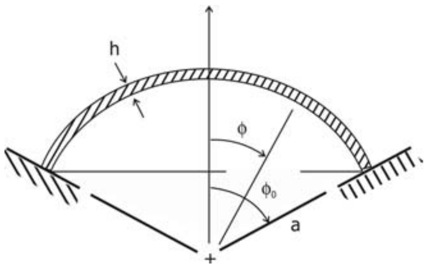


Figure 8-16. Spherical dome. ϕ_0 is the "edge angle" for the dome.

Suppose we define the following parameter:

$$4\lambda_0^4 = \frac{Eha^2}{D} \quad \text{or} \quad \lambda_0 = [3(1-\mu^2)]^{1/4}\sqrt{\frac{a}{h}} \tag{8-144}$$

The homogeneous equations corresponding to eqs. (8-143) may be written as

$$\frac{d^4 Q_\phi}{d\phi^4} + 4\lambda_0^4 Q_\phi = 0 \tag{8-145a}$$

$$\frac{d^4 V}{d\phi^4} + 4\lambda_0^4 V = 0 \tag{8-145b}$$

The solutions are of the following form:

$$V_{\text{hom}} = e^{\lambda_0\phi}(C_1 \cos\lambda_0\phi + C_2 \sin\lambda_0\phi) + e^{-\lambda_0\phi}(C_3 \cos\lambda_0\phi + C_4 \sin\lambda_0\phi) \tag{8-146}$$

In order that the solution for the slope or transverse shear stress resultant damp out toward the interior of the shell (where ϕ approaches 0), the last set of terms must drop out, or $C_3 = C_4 = 0$. The solution of the homogeneous equation, eq. (8-146), reduces to

$$V_{\text{hom}} = e^{\lambda_0\phi}(C_1 \cos\lambda_0\phi + C_2 \sin\lambda_0\phi) \tag{8-146}$$

The constants of integration are determined from the edge conditions for the spherical shell.

In order that the edge effects damp out sufficiently toward the center ($\phi = 0$), the exponential term must be much larger at the edge of the shell (at $\phi = \phi_0$) than at the center. For practical purposes we may (arbitrarily) take the ratio to be 100 or more:

$$\frac{\exp(\lambda_0\phi_0)}{\exp(0)} \geq 100 \quad \text{or} \quad \lambda_0\phi_0 \geq \ln(100) = 4.6$$

This condition is met when the rim angle ϕ_0 is

$$\phi_0 \geq \frac{4.6}{\lambda_0} = \frac{4.6}{[3(1-\mu^2)]^{1/4}\sqrt{a/h}} = \frac{3.5}{(1-\mu^2)^{1/4}\sqrt{a/h}} \tag{8-147}$$

For example, for $(a/h) = 50$ and $\mu = 0.3$, the rim angle required for satisfactory accuracy of this approximate solution according to eq. (8-147) would be $\phi_0 \geq 0.507 = 29^\circ$.

In many practical applications, the displacement δ of the shell in the planes of the parallel circles ($\theta =$ constant) is important:

$$\delta = v \cos\phi + w \sin\phi = (v \cot\phi + w)\sin\phi = (a\overline{\varepsilon}_\theta)\sin\phi \tag{8-148}$$

If we use eq. (8-9b), the deflection may be written in terms of the stress resultants:

$$\delta = \frac{a \sin\phi}{Eh}(N_\theta - \mu N_\phi + N_T) \tag{8-149}$$

The application of this solution approach may be best illustrated by an example.

Example 8-5 The spherical dome shown in Figure 8-16 has the edge rigidly fixed. The dome is uniformly heated from a temperature T_0 to a final temperature of T_1, such that the thermal membrane stress resultant is

$$N_T = \alpha E \Delta T_1 h$$

where $\Delta T_1 = T_1 - T_0$. Determine the stresses at the edge of the dome.

The thermal moment is $M_T = 0$ for a uniform temperature across the thickness of the dome, so the right side of eqs. (8-143) reduces to zero. The solution for the rotation is that of the homogeneous equation, eq. (8-146), which may be written in the following form:

$$V(\phi) = \exp(\lambda_0\phi)(C_1 \cos\lambda_0\phi + C_2 \sin\lambda_0\phi) \tag{8-150}$$

At the fixed edge, the rotation $V(\phi_0) = 0$. The constants are then related as follows:

$$C_2 = -C_1 \frac{\cos\lambda_0\phi_0}{\sin\lambda_0\phi_0} \tag{8-151}$$

With this result, the rotation may be written as

$$V(\phi) = C_1 \exp(\lambda_0\phi)\left(\frac{\sin\lambda_0\phi_0 \cos\lambda_0\phi - \cos\lambda_0\phi_0 \sin\lambda_0\phi}{\sin\lambda_0\phi_0}\right) \tag{8-152}$$

The trig identity may be used to simplify the numerator in the last term:

$$V(\phi) = C_1 \exp(\lambda_0\phi)\frac{\sin[\lambda_0(\phi_0 - \phi)]}{\sin\lambda_0\phi_0} \tag{8-153}$$

From eq. (8-142b), the transverse shear stress resultant may be related to the rotation as follows, for $M_T = 0$ in this example.

$$Q_\phi = -\frac{D}{a^2}\frac{d^2V}{d\phi^2} = -C_1\lambda_0^2 \frac{2D \exp(\lambda_0\phi)\sin[\lambda_0(\phi_0 - \phi)]}{a^2 \sin\lambda_0\phi_0} \tag{8-154}$$

The azimuth and polar membrane stress resultants N_ϕ and N_θ are evaluated from eqs. (8-129), with $F(\phi) = p_r = 0$:

$$N_\phi = Q_\phi \cot\phi = -2C_1\lambda_0^2 \frac{D\cot\phi\exp(\lambda_0\phi)\sin[\lambda_0(\phi_0-\phi)]}{a^2\sin\lambda_0\phi_0} \tag{8-155a}$$

$$N_\theta = \frac{dQ_\phi}{d\phi} = 2C_1\lambda_0^3 \frac{D\exp(\lambda_0\phi)\,\{\cos[\lambda_0(\phi_0-\phi)] + \sin[\lambda_0(\phi_0-\phi)]\}}{a^2\sin\lambda_0\phi_0} \tag{8-155b}$$

The deflection δ is given by eq. (8-149):

$$\delta = \frac{a\sin\phi}{Eh}(N_\theta - \mu N_\phi + N_T)$$

$$\delta = \frac{a\sin\phi}{Eh}\left\{\left(\frac{2C_1\lambda_0^3 D\exp\lambda_0\phi}{a^2\sin\lambda_0\phi_0}\right)\left[\cos\lambda_0(\phi_0-\phi) - \frac{\mu\cot\phi\sin\lambda_0(\phi_0-\phi)}{\lambda_0}\right] + N_T\right\} \tag{8-156}$$

At the edge ($\phi = \phi_0$), the deflection δ is zero:

$$\frac{2C_1\lambda_0^3 D\exp\lambda_0\phi_0}{a^2\sin\lambda_0\phi_0} + N_T = 0 \tag{8-157a}$$

$$C_1 = -\frac{N_T a^2\sin\lambda_0\phi_0\exp(-\lambda_0\phi_0)}{2D\lambda_0^3} \tag{8-157b}$$

Making this substitution into eq. (8-153), the following expression is obtained for the rotation:

$$V(\phi) = -\frac{N_T a^2}{2D\lambda_0^3}\exp[-\lambda_0(\phi_0-\phi)]\sin[\lambda_0(\phi_0-\phi)] \tag{8-158}$$

Using the definitions of the parameter λ_0, eq. (8-144), the flexural rigidity D, and the thermal membrane stress resultant N_T for this problem, the coefficient in eq. (8-158) may be written as

$$\frac{N_T a^2}{2D\lambda_0^3} = 2[3(1-\mu^2)]^{3/4}\alpha\Delta T_1\left(\frac{a}{h}\right)^{3/2} \tag{8-159}$$

The membrane stress resultants may now be written using eqs. (8-155):

$$N_\phi = \frac{N_T\cot\phi}{\lambda_0}\exp[-\lambda_0(\phi_0-\phi)]\sin[\lambda_0(\phi_0-\phi)] \tag{8-160a}$$

$$N_\theta = N_T\exp[-\lambda_0(\phi_0-\phi)]\cos[\lambda_0(\phi_0-\phi)] \tag{8-160b}$$

The bending stress resultants may be written using eqs. (8-125), neglecting the lower-order terms involving V and noting that the thermal moment $M_T = 0$ in this problem.

$$M_\phi = -\frac{D}{a}\frac{dV}{d\phi} - \frac{M_T}{1-\mu} = -\frac{D}{a}\frac{dV}{d\phi} \tag{8-161a}$$

$$M_\theta = -\frac{\mu D}{a}\frac{dV}{d\phi} - \frac{M_T}{1-\mu} = \mu M_\phi - M_T = \mu M_\phi \tag{8-161b}$$

Using the relationship for the rotation, eq. (8-158), the azimuth bending stress resultant may be written as

$$M_\phi(\phi) = \frac{N_T a}{2\lambda_0^2}\exp[-\lambda_0(\phi_0 - \phi)]\{\sin[\lambda_0(\phi_0 - \phi)] + \cos[\lambda_0(\phi_0 - \phi)]\} \tag{8-162}$$

The coefficient term may be written as

$$\frac{N_T a}{2\lambda_0^2} = \frac{\alpha E \Delta T_1 h^2}{2[3(1-\mu^2)]^{1/2}} \tag{8-163}$$

At the edge of the dome ($\phi = \phi_0$), the azimuth and polar stresses, σ_ϕ and σ_θ, are

$$(\sigma_\phi)_{\phi 0} = \frac{N_\phi(\phi_0)}{h} \pm \frac{6M_\phi(\phi_0)}{h^2} = \pm \frac{\alpha E \Delta T_1}{\sqrt{\frac{1}{3}(1-\mu^2)}} \tag{8-164a}$$

$$(\sigma_\theta)_{\phi 0} = \frac{N_\theta(\phi_0)}{h} \pm \frac{6M_\theta(\phi_0)}{h^2} = \frac{\alpha E \Delta T_1 h}{a} \pm \frac{\mu \alpha E \Delta T_1}{\sqrt{\frac{1}{3}(1-\mu^2)}}$$

$$(\sigma_\theta)_{\phi 0} = \alpha E \Delta T_1 \left[\frac{h}{a} \pm \frac{\mu}{\sqrt{\frac{1}{3}(1-\mu^2)}}\right] \tag{8-164b}$$

8.7.2 Shallow Shells

For spherical shells for which the condition given by eq. (8-147) for the rim angle ϕ_0 is no longer valid, the bending stresses do not decay sufficiently for the approximate solution of the previous section to be applicable for accurate engineering purposes. In this case, an alternate approach, called *shallow shell theory*, may be applied.

8.7.2.1 Governing relations for shallow shells. The cotangent function may be expanded in an infinite series:

$$\cot\phi = \frac{1}{\phi} - \frac{\phi}{3} - \frac{\phi^3}{45} - \frac{2\phi^5}{945} - \cdots \tag{8-165}$$

For angles less than $\phi = 0.375$ rad (21.5°), the cotangent may be approximated by the first term only, within an error of less than 5%:

$$\cot\phi \approx \frac{1}{\phi} \quad \text{for} \quad \phi \leq 0.375 \text{ rad} \tag{8-166}$$

Using this approximation, the linear operator, eq. (8-134), may be written as

$$L = \frac{d^2}{d\phi^2} + \frac{1}{\phi}\frac{d}{d\phi} - \frac{1}{\phi^2} \tag{8-167}$$

The homogeneous equation for this case is given by eq. (8-140) or eq. (8-141). If we write out the first of the differential equations of eq. (8-41), the following is obtained:

$$\frac{d^2 Q_\phi}{d\phi^2} + \frac{1}{\phi}\frac{dQ_\phi}{d\phi} + \left(2i\lambda^2 - \frac{1}{\phi^2}\right) Q_\phi = 0 \tag{8-168}$$

Let us make the following substitution into eq. (8-168):

$$\phi = \frac{x}{\sqrt{2}\,\lambda} \tag{8-169}$$

Then, eq. (8-168) reduces to

$$\frac{d^2 Q_\phi}{dx^2} + \frac{1}{x}\frac{dQ_\phi}{dx} + \left(i - \frac{1}{x^2}\right) Q_\phi = 0 \tag{8-170}$$

This expression is identical with eq. (E-14) in Appendix E for the Kelvin functions of order $n = 1$. Because the Kelvin functions of order 1 are linearly related to the first derivatives of the Kelvin functions of order 0, as shown in eqs. (E-25) through (E-28), the solution may be written in terms of the derivatives:

$$Q_\phi = C_1 \text{ber}' x + C_2 \text{bei}' x + C_3 \text{ker}' x + C_4 \text{kei}' x \tag{8-171}$$

The variable x is

$$x = \sqrt{2}\lambda\phi \tag{8-172}$$

The derivative is taken with respect to the variable x:

$$\text{ber}' x = \frac{d(\text{ber} x)}{dx}$$

8.7.2.2 Shallow shell example. As an example of the application of the shallow shell concept, let us consider the case of the shallow shell shown in Figure 8-17, for which the temperature distribution is given by

$$\Delta T = T - T_0 = \Delta T_1 \left(\frac{z}{h}\right)\left[1 - \left(\frac{\phi}{\phi_0}\right)^2\right] \tag{8-173}$$

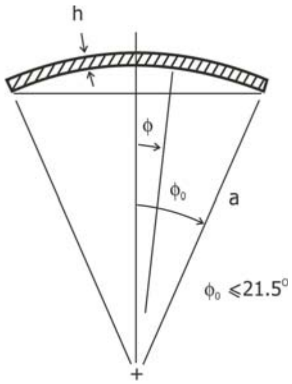


Figure 8-17. Shallow spherical shell or spherical cap. The edge angle ϕ_0 must be $\leq 21.5^\circ$ for the "shallow shell" approximation to be valid within 5%.

The thermal stress resultant and thermal moment for this temperature distribution are

$$N_T = 0 \quad \text{and} \quad M_T = \frac{1}{12}\alpha E \Delta T_1 h^2 \left[1 - \left(\frac{\phi}{\phi_0}\right)^2\right] \tag{8-174}$$

The edge of the cap is free, so there are no applied forces or moments along the edge of the spherical cap.

The governing differential equation, eq. (8-137), reduces to

$$L^2 Q_\phi + 4\lambda^4 Q_\phi = -\frac{E\,h\,a}{(1-\mu)D}\left(-\frac{\alpha E \Delta T_1 h^2 \phi}{6\phi_0^2}\right) = \left[\frac{2(1+\mu)\alpha E \Delta T_1 a}{\phi_0^2}\right]\phi = B\frac{\phi}{\phi_0} \tag{8-175}$$

The quantity B is defined as

$$B = \frac{2(1+\mu)\alpha E \Delta T_1 a}{\phi_0} \tag{8-176}$$

8.7.2.3 Solution of the governing equations. Because the right side of eq. (8-175) is a linear function, let us try a particular solution of the form $(Q_\phi)_{\text{part}} = C\phi$, where C is a constant to be determined. Applying the linear operator $L(\phi)$ given by eq. (8-167), we find

$$L(C\phi) = \frac{d^2(C\phi)}{d\phi^2} + \frac{1}{\phi}\frac{d(C\phi)}{d\phi} - \frac{1}{\phi^2}(C\phi) = 0 + \frac{C}{\phi} - \frac{C}{\phi} = 0$$

Also,

$$L^2(C\phi) = L[L(C\phi)] = 0$$

The constant C is

$$0 + 0 + 4\lambda^4(C\phi) = B\frac{\phi}{\phi_0} \quad \text{or} \quad C = \frac{B}{4\lambda^4\phi_0} \tag{8-177}$$

The general solution for the transverse shear stress resultant is the sum of the solution of the homogeneous equation, eq. (8-.171), and the particular solution, eq. (8-177):

$$Q_\phi = C_1 \text{ber}' x + C_2 \text{bei}' x + C_3 \text{ker}' x + C_4 \text{kei}' x + \frac{B}{4\lambda^4}\left(\frac{\phi}{\phi_0}\right) \tag{8-178}$$

From Table E.2 of Appendix E, we note that $\text{ker}'(0) = -\infty$; therefore, we must have $C_4 = 0$ for the solution to yield a finite transverse shear at the center ($\phi = 0$) of the spherical cap.

With the approximation $\cot\phi \approx 1/\phi$, the azimuth bending stress resultant may be written as follows from eq. (8-125a):

$$M_\phi = -\frac{D}{a}\left(\frac{dV}{d\phi} + \frac{\mu V}{\phi}\right) - \frac{M_T}{1-\mu} \tag{8-179}$$

Similarly, for $N_T = p_r = 0$, the rotation may be written as follows from eq. (8-132):

$$EhV = \frac{d^2 Q_\phi}{d\phi^2} + \frac{1}{\phi}\frac{dQ_\phi}{d\phi} - \left(\frac{1}{\phi^2} - \mu\right) Q_\phi \tag{8-180}$$

The first derivative of the transverse shear resultant involves the second derivative of the Kelvin function $\text{ker}(x)$, $\text{ker}''(x) = -\text{kei}(x) - (1/x)\text{ker}'(x)$. This function is infinite at $x = 0$, because $\text{kei}(0) = \infty$, as given in Table E.2. In order that the azimuth bending stress resultant be finite at the center of the spherical cap, we must have $C_3 = 0$ in the general solution. The transverse shear stress resultant may be written as

$$Q_\phi = C_1 \text{ber}' x + C_2 \text{bei}' x + \frac{B}{4\lambda^4}\left(\frac{\phi}{\phi_0}\right) \tag{8-181}$$

The transverse force at the edge of the spherical cap ($\phi = \phi_0$) is zero for this example:

$$Q_\phi(\phi_0) = 0 = C_1 \text{ber}'(x_0) + C_2 \text{bei}'(x_0) + \frac{B}{4\lambda^4} \tag{8-182}$$

The following expression is obtained for the constant of integration C_2.

$$C_2 = -C_1 \frac{\text{bei}'(x_0)}{\text{ber}'(x_0)} - \frac{B}{4\lambda^4 \text{ber}'(x_0)} \tag{8-183}$$

The quantity x_0 is defined as

$$x_0 = \sqrt{2}\lambda\ \phi_0 \tag{8-184}$$

Making the substitution from eq. (8-182) for the transverse shear stress resultant into eq. (8-180), the following expression is found for the rotation:

$$EhV = -C_1(2\lambda^2 \text{bei}' x - \mu \text{ber}' x) + C_2(2\lambda^2 \text{ber}' x + \mu \text{bei}' x) + \frac{\mu B \phi}{4\lambda^4 \phi_0} \tag{8-185}$$

Making the substitution for the rotation from eq. (8-185) into eq. (8-179) for the azimuth bending stress resultant, the following is obtained:

$$M_\phi = \frac{\sqrt{2}\lambda h^2}{12(1-\mu^2)a}\{C_1[2\lambda^2\varphi_1(x) + \mu\varphi_2(x)] + C_2[2\lambda^2\varphi_2(x) - \mu\varphi_1(x)]\}$$
$$- \frac{\mu(1+\mu)\alpha E\Delta T_1 h^2}{2\lambda^4(1-\mu)\phi_0^2} - \frac{\alpha E\Delta T_1 h^2}{12(1-\mu)}\left[1 - \left(\frac{\phi}{\phi_0}\right)^2\right] \tag{8-186}$$

The functions in eq. (8-186) are

$$\varphi_1(x) = \text{ber } x - \frac{(1-\mu)}{x}\text{bei}' x \tag{8-187a}$$

$$\varphi_2(x) = \text{bei } x + \frac{(1-\mu)}{x}\text{ber}' x \tag{8-187b}$$

Making the substitution for C_2 from eq. (8-183) into the azimuth bending stress resultant expression, eq. (8-186), the following is obtained:

$$M_\phi = \frac{\sqrt{2}\lambda h^2\varphi_3(x)}{12(1-\mu^2)a}C_1 - \frac{\alpha E\Delta T_1 h^2}{12(1-\mu)\lambda^2 x_0 \text{ber}'(x_0)}[2\lambda^2\varphi_2(x) - \mu\varphi_1(x)]$$
$$- \frac{\alpha E\Delta T_1 h^2}{12(1-\mu)}\left[\frac{12\mu(1+\mu)}{\lambda^2 x_0^2} + 1 - \left(\frac{\phi}{\phi_0}\right)^2\right] \tag{8-188}$$

The quantity $\varphi_3(x)$ is defined as

$$\varphi_3(x) = [2\lambda^2\varphi_1(x) + \mu\varphi_2(x)] - \frac{\text{bei}'(x_0)}{\text{ber}'(x_0)}[2\lambda^2\varphi_2(x) - \mu\varphi_1(x)] \tag{8-189}$$

The constant of integration C_1 is determined from the condition that the azimuth bending stress resultant is zero at the edge of the cap ($\phi = \phi_0$):

$$\frac{\sqrt{2}\lambda h^2\varphi_3(x_0)}{12(1-\mu^2)a}C_1 = \frac{\alpha E\Delta T_1 h^2}{12(1-\mu)}\left\{\frac{[2\lambda^2\varphi_2(x_0) - \mu\varphi_1(x_0)]}{\lambda^2 x_0 \text{ber}'(x_0)} + \frac{12\mu(1+\mu)}{\lambda^2 x_0^2}\right\} \tag{8-190}$$

or

$$C_1 = \frac{(1+\mu)\alpha E\Delta T_1 a}{\sqrt{2}\lambda^3 x_0^2\varphi_3(x_0)}\varphi_4(x_0) \tag{8-191}$$

The quantity $\varphi_4(x_0)$ is defined as

$$\varphi_4(x_0) = \frac{[2\lambda^2\varphi_2(x_0) - \mu\varphi_1(x_0)]x_0}{\text{ber}'(x_0)} + 12\mu(1+\mu) \tag{8-192}$$

Making the substitution for the constant of integration C_1 from eq. (8-190) into the azimuth bending stress resultant expression, eq. (8-188), the following expression is obtained:

$$M_\phi = \frac{\alpha E \Delta T_1 h^2}{12(1-\mu)} \left\{ \varphi_5(x) + \frac{12\mu(1+\mu)}{\lambda^2 x_0^2} \left[\frac{\varphi_3(x)}{\varphi_3(x_0)} - 1 \right] + 1 - \left(\frac{\phi}{\phi_0} \right)^2 \right\} \tag{8-193}$$

The quantity $\varphi_5(x)$ is defined as

$$\varphi_5(x) = \frac{[2\lambda^2\varphi_2(x_0) - \mu\varphi_1(x_0)]\dfrac{\varphi_3(x)}{\varphi_3(x_0)} - [2\lambda^2\varphi_2(x) - \mu\varphi_1(x)]}{\lambda^2 x_0 \operatorname{ber}'(x_0)} \tag{8-194}$$

The azimuth membrane stress resultant N_ϕ may be evaluated from eq. (8-129b) with no mechanical loads, $F(\phi) = 0$ and approximating $\cot\phi \approx 1/\phi$:

$$N_\phi = \frac{Q_\phi}{\phi} = C_1\sqrt{2}\lambda \frac{\operatorname{ber}' x}{x} + C_2\sqrt{2}\lambda \frac{\operatorname{bei}' x}{x} + \frac{(1+\mu)\,\alpha E \Delta T_1 a}{\lambda^2 x_0^2} \tag{8-195}$$

If we make the substitutions for the constants of integration C_1 and C_2, the following is obtained:

$$N_\phi = \frac{(1+\mu)\alpha E \Delta T_1 a}{\lambda^2 x_0^2} \left[\frac{\varphi_4(x_0)}{\varphi_3(x_0)} \varphi_6(x) - \frac{\sqrt{2}\lambda \operatorname{bei}' x}{x \operatorname{ber}' x_0} + 1 \right] \tag{8-196}$$

The quantity $\varphi_6(x)$ is defined as

$$\varphi_6(x) = \frac{\operatorname{ber}' x}{x} - \frac{\operatorname{bei}' x_0 \operatorname{bei}' x}{x \operatorname{ber}' x_0} \tag{8-197}$$

The polar membrane stress resultant N_θ may be evaluated from eq. (8-129c) with no mechanical loads, $F(\phi) = p_r = 0$:

$$N_\theta = \frac{dQ_\phi}{d\phi} = -C_1\sqrt{2}\lambda \left(\operatorname{bei} x + \frac{\operatorname{ber}' x}{x} \right) + C_2\sqrt{2}\lambda \left(\operatorname{ber} x - \frac{\operatorname{bei}' x}{x} \right) + \frac{(1+\mu)\,\alpha E \Delta T_1 a}{\lambda^2 x_0^2} \tag{8-198}$$

If we make the substitutions for the constants of integration C_1 and C_2, the following is obtained:

$$N_\theta = -\frac{(1+\mu)\alpha E \Delta T_1 a}{\lambda^2 x_0^2} \left\{ \frac{\varphi_4(x_0)}{\varphi_3(x_0)} \varphi_7(x) + \sqrt{2}\lambda \left(\operatorname{ber} x - \frac{\operatorname{bei}' x}{x} \right) - 1 \right\} \tag{8-199}$$

The quantity $\varphi_7(x)$ is defined as

$$\varphi_7(x) = \operatorname{bei} x + \frac{\operatorname{ber}' x}{x} - \left(\operatorname{ber} x - \frac{\operatorname{bei}' x}{x} \right) \frac{\operatorname{bei}' x_0}{\operatorname{ber}' x_0} \tag{8-200}$$

The polar bending stress resultant M_θ is given by eq. (8-125b), with the approximation $\cot\phi \approx 1/\phi$:

$$M_\theta = -\frac{D}{a}\left(\frac{V}{\phi} + \mu\frac{dV}{d\phi}\right) - \frac{M_T}{1-\mu} \tag{8-201}$$

Using eq. (8-185) for the rotation V, the following result is obtained:

$$\begin{aligned} M_\theta = {} & \frac{\sqrt{2}\lambda h^2}{12(1-\mu^2)a}\{C_1[2\lambda^2\varphi_8(x) + \mu\varphi_9(x)] + C_2[2\lambda^2\varphi_9(x) - \mu\varphi_8(x)]\} \\ & - \frac{\alpha E \Delta T_1 h^2}{12(1-\mu)}\left[\frac{12\mu(1+\mu)}{\lambda^2 x_0^2} + 1 - \left(\frac{\phi}{\phi_0}\right)^2\right] \end{aligned} \tag{8-202}$$

The quantities $\varphi_8(x)$ and $\varphi_9(x)$ are defined as

$$\varphi_8(x) = \mu \operatorname{ber} x + (1-\mu)\frac{\operatorname{bei}' x}{x} \tag{8-203a}$$

$$\varphi_9(x) = \mu \operatorname{bei} x - (1-\mu)\frac{\operatorname{ber}' x}{x} \tag{8-203b}$$

Making the substitution for the constant of integration C_1 from eq. (8-191) and C_2 from eq. (8-183) into the polar bending stress resultant expression, eq. (8-202), the following expression is obtained:

$$\begin{aligned} M_\theta = {} & \frac{\alpha E \Delta T_1 h^2}{12(1-\mu)}\left\{\frac{1}{\lambda^2 x_0^2}\left[\frac{\varphi_4(x_0)\varphi_{10}(x)}{\varphi_3(x_0)} + \frac{\lambda}{\sqrt{2}\operatorname{ber}' x_0} - 12\mu(1+\mu)\right]\right. \\ & \left. - \left[1 - \left(\frac{\phi}{\phi_0}\right)^2\right]\right\} \end{aligned} \tag{8-204}$$

The function $\varphi_{10}(x)$ is defined as

$$\varphi_{10}(x) = [2\lambda^2\varphi_8(x) + \mu\varphi_9(x)] - [2\lambda^2\varphi_9(x) - \mu\varphi_8(x)]\frac{\operatorname{bei}'(x_0)}{\operatorname{ber}'(x_0)} \tag{8-205}$$

The azimuth and polar stresses may be determined from the stress resultants and equations similar to eqs. (8-19):

$$\sigma_\phi = \frac{N_\phi}{h} \pm \frac{6M_\phi}{h^2} \tag{8-206a}$$

$$\sigma_\theta = \frac{N_\theta}{h} \pm \frac{6M_\theta}{h^2} \tag{8-206b}$$

$$\tau_{\phi z,\max} = \frac{3Q_\phi}{2h} \tag{8-206c}$$

The first term in eqs. (8-206a, b) represents the membrane stress and the second term represents the bending stress.

Although the resulting expressions for the stresses are too algebraically complex to solve directly for the shell thickness in design calculations, iterative computer-aided calculations may be used for design purposes. The numerical calculations are illustrated in the following example.

Example 8-6 A spherical cap is constructed of C1020 carbon steel having the following properties: $\alpha = 12 \times 10^{-6}\,^\circ\mathrm{C}^{-1}$ ($6.7 \times 10^{-6}\,^\circ\mathrm{F}^{-1}$), $E = 205$ GPa (29.7×10^6 psi), $\mu = 0.28$, $S_y = 324$ MPa (47,000 psi). The temperature distribution is given by eq. (8-173) with $\Delta T_1 = 50^\circ$C (90°F). The subtended angle of the cap is $\phi_0 = 10^\circ = 0.1745$ rad, the shell thickness is $h = 25$ mm (0.984 in.), and the radius of the cap is $a = 1.00$ m (3.281 ft = 39.37 in.). The edge of the cap is free (no forces applied). Determine the azimuth stress σ_ϕ at the center of the cap ($\phi = 0$) and the polar stress σ_θ at the edge of the cap ($\phi = \phi_0 = 10^\circ$).

First, we note that the shallow shell approach is valid for this problem within an error of

$$\frac{(1/\phi_0) - \cot\phi_0}{\cot\phi_0} = \frac{(1/0.1745) - \cot(10^\circ)}{\cot(10^\circ)} = \frac{5.730 - 5.671}{5.671} = 0.010 = 1\%$$

The shell parameter λ may be found from it definition, eq. (8-139):

$$\lambda^4 = 3(1-\mu^2)\left(\frac{a}{h}\right)^2 - \frac{1}{4}\mu^2 = (3)(1-0.28^2)\left(\frac{1.00}{0.025}\right)^2 - \frac{1}{4}(0.28) = 4423.61$$

$$\lambda = 8.1554$$

The edge variable x_0 is

$$x_0 = \sqrt{2}\,\lambda\phi_0 = \sqrt{2}(8.1554)(0.1745) = 2.013$$

Let us determine the various dimensionless functions related to this problem at the center of the spherical cap. From the series expansion of the Kelvin functions for small x, eqs. (E-7) and (E-8), we find

$$\frac{\mathrm{bei}'x}{x} = \frac{1}{2} - \cdots \quad \text{and} \quad \frac{\mathrm{ber}'x}{x} = \frac{1}{4}x^2 + \cdots$$

From eqs. (8-187), for $x = 0$,

$$\varphi_1(0) = \mathrm{ber}(0) - (1-0.28)\frac{\mathrm{bei}'(x)}{(x)} = 1 - (0.72)\left(\frac{1}{2}\right) = 0.640$$

$$\varphi_2(0) = \mathrm{bei}(0) - (1-0.28)\frac{\mathrm{ber}'(x)}{(x)} = 0 - (0.72)(0) = 0$$

For $x = x_0 = 2.013$,

$$\varphi_1(2.013) = \mathrm{ber}(2.013) - (1-0.28)\frac{\mathrm{bei}'(2.013)}{2.013}$$

$$\varphi_1(2.013) = 0.74483 - (0.72)\frac{0.92054}{2.013} = 0.41558$$

$$\varphi_2(2.013) = \text{bei}(2.013) - (1 - 0.28)\frac{\text{ber}'(2.013)}{2.013}$$

$$\varphi_2(2.013) = 0.98439 - (0.72)\frac{-0.50295}{2.013} = 1.16428$$

Using eq. (8-189),

$$\varphi_3(0) = [(2)(9.1554)^2(0.640) + 0] - \frac{\text{bei}'(2.013)}{\text{ber}'(2.013)}[0 - (0.28)(0.640)]$$

$$\varphi_3(0) = 107.291 - \frac{0.92054}{-0.50295}(-0.1792) = 106.963$$

$$\begin{aligned}\varphi_3(2.013) &= (2)(8.1554)^2(0.41558) + (0.28)(1.16428)\\ &\quad - \frac{0.92054}{-0.50295}[(2)(8.1554)^2(1.26428) - (0.28)(0.41558)]\end{aligned}$$

$$\varphi_3(2.013) = 55.6069 - \frac{0.92054}{-0.50295}(168.0596) = 363.203$$

Using eq. (8-192),

$$\begin{aligned}\varphi_4(2.013) &= \frac{[(2)(8.1554)^2(1.16428) - (0.28)(0.41558)](2.013)}{\text{ber}'(2.013)}\\ &\quad + (12)(0.28)(1.28)\end{aligned}$$

$$\varphi_4(2.013) = \frac{(154.7574)(2.013)}{-0.50295} + 4.3008 = -615.0982$$

Using eq. (8-194),

$$\varphi_5(0) = \frac{[(2)(8.1554)^2(1.16428) - (0.28)(0.41558)]\frac{106.963}{363.203} - [0 - (0.28)(0.640)]}{(8.1554)^2(2.013)\text{ber}'(2.013)}$$

$$\varphi_5(0) = \frac{45.5759 + 0.1792}{(8.1554)^2(2.013)(-0.50295)} = -0.6795$$

The bending stress resultant at the center of the cap may be determined from eq. (8-193):

$$\begin{aligned}M_\phi(0) &= \frac{(12 \times 10^{-6})(205 \times 10^9)(50^\circ)(0.025)^2}{(12)(1 - 0.28)}\\ &\quad \times \left[-0.6795 + \frac{(12)(0.28)(1.28)}{(8.1554)^2(2.013)^2}\left(\frac{106.963}{363.203} - 1\right) + 1\right]\end{aligned}$$

$$M_\phi(0) = (8897.55)(0.3092) = 2751.12\,\text{N} = 2.75112\,\text{kN}$$

From eq. (8-197),

$$\varphi_6(0) = 0 - 0 = 0$$

The azimuth membrane stress resultant at the center of the cap may be determined from eq. (8-196):

$$N_\phi(0) = \frac{(1+0.28)(12\times10^{-6})(205\times10^{9})(50^\circ)(1.00)}{(8.1554)^2(2.013)^2} \times\left[0+\sqrt{2}(8.1554)(1.00-\frac{1}{2})-1\right]$$

$$N_\phi(0) = (584,167)(4.7667) = 2.7846\times10^{6}\ \text{N/m} = 2.7846\ \text{MN/m}$$

The azimuth stress at the center of the spherical cap may be determined from eq. (8-206a):

$$\sigma_\phi(0) = \frac{2.7846\times10^{6}}{0.025} \pm \frac{(6)(2.75112\times10^{3})}{(0.025)^2} = 111.384\times10^{6} + 26.411\times10^{6}$$

$$\sigma_\phi(0) = 137.79\times10^{6}\ \text{Pa} = 137.79\ \text{MPa}\quad (19{,}980\ \text{psi})$$

In this problem, the membrane stress at the center is about (111.384/26.411) = 4.2 times the bending stress at the same point.

The polar stress at the edge of the cap ($\phi = \phi_0$ or $x = x_0$) may be determined as follows. From eq. (8-200),

$$\varphi_7(x_0) = \text{bei}(2.013) + \frac{\text{ber}'(2.013)}{2.013} - \left[\text{ber}(2.013) - \frac{\text{bei}'(2.013)}{2.013}\right]\left[\frac{\text{bei}'(2.013)}{\text{ber}'(2.013)}\right]$$

$$\varphi_7(2.013) = 0.98439 + \frac{-0.50295}{2.013} - \left[0.74483 - \frac{0.92054}{2.013}\right]\left[\frac{0.92054}{-0.50295}\right]$$

$$\varphi_7(2.013) = 0.73454 - (-0.52626) = 1.26080$$

The polar membrane stress resultant N_θ may be found from eq. (8-199):

$$N_\theta(\phi_0) = -\frac{(1.28)(12\times10^{-6})(205\times10^{9})(50^\circ)(1.00)}{(8.1554)^2(2.013)^2}\left\{\frac{-615.0982}{363.203}(1.26080) + \sqrt{2}(8.1554)\left(0.74483 - \frac{0.92054}{2.013}\right) - 1\right\}$$

$$N_\theta(\phi_0) = -(570{,}476)(-2.13521 + 3.31625 - 1) = 103.28\times10^{3}\ \text{N/m}$$

$$N_\theta(\phi_0) = -103.28\ \text{kN/m}$$

The functions $\varphi_8(x)$ and $\varphi_9(x)$ may be found from eqs. (8-203):

$$\varphi_8(x_0) = (0.28)(0.74483) + (0.72)\frac{0.92054}{2.013} = 0.53781$$

$$\varphi_9(x) = (0.28)0.98439) - (0.72)\frac{-0.50295}{2.013} = 0.45552$$

The function $\varphi_{10}(x)$ may be found from eq. (8-205):

$$\begin{aligned}\varphi_{10}(x_0) &= [(2)(8.1554)^2(0.53781) + (0.28)(0.45552)] \\ &\quad - [(2)(8.1554)^2(0.45552) - (0.28)(0.53781)]\frac{0.92054}{-0.50295} \\ \varphi_{10}(x_0) &= 71.66762 - (-110.62803) = 182.29565\end{aligned}$$

The polar bending stress resultant may be found from eq. (8-204):

$$\begin{aligned}M_\theta(x_0) &= \frac{(12\times10^{-6})(205\times10^9)(50^\circ)(0.025)^2}{(12)(0.72)}\left\{\frac{1}{(8.1554)^2(2.013)^2}\right. \\ &\quad \times\left.\left[\frac{(-615.0982)(182.296)}{363.203}+\frac{8.1554}{\sqrt{2}(-0.50295)}-(12)(0.28)(1.28)\right]-0\right\} \\ M_\theta(x_0) &= (8897.6)\left[\frac{1}{269.5120}(-308.725-11.466-4.200)\right] \\ M_\theta(x_0) &= (8897.6)(-1.2036) = -10{,}709.3\,\mathrm{N} = -10.7093\,\mathrm{kN}\end{aligned}$$

The polar stress at the edge of the spherical cap is found from eq. (8-206b):

$$\sigma_\theta(\phi_0) = \frac{N_\theta}{h} \pm \frac{6M_\theta}{h^2} = \frac{-103.28\times10^3}{0.025} \pm \frac{(6)(-10.7093)}{(0.025)^2}$$

$$\sigma_\theta(\phi_0) = -4.1312\times10^6 - 0.1028\times10^6 = -4.234\times10^6\ \mathrm{Pa} = -4.234\ \mathrm{MPa}$$

In this problem, the polar membrane stress at the edge is about $(4.1212/0.1032) \approx$ 40 times the polar bending stress at the same point.

8.8 HISTORICAL NOTE

Augustus Edward Hough Love (Figure 8-18) was a mathematician and Sedleian Professor of Natural Philosophy at Oxford University and was well known for his work in the area of theory of elasticity and geophysics. He began his original technical contributions in theory of elasticity with a study of thin shells, in which he made an extensive study of bending of thin shells by using Kirchhoff's

Figure 8-18. Augustus Edward Hough Love.

model. His most prominent contribution was the two-volume text, *A Treatise on the Mathematical Theory of Elasticity*, the first edition of which appeared in 1882–1883. In 1906, a second edition in German was published that was a little less mathematically abstract and more useful for engineering work.

In addition to his studies in theory of elasticity and thin shells, A.E.H. Love was interested in geophysics. In 1911 he published a book *Some Problems of Geodynamics* that contained his original solutions to several problems involving seismic waves and structure of the Earth. He suggested the existence of waves in stratified materials that give rise to vibrations transverse to the direction of propagation of the waves and in the boundary of the material transmitting the waves. These waves have been called *Love waves* [Timoshenko, 1983].

Wilhelm Flügge was a German civil engineer who contributed extensively to the literature on plates and shells. In 1934 he published the book for which he is widely recognized, *Statik und Dynamik der Schalen*, which was the first text on the stress analysis of shells. The updated English version, *Stresses in Shells*, appeared in 1960 [Flügge, 1960]. An analysis of thermal stresses in spherical shells was presented in this text.

Shortly after the political turnover in Germany in 1933, Flügge was branded as "politically unreliable," despite all of his technical accomplishments, and he lost all chance of a university career in pre-war Germany. In 1938 because Göring was more interested in technical competence than political purity, however, he offered Flügge a department head position in the Deutschland Versuchsanstalt Luftfahrt (DVL), which was the German equivalent of NASA. About the same time, Flügge married Irmgard Lotz, who was an engineer and head of the fluid mechanics division of DVL. After the end of World War II in 1947, they worked in Paris with the Office National d'Études et de Recherches Aéronautiques (ONERA).

Although they enjoyed life in Paris, they felt that their professional future was somewhat limited. After a casual letter of inquiry to Professor Stephen Timoshenko, Flügge was offered a position as professor at Stanford University and his wife was offered a lecturer position, because Stanford had the regulation that a husband and wife could not hold professorial positions in the same department. While Wilhelm was busy with his lecturing and research in plates and shells, Irmgard became one of the pioneers in the field of discontinuous automatic control, in addition to her work in fluid mechanics.

Flügge and his wife taught at Stanford for about twenty years. There were about seventy Ph.D. dissertations completed under their direction, so they had a significant impact on students and faculty at Stanford and on the engineering profession, as well.

PROBLEMS

8-1. A very long cylindrical shell (λL >> 9) of mean radius a and wall thickness h, as shown in Figure 8-19, has a temperature distribution as follows:

$$\Delta T = T(x) - T_0 = (T_1 - T_0)\left(\frac{2z}{h}\right) + (T_2 - T_0) = \Delta T_1\left(\frac{2z}{h}\right) + \Delta T_2$$

The end of the shell (at $x = 0$) is built-in (rigidly fixed) such that the radial deflection and axial rotation at the end are zero. Determine the expressions for (a) the transverse deflection w, (b) the axial and circumferential bending stress resultants, M_x and M_θ, and (c) the axial and circumferential stresses at the outermost fiber (surface) of the shell, σ_x and σ_θ.

8-2. Using the results of Problem 8-1, determine the numerical values for (a) the location and magnitude of the maximum transverse deflection, and (b) the maximum axial stress and maximum circumferential stress. The shell is constructed of C1020 carbon steel with the following properties: $\alpha = 11.9 \times 10^{-6}\ ^\circ\mathrm{C}$ ($6.6 \times 10^{-6\circ}\mathrm{F}^{-1}$), $E = 205$ GPa (29.7×10^6 psi), and

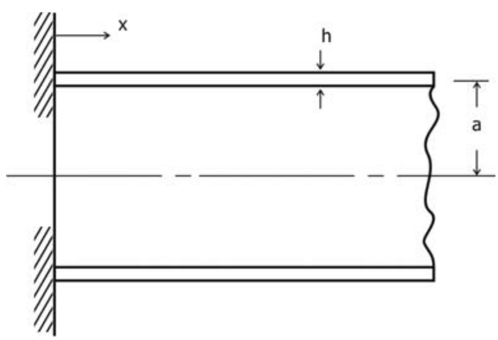


Figure 8-19. Sketch for Problem 8-1.

$\mu = 0.28$. The shell diameter $d = 2a = 3.00$ m (118.1 in.), and the shell wall thickness is $h = 10$ mm (0.394 in.). The temperature parameters are $T_1 = 40°\text{C}$ (104°F), $T_2 = 55°\text{C}$ (131°F), and $T_0 = 25°\text{C}$ (77°F), such that $\Delta T_1 = 15°\text{C}$ (27°F) and $\Delta T_2 = 30°\text{C}$ (54°F).

8-3. Two pieces of long pipe, as shown in Figure 8-20, are to be joined together. The pipe material is 304 stainless steel with the following properties: $\alpha = 16 \times 10^{-6}{}°\text{C}^{-1}$ (8.9×10^{-6} °F^{-1}), $E = 195$ GPa (28.3×10^6 psi), and $\mu = 0.30$. The pipe has a radius a and wall thickness h. At the joint between the two pieces of pipe, the transverse deflection due to heating at the joint is

$$w(0) = w_0 = \alpha(T_1 - T_0)a = \alpha\Delta T_1 a$$

No other mechanical loads are applied, and the remainder of the pipe remains at a temperature T_0 (no temperature change). Determine (a) the expression for the transverse deflection at any location in the pipe ($x \geq 0$), (b) the expressions for the axial and circumferential stresses at any location in the pipe ($x \geq 0$), (c) the maximum temperature change ΔT_1 such that the maximum stress within the pipe does not exceed 200 MPa (29,000 psi). The pipe mean radius is 159.6 mm (6.285 in.) and the pipe wall thickness is 4.57 mm (0.180 in.).

8-4. A more "thermally accurate" temperature distribution for the case given in Problem 8-3 would be that of a long fin,

$$\Delta T = T - T_0 = (T_1 - T_0)e^{-x/b} = \Delta T_1 e^{-x/b}$$

The quantity b is defined as

$$b = \sqrt{\frac{k_t h}{h_c\left(1 + \dfrac{2h}{a}\right)}}$$

The quantity k_t is the thermal conductivity (18 W/m-°C = 10.4 Btu/hr-ft-°F for 304 stainless steel) and h_c is the convective heat transfer coefficient on the outside of the pipe. For this problem, $h_c = 40$ W/m^2-°C (7.04 Btu/hr-ft^2-°F), which is typical for a pipe in air flowing at about 9 m/s (20 mph)

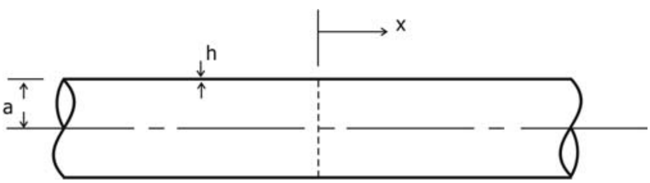


Figure 8-20. Sketch for Problem 8-3.

over the surface. For the same information (except for the temperature distribution) as in Problem 4-3, rework the problem for the temperature distribution given above.

8-5. A ring-stiffened cryogenic fluid storage vessel is constructed of 304 stainless steel (both the shell and the stiffening rings), with the following properties: density, $\rho = 7750$ kg/m^3 (483.8 lb$_m$/ft^3 = 0.280 lb$_m$/in^3) and specific heat, $c = 0.460$ kJ/kg-°C (0.110 Btu/lb$_m$-°F). The shell thickness $h = 20$ mm (0.787 in.), the shell diameter $d = 2a = 1.80$ m (5.905 ft), and the distance between the rings $L = 1.50$ m (4.921 ft). The volume of the shell associated with a ring is $V_{sh} = \pi d L h$, and the surface area of the shell exposed to the fluid is $A_{sh} = \pi(d - h)L$. The cross-sectional area of the ring (an I-beam structural shape) $A_r = 32.5$ cm^2 (5.04 in^2), and the surface area of the ring exposed to the fluid $A_{rg} = 140$ cm^2 (21.7 in^2). The volume of the ring is $V_{rg} = 2\pi r_r A_r$, where the mean radius of the ring $r_r = 0.816$ m (2.677 ft). The vessel is initially at a temperature $T_0 = 22$°C (72°F), and the vessel is suddenly filled with liquid oxygen at $T_f = -183$°C (90.2 K or –297°F). The heat transfer coefficient between the vessel and the LOX $h_c = 95$ W/m^2-°C (16.7 Btu/hr-ft^2-°F). Determine (a) the time constants for the ring and shell section, (b) the time at which the maximum temperature difference occurs, and (c) the maximum temperature difference between the shell and the ring.

8-6. A ring-stiffened cylindrical vessel is constructed of C1020 carbon steel with the following properties: $\alpha = 16 \times 10^{-6}$ °C^{-1} (8.89×10^{-6} °F^{-1}), $E = 200$ GPa (29.0×10^6 psi), and $\mu = 0.270$. The vessel diameter $d = 2a = 2.40$ m (7.874 ft), and the distance between stiffening rings is 3.00 m (9.843 ft). The vessel wall thickness $h = 12.5$ mm (0.492 in.), and the cross sectional area of the ring $A_r = 67.5$ cm^2 (10.46 in^2). The temperature difference between the shell and the ring is $(T_{sh} - T_{rg}) = 30$°C (54°F). Determine (a) the stress in the ring, (b) the axial bending stress at the junction of the ring and shell, (c) the circumferential stress at the junction of the ring and shell, and (d) the axial bending stress at a distance $x =$ 27.5 mm (1.083 in.) from the junction of the ring and shell.

8-7. A cylindrical shell having a diameter $d = 2a$, a wall thickness h, and a length L has the following temperature distribution in the shell wall.

$$\Delta T = T - T_0 = (T_1 - T_0)\left(\frac{2z}{h}\right)\left(1 - \frac{2x}{L}\right)^2$$

The coordinate x is measured from one edge of the shell, and the shell may be treated as a long cylinder. Both edges of the shell are rigidly fixed, such that the radial displacement and axial rotation are zero at the edge ($x = 0$). There is no axial mechanical force applied ($N_x = 0$). Determine the expressions for (a) the transverse deflection of the left half ($0 \le x < \frac{1}{2}L$) of the shell, and (b) the axial stress and circumferential stress in the

same portion of the shell. If the vessel is constructed of C1020 carbon steel, with $\alpha = 16 \times 10^{-6}$ °C^{-1} (8.9×10^{-6} °F^{-1}), $E = 200$ GPa (2.9×10^6 psi), and $\mu = 0.260$, with a mean diameter $d = 2a = 1.20$ m (3.937 ft), a shell wall thickness $h = 10$ mm (0.394 in.), and length $L = 1.50$ m (4.921 ft), determine (c) the numerical value for the transverse deflection at the center of the vessel ($x = \frac{1}{2}$ L), (d) the maximum axial stress at the edge ($x = 0$) of the vessel, and (e) the maximum circumferential stress at the center of the vessel ($x = \frac{1}{2}L$). The temperature difference parameter is $\Delta T_1 = (T_1 - T_0) = 30$°C (54°F).

8-8. A short cylindrical shell of mean radius a and wall thickness h has the following temperature distribution in the wall:

$$\Delta T = T(x) - T_0 = (T_1 - T_0)\left(\frac{2z}{h}\right) = \Delta T_1\left(\frac{2z}{h}\right)$$

The ends are free of any applied force or moment. Select to origin $x = 0$ at the center of the shell and use $2L$ as the total length of the shell, such that the transverse deflection is an even function, $w(+x) = w(-x)$. Determine the expression for the axial bending stress, $\sigma_{x,b} = \pm(6M_x/h^2)$, and circumferential stress, $\sigma_\phi = (N_\phi/h) \pm (6M_\phi/h^2)$, as a function of x. Suppose that the shell is constructed of 304 stainless steel having the following properties: $\alpha = 16.0 \times 10^{-6}$ °C^{-1} (8.89×10^{-6} °F^{-1}), $E = 193$ GPa (28.0×10^6 psi), and $\mu = 0.305$. If the shell dimensions are diameter $2a = 1.200$ m (47.24 in.), shell wall thickness $h = 20$ mm (0.787 in.), and length $2L = 300$ mm (11.81 in.), determine the maximum axial and circumferential stresses σ_x and σ_ϕ at the center ($x = 0$) of the shell. The temperature difference parameter is $\Delta T_1 = (T_1 - T_0) = 40$°C (72°F).

8-9. A hemispherical dome ($\phi_0 = 90° = \frac{1}{2}\pi$ rad) of radius a and wall thickness h has its edge rigidly fixed and has the following temperature distribution:

$$\Delta T = T - T_0 = (T_1 - T_0)\left(\frac{2z}{h}\right)\left(1 - \frac{\phi^2}{\phi_0^2}\right) = \Delta T_1\left(\frac{2z}{h}\right)\left(1 - \frac{\phi^2}{\phi_0^2}\right)$$

No other mechanical loads are applied. Determine the expressions for (a) the azimuth bending stress $\sigma_{\phi,b}$ and (b) the total polar stress σ_θ. Suppose that the shell material is 4340 steel with the following properties: $\alpha = 11.2 \times 10^{-6}$ °C^{-1} (6.22×10^{-6} °F^{-1}), $E = 214$ GPa (31×10^6 psi), and $\mu = 0.27$. Determine the numerical values for (c) the axial bending stress at the center ($\phi = 0$) of the shell and (d) the total polar stress at the edge ($\phi = \phi_0$) of the shell. The temperature difference parameter $\Delta T_1 = 50$°C (90°F) and $a = 1.50$ m, $h = 20$ mm.

8-10. A shallow spherical shell having an edge angle $\phi_0 = 12° = \pi/15$ rad is subjected to a temperature distribution such that the thermal moment is zero, $N_T =$ constant, and the transverse shear resultant at the edge is given by

$$(Q_\phi)_{\phi=\phi o} = \alpha E \Delta T_1 h \quad \text{with} \quad \Delta T_1 = T_1 - T_0$$

The edge of the shell is simply-supported such that the edge bending moment is zero. Determine the expressions for (a) the azimuth bending stress $\sigma_{\phi,b}$ and (b) the total polar stress σ_θ. If the shell material is C1020 carbon steel, with properties given in Problem 8-7, determine the numerical values for (c) the azimuth bending stress at the edge ($\phi = 0$) of the shell and (d) the total polar stress at the edge ($\phi = \phi_0$) of the shell. The temperature difference parameter is $\Delta T_1 = 25^\circ\text{C}$ (45°F). The shell radius is $a = 1.25\,\text{m}$ (4.10 ft), and the shell wall thickness is $h = 12\,\text{mm}$ (0.472 in.)

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9

THICK-WALLED CYLINDERS AND SPHERES

9.1 INTRODUCTION

In cases in which the thickness of the element is not small compared with the other dimensions, the plane-stress model and the thin-shell model are no longer accurate. There are many engineering applications that fall in this category, including thick-walled pressure vessels, such as a nuclear containment vessel; a cylindrical roller, such as a roller bearing; and a long culvert or thick-walled tunnel structure. In these cases, the external force may not vary significantly in the axial direction, except possibly in the immediate vicinity of the ends of the structure.

One model that can be used in these cases is the *plane strain* model. The fundamental component of this model is that the axial strain may be treated as constant. The displacement in the axial direction (z-direction, for example) is taken to be a linear function of the axial coordinate for cylindrical coordinates:

$$w = \varepsilon_0 z \tag{9-1}$$

The strain in the axial direction is given by eq. (5-3):

$$\varepsilon_z = \frac{\partial w}{\partial z} = \varepsilon_0 = \text{constant} \tag{9-2}$$

If the constant axial strain $\varepsilon_0 = 0$, the problem is called *zero plane strain* or *totally constrained plane strain*, and if the axial stress $\sigma_z = 0$, the problem is called *free plane strain* [Burgreen, 1971].

The restrictions on the accurate application of the plane strain approach include the following:

(a) The surface forces must be constant (uniform) in the axial direction. This means that the stresses and strains are independent of the axial coordinate, and the problem is a two-dimensional one, at most.
(b) Torsional forces are not present on the ends of the structure. There are no twisting moments, so the shear stresses and shear strains in the axial direction are zero.

$$\tau_{xz} = \tau_{yz} = \gamma_{xz} = \gamma_{yz} = 0 \qquad \text{for Cartesian coordinates} \tag{9-3a}$$

$$\tau_{rz} = \tau_{\theta z} = \gamma_{rz} = \gamma_{\theta z} = 0 \qquad \text{for cylindrical coordinates} \tag{9-3b}$$

(c) The temperature within the element does not vary along the axis, $T \neq f(z)$.
(d) The stresses are below the elastic limit.

9.2 GOVERNING EQUATIONS FOR PLANE STRAIN

9.2.1 Stress and Strain Relationships

The strain components in cylindrical coordinates, as shown in Figure 9-1, for general plane strain problems may be obtained from eq. (5-12):

$$\varepsilon_r = \frac{\partial u}{\partial r} \quad \text{and} \quad \varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \varepsilon_z = \varepsilon_0 = \text{constant} \tag{9-4a}$$

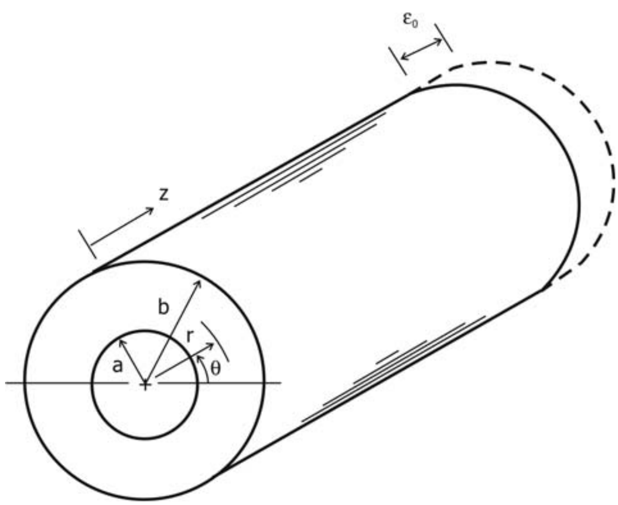

Figure 9-1. Long, thick-wall, hollow cylinder.

For one-dimensional problems, in which the displacements are functions of the radial coordinate r only, eq. (9-4a) reduces to

$$\varepsilon_r = \frac{du}{dr} \quad \text{and} \quad \varepsilon_\theta = \frac{u}{r} \quad \text{and} \quad \varepsilon_z = \varepsilon_0 = \text{constant} \tag{9-4b}$$

The shear strains are obtained from eq. (5-13) in cylindrical coordinates for general plane strain problems:

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \quad \text{and} \quad \gamma_{rz} = \gamma_{\theta z} = 0 \tag{9-5a}$$

For one-dimensional problems, in which the displacements are functions of the radial coordinate r only, all of the shear strains are zero:

$$\gamma_{r\theta} = \gamma_{rz} = \gamma_{\theta z} = 0 \tag{9-5b}$$

The stress equilibrium relationships for general plane strain problems are obtained from eqs. (5-13):

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{9-6a}$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{r\theta}}{r} = 0 \tag{9-6b}$$

For one-dimensional problems, in which the displacements are functions of the radial coordinate r only, eq. (9-6a) reduces to

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad \text{or} \qquad \frac{d(r\sigma_r)}{dr} - \sigma_\theta = 0 \tag{9-6c}$$

Using the Hooke's law relationships, eqs. (5-33), the stress–strain relations for plane strain nay be written as

$$\varepsilon_r = \frac{1}{E}[\sigma_r - \mu(\sigma_\theta + \sigma_z)] + \alpha \Delta T \tag{9-7a}$$

$$\varepsilon_\theta = \frac{1}{E}[\sigma_\theta - \mu(\sigma_r + \sigma_z)] + \alpha \Delta T \tag{9-7b}$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \mu(\sigma_r + \sigma_\theta)] + \alpha \Delta T = \varepsilon_0 \tag{9-7c}$$

The value of the axial strain ε_0 is generally a known quantity or the axial stress σ_z is known. In either case, the other variable may be found from eq. (9-7c). For example,

$$\sigma_z = \mu(\sigma_r + \sigma_\theta) - \alpha E \Delta T + \varepsilon_0 E \tag{9-8}$$

If we make the substitution for the axial stress from eq. (9-8) into eqs. (9-7a) and (9-7b), the following stress–strain relationships are obtained from the plane strain conditions in cylindrical coordinates:

$$\varepsilon_r = \frac{1 - \mu^2}{E}\left(\sigma_r - \frac{\mu}{1-\mu}\sigma_\theta\right) + (1+\mu)\alpha \Delta T - \mu \varepsilon_0 \tag{9-9a}$$

$$\varepsilon_\theta = \frac{1-\mu^2}{E}\left(\sigma_\theta - \frac{\mu}{1-\mu}\sigma_r\right) + (1+\mu)\alpha\Delta T - \mu\varepsilon_0 \qquad (9\text{-}9b)$$

The stresses may also be expressed in terms of the strains for plane strain in cylindrical coordinates:

$$\sigma_r = \frac{(1-\mu)E}{(1+\mu)(1-2\mu)}\left[\varepsilon_r + \left(\frac{\mu}{1-\mu}\right)(\varepsilon_\theta + \varepsilon_0)\right] - \frac{\alpha E\Delta T}{1-2\mu} \qquad (9\text{-}10a)$$

$$\sigma_\theta = \frac{(1-\mu)E}{(1+\mu)(1-2\mu)}\left[\varepsilon_\theta + \left(\frac{\mu}{1-\mu}\right)(\varepsilon_r + \varepsilon_0)\right] - \frac{\alpha E\Delta T}{1-2\mu} \qquad (9\text{-}10b)$$

The stress relations may be written in an alternate form using the Lamè elastic constant λ_L, given by eq. (1-22),

$$\lambda_L = \frac{\mu E}{(1+\mu)(1-2\mu)} \qquad (9\text{-}11a)$$

the shear modulus G, given by eq. (1-18),

$$G = \frac{E}{2(1+\mu)} \qquad (9\text{-}11b)$$

the volumetric coefficient of thermal expansion, $\beta_t = 3\alpha$, and the volumetric modulus of elasticity B, given by eq. (1-20),

$$B = \frac{E}{3(1-2\mu)} \qquad (9\text{-}11c)$$

$$\sigma_r = (\lambda_L + 2G)\varepsilon_r + \lambda_L(\varepsilon_\theta + \varepsilon_0) - \beta_t B\Delta T \qquad (9\text{-}12a)$$

$$\sigma_\theta = (\lambda_L + 2G)\varepsilon_\theta + \lambda_L(\varepsilon_r + \varepsilon_0) - \beta_t B\Delta T \qquad (9\text{-}12b)$$

9.2.2 Displacement Formulation

The expressions for the stresses from eqs. (9-10) may be substituted into the stress equilibrium relationship, eq. (9-6c), to obtain the following for the one-dimensional problem:

$$\frac{d}{dr}\left[\varepsilon_r + \frac{\mu}{1-\mu}(\varepsilon_\theta + \varepsilon_0)\right] - \left(\frac{1+\mu}{1-\mu}\right)\alpha\frac{d(\Delta T)}{dr} + \left(\frac{1-2\mu}{1-\mu}\right)\frac{\varepsilon_r - \varepsilon_\theta}{r} = 0 \qquad (9\text{-}13a)$$

Using eq. (9-4b) for the displacement components, eq. (9-13a) may be written as

$$\frac{d}{dr}\left[\frac{du}{dr} + \frac{\mu}{1-\mu}\left(\frac{u}{r} + \varepsilon_0\right)\right] - \left(\frac{1+\mu}{1-\mu}\right)\alpha\frac{d(\Delta T)}{dr} + \left(\frac{1-2\mu}{1-\mu}\right)\left(\frac{1}{r}\frac{du}{dr} - \frac{u}{r^2}\right) = 0 \qquad (9\text{-}13b)$$

Equation (9-13b) may be simplified to obtain the governing expression for the displacement formulation of the one-dimensional plane strain problem in cylindrical coordinates:

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d(ru)}{dr}\right] - \left(\frac{1+\mu}{1-\mu}\right)\alpha\frac{d(\Delta T)}{dr} = 0 \tag{9-14}$$

Separating variables and integrating eq. (9-14), the following is obtained:

$$\frac{1}{r}\frac{d(ru)}{dr} = \left(\frac{1+\mu}{1-\mu}\right)\alpha\Delta T + 2C_1 - 2\mu\varepsilon_0 \tag{9-15}$$

Separating variables and integrating eq. (9-15), the final general expression is obtained for the radial displacement u:

$$u = \left(\frac{1+\mu}{1-\mu}\right)\frac{\alpha}{r}\int \Delta T r\, dr + C_1 r + \frac{C_2}{r} - \mu r\varepsilon_0 \tag{9-16}$$

Equation (9-16) may be used in eqs. (9-10) to obtain the expressions for the stresses in the cylinder:

$$\sigma_r = -\frac{\alpha E}{(1-\mu)r^2}\int \Delta T r\, dr + \frac{E}{1+\mu}\left(\frac{C_1}{1-2\mu} - \frac{C_2}{r^2} - \frac{\mu\varepsilon_0}{1-2\mu}\right) \tag{9-17a}$$

$$\sigma_\theta = -\frac{\alpha E}{(1-\mu)r^2}\int \Delta T r\, dr - \frac{\alpha E\Delta T}{1-\mu} + \frac{E}{1+\mu}\left(\frac{C_1}{1-2\mu} + \frac{C_2}{r^2} - \frac{\mu\varepsilon_0}{1-2\mu}\right) \tag{9-17b}$$

If the ends of the cylinder are rigidly fixed, the axial strain $\varepsilon_0 = 0$. On the other hand, if the ends are free, the average axial stress $(\sigma_z)_{\text{ave}} = 0$. In general, we may write the average axial stress in terms of an end reaction per unit circumferential length R_e as follows:

$$(\sigma_z)_{\text{ave}} = \frac{R_e[2\pi(a+b)/2]}{\pi(b^2-a^2)} = \frac{R_e}{b-a} = \frac{1}{\pi(b^2-a^2)}\int_a^b \sigma_z(2\pi r\, dr) \tag{9-18a}$$

$$(\sigma_z)_{\text{ave}} = \frac{2}{b^2-a^2}\int_a^b \sigma_z\, r\, dr \tag{9-18b}$$

The axial stress is given by eq. (9-8):

$$\sigma_z = E\varepsilon_0 + \frac{2\mu E C_1}{(1+\mu)(1-2\mu)} - \frac{\alpha E\Delta T}{1-\mu} \tag{9-19}$$

Making the substitution for the axial stress σ_z from eq. (9-19) into eq. (9-18), the following relationship is obtained between the end reaction per unit circumferential length and the axial strain ε_0:

$$\varepsilon_0 = \frac{2\alpha}{(1-\mu)(b^2-a^2)}\int_a^b \Delta T r\, dr - \frac{2\mu C_1}{(1+\mu)(1-2\mu)} + \frac{R_e}{E(b-a)} \tag{9-20}$$

9.2.3 Stress Formulation

The plane strain relationships may also be expressed in terms of the radial stress as the primary variable. For cases in which the stresses are known at the boundaries of the system, the stress formulation is usually more advantageous to use than the displacement formulation, although both approaches are equivalent.

The following relationship may be found between the strain relations, eq. (9-4b):

$$\varepsilon_r = \frac{du}{dr} \quad \text{and} \quad u = r\varepsilon_\theta \tag{9-21}$$

Therefore,

$$\varepsilon_r = \frac{d(r\varepsilon_\theta)}{dr} \tag{9-22}$$

For one-dimensional problems, the strain relation, eq. (9-22), may be combined with the Hooke's law expressions, eqs. (9-9), to yield

$$\sigma_r = \frac{d}{dr}\left[r\frac{d(r\sigma_r)}{dr}\right] + \frac{\alpha E r}{1-\mu}\frac{d(\Delta T)}{dr} \tag{9-23}$$

Equation (9-23) may be simplified as follows to obtain the governing relationship for the stress formulation of the one-dimensional plane strain problem in cylindrical coordinates:

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d(r^2\sigma_r)}{dr}\right] + \frac{\alpha E}{1-\mu}\frac{d(\Delta T)}{dr} = 0 \tag{9-24}$$

Separating variables and integrating eq. (9-24), we get

$$\frac{1}{r}\frac{d(r^2\sigma_r)}{dr} = 2C_1 - \frac{\alpha E \Delta T}{1-\mu} \tag{9-25}$$

Separating variables and integrating eq. (9-25), we obtain the general solution for the radial stress:

$$\sigma_r = C_1 + \frac{C_2}{r^2} - \frac{\alpha E}{(1-\mu)r^2}\int \Delta T r\, dr \tag{9-26a}$$

The expression for the circumferential stress may be found by combining the stress equilibrium relation, eq. (9-6c), and the expression for the radial stress, eq. (9-26a):

$$\sigma_\theta = \frac{d(r\sigma_r)}{dr} = C_1 - \frac{C_2}{r^2} - \frac{\alpha E \Delta T}{1-\mu} + \frac{\alpha E}{(1-\mu)r^2}\int \Delta T r\, dr \tag{9-26b}$$

Note that the constants of integration C_1 and C_2 in eqs. (9-26), the stress formulation, are not the same as the constants in eqs. (9-17), the displacement formulation.

The axial stress may be found from eq. (9-8):

$$\sigma_z = 2\mu C_1 - \frac{\alpha E \Delta T}{1-\mu} + E\varepsilon_0 \tag{9-27}$$

If the ends of the cylinder are rigidly fixed, the strain $\varepsilon_0 = 0$, and if the ends are free, the average axial stress $(\sigma_r)_{\text{ave}} = 0$. In general, the average stress may be related to the end reaction force per unit circumferential length, eq. (9-18):

$$(\sigma_z)_{\text{ave}} = \frac{R_e}{b-a} = \frac{2}{b^2 - a^2} \int_a^b \sigma_z \, r \, dr \tag{9-28}$$

If we make the substitution for the axial stress from eq. (9-27) into eq. (9-28), the following expression is obtained for the axial strain:

$$E\varepsilon_0 = \frac{R_e}{b-a} - 2\mu C_1 + \frac{2E}{(1-\mu)(b^2 - a^2)} \int_a^b \Delta T r \, dr \tag{9-29}$$

9.3 HOLLOW CYLINDER WITH STEADY-STATE HEAT TRANSFER

9.3.1 Thermal Analysis

Let us consider the thermal stresses arising in a hollow cylinder in which steady-state heat transfer occurs, as shown in Figure 9-2. For this example, there is no energy dissipation (heat “generation”) and the temperature distribution is a function of the radial coordinate only (one-dimensional steady-state heat transfer). Let us also consider the case in which the material properties are not functions of temperature. For this case, the temperature field equation in cylindrical coordinates, eq. (5-64), reduces to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0 \tag{9-30}$$

The two boundary conditions for the temperature are as follows:

At the outer surface, $r = b$, the temperature $T = T_0$
At the inner surface, $r = a$, the temperature $T = T_1$

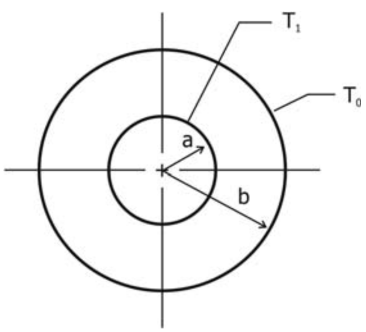

Figure 9-2. Surface temperatures in a hollow cylinder with steady-state heat transfer.

The temperature field equation, eq. (9-30), may be integrated twice to obtain the temperature distribution in the cylinder wall:

$$T = B_1 \ln r + B_2 \tag{9-31}$$

The quantities B_1 and B_2 are constants of integration to be determined by the boundary conditions. Using the two boundary conditions in eq. (9-31), the following expressions are obtained:

$$T_1 = B_1 \ln a + B_2 \tag{9-32a}$$

$$T_0 = B_1 \ln b + B_2 \tag{9-32b}$$

The constants of integration may be found from the solutions of eq. (9-32):

$$B_1 = \frac{T_1 - T_0}{\ln(a/b)} = \frac{\Delta T_1}{\ln(a/b)} \tag{9-33a}$$

$$B_2 = T_0 - \Delta T_1 \frac{\ln b}{\ln(a/b)} \tag{9-33b}$$

The temperature distribution is obtained by making the substitutions for the constants of integration from eqs. (9-33) into eq. (9-31).

$$\Delta T = T - T_0 = \Delta T_1 \frac{\ln(r/b)}{\ln(a/b)} \tag{9-34}$$

9.3.2 Stress Analysis

Because the stress is known at both the inner and outer surfaces of the cylinder, it is more mathematically convenient to use the stress formulation in this example. The integral in eqs. (9-26) may be evaluated for the temperature distribution given by eq. (9-34):

$$\int \Delta T r\, dr = \frac{\Delta T_1}{\ln(a/b)} \int \ln(r/b)\, r\, dr = \frac{\Delta T_1 r^2}{4 \ln(a/b)} \left[2 \ln\left(\frac{r}{b}\right) - 1 \right] \tag{9-35}$$

The radial stress may be found from eq. (9-26a):

$$\sigma_r = C_1 + \frac{C_2}{r^2} - \frac{\alpha E \Delta T_1}{4(1-\mu)\ln(a/b)} \left[2 \ln\left(\frac{r}{b}\right) - 1 \right] \tag{9-36}$$

The two boundary conditions for the radial stress are as follows (no force is applied on the surfaces):

At the outer surface, $r = b$, the stress $\sigma_r = 0$
At the inner surface, $r = a$, the stress $\sigma_r = 0$

Applying these boundary conditions in eq. (9-36), the following expressions are obtained:

$$0 = C_1 + \frac{C_2}{a^2} - \frac{\alpha E \Delta T_1}{4(1-\mu)\ln(a/b)}\left[2\ln\left(\frac{a}{b}\right) - 1\right] \tag{9-37a}$$

$$0 = C_1 + \frac{C_2}{b^2} + \frac{\alpha E \Delta T_1}{4(1-\mu)\ln(a/b)} \tag{9-37b}$$

The expressions for the two constants of integration from eqs. (9-37) are as follows:

$$C_1 = -\frac{\alpha E \Delta T_1}{2(1-\mu)}\left[\frac{1}{2\ln(a/b)} + \frac{1}{(b/a)^2 - 1}\right] \tag{9-38a}$$

$$C_2 = -\frac{\alpha E \Delta T_1}{2(1-\mu)}\frac{b^2}{[(b/a)^2 - 1]} \tag{9-38b}$$

By making the substitutions for the constants from eqs. (9-38) into the radial stress expression, eq. (9-36), the following result is obtained:

$$\sigma_r = -\frac{\alpha E \Delta T_1}{2(1-\mu)}\left[\frac{\ln(r/b)}{\ln(a/b)} - \frac{(b/r)^2 - 1}{(b/a)^2 - 1}\right] \tag{9-39}$$

Similarly, the circumferential stress distribution may be obtained from eq. (9-26b).

$$\sigma_\theta = -\frac{\alpha E \Delta T_1}{2(1-\mu)}\left[\frac{1+\ln(r/b)}{\ln(a/b)} + \frac{(b/r)^2 + 1}{(b/a)^2 - 1}\right] \tag{9-40}$$

These stress distributions are illustrated in Figure 9-3.

Generally, at any location, the circumferential stress is larger than the radial stress. The maximum stress in this example is the circumferential stress at the inner surface ($r = a$) of the cylinder:

$$(\sigma_\theta)_{r=a} = -\frac{\alpha E \Delta T_1}{2(1-\mu)}\left[\frac{1}{\ln(a/b)} + \frac{2}{1-(a/b)^2}\right] \tag{9-41}$$

The axial stress distribution may be found from eq. (9-7c), using the stresses from eqs. (9-39) and (9-40) for zero axial strain:

$$\sigma_z = -\frac{\alpha E \Delta T_1}{2(1-\mu)}\left[\frac{1}{\ln(a/b)} + \frac{2\ln(r/b)}{\ln(a/b)} + \frac{2}{(b/a)^2 - 1}\right] \tag{9-42}$$

The radial displacement may be found from eqs. (9-21b) and (9-22), using the stress distribution given by eqs. (9-9). The expression for the radial displacement u may be written as follows for zero axial strain:

$$u = \varepsilon_r r = \frac{1}{2}(1+\mu)\alpha \Delta T_1 r\left\{\frac{1}{\ln(b/a)} + \frac{1}{1-\mu}\frac{\ln(b/r)}{\ln(b/a)} - \frac{(b/r)^2 + (1-2\mu)}{(1-\mu)[(b/a)^2 - 1]}\right\} \tag{9-43}$$

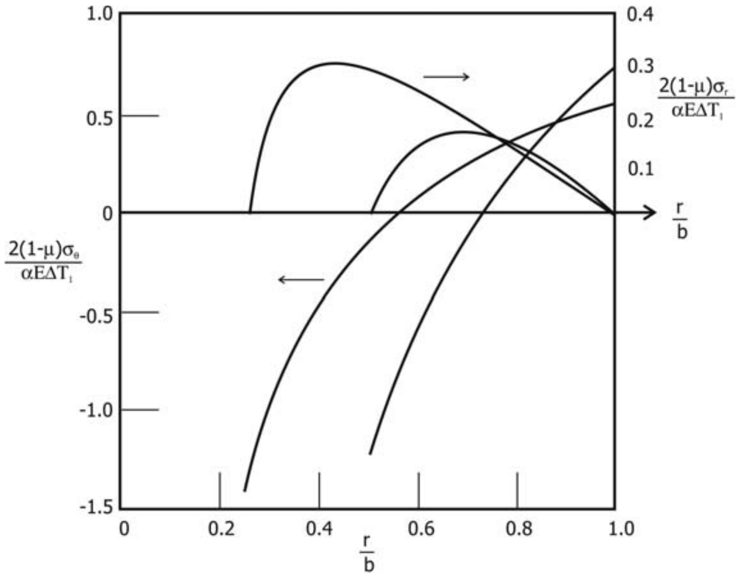

Figure 9-3. Plot of the radial and tangential stress distribution in a hollow cylinder with steady-state heat transfer.

The radial displacement at the outer surface ($r = b$) is

$$u(b) = \frac{1}{2}(1+\mu)\alpha\Delta T_1 b\left\{\frac{1}{\ln(b/a)} - \frac{2}{(b/a)^2 - 1}\right\} \tag{9-44a}$$

At the inner surface ($r = a$) the radial displacement is

$$u(a) = \frac{1}{2}(1+\mu)\alpha\Delta T_1 a\left\{\frac{1}{\ln(b/a)} + \frac{1}{1-\mu} - \frac{(b/a)^2 + (1-2\mu)}{(1-\mu)[(b/a)^2 - 1]}\right\} \tag{9-44b}$$

Example 9-1 A cylindrical container is made of polystyrene with the following properties: $\alpha = 7.2 \times 10^{-5}\ {}^\circ\mathrm{C}^{-1}$ ($4.0 \times 10^{-5}\ {}^\circ\mathrm{F}^{-1}$), $E = 3.1$ GPa (4.5×10^6 psi), $\mu = 0.32$. The inside diameter of the container is 300 mm ($a = 150\,\mathrm{mm} = 5.91$ in.), and the outside diameter is 400 mm ($b = 200\,\mathrm{mm} = 7.88$ in.). The container is filled with liquid nitrogen, such that the temperature of the inner surface $T_1 = -196^\circ$C (-321°F), and the outer surface temperature $T_0 = +14^\circ$C ($+57^\circ$F). Determine the maximum stress in the polystyrene container wall if the ends of the cylinder are rigidly fixed.

The numerical value of the coefficient term in eq. (9-41) is

$$\frac{\alpha E\Delta T_1}{2(1-\mu)} = \frac{(7.2\times10^{-5})(3.1\times10^{9})(-196^\circ - 14^\circ)}{(2)(1-0.32)} = -34.46\times10^{6}\,\mathrm{Pa}$$
$$= -34.46\,\mathrm{MPa}$$

The radius ratio for this example is

$$\frac{a}{b} = \frac{150}{200} = 0.750$$

The circumferential stress at the inner surface (the maximum stress) is found from eq. (9-41):

$$(\sigma_\theta)_{r=a} = -(-34.46)\left[\frac{1}{\ln(0.750)} + \frac{2}{1-(0.750)^2}\right] = (34.46)(1.0954)$$

$$(\sigma_\theta)_{r=a} = \sigma_{\max} = +37.75\,\text{MPa}\ (+5475\,\text{psi}) \quad \text{a } \textit{tensile} \text{ stress}$$

The yield strength of polystyrene foam in tension is about $S_y = 50\,\text{MPa}$ (7250 psi), so the cylinder will not fail as a result of thermal stresses.

The circumferential stress at the outer surface is found from eq. (9-40) with $r = b$:

$$(\sigma_\theta)_{r=b} = -(-34.46)\left[\frac{1+0}{\ln(0.750)} + \frac{1+1}{(1/0.750)^2 - 1}\right] = (34.46)(-0.9046)$$

$$(\sigma_\theta)_{r=b} = -31.17\,\text{MPa}\ \ (-4520\,\text{psi}) \quad \text{a } \textit{compressive} \text{ stress}$$

The radial stress is zero at both inner and outer surfaces (no mechanical forces applied). The radial stress at $r = 175$ mm (6.90 in.) may be calculated from eq. (9-39):

$$(\sigma_r)_{r=350} = -(-34.46)\left[\frac{\ln(175/200)}{\ln(150/200)} - \frac{(200/175)^2 - 1}{(200/150)^2 - 1}\right] = (34.46)(0.07058)$$

$$(\sigma_r)_{r=350} = +2.43\,\text{MPa} \quad (353\,\text{psi})$$

The radial stress at this point is only about 7% of the maximum circumferential stress.

The displacement of the outside surface of the container may be found from eq. (9-44a):

$$u(b) = \frac{1}{2}(1.32)(7.2\times 10^{-5})(-210^\circ)(0.200)\left\{\frac{1}{\ln(1.333)} - \frac{2}{(1.333)^2 - 1}\right\}$$

$$u(b) = (-0.001996)(0.9046) = -1.805\times 10^{-3}\text{m} = -1.805\,\text{mm} \qquad (-0.071\text{ in.})$$

9.4 SOLID CYLINDER

There are many engineering applications in which thermal stresses arise due to transient heating or cooling a solid rod, such as quenching of a machine element rod. In this section, we consider the transient heat transfer and resulting thermal stresses for a long rod for which the axial strain is negligible. The rod is initially at a uniform temperature T_0, and at the initial time ($t = 0$), the rod is suddenly exposed to a fluid at T_f with convective heat transfer taking place at the surface, as shown in Figure 9-4.

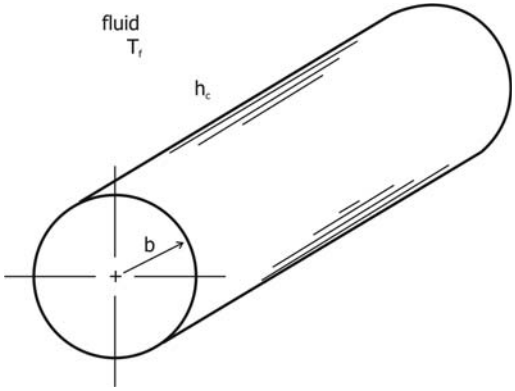

Figure 9-4. Solid cylinder under transient heat transfer conditions with convection at the surface.

9.4.1 Thermal Analysis

The temperature field equation for one-dimensional transient heat transfer in cylindrical coordinates is obtained from eq. (5-64):

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\kappa}\frac{\partial T}{\partial t} \tag{9-45}$$

The quantity κ {units, m^2/s or ft^2/hr} is the *thermal diffusivity* of the material. The boundary conditions on the problem are as follows:

At the outer surface, $r = b$, the heat conducted at the surface is equal to the heat transfer by convection at the surface,

$$-k_t\left(\frac{\partial T}{\partial r}\right)_{r=b} = h_c(T_{r=b} - T_f)$$

At the center, $r = 0$, the temperature is finite.

The initial condition is that, at $t = 0$, the temperature $T = T_0$ throughout the cylinder.

This problem was solved by French mathematician Joseph B. Fourier (who also developed the Fourier series analysis and the Fourier transform) in his pioneering work *Théorie Analytique de la Chaleur* (Analytical Theory of Heat), published in 1822 [Fourier, 1955]. Fourier presented his solution as an infinite series, which is now called a *Bessel function*.

One mathematical technique for solving a partial differential equation, such as the temperature field equation, is the method of *separation of variables*. Let us try a solution of eq. (9-45) of the following form:

$$T(r, t) - T_f = R(r)F(t) \tag{9-46}$$

The quantities $R(r)$ and $F(t)$ are functions of only r or only t, respectively, so that the dependence on the variables r and t is separated into two different functions. The following result is obtained after substitution of the trial solution into eq. (9-45):

$$F\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = R\frac{1}{\kappa}\frac{dF}{dt} \tag{9-47}$$

or

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \frac{1}{\kappa F}\frac{dF}{dt} = -\lambda^2 \tag{9-48}$$

The quantity λ is a constant, called the *eigenvalue* for the problem. Actually, we will find that there are an infinite number of eigenvalues for this problem. The negative sign is inserted because the temperature difference must decrease with time (an *exponential* function will be found as the solution for F) instead of increasing without limit.

It is possible to separate variables in the temperature field equation to obtain two total differential equations:

$$\frac{d^2(T - T_f)}{dr^2} + \frac{1}{r}\frac{d(T - T_f)}{dr} + \lambda^2 R = 0 \tag{9-49a}$$

$$\frac{dF}{dt} + \lambda^2 \kappa F = 0 \tag{9-49b}$$

The first expression, eq. (9-49a), is a form of the Bessel equation, as shown in Appendix D.

The solutions of eqs. (9-49) are as follows:

$$R(r) = B_1 J_0(\lambda r) + B_2 Y_0(\lambda r) \tag{9-50a}$$

$$F(t) = B_3 \exp(-\lambda^2 \kappa t) \tag{9-50b}$$

$$T(r,t) - T_f = B_3 \exp(-\lambda^2 \kappa t)[B_1 J_0(\lambda r) + B_2 Y_0(\lambda r)] \tag{9-50c}$$

As discussed in Appendix D, the Bessel function of the second kind $Y_0(r)$ is infinite at $r = 0$ (the center of the cylinder); therefore, the constant $B_2 = 0$ to meet the condition that the temperature must be finite.

The first boundary condition (at the surface of the cylinder) may be expressed as follows, using eq. (D-29):

$$+\lambda J_1(\lambda b) = \frac{h_c}{k_t} J_0(\lambda b)$$

or

$$\frac{(\lambda b) J_1(\lambda b)}{J_0(\lambda b)} = \frac{h_c b}{k_t} = \text{Bi} = \text{Biot number} \tag{9-51}$$

The dimensionless Biot number Bi gives a measure of the ratio of the thermal resistance to conduction within the cylinder to the thermal resistance to convection at the surface of the cylinder. If the Biot number is small (less than

about 0.2), the temperature within the cylinder is approximately uniform (within 10% or so); if the Biot number is very large (greater than about 20), the surface temperature of the cylinder rapidly approaches the fluid temperature T_f.

There are an infinite number of solutions for eq. (9-51). If the solutions are denoted by $\beta_n = \lambda_n b$ ($n = 1, 2, 3, \ldots$), the solution for the temperature distribution may be written as follows:

$$T(r,\ t) - T_f = B_n \exp(-\beta_n^2 \text{Fo}) J_0(\beta_n r/b) \tag{9-52}$$

The quantity

$$\text{Fo} = \frac{\kappa t}{b^2} \tag{9-53}$$

is the *Fourier number*, which is a dimensionless time quantity.

No one value of β_n will satisfy the initial condition, so we may try the linear combination of all solutions, which results in a Fourier series in terms of the Bessel functions.

$$T(r,\ t) - T_f = \sum_{n=1}^{\infty} B_n \exp(-\beta_n^2 \text{Fo}) J_0(\beta_n r/b) \tag{9-54}$$

Applying the initial condition to eq. (9-54), the following is obtained:

$$T(r,\ 0) - T_f = T_0 - T_f = \sum_{n=1}^{\infty} B_n J_0(\beta_n r/b) \tag{9-55}$$

The constants B_n may be found by multiplying both sides of eq. (9-55) by $r J_0(\beta_m r/b)$ and integrating from the center to the surface of the cylinder:

$$(T_0 - T_f) \int_0^b r J_0(\beta_m r/b)\, dr = \sum_{n=1}^{\infty} B_n \int_0^b r J_0(\beta_m r/b) J_0(\beta_n r/b)\, dr \tag{9-56}$$

Because Bessel functions have an "orthogonality" property, all of the integrals of the series are zero except the one for which $m = n$ [Arfken, 1966]:

$$\int_0^b r J_0(\beta_m r/b) J_0(\beta_n r/b)\, dr = \begin{cases} 0 & \text{if } \ m \neq n \\ \frac{1}{2} b^2 [J_0^2(\beta_n) + J_1^2(\beta_n)] & \text{if } \ m = n \end{cases} \tag{9-57}$$

The integral of the initial temperature distribution may be found from eq. (D-27):

$$\int_0^b r J_0(\beta_n r/b) dr = \frac{b^2}{\beta_n} J_1(\beta_n) \tag{9-58}$$

Substituting the expressions for the integrals, eqs. (9-57) and (9-58), into eq. (9-56) we gives

$$B_n = \frac{2 J_1(\beta_n)}{\beta_n [J_0^2(\beta_n) + J_1^2(\beta_n)]} (T_0 - T_f) = A_n (T_0 - T_f) \tag{9-59}$$

The temperature distribution may be written in the following series form:

$$T(r,t) - T_f = T(r,t) - T_f = (T_0 - T_f)\sum_{n=1}^{\infty} A_n \exp(-\beta_n^2 \mathrm{Fo}) J_0(\beta_n r/b) \quad (9\text{-}60)$$

The coefficient A_n is given by

$$A_n = \frac{2J_1(\beta_n)}{\beta_n[J_0^2(\beta_n) + J_1^2(\beta_n)]} \quad (9\text{-}61)$$

A plot of the center temperature change as a function of the Fourier number and Biot number is shown in Figure 9-5.

Although an infinite series may seem mathematically formidable, generally, only the first term in eq. (9-60) is sufficient to represent the series for Fourier numbers larger than about 0.20 within 1% accuracy, because of the exponential function in the series. For example, for Bi = 5 and Fo = 0.2, we find the following values for the constants:

$$\beta_1 = 1.9898 \quad \text{and} \quad \beta_2 = 4.7131$$

$$A_1 = 1.5029 \quad \text{and} \quad A_2 = -0.3988$$

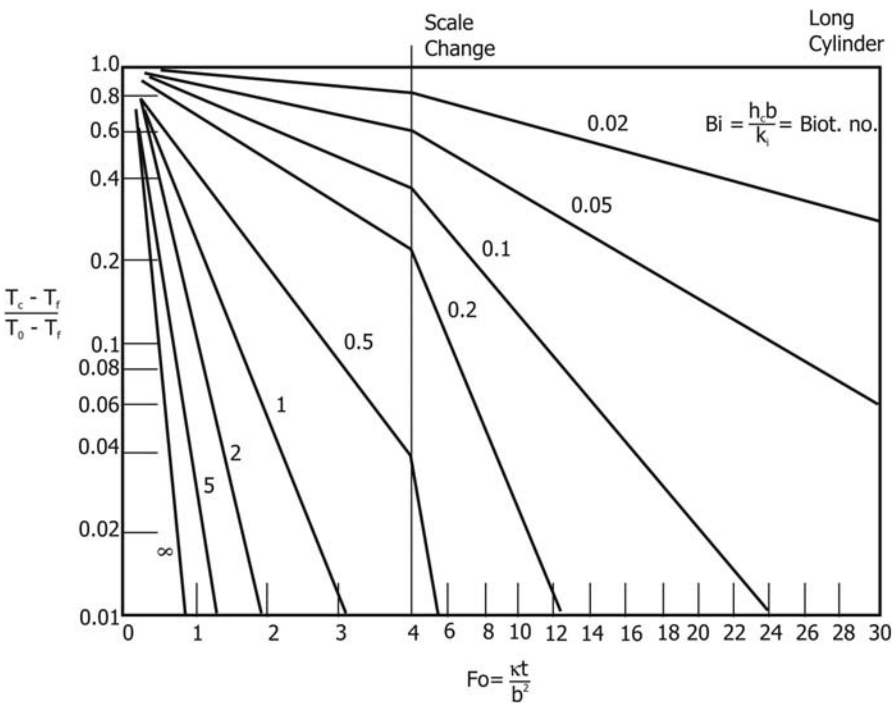

Figure 9-5. Plot of the center temperature ratio, $(T_c - T_f)/(T_0 - T_f)$, as a function of the Fourier number $\mathrm{Fo} = \alpha t/b^2$ for various values of the Biot number $\mathrm{Bi} = h_c b/k_t$ for a very long (infinite) solid cylinder initially at T_0 and suddenly exposed to a fluid at T_f.

TABLE 9-1. Values of the Constants A_1 and β_1 for the Transient Temperature Distribution in a Solid Cylinder, Eq. (9-62)

$\mathrm{Bi} = \dfrac{h_c b}{k_f}$	A_1	β_1	$\mathrm{Bi} = \dfrac{h_c b}{k_f}$	A_1	β_1
0.00	0.0000	0.0000	2.0	1.3384	1.5994
0.01	0.1412	0.1412	3.0	1.4191	1.7887
0.02	1.0050	0.1995	4.0	1.4698	1.9081
0.04	1.0099	0.2814	5.0	1.5029	1.9898
0.06	1.0148	0.3438	6.0	1.5253	2.0490
0.08	1.0197	0.3960	7.0	1.5411	2.0937
0.10	1.0246	0.4417	8.0	1.5526	2.1286
0.20	1.0483	0.6170	9.0	1.5611	2.1566
0.30	1.0712	0.7465	10.0	1.5677	2.1795
0.40	1.0931	0.8516	15.0	1.5861	2.2509
0.50	1.1143	0.9408	20.0	1.5919	2.2880
0.60	1.1345	1.0184	30.0	1.5973	2.3261
0.70	1.1539	1.0873	40.0	1.5993	2.3455
0.80	1.1724	1.1490	50.0	1.6002	2.3572
0.90	1.1902	1.2048	100.0	1.6015	2.3809
1.00	1.2071	1.2558	∞	1.6021	2.4048

The temperature ratio for the center temperature in this case is

$$\frac{T(0,t) - T_f}{T_0 - T_f} = 1.5029 \exp[-(1.9898)^2(0.20)] - 0.3988 \exp[-(4.7131)^2(0.20)] + \cdots$$

$$\frac{T(0,t) - T_f}{T_0 - T_f} = 0.68082 - 0.00469 + \cdots$$

The second term is about 0.7% that of the first term, in this case.

For a Fourier number $\mathrm{Fo} \geq 0.20$, the only first term may be used in eq. (9-60) for accuracy better than 1%:

$$\frac{T(r,t) - T_f}{T_0 - T_f} \approx A_1 \exp(-\beta_1^2 \mathrm{Fo}) J_0(\beta_1 r/b) \quad (\text{for Fo } \geq 0.20) \tag{9-62}$$

Representative values of the constants A_1 and β_1 are given in Table 9-1.

Example 9-2 A rod is constructed of 304 stainless steel with the following properties: $k_t = 16\,\mathrm{W/m\text{-}^\circ C}$ (9.24 Bru/hr-ft-°F) and $\kappa = 4.24\,\mathrm{mm^2/s}$ ($0.164\,\mathrm{ft^2/hr}$). The rod has a diameter of 40 mm ($b = 20\,\mathrm{mm} = 0.787$ in.), and an initial temperature $T_0 = 25^\circ\mathrm{C}$ (77°F). The rod is suddenly plunged into liquid nitrogen at $T_f = -195^\circ\mathrm{C}$ ($-319^\circ\mathrm{F}$), and the convective heat transfer coefficient between the rod and the liquid nitrogen is $h_c = 120\,\mathrm{W/m^2\text{-}^\circ C}$ (21.1 Btu/hr-ft^2-°F). Determine the time required for the center of the rod to reach a temperature of $-173^\circ\mathrm{C}$ ($-279^\circ\mathrm{F}$). Determine the surface temperature of the rod at this time.

The temperature ratio for the center temperature is

$$\frac{T(0,\ t) - T_f}{T_0 - T_f} = \frac{-173^\circ + 195^\circ}{25^\circ + 195^\circ} = 0.100$$

The Biot number for this example is

$$\mathrm{Bi} = \frac{h_c b}{k_t} = \frac{(120)(0.020)}{(16)} = 0.150$$

From Table 9-1, we find the following values for the constants A_1 and β_1.

$$A_1 = 1.0364 \quad \text{and} \quad \beta_1 = 0.5294$$

Let us use eq. (9-62) to determine the Fourier number, and then verify that the resulting Fourier number is large enough for the equation to be valid:

$$\frac{T(0, t) - T_f}{T_0 - T_f} = 0.100 = (1.0364)\exp[-(0.5294)^2\mathrm{Fo}]J_0(0)$$

Solving for the Fourier number, the following is obtained:

$$\mathrm{Fo} = \frac{\kappa t}{b^2} = 8.343 > 0.20$$

The corresponding time for the temperature at the center to reach -173°C is

$$t = \frac{(8.343)(0.020)^2}{(4.24 \times 10^{-6})} = 787\ \mathrm{s} = 13.1\ \mathrm{min}\,.$$

For a Fourier number $\mathrm{Fo} = 8.343$, the corresponding surface temperature of the rod is found from eq. (9-62). Note that $J_0(\beta_1) = J_0(0.5294) = 0.9307$.

$$\frac{T(b,\ t) - T_f}{T_0 - T_f} = A_1 \exp(-\beta_1^2 \mathrm{Fo}) J_0(\beta_1) = (0.100)(0.9307) = 0.09307$$

The surface temperature at the end of 13.1 min is

$$T(b, t) = -195^\circ + (25^\circ + 195^\circ)(0.09307) = -195^\circ + 20.5^\circ = -174.5^\circ\mathrm{C}\ (282^\circ\mathrm{F})$$

9.4.2 Stress Analysis

The general expression for the radial stress in this case is eq. (9-26a); however, the constant $C_2 = 0$, because the stress must be finite at the center of the cylinder.

$$\sigma_r = C_1 - \frac{\alpha E}{(1-\mu)r^2}\int \Delta T r\ dr \tag{9-63}$$

In eq. (9-63), the temperature difference for stress analysis is the difference between the temperature $T(r,\ t)$ and the stress-free temperature T_0:

$$\Delta T = T(r, t) - T_0 = T(r, t) - T_f + (T_f - T_0) = (T_f - T_0)\left[1 - \frac{T(r, t) - T_f}{T_0 - T_f}\right]$$

The integral of the Bessel function in the temperature distribution may be determined from eq. (D-27):

$$\int r J_0(\beta_n r/b)\,dr = \left(\frac{br}{\beta_n}\right) J_1(\beta_n r/b) \tag{9-64}$$

The constant C_1 may be found from the condition that, at the surface ($r = b$), the radial stress is zero (no applied mechanical loads at the surface):

$$C_1 = \frac{\alpha E(T_f - T_0)}{1-\mu}\left[\frac{1}{2} - \sum_{n=1}^{\infty} A_n \exp(-\beta_n^2 \mathrm{Fo})\frac{J_1(\beta_n)}{\beta_n}\right] \tag{9-65}$$

The radial stress distribution may be found using this value of C_1 in eq. (9-63):

$$\sigma_r = -\frac{\alpha E(T_f - T_0)}{1-\mu}\sum_{n=1}^{\infty}\left(\frac{A_n}{\beta_n}\right)\exp(-\beta_n^2 \mathrm{Fo})\left[J_1(\beta_n) - \frac{J_1(\beta_n r/b)}{r/b}\right] \tag{9-66}$$

For Fo ≥ 0.20, the first term is sufficiently accurate for the series.

The circumferential stress may be found from eq. (9-26b) with $C_2 = 0$:

$$\sigma_\theta = -\frac{\alpha E(T_f - T_0)}{1-\mu}\sum_{n=1}^{\infty}\left(\frac{A_n}{\beta_n}\right)\exp(-\beta_n^2 \mathrm{Fo})\left[J_1(\beta_n) + \frac{J_1(\beta_n r/b)}{r/b} - \beta_n J_0(\beta_n r/b)\right] \tag{9-67}$$

The axial stress distribution is

$$\sigma_z = \frac{\alpha E(T_0 - T_f)}{1-\mu}\sum_{n=1}^{\infty} A_n \exp(-\beta_n^2 \mathrm{Fo})\left[\frac{2\mu J_1(\beta_n)}{\beta_n} - J_0(\beta_n r/b)\right] \tag{9-68}$$

Example 9-3 For the 304 stainless steel rod given in Example 9-2, the mechanical properties are as follows: $\alpha = 16 \times 10^{-6}\ {}^\circ\mathrm{C}^{-1}$ $(8.89 \times 10^{-6}\ {}^\circ\mathrm{F}^{-1})$, $E =$ 193 GPa (28.0×10^6 psi), and $\mu = 0.305$. Determine the radial stress at the center of the rod ($r = 0$) and the circumferential stress at the surface of the rod ($r = b$) after 13.1 min has elapsed. The initial temperature of the rod is $T_0 = 25^\circ$C (77°F), and the surrounding fluid temperature is $T_f = -195^\circ$C (-319°F).

From Example 9-2, the following quantities were determined:

$$A_1 = 1.0364 \quad \text{and} \quad \beta_1 = 0.5294$$

From eq. (D-3) for the series expression of the Bessel function $J_1(x)$, we may determine the ratio in eq. (9-66) for the radial stress at the center of the rod:

$$\lim_{r\to 0}\left[\beta_1 \frac{J_1(\beta_1 r/b)}{(\beta_1 r/b)}\right] = \lim_{r\to 0}\left[\beta_1 \frac{\frac{1}{2}(\beta_1 r/b) - \cdots}{(\beta_1 r/b)}\right] = \frac{1}{2}\beta_1$$

Values for the Bessel functions may be found from Table D-1:

$$J_0(\beta_1) = J_0(0.5294) = 0.93069$$
$$J_1(\beta_1) = J_1(0.5294) = 0.25533$$

The numerical value of the coefficient in eqs. (9-66) and (9-67) is

$$\frac{\alpha E(T_0 - T_f)}{1-\mu} = \frac{(16\times 10^{-6})(193\times 10^9)(25^\circ + 195^\circ)}{1-0.305} = 977.5\times 10^6\,\text{Pa} = 977.5\,\text{MPa}$$

The radial stress at the center may be found from eq. (9-66):

$$\sigma_r(0,t) = \frac{\alpha E(T_0 - T_f)}{1-\mu}\sum_{n=1}^{\infty}\left(\frac{A_n}{\beta_n}\right)\exp(-\beta_n^2\text{Fo})\left[J_1(\beta_n) - \frac{1}{2}\beta_n\right]$$

Using only the first term in the series, the following is obtained:

$$\sigma_r(0,t) = (977.5)\left(\frac{1.0364}{0.5294}\right)\exp[-(0.5294)^2(8.343)]\left[0.25533 - \frac{1}{2}(0.5294)\right]$$
$$\sigma_r(0,t) = (977.5)(-0.001770) = -1.730\,\text{MPa}\ (-251\,\text{psi})$$

The circumferential stress at the outer surface of the cylinder may be determined from eq. (9-67):

$$\sigma_\theta(b,t) = \frac{\alpha E(T_0 - T_f)}{1-\mu}\sum_{n=1}^{\infty}\left(\frac{A_n}{\beta_n}\right)\exp(-\beta_n^2\text{Fo})\,[2J_1(\beta_n) - \beta_n J_0(\beta_n)]$$

Using only the first term in the series, the following is obtained:

$$\sigma_\theta(b,t) = (977.5)\left(\frac{1.0364}{0.5294}\right)\exp(-2.3382)[(2)(0.25533 - (0.5294)(0.93069)]$$
$$\sigma_\theta(b,t) = (977.5)(0.003392) = 3.316\,\text{MPa}\quad(481\,\text{psi})$$

We observe that the stresses at this time are relatively small. The convective heat transfer coefficient in this example is not large, so the cooldown rate is much slower than would occur for a very large heat transfer coefficient. If the convective heat transfer coefficient were extremely large, the maximum circumferential stress (which occurs near the initial time, $t = 0$) would be

$$(\sigma_\theta)_{\max}\,(for\ \ h_c \to \infty) = \frac{\alpha E(T_0 - T_f)}{1-\mu}$$

For this example (if $h_c = \infty$), the maximum stress would be 977.5 MPa (141,800 psi), which exceeds the yield strength of the stainless steel ($S_y =$ 232 MPa).

9.5 THICK-WALLED SPHERICAL VESSELS

In this section, we consider the case of spheres with an axisymmetric temperature distribution, such that the temperature distribution and stress distribution are both one dimensional or are functions of the radial coordinate only.

9.5.1 Thermal Analysis

For the case of no energy dissipation (no heat "generation"), steady-state heat transfer with constant material properties, the temperature field equation in spherical coordinates, eq. (5-65), reduces to

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0 \tag{9-69}$$

Let us consider the case of a hollow sphere, as shown in Figure 9-6. The boundary conditions on the temperature are as follows:

At the outer surface, $r = b$, the temperature is $T = T_0$.
At the inner surface, $r = a$, the temperature is $T = T_1$.

The temperature field equation may be integrated twice to obtain

$$T(r) = -\frac{B_1}{r} + B_2 \tag{9-70}$$

The constants of integration may be found by using the boundary conditions to obtain

$$B_1 = -\frac{T_1 - T_0}{\dfrac{1}{a} - \dfrac{1}{b}} \tag{9-71a}$$

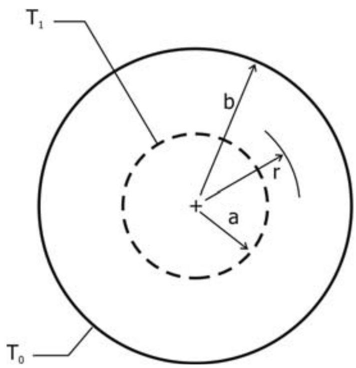

Figure 9-6. Surface temperatures in a hollow sphere with steady-state heat transfer.

$$B_2 = T_0 - \frac{(T_1 - T_0)\left(\dfrac{1}{b}\right)}{\dfrac{1}{a} - \dfrac{1}{b}} \tag{9-71b}$$

The temperature distribution in a hollow sphere in steady-state heat transfer is

$$\Delta T = T - T_0 = (T_1 - T_0)\frac{\dfrac{b}{r} - 1}{\dfrac{b}{a} - 1} \tag{9-72}$$

9.5.2 Stress Analysis

Let us consider the element in spherical coordinates shown in Figure 9-7. The equilibrium relationship for forces in the radial direction is

$$\begin{aligned}&(\sigma_r + d\sigma_r)(r + dr)^2(\sin\phi d\phi d\theta) - \sigma_r(r^2 \sin\phi d\phi d\theta)\\&\quad -2\sigma_t(r\sin\phi dr d\theta)\cos(\tfrac{1}{2}d\phi) - 2\sigma_t(r\sin\phi d\theta dr)\cos(\tfrac{1}{2}d\theta) = 0\end{aligned}$$

The stress σ_t is the tangential stress. By neglecting higher-order differentials (such as $dr\, d\phi$, etc.) and noting that the cosine for very small angles is essentially equal to 1, the radial equilibrium expression reduces to

$$2r\sigma_r + r^2\frac{d\sigma_r}{dr} - 2r\sigma_t = 0 \tag{9-73}$$

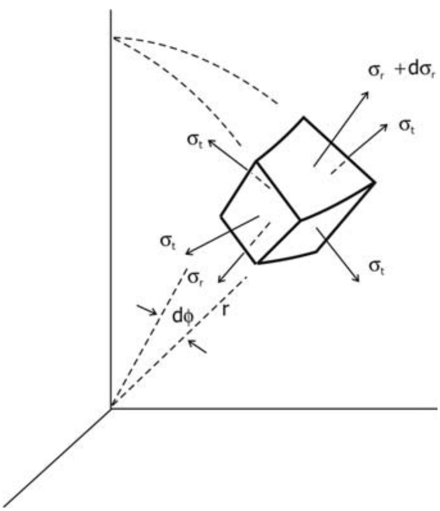

Figure 9-7. Forces on a differential element in spherical coordinates for axisymmetric conditions.

The radial equilibrium expression may be written in an alternate form:

$$\frac{1}{r}\frac{d(r^2\sigma_r)}{dr} - 2\sigma_t = 0 \tag{9-74}$$

The stress–strain relations are as follows:

$$\varepsilon_r = \frac{1}{E}(\sigma_r - 2\mu\sigma_t) + \alpha\Delta T = \frac{du}{dr} \tag{9-75a}$$

$$\varepsilon_t = \frac{1}{E}[(1-\mu)\sigma_t - \mu\sigma_r] + \alpha\Delta T = \frac{u}{r} \tag{9-75b}$$

From eqs. (9-75), we note that the strains are related by

$$\varepsilon_r = \frac{du}{dr} = \frac{d(r\varepsilon_t)}{dr} \tag{9-76}$$

Making the substitution of the stresses from eqs. (9-75) into eq. (9-76), we obtain

$$\sigma_r - 2\mu\sigma_t + \alpha E\Delta T = \frac{d}{dr}[(1-\mu)r\sigma_r - \mu r\sigma_t] + \alpha E\frac{d(r\Delta T)}{dr} \tag{9-77}$$

The tangential stress and the derivative of the tangential stress may be eliminated from eq. (9-77) by using the radial equilibrium expression, eq. (9-74):

$$\frac{d(r\sigma_t)}{dr} = \frac{1}{2}\frac{d^2(r^2\sigma_r)}{dr^2} \tag{9-78}$$

$$\sigma_r - \frac{\mu}{r}\frac{d(r^2\sigma_r)}{dr} = (1-\mu)\frac{d(r\sigma_r)}{dr} - \frac{\mu}{2}\frac{d^2(r^2\sigma_r)}{dr^2} + \alpha Er\frac{d(\Delta T)}{dr} \tag{9-79}$$

Simplifying eq. (9-79), gives

$$(1-\mu)\sigma_r - \frac{1}{2}(1-\mu)\frac{d(r^2\sigma_r)}{dr} = \alpha Er\frac{d(\Delta T)}{dr} \tag{9-80}$$

Equation (9-80) may be further simplified to obtain the governing equation for axisymmetric loading of a sphere:

$$\frac{d}{dr}\left[\frac{1}{r^2}\frac{d(r^3\sigma_r)}{dr}\right] = -\frac{2\alpha E}{(1-\mu)}\frac{d(\Delta T)}{dr} \tag{9-81}$$

If we separate variables and integrate eq. (9-81) twice, the following general result is obtained for the radial stress:

$$\sigma_r = C_1 + \frac{C_2}{r^3} - \frac{2\alpha E}{(1-\mu)r^3}\int \Delta T r^2 dr \tag{9-82a}$$

The tangential stress may be found from the radial equilibrium expression, eq. (9-74):

$$\sigma_t = C_1 - \frac{C_2}{2r^3} - \frac{\alpha E\Delta T}{1-\mu} + \frac{\alpha E}{(1-\mu)r^3}\int \Delta T r^2 dr \tag{9-82b}$$

The radial displacement for the sphere may be found from eq. (9-75b):

$$u = r\varepsilon_t = \frac{(1-2\mu)C_1 r}{E} - \frac{(1+\mu)C_2}{2Er^2} + \frac{(1+\mu)\alpha}{(1-\mu)r^2}\int \Delta T r^2 dr \tag{9-82c}$$

The integral of the temperature distribution given by eq. (9-72) is

$$\int \Delta T r^2\, dr = \frac{\Delta T_1 r^3 \left(\dfrac{3b}{r} - 2\right)}{6\left(\dfrac{b}{a} - 1\right)} \tag{9-83}$$

The constants of integration C_1 and C_2 may be found from the boundary condition that the radial stress is zero at the inner surface ($r = a$) and also zero at the outer surface ($r = b$):

$$C_1 = \frac{\alpha E \Delta T_1}{1-\mu}\left[\frac{1}{3\left(\dfrac{b}{a} - 1\right)} - \frac{1}{\left(\dfrac{b}{a}\right)^3 - 1}\right] \tag{9-84a}$$

$$C_2 = \frac{\alpha E \Delta T_1}{1-\mu}\frac{b^3}{\left(\dfrac{b}{a}\right)^3 - 1} \tag{9-84b}$$

The stresses may be determined by substituting the expressions for the constants of integration into eqs. (9-82a) and (9-82b):

$$\sigma_r = \frac{\alpha E \Delta T_1}{1-\mu}\frac{1}{(b/a)^3 - 1}\left[\left(\frac{b}{a}\right)\left(1+\frac{b}{a}\right) - \left(\frac{b^2}{a^2}+\frac{b}{a}+1\right)\left(\frac{b}{r}\right) + \left(\frac{b}{r}\right)^3\right] \tag{9-85a}$$

$$\sigma_t = \frac{\alpha E \Delta T_1}{1-\mu}\frac{1}{(b/a)^3 - 1}\left[\left(\frac{b}{a}\right)\left(1+\frac{b}{a}\right) - \frac{1}{2}\left(\frac{b^2}{a^2}+\frac{b}{a}+1\right)\left(\frac{b}{r}\right) - \frac{1}{2}\left(\frac{b}{r}\right)^3\right] \tag{9-85b}$$

The maximum tangential stress occurs at the inner surface, $r = a$. The maximum absolute value for the radial stress occurs at a radial position:

$$\frac{r}{b} = \sqrt{\frac{3}{(b/a)^2 + (b/a) + 1}} \tag{9-86}$$

Example 9-4 A spherical vessel is constructed of 9% Ni steel, with the following properties: $\alpha = 11.8 \times 10^{-6}$ °C (6.56×10^{-6} °F), $E = 189$ GPa (27.4×10^6 psi), and $\mu = 0.286$. The inside diameter of the vessel is 240 mm

($a = 120\,\text{mm} = 4.72\,\text{in.}$), and the outside diameter is 300 mm ($b = 150\,\text{mm} = 5.91\,\text{in.}$). The container is filled with an ice-water mixture, such that the inner surface is maintained at $T_1 = 0^\circ\text{C}$ (32°F) and the outer surface is maintained at $T_0 = 25^\circ\text{C}$ (77°F). Determine the tangential stress at the inner and outer surfaces and the maximum radial stress.

From eq. (9-85b), the tangential stress at the inner surface ($r = a$) is

$$\sigma_t(a) = -\frac{\alpha E \Delta T_1}{2(1-\mu)} \frac{\left(\dfrac{b}{a}\right)\left(\dfrac{b}{a} - 1\right)\left(1 + 2\dfrac{b}{a}\right)}{\left(\dfrac{b}{a}\right)^3 - 1} \tag{9-87a}$$

The tangential stress at the outer surface ($r = b$) is

$$\sigma_t(b) = \frac{\alpha E \Delta T_1}{2(1-\mu)} \frac{\left(\dfrac{b}{a} - 1\right)\left(2 + \dfrac{b}{a}\right)}{\left(\dfrac{b}{a}\right)^3 - 1} \tag{9-87b}$$

In this example, the radius ratio is $b/a = 150/120 = 1.250$. The numerical value of the coefficient term is

$$\frac{\alpha E \Delta T_1}{2(1-\mu)} = \frac{(11.8 \times 10^{-6})(189 \times 10^9)(0^\circ - 25^\circ)}{(2)(1-0.286)} = -39.04 \times 10^6\,\text{Pa} = -39.04\,\text{MPa}$$

The numerical value of the tangential stress at the inner surface is

$$\sigma_t(a) = -(-39.04)\frac{(1.250)(1.250-1)(1+2.50)}{(1.250)^3 - 1} = +(39.04)(1.1475)$$

$$\sigma_t(a) = +44.80\,\text{MPa} \qquad (6500\,\text{psi})$$

The numerical value of the tangential stress at the outer surface is

$$\sigma_t(b) = (-39.04)\frac{(1.250-1)(2+1.250)}{(1.250)^3 - 1} = -(39.04)(0.8525)$$

$$\sigma_t(b) = -33.28\,\text{MPa} \qquad (-4827\,\text{psi})$$

The location at which the maximum radial stress occurs may be found from eq. (9-86):

$$\frac{r}{b} = \sqrt{\frac{3}{(1.250)^2 + 1.250 + 1}} = 0.8871 \quad \text{or} \quad \frac{b}{r} = 1.127$$

The radius at which the maximum radial stress occurs is

$$r = (150)(0.8871) = 133\,\text{mm}$$

The maximum radial stress occurs at a location $(a - r) = 13\,\text{mm}$ from the inner surface. The maximum radial stress may be found from eq. (9-85a):

$$(\sigma_r)_{\max} = (2)(-39.04)$$
$$\times \left[\frac{(1.250)(1 + 1.250) - (1.250^2 + 1.250 + 1)(1.127) + (1.127)^3}{(1.250)^3 - 1} \right]$$
$$(\sigma_r)_{\max} = (-78.08)(-0.05275) = +4.12\,\text{MPa} \quad (597\,\text{psi})$$

The maximum radial stress is less that 10% of the maximum tangential stress in the sphere.

9.6 SOLID SPHERES

In this section, thermal stresses arising during the transient cooling or heating of a sphere are considered. The solid sphere is initially at a uniform temperature T_0, and at the initial time ($t = 0$), the sphere of radius b is suddenly exposed to a fluid at T_f with a convective heat transfer coefficient h_c at the surface, as shown in Figure 9-8.

9.6.1 Thermal Analysis

The temperature field equation for one-dimensional transient heat transfer in a sphere is obtained from eq. (5-65). The temperature is independent of the angular coordinates, and there is no energy dissipation (no "heat generation"):

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) = \frac{1}{\kappa} \frac{\partial T}{\partial t} \tag{9-88}$$

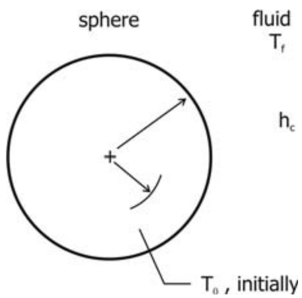

Figure 9-8. Solid sphere under transient heat transfer conditions with convection at the surface.

The boundary conditions on the problem are as follows:

At the outer surface, $r = b$, the heat conducted at the surface is equal to the heat transfer by convection at the surface,

$$-k_t \left(\frac{\partial T}{\partial r} \right)_{r=b} = h_c (T_{r=b} - T_f)$$

At the center, $r = 0$, the temperature is finite.

The initial condition is that, at $t = 0$, the temperature $T = T_0$ throughout the sphere.

This problem was of interest in predicting the time lag in thermometers with a spherical element [Carslaw and Jaeger, 1959].

Let us make the following change in variable:

$$y(r, t) = r[T(r, t) - T_f] \tag{9-89}$$

If we make this change of variable in eq. (9-88), the following result is obtained:

$$\frac{\partial^2 y}{\partial r^2} = \frac{1}{\kappa} \frac{\partial y}{\partial t} \tag{9-90}$$

The boundary conditions in terms of the y-variable are as follows:

(a) At the outer surface, $r = b$, the heat conducted at the surface is equal to the heat transfer by convection at the surface,

$$\left(\frac{\partial y}{\partial r} \right)_{r=b} = -\left(\frac{h_c b}{k_t} - 1 \right) \frac{y(b, t)}{b} = -(\mathrm{Bi} - 1) \frac{y(b, t)}{b} \tag{9-91}$$

The quantity Bi is the *Biot number*, defined as

$$\mathrm{Bi} = \frac{h_c b}{k_t} \tag{9-92}$$

(b) At the center, $r = 0$, the variable $y(0, t)$ must be finite.

The initial condition is that, at $t = 0$, the temperature $y(r, 0) = r(T_0 - T_f)$ throughout the sphere.

Let us try a product solution of eq. (9-90) in the following form, $y(r, t) = R(r)F(t)$. If we substitute this product solution into eq. (9-90), we obtain

$$\frac{1}{R} \frac{d^2 R}{dr^2} = \frac{1}{\kappa F} \frac{dF}{dt} = -\lambda^2 \tag{9-93}$$

The two differential equations from eq. (9-93) are

$$\frac{d^2 R}{dr^2} + \lambda^2 R = 0 \quad \text{and} \quad \frac{dF}{dt} + \lambda^2 F = 0 \tag{9-94}$$

The solutions are

$$R(r) = B_1 \sin \lambda r + B_2 \cos \lambda r \tag{9-95a}$$

$$F(t) = B_3 \exp(-\lambda^2 \kappa t) \tag{9-95b}$$

The temperature distribution is

$$T(r, t) - T_f = \frac{y(r, t)}{r} = \left(B_1 \frac{\sin \lambda r}{r} + B_2 \frac{\cos \lambda r}{r} \right) \exp(-\lambda^2 \kappa t) \tag{9-96}$$

For a finite temperature at the center of the sphere ($r = 0$), we must have $B_2 = 0$.

Using the convection boundary condition, eq. (9-91), the following expression is obtained:

$$\lambda b \cos \lambda b = -(\mathrm{Bi} - 1) \sin \lambda b \tag{9-97}$$

The eigenvalues for this problem are the solutions of eq. (9-97), which may be written in the following alternate form:

$$1 - \lambda b \cot \lambda b = \mathrm{Bi} \tag{9-98}$$

There are an infinite number of solutions to eq. (9-98). Let us denote the solutions by $\beta_n = \lambda_n b (n = 1, 2, 3, \ldots)$. Because the initial condition cannot be met using any one of these solutions, we may try the linear combination of all the solutions:

$$T(r, t) - T_f = \sum_{n=1}^{\infty} B_n \frac{\sin(\beta_n r/b)}{r} \exp(-\beta_n^2 \mathrm{Fo}) \tag{9-99}$$

At the initial time ($t = 0$), the temperature is uniform and eq. (9-99) reduces to

$$r[T(r, 0) - T_f] = r(T_0 - T_f) = \sum_{n=1}^{\infty} B_n \sin(\beta_n r/b) \tag{9-100}$$

Using Fourier series analysis techniques, the constants B_n may be evaluated as follows:

$$B_n = \frac{4\beta_n(\sin \beta_n - \beta_n \cos \beta_n)}{[2\beta_n - \sin(2\beta_n)]b}(T_0 - T_f) = A_n(T_0 - T_f)(\beta_n/b) \tag{9-101}$$

The coefficients A_n are given by

$$A_n = \frac{4(\sin \beta_n - \beta_n \cos \beta_n)}{2\beta_n - \sin(2\beta_n)} \tag{9-102}$$

The temperature distribution may be written in the following series form:

$$T(r, t) - T_f = (T_0 - T_f) \sum_{n=1}^{\infty} A_n \frac{\sin(\beta_n r/b)}{\beta_n r/b} \exp(-\beta_n^2 \mathrm{Fo}) \tag{9-103}$$

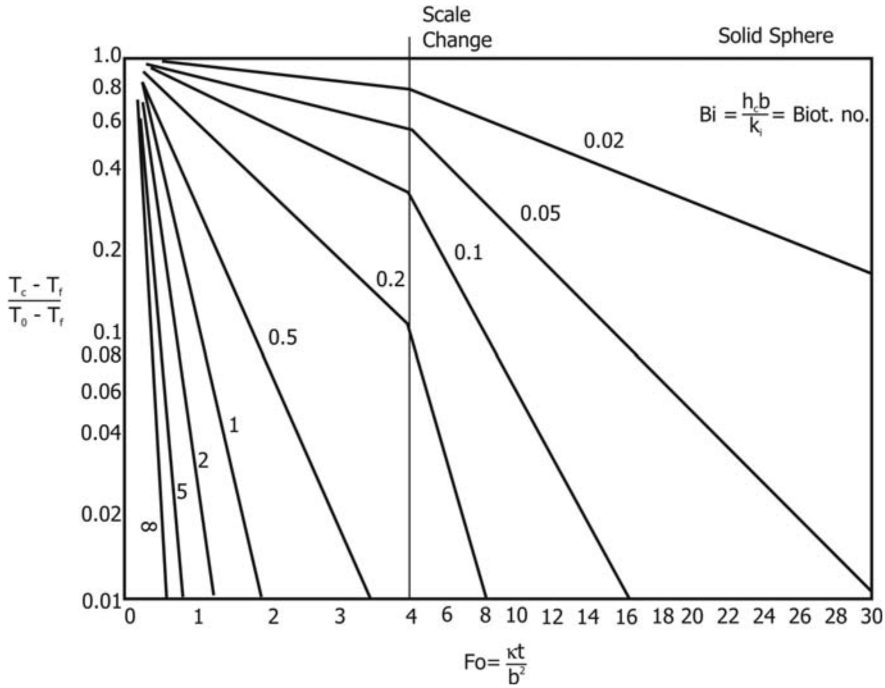

Figure 9-9. Plot of the center temperature ratio, $(T_c - T_f)/(T_0 - T_f)$, as a function of the Fourier number $\mathrm{Fo} = \alpha t/b^2$ for various values of the Biot number $\mathrm{Bi} = h_c b/k_t$ for a solid sphere initially at T_0 and suddenly exposed to a fluid at T_f.

A plot of the center temperature change as a function of the dimensionless time (Fourier number, Fo) and the Biot number Bi is shown in Figure 9-9.

As was shown to be the case for the solid cylinder in Section 9.4.1, generally, the first term in eq. (9-103) is sufficiently accurate to represent the series for Fourier numbers larger than about 0.20. For example, for Bi = 5 and Fo = 0.2, we find the following values for the constants:

$$\beta_1 = 2.5704 \quad \text{and} \quad \beta_2 = 5.3540$$
$$A_1 = 1.7870 \quad \text{and} \quad A_2 = -1.3921$$

The temperature ratio for the center temperature in this case is

$$\frac{T(0,t) - T_f}{T_0 - T_f} = (1.8780)\left(\frac{\sin 2.5704}{2.5704}\right)\exp[-(2.5704)^2(0.20)] + (-1.3921)\left(\frac{\sin 5.3540}{5.3540}\right)\exp[-(5.3540)^2(0.20)] + \cdots$$

$$\frac{T(0,\ t) - T_f}{T_0 - T_f} = 0.105372 + 0.000674 + \cdots$$

The second term is about 0.6% of the first term, in this case.

TABLE 9-2. Values of the Constants A_1 and β_1 for the Transient Temperature Distribution in a Solid Sphere, Eq. (9-104)

$\mathrm{Bi} = \dfrac{h_c b}{k_f}$	A_1	β_1	$\mathrm{Bi} = \dfrac{h_c b}{k_f}$	A_1	β_1
0.00	1.0000	0.0000	2.0	1.4793	2.0288
0.01	1.0030	0.1730	3.0	1.6227	2.2889
0.02	1.0060	0.2445	4.0	1.7202	2.4557
0.04	1.0120	0.3450	5.0	1.7870	2.5704
0.06	1.0179	0.4217	6.0	1.8338	2.6537
0.08	1.0239	0.4860	7.0	1.8673	2.7165
0.10	1.0298	0.5423	8.0	1.8920	2.7654
0.20	1.0592	0.7593	9.0	1.9106	2.8044
0.30	1.0880	0.9208	10.0	1.9249	2.8363
0.40	1.1164	1.0528	15.0	1.9629	2.9349
0.50	1.1441	1.1656	20.0	1.9781	2.9857
0.60	1.1713	1.2644	30.0	1.9898	3.0372
0.70	1.1978	1.3525	40.0	1.9942	3.0632
0.80	1.2236	1.4320	50.0	1.9962	3.0788
0.90	1.2488	1.5044	100.0	1.9990	3.1102
1.00	1.2732	1.5708	∞	2.0000	3.1416

For $\mathrm{Fo} \geq 0.20$, only the first term may be used in eq. (9-103) for accuracy better than 1%:

$$\frac{T(r,\,t) - T_f}{T_0 - T_f} \approx A_1 \left[\frac{\sin(\beta_1 r/b)}{(\beta_1 r/b)} \right] \exp[-\beta_1^2 \mathrm{Fo}] \tag{9-104}$$

Representative values of the constants A_1 and β_1 for the sphere are given in Table 9-2.

Example 9-5 An orange (assumed to be a solid sphere) has a 100-mm diameter ($b = 50\,\mathrm{mm} = 1.969\,\mathrm{in.}$). The thermal properties of the orange are as follows: thermal conductivity, $k_t = 0.624\,\mathrm{W/m\text{-}^\circ C}$ (0.361 Btu/hr-ft-°F), and thermal diffusivity, $\kappa = 0.150\,\mathrm{mm^2/s}$ ($0.00581\,\mathrm{ft^2/hr}$). The orange is initially at a uniform temperature $T_0 = 26.5^\circ\mathrm{C}$ (79.7°F), and a really sudden cold snap drops the ambient air temperature down to −3.5°C (25.7°F). The convective heat transfer coefficient between the orange and the ambient air is $h_c = 8.736\,\mathrm{W/m^2\text{-}^\circ C}$ (1.538 Btu/hr-ft^2-°F). Determine the time elapsed until the surface temperature of the orange reaches 1°C (33.8°F). At this time, determine the center temperature of the orange.

The Biot number is

$$\mathrm{Bi} = \frac{h_c b}{k_t} = \frac{(8.736)(0.050)}{0.624} = 0.700$$

The constants A_1 and β_1 are found from Table 9-2:

$$A_1 = 1.1978 \quad \text{and} \quad \beta_1 = 1.3525$$

The surface temperature ratio is

$$\frac{T(b,\ t) - T_f}{T_0 - T_f} = \frac{1^\circ - (-3.5^\circ)}{26.5^\circ - (-3.5^\circ)} = 0.150$$

The time required for the surface ($r = b$) to reach this condition may be found from eq. (9-104), if the Fourier number is greater than 0.20. Note that the β_1 term is in *radians*.

$$\frac{T(r,\ t) - T_f}{T_0 - T_f} = 0.150 = (1.1978)\left[\frac{\sin(1.3525)}{1.3525}\right]\exp[-(1.3525)^2\mathrm{Fo}]$$

$$\exp[-(1.3525)^2\mathrm{Fo}] = 0.1735$$

The Fourier number Fo is

$$\mathrm{Fo} = \frac{\kappa t}{b^2} = -\frac{\ln(0.1735)}{(1.3525)^2} = 0.9576$$

The corresponding time t is

$$t = \frac{(0.95760)(0.050)^2}{0.150 \times 10^{-6}} = 15{,}960\ \mathrm{s} = 266\ \mathrm{min} = 4.433\,\mathrm{hrs}$$

The temperature at the center of the orange at this time may be found from eq. (9-104) with $r = 0$. Note that

$$\lim_{r\to 0}\left[\frac{\sin(\beta_1 r/b)}{\beta_1 r/b}\right] = \lim_{r\to 0}\left[\frac{(\beta_1 r/b) - \frac{1}{6}(\beta_1 r/b)^3 + \cdots}{\beta_1 r/b}\right] = 1$$

$$\frac{T(0,t) - T_f}{T_0 - T_f} = A_1\exp[-\beta_1^2\mathrm{Fo}] = (1.1978)\exp[-(1.3525)^2(0.9576)] = 0.2078$$

The center temperature of the orange is

$$T(0,\ t) = -3.5^\circ + (26.5^\circ + 3.5^\circ)(0.2078) = -3.5^\circ + 6.2^\circ = 2.7^\circ\mathrm{C} \quad (36.9^\circ\mathrm{F})$$

9.6.2 Stress Analysis

The general expression for the radial stress in the solid sphere is eq. (9-82a). The stress must be finite at the center ($r = 0$), so the constant C_2 must be zero:

$$\sigma_r = C_1 - \frac{2\alpha E}{(1-\mu)r^3}\int \Delta T r^2 dr \tag{9-105}$$

In eq. (9-105), the temperature difference is

$$\Delta T(r,t) = T(r,\ t) - T_0 = [T(r,\ t) - T_f] - (T_0 - T_f) = (T_f - T_0)\left[1 - \frac{T(r,t) - T_f}{T_0 - T_f}\right]$$

The integral of the temperature distribution is

$$\int \Delta T r^2 dr = (T_f - T_0)\left\{\frac{1}{3}r^3 - \sum_{n=1}^{\infty} A_n \exp[-\beta_n^2 \mathrm{Fo}] \int r^2 \frac{\sin(\beta_n r/b)}{\beta_n r/b} dr\right\}$$

$$\int \Delta T r^2 dr = (T_f - T_0) \times \left\{\frac{1}{3}r^3 - \sum_{n=1}^{\infty} A_n \exp[-\beta_n^2 \mathrm{Fo}] \frac{[\sin(\beta_n r/b) - (\beta_n r/b)\cos(\beta_n r/b)]}{(\beta_n/b)^3}\right\} \tag{9-106}$$

If we make this substitution into eq. (9-105), we get

$$\sigma_r = C_1 - \frac{2\alpha E(T_f - T_0)}{(1-\mu)} \times \left\{\frac{1}{3} - \sum_{n=1}^{\infty} A_n \exp[-\beta_n^2 \mathrm{Fo}] \frac{[\sin(\beta_n r/b) - (\beta_n r/b)\cos(\beta_n r/b)}{(\beta_n r/b)^3}\right\} \tag{9-107}$$

Applying the boundary condition that the radial stress at the surface of the solid sphere is zero (no mechanical forces applied), the constant c_1 may be evaluated. The final expression for the radial stress is

$$\sigma_r = -\frac{2\alpha E(T_f - T_0)}{1-\mu} \sum_{n=1}^{\infty} A_n \exp[-\beta_n^2 \mathrm{Fo}] \left\{\left[\frac{\sin\beta_n - \beta_n \cos\beta_n}{\beta_n^3}\right] - \left[\frac{\sin(\beta_n r/b) - (\beta_n r/b)\cos(\beta_n r/b)}{(\beta_n r/b)^3}\right]\right\} \tag{9-108a}$$

The tangential stress may be found from eq. (9-82b):

$$\sigma_t = -\frac{\alpha E(T_f - T_0)}{1-\mu} \sum_{n=1}^{\infty} A_n \exp[-\beta_n^2 \mathrm{Fo}] \left\{2\left[\frac{\sin\beta_n - \beta_n \cos\beta_n}{\beta_n^3}\right] - \frac{\sin(\beta_n r/b)}{\beta_n r/b} + \left[\frac{\sin(\beta_n r/b) - (\beta_n r/b)\cos(\beta_n r/b)}{(\beta_n r/b)^3}\right]\right\} \tag{9-108b}$$

As discussed previously, for $\mathrm{Fo} \geq 0.20$, using only the first term in the series in eqs. (9-108) is sufficiently accurate for most engineering applications. The stress is finite at the center, because the last term has a finite value at $r = 0$:

$$\lim_{r\to 0}\left[\frac{\sin(\xi) - \xi\cos(\xi)}{\xi^3}\right] = \lim_{r\to 0} \frac{(\xi - \frac{1}{6}\xi^3 + \cdots) - \xi(1 - \frac{1}{2}\xi^2 + \cdots)}{\xi^3} = \frac{1}{3} \tag{9-109}$$

The tangential and radial stress at the center is

$$\sigma_r(0, t) = \sigma_t(0, t) = -\frac{2\alpha E(T_f - T_0)}{1-\mu}\sum_{n=1}^{\infty} A_n \exp[-\beta_n^2 \mathrm{Fo}]\left[\frac{\sin\beta_n - \beta_n\cos\beta_n}{\beta_n^3} - \frac{1}{3}\right] \tag{9-110}$$

The radial displacement may be found from eq. (9-82c):

$$u = \alpha(T_f - T_0)r - \frac{(1+\mu)\alpha(T_f - T_0)r}{1-\mu}\sum_{n=1}^{\infty} A_n \exp[-\beta_n^2 \mathrm{Fo}]$$
$$\times\left\{\frac{2(1-2\mu)}{(1+\mu)}\left[\frac{\sin\beta_n - \beta_n\cos\beta_n}{\beta_n^3}\right] + \left[\frac{\sin(\beta_n r/b) - (\beta_n r/b)\cos(\beta_n r/b)}{(\beta_n r/b)^3}\right]\right\} \tag{9-111a}$$

At the surface of the sphere ($r = b$), the radial displacement is found from eq. (9-111a):

$$u(b) = \alpha(T_f - T_0)b\left[1 - 3\sum_{n=1}^{\infty} A_n \exp(-\beta_n^2 \mathrm{Fo})\left(\frac{\sin\beta_n - \beta_n\cos\beta_n}{\beta_n^3}\right)\right] \tag{9-111b}$$

For the limiting case of very large convective heat transfer coefficients ($\mathrm{Bi} \to \infty$), it has been shown [Timoshenko and Goodier, 1970] that the maximum stress at the center of the sphere occurs at a Fourier number (dimensionless time)

$$\mathrm{Fo}^* = \frac{\kappa t^*}{b^2} = 0.0574 \tag{9-112}$$

The magnitude of the maximum stress in this case is

$$\sigma_r(0, t^*) = \sigma_t(0, t^*) = 0.386\frac{\alpha E(T_f - T_0)}{1-\mu} \tag{9-113}$$

For the case of very large convective heat transfer coefficients, the surface temperature of the sphere is suddenly changed from the initial temperature T_0 to the fluid temperature T_f, and the corresponding tangential stress at the surface is

$$\sigma_t(b,\ 0) = -\frac{\alpha E(T_f - T_0)}{1-\mu} \tag{9-114}$$

Example 9-6 A cast iron sphere has the following properties: $\alpha = 10.67 \times 10^{-6}\ ^\circ\mathrm{C}^{-1}$ ($5.93 \times 10^{-6}\ ^\circ\mathrm{F}^{-1}$), $E = 80\,\mathrm{GPa}$ (11.6×10^6 psi), $\mu = 0.25$, $\kappa = 16.67\,\mathrm{mm}^2/\mathrm{s}$ ($0.646\,\mathrm{ft}^2/\mathrm{hr}$), $k_t = 20\,\mathrm{W/m}\text{-}^\circ\mathrm{C}$ (11.6 Btu/hr-ft-°F). The sphere diameter is 160 mm ($b = 80\,\mathrm{mm} = 3.15$ in.). The sphere is initially at a uniform temperature $T_0 = 325^\circ\mathrm{C}$ (617°F), and the sphere is suddenly plunged into a water bath at $T_f = 25^\circ\mathrm{C}$ (77°F). The convective heat transfer coefficient between the sphere and the water is $h_c = 125\,\mathrm{W/m^2}\text{-}^\circ\mathrm{C}$ (22 Btu/hr-ft²-°F).

Determine the surface tangential stress in the sphere after a time $t = 12$ min has elapsed.

The Fourier number is

$$\mathrm{Fo} = \frac{(10.67 \times 10^{-6})(12)(60)}{(0.080)^2} = 1.200 > 0.20$$

The Biot number is

$$\mathrm{Bi} = \frac{h_c b}{k_t} = \frac{(125)(0.080)}{20} = 0.500$$

From Table 9-2, we find the following constants:

$$A_1 = 1.1441 \quad \text{and} \quad \beta_1 = 1.1656$$

Before we determine the stress, let us find the center and surface temperatures at the end of 12 min for the sphere. The center temperature is found from eq. (9-104) with $r = 0$:

$$\frac{T(0,\ t) - T_f}{T_0 - T_f} = A_1 \exp[-\beta_1^2 \mathrm{Fo}] = (1.1441)\exp[-(1.1656)^2(1.200)] = 0.2241$$

$$T(0,\ t) = 25^\circ + (0.2241)(325^\circ - 25^\circ) = 25^\circ + 67.2^\circ = 92.2^\circ\mathrm{C} \quad (198^\circ\mathrm{F})$$

The surface temperature is found as follows:

$$\frac{T(b,\ t) - T_f}{T_0 - T_f} = A_1 \left[\frac{\sin\beta_1}{\beta_1}\right] \exp[-\beta_1^2 \mathrm{Fo}] = (0.2241)\left(\frac{\sin 1.1656}{1.1656}\right) = 0.1769$$

$$T(b, t) = 25^\circ + (0.0.1769)(325^\circ - 25^\circ) = 25^\circ + 53.0^\circ = 78.0^\circ\mathrm{C} \quad (172.4^\circ\mathrm{F})$$

The tangential stress at the surface of the sphere is found from eq. (9-108b).

$$\sigma_t = -\frac{\alpha E(T_f - T_0)}{1-\mu} A_1 \exp[-\beta_1^2 \mathrm{Fo}] \left[3\left(\frac{\sin\beta_1 - \beta_1\cos\beta_1}{\beta_1^3}\right) - \frac{\sin\beta_1}{\beta_1}\right]$$

The numerical value for the coefficient term is

$$\frac{\alpha E(T_f - T_0)}{1-\mu} = \frac{(10.67 \times 10^{-6})(80 \times 10^9)(25^\circ - 325^\circ)}{1 - 0.25} = -341.4 \times 10^6\ \mathrm{Pa}$$

$$= -341.4\ \mathrm{MPa}$$

The surface tangential stress is

$$\sigma_t = -(-341.4)(0.2241)\left[3\left(\frac{\sin 1.1656 - 1.1656\cos 1.1656}{1.1656^3}\right) - \frac{\sin 1.1656}{1.1656}\right]$$

$$\sigma_t = +(341.4)(0.2241)(0.08211) = 6.28\ \mathrm{MPa} \quad (911\ \mathrm{psi})$$

The tangential stress at the surface is somewhat low at this time, because the temperature difference between the center and surface of the sphere is only about 14°C. If the convective heat transfer coefficient had been 250,000 W/m^2-°C, instead of 125 W/m^2-°C (Biot number Bi = 1000, the initial surface stress would be approximately given by eq. (9-114):

$$\sigma_t(b,\ 0) = -\frac{\alpha E(T_f - T_0)}{1-\mu} = +341.4\,\text{MPa} \qquad (49{,}500\,\text{psi})$$

Because the ultimate strength of the cast iron is about $S_u = 179$ MPa (26,000 psi), the sphere would fail for this condition.

9.7 HISTORICAL NOTE

Although the French mathematician and physicist Jean Baptiste Joseph Fourier (Figure 9-10) did no specific work in the thermal stress field, he was one of the pioneers in the conduction heat transfer area. A thermal analysis is necessary before thermal stresses may be predicted. In addition to his work in heat transfer, Fourier also developed the foundations for Fourier series analysis and introduced Fourier transforms. Fourier's doctoral advisor was the famous mathematician Joseph Lagrange, and Gustav Dirichlet and Claude-Louis Navier were two of Fourier's doctoral students while he was at the École Polytechnique.

Fourier was also active in the French political arena. He accompanied Napoleon Bonaparte on his Egyptian expedition in 1798 and was appointed

Figure 9-10. Jean Baptiste Joseph Fourier.

governor of Lower Egypt and secretary of the Institut d'Egypte. He returned to France in 1801 after the French were defeated by the British army.

The Fourier rate equation for conduction was originally proposed by Jean-Baptiste Biot [Jacob, 1949] in 1804; however, the relationship is generally called *Fourier's law* because it was a fundamental relationship in his pioneering book, *Théorie Analytical de la Chaleur* (Analytical Theory of Heat), published in 1822. The Fourier rate equation may be written as follows.

$$\frac{\dot{Q}}{A} = -k_t \frac{dT}{dx} \quad \text{or} \quad \textit{heat flux} \text{ is proportional to the } \textit{temperature gradient}$$

Actually, the Fourier rate equation for conduction is not a "law" of nature; instead, it is the defining relationship for the quantity *thermal conductivity* k_t. There are several relationships of this type, including Hooke's law (which defines *Young's modulus*), Ohm's law (defines *electrical resistance*), and Fick's law (defines *mass diffusivity*).

Fourier was motivated to develop an expression in terms of quantities with physical units, rather than simply using a ratio of quantities. In fact, Fourier performed some of the initial work in the area of *dimensional analysis*, and he insisted that equations be dimensionally consistent (same dimensions on both sides of the equation).

Fourier was erroneously credited with discovery of the "greenhouse effect" in 1896, long before it became of considerable interest in the scientific and political community. [Cowie, 2007] Svante Arrhenius misunderstood Fourier's explanation of an experiment conducted in 1827 by G. B. de Saussure, and Arrhenius erroneously credited Fourier with an explanation of how greenhouses operate. In the experiment, a wooden box lined with black cork was exposed to sunlight. Three panes of glass were placed at the top of the box, and it was observed that the temperature within the box was higher with the glass panes than without them. Fourier actually explained that this effect could not happen in the atmosphere, because of turbulent convection in atmospheric air. Fourier did not know about the mechanism of thermal radiation, because it was not quantified until about 50 years after the experiment, and the Stefan-Boltzmann relationship was not discovered until about 20 years later than that.

PROBLEMS

9-1. A cylindrical container is constructed of Pyrex glass, which has the following properties: $\alpha = 3.306 \times 10^{-6}\ {}^\circ\mathrm{C}^{-1}$ $(1.837 \times 10^{-6}\ {}^\circ\mathrm{F}^{-1})$, $E = 62\,\mathrm{GPa}$ $(9.0 \times 10^6\,\mathrm{psi})$, $\mu = 0.24$. The container is filled with boiling water, such that the inner surface is maintained at 100°C (212°F) while the outer surface is maintained at 20°C (68°F). The outer diameter of the container is 228 mm ($b = 114\,\mathrm{mm} = 4.49\,\mathrm{in.}$), and the inner diameter is 190 mm ($a = 95\,\mathrm{mm} = 3.74\,\mathrm{in.}$). The ends of the cylinder are fixed (zero axial strain). Determine the circumferential stress in the container wall at the outer surface and at the inner surface.

9-2. A smoker's favorite pipe has a long cylindrical bowl with the following properties: $\alpha = 5.40 \times 10^{-6}\ {}^\circ\mathrm{C}^{-1}$ ($3.00 \times 10^{-6}\ {}^\circ\mathrm{F}^{-1}$), $E = 11$ GPa (1.6×10^6 psi), $\mu = 0.20$, ultimate strength $S_{\mathrm{ut}} = 24$ MPa (3480 psi) in tension and $S_{\mathrm{uc}} = 41$ MPa (5950 psi) in compression. The OD of the pipe bowl is 40 mm ($b = 20$ mm $= 0.788$ in.) and the ID is 20 mm ($a = 10$ mm $= 0.394$ in.). If the outer surface is maintained at 45°C (113°F), determine the maximum inner surface temperature to avoid cracking his favorite pipe. The axial strain is essentially zero. Use the *Principal Shear Stress* failure theory to determine failure. According to this failure theory, failure will occur when one of the following criteria are met:

$$\sigma_r = S_{\mathrm{ut}} \quad \text{or} \quad \sigma_t = S_{\mathrm{ut}} \quad \text{for} \quad \sigma_r > 0 \quad \text{and} \quad \sigma_t > 0 \ \text{(both } \textit{tensile}\text{)}$$

$$\frac{\sigma_t}{S_{\mathrm{ut}}} - \frac{\sigma_r}{S_{\mathrm{uc}}} = 1 \quad \text{for} \quad \sigma_r < 0 \ (\textit{compressive}) \quad \text{and} \quad \sigma_t > 0 \ (\textit{tensile})$$

$$\sigma_r = -S_{\mathrm{uc}} \ \text{or} \ \sigma_t = -S_{\mathrm{uc}} \ \text{for} \ \sigma_r < 0 \ \text{and} \ \sigma_t < 0 \ \text{(both } \textit{compressive}\text{)}$$

$$\frac{\sigma_r}{S_{\mathrm{ut}}} - \frac{\sigma_t}{S_{\mathrm{uc}}} = 1 \quad \text{for} \quad \sigma_r > 0 \ (\textit{tensile}) \quad \text{and} \quad \sigma_t < 0 \ (\textit{compressive})$$

9-3. A long cylindrical electrical conductor of radius b (4.8 mm) and thermal conductivity k_t has a uniform (constant) energy dissipation rate per unit volume (heat "generation" rate) q_g. The temperature field equation for steady-state heat transfer in this case is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{q_g}{k_t} = 0$$

Determine the expression for the temperature distribution $T(r)$ if the outer surface is maintained at $T = T_0$ and the temperature at the center is finite. Using this relationship, determine the expression for the temperature T_c at the center of the cylinder. If the cylinder is constructed of Nichrome V (80% Ni, 20% Cr) with a thermal conductivity $k_t = 12.5$ W/m-°C (7.22 Btu/hr-ft-°F) and the surface temperature is $T_0 = 30$°C (86°F), determine the numerical value of the center temperature for an energy dissipation rate of $q_g = 15$ MW/m^3(839 Btu/hr-in^3).

9-4. A long cylindrical electrical conductor of radius b has a uniform (constant) energy dissipation per unit volume (heat "generation") q_g. The temperature distribution for steady-state heat transfer in this case is

$$T - T_0 = \left(\frac{q_g b^2}{4k_t}\right)\left[1 - \left(\frac{r}{b}\right)^2\right] = (T_c - T_0)\left[1 - \left(\frac{r}{b}\right)^2\right]$$

If the cylindrical conductor is constrained such that the axial strain is zero and the outer surface has no mechanical loads applied, determine the expressions for (a) the radial stress σ_r, and (b) the circumferential stress σ_θ. Using this relationship, determine the maximum circumferential stress in the cylinder, if the cylinder is constructed of Nichrome V

(80% Ni, 20% Cr) with the following properties: $\alpha = 13.2 \times 10^{-6}\ ^\circ\mathrm{C}^{-1}$ ($7.33 \times 10^{-6}\ ^\circ\mathrm{F}^{-1}$), $E = 206$ GPa (29.88×10^6 psi) and $\mu = 0.30$. The surface temperature of the cylinder is $T_0 = 30^\circ$C (86°F), the center temperature is $T_c = 60^\circ$C (140°F), and $b = 4.8$ mm.

9-5. A cylindrical wooden support column has the following properties (disregard anisotropic effects for this problem): $\alpha = 30.0 \times 10^{-6}\ ^\circ\mathrm{C}^{-1}$ ($16.7 \times 10^{-6}\ ^\circ\mathrm{F}^{-1}$), $E = 12$ GPa (1.74×10^6 psi), $\mu = 0.20$, $k_t = 0.1725$ W/m-°C (0.100 Btu/hr-ft-°F), and $\kappa = 0.120\ \mathrm{mm}^2/\mathrm{s}$ ($0.670\ \mathrm{ft}^2/\mathrm{hr}$). The column has a diameter of 230 mm ($b = 115$ mm = 4.53 in.) and the axial strain is zero for the column. The cylinder is initially at a uniform temperature of 26°C (79°F), and suddenly a hot wind at 56°C (133°F) from a nearby forest fire blows across the column. The convective heat transfer coefficient between the air and the column is 3.0 W/m^2-°C (0.528 Btu/hr-ft^2-°F). Determine the center temperature and the surface temperature of the cylinder at the end of 12 hours. Determine the circumferential stress at the surface of the cylinder at the end of 12 hours.

9-6. An experimental bathysphere (which may be treated as a hollow sphere for this problem) is constructed of ASME SB 168 (Inconel-600) Ni-Cr-Fe alloy with the following properties: $\alpha = 13.0 \times 10^{-6}\ ^\circ\mathrm{C}^{-1}$ ($7.2 \times 10^{-6}\ ^\circ\mathrm{F}^{-1}$), $E = 215$ GPa (31.2×10^6 psi), and $\mu = 0.325$. The ID of the sphere is 1.80 m ($a = 0.900$ m = 2.953 ft), and the OD is 2.10 m ($b = 1.050$ m = 3.445 ft). The sphere is operating at a depth of 1488 m (4882 ft) in the ocean, at which point the pressure of the outside of the sphere is $p_0 = 15$ MPa (2176 psi). The temperature of the inside surface of the sphere is $T_1 = 25^\circ$C (77°F), and because the sphere is operating near an underwater hot plume, the outside temperature is T_0. The stress due the pressure acting on the outer surface (mechanical loading) is given by

$$(\sigma_r)_{\mathrm{mech}} = -p_0 \left[\frac{(b/a)^3 - (b/r)^3}{(b/a)^3 - 1} \right]$$

$$(\sigma_t)_{\mathrm{mech}} = -\frac{1}{2} p_0 \left[\frac{2(b/a)^3 + (b/r)^3}{(b/a)^3 - 1} \right]$$

Determine the maximum temperature T_0 of the outer surface for the combined mechanical and thermal stresses ($\sigma = \sigma_{\mathrm{mech}} + \sigma_{\mathrm{th}}$) not to exceed an absolute value of 135 MPa (19,580 psi). Note that the critical stress may occur either at the outer surface or at the inner surface.

9-7. A billiard ball has the following properties: $\alpha = 66.7 \times 10^{-6}\ ^\circ\mathrm{C}^{-1}$ ($37 \times 10^{-6}\ ^\circ\mathrm{F}^{-1}$), $E = 2.4$ GPa (0.35×10^6 psi), $\mu = 0.15$, $k_t = 0.750$ W/m-°C (0.433 Btu/hr-ft-°F), and $\kappa = 4.50\ \mathrm{mm}^2/\mathrm{s}$ ($0.174\ \mathrm{ft}^2/\mathrm{hr}$). The ball has a diameter of 60 mm ($b = 30$ mm = 1.18 in.). The ball is initially at a uniform temperature $T_0 = 20^\circ$C (68°F) and it is suddenly plunged into hot water at 95°C (203°F) by accident. The convective heat transfer coefficient

between the water and the ball is 250 W/m^2-°C (44 Btu/hr-ft^2-°F). Determine the center temperature and the surface temperature of the solid sphere at the end of 1 minute. Determine the circumferential stress at the surface of the cylinder at the end of 1 minute.

9-8. In a containment system for nuclear waste, a spherical cavity of radius a is formed in a large concrete structure. The temperature far from the cavity ($r \to \infty$) is T_0, and the surface of the cavity is maintained at T_1. There are no mechanical radial forces applied at the surface of the cavity, and the radial stress is zero at distances far from the cavity. Show that the steady-state temperature distribution outside the cavity (for constant thermal conductivity) is

$$\frac{T(r) - T_0}{T_1 - T_0} = \frac{a}{r}$$

Using this expression for the temperature distribution, determine the relationships for the radial and tangential stresses outside the cavity. If the cavity surface temperature is $T_1 = 35°\mathrm{C}$ (95°F) and the temperature far away from the cavity is $T_0 = 15°\mathrm{C}$ (59°F), determine the maximum tangential stress and maximum radial stress. For concrete, $\alpha = 10.8 \times 10^{-6}\ °\mathrm{C}^{-1}$ $(6.0 \times 10^{-6}\ °\mathrm{F}^{-1})$, $E = 20.7\,\mathrm{GPa}$ $(3.0 \times 10^6\,\mathrm{psi})$, $\mu = 0.15$, and $k_t = 1.385\,\mathrm{W/m}$-°C (0.80 Btu/hr-ft-°F). The cavity has a diameter of 600 mm ($a = 300$ mm $= 11.81$ in.).

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10

THERMOELASTIC STABILITY

10.1 INTRODUCTION

Because thermal expansion may result in compressive stresses within a member, there exists the possibility that *thermal buckling* or *elastic instability* may occur due to the induced compressive stresses. The problem is somewhat more complex, in general, because the thermal effects and mechanical effects may interact. For example, a column that would be safe under a given mechanical loading may become unstable (may buckle) under the same mechanical load with an additional thermal loading.

If mechanical loads are not applied, the thermal buckling may not be as catastrophic as is the case with mechanical loading and no thermal loads. In fact, the greatly increased flexibility of the buckled member in the thermal case results in small increases in the stress in the member after thermal buckling has occurred. This behavior may be acceptable from strength considerations; however, the lateral deflection after thermal buckling may not be acceptable for other reasons. For example, the waviness of the thermally buckled skin of an aircraft wing may interfere with the aerodynamic performance of the wing structure [Van der Neut, 1958].

10.2 THERMAL BUCKLING OF COLUMNS

Let us consider the case of a beam column that has pinned ends, as shown in Figure 10-1. For this end condition, there is no mechanical moment at the ends

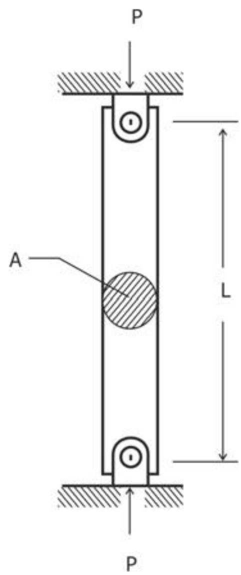

Figure 10-1. Pinned-end column with axial constraint.

(the ends are free to rotate), and the deflection of the ends is zero. For fixed ends, the axial load induced by a temperature change $\Delta T = T - T_0$ may be obtained from eq. (2-8):

$$P = -\sigma A = AE\alpha\Delta T \tag{10-1}$$

The constraint on the ends of the column may be reduced either by providing an expansion gap or by an axial elastic support, instead of a rigid one. If the column is allowed to expand due to a gap of thickness δ between the end of the column and the support, the axial load may be obtained from eq. (2-12):

$$P = -\sigma A = AE\alpha\Delta T\left[1 - \left(\frac{\delta/L}{\alpha\,|\Delta T|}\right)\right] \tag{10-2}$$

If the column is constrained by an axial elastic support with a spring constant k_{sp}, the axial load induced by the temperature change may be obtained from eq. (2-16):

$$P = -\sigma A = \frac{AE\alpha\Delta T}{1 + \left(\dfrac{AE}{k_{sp}L}\right)} \tag{10-3}$$

For a long column with pinned ends, the mechanical load that will cause mechanical instability (buckling) is the *Euler load* [Timoshenko and Gere, 1961]:

$$P_{cr} = \frac{\pi^2 EI}{L^2} \tag{10-4}$$

The quantity I is the smaller area moment of inertia about the centroid axis of the column. If we equate the critical end load from eq. (10-4) to the axial load with thermal effects, the following expression is obtained for the critical or buckling temperature difference for a column with pinned ends:

$$\Delta T_{\text{cr}} = \frac{\pi^2 I}{\alpha A L^2} = \frac{\pi^2}{\alpha (L/r_g)^2} \tag{10-5}$$

The quantity $r_g = \sqrt{I/A}$ is the minimum *radius of gyration* for the cross section.

For a column with pinned ends with a gap of width δ at one end, the critical temperature increase is

$$\Delta T_{\text{cr}} = \frac{\pi^2 I}{\alpha A L^2} + \frac{\delta}{\alpha L} \tag{10-6}$$

For a column with pinned ends with an elastic support (spring with a spring constant k_{sp}), the critical temperature increase is

$$\Delta T_{\text{cr}} = \frac{\pi^2 I}{\alpha A L^2}\left[1 + \left(\frac{AE}{k_{\text{sp}} L}\right)\right] \tag{10-7}$$

Except for the case of a column constrained axially by an elastic support, it is noted that the critical temperature difference causing thermal buckling is independent of Young's modulus, in contrast for the case for buckling caused by a mechanical load, in which the critical load is directly proportional to Young's modulus.

For other end conditions, as shown in Figure 10-2, the critical temperature difference may be written in the following form:

$$\Delta T_{\text{cr}} = K_b \frac{\pi^2 I}{\alpha A L^2} = K_b \frac{\pi^2}{\alpha (L/r_g)^2} \tag{10-8}$$

Values of the coefficient K_b for various end conditions are given in Table 10-1. Note that $K_b = 1$ for both ends pinned.

Example 10-1 A 100-mm nominal (4-in. nominal) SCH 40 pipe is used in a steam system. The pipe outside diameter is 114.3 mm (4.500 in.), the wall thickness is 6.0 mm (0.237 in.), and the length of the pipe between supports is 4.572 m (15 ft). The pipe is rigidly anchored at each end. The pipe material is a Cr–Mo alloy steel having the following properties: $\alpha = 12.5 \times 10^{-6\circ}\text{C}^{-1}$ $(6.94^{-6\circ}\text{F}^{-1})$, $E = 200\,\text{GPa}$ $(29 \times 10^6\,\text{psi})$, yield strength $S_y = 270\,\text{MPa}$ (39,200 psi). The pipe is initially stress-free at $T_0 = 25^\circ\text{C}$ (77°F). Determine the temperature T_1 at which the pipe will buckle.

The cross-sectional area of the pipe material is given by

$$A = \pi(D_0 - t)t = (\pi)(0.1143 - 0.0060)(0.006) = 20.41 \times 10^{-4}\,\text{m}^2$$
$$= 20.41\ \text{cm}^2$$

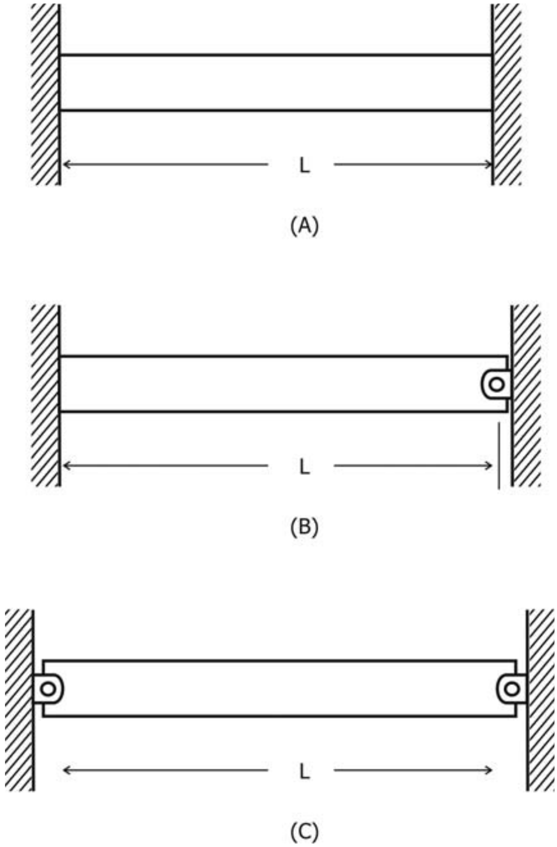

Figure 10-2. End conditions for columns with a uniform temperature change. (A) Both ends fixed, $K_b = 4$. (B) One end fixed, the other end pinned, $K_b = 2.0457 = 20.1906/\pi^2$. (C) Both ends pinned (simply-supported), $K_b = 1$.

TABLE 10-1. Values for the Coefficient K_b in Eq. (10-8) for Beam-Column Buckling

$$\Delta T_{cr} = K_b \frac{\pi^2 I}{\alpha A L^2}$$

End conditions	K_b
Both ends pinned (simply-supported)	1
Both ends fixed (no rotation)	4
One end fixed (no rotation) and the other end pinned.	2.0457
	Solution of $\tan \xi = \xi$, with $K_b = \left(\dfrac{\xi}{\pi}\right)^2$

The area moment of inertia I for the pipe is

$$I = \frac{\pi}{8}(D_0 - t)^3 t \left[1 + \left(\frac{t}{D_0 - t}\right)^2\right]$$

$$I = \frac{\pi}{8}(0.1143 - 0.0060)^3(0.006)\left[1 + \left(\frac{6.0}{114.3 - 6.0}\right)^2\right]$$

$$I = (2.99929 \times 10^{-6})(1.00307) = 3.002 \times 10^{-6}\,\text{m}^4 = 300.2\,\text{cm}^4$$

The radius of gyration r_g for the column is

$$r_g = \frac{I}{A} = \frac{(D_0 - t)}{2\sqrt{2}}\sqrt{1 + \left(\frac{t}{D_0 - t}\right)^2}$$

$$r_g = \frac{0.1143 - 0.0060}{2\sqrt{2}}\sqrt{1 + \left(\frac{6}{114.3 - 6}\right)^2}$$

$$= (0.03829)(1.0015) = 0.03835 \text{ m}$$

For a column with both ends fixed, $K_b = 4$. The critical temperature difference is given by eq. (10-8):

$$\Delta T_{\text{cr}} = \frac{4\pi^2}{\alpha(L/r_g)^2} = \frac{(4)(\pi^2)}{(12.5 \times 10^{-6})(4.572/0.03835)^2} = 222.2^\circ\text{C} \quad (400^\circ\text{F})$$

The pipe would buckle when the temperature reaches

$$T_1 = 25^\circ + 222.2^\circ = 247.2^\circ\text{C} \quad (477^\circ\text{F})$$

Let us check the stress at this condition, because the column could yield before buckling can occur:

$$\sigma = -\alpha E \Delta T = -(12.5 \times 10^{-6})(200 \times 10^9)(222.2^\circ)$$

$$= -55.55 \times 10^6 \text{ Pa} = -55.55 \text{ MPa}$$

The yield strength is $S_y = 270\,\text{MPa}$, so the column will buckle in the elastic region at 247.2°C.

10.3 GENERAL FORMULATION FOR BEAM COLUMNS

Problems involving thermal buckling of beam columns may be treated in a manner similar to that for buckling due to mechanical compressive loads. The

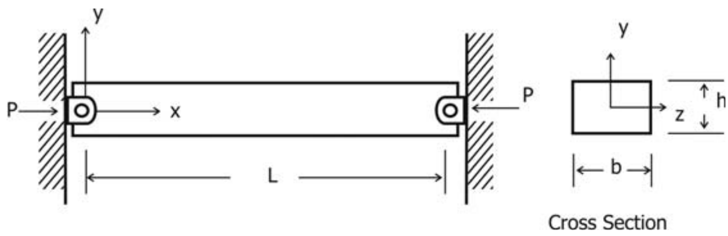

Figure 10-3. Pinned-end beam column of rectangular cross section.

governing expression is the moment balance expression for beams, eq. (3-25):

$$\frac{d^2v}{dx^2} = -\frac{M + M_T}{EI} \tag{10-9}$$

The quantity I is the area moment of inertia of the beam cross section about the z-axis.

Let us consider the beam column with pinned ends, as shown in Figure 10-3. To illustrate the general principle of analysis, let us consider a specific example for the thermal moment. Suppose the temperature distribution varies linearly in the span-wise direction (y-direction) and varies sinusoidally in the axial direction (x-direction):

$$\Delta T = T(x, y) - T_0 = \Delta T_1 \sin(\pi x/L)\left(\frac{2y}{h} - 1\right) \tag{10-10}$$

The quantity $\Delta T_1 = T_1 - T_0$, where T_1 is the temperature at the top of the beam $(y = \frac{1}{2}h)$ and T_0 is the temperature at the center (*centroid axis*) of the beam $(y = 0)$. The sinusoidal function is an important one, because any piece-wise continuous function can be expanded as a Fourier series involving sine functions.

The thermal moment may be found from its definition, eq. (3-10):

$$M_T = \alpha E \int_{-h/2}^{h/2} \Delta T \; y \, dA \tag{10-11}$$

$$M_T = \alpha E \Delta T_1 b \sin(\pi x/L) \int_{-h/2}^{h/2} \left(\frac{2y}{h} - 1\right) y \, dy = \frac{1}{6}\alpha E \Delta T_1 b h^2 \sin(\pi x/L)$$

For the rectangular cross section in this example, the area moment of inertia $I = \frac{1}{12}bh^3$:

$$M_T = \frac{2\alpha EI \Delta T_1}{h} \sin(\pi x/L) \tag{10-12}$$

The average temperature change across any cross section is

$$(\Delta T)_{\text{ave}} = \frac{1}{A}\int_{-h/2}^{h/2} \Delta T \, dA = \frac{\Delta T_1 \sin(\pi x/L)}{bh} \int_{-h/2}^{h/2} \left(\frac{2y}{h} + 1\right) b \, dy$$
$$= \Delta T_1 \sin(\pi x/L)$$

The axial load is

$$P = \alpha E \int_{-h/2}^{h/2} (\Delta T)_{\text{ave}}\, A(dx/L) \tag{10-13}$$

$$P = \alpha E \Delta T_1 A \int_{-h/2}^{h/2} \sin(\pi x/L)\, dx = \frac{2\alpha E A \Delta T_1}{\pi} \tag{10-14}$$

An important distinction between the elastic stability analysis and an elementary "beam-bending" analysis is that the bending moment of the axial force cannot be neglected in the elastic stability analysis. In fact, it is the "negative feedback" of this ordinarily small and negligible bending moment that gives rise to the elastic instability:

$$M = Pv \tag{10-15}$$

The quantity v is the displacement of the centroid axis in the y-direction. For fixed axial load P, as the displacement increases, the bending moment causing displacement increases, which causes more displacement, etc. It is this "runaway" displacement that produces the instability or buckling at some critical axial load.

Making the substitutions for the mechanical and thermal moments into the moment balance expression, eq. (10-9), the following differential equation is obtained for this problem:

$$\frac{d^2 v}{dx^2} + \left(\frac{P}{EI}\right) v = -\left(\frac{2\alpha \Delta T_1}{h}\right) \sin(\pi x/L) \tag{10-16}$$

Suppose we define the following buckling parameter:

$$k = \sqrt{\frac{P}{EI}} \tag{10-17}$$

The solution of eq. (10-16) is

$$v(x) = C_1 \sin kx + C_2 \cos kx - \frac{\sin(\pi x/L)}{k^2 - (\pi/L)^2} \left(\frac{2\alpha \Delta T_1}{h}\right) \tag{10-18}$$

At one end of the column ($x = 0$), the displacement is zero; therefore, the constant $C_2 = 0$. Similarly, at the other end of the column ($x = L$), the displacement is also zero:

$$v(L) = C_1 \sin kL \tag{10-19}$$

To meet this condition, there are two possibilities.

(a) We could take $C_1 = 0$. This choice would yield

$$v(x) = -\frac{\sin(\pi x/L)}{k^2 - (\pi/L)^2} \left(\frac{2\alpha \Delta T_1}{h}\right) \tag{10-20}$$

When the axial load is such that $k^2 = (\pi/L)^2$, the displacement becomes indeterminate.

(b) We could take $\sin kL = 0$ or $(kL) = \pi$ or $k^2 = (\pi/L)^2$. This choice would yield

$$v(x) = \left[C_1 - \frac{2\alpha\Delta T_1/h}{(\pi/L)^2 - (\pi/L)^2}\right]\sin(\pi x/L) \to \text{indeterminate} \quad (10\text{-}21)$$

One of the characteristics of thermal buckling or thermal elastic instability is the *bifurcation* of the solutions (splitting into two possibilities) at the critical temperature difference. If there were no thermal bending moment in this problem, the beam could either (a) remain straight or (b) deflect with a sinusoidal distribution at the critical temperature difference point. Another characteristic of the thermal instability problem is that the solution becomes indeterminate at the critical temperature difference point.

For either of the choices the critical or buckling temperature difference occurs when the following condition is met:

$$k^2 = \frac{\pi^2}{L^2} = \frac{2\alpha\Delta T_1 A}{\pi I}$$

The critical temperature difference in this example is

$$\Delta T_{cr} = T_{cr} - T_0 = \frac{\pi^3 I}{2\alpha L^2 A} = \frac{\pi^3}{2\alpha(L/r_g)^2} \quad (10\text{-}22)$$

We observe that the critical temperature difference is a function of the thermal expansion coefficient α and the slenderness ratio (L/r_g), and is independent of Young's modulus E.

10.4 POSTBUCKLING BEHAVIOR OF COLUMNS

It was mentioned in Section 10.1 that thermal buckling may not be as catastrophic as is the case with mechanical loading and no thermal loads. Let us examine the deflection of a beam column after thermal buckling has occurred. As an example, let us consider a pinned-end column, as shown in Figure 10-4, with no thermal moment (only axial forces due to thermal expansion). The beam equation for this case reduces to

$$\frac{d^2v}{dx^2} = -\frac{M}{EI} = -\frac{Pv}{EI} \quad (10\text{-}23)$$

If we define $k^2 = P/EI$, the solution of eq. (10-23) may be written as

$$v(x) = C_1 \sin kx + C_2 \cos kx \quad (10\text{-}24)$$

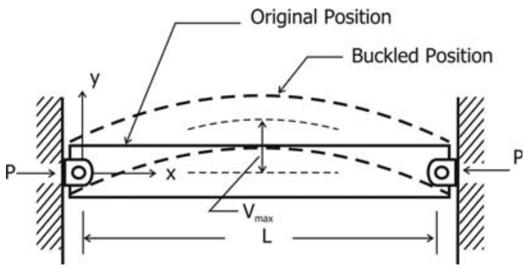

Figure 10-4. Postbuckling behavior of a beam column with pinned ends.

At $x = 0$, the deflection is zero; therefore, $C_2 = 0$. Similarly, at $x = L$, the deflection is zero, and we may take $\sin(kL) = 0$, or $k = \pi/L$. The deflection may be written as

$$v(x) = C_1 \sin \pi x/L \tag{10-25}$$

The coefficient C_1 is arbitrary. The maximum deflection $v_{\max}$ occurs at the center of the column ($x = \frac{1}{2}L$), which makes $C_1 = v_{\max}$. The deflection may be written as

$$v(x) = v_{\max} \sin \pi x/L \tag{10-26}$$

After the column has buckled, eq. (10-26) applies for the deflection, and the change in the overall length of the column from the original length L may be determined from

$$\Delta L = \int_0^L ds - L \tag{10-26}$$

The arc length differential ds may be written as follows for small deflections:

$$ds = \sqrt{1 + \left(\frac{dv}{dx}\right)^2} = 1 + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 + \cdots \tag{10-27}$$

Making this substitution into eq. (10-26) and taking the first two terms, the result for the change in length ΔL of the column is

$$\Delta L = \frac{1}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx = \tfrac{1}{2}\left(\frac{\pi v_{\max}}{L}\right)^2 \int_0^L \sin^2(\pi x/L)\, dx$$

$$\Delta L = \frac{\pi^2 v_{\max}^2}{4L} \tag{10-28}$$

The total strain along the centroid axis of the beam after buckling may be written as

$$\frac{\Delta L}{L} = \alpha(T - T_{\text{cr}}) = \frac{\pi^2 v_{\max}^2}{4L^2} \tag{10-29}$$

The temperature difference ratio may be written as

$$\frac{T - T_{cr}}{T_{cr} - T_0} = \frac{(T - T_0) - (T_{cr} - T_0)}{T_{cr} - T_0} = \frac{\Delta T}{\Delta T_{cr}} - 1 = \frac{\left(\dfrac{\pi^2 v_{max}^2}{4\alpha L^2}\right)}{\left(\dfrac{\pi^2 r_g^2}{\alpha L^2}\right)} = \frac{v_{max}^2}{4r_g^2} \quad (10\text{-}30)$$

We may solve for the maximum deflection after buckling as follows:

$$v_{max} = 2r_g\sqrt{\left(\frac{\Delta T}{\Delta T_{cr}}\right) - 1} \quad (10\text{-}31)$$

The expression given by eq. (10-31) is valid until the outermost fibers begin to yield. The stress at the outermost fiber may be written as follows, noting that the axial force P remains essentially constant at the critical load P_{cr}:

$$\sigma_{max} = \frac{P_{cr}}{A} \pm \frac{Mh}{2I} = \left(\frac{P_{cr}}{A}\right)\left(1 + \frac{v_{max}Ah}{2I}\right) \quad (10\text{-}32)$$

Making the substitution from eq. (10-4) for the critical axial load P_{cr}, eq. (10-32) may be written as

$$\sigma_{max} = \frac{\pi^2 E}{(L/r_g)^2}\left(1 + \frac{v_{max}h}{2r_g^2}\right) = \frac{\pi^2 E}{(L/r_g)^2}\left[1 + \left(\frac{h}{r_g}\right)\sqrt{\left(\frac{T - T_0}{T_{cr} - T_0}\right) - 1}\right] \quad (10\text{-}33)$$

The temperature change $(T_y - T_0)$ that results in yielding in the outermost fiber may be found from eq. (10-33) by setting $\sigma_{max} = S_y$ and solving for the temperature difference $(T - T_0)$:

$$\frac{T_y - T_0}{T_{cr} - T_0} = 1 + \left(\frac{r_g}{h}\right)^2\left[\left(\frac{L}{\pi r_g}\right)^2\left(\frac{S_y}{E}\right) - 1\right]^2 \quad (10\text{-}34)$$

Example 10-2 Suppose the pipe in Example 10-1 were pinned at both ends. Determine the temperature at which the column would buckle. Also, determine the temperature at which the buckled column would begin to yield.

The critical or buckling temperature difference may be found from eq. (10-5):

$$\Delta T_{cr} = \frac{\pi^2}{\alpha(L/r_g)^2} = \frac{\pi^2}{(12.5 \times 10^{-6})(4.572/0.03835)^2} = 55.6^\circ\text{C}$$

The temperature at which thermal buckling will occur is

$$T_{cr} = 55.6^\circ + 25^\circ = 80.6^\circ\text{C} \quad (177.1^\circ\text{F})$$

The temperature at which the outermost fiber will begin to yield after buckling may be found from eq. (10-34), with the distance from the centroid axis to the outermost fiber $h = \frac{1}{2}D_0 = \frac{1}{2}(114.3) = 57.15\,\text{mm}$

$$\frac{T_y - T_0}{T_{cr} - T_0} = 1 + \left(\frac{38.35}{57.15}\right)^2 \left[\left(\frac{4.572}{0.03835\pi}\right)^2 \left(\frac{270 \times 10^6}{200 \times 10^9}\right) - 1\right]^2$$

$$\frac{T_y - T_0}{T_{cr} - T_0} = 1 + (0.4503)(1.9441 - 1) = 1.4251$$

The temperature at which the column will first begin to yield is

$$T_y = 25^\circ + (1.4251)(55.6^\circ) = 25^\circ + 79.2^\circ = 104.2^\circ\text{C} \quad (220^\circ\text{F})$$

The yield temperature is only about 24°C above the buckling temperature:

$$T_y - T_{cr} = 104.2^\circ - 80.6^\circ = 23.6^\circ\text{C} \quad (42.5^\circ\text{F})$$

From this example, we may observe that, although thermal buckling does not result in catastrophic collapsing failure, the column will generally begin to yield if it is heated very much above the critical temperature difference. Another consideration is that, when the buckled column is cooled back to the initial temperature after yielding, some permanent set will have occurred and the column will no longer be straight.

10.5 LATERAL THERMAL BUCKLING OF BEAMS

If a beam is subjected to a temperature distribution that produces a nonzero thermal moment, usually there is no problem with elastic stability, except in cases in which the beam is narrow in cross section, such as a deep I-beam. In this case, the beam may buckle sideways (laterally) when the thermal moment reaches a critical value.

Let us consider the case of a thin rectangular beam, as shown in Figure 10-5. In general, there could be thermal moments in both the spanwise and lateral direction:

$$M_{Tz} = \alpha E \int \Delta T\, y\, dA \tag{10-35a}$$

$$M_{Ty} = \alpha E \int \Delta T\, z\, dA \tag{10-35b}$$

If the y- and z-axes are the principal axes for the cross section, the area product of inertia is zero, and the expressions relating the deflections and moments are

$$\frac{d^2 v}{dx^2} = -\frac{M_z + M_{Tz}}{EI_z} \tag{10-36a}$$

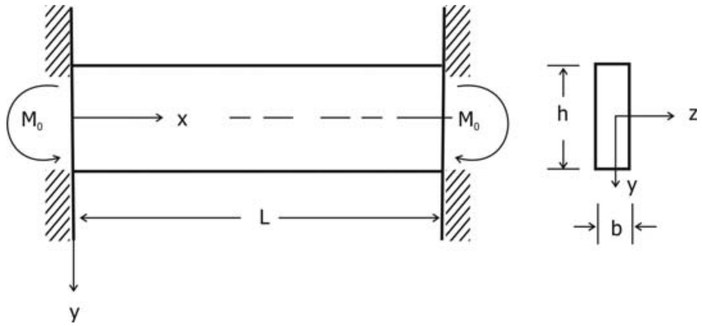

Figure 10-5. Lateral thermal buckling of a beam with a rectangular cross section.

$$\frac{d^2w}{dx^2} = -\frac{M_y + M_{Ty}}{EI_y} \tag{10-36b}$$

The quantities I_z amd I_y are the area moments of inertia about the z- and y-axes, respectively. Let us consider a beam with a rectangular cross section $h \times b$:

$$I_z = \int y^2\,dA = \tfrac{1}{12}bh^3 \tag{10-37a}$$

$$I_y = \int z^2\,dA = \tfrac{1}{12}b^3h \tag{10-37b}$$

Let us consider a beam with fixed ends subjected to a linear temperature distribution as follows:

$$\Delta T = T - T_0 = (T_1 - T_0)\left(\frac{2y}{h}\right) \tag{10-38}$$

For this temperature distribution, the thermal moment for a beam having a rectangular cross section $h \times b$ is

$$M_{Tz} = \frac{1}{6}\alpha E \Delta T_1 bh^2 = \frac{2\alpha E \Delta T_1 I_z}{h} = M_{T0} \qquad \text{(constant)} \tag{10-39}$$

For this example, $M_{Ty} = 0$. If the only mechanical moment applied is the end reaction M_0, the solution of eq. (10-37a) is

$$v(x) = -\frac{(M_0 + M_{T0})x^2}{2EI_z} + C_1x + C_2 \tag{10-40}$$

From the fixed-end boundary condition at $x = 0$,

$$v(0) = 0 \quad \text{so} \quad C_2 = 0$$

$$\frac{dv(0)}{dx} = 0 \quad \text{so} \quad C_1 = 0$$

From the fixed-end boundary condition at $x = L$,

$$v(L) = 0 \quad \text{so} \quad M_0 = -M_{T0}$$

Thus, the beam will experience no deflection before buckling.

Suppose the beam buckles in the z-direction, as shown in Figure 10-6. For the buckled configuration, the differential equations are

$$\frac{d^2w}{dx^2} = -\frac{M_y + M_{Ty}}{EI_y} = -\frac{-\sin\phi M_{T0} + 0}{EI_y} = \frac{\phi M_{T0}}{EI_y} \tag{10-41a}$$

$$\frac{d\phi}{dx} = -\frac{M_{yz}}{GJ} = -\frac{\sin\left(\dfrac{dw}{dx}\right) M_0}{G(I_z + I_y)} = \frac{M_{T0}}{G(I_z + I_y)}\frac{dw}{dx} \tag{10-41b}$$

In eqs. (10-41), we have used the approximations that, for small angles, $\sin\phi \approx \phi$. If we take the derivative of both sides of eq. (10-41b), the following is obtained:

$$\frac{d^2\phi}{dx^2} = \frac{M_{T0}}{G(I_z + I_y)}\frac{d^2w}{dx^2} \tag{10-41c}$$

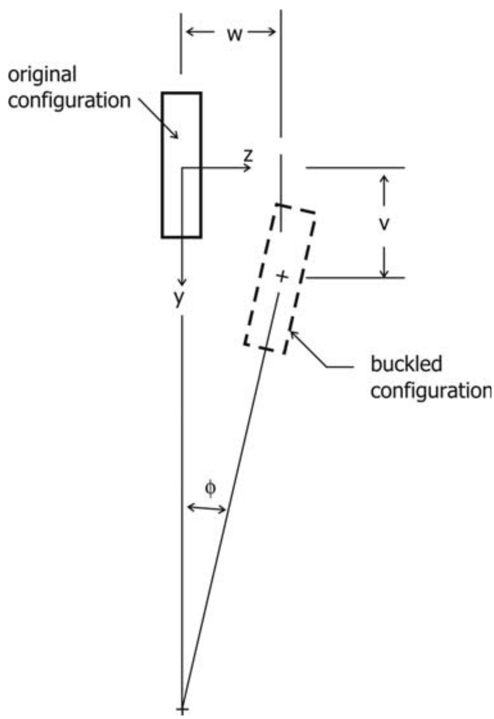

Figure 10-6. Laterally buckled configuration for a beam with rectangular cross section.

If we use eq. (10-41c) to eliminate $\left(d^2w/dx^2\right)$ in eq. (10-41a), the following is obtained:

$$\frac{d^2\phi}{dx^2} + \frac{M_{T0}^2}{G(I_z + I_y)EI_y}\phi = 0 \tag{10-41d}$$

Suppose we define the following buckling parameter:

$$k^2 = \frac{M_{T0}^2}{G(I_z + I_y)EI_y} \tag{10-42a}$$

If the beam has a rectangular cross section, we may substitute for the area moment of inertias from eqs. (10-37) to obtain

$$k^2 = \frac{288(1+\mu)M_{T0}^2}{E^2b^4h^4[1+(b/h)^2]} \tag{10-42b}$$

Equation (1-18) was also used to eliminate the shear modulus G.

In terms of the parameter k, eq. (10-41d) may be written as

$$\frac{d^2\phi}{dx^2} + k^2\phi = 0 \tag{10-43}$$

The solution of eq. (10-43) is

$$\phi(x) = C_1 \sin kx + C_2 \cos kx$$

For the beam with rigidly fixed ends, the rotation ϕ is zero at each end:

$$\phi(0) = 0 \quad \text{therefore,} \quad C_2 = 0$$

$$\phi(L) = 0 \quad \text{therefore, for buckling,} \quad \sin(kL) = 0 \quad \text{or} \quad kL = \pi$$

The critical thermal moment may be found from the definition of the k parameter, eq. (10-42a):

$$M_{T,\mathrm{cr}} = \frac{\pi EI_z}{\sqrt{2(1+\mu)}L}\sqrt{\left(1+\frac{I_y}{I_z}\right)\left(\frac{I_y}{I_z}\right)} \tag{10-44a}$$

For a beam with a rectangular cross section, the critical thermal moment may be written as

$$M_{T,\mathrm{cr}} = \frac{\pi Eb^2h^2\sqrt{1+(b/h)^2}}{12\sqrt{2(1+\mu)}L} \tag{10-44b}$$

The critical temperature difference may be found by using eq. (10-39) for the critical thermal moment. For a beam with area moments of inertia I_z and

I_y, the critical temperature difference for the linear temperature difference of eq. (10-39) is

$$\Delta T_{cr} = T_{cr} - T_0 = \frac{\pi h}{2\sqrt{2(1+\mu)}\alpha L}\sqrt{\left(1 + \frac{I_y}{I_z}\right)\left(\frac{I_y}{I_z}\right)} \tag{10-45a}$$

For a beam with a rectangular cross section, the critical temperature change is

$$\Delta T_{cr} = T_{cr} - T_0 = \frac{\pi b}{2\sqrt{2(1+\mu)}\alpha L}\sqrt{\left(1 + \frac{b^2}{h^2}\right)} \tag{10-45b}$$

The critical or buckling temperature difference for the beam with a rectangular cross section is a function of the "slenderness ratio" (b/L).

Example 10-3 A beam is constructed of structural steel with the following properties: $\alpha = 12 \times 10^{-6\circ}\mathrm{C}^{-1}$ ($6.67 \times 10^{-6\circ}\mathrm{F}^{-1}$), $E = 200$ GPa (29×10^6 psi), $\mu = 0.28$, $S_y = 225$ MPa (32,600 psi), and allowable stress $S_a = 138$ MPa (20,000 psi). The beam is a 150×85 mm nominal 18.6 kg/m ($6 \times 3\frac{3}{8}$ in. nominal, 12.5 $\mathrm{lb_m}$/ft) I-beam, as shown in Figure 10-7. The area moment of inertia of the cross section around the horizontal axis is $I_z = 907.4\ \mathrm{cm}^4$ (21.8 in^4), the area moment of inertia around the vertical axis is $I_y = 74.9\ \mathrm{cm}^4$ (1.8 in^4), and the cross-sectional area is $A = 23.29\ \mathrm{cm}^2$ (3.61 in^2). The depth of the I-beam is $h = 152.4$ mm (6.00 in.), and the length of the beam is $L = 14.63$ m (48.0 ft). The lower surface of the beam is maintained at 10°C (50°F). Determine the maximum temperature of the top surface to avoid thermal buckling.

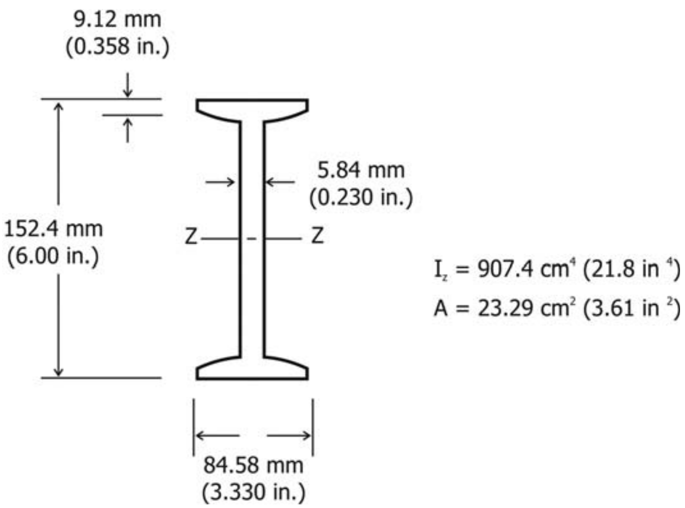

Figure 10-7. I-beam cross section, Example 10-3.

The numerical value of the term involving the moment of inertia ratio in eq. (10-45a) is

$$\sqrt{\left(1+\frac{I_y}{I_z}\right)\left(\frac{I_y}{I_z}\right)} = \sqrt{\left(1+\frac{74.9}{907.4}\right)\left(\frac{74.9}{907.4}\right)} = 0.2989$$

From eq. (10-45a), we find the critical temperature difference:

$$\Delta T_{\text{cr}} = \frac{\pi(0.1524)(0.2989)}{2\sqrt{2(1+0.28)}(12\times 10^{-6})(14.63)} = 254.7^{\circ}\text{C} \qquad (458.5^{\circ}\text{F})$$

The maximum allowable center temperature for the beam is

$$T_0 = 10^{\circ} + 254.7^{\circ} = 264.7^{\circ}\text{C}$$

The maximum allowable temperature for the upper surface of the beam to avoid thermal buckling is

$$T_1 = 264.7^{\circ} + 254.7^{\circ} = 519.4^{\circ}\text{C} \quad (967^{\circ}\text{F})$$

Let us check the bending stress in the beam that would result from this temperature difference. The thermal moment ($M_{T0} = -M_0$) is found from eq. (10-39):

$$M_{T0} = \frac{2\alpha E \Delta T_1 I_z}{h} = \frac{(2)(12\times 10^{-6})(200\times 10^{9})(254.7^{\circ})(907.4\times 10^{-8})}{0.1524}$$

$$M_{T0} = -M_0 = 72{,}792 \text{ N-m} = 72.792 \text{ kN-m} \quad (0.6442\times 10^{6}\text{ in-lbf})$$

The maximum bending stress is

$$\sigma_b = \pm\frac{M_0 h}{2I_z} = \pm\frac{(72{,}792)(0.1524)}{(2)(907.4\times 10^{-8})}$$
$$= \pm 611.3\times 10^{6}\text{ Pa} = 611.3\text{ MPa} \quad (88{,}700\,\text{psi})$$

The yield strength of the steel is $S_y = 228\,\text{MPa}$, so the given thermal condition would cause yielding before thermal buckling would occur. The "allowable" temperature change, resulting in the allowable stress of $S_a = 138\,\text{MPa}$, would be

$$\Delta T_a = \frac{(254.7^{\circ})(138)}{(611.3)} = 57.5^{\circ}\text{C} \quad (103.5^{\circ}\text{F})$$

The allowable temperature at the top of the beam would be

$$T_a = 10^{\circ} + (2)(57.5^{\circ}) = 125^{\circ}\text{C} \quad (257^{\circ}\text{F})$$

10.6 SYMMETRICAL BUCKLING OF CIRCULAR PLATES

10.6.1 General Relationships

As was the case for beam-column buckling, the axisymmetric buckling of circular plates is produced by small bending moments that were negligible in the problems dealing with plate bending without elastic instability considerations, as given by eq. (7-93). The forces and moments acting on a circular plate element in cylindrical coordinates are shown in Figure 10-8.

The additional force components produced by the in-plane or membrane stress resultants, as shown in Figure 10-9, is

$$rN_r d\omega_r d\theta + N_\theta \omega_r dr$$

If these terms are added in the moment balance, eq. (7-88), we find the following governing relationship:

$$\nabla^4 w = \frac{1}{D}\left(p - \frac{1}{1-\mu}\nabla^2 M_T + N_r \frac{d\omega_r}{dr} + N_\theta \frac{\omega_r}{r}\right) \tag{10-46}$$

The biharmonic operator may be written as

$$\nabla^4 w = \nabla^2(\nabla^2 w) \tag{10-47}$$

The Laplacian operator in cylindrical coordinates is

$$\nabla^2 w = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2} \tag{10-48}$$

For axisymmetric conditions (deflection a function of the radial coordinate only), the biharmonic operator may be written as

$$\nabla^4 w = \frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\} \tag{10-49}$$

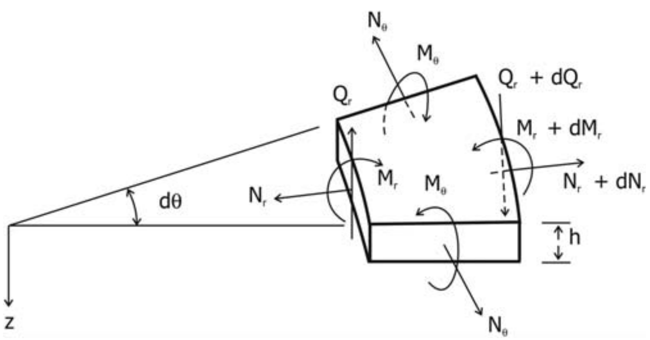

Figure 10-8. Forces and moments acting on an element in a circular plate.

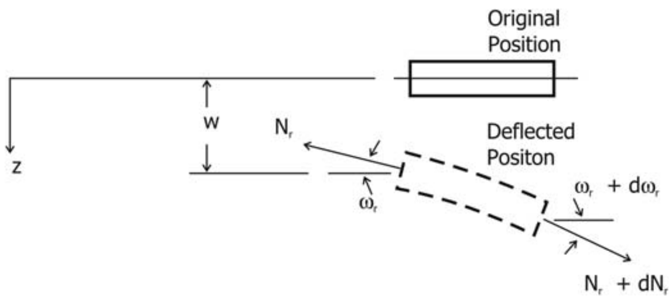

Figure 10-9. Transverse deflection and rotation of an element in a circular plate.

The rotation is related to the transverse deflection as follows:

$$\omega_r = \frac{dw}{dr} \tag{10-50}$$

If we use eq. (10-50) to eliminate the rotation in the moment balance expression, eq. (10-46), the following is obtained for axisymmetric loading:

$$\nabla^4 w = \frac{1}{D}\left(p - \frac{1}{1-\mu}\nabla^2 M_T + N_r \frac{d^2 w}{dr^2} + N_\theta \frac{1}{r}\frac{dw}{dr}\right) \tag{10-51}$$

If we repeat the development for the case of general loading (two-dimensional) in cylindrical coordinates, we get

$$\begin{aligned}\nabla^4 w = \frac{1}{D}\Bigg[p &- \frac{1}{1-\mu}\nabla^2 M_T \\ &+ N_r \frac{\partial^2 w}{\partial r^2} + N_\theta\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) + 2N_{r\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\Bigg]\end{aligned} \tag{10-52}$$

10.6.2 Circular Plate with a Uniform Temperature Change

Let us consider the circular plate having a radius b and a thickness h, as shown in Figure 10-10. Suppose the plate has no mechanical load applied and is subject to a uniform temperature change, $\Delta T = (T_1 - T_0) = \Delta T_1 =$ constant. The thermal membrane stress resultant N_T and thermal bending moment M_T are as follows for this case:

$$N_T = \alpha E \int \Delta T \, dz = \alpha E \Delta T_1 h \quad \text{and} \quad M_T = 0 \tag{10-53}$$

Let us consider the case in which the plate is radially constrained around the edge, such that the radial displacement $u = 0$. The radial and circumferential

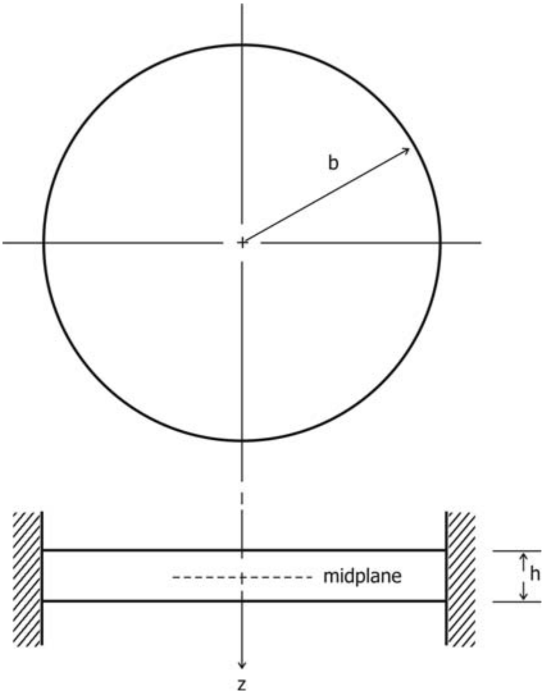

Figure 10-10. Circular plate with a rigidly clamped edge.

membrane stress resultants are given by eqs. (6-22). For zero radial displacement, these relationships reduce to

$$N_r = N_\theta = -\frac{1}{1-\mu} N_T = -\frac{\alpha E \Delta T_1 h}{1-\mu} \tag{10-54}$$

If we make these substitutions into eq. (10-51), we obtain

$$\nabla^4 w = -\frac{\alpha E \Delta T_1 h}{(1-\mu)D} \frac{1}{r} \frac{d}{dr}\left(r \frac{dw}{dr}\right) \tag{10-55}$$

Equation (10-55) may be written in the following alternate form:

$$\nabla^2 \left[\nabla^2 w + k^2 w\right] = 0 \tag{10-56}$$

The quantity k is defined as

$$k^2 = \frac{\alpha E \Delta T_1 h}{(1-\mu)D} = \frac{12(1+\mu)\alpha \Delta T_1}{h^2} \tag{10-57}$$

If we separate variables and integrate eq. (10-56) twice, we get

$$\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + k^2 w = C_1 \ln r + C_2 \tag{10-58}$$

The deflection and its derivatives must be finite at the center of the plate, if the solution is to represent the physical condition. This requires that $C_1 = 0$. With this substitution, eq. (10-58) is a form of the Bessel equation, and the solution for the deflection may be written as

$$w(r) = C_3 J_0(kr) + C_4 Y_0(kr) + \frac{C_2}{k^2} \tag{10-59}$$

As noted in Appendix D, eq. (D-14), the Bessel function of the second kind $Y_0(0) \to \infty$, so the constant $C_4 = 0$ to make the deflection given by eq. (10-59).

Let us consider two cases for the edge restraint. For a *built-in edge*, both the deflection and the slope are zero at the edge of the plate, $w(b) = 0$ and $\omega_r(b) = 0$. If we set the deflection at the edge to zero, the following result is obtained:

$$\frac{C_2}{k^2} = -C_3 J_0(kb) \tag{10-60}$$

$$w(r) = C_3[J_0(kr) - J_0(kb)] \tag{10-61}$$

$$\omega_r(r) = \frac{dw}{dr} = -C_3 k J_1(kr) \tag{10-62}$$

To meet the condition that the slope is zero at the edge, there are two choices: $C_3 = 0$ (no deflection at all) or $J_1(kb) = 0$. The buckled configuration corresponds to the second choice, which results in the following, as given in Table D-3:

$$kb = j_{0,1} = 3.83171 \tag{10-63}$$

The critical temperature change for a circular plate with a built-in edge is

$$\Delta T_{\text{cr}} = \frac{(3.83171)^2}{12(1+\mu)\alpha}\left(\frac{h}{b}\right)^2 = \frac{1.22350}{(1+\mu)\alpha}\left(\frac{h}{b}\right)^2 \tag{10-64}$$

For a *simply-supported edge*, both the deflection and the mechanical moment are zero at the edge of the plate, $w(b) = 0$ and $M_r(b) = 0$. If we use the condition for the edge deflection, the result is

$$w(r) = C_3[J_0(kr) - J_0(kb)] \tag{10-65}$$

$$\omega_r(r) = \frac{dw}{dr} = -C_3 k J_1(kr) \tag{10-66}$$

$$\frac{d^2w}{dr^2} = -C_3 k^2 \left[J_0(kr) - \frac{1}{kr} J_1(kr)\right] \tag{10-67}$$

The boundary condition for zero bending moment at the edge is given by eq. (7-96b):

$$\frac{d^2w(b)}{dr^2} + \frac{\mu}{b}\frac{dw(b)}{dr} = -\frac{M_T}{(1-\mu)D} = 0 \tag{10-68}$$

If we make the substitutions from eqs. (10-65) and (10-66) into eq. (10-68), the result is

$$C_3k^2\left[J_0(kb) - \frac{1-\mu}{kb}J_1(kb)\right] = 0 \tag{10-69}$$

Again, there are two choices for the constants, and buckling corresponds to

$$kb\,J_0(kb) - (1-\mu)J_1(kb) = 0 \tag{10-70}$$

If we let $\beta = kb$, the value of β is found from the solution of eq. (10-70) for a specific value of Poisson's ratio μ,

$$\beta J_0(\beta) - (1-\mu)J_1(\beta) = 0 \tag{10-71}$$

Values for the constant β are given in Table 10-2 as a function of Poisson's ratio.

If we use the definition of the buckling parameter k from eq. (10-57), the following expression is obtained for the critical or buckling temperature difference for a circular plate with a simply-supported edge:

$$\Delta T_{\text{cr}} = \frac{\beta^2}{12(1+\mu)\alpha}\left(\frac{h}{b}\right)^2 \tag{10-72}$$

The critical temperature difference depends on the "slenderness ratio" (*b/h*), similar to that of the circular plate with a built-in edge.

TABLE 10-2. Solutions of the Equation, $\beta J_0(\beta) - (1-\mu)J_1(\beta) = 0$, As a Function of Poisson's Ratio, μ

μ	β
0.00	1.841
0.10	1.915
0.15	1.950
0.20	1.984
0.25	2.017
0.30	2.049
$\frac{1}{3}$	2.069
0.40	2.109
0.50	2.166

Example 10-4 A circular glass window is made of Pyrex glass with the following properties: $\alpha = 3.3 \times 10^{-6}\ ^\circ\mathrm{C}^{-1}$ ($1.83 \times 10^{-6}\ ^\circ\mathrm{F}^{-1}$), $E = 62.1$ GPa (9.00×10^6 psi), $\mu = 0.24$, and ultimate strength, $S_u = 138$ MPa (20,000 psi). The window diameter is 450 mm ($b = 225$ mm $= 8.86$ in.), and the thickness is $h = 6$ mm (0.236 in.). The edge of the window is simply-supported. The stress-free temperature $T_0 = 25^\circ$C (77°F). Determine the temperature to which the window could be heated such that buckling would result.

The value of the β parameter for Poisson's ratio $\mu = 0.24$ is $\beta = 2.010$ from Table 10-2. The critical or buckling temperature difference is given by eq. (10-72):

$$\Delta T_{\mathrm{cr}} = \frac{(2.010)^2}{12(1+0.24)(3.3\times 10^{-6})}\left(\frac{6}{225}\right)^2 = 58.5^\circ\mathrm{C} \quad (105.3^\circ\mathrm{F})$$

The corresponding temperature $T_{1,\mathrm{cr}}$ that would result in buckling is

$$T_{1,\mathrm{cr}} = 25^\circ + 58.5^\circ = 83.5^\circ\mathrm{C} \quad (182.3^\circ\mathrm{F})$$

If the edge were rigidly clamped, the window could resist buckling a little more effectively. For this case, the constant corresponding to β is 3.83171 for a clamped edge. The corresponding critical temperature difference for a clamped edge is

$$\Delta T_{\mathrm{cr}} = \left(\frac{3.83171}{2.010}\right)^2 (58.5^\circ) = 212.6^\circ\mathrm{C} \quad (382.7^\circ\mathrm{F})$$

The window would buckle when heated to a temperature of

$$T_{1,\mathrm{cr}} = 25^\circ + 212.6^\circ = 237.6^\circ\mathrm{C} \quad (459.7^\circ\mathrm{F})$$

At the inception of buckling, the bending moment is small, so the main stress is the membrane stress, given by eq. (10-54):

$$\sigma_0 = \frac{N_r}{h} = -\frac{\alpha E \Delta T_1}{1-\mu} = -\frac{(3.3\times 10^{-6})(62.1\times 10^9)(58.5^\circ)}{1-0.24}$$

$$\sigma_0 = -15.774 \times 10^6\,\mathrm{Pa} = -15.774\,\mathrm{MPa} \quad (-2288\,\mathrm{psi})$$

This stress is about 11% of the ultimate strength, so the glass will not break immediately after buckling; however, as mention in Section 10.4, the bending stresses begin to increase rapidly as the temperature is increased above the critical temperature.

10.7 THERMAL BUCKLING OF RECTANGULAR PLATES

10.7.1 General Relationships

There are bending moments, ordinarily negligibly small, produced by in-plane (membrane) forces in a rectangular flat plate that are responsible for elastic

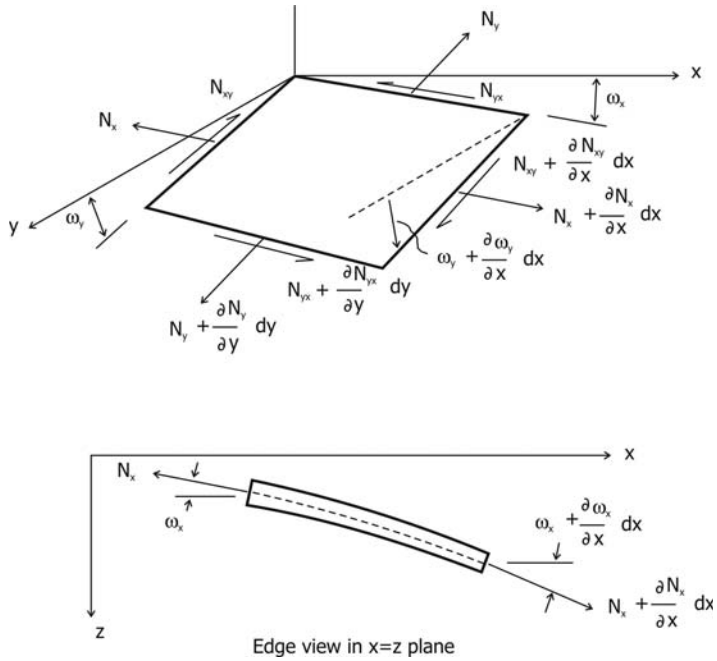

Figure 10-11. Membrane forces acting on a rectangular plate after bending.

instability under compressive loads. If we consider the plate differential element, as shown in Figure 10-11, the expressions for these forces may be developed.

The additional force components produced by the membrane forces N_x are

$$
\begin{aligned}
& -N_x\, dy \sin\omega_x + \left(N_x + \frac{\partial N_x}{\partial x} dx\right) dy \sin\left(\omega_x + \frac{\partial \omega_x}{\partial x} dx\right) \\
& = -N_x\omega_x\, dy + \left(N_x + \frac{\partial N_x}{\partial x} dx\right)\left(\omega_x + \frac{\partial \omega_x}{\partial x} dx\right) dy \qquad (10\text{-}73)
\end{aligned}
$$

In eq. (10-73), the approximation that $\sin\omega \approx \omega$ for small angles ω has been used. If the expression is expanded and the term containing higher-order differentials $(dx)^2 dy$ is neglected, the following result is obtained:

$$
\begin{aligned}
& -N_x\omega_x\, dy + \left(N_x + \frac{\partial N_x}{\partial x}\, dx\right)\left(\omega_x + \frac{\partial \omega_x}{\partial x}\, dx\right) dy \\
& = \frac{\partial N_x}{\partial x}\omega_x\, dx\, dy + \frac{\partial \omega_x}{\partial x} N_x\, dx\, dy \qquad (10\text{-}74)
\end{aligned}
$$

If we introduce the definition of the rotation,

$$
\omega_x = \frac{\partial w}{\partial x},
$$

then eq. (10-74) may be written as

$$\frac{\partial N_x}{\partial x}\omega_x\,dx\,dy + \frac{\partial \omega_x}{\partial x}N_x\,dx\,dy = \left(\frac{\partial N_x}{\partial x}\right)\left(\frac{\partial w}{\partial x}\right)dx\,dy + N_x\frac{\partial^2 \omega_x}{\partial x^2}\,dx\,dy \tag{10-75}$$

In addition to the two terms involving the x-stress resultant N_x, there are 6 additional terms involving N_y, N_{xy}, and $N_{yx} = N_{xy}$. The complete set of additional terms is as follows:

$$\begin{aligned}&\left(\frac{\partial N_x}{\partial x}\right)\left(\frac{\partial w}{\partial x}\right) + N_x\frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial N_y}{\partial y}\right)\left(\frac{\partial w}{\partial y}\right) + N_y\frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x\partial y}\\ &\quad + \left(\frac{\partial N_{xy}}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) + \left(\frac{\partial N_{xy}}{\partial y}\right)\left(\frac{\partial w}{\partial x}\right)\end{aligned} \tag{10-76}$$

The group of terms in eq. (10-76) may be simplified by using the force-balance equations, eqs. (6-85):

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad \text{and} \quad \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \tag{10-77}$$

The terms that remain after this substitution are

$$N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x\partial y} \tag{10-78}$$

The terms in eq. (10-78) are ones that should be added to the main terms in the force equilibrium relationship in the z-direction, eq. (7-20), to obtain

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p + N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x\partial y} = 0 \tag{10-79}$$

If the moment-balance relations, eqs. (7-21) are used to eliminate the shear resultants, the following is obtained:

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\partial y} + \frac{\partial^2 M_y}{\partial y^2} + p + N_x\frac{\partial^2 w_x}{\partial x^2} + N_y\left(\frac{\partial^2 w}{\partial^2 y}\right) + 2N_{xy}\frac{\partial^2 w}{\partial x\partial y} = 0 \tag{10-80}$$

The bending stress and twisting stress resultants may be written in terms of the displacements, as given in eqs. (7-19):

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right) - \frac{M_T}{(1-\mu)} \tag{10-81a}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right) - \frac{M_T}{(1-\mu)} \tag{10-81b}$$

$$M_{xy} = -(1-\mu)D\frac{\partial^2 w}{\partial x\partial y} \tag{10-81c}$$

If we make the substitutions for the stress results from eqs. (8-81) into eq. (10-80), the following governing equation for buckling of flat plates is obtained:

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)=p+N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}$$
$$+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}-\frac{1}{1-\mu}\left(\frac{\partial^2 M_T}{\partial x^2}+\frac{\partial^2 M_T}{\partial y^2}\right) \tag{10-82}$$

The governing equation may be written in more compact form using the biharmonic operator and the Laplacian operator:

$$D\nabla^4 w=p+N_x\frac{\partial^2 w}{\partial x^2}+2N_{xy}\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}-\frac{1}{1-\mu}\nabla^2 M_T \tag{10-83}$$

The force-balance equation, eq. (6-98), may be written in the following form:

$$\frac{\partial^2 N_x}{\partial y^2}+\frac{\partial^2 N_y}{\partial x^2}-\mu\left(\frac{\partial^2 N_x}{\partial x^2}+\frac{\partial^2 N_y}{\partial y^2}\right)$$
$$-2(1+\mu)\frac{\partial^2 N_{xy}}{\partial x\partial y}+\left(\frac{\partial^2 N_T}{\partial x^2}+\frac{\partial^2 N_T}{\partial y^2}\right)=0 \tag{10-84}$$

Let us define the Airy stress function $\Psi(x,\ y)$ by eqs. (6-103):

$$N_x=\frac{\partial^2\Psi}{\partial y^2} \tag{10-85a}$$

$$N_y=\frac{\partial^2\Psi}{\partial x^2} \tag{10-85b}$$

$$N_{xy}=-\frac{\partial^2\Psi}{\partial x\partial y} \tag{10-85c}$$

The force-balance equation can be written in terms of the Airy stress function:

$$\nabla^2(\nabla^2\Psi+N_T)=0 \tag{10-86}$$

10.7.2 Thermal Buckling of a Simply-supported Plate

Let us consider the case of a rectangular plate having dimensions $a\times b$, with simply-supported edges and, as shown in Figure 10-12. The plate is loaded at the edges with a uniform force per unit length N_0 in the x-direction, but is unconstrained and unloaded in the y-direction.

The plate is subjected to the following temperature distribution:

$$\Delta T(y)=T(y)-T_0=(T_1-T_0)\cos(2\pi y/b)=\Delta T_1\cos(2\pi y/b) \tag{10-87}$$

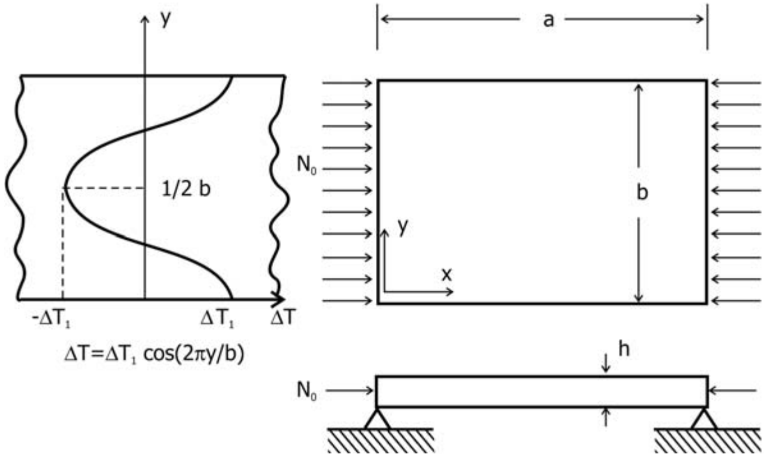

Figure 10-12. Thermal and mechanical loading on a rectangular plate. The mechanical force per unit length $N_0 = P/b$ is applied parallel to the x-axis, and the temperature change is a function of the y-coordinate only, $\Delta T = \Delta T_1 \cos(2\pi y/b)$.

The plate temperature at the upper and lower edges ($y = b$ and $y = 0$) is T_1, the plate temperature at the center ($y = \frac{1}{2}b$) is $2T_0 - T_1$, and the plate temperature at $\frac{1}{4}b$ and at $\frac{3}{4}b$ is T_0. Any arbitrary temperature distribution may be developed by expanding the function describing the temperature distribution as a Fourier cosine series. This example uses only the first term in such a series.

For the temperature distribution given by eq. (10-87), the governing relationship for the membrane (in-plane) stress resultants, eq. (10-86), reduces to

$$\nabla^4\Psi = -\nabla^2 N_T = -\left(\frac{2\pi}{b}\right)^2 \alpha E \Delta T_1 h \cos(2\pi y/b) \tag{10-88}$$

There are no forces applied in the y-direction and the plate undergoes no twisting, so the stress resultants $N_y = N_{xy} = 0$. This restriction requires that the Airy stress function be a function of the y-coordinate only. Under this condition, the solution for eq. (10-88) is

$$\Psi = \frac{1}{2}y^2 N_0 - \left(\frac{b}{2\pi}\right)^2 \alpha E \Delta T_1 h \cos(2\pi y/b) \tag{10-89}$$

The membrane stress resultant in the x-direction is found from eq. (10-85a):

$$N_x = \frac{\partial^2\Psi}{\partial y^2} = N_0 + \alpha E \Delta T_1 h \cos(2\pi y/b) = N_0 + N_T \tag{10-90}$$

For the case of $N_y = N_{xy} = M_T = p = 0$, eq. (10-83) reduces to

$$D\nabla^4 w = N_x \frac{\partial^2 w}{\partial x^2} \tag{10-91a}$$

We may expand eq. (10-91a) as follows:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D}(\alpha E \Delta T_1 h + N_0)\frac{\partial^2 w}{\partial x^2} \tag{10-91b}$$

The plate is simply-supported at the edges, so the boundary conditions are

$$w(0, y) = w(a, y) = 0 \quad \text{and} \quad w(x, 0) = w(x, b) = 0$$

$$\frac{\partial^2 w(0, y)}{\partial x^2} = \frac{\partial^2 w(a, y)}{\partial x^2} = 0 \quad \text{and} \quad \frac{\partial^2 w(x, 0)}{\partial y^2} = \frac{\partial^2 w(x, b)}{\partial y^2} = 0$$

Let us try a solution of the following form:

$$w(x, y) = Y(y)\sin(m\pi x/a) \qquad (m = 1, 2, 3, \ldots) \tag{10-92}$$

This solution form automatically satisfies all four boundary condition on the x-coordinate. If we make the substitution from eq. (10-92) into eq. (10-91), the following differential equation is obtained for the function $Y(y)$:

$$\frac{d^4 Y}{dy^4} - 2\left(\frac{m\pi}{a}\right)^2 \frac{d^2 Y}{dy^2} + \left(\frac{m\pi}{a}\right)^4 Y - \frac{1}{D}[K_T \cos(2\pi y/b) + N_0]\left(\frac{m\pi}{a}\right)^2 Y = 0 \tag{10-93}$$

The quantity K_T is defined as

$$K_T = \alpha E \Delta T_1 h \tag{10-94}$$

For the function $Y(y)$, let us try a Fourier sine series, which satisfies the boundary conditions on the coordinate y term-by-term:

$$Y(y) = \sum_{n=1}^{\infty} C_n \sin(n\pi y/b) \tag{10-95}$$

If we make the substitution from eq. (10-95) into eq. (10-93), we get

$$\sum_{n=1}^{\infty} C_n \left[\left(\frac{m\pi}{a}\right)^2 (K_n - N_0)\sin(n\pi y/b) - K_T\left(\frac{m\pi}{a}\right)^2 \cos(2\pi y/b)\sin(n\pi y/b)\right] = 0 \tag{10-96}$$

The quantity K_n is defined as

$$K_n = D\left(\frac{m\pi}{a}\right)^2\left[1 + \left(\frac{na}{mb}\right)^2\right]^2 \qquad (n = 1, 2, 3, \ldots) \tag{10-97}$$

The last term in eq. (10-96) may be somewhat simplified by using the trig identity,

$$\cos(2\pi y/b)\sin(n\pi y/b) = \frac{1}{2}\left[\sin\frac{(n+2)\pi y}{b} + \sin\frac{(n-2)\pi y}{b}\right] \tag{10-98}$$

For eq. (10-96) to be satisfied in general, the coefficients on each of the $\sin(n\pi y/a)$ terms must be equal to zero. The result is

For $n = 1$:

$$C_1(K_1 - N_0) + \tfrac{1}{2}C_1K_T - \tfrac{1}{2}C_3K_T = 0 \tag{10-99a}$$

For $n = 2$:

$$C_2(K_2 - N_0) - \tfrac{1}{2}C_4K_T = 0 \tag{10-99b}$$

For $n = 3, 4, 5, \ldots$:

$$C_n(K_n - N_0) - \tfrac{1}{2}(C_{n-2} + C_{n+2})K_T = 0 \qquad (n \geq 3) \tag{10-99c}$$

The critical or buckling compressive load per unit width N_{cr} and the critical temperature difference $\Delta T_{cr} = (T_{1,cr} - T_0)$ are found by equating the determinant of the coefficients to zero. It is observed that the expressions involving even values of $n(n = 2, 4, 6, \ldots)$ are independent of the equations involving odd values of $n(n = 1, 3, 5, \ldots)$. This characteristic has two consequences. First, only a single value of m is needed to satisfy eq. (10-92), and this value is the one that yields the lowest critical combination of mechanical load and thermal load. Second, two independent determinants may be set up for the solution, one involving only even values of n and the other involving only odd values of n.

From a physical viewpoint, the coefficients involving *odd* values of n correspond to *symmetric* buckling configurations, whereas the coefficients involving *even* values of n correspond to *antisymmetric* buckling configurations. These configurations are illustrated in Figure 10-13 for $n = 1$ and for $n = 2$. Calculations have shown [Boley and Weiner, 1960] that the symmetric configuration corresponds to the lower buckling load, so only the symmetric solution will be considered.

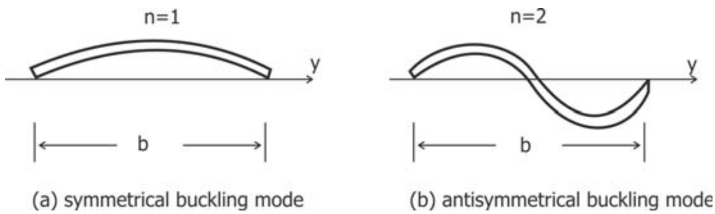

Figure 10-13. Buckled configuration of the plate in the y-direction. (a) The symmetric buckling configuration corresponds to odd integers $n = 1, 3, 5, \ldots$ and (b) the antisymmectric buckling configuration corresponds to even integers $n = 2, 4, 6, \ldots$. The cases for $n = 1$ and $n = 2$ are illustrated.

The determinant of the coefficients of the constants with odd subscripts (the symmetrical buckling mode) is

$$\begin{vmatrix} (K_1 - N_0 + \frac{1}{2}K_T) & -\frac{1}{2}K_T & 0 & 0 & \cdots \\ -\frac{1}{2}K_T & (K_3 - N_0) & -\frac{1}{2}K_T & 0 & \cdots \\ 0 & -\frac{1}{2}K_T & (K_5 - N_0) & -\frac{1}{2}K_T & \cdots \\ 0 & 0 & -\frac{1}{2}K_T & (K_7 - N_0) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 \tag{10-100}$$

Before we consider the combined condition, let us examine the case of only mechanical loading and only thermal loading.

10.7.2.1 Mechanical loading only ($K_T = 0$). For this case, all of the terms in the determinant are zero, except for the ones on the diagonal. The value of the determinant is

$$(K_1 - N_0)(K_3 - N_0)(K_5 - N_0)\cdots = 0 \tag{10-101}$$

As noted from eq. (10-97), the smallest value of K_n corresponds to $n = 1$, and the critical external mechanical load N_{cr}^* for zero temperature change is

$$N_{\mathrm{cr}}^* = K_1 = k_0 \frac{\pi^2 Eh}{12(1-\mu^2)}\left(\frac{h}{b}\right)^2 = \frac{P_{\mathrm{cr}}}{b} \tag{10-102}$$

The quantity k_0 is

$$k_0 = \left(\frac{bm}{a} + \frac{a}{bm}\right)^2 \tag{10-103}$$

This function is plotted in Figure 10-14.

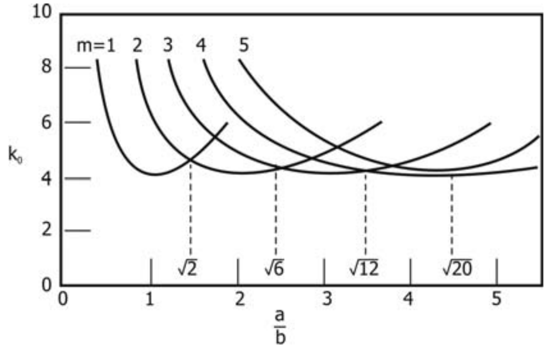

Figure 10-14. Critical load factor k_0 for mechanical loading only ($K_T = 0$).

For any given value of the index m, the minimum value for k_0 is 4, as observed from Figure 10-14. A value of $m = 1$ yields the smallest value of k_0 up to an aspect ratio $(a/b) = \sqrt{2}$. For $(\sqrt{2})$ between $\sqrt{a/b}$ and $\sqrt{6}$, $m = 2$ corresponds to the smaller value of k_0, etc. We observe also that between $(a/b) = 1$ and $(a/b) = 2$, the constant k_0 differs from $k_0 = 4$ by 12.5 percent or less. Between $(a/b) = 2$ and $(a/b) = 3$, the constant k_0 differs from $k_0 = 4$ by 4.17 percent or less, and the deviation is even smaller as the aspect ratio is increased further.

Based on these observations, for (conservative) engineering calculations, the expression for the coefficient k_0 may be simplified as follows:

$$k_0 = \begin{cases} \left(\dfrac{b}{a} + \dfrac{a}{b}\right)^2 & \text{for } (a/b) \le 1 \\ 4 & \text{for } (a/b) > 1 \end{cases} \tag{10-104}$$

10.7.2.2 Thermal loading only ($N_0 = 0$). The second case ($N_0 = 0$) is somewhat more complicated. The determinant for zero mechanical loading is

$$\begin{vmatrix} (K_1 + \frac{1}{2}K_T) & -\frac{1}{2}K_T & 0 & 0 & \dots \\ -\frac{1}{2}K_T & K_3 & -\frac{1}{2}K_T & 0 & \dots \\ 0 & -\frac{1}{2}K_T & K_5 & -\frac{1}{2}K_T & \dots \\ 0 & 0 & -\frac{1}{2}K_T & K_7 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = 0 \tag{10-105}$$

Again, a large number of calculations [Boley and Weiner, 1960] have shown that the solution using the first two rows and first two columns of the determinant, eq. (10-105), yield results for the thermal buckling load that are very good approximations for the solutions using 3 or more rows and columns. Let us approximate the infinite determinant by using only the first 2 rows and columns:

$$\begin{vmatrix} (K_1 + \frac{1}{2}K_T) & -\frac{1}{2}K_T \\ -\frac{1}{2}K_T & K_3 \end{vmatrix} = 0 \tag{10-106}$$

If the determinant eq. (10-106) is expanded, the following is obtained:

$$(K_1 + \tfrac{1}{2}K_T)K_3 - \tfrac{1}{4}K_T^2 = 0 \tag{10-107}$$

Solving for the critical thermal parameter $K_{T,\text{cr}}^*$ for zero mechanical load, the following is obtained from eq. (10-107):

$$K_{T,\text{cr}}^* = \frac{4K_1K_3}{1 - 2K_3} = \alpha E h \Delta T_{1,\text{cr}}^* \tag{10-108}$$

If the values for K_1 and K_3 are used from eq. (10-97), the following expression is obtained for the critical or buckling thermal load in the absence of mechanical loading ($N_0 = 0$):

$$\Delta T^*_{1,\mathrm{cr}} = -k_1 \frac{\pi^2}{6(1-\mu^2)\alpha}\left(\frac{h}{b}\right)^2 \tag{10-109}$$

The expression for the coefficient k_1 is

$$k_1 = \frac{1}{2}\left\{\sqrt{\left[\left(\frac{mb}{a}\right)^2 + 9\right]^4 + 4\left[\left(\frac{mb}{a}\right)^2 + 1\right]^2\left[\left(\frac{mb}{a}\right)^2 + 9\right]^2} - \left[\left(\frac{mb}{a}\right)^2 + 9\right]^2\right\}\left(\frac{a}{mb}\right)^2 \tag{10-110}$$

A plot of the thermal buckling parameter k_1 is shown in Figure 10-15.

For any value of m, the minimum in the curves for the coefficient is $k_1 = 3.848$. As was the case for mechanical loading only, the deviation of the smallest value of k_1 for $m = 2, 3, \ldots$ from the value of 3.848 is small (and on the conservative or "safe" side). For engineering design purposes, we may write the expression for the coefficient as

$$k_1 = \frac{1}{2}\left\{\sqrt{\left[\left(\frac{b}{a}\right)^2 + 9\right]^4 + 4\left[\left(\frac{b}{a}\right)^2 + 1\right]^2\left[\left(\frac{b}{a}\right)^2 + 9\right]^2} - \left[\left(\frac{b}{a}\right)^2 + 9\right]^2\right\}\left(\frac{a}{b}\right)^2 \quad \text{for } (a/b) \le 1$$

$$k_1 = 3.848 \quad \text{for } (a/b) \ge 1 \tag{10-111}$$

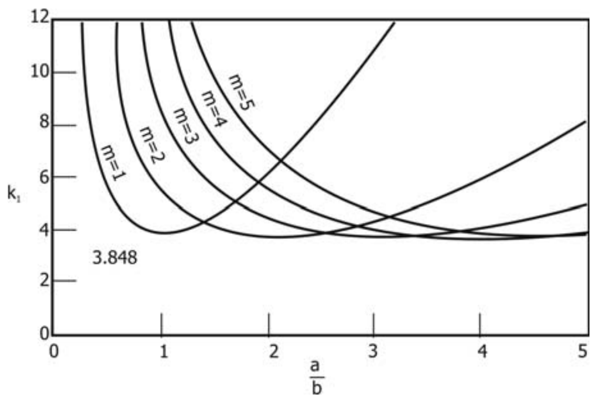

Figure 10-15. Critical load factor k_1 for thermal loading only ($N_0 = 0$).

10.7.2.3 Both mechanical and thermal loading. The general case in which both mechanical and thermal loads are present may be analyzed in a manner similar to the previous cases by using the first two rows and first two columns of the determinant for a satisfactory solution. The calculations are somewhat lengthy, and the resulting expression is quite algebraically complex. An excellent approximation for this case is as follows [Boley and Weiner, 1960]:

$$\frac{N_{\text{cr}}}{N_{\text{cr}}^*} + \frac{\Delta T_{1.\text{cr}}}{\Delta T_{1,\text{cr}}^*} = 1 \tag{10-112}$$

This expression is accurate within 5% for the following range of variables:

$$-1 \leq \frac{N_{\text{cr}}}{N_{\text{cr}}^*} \leq +2 \quad \text{and} \quad -1 \leq \frac{\Delta T_{1,\text{cr}}}{\Delta T_{1,\text{cr}}^*} \leq +2 \tag{10-113}$$

The expression, eq. (10-112), may be written as a “design” equation by including a factor of safety for the mechanical loading f_{mech} and a factor of safety for the thermal loading f_{th}, defined as follows:

$$\frac{P}{b} = \frac{N_{\text{cr}}}{f_{\text{mech}}} \quad \text{and} \quad \Delta T_1 = \frac{\Delta T_{1,\text{cr}}}{f_{\text{th}}} \tag{10-114}$$

The quantity P is the applied (design) mechanical force on the edge of the plate, and $\Delta T_1 = (T_1 - T_0)$ is the design temperature change parameter for the plate. If the substitutions are made from eq. (10-114) into eq. (10-112), the following design relationship is obtained:

$$\frac{f_{\text{mech}} P}{b N_{\text{cr}}^*} + \frac{f_{\text{th}} \Delta T_1}{\Delta T_{1,\text{cr}}^*} = 1 \tag{10-115}$$

If eqs. (10-102) and (10-109) are used for the terms N_{cr}^* and $\Delta T_{1,\text{cr}}^*$, the final design equation is obtained:

$$\frac{P}{bh} - \frac{k_0 f_{\text{th}} \alpha E \Delta T_1}{2 k_1 f_{\text{mech}}} = \frac{\pi^2 k_0 E}{12(1-\mu^2) f_{\text{mech}}} \left(\frac{h}{b}\right)^2 \tag{10-116}$$

Equation (10-116) applies only if the failure mode is elastic instability or buckling. If the failure mode is yielding of the material, the design relationship is

$$\sigma_x = \frac{N_x}{h} = \frac{S_y}{f_{\text{mech}}} \tag{10-117a}$$

Using eq. (10-90) at $y = 0$, eq. (10-117a) may be written in the final form:

$$\frac{P}{bh} + \alpha E \left|\Delta T_1\right| = \frac{S_y}{f_{\text{mech}}} \tag{10-117b}$$

10.7.3 Thermal Buckling for Other Edge Conditions

The expression for the critical thermal buckling load for rectangular plates, as shown in Figure 10-16, with the edges ($x = 0$ and $x = a$) simply supported, but constrained in the x-direction, and various boundary conditions for the other two edges is given by

$$\Delta T_{cr} = T_{cr} - T_0 = k_1 \frac{\pi^2}{12(1-\mu^2)\alpha} \left(\frac{h}{b}\right)^2 \quad (10\text{-}118)$$

The values for the coefficient k_1 are given in Table 10-3.

Case A involves one edge simply-supported and the other edge free. This condition approximates the condition in the legs of an angle under longitudinal compression. When the thermal load reaches the critical temperature difference for an axially constrained angle, the free edges buckle, but the common edge remains straight, as would be the case for a simply-supported edge. For long angles, one must also check the buckling mode for the angle as a column, as given by eq. (10-7) or (10-8) in Section 10-2.

Case B involves one edge built-in and the other edge free. This condition corresponds to a plate with one edge welded or rigidly attached to a very stiff support structure and constrained to move parallel to this edge.

Case C involves both edges simply-supported. This condition corresponds to the side panels of a thin tube of square cross section that is uniformly heated, with the ends of the tube constrained.

Example 10-5 A plate made of 6061-T6 aluminum has the following properties: $\alpha = 23.4 \times 10^{-6}{}^\circ\text{C}^{-1}$ ($13.0 \times 10^{-6}{}^\circ\text{F}^{-1}$), $E = 69$ GPa (10.0×10^6 psi), $\mu = 0.30$, and $S_y = 275$ MPa (39,900 psi). The plate is 750 mm long (29.53 in.) and 250 mm wide (9.84 in.). The plate is loaded with a 25-kN

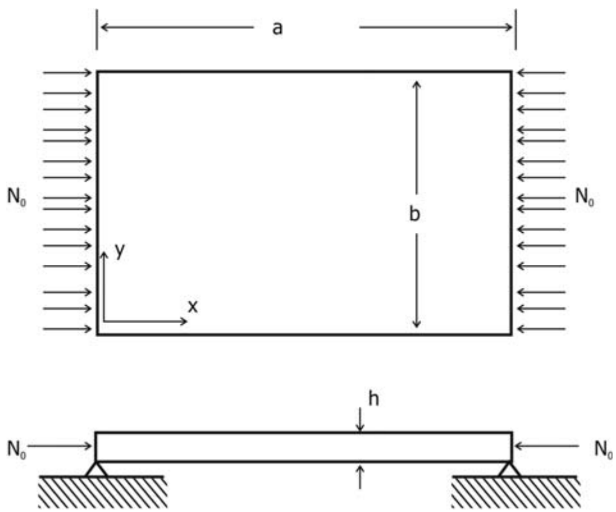

Figure 10-16. Thermal buckling of plates with simply-supported edges.

TABLE 10-3. Values for the Coefficient k_1 in eq. (10-118) for Buckling of Constrained Rectangular Plates with Various Edge Conditions, As Shown in Figure 10-16, for a Uniform Temperature Change, $\Delta T_{cr} = T_{cr} - T_0$

$$\Delta T_{cr} = k_1 \frac{\pi^2}{12(1-\mu^2)\alpha}\left(\frac{h}{b}\right)^2$$

a/b	CASE A	CASE B	CASE C
0.5	4.400		
Less than 1			$k_1 = \left(\frac{b}{a} + \frac{a}{b}\right)^2$
1.0	1.440	1.700	4.000
1.2	1.135	1.467	4.134
1.4	0.952	1.360	4.470
1.6	0.835	1.328	4.203
1.8	0.755	1.337	4.045
2.0	0.698	1.375	4.000
2.5	0.610	1.432	4.134
3.0	0.564	1.335	4.000
4.0	0.516	1.328	4.000
5.0	0.506	1.328	4.000
Greater than 5	$0.456 + \frac{1}{(a/b)^2}$	1.328	4.000

CASE A. One side ($y = 0$) simply-supported; the other side ($y = b$) free.
CASE B. One side ($y = 0$) built-in; the other side ($y = b$) free.
CASE C. Both edges ($y = 0$; $y = b$) simply-supported.

(5620-lb$_f$) compressive load distributed along the width. The temperature parameter $T_1 = -45^\circ$C (-49°F), and the stress-free temperature of the plate $T_0 = +25^\circ$C (77°F). The factor of safety is $f_s = f_{mech} = f_{th} = 2.0$. Determine the required plate thickness to avoid buckling of the plate.

The aspect ratio for the plate is

$$\frac{a}{b} = \frac{750}{250} = 3.00 > 1$$

In this case, we may take the values of the coefficients as $k_0 = 4$ and $k_1 = 3.848$. The temperature difference parameter is

$$\Delta T_1 = T_1 - T_0 = -45^\circ - 25^\circ = -70^\circ\text{C} \quad (-126^\circ\text{F})$$

Using the design relationship, eq. (10-116), the following is obtained:

$$\frac{25 \times 10^3}{(0.250)h} - \frac{(4)(23.4 \times 10^{-6})(69 \times 10^9)(-70^\circ)}{(2)(3.848)}$$
$$= \frac{\pi^2(4)(69 \times 10^9)}{(12)(1 - 0.30^2)(2.0)}\left(\frac{h}{0.250}\right)^2$$

Simplifying, we find

$$\frac{1}{h} + 587.43 = (1.9956 \times 10^7)h^2$$

Solving for the design plate thickness, the following value is obtained:

$$h = 0.00613 \text{ m} = 6.13 \text{ mm} \quad (0.241 \text{ in.})$$

Let us check the stress level for this thickness dimension:

$$(\sigma_x)_{\max} = \frac{P}{bh} + \alpha E\,|\Delta T_1| = \frac{25{,}000}{(0.250)(0.00613)} + (23.4 \times 10^{-6})(69 \times 10^9)(70^\circ)$$

$$(\sigma_x)_{\max} = (16.31 + 113.02)(10^6) = 129.33 \times 10^6 \text{ Pa}$$

$$= 129.33\,\text{MPa (18,760 psi)}$$

The allowable stress is

$$S_a = \frac{S_y}{f_{\text{mech}}} = \frac{275}{2.0} = 137.5 \text{ MPa}$$

Because the maximum stress is less than the allowable stress, the critical failure mode is buckling in this example.

Let us determine the design thickness of the plate to prevent buckling if there were no thermal load, $\Delta T_1 = 0$. In this case, the thickness can be solved directly from eq. (10-102):

$$h = \left[\frac{12(1-\mu^2)Pbf_{\text{mech}}}{\pi^2 k_0 E}\right]^{1/3}$$

$$h = \left[\frac{12(1-0.30^2)(25 \times 10^3)(0.250)(2.0)}{\pi^2(4)(69 \times 10^9)}\right]^{1/3} = 0.00369 \text{ m} = 3.69 \text{ mm}$$

The presence of the temperature variation results in a required plate thickness that is $(6.13/3.69) = 1.66$ times larger than the thickness required for the isothermal plate.

10.8 THERMAL BUCKLING OF CYLINDRICAL SHELLS

Thermal buckling of cylindrical shells due to a thermal moment usually does not occur unless certain types of external constraints are applied [Johns, 1965]. For this reason, only thermal buckling due to membrane stresses is considered in this section.

Cylindrical shells having an axial temperature change may be subject to buckling in various modes, including axial buckling and circumferential buckling. Let us consider the case of a circular cylindrical shell that is constrained at the ends such that no axial displacement occurs and has a uniform temperature

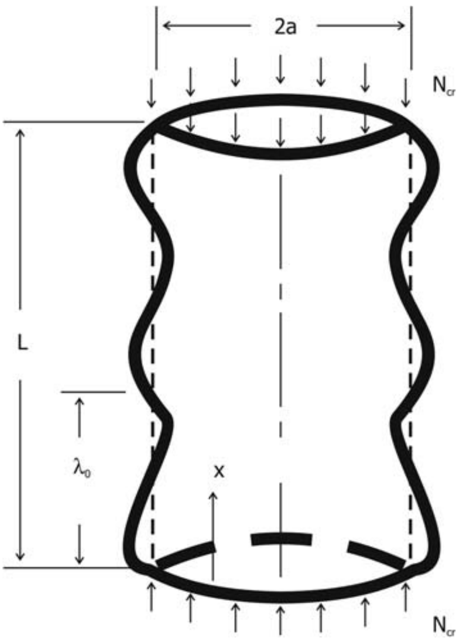

Figure 10-17. Thermal buckling of a cylinder with axial constraint.

change $\Delta T_1 = T_1 - T_0$ imposed on the shell. In this case, the shell may buckle in the configuration shown in Figure 10-17, in which the relationship for the radial displacement during buckling has the following form for simply-supported ends:

$$w = C_1 \sin(m\pi x/L) = C_1 \sin(\pi x/\lambda_b) \qquad (m = 1, 2, 3, \ldots) \tag{10-119}$$

The integer m is the number of buckled segments, and $\lambda_b = L/m$ is the wavelength or length of each buckled segment.

In determining the critical or buckling thermal load, let us measure the transverse deflection from the position of the midplane surface after the axial thermal load has been applied, instead of measuring from the original unstressed position. For this condition, the axial stress resultant and the membrane stress resultant are related from eq. (8-11a) with $u = 0$ as follows:

$$N_x = -\frac{N_T}{1-\mu} = -\frac{\alpha E \Delta T_1 h}{1-\mu} \tag{10-120}$$

The additional term responsible for elastic instability that must be added to eq. (8-17) is

$$N_x \frac{d\omega_x}{dx} = N_x \frac{d^2 w}{dx^2} \tag{10-121}$$

With this addition and setting $M_T = p_r = 0$ in eq. (8-17), the governing equation for buckling of the cylindrical shell is

$$\frac{d^4w}{dx^4} + \frac{N_x}{D}\frac{d^2w}{dx^2} + \frac{Eh}{a^2D}w = 0 \tag{10-122}$$

The axial stress resultant may be eliminated by using eq. (10-120):

$$\frac{d^4w}{dx^4} - \frac{N_T}{(1-\mu)D}\frac{d^2w}{dx^2} + \frac{Eh}{a^2D}w = 0 \tag{10-123}$$

The following result is obtained by substituting the expression from eq. (10-119) into eq. (10-123):

$$C_1 \sin(m\pi x/L)\left[\left(\frac{m\pi}{L}\right)^4 - \frac{N_T}{(1-\mu)D}\left(\frac{m\pi}{L}\right)^2 + \frac{Eh}{a^2D}\right] = 0 \tag{10-124}$$

Equation (10-124) could be satisfied by taking $C_1 = 0$ or $m = 0$. These choices correspond to the unbuckled configuration, $w = 0$. The buckled configuration corresponds to the case in which the bracketed term is equal to zero. In this case, the critical thermal stress resultant is

$$N_{T,\mathrm{cr}} = (1-\mu)D\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{Eh}{a^2D}\right)\left(\frac{L}{m\pi}\right)^2\right] \tag{10-125}$$

The minimum value of the term in brackets in eq. (10-125) is

$$\frac{m\pi}{L} = \frac{\pi}{\lambda_b} = \left(\frac{Eh}{a^2D}\right)^{1/4} = \left[\frac{12(1-\mu^2)}{a^2h^2}\right]^{1/4} \tag{10-126}$$

The wavelength of the buckled segment is

$$\lambda_b = \frac{L}{m} = \frac{\pi\sqrt{ah}}{[12(1-\mu^2)]^{1/4}} \tag{10-127}$$

Making the substitution from eq. (10-126) into eq. (10-125) yields the following expression for the minimum critical thermal stress resultant:

$$N_{T,\mathrm{cr}} = 2(1-\mu)D\left(\frac{Eh}{a^2D}\right)^{1/2} = \left[\frac{1-\mu}{3(1+\mu)}\right]^{1/2}\left(\frac{Eh^2}{a}\right) = \alpha E\,\Delta T_{\mathrm{cr}}h \tag{10-128}$$

The critical or buckling temperature change is

$$\Delta T_{\mathrm{cr}} = \left[\frac{1-\mu}{3(1+\mu)}\right]^{1/2}\left(\frac{h}{a}\right)\frac{1}{\alpha} \tag{10-129}$$

Example 10-6. A thin-walled tube is constructed of cupro-nickel (70% Cu/30% Ni) with the following properties: $\alpha = 16.2 \times 10^{-6}\ {}^\circ\mathrm{C}^{-1}$ ($9.0 \times 10^{-6}\,{}^\circ\mathrm{F}^{-1}$), $E = 150$ GPa (21.8×10^6 psi), $\mu = 0.33$, and $S_y = 480$ MPa (69,600 psi). The tube has a length of 914.4 mm (36 in), an OD of 609.6 mm (24.0 in.), and is rigidly fixed at both ends. The tube is subjected to a temperature change from 25°C (77°F) to 60°C (140°F). A factor of safety for thermal buckling $f_{\mathrm{th}} = 2.0$ and a factor of safety for mechanical stresses $f_{\mathrm{mech}} = 3.0$ are to be used. Determine the required thickness h of the tube.

The design buckling temperature difference is

$$\Delta T_{1,\mathrm{cr}} = f_{\mathrm{th}} \Delta T_1 = (2.0)(60^\circ - 25^\circ) = 70^\circ\mathrm{C} \qquad (126^\circ\mathrm{F})$$

The mean tube radius a is related to the tube OD d_0 and wall thickness h as follows:

$$a = \tfrac{1}{2}(d_0 - h)$$

Equation (10-129) may be written as

$$\Delta T_{\mathrm{cr}} = \left[\frac{1-\mu}{3(1+\mu)}\right]^{1/2} \left[\frac{2}{(d_0/h) - 1}\right] \frac{1}{\alpha}$$

Solving for the diameter-to-thickness ratio, we get

$$\frac{d_0}{h} = 1 + \frac{2}{\alpha \Delta T_{1,\mathrm{cr}}} \left[\frac{1-\mu}{3(1+\mu)}\right]^{1/2} = 1 + \frac{2}{(16.2 \times 10^{-6})(70^\circ)} \left[\frac{1-0.33}{(3)(1+0.33)}\right]^{1/2}$$

$$\frac{d_0}{h} = 1 + 722.7 = 723.7$$

The minimum design thickness of the tube to avoid buckling is

$$h = \frac{609.6}{723.7} = 0.844 \text{ mm} \qquad (0.033 \text{ in.})$$

The next larger standard thickness for the tubing is 20 gauge, which corresponds to a thickness $h = 0.89$ mm $= 0.035$ in.

The allowable compressive stress for the tube is

$$S_a = \frac{S_y}{f_{\mathrm{mech}}} = \frac{480}{3.0} = 160 \text{ MPa} \qquad (23{,}200 \text{ psi})$$

The axial stress in the tube is found from eq. (10-120):

$$\sigma_x = \frac{N_x}{h} = -\frac{\alpha E \Delta T_1}{1-\mu} = -\frac{(16.2 \times 10^{-6})(150 \times 10^9)(35^\circ)}{1-0.33}$$

$$\sigma_x = -126.94 \times 10^6 \text{ Pa} = -126.94 \text{ MPa} \quad (18{,}400 \text{ psi})$$

This stress level is less than the allowable stress; therefore, the tube will not fail by yielding.

The wavelength of the buckled segments (if buckling were to occur) in this example may be found from eq. (10-127). The mean radius of the tube is

$$a = \tfrac{1}{2}(609.6 - 0.89) = 304.4 \text{ mm} \quad (11.98 \text{ in.})$$

$$\lambda_b = \frac{\pi\sqrt{(0.3044)(0.00089)}}{[12(1 - 0.33^2)]^{1/4}} = 0.0286 \text{ m} = 28.6 \text{ mm} \quad (1.13 \text{ in.})$$

There would be space along the length of the tube for as many as 31 "wrinkles" along the tube length after the tube had buckled.

10.9 HISTORICAL NOTE

Leonard Euler (Figure 10-18) published his analysis of column buckling, which is now called *Euler buckling analysis*, in 1757 when he was 50 years old [Timoshenko, 1983]. In addition to his work in solid mechanics, Euler was a pioneering Swiss mathematician and physicist who made contributions in fluid dynamics, optics, and astronomy. At the University of Basel, his doctoral advisor was Johann Bernoulli, who was one of his father's friends, and his doctoral dissertation, entitled *De Sono*, dealt with the propagation of sound.

In 1727, Euler accepted a position in the mathematics/physics department at the Imperial Russian Academy of Sciences in St. Petersburg, upon the recommendation of Daniel Bernoulli, one of Johann Bernoulli's sons. While he was

Figure 10-18. Leonard Euler.

at the Russian Academy, he wrote his famous book on mechanics, *Mechanica Sive Motus Scientia Analysice Exposita* (1736). In this book, he used analytical methods (differential equations) for problem solution, in contrast to Newton, who preferred geometrical methods.

In 1740 Frederick the Great became King of Prussia, and he wanted to have the best scientists at the Prussian Academy. Frederick invited Euler to become a member of the Academy. Because of unrest in Russia at the time, Euler decided to move to the Berlin Academy in 1741. During this time, he wrote two mathematics (calculus) books, *Introducto in Analysin Infinitorium* (1748) and *Institutiones Calculi Differentialis* (1775), and was asked to tutor the niece of Frederick the Great of Prussia. Euler lived in Berlin for about 25 years.

The political climate in Russia improved in its treatment of scientific research and technology when Catherine II became empress of Russia in 1762. She made an offer that Euler couldn't refuse, and he returned to St. Petersburg in 1766, where he worked until his death in 1783. During the 17 years after his return to Russia, Euler wrote more than 400 scientific papers, despite his almost complete blindness due to cataract formation. He gave complete explanations of his problems and methods of solution to his assistants, who completed the work, discussed the final results with Euler, and wrote the papers for Euler's final approval.

Euler introduced several mathematical notations that we accept and take for granted today. He was the first to use the function notation $f(x)$ to describe a function of the variable x. He also introduced the present-day notation for the trigonometric functions ($\sin x$, $\cos x$, etc.), the letter e for the base of the natural logarithms, the Greek letter Σ for summations, and the letter i for the imaginary unit $\sqrt{-1}$. He was also an originator of an area in mathematics, called the *calculus of variations*, including the Euler-Lagrange equation.

Euler also solved the famous mathematical problem, called the *Seven Bridges of Königsberg*, in 1736. The city of Königsberg, Prussia, is located on the Pregel River and has two large islands that are connected to each other and to the mainland by seven bridges. The problem was to determine whether it was possible to follow a path that crossed each of the seven bridges only once and afterwards return to the starting point. Euler showed that it was not possible, and his solution is often considered to be the first theorem of the present-day mathematical area, called *graph theory*.

PROBLEMS

10-1. A 6061-T6 aluminum column has both ends pinned. The properties of the aluminum are as follows: $\alpha = 23.4 \times 10^{-6}\ {}^\circ C^{-1}$ ($13.0 \times 10^{-6}\ {}^\circ F^{-1}$), $E = 68.9$ GPa (10.0×10^6 psi), $\mu = 0.30$, and $S_y = 275$ MPa (40,000 psi). The column has a length of 1.150 m (3.773 ft) and a rectangular cross section, 50 mm by 75 mm (1.969 in. $\times$ 2.953in.). If the column is stress-free at $T_0 = +26^\circ C$, determine the temperature to

which the column must be heated to cause buckling, if the temperature is uniform in the column. Determine the mechanical stress in the column at this condition.

10-2. For the column given in Problem 10-1, determine the gap width required at one end such that the critical or buckling temperature increase is $\Delta T_{cr} = 100^\circ C$ (180°F).

10-3. A column is constructed by welding two channels together to form a box cross section. The area properties of the column are given in Table 10-4. The column is made of C1020 carbon steel, with the following properties: $\alpha = 12.0 \times 10^{-6\circ}C^{-1}$ $(6.67 \times 10^{-6\circ}F^{-1})$, $E = 205$ GPa $(29.7 \times 10^6$ psi), $\mu = 0.28$, and $S_y = 324$ MPa (47,000 psi). One end of the column is built-in and the other is pinned, and the length of the column is 8.25 m (27.07 ft). Determine the size of the standard channel box section that is required such that the critical or buckling temperature T_{cr} is equal to or less than 95°C (203°F), if the stress-free temperature is $T_0 = 25^\circ C$ (77°F).

10-4. A column having a rectangular cross section $h \times b$, with $h < b$, has one end rigidly fixed and the other end pinned, as shown in Figure 10-19. The column is subjected to the following temperature distribution:

$$\Delta T = T - T_0 = \Delta T_1 + \Delta T_2 \left(\frac{2y}{h}\right)$$

TABLE 10-4. Properties of a Cross Section Made Up of Two Channels Welded Together to Form a Box Section

Channel in. × in × lb/ft mm × mm × kg/m	Area, A cm^2	Area, A in^2	Min. I cm^4	Min. I in^4	r_g mm	r_g in
$3 \times 1\frac{7}{8} \times 7.1$	26.8	4.16	338	8.11	35.5	1.396
$80 \times 48 \times 10.6$						
$4 \times 2\frac{1}{2} \times 13.8$	51.6	8.00	1079	25.92	45.7	1.800
$100 \times 65 \times 20.5$						
$6 \times 3 \times 15.1$	56.4	8.74	1709	41.05	55.0	2.167
$150 \times 80 \times 22.5$						
$7 \times 3\frac{1}{2} \times 19.1$	71.0	11.00	3031	72.83	65.2	2.567
$180 \times 90 \times 28.4$						
$8 \times 3 \times 18.7$	70.1	10.86	2350	56.47	57.9	2.280
$200 \times 80 \times 27.8$						
$10 \times 3\frac{1}{2} \times 21.9$	82.3	12.76	3823	91.85	68.1	2.683
$250 \times 90 \times 35.6$						
$10 \times 4 \times 28.5$	107.1	16.60	5417	130.14	71.1	2.800
$250 \times 100 \times 42.4$						

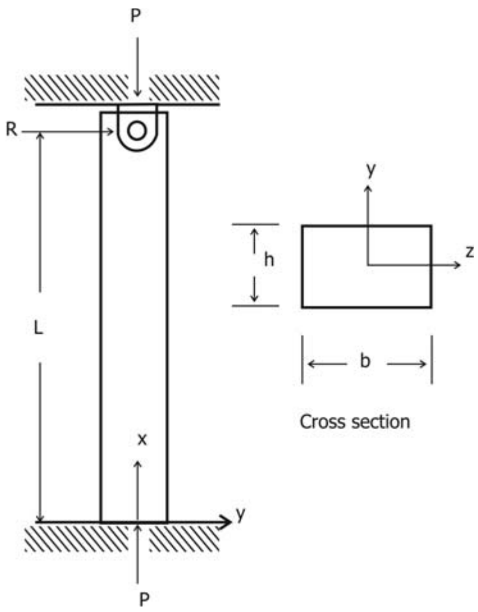

Figure 10-19. Pinned-end column with transverse mechanical force R, Problem 10-4.

The boundary conditions are

$$v(0) = v(L) = \frac{dv(0)}{dx} = 0$$

Note that there will be an axial mechanical forced developed, $P = \alpha E A \Delta T_1$, and a transverse reaction R at the upper support point. The mechanical moment at any point is

$$M = Pv - R(L - x)$$

It should be found that the deflection becomes infinite when the following is valid:

$$\frac{\sin kL}{kL} - \cos kL = 0 \quad \text{or} \quad \tan kL = kL$$

where $k^2 = \dfrac{P}{EI} = \dfrac{\alpha A \Delta T_1}{I}$

Determine the expression for the critical or buckling temperature change $\Delta T_{1,\mathrm{cr}}$.

If the column is constructed of steel with $\alpha = 16 \times 10^{-6}{}^\circ\mathrm{C}^{-1}$ ($8.89 \times 10^{-6}{}^\circ\mathrm{F}^{-1}$), $E = 200\,\mathrm{GPa}$ (29.0×10^6 psi), and $\mu = 0.28$, determine the numerical value of the critical temperature change $\Delta T_{1,\mathrm{cr}}$. The dimensions of the column are: depth $h = 90\,\mathrm{mm}$ (3.543 in.), width $b = 120\,\mathrm{mm}$ (4.724 in.), and length $L = 2.00\,\mathrm{m}$ (78.74 in.).

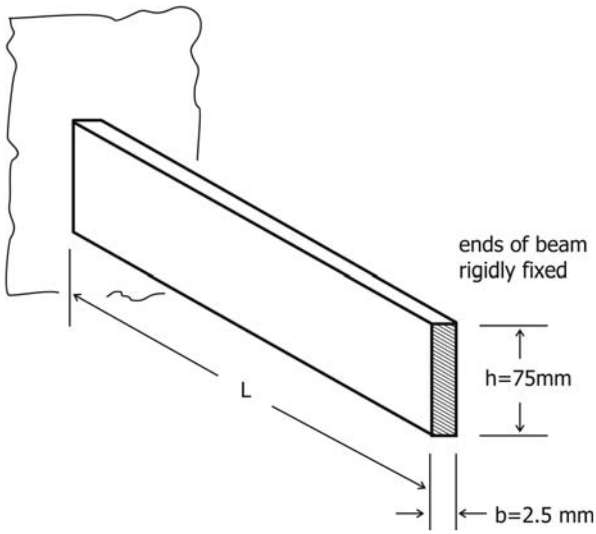

Figure 10-20. Beam subject to lateral thermal buckling, Problem 10-6.

10-5. For the column given in Problem 10-1, determine the temperature T_y at which the column would yield after the buckling has occurred.

10-6. A 6061-T6 beam, as shown in Figure 10-20, has a thickness $b = 2.5\,\text{mm}$ (0.10 in.), a depth $h = 75\,\text{mm}$ (2.953 in.), and a length L. The beam is rigidly fixed at both ends. The beam has a linear temperature distribution through its depth, with the top surface of the beam at 140°C (284°F) and the bottom surface at 0°C (32°F). The properties of the aluminum are as follows: $\alpha = 23.4 \times 10^{-6\circ}\text{C}^{-1}$ $(13.0 \times 10^{-6\circ}\text{F}^{-1})$, $E = 68.9\,\text{GPa}$ $(10.0 \times 10^6\,\text{psi})$, $\mu = 0.30$, $S_y = 275\,\text{MPa}$ (40,000 psi). Determine the maximum length L of the beam so that the beam will not buckle laterally for the given temperature distribution.

10-7. A circular plate of radius $b = 375$ mm (14.76 in.) has a built-in edge and is subjected to a uniform temperature change $\Delta T_1 = 125^\circ\text{C}$ (225°F). The plate is constructed of C1020 steel with the following properties: $\alpha = 12 \times 10^{-6\circ}\text{C}^{-1}$ $(6.67 \times 10^{-6\circ}\text{F}^{-1})$, $E = 205\,\text{GPa}$ $(29.7 \times 10^6\,\text{psi})$, $\mu = 0.28$, and $S_y = 324\,\text{MPa}$ (47,000 psi). Determine the required thickness of the plate to prevent buckling.

10-8. A circular plate of radius b and thickness h has the edge restrained, such that the radial displacement $u = 0$, and the rotation of the edge is elastically restrained in bending, such that the moment stress resultant is related to the rotation at the edge as follows:

$$M_r(b) = -\frac{Dk_\theta}{b}\omega_r(b) = -\frac{Eh^3 k_\theta}{12(1-\mu^2)b}\frac{dw(b)}{dr}$$

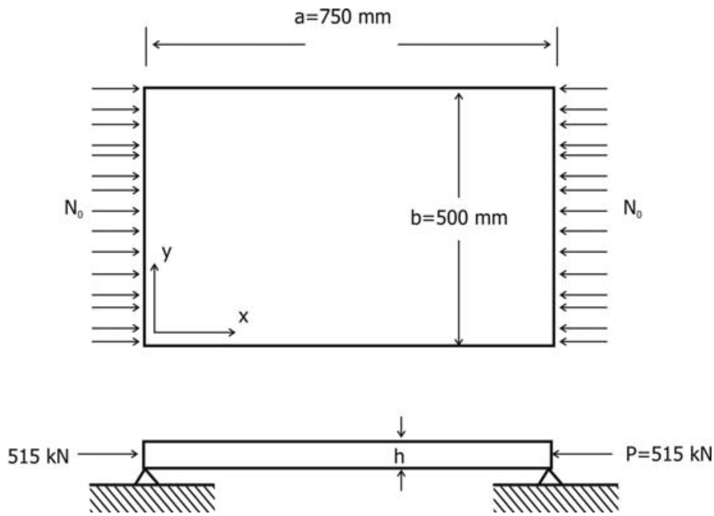

Figure 10-21. Rectangular plate with simply-supported edges, Problem 10-10.

The quantity k_θ is a dimensionless constant. The plate is subjected to a uniform temperature change $\Delta T_1 = T_1 - T_0$. Show that the critical or buckling temperature change is

$$\Delta T_{1,\mathrm{cr}} = \frac{\lambda^2}{12(1+\mu)\alpha}\left(\frac{h}{b}\right)^2$$

The quantity λ is the solution of the transcendental equation,

$$\lambda J_0(\lambda) = (1 - \mu + k_\theta) J_1(\lambda)$$

10-9. Suppose the plate in Problem 10-8 is constructed of steel with the following properties: $\alpha = 11 \times 10^{-6\circ}\mathrm{C}^{-1}$ ($6.10 \times 10^{-6\circ}\mathrm{F}^{-1}$), $E = 200\,\mathrm{GPa}$ (29.0×10^6 psi), $\mu = 0.28$, and $S_y = 330\,\mathrm{MPa}$ (47,900 psi). The plate diameter is 2.50 m ($b = 1.250\,\mathrm{m} = 49.2$ in.), and the elastic restraint constant $k_\theta = 0.280$. If the plate buckles at a temperature difference $\Delta T_{1,\mathrm{cr}} = 32^\circ\mathrm{C}$ ($57.6^\circ\mathrm{F}$), determine numerical value of the thickness h of the plate.

10-10. A rectangular plate with simply-supported edges has a force of 515 kN (115,800 $\mathrm{lb_f}$) applied to the two shorter edges, as shown in Figure 10-21. The temperature distribution in the plate is

$$\Delta T = T - T_0 = \Delta T_1 \cos(2\pi y/b) \quad \text{with} \quad \Delta T_1 = -50^\circ\mathrm{C}\ (-90^\circ\mathrm{F})$$

The plate is constructed of stainless steel with the following properties: $\alpha = 16.4 \times 10^{-6\circ}\mathrm{C}^{-1}$ ($9.11 \times 10^{-6\circ}\mathrm{F}^{-1}$), $E = 193\,\mathrm{GPa}$ (28×10^6 psi),

$\mu = 0.30$, and $S_y = 235\,\text{MPa}$ (34,100 psi). The plate has a width $b =$ 500 mm (19.69 in.) and a length $a = 750\,\text{mm}$ (29.53 in.). Determine the required thickness of the plate, based on buckling as the failure mode, for a factor of safety $f_s = f_{\text{th}} = f_{\text{mech}} = 1.50$.

10-11. If the in-plane mechanical force for the plate in Problem 10-10 were zero, determine the required plate thickness for the thermal loading only.

10-12. Hot benzene is to be stored in an insulated cylindrical container having a mean diameter of 2.440 m ($a = 1.220\,\text{m} = 4.00\,\text{ft}$), a wall thickness of 2.0 mm (0.079 in.), and a length of 3.05 m (10.0 ft). The ends of the cylinder are simply-supported and restrained from moving in the axial direction. The tank is constructed of 304 stainless steel with the following properties: $\alpha = 16.0 \times 10^{-6}\,{}^\circ\text{C}^{-1}$ ($8.89 \times 10^{-6}\,{}^\circ\text{F}^{-1}$), $E = 193\,\text{GPa}$ (28×10^6 psi), $\mu = 0.305$, and $S_y = 232\,\text{MPa}$ (33,600 psi). The vessel is stress-free at a temperature $T_0 = 25^\circ\text{C}$ (77°F). When the hot benzene is added to the vessel, the vessel walls are warmed to the benzene temperature. Determine the maximum temperature of the benzene (or the vessel wall) such that the vessel will not buckle under the thermal loading. Determine the thermal stress in the vessel wall for this temperature.

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APPENDIX A

PREFERRED PREFIXES IN THE SI SYSTEM OF UNITS

Preferred Prefixes in the SI (International Metric) System of Units

Prefix	Abbreviation	Multiplier	Example	Name	Value
yocto	y	E−24	ym	yoctometer	10^{-24} m
zepto	z	E−21	zm	zeptometer	10^{-21} m
atto	a	E−18	am	attometer	10^{-18} m
femto	f	E−15	fm	femtometer	10^{-15} m
pico	p	E−12	pm	picometer	10^{-12} m
nano	n	E−09	nm	nanometer	10^{-9} m
micro	μ	E−06	μm	micrometer	10^{-6} m
milli	m	E−03	mm	millimeter	10^{-3} m
Base Unit			m	meter	
kilo	k	E+03	km	kilometer	10^{3} m
mega	M	E+06	Mm	megameter	10^{6} m
giga	G	E+09	Gm	gigameter	10^{9} m
tera	T	E+12	Tm	terameter	10^{12} m
peta	P	E+15	Pm	petameter	10^{15} m
exa	E	E+18	Em	exameter	10^{18} m
zetta	Z	E+21	Zm	zettameter	10^{21} m
yotta	Y	E+24	Ym	yottameter	10^{24} m

The prefixes *centi* (c) multiplier of 10^{-2} and *deci* (d) multiplier 10^{-1} are recommended only for area and volume units (cm^2 and dm^3, for example). According to SI usage, the prefixes should be applied only to a unit in the numerator of a set of units (MN/m^2, for example, and *not* N/mm^2). When an exponent is involved with a unit with a prefix, the exponent applies to the entire unit. For example, mm^2 is a *square millimeter*, or 10^{-6} m^2, and not a milli-(square meter). Double prefixes should not be used (picometer, pm, for example, and not micro-micrometer, $\mu\mu$m).

Reference: Standard for Use of the International System of Units (SI): The Modern Metric System, IEEE/ASTM SI 10-1997, Institute of Electrical and Electronics Engineers and American Society for Testing and Materials (1997).

APPENDIX B

PROPERTIES OF MATERIALS AT 300 K

Properties of Materials at 300 K (80°F or 27°C). S_u, Ultimate Strength; S_y, Yield Strength; ρ, Density; E, Young's Modulus; μ, Poisson's Ratio; α, Thermal Expansion Coefficient; $\kappa = k_t/\rho c$, Thermal Diffusivity

Material	S_u, MPa	S_y, MPa	ρ, kg/m^3	E, GPa	μ	α, K^{-1}	κ, mm^2/s
Aluminum, 2024-T3	462	316	2770	73.4	0.344	22.5×10^{-6}	48.0
Aluminum, 3003-H12	130	125	2740	69.0	0.33	22.5×10^{-6}	66.6
Aluminum, 6061-T6	310	275	2710	69.0	0.30	23.4×10^{-6}	66.6
Beryllium copper	1150	1010	8250	122	0.357	18.1×10^{-6}	24.7
Brass, 70/30	352	262	8520	110	0.331	19.8×10^{-6}	34.1
Bronze, UNS-22000	370	250	8860	103	0.30	17.8×10^{-6}	8.8
Copper/10% nickel	360	250	8940	124	0.33	16.2×10^{-6}	13.9
Inconel, 600	640	303	8420	215	0.325	13.0×10^{-6}	8.72
Invar	643	615	8140	149	0.310	2.6×10^{-6}	2.86
Monel, K-500	957	690	8560	181	0.315	13.9×10^{-6}	4.87
Gray cast iron, class 20	141	...	6950	81.5	0.25	10.8×10^{-6}	16.3
Gray cast iron, class 40	292	...	7200	124	0.25	10.8×10^{-6}	16.3
Steel, C1020, annealed	439	324	8000	205	0.28	11.9×10^{-6}	18.2
Steel, 4340	1012	934	7840	214	0.27	11.2×10^{-6}	8.0
Steel, 9% nickel	856	645	7950	189	0.286	11.8×10^{-6}	7.2
Steel, 304 stainless	516	232	7750	193	0.305	16.0×10^{-6}	4.26
Steel, 416 stainless	510	276	7800	200	0.31	9.9×10^{-6}	7.25

(*Continued*)

Material	S_u MPa	S_y, MPa	ρ, kg/m^3	E, GPa	μ	α, K^{-1}	κ, mm^2/s
Titanium, Ti-5Al-2.5Sn	873	801	4480	111	0.325	9.6×10^{-6}	3.68
Brick, common	24.1(1)	...	1920	24.1	0.20	5.4×10^{-6}	0.46
Concrete, 1:$2\frac{1}{2}$:$3\frac{1}{4}$	17.2(1)	...	2410	20.7	0.13	10.8×10^{-6}	0.68
Marble	24.1(1)	...	2600	55.2	0.26	6.8×10^{-6}	1.39
Glass, silicate 7740	50	...	2350	62.7	0.20	4.5×10^{-6}	0.34
Glass, Pyrex	138	...	2230	62.1	0.24	3.3×10^{-6}	0.47
Nylon	60.7	55.1	1140	3.0	0.40	84×10^{-6}	1.83
Rubber, molded	14	...	913	...	0.50	162×10^{-6}	0.062
Teflon	18.3	10.7	2210	0.40	0.40	100×10^{-6}	0.12

(1) Ultimate tensile strength in compression, S_{uc}. The ultimate tensile strength in tension, $S_{ut} \approx S_{uc}/10$ Units conversion factors:

1 GPa = 10^9 Pa = 10^9 N/m^2

Multiply {MPa} by 145.038 to obtain {lb$_f$/in^2 or psi}

Multiply {kg/m^3} by 3.61273×10^{-5} to obtain {lb$_m$/in^3}

Multiply {K^{-1}} by 5/9 to obtain {^{o}F^{-1})

Multiply {mm^2/s} by 0.0387501 to obtain {ft^2/hr}

APPENDIX C

PROPERTIES OF SELECTED MATERIALS AS A FUNCTION OF TEMPERATURE

TABLE C-1. Properties of 2024-T3 Aluminum. S_u, Ultimate Strength; S_y, Yield Strength; E, Young's Modulus; α, Thermal Expansion Coefficient; $\kappa = k_t/\rho c$, Thermal Diffusivity

Temperature, K	S_u, MPa	S_y, MPa	E, GPa	α, K^{-1}	$\int \alpha dT$	κ, mm^2/s
4	774	528	87.6	0.0017×10^{-6}	0.00×10^{-5}	4120
10	751	509	87.3	0.027×10^{-6}	0.01×10^{-5}	2120
20	738	491	86.9	0.212×10^{-6}	0.11×10^{-5}	664
30	709	473	86.4	0.717×10^{-6}	0.54×10^{-5}	273
40	683	456	85.9	2.09×10^{-6}	2.04×10^{-5}	148
50	661	441	85.4	4.15×10^{-6}	5.33×10^{-5}	95.8
60	642	426	84.9	6.11×10^{-6}	10.5×10^{-5}	75.3
70	626	412	84.4	7.89×10^{-6}	17.5×10^{-5}	63.6
80	612	399	84.0	9.52×10^{-6}	26.3×10^{-5}	56.2
90	600	387	83.5	11.0×10^{-6}	36.6×10^{-5}	51.8
100	589	376	83.0	12.3×10^{-6}	48.2×10^{-5}	48.5
120	570	357	82.0	14.5×10^{-6}	75.1×10^{-5}	45.2
140	554	342	81.1	16.2×10^{-6}	105.9×10^{-5}	43.9
160	541	331	80.1	17.7×10^{-6}	139.9×10^{-5}	43.8
180	529	323	79.2	18.8×10^{-6}	176.4×10^{-5}	44.0
200	519	318	78.2	19.7×10^{-6}	215×10^{-5}	44.2
250	490	318	75.8	21.3×10^{-6}	319×10^{-5}	46.1
300	462	316	73.4	22.5×10^{-6}	429×10^{-5}	48.0
350	432	303	71.0	23.5×10^{-6}	554×10^{-5}	50.3
400	401	285	68.6	24.3×10^{-6}	664×10^{-5}	52.1
500	336	231	64.2	25.5×10^{-6}	914×10^{-5}	53.9

TABLE C-2. Properties of C1020 Carbon Steel, Annealed. S_u, Ultimate Strength; S_y, Yield Strength; E, Young's Modulus; α, Thermal Expansion Coefficient; $\kappa = k_t/\rho c$, Thermal Diffusivity

Temperature, K	S_u, MPa	S_y, MPa	E, GPa	α, K^{-1}	$\int \alpha\, dT$	κ, mm^2/s
60	780	561	216	2.3×10^{-6}	4×10^{-5}	76.5
70	756	544	216	3.1×10^{-6}	7×10^{-5}	59.0
80	732	528	215	4.0×10^{-6}	10×10^{-5}	48.4
90	710	513	215	4.8×10^{-6}	15×10^{-5}	41.3
100	689	499	214	5.5×10^{-6}	20×10^{-5}	36.2
120	650	472	213	6.8×10^{-6}	32×10^{-5}	29.8
140	616	448	213	7.8×10^{-6}	47×10^{-5}	26.2
160	584	427	212	8.7×10^{-6}	64×10^{-5}	23.8
180	556	407	211	9.4×10^{-6}	82×10^{-5}	22.2
200	531	389	210	9.9×10^{-6}	101×10^{-5}	21.1
250	479	352	207	11.3×10^{-6}	155×10^{-5}	19.3
300	439	324	205	11.9×10^{-6}	210×10^{-5}	18.1
350	410	303	202	12.5×10^{-6}	271×10^{-5}	17.1
400	390	288	200	13.0×10^{-6}	335×10^{-5}	15.8
500	373	273	195	13.3×10^{-6}	466×10^{-5}	13.8

TABLE C-3. Properties of 9% Nickel Steel, Normalized. S_u, Ultimate Strength; S_y, Yield Strength; E, Young's Modulus; α, Thermal Expansion Coefficient; $\kappa = k_t/\rho c$, Thermal Diffusivity

Temperature, K	S_u, MPa	S_y, MPa	E, GPa	α, K^{-1}	$\int \alpha\, dT$	κ, mm^2/s
30	1410	1100	212	0.34×10^{-6}	0.3×10^{-5}	82.8
40	1360	1010	211	0.73×10^{-6}	0.8×10^{-5}	42.0
50	1320	950	210	1.28×10^{-6}	1.8×10^{-5}	28.1
60	1290	909	209	1.95×10^{-6}	3.5×10^{-5}	20.5
70	1260	878	208	2.72×10^{-6}	5.8×10^{-5}	16.6
80	1230	854	208	3.58×10^{-6}	7.9×10^{-5}	14.1
90	1200	834	207	4.33×10^{-6}	11.9×10^{-5}	12.3
100	1180	817	206	5.07×10^{-6}	16.6×10^{-5}	11.1
120	1140	789	204	6.44×10^{-6}	28.1×10^{-5}	9.83
140	1100	766	202	7.64×10^{-6}	42.2×10^{-5}	8.92
160	1060	746	201	8.66×10^{-6}	58.5×10^{-5}	8.44
180	1030	729	199	9.50×10^{-6}	77.7×10^{-5}	8.02
200	994	713	197	10.1×10^{-6}	97.3×10^{-5}	7.75
250	921	677	193	11.1×10^{-6}	150×10^{-5}	7.35
300	856	645	189	11.8×10^{-6}	208×10^{-5}	7.20
350	797	615	184	12.3×10^{-6}	268×10^{-5}	7.20
400	744	586	180	12.6×10^{-6}	302×10^{-5}	7.25
500	650	530	171	13.1×10^{-6}	431×10^{-5}	7.30

TABLE C-4. Properties of 304 Stainless Steel, Annealed. S_u, Ultimate Strength; S_y, Yield Strength; E, Young's Modulus; α, Thermal Expansion Coefficient; $\kappa = k_t/\rho c$, Thermal Diffusivity

Temperature, K	S_u, MPa	S_y, MPa	E, GPa	α, K^{-1}	$\int \alpha\, dT$	κ, mm^2/s
4	1030	670	215	0.001×10^{-6}	0.0×10^{-5}	270
10	1020	662	214	0.023×10^{-6}	0.0×10^{-5}	99
20	995	647	214	0.18×10^{-6}	0.1×10^{-5}	35
30	974	627	213	0.62×10^{-6}	0.4×10^{-5}	29
40	953	606	212	1.1×10^{-6}	1.2×10^{-5}	16.5
50	932	584	211	2.3×10^{-6}	2×10^{-5}	9.45
60	912	562	211	4.3×10^{-6}	5×10^{-5}	7.20
70	892	541	210	6.1×10^{-6}	11×10^{-5}	5.87
80	872	521	209	7.5×10^{-6}	17×10^{-5}	5.10
90	852	501	208	8.7×10^{-6}	25×10^{-5}	4.58
100	833	482	208	9.6×10^{-6}	35×10^{-5}	4.39
120	795	445	206	10.9×10^{-6}	55×10^{-5}	4.03
140	758	411	205	12.0×10^{-6}	78×10^{-5}	3.90
160	722	380	203	12.8×10^{-6}	103×10^{-5}	3.89
180	688	351	202	13.4×10^{-6}	129×10^{-5}	3.95
200	656	325	200	14.0×10^{-6}	157×10^{-5}	4.07
250	581	270	197	15.1×10^{-6}	229×10^{-5}	4.18
300	516	232	193	16.0×10^{-6}	307×10^{-5}	4.26
350	467	210	189	16.7×10^{-6}	389×10^{-5}	4.29
400	430	188	186	17.3×10^{-6}	471×10^{-5}	4.30
500	390	157	178	18.2×10^{-6}	639×10^{-5}	4.31

TABLE C-5. Properties of Beryllium Copper (ASTM B194), Solution Treated, Precipitation Hardened. S_u, Ultimate Strength; S_y, Yield Strength; E, Young's Modulus; α, Thermal Expansion Coefficient; $\kappa = k_t/\rho c$, Thermal Diffusivity

Temperature, K	S_u, MPa	S_y, MPa	E, GPa	α, K^{-1}	$\int \alpha\, dT$	κ, mm^2/s
4	1310	1160	136	0.001×10^{-6}	0.00×10^{-5}	2500
10	1310	1160	136	0.021×10^{-6}	0.01×10^{-5}	670
20	1310	1160	135	0.17×10^{-6}	0.08×10^{-5}	165
30	1310	1150	135	0.57×10^{-6}	0.43×10^{-5}	72
40	1300	1150	134	1.4×10^{-6}	1.4×10^{-5}	42
50	1300	1150	134	2.7×10^{-6}	3.3×10^{-5}	32
60	1300	1140	133	4.3×10^{-6}	6.8×10^{-5}	26.3
70	1300	1140	133	6.5×10^{-6}	12×10^{-5}	23.4
80	1290	1130	132	8.4×10^{-6}	20×10^{-5}	21.7
90	1290	1130	132	9.6×10^{-6}	29×10^{-5}	20.8
100	1280	1130	131	10.4×10^{-6}	39×10^{-5}	20.3
120	1270	1120	130	11.6×10^{-6}	61×10^{-5}	20.2
140	1260	1110	129	12.4×10^{-6}	85×10^{-5}	20.5
160	1250	1090	129	13.2×10^{-6}	110×10^{-5}	20.9
180	1240	1080	128	13.8×10^{-6}	137×10^{-5}	21.4

TABLE C-5. ***Continued***

Temperature, K	S_u, MPa	S_y, MPa	E, GPa	α, K^{-1}	$\int \alpha\, dT$	κ, mm^2/s
200	1220	1070	127	14.5×10^{-6}	165×10^{-5}	22.0
250	1180	1040	124	16.3×10^{-6}	242×10^{-5}	23.5
300	1150	1010	122	18.1×10^{-6}	329×10^{-5}	24.7
350	1130	980	120	19.9×10^{-6}	424×10^{-5}	25.8
400	1100	955	117	21.6×10^{-6}	528×10^{-5}	26.9
500	1070	904	113	25.2×10^{-6}	762×10^{-5}	29.0

TABLE C-6. Properties of Titanium Alloy, Ti-5Al-2.5Sn, Annealed. S_u, Ultimate Strength; S_y, Yield Strength; E, Young's Modulus; α, Thermal Expansion Coefficient; $\kappa = k_t/\rho c$, Thermal Diffusivity

Temperature, K	S_u, MPa	S_y, MPa	E, GPa	α, K^{-1}	$\int \alpha\, dT$	κ, mm^2/s
20	1710	1690	118	0.8×10^{-6}	0.4×10^{-5}	56
30	1680	1630	118	2.6×10^{-6}	2.0×10^{-5}	25
40	1660	1580	118	3.9×10^{-6}	5.2×10^{-5}	12.4
50	1640	1530	118	4.6×10^{-6}	9.5×10^{-5}	8.28
60	1610	1480	118	5.2×10^{-6}	14.4×10^{-5}	6.05
70	1570	1440	118	5.5×10^{-6}	19.7×10^{-5}	5.05
80	1530	1400	118	5.9×10^{-6}	25.4×10^{-5}	4.44
90	1480	1360	118	6.2×10^{-6}	31.5×10^{-5}	3.99
100	1440	1320	118	6.5×10^{-6}	37.8×10^{-5}	3.78
120	1360	1260	118	6.9×10^{-6}	51×10^{-5}	3.41
140	1280	1190	118	7.4×10^{-6}	66×10^{-5}	3.24
160	1210	1140	118	7.8×10^{-6}	81×10^{-5}	3.16
180	1150	1080	118	8.1×10^{-6}	97×10^{-5}	3.14
200	1100	1030	118	8.5×10^{-6}	113×10^{-5}	3.11
250	972	910	114	9.1×10^{-6}	157×10^{-5}	3.31
300	873	801	111	9.6×10^{-6}	204×10^{-5}	3.68
350	790	713	108	10.1×10^{-6}	253×10^{-5}	3.87
400	722	638	104	10.6×10^{-6}	305×10^{-5}	4.07
500	613	513	98	11.6×10^{-6}	416×10^{-5}	4.50

TABLE C-7. Properties of Polytetrafluoroethylene (Teflon), Extruded. S_u, Ultimate Strength; S_y, Yield Strength; E, Young's Modulus; α, Thermal Expansion Coefficient; $\kappa = k_t/\rho c$, Thermal Diffusivity

Temperature, K	S_u, MPa	S_y, MPa	E, GPa	α, K^{-1}	$\int \alpha\, dT$	κ, mm^2/s
4	157	127	5.90	0.12×10^{-6}	0	7.67
10	151	123	5.85	0.76×10^{-6}	0	2.28
20	141	117	5.76	3.14×10^{-6}	6×10^{-5}	0.79
30	131	110	5.65	7.16×10^{-6}	22×10^{-5}	0.59

TABLE C-7. *Continued*

Temperature, K	S_u, MPa	S_y, MPa	E, GPa	α, K^{-1}	$\int \alpha\, dT$	κ, mm^2/s
40	122	104	5.54	12.9×10^{-6}	40×10^{-5}	0.51
50	114	98	5.41	20.3×10^{-6}	62×10^{-5}	0.44
60	106	92	5.28	29.4×10^{-6}	75×10^{-5}	0.39
70	98	87	5.13	40.3×10^{-6}	125×10^{-5}	0.36
80	91	81	4.98	43.3×10^{-6}	170×10^{-5}	0.32
90	85	76	4.81	46.9×10^{-6}	219×10^{-5}	0.295
100	79	71	4.71	50.4×10^{-6}	272×10^{-5}	0.273
120	68	61	4.50	57.6×10^{-6}	386×10^{-5}	0.237
140	59	53	3.88	64.6×10^{-6}	514×10^{-5}	0.210
160	51	45	3.21	71.6×10^{-6}	655×10^{-5}	0.187
180	44	37	2.56	78.8×10^{-6}	807×10^{-5}	0.166
200	38	31	1.95	86.2×10^{-6}	975×10^{-5}	0.153
250	26	18.0	0.84	104.8×10^{-6}	1449×10^{-5}	0.131
300	18.3	10.7	0.40	125×10^{-6}	2004×10^{-5}	0.121
350	12.7	6.3	0.186	147×10^{-6}	2640×10^{-5}	0.117
400	8.8	3.8	0.119	170×10^{-6}	3356×10^{-5}	0.119

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APPENDIX D

BESSEL FUNCTIONS

D.1 INTRODUCTION

Bessel functions are not classified as one of the “elementary functions” in mathematics; however, Bessel functions appear in the solution of many physical problems in heat transfer, mechanical vibrations and acoustics, electromagnetic systems, and elasticity. In fact, Bessel functions appear particularly in a wide variety of problems involving cylindrical coordinates, such as vibration of a circular membrane, buckling of a tapered cylindrical column, and transient cooling of a cylinder.

The Bessel functions have been tabulated [Abramowitz and Stegun, 1964], similar to the trigonometric sine and cosine functions. The extensive tables by Abramowitz and Stegun may be used to obtain numerical values of the Bessel functions. In addition, computer “libraries” or “recipes” are available for evaluation of the Bessel functions in numerical computer solutions.

Bessel functions were first discovered in 1732 by Daniel Bernoulli [Gray and Mathews, 1922]. He obtained a series solution (representing one of the Bessel functions) for the oscillation of a uniform heavy flexible chain, fixed at the upper end and free at the lower end, after the chain is slightly moved from its vertical equilibrium position. In 1781 Leonard Euler developed a series solution of the heavy hanging chain problem that was similar to that of Bernoulli. Euler had developed a differential equation, which was the Bessel equation, dealing with the vibrations of a circular drumhead or membrane in 1764 [Andrews, 1992].

Bessel functions appeared again in 1822 in the classic book by Joseph B. Fourier on heat transfer [Fourier, 1822]. Fourier solved the problem of the transient temperature distribution in a long circular cylinder using an infinite series

similar to that used by Bernoulli. As a result of the nature of the boundary conditions that Fourier used in this problem, he also investigated the zeros of the Bessel function of the first kind.

Friedrich W. Bessel, an astronomer who first computed the orbit of Haley's comet, examined the problem of planetary motion and discovered the functions that we call "Bessel functions" today in connection with the mathematical description of elliptic motion. He presented his results in 1824 in his memoir of that date, in which he derived the Bessel differential equation and presented a systematic study of the properties of the solution of the equation.

D.2 BESSEL FUNCTIONS OF THE FIRST KIND

The Bessel functions may be defined in several different ways (as solutions of a differential equation, from generating functions, etc.), but let us express the Bessel functions as an infinite series, as was done originally by Bernoulli and Fourier. If n is a nonnegative integer $(0, 1, 2, \ldots)$, the Bessel function of the first kind and of order n may be written as

$$J_n(x) = \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{1}{2}x\right)^{n+2j}}{j!\,(n+j)!} \tag{D-1}$$

In particular, the series for the Bessel function of the first kind and of order 0 and of order 1 are

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \cdots = \sum_{j=0}^{\infty} \frac{(-1)^j (x)^{2j}}{2^{2j} (j!)^2} \tag{D-2}$$

$$J_1(x) = \frac{1}{2}x - \frac{x^3}{2^2 4} + \frac{x^5}{2^2 4^2 6} - \frac{x^7}{2^2 4^2 6^2 8} + \cdots = \sum_{j=0}^{\infty} \frac{(-1)^j (x)^{2j+1}}{2(j+1)\, 2^{2j} (j!)^2} \tag{D-3}$$

A plot of these two functions is shown in Figure D-1. We note that, at the origin, the Bessel functions of the first kind and of integer order have the following values:

$$J_0(0) = 1$$

$$J_1(0) = J_2(0) = J_3(0) = \cdots = 0$$

D.3 BESSEL FUNCTIONS OF NONINTEGER ORDER

If the order n of the Bessel function is not an integer, we have a problem evaluating the factorial. For example, what is $(\pi!)$? This problem is eliminated by the introduction of the *gamma function* $\Gamma(x)$, which is sometimes called

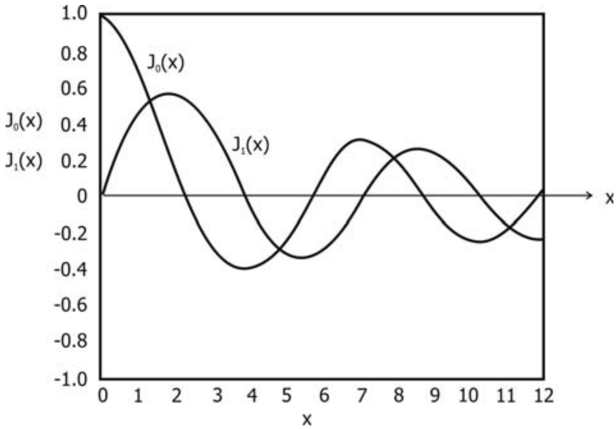

Figure D-1. Plot of the Bessel functions of the first kind of order 0 and order 1.

the *generalized factorial function*. The gamma function is defined by a definite integral as follows [Bell, 1968]:

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt \qquad \text{(where:} \quad x > 0) \tag{D-4}$$

By directly evaluating the integral, we find the following value for $x = 1$:

$$\Gamma(1) = \int_0^{\infty} e^{-t}\, dt = \left[-e^{-t}\right]_0^{\infty} = 1 \tag{D-5}$$

Also, let us determine the following value by integration by parts:

$$\Gamma(x+1) = \int_0^{\infty} t^x e^{-t} dt = \left[-t^x e^{-t}\right]_0^{\infty} + x \int_0^{\infty} t^{x-1} e^{-t} dt = x\Gamma(x) \tag{D-6}$$

From this result, we see that it is not necessary to tabulate the gamma function for all values of the argument x. If we have tabular values of $\Gamma(x)$ for x between 1 and 2, then we may use eq. (D-6) to determine all other values. For example,

$$\Gamma(\pi) = (\pi - 1)\,\Gamma(\pi - 1) = (\pi - 1)\,(\pi - 2)\,\Gamma(\pi - 2)$$

Using the numerical tables to evaluate the gamma function, we obtain

$$\Gamma(\pi) = (\pi - 1)\,\Gamma(\pi - 2)\,(0.935873) = 2.288047$$

Let us consider the case for $\Gamma(x + n)$, where n is a positive integer (1, 2, 3, ...). Using eq. (D-6), we may write the following expression for this case:

$$\Gamma(x+n) = (x+n-1)\,\Gamma(x+n-1) \tag{D-7}$$

If we repeatedly use eq. (D-6) to evaluate the gamma function on the right side of the equation, we obtain the final result:

$$\Gamma(x+n) = (x+n-1)(x+n-2)(x+n-3)\cdots(x+2)(x+1)\,x\Gamma(x) \tag{D-8}$$

If we substitute $x = 1$ into eq. (D-8), we obtain

$$\Gamma(n+1) = n(n-1)(n-2)\cdots(3)(2)(1)$$

or

$$\Gamma(n+1) = (1)(2)(3)\cdots(n-1)(n) = n! \tag{D-9}$$

Because of the expression given by eq. (D-9), we see why the gamma function is sometimes called the generalized factorial function.

We may now define the series for the Bessel function of the first kind and order ν, where ν is not an integer:

$$J_\nu(x) = \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{1}{2}x\right)^{\nu+2j}}{j!\,\Gamma(\nu+j+1)} \tag{D-10}$$

D.4 BESSEL FUNCTIONS OF THE SECOND KIND

Because the Bessel functions are solutions of a second-order differential equation, there must be a second independent solution, in addition to the $J_n(x)$ function. When the order ν is not an integer, the functions $J_\nu(x)$ and $J_{-\nu}(x)$ are independent functions, so $J_{-\nu}(x)$ can be taken as the second independent solution. On the other hand, when the order n is an integer, then $J_{-n}(x)$ is a linear function of $J_n(x)$. In this case,

$$J_{-n}(x) = (-1)^n J_n(x) \qquad (n = 1, 2, 3, \ldots) \tag{D-11}$$

In the case for $n =$ *integer* $(1, 2, 3, \ldots)$, the following form for the second solution of Bessel's equation is often used. This function is called the *Weber form* of the Bessel function of the second kind and of order n [Watson, 1958].

$$\begin{aligned} Y_n(x) = {} & \frac{2}{\pi}\left[\ln\left(\frac{1}{2}x\right) + \gamma\right] J_n(x) - \frac{1}{\pi}\sum_{j=0}^{n-1} \frac{(n-j-1)!}{j!}\left(\frac{1}{2}x\right)^{2j-n} \\ & -\frac{1}{\pi}\sum_{j=0}^{\infty} (-1)^j \left[\varphi(j) + \varphi(n+j)\right] \frac{\left(\frac{1}{2}x\right)^{2j+n}}{j!\,(n+j)!} \end{aligned} \tag{D-12}$$

where

$$\varphi(p) = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots + \tfrac{1}{p} = \sum_{k=0}^{p} \frac{1}{k} \qquad \text{with} \quad \varphi(0) \equiv 0 \tag{D-13}$$

and $\gamma = 0.5772156649\ldots =$ Euler's constant

In particular, the series for the Bessel function of the second kind and of order 0 and of order 1 are

$$Y_0(x) = \frac{2}{\pi}\left[\ln\left(\frac{1}{2}x\right) + \gamma\right] J_0(x) + \frac{2}{\pi}\left[\frac{x^2}{2^2} - \left(1 + \frac{1}{2}\right)\frac{x^4}{2^2 4^2} + \left(1 + \frac{1}{2} + \frac{1}{3}\right)\frac{x^6}{2^2 3^2 6^2} - \cdots\right] \quad \text{(D-14)}$$

$$Y_1(x) = \frac{2}{\pi}\left[\ln\left(\frac{1}{2}x\right) + \gamma\right] J_1(x) - \frac{1}{\pi}\left[\frac{2}{x} + \frac{x}{2} - \left(1 + 1 + \frac{1}{2}\right)\frac{x^3}{2^3 2!} + \left(1 + \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{3}\right)\frac{x^5}{2^5 2! 3!} - \cdots\right] \quad \text{(D-15)}$$

A plot of $Y_0(x)$ and $Y_1(x)$ is given in Figure D-2.

There is an alternate form for the Bessel function of the second kind, called the *Neumann form*, which is related to the Weber form as follows:

$$N_n(x) = \frac{\pi}{2} Y_n(x) + (\ln 2 - \gamma) J_n(x) \quad \text{(D-16)}$$

When using numerical values from a table, it is important to identify which form is used for the Bessel function of the second kind in the table. If the form is not identified in the table caption, a simple check may be made to determine the form:

$$Y_0(1) = 0.088257\ldots \quad \text{and} \quad Y_1(1) = -0.781213\ldots \quad \text{(Weber form)}$$

$$N_0(1) = 0.227344\ldots \quad \text{and} \quad N_1(1) = -1.176110\ldots \quad \text{(Neumann form)}$$

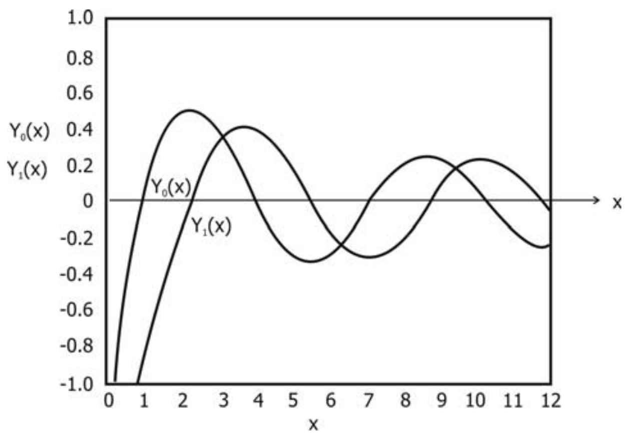

Figure D-2. Plot of the Bessel functions of the second kind of order 0 and order 1.

The Bessel function of the second kind for a noninteger order ν is defined by

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu x) - J_{-\nu}(x)}{\sin(\nu x)} \tag{D-17}$$

D.5 BESSEL'S EQUATION

The Bessel functions may also be defined as the solutions of a differential equation, called *Bessel's equation*:

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - n^2\right) y = 0 \tag{D-18}$$

The general solution of Bessel's equation is

$$y(x) = C_1 J_n(x) + C_2 Y_n(x) \tag{D-19}$$

If $n = \nu$, where ν is not an integer, an alternate form for the solution may be written:

$$y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x) \quad (\nu \text{ is not an integer}) \tag{D-20}$$

Bessel's equation may appear in many disguised forms. One of the useful general forms for Bessel's equation is

$$\frac{d^2 y}{dx^2} + \frac{(1-2a)}{x}\frac{dy}{dx} + \left[b^2 c^2 x^{2(c-1)} + \frac{\left(a^2 - n^2 c^2\right)}{x^2}\right] y = 0 \tag{D-21}$$

The quantities a, b, c, and n are constants. The general solution of eq. (D-21) is

$$y(x) = x^a \left[C_1 J_n\left(bx^c\right) + C_2 Y_n\left(bx^c\right)\right] \tag{D-22}$$

If $n = \nu$ (not an integer), then $J_{-\nu}(bx^c)$ may be used instead of $Y_n(bx^c)$.

If a differential equation is suspected to be a disguised Bessel's equation, we may attempt to determine the constants in eq. (D-21). If we are successful, then the equation *is* a form of Bessel's equation. The steps in the solution are as follows:

(a) First, arrange the differential equation such that the leading term is the second derivative with a coefficient of unity (1). If the coefficient of the first derivative is not of the form (constant/x) after this step, then the equation is probably not Bessel's equation. Note that the constant may be zero.

(b) If the coefficient of the first derivative term is of the correct form, the next step is to set (constant $= 1 - 2a$) and solve for the constant a.

(c) Next, examine the term multiplying y and containing x raised to any power *other than* (–2). Solve for the constant c by equating exponent of x in this term to $2(c - 1)$.

(d) Solve for the constant n by equating the numerical coefficient of the term multiplying y and containing x^{-2} to $(a^2 - n^2c^2)$.

(e) Finally, solve for the constant b by equating the numerical coefficient of the term containing $x^{2(c-1)}$ to $(bc)^2$. If this term is negative, see Section D-9.

Example D.1 Determine the general solution for the following differential equation:

$$4x^2\frac{d^2y}{dx^2} + \left(4x^4 - 3\right) y = 0$$

We may divide through by the coefficient of the second derivative term to put the equation in the standard form:

$$\frac{d^2y}{dx^2} + \left(x^2 - \frac{3}{4x^2}\right) y = 0$$

The numerical coefficient multiplying the first derivative is zero:

$$(1 - 2a) = 0 \quad \text{therefore,} \quad a = \tfrac{1}{2}$$

The exponent on the term multiplying y and containing powers other than x^{-2} is 2:

$$2\,(c - 1) = 2 \quad \text{therefore} \quad c = 1 + 1 = 2$$

The numerical coefficient of the term multiplying y and the term x^{-2} is $(-\frac{3}{4})$:

$$a^2 - n^2c^2 = -\tfrac{3}{4} \quad \text{therefore,} \quad n^2 = \frac{\frac{1}{4} + \frac{3}{4}}{2^2} = \frac{1}{4} \quad \text{or} \quad n = \tfrac{1}{2}$$

We note that $n = \frac{1}{2} = \nu$ is not an integer in this problem.

Finally, the coefficient of the term multiplying both y and the term containing x raised to a power other than (–2) is positive and is equal to 1:

$$(bc)^2 = 1 \quad \text{therefore,} \quad b = \tfrac{1}{2}$$

The solution of the differential equation involves Bessel functions and may be written as

$$y\,(x) = x^{1/2}\left[C_1 J_{1/2}\left(\tfrac{1}{2}x^2\right) + C_2 J_{-1/2}\left(\tfrac{1}{2}x^2\right)\right]$$

D.6 RECURRENCE RELATIONSHIPS FOR $J_n(x)$ AND $Y_n(x)$

In solving problems involving Bessel functions, we often need expressions for the derivatives and integrals of the functions. In the following recurrence relations, the first gives a relationship between Bessel functions of different orders. This is used in a manner similar to the "trig identities." The next three relations give expressions for the first derivative in terms of Bessel functions of various orders.

The last two relations are expressions that may be used in integration of the Bessel functions.

(a) $$J_{n+1}(x) = (2n/x)\,J_n(x) - J_{n-1}(x) \tag{D-23}$$

(b) $$\frac{dJ_n(x)}{dx} = \frac{1}{2}J_{n-1}(x) - \frac{1}{2}J_{n+1}(x) \tag{D-24}$$

(c) $$\frac{dJ_n(x)}{dx} = \frac{n}{x}J_n(x) - J_{n+1}(x) \tag{D-25}$$

(d) $$\frac{dJ_n(x)}{dx} = J_{n-1}(x) - \frac{n}{x}J_n(x) \tag{D-26}$$

(e) $$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x) \tag{D-27}$$

(f) $$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x) \tag{D-28}$$

If the Weber form is used for the Bessel functions of the second kind, then the recurrence relationships for $Y_n(x)$ are identical in form to those for $J_n(x)$ given by eqs. (D-23) through (D-28).

If we use eq. (D-25), the derivatives for the Bessel functions of order 0 and of order 1 may be written as

$$\frac{dJ_0(x)}{dx} = -J_1(x) \tag{D-29}$$

$$\frac{dJ_1(x)}{dx} = \frac{1}{x}J_1(x) - J_2(x) = -\frac{d^2 J_0(x)}{dx^2} \tag{D-30}$$

If we use eq. (D-26), we obtain an alternate result for the derivative of $J_1(x)$:

$$\frac{dJ_1(x)}{dx} = J_0(x) - \frac{1}{x}J_1(x) \tag{D-31}$$

If we use eq. (D-27) with $n = 1$, we may evaluate the following integral:

$$\int x J_0(x)\,dx = \int \frac{d}{dx}\left[x J_1(x)\right]dx = x J_1(x) + C \tag{D-32}$$

D.7 ASYMPTOTIC RELATIONS AND ZEROS FOR $J_n(x)$ AND $Y_n(x)$

In some problems, we need to evaluate the Bessel functions for large values of the argument of the function. The following asymptotic expressions may be used to evaluate the Bessel functions for arguments x greater than about 20 with an accuracy better than 10 percent:

$$J_n(x) \to \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{1}{4}\pi - \frac{1}{2}n\pi\right) \tag{D-33}$$

$$Y_n(x) \to \left(\frac{2}{\pi x}\right)^{1/2} \sin\left(x - \frac{1}{4}\pi - \frac{1}{2}n\pi\right) \tag{D-34}$$

In some problems, we need to determine the value of the argument for which the Bessel function is equal to zero. These values are called the *zeros* of the Bessel functions, $J_n(j_{n,k}) = 0$ and $Y_n(y_{n,k}) = 0$. There are an infinite number of zeros for the ordinary Bessel functions. The first five zeros for the Bessel functions of order 0 and 1 are given in Table D-3 at the end of this appendix.

D.8 MODIFIED BESSEL FUNCTIONS

We note that the trigonometric sine function may be expressed in terms of an infinite series as follows:

$$\sin x = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(2j+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{D-35}$$

The hyperbolic sine has a similar series expansion, except that all of the coefficients are positive numbers:

$$\sinh x = \sum_{j=0}^{\infty} \frac{x^{2j+1}}{(2j+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \tag{D-36}$$

In a corresponding manner, the modified Bessel functions of the first kind and of order ν (which may not be an integer) are given by a series similar to that of the ordinary Bessel functions of the first kind, except all of the coefficients in the series are positive numbers [Bowman, 1958]:

$$I_\nu(x) = \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}x\right)^{\nu+2j}}{j!\,\Gamma(\nu+j+1)} \tag{D-37}$$

In particular, the series for the modified Bessel functions of the first kind and of order 0 and of order 1 are given by the following infinite series:

$$I_0(x) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} + \frac{x^6}{2^2 4^2 6^2} + \cdots = \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}x\right)^{2j}}{(j!)^2} \tag{D-38}$$

$$I_1(x) = \frac{1}{2}x + \frac{x^3}{2^2 4} + \frac{x^5}{2^2 4^2 6} + \frac{x^7}{2^2 4^2 6^2 8} + \cdots = \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}x\right)^{2j+1}}{(j+1)(j!)^2} \tag{D-39}$$

These functions are shown in Figure D-3. We observe that these functions behave a lot like the hyperbolic cosine and hyperbolic sine functions.

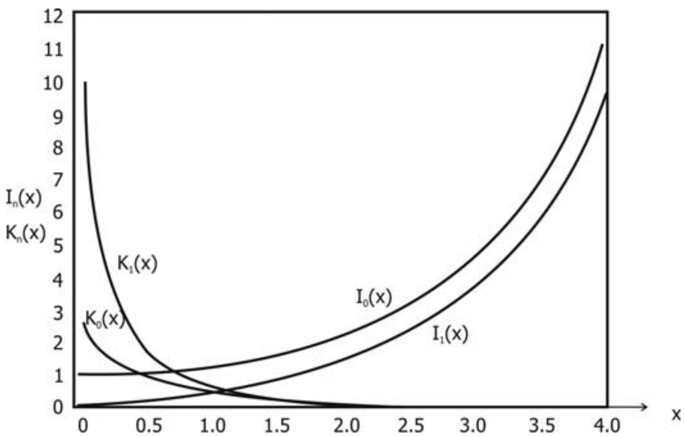

Figure D-3. Plot of the modified Bessel functions of the first kind and modified Bessel functions of the second kind of order 0 and order 1.

If the order ν is not an integer, the modified Bessel functions $I_\nu(x)$ and $I_{-\nu}(x)$ are independent functions. The modified Bessel functions of the second kind and of order ν are usually used, however, instead of $I_{-\nu}(x)$. These functions are defined by the following relationship:

$$K_\nu (x) = \frac{1}{2}\pi \left[\frac{I_{-\nu} (x) - I_\nu (x)}{\sin (\nu x)} \right] \tag{D-40}$$

If $\nu = n$ (an *integer*), the modified Bessel functions of the second kind and integral order may be expressed in terms of a series as follows:

$$K_n (x) = (-1)^{n+1} \left[\gamma + \ln \left(\frac{1}{2} x \right) \right] I_n (x) + \frac{1}{2} \sum_{j=0}^{n-1} \frac{(-1)^j (n - j - 1)!}{j!} \left(\frac{1}{2} x \right)^{2j-n}$$
$$+ \frac{1}{2} (-1)^n \sum_{j=1}^{\infty} \frac{\left(\frac{1}{2} x\right)^{2j+n}}{j! (n + j)!} \left[\varphi (j) + \varphi (n + j) \right] \tag{D-41}$$

The parameter $\varphi (p)$ is given by eq. (D-13).

A plot of $K_0(x)$ and $K_1(x)$ is shown in Figure D-3. We note that the function $K_0(x)$ resembles the negative exponential function, e^{-x}.

D.9 MODIFIED BESSEL EQUATION

The basic form of the differential equation satisfied by the modified Bessel functions is similar to the basic Bessel equation, except that the algebraic sign of the

last term involving x^2 is negative, instead of being positive:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - \left(x^2 + \nu^2\right) y = 0 \tag{D-42}$$

The general solution of the modified Bessel equation is

$$y(x) = C_1 J_\nu(x) + C_2 K_\nu(x) \tag{D-43}$$

The modified Bessel equation may also appear in various disguised forms. One of the general forms for the modified Bessel equation is

$$\frac{d^2 y}{dx^2} + \frac{(1-2a)}{x}\frac{dy}{dx} + \left[-\left(bcx^{c-1}\right)^2 + \frac{a^2 - n^2c^2}{x^2}\right] y = 0 \tag{D-44}$$

The general solution of eq. (D-44) is

$$y(x) = x^a \left[C_1 I_n\left(bx^c\right) + C_2 K_n\left(bx^c\right)\right] \tag{D-45}$$

We observe that eq. (D-44) for the modified Bessel functions is identical to eq. (D-21) for the ordinary Bessel functions, except that the coefficient of the term involving $x^{2(c-1)}$ is negative. To determine whether the solution of a Bessel equation involves ordinary or modified functions, examine the algebraic sign of the term $(bcx^{c-1})^2$. If the term is *positive*, the solution involves *ordinary* Bessel functions; if the term is *negative*, the solution involves *modified* Bessel functions.

D.10 RECURRENCE RELATIONS FOR THE MODIFIED BESSEL FUNCTIONS

The recurrence relations for the modified Bessel functions may be used for differentiation and integration of the functions. The expressions for $I_n(x)$ are slightly different from those for $K_n(x)$:

(a) $$I_{n+1}(x) = I_{n-1}(x) - (2n/x) I_n(x) \quad \text{and} \quad K_{n+1}(x) = K_{n-1}(x) + (2n/x) K_n(x) \tag{D-46}$$

(b) $$\frac{dI_n(x)}{dx} = \frac{1}{2} I_{n-1}(x) + \frac{1}{2} I_{n+1}(x) \quad \text{and} \quad \frac{dK_n(x)}{dx} = -\frac{1}{2} K_{n-1}(x) - \frac{1}{2} K_{n+1}(x) \tag{D-47}$$

(c) $$\frac{dI_n(x)}{dx} = \frac{n}{x} I_n(x) + I_{n+1}(x) \quad \text{and} \quad \frac{dK_n(x)}{dx} = \frac{n}{x} K_n(x) - K_{n+1}(x) \tag{D-48}$$

(d) $$\frac{dI_n(x)}{dx} = -\frac{n}{x}I_n(x) + I_{n-1}(x) \quad \text{and} \quad \frac{dK_n(x)}{dx} = -\frac{n}{x}K_n(x) - K_{n-1}(x) \tag{D-49}$$

(e) $$\frac{d}{dx}\left[x^n I_n(x)\right] = x^n I_{n-1}(x) \quad \text{and} \quad \frac{d}{dx}\left[x^n K_n(x)\right] = -x^n K_{n-1}(x) \tag{D-50}$$

(f) $$\frac{d}{dx}\left[x^{-n} I_n(x)\right] = x^{-n} I_{n+1}(x) \quad \text{and} \quad \frac{d}{dx}\left[x^{-n} K_n(x)\right] = -x^{-n} K_{n+1}(x) \tag{D-51}$$

For example, if we use eq. (D-48), we may evaluate the derivatives of $I_0(x)$ and $K_0(x)$ as follows:

$$\frac{dI_0(x)}{dx} = I_1(x) \quad \text{and} \quad \frac{dK_0(x)}{dx} = -K_1(x) \tag{D-52}$$

The derivatives of $I_1(x)$ and $K_1(x)$ may be found from eq. (D-49):

$$\frac{dI_1(x)}{dx} = -\frac{1}{x}I_1(x) + I_0(x) \quad \text{and} \quad \frac{dK_1(x)}{dx} = -\frac{1}{x}K_1(x) - K_0(x) \tag{D-53}$$

If we use eq. (D-50) with $n = 1$, we may evaluate the following integral:

$$\int xI_0(x)\,dx = \int \frac{d}{dx}\left[xI_1(x)\right]dx = xI_1(x) + C \tag{D-54}$$

D.11 ASYMPTOTIC RELATIONS FOR $I_n(x)$ AND $K_n(x)$

The modified Bessel functions may be approximated by the following asymptotic expressions for large values of the argument x. The error in using the asymptotic expressions for $x \geq 10$ is less than 1 percent.

$$I_n(x) \to \frac{e^x}{\sqrt{2\pi x}}\left(1 - \frac{4n^2 - 1}{8x}\right) \tag{D-55}$$

$$K_n(x) \to \sqrt{\frac{\pi}{2x}}e^{-x}\left(1 + \frac{4n^2 - 1}{8x}\right) \tag{D-56}$$

In many of the tables for modified Bessel functions, the quantities actually tabulated are $[e^{-x}I_n(x)]$ and $[e^x K_n(x)]$.

TABLE D-1. Bessel functions of the first and second kind, for order 0 and 1

x	$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$
0.0	1.00000	0.00000	$-\infty$	$-\infty$
0.1	0.99750	0.04994	−1.53424	−6.45895
0.2	0.99002	0.09950	−1.08111	−3.32382
0.3	0.97763	0.14832	−0.80727	−2.29311
0.4	0.96040	0.19603	−0.60602	−1.78087
0.5	0.93847	0.24227	−0.44452	−1.47147
0.6	0.91200	0.28670	−0.30851	−1.26039
0.7	0.88120	0.32900	−0.19066	−1.10325
0.8	0.84629	0.36884	−0.08680	−0.97814
0.9	0.80752	0.40595	+0.00563	−0.87313
1.0	0.76520	0.44005	0.08826	−0.78121
1.1	0.71962	0.47090	0.16216	−0.69812
1.2	0.67113	0.49829	0.22808	−0.62114
1.3	0.62009	0.52202	0.28654	−0.54852
1.4	0.56686	0.54195	0.33790	−0.47915
1.5	0.51183	0.55794	0.38245	−0.41231
1.6	0.45540	0.56990	0.42043	−0.34758
1.7	0.39798	0.57777	0.45203	−0.28473
1.8	0.33999	0.58152	0.47743	−0.22366
1.9	0.28182	0.58116	0.49682	−0.16441
2.0	0.22389	0.57672	0.51038	−0.10703
2.1	0.16661	0.56829	0.51829	−0.05168
2.2	0.11036	0.55596	0.52078	+0.00149
2.3	0.05554	0.53987	0.51808	0.05228
2.4	0.00251	0.52019	0.51041	0.10049
2.5	−0.04838	0.49709	0.49807	0.14592
2.6	−0.09680	0.47082	0.48133	0.18836
2.7	−0.14245	0.44160	0.46050	0.22763
2.8	−0.18504	0.40971	0.43592	0.26355
2.9	−0.22431	0.37543	0.40791	0.29594
3.0	−0.26005	0.33906	0.37685	0.32467
3.2	−0.32019	0.26134	0.30705	0.37071
3.4	−0.36430	0.17923	0.22962	0.40102
3.6	−0.39177	0.09547	0.14771	0.41539
3.8	−0.40256	0.01282	0.06450	0.41411
4.0	−0.39715	−0.06604	−0.01694	0.39793

TABLE D-1. *Continued*

x	$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$
4.5	−0.32054	−0.23106	−0.19471	0.30100
5.0	−0.17760	−0.32758	−0.30852	0.14786
5.5	−0.04121	−0.34144	−0.33948	−0.02376
6.0	+0.15065	−0.27668	−0.28819	−0.17501

For n = integer,

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

$$Y_{n+1}(x) - \frac{2n}{x} Y_n(x) - Y_{n-1}(x)$$

From M. Abramowitz and I. A. Stegun (1964). *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*, Applied Mathematics Series 55, U. S. Dept. of Commerce, Washington, DC.

TABLE D-2. Modified Bessel functions of the first and second kind, for order 0 and 1

x	$e^{-x} I_0(x)$	$e^{-x} I_1(x)$	$e^x K_0(x)$	$e^x K_1(x)$
0.0	1.00000	0.00000	∞	∞
0.1	0.90710	0.04530	2.68233	10.89018
0.2	0.82694	0.08228	2.14076	5.83339
0.3	0.75758	0.11238	1.85263	4.12516
0.4	0.69740	0.13676	1.66268	3.25867
0.5	0.64504	0.15642	1.52411	2.73101
0.6	0.59933	0.17216	1.41674	2.37392
0.7	0.55931	0.18467	1.33012	2.11501
0.8	0.52415	0.19450	1.25820	1.91793
0.9	0.49316	0.20212	1.19716	1.76239
1.0	0.46576	0.20791	1.14446	1.63615
1.1	0.44144	0.21220	1.09833	1.53140
1.2	0.41978	0.21526	1.05748	1.44290
1.3	0.40042	0.21730	1.02097	1.36699
1.4	0.38306	0.21851	0.98807	1.30105
1.5	0.36743	0.21904	0.95821	1.24317
1.6	0.35331	0.21902	0.93095	1.19187
1.7	0.34052	0.21855	0.90592	1.14604
1.8	0.32887	0.21773	0.88283	1.10481
1.9	0.31824	0.21661	0.86145	1.06747
2.0	0.30851	0.21527	0.84157	1.03348
2.1	0.29956	0.21375	0.82402	1.00237
2.2	0.29132	0.21209	0.80565	0.97377
2.3	0.28369	0.21032	0.78936	0.94737
2.4	0.27662	0.20848	0.77402	0.92291
2.5	0.27005	0.20658	0.75955	0.90017
2.6	0.26391	0.20465	0.74587	0.87897

TABLE D-2. ***Continued***

x	$e^{-x} I_0(x)$	$e^{-x} I_1(x)$	$e^x K_0(x)$	$e^x K_1(x)$
2.7	0.25818	0.20270	0.73291	0.85913
2.8	0.25281	0.20074	0.72060	0.84054
2.9	0.24776	0.19878	0.70890	0.82304
3.0	0.24300	0.19683	0.69776	0.80656
3.2	0.23427	0.19298	0.67698	0.77628
3.4	0.22643	0.18923	0.65795	0.74907
3.6	0.21935	0.18560	0.64046	0.72446
3.8	0.21290	0.18211	0.62429	0.70206
4.0	0.20700	0.17875	0.60930	0.68158
4.5	0.19420	0.17096	0.57610	0.63715
5.0	0.18354	0.16397	0.54781	0.60027
5.5	0.17448	0.15770	0.52332	0.56905
6.0	0.16666	0.15205	0.50186	0.54218

For n = integer,

$$I_{n+1}(x) = -\frac{2n}{x} I_n(x) + I_{n-1}(x)$$

$$K_{n+1}(x) = \frac{2n}{x} K_n(x) + K_{n-1}(x)$$

From M. Abramowitz and I. A. Stegun (1964). *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*, Applied Mathematics Series 55, U. S. Dept. of Commerce, Washington, DC.

TABLE D-3. First five of the zeros of the Bessel functions of the first and second kinds. $J_0(j_{0,k}) = 0$, $Y_0(y_{0,k}) = 0$, $J_1(j_{1,k}) = 0$, and $Y_1(y_{1,k}) = 0$

ZERO	$j_{0,k}$	$j_{1,k}$	$y_{0,k}$	$y_{1,k}$
$k = 1$	2.40483	3.83171	0.89358	2.19714
2	5.52008	7.01559	3.95768	5.59601
3	8.65373	10.17347	7.08605	8.59601
4	11.79153	13.32369	10.22235	11.74915
5	14.93092	16.47063	13.36110	14.89744

From M. Abramowitz and I. A. Stegun (1964). *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*, Applied Mathematics Series 55, U. S. Dept. of Commerce, Washington, DC.

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APPENDIX E

KELVIN FUNCTIONS

E.1 INTRODUCTION

In an investigation of the so-called *skin effect* in a wire carrying an alternating current, Lord Kelvin tried to solve the governing equation for the current density J (amp/m^2) in a circular wire:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial J}{\partial r}\right) = \frac{4\pi\mu_m}{\rho_e}\frac{\partial J}{\partial t} \tag{E-1}$$

where t = time, μ_e = magnetic permeability, and r_e = electrical resistivity of the wire. Kelvin considered an alternating electrical current, which could be expressed in the following mathematical form:

$$J(r,t) = R(r)\exp(i\omega t) \tag{E-2}$$

where $R(r)$ is the amplitude function, ω = circular frequency, and $i = \sqrt{-1}$.

When Kelvin made this substitution into the governing equation, he obtained a solution in the form of Bessel's equation that involved the imaginary number i:

$$R(r) = C_1 I_0(kr\sqrt{i}) + C_2 K_0(kr\sqrt{i}) \tag{E-3}$$

where k is a characteristic constant, and the constants of integration C_1 and C_2 could be complex numbers.

The series expansion for the modified Bessel function of the first kind and order zero is

$$I_0(z) = 1 + \left(\frac{1}{2}z\right)^2 + \frac{(\frac{1}{2}z)^4}{(2!)^2} + \frac{(\frac{1}{2}z)^6}{(3!)^2} + \frac{(\frac{1}{2}z)^8}{(4!)^2} + \frac{(\frac{1}{2}z)^{10}}{(5!)^2} + \cdots \tag{E-4}$$

If we make the direct substitution, $z = x\sqrt{i}$, the following complex expression is obtained:

$$I_0(x\sqrt{i}) = \left[1 - \frac{(\frac{1}{2}x)^4}{(2!)^2} + \frac{(\frac{1}{2}x)^8}{(4!)^2} - \cdots\right] + i\left[\left(\frac{1}{2}x\right)^2 - \frac{(\frac{1}{2}x)^6}{(3!)^2} + \frac{(\frac{1}{2}x)^{10}}{(5!)^2} - \cdots\right] \tag{E-5}$$

Kelvin wrote the expression in the following form:

$$I_0(x\sqrt{i}) = \mathrm{ber}(x) + i\,\mathrm{bei}(x) \tag{E-6}$$

E.2 KELVIN FUNCTIONS

The functions given in eq. (E-6) are the first pair of Kelvin functions. The notations ber(x) and bei(x) are abbreviations for *Bessel-real* and *Bessel-imaginary*, because these functions are the real and imaginary parts of the complex function:

$$\mathrm{ber}(x) = 1 - \frac{(\frac{1}{2}x)^4}{(2!)^2} + \frac{(\frac{1}{2}x)^8}{(4!)^2} - \cdots = \sum_{j=0}^{\infty} \frac{(-1)^j(\frac{1}{2}x)^{4j}}{[(2j)!]^2} \tag{E-7}$$

$$\mathrm{bei}(x) = \left(\frac{1}{2}x\right)^2 - \frac{(\frac{1}{2}x)^6}{(3!)^2} + \frac{(\frac{1}{2}x)^{10}}{(5!)^2} - \cdots = \sum_{j=0}^{\infty} \frac{(-1)^j(\frac{1}{2}x)^{4j+2}}{[(2j+1)!]^2} \tag{E-8}$$

Another pair of functions, called the *Kelvin-real* and *Kelvin-imaginary*, can be defined in a similar manner from the modified Bessel functions of the second kind and of order zero:

$$K_0(x) = \mathrm{ker}(x) + i\,\mathrm{kei}(x) \tag{E-9}$$

The series expansion for these functions may be found in a similar manner:

$$\begin{aligned}\mathrm{ker}(x) = &-\left(\ln\tfrac{1}{2}x + \gamma\right)\mathrm{ber}(x) + \tfrac{1}{4}\pi\,\mathrm{bei}(x)\\ &-\left(\frac{x^4}{2^2 4^2}\right)\left(1 + \frac{1}{2}\right) + \left(\frac{x^8}{2^2 4^2 6^2 8^2}\right)\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) - \cdots\end{aligned} \tag{E-10}$$

$$\begin{aligned}\mathrm{kei}(x) = &-\left(\ln\frac{1}{2}x + \gamma\right)\mathrm{bei}(x) - \tfrac{1}{4}\pi\,\mathrm{ber}(x) + \frac{x^2}{2^2} - \left(\frac{x^6}{2^2 4^2 6^2}\right)\\ &\times\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \cdots\end{aligned} \tag{E-11}$$

The quantity $\gamma = 0.57721\ 56649\ \ldots =$ Euler's constant.

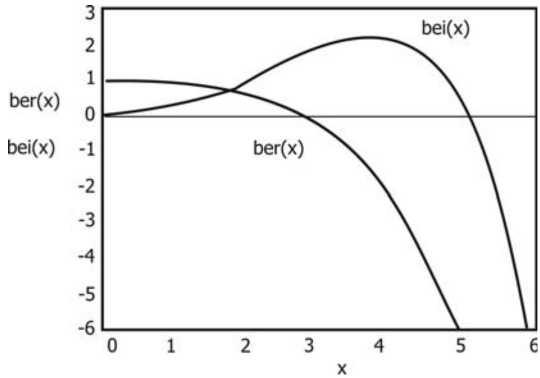

Figure E-1. Plot of the Kelvin functions ber(x) and bei(x).

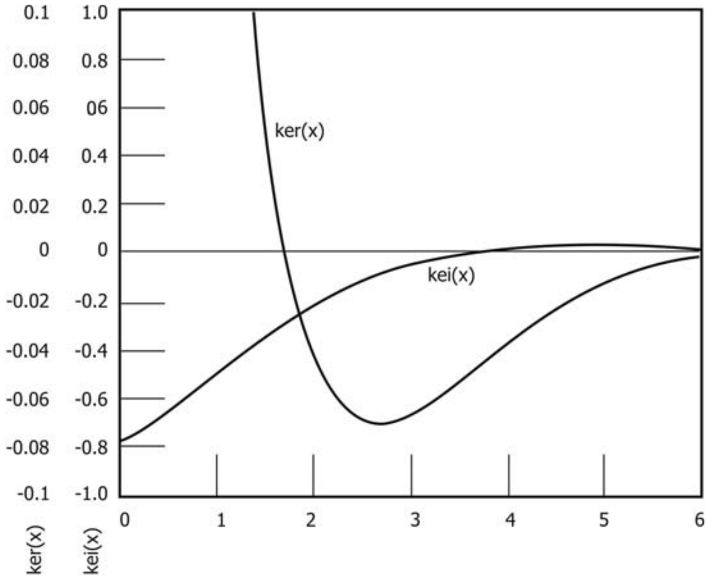

Figure E-2. Plot of the Kelvin functions ker(x) and kei(x).

A plot of the Kelvin functions is given in Figures E-1 and E-2. The Kelvin functions are tabulated in Table E-1 and Table E-2. It is noted that the functions ber(x), bei(x), and kei(x) are finite at the origin, but the other function ker(x) is infinite at the origin:

$$\text{ber}(0) = 1.00000$$
$$\text{bei}(0) = 0.00000$$
$$\text{kei}(0) = -0.78540$$
$$\text{ker}(0) = -\infty$$

TABLE E-1. Values of the Kelvin Functions ber(x) and bei(x), and the First Derivatives of These Functions

x	ber(x)	bei(x)	ber′(x)	bei′(x)
0.0	1.00000	0.00000	0.00000	0.00000
0.1	1.00000	0.00250	−0.00006	0.03536
0.2	0.99998	0.01000	−0.00050	0.10000
0.3	0.99987	0.02250	−0.00169	0.14999
0.4	0.99960	0.04000	−0.00400	0.19997
0.5	0.99902	0.06249	−0.00781	0.24992
0.6	0.99798	0.08998	−0.01350	0.29980
0.7	0.99625	0.12245	−0.02143	0.34956
0.8	0.99360	0.15989	−0.03199	0.39915
0.9	0.98975	0.20227	−0.04554	0.44846
1.0	0.98438	0.24957	−0.06245	0.49740
1.1	0.97714	0.30172	−0.08678	0.54211
1.2	0.96763	0.35870	−0.10781	0.59352
1.3	0.95543	0.42041	−0.13697	0.64034
1.4	0.94008	0.48673	−0.17093	0.68601
1.5	0.92107	0.55756	−0.21001	0.73025
1.6	0.89789	0.63273	−0.25454	0.77274
1.7	0.86997	0.71204	−0.30484	0.81310
1.8	0.83672	0.79526	−0.36118	0.85093
1.9	0.79752	0.88212	−0.42384	0.88574
2.0	0.75173	0.97229	−0.49307	0.91701
2.1	0.69869	1.06539	−0.56906	0.94418
2.2	0.63769	1.16097	−0.65200	0.96661
2.3	0.56805	1.25853	−0.74202	0.98361
2.4	0.48905	1.35749	−0.83920	0.99443
2.5	0.39997	1.45718	−0.94358	0.99827
2.6	0.30009	1.55688	−1.05513	0.99426
2.7	0.18871	1.65574	−1.17375	0.98149
2.8	0.06511	1.75285	−1.29926	0.95897
2.9	−0.07137	1.84718	−1.43141	0.92566
3.0	−0.22138	1.93759	−1.56985	0.88048
3.2	−0.56438	2.10157	−1.86362	0.74992
3.4	−0.96804	2.23345	−2.17550	0.55769
3.6	−1.43531	2.31986	−2.49825	0.29366
3.8	−1.96742	2.34543	−2.82216	−0.05253
4.0	−2.56342	2.29269	−3.13465	−0.49114
4.5	−4.29909	1.68602	−3.75368	−2.05263
5.0	−6.23008	0.11603	−3.84534	−4.35414
5.5	−7.98927	−2.79446	−2.91281	−7.38787
6.0	−8.93846	−7.23686	−0.29308	−10.84622

Handbook of Mathematical Functions, M. Abramowitz and I. Stegun, eds., National Bureau of Standards Applied Mathematics From Series 55, U.S. Government Printing Office, Washington, DC (1964), p. 381.

TABLE E-2. Values of the Kelvin Functions ker(x) and kei(x) and the First Derivatives of These Functions

x	$\ker(x)$	$\mathrm{kei}(x)$	$\ker'(x)$	$\mathrm{kei}'(x)$
0.0	∞	−0.78540	$-\infty$	0.00000
0.1	2.42047	−0.77685	−9.96096	0.14797
0.2	1.73314	−0.75812	−4.92295	0.22293
0.3	1.33722	−0.73310	−3.21987	0.27429
0.4	1.06262	−0.70380	−2.35207	0.30951
0.5	0.85591	−0.67158	−1.81980	0.33320
0.6	0.69312	−0.63745	−1.45654	0.34816
0.7	0.56138	−0.60218	−1.19094	0.35631
0.8	0.45288	−0.56637	−0.98734	0.35904
0.9	0.36251	−0.53051	−0.82587	0.35744
1.0	0.28671	−0.49499	−0.69460	0.35237
1.1	0.22284	−0.46013	−0.58591	0.34452
1.2	0.16895	−0.42616	−0.49464	0.33447
1.3	0.12346	−0.39329	−0.41723	0.32271
1.4	0.08513	−0.36166	−0.35106	0.30964
1.5	0.05293	−0.33140	−0.29418	0.29561
1.6	0.02603	−0.30257	−0.24511	0.28090
1.7	0.00369	−0.27523	−0.20268	0.26578
1.8	−0.01470	−0.24942	−0.16592	0.25044
1.9	−0.02966	−0.22514	−0.13413	0.23507
2.0	−0.04166	−0.20240	−0.10660	0.21981
2.1	−0.05111	−0.18117	−0.08282	0.20479
2.2	−0.05834	−0.16143	−0.06234	0.19011
2.3	−0.06367	−0.14314	−0.04475	0.17586
2.4	−0.06737	−0.12624	−0.02971	0.16211
2.5	−0.06969	−0.11070	−0.01693	0.14890
2.6	−0.07083	−0.09644	−0.00614	0.13627
2.7	−0.07097	−0.08342	0.00290	0.12426
2.8	−0.07030	−0.07157	0.01040	0.11287
2.9	−0.06894	−0.06083	0.01653	0.10214
3.0	−0.06703	−0.05112	0.02148	0.09204
3.2	−0.06198	−0.03458	0.02836	0.07378
3.4	−0.05590	−0.02145	0.03207	0.05799
3.6	−0.04932	−0.01123	0.03341	0.04454
3.8	−0.04365	−0.00349	0.03304	0.03325
4.0	−0.03618	0.00220	0.03148	0.02391
4.5	−0.02200	0.00972	0.02481	0.00772
5.0	−0.01151	0.01119	0.01719	−0.00082
5.5	−0.00465	0.00975	0.01060	−0.00440
6.0	−0.00065	0.00722	0.00563	−0.00522

Handbook of Mathematical Functions, M. Abramowitz and I. Stegun, eds., National Bureau of Standards Applied Mathematics Series 55, U.S. Government Printing Office, Washington, DC (1964), p. 381.

The Kelvin functions may be generalized to other orders; however, when the order is zero, the subscript is usually omitted, i.e., $\text{ber}_0(x) \equiv \text{ber}(x)$. For any order n, the following series functions apply:

$$\text{ber}_n(x) = \sum_{j=0}^{\infty} \frac{\cos[(\frac{3}{4}n + \frac{1}{2}j)\pi](\frac{1}{2}x)^{2j+n}}{j!\Gamma(n+j+1)} \tag{E-12}$$

$$\text{bei}_n(x) = \sum_{j=0}^{\infty} \frac{\cos[(\frac{3}{4}n + \frac{1}{2}j)\pi](\frac{1}{2}x)^{2j+n}}{j!\Gamma(n+j+1)} \tag{E-13}$$

E.3 DIFFERENTIAL EQUATION FOR KELVIN FUNCTIONS

The Kelvin functions are solutions of the Bessel differential equation with a complex term:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \left(i + \frac{n^2}{x^2}\right)y = 0 \quad \text{where} \quad i = \sqrt{-1} \tag{E-14}$$

The general solution is as follows, where the constants of integration may be complex:

$$y(x) = C_1[\text{ber}_n(x) + i\text{bei}_n(x)] + C_2[\text{ker}_n(x) + i\text{kei}_n(x)] \tag{E-15}$$

The four Kelvin functions are also solutions of the following fourth-order differential equation:

$$\frac{d^4y}{dx^4} + \frac{2}{x}\frac{d^3y}{dx^3} - \frac{(1+2n^2)}{x^2}\left(\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx}\right) + \left[\frac{n^2(n^2-4)}{x^4} + 1\right]y = 0 \tag{E-16}$$

The differential equation may be written in the following alternate form:

$$\frac{1}{x}\frac{d}{dx}\left\{x\frac{d}{dx}\left[\frac{1}{x}\frac{d}{dx}\left(x\frac{dy}{dx}\right) - \frac{n^2}{x^2}y\right]\right\} - \frac{n^2}{x^2}\left[\frac{1}{x}\frac{d}{dx}\left(x\frac{dy}{dx}\right) - \frac{n^2}{x^2}y\right] = 0 \tag{E-17}$$

This equation is encountered in several problems in analysis of plates and shells, including the evaluation of displacements of a shallow spherical shell, a conical shell, and a circular plate on an elastic foundation.

The general solution of the differential equation, involving real constants of integration is

$$y(x) = C_1\text{ber}_n(x) + C_2\text{bei}_n(x) + C_3\,\text{ker}_n(x) + C_4\text{kei}_n(x) \tag{E-18}$$

E.4 RECURRENCE RELATIONSHIPS FOR THE KELVIN FUNCTIONS

The derivative and integral relationships for the Kelvin functions of general order are more complicated than those for the other Bessel functions. The six recurrence relationships for the Kelvin functions are as follows:

(a)
$$f_{n+1} = -\frac{n\sqrt{2}}{x}(f_n - g_n) - f_{n-1} \tag{E-19}$$

(b)
$$\frac{df_n}{dx} = \frac{1}{2\sqrt{2}}(f_{n+1} + g_{n+1} - f_{n-1} - g_{n-1}) \tag{E-20}$$

(c)
$$\frac{df_n}{dx} = \frac{n}{x}f_n + \frac{1}{\sqrt{2}}(f_{n+1} + g_{n+1}) \tag{E-21}$$

(d)
$$\frac{df_n}{dx} = -\frac{n}{x}f_n - \frac{1}{\sqrt{2}}(f_{n-1} + g_{n-1}) \tag{E-22}$$

(e)
$$\int x^{n+1} f_n dx = -\frac{x^{1+n}}{\sqrt{2}}(f_{n+1} - g_{n+1}) \tag{E-23}$$

(f)
$$\int x^{1-n} f_n dx = \frac{x^{1-n}}{\sqrt{2}}(f_{n-1} - g_{n-1}) \tag{E-24}$$

In the recursion relationships, the f_n and g_n are any two combinations:

$$\begin{cases} f_n = \mathrm{ber}_n(x) \\ g_n = \mathrm{bei}_n(x) \end{cases} \qquad \begin{cases} f_n = \mathrm{bei}_n(x) \\ g_n = -\mathrm{ber}_n(x) \end{cases} \qquad \begin{cases} f_n = \mathrm{ker}_n(x) \\ g_n = \mathrm{kei}_n(x) \end{cases} \qquad \begin{cases} f_n = \mathrm{kei}_n(x) \\ g_n = -\mathrm{ker}_n(x) \end{cases}$$

In particular, we find the following relationships for the Kelvin functions of zero order:

$$\mathrm{ber}'(x) = \frac{d[\mathrm{ber}(x)]}{dx} = [\mathrm{ber}_1(x) + \mathrm{bei}_1(x)]/\sqrt{2} \tag{E-25}$$

$$\mathrm{bei}'(x) = \frac{d[\mathrm{bei}(x)]}{dx} = [-\mathrm{ber}_1(x) + \mathrm{bei}_1(x)]/\sqrt{2} \tag{E-26}$$

$$\mathrm{ker}'(x) = \frac{d[\mathrm{ker}(x)]}{dx} = [\mathrm{ker}_1(x) + \mathrm{kei}_1(x)]/\sqrt{2} \tag{E-27}$$

$$\mathrm{kei}'(x) = \frac{d[\mathrm{kei}(x)]}{dx} = [-\mathrm{ker}_1(x) + \mathrm{kei}_1(x)]/\sqrt{2} \tag{E-28}$$

The second-derivative relationships are

$$\mathrm{ber}''(x) = -\mathrm{bei}(x) - \frac{\mathrm{ber}'(x)}{x} \tag{E-29}$$

$$\mathrm{bei}''(x) = \mathrm{ber}(x) - \frac{\mathrm{bei}'(x)}{x} \tag{E-30}$$

$$\mathrm{ker}''(x) = -\mathrm{kei}(x) - \frac{\mathrm{ker}'(x)}{x} \tag{E-31}$$

$$\mathrm{kei}''(x) = \mathrm{ker}(x) - \frac{\mathrm{kei}'(x)}{x} \tag{E-32}$$

The integral expressions are

$$\int x\,\mathrm{ber}(x)\,dx = x\,\mathrm{bei}'(x) \tag{E-33}$$

$$\int x\,\mathrm{bei}(x)\,dx = -x\,\mathrm{ber}'(x) \tag{E-34}$$

$$\int x\,\mathrm{ker}(x)\,dx = x\,\mathrm{kei}'(x) \tag{E-35}$$

$$\int x\,\mathrm{kei}(x)\,dx = -x\,\mathrm{ker}'(x) \tag{E-36}$$

E.5 ASYMPTOTIC RELATIONS FOR THE KELVIN FUNCTIONS

For large values of the augment ($x \geq 6$), the following limiting (asymptotic) relationships may be used for the Kelvin functions and their derivatives:

$$\mathrm{ber}(x) \to \frac{\exp(x/\sqrt{2})}{\sqrt{2\pi x}} \cos\varphi_1 \tag{E-37}$$

$$\mathrm{bei}(x) \to \frac{\exp(x/\sqrt{2})}{\sqrt{2\pi x}} \sin\varphi_1 \tag{E-38}$$

$$\mathrm{ker}(x) \to \sqrt{\frac{\pi}{2x}} \exp(-x/\sqrt{2}) \cos\varphi_2 \tag{E-39}$$

$$\mathrm{kei}(x) \to -\sqrt{\frac{\pi}{2x}} \exp(-x/\sqrt{2}) \sin\varphi_2 \tag{E-40}$$

$$\mathrm{ber}'(x) \to \frac{\exp(x/\sqrt{2})}{\sqrt{2\pi x}} \cos\varphi_2 \tag{E-41}$$

$$\mathrm{bei}'(x) \to \frac{\exp(x/\sqrt{2})}{\sqrt{2\pi x}} \sin\varphi_2 \tag{E-42}$$

$$\mathrm{ker}'(x) \to -\sqrt{\frac{\pi}{2x}}\exp(-x/\sqrt{2})\cos\varphi_1 \tag{E-43}$$

$$\mathrm{kei}'(x) \to \sqrt{\frac{\pi}{2x}}\exp(-x/\sqrt{2})\sin\varphi_1 \tag{E-44}$$

The angles in these expressions are

$$\varphi_1 = \frac{x}{\sqrt{2}} - \frac{\pi}{8} \qquad \text{and} \qquad \varphi_2 = \frac{x}{\sqrt{2}} + \frac{\pi}{8} \tag{E-45}$$

E.6 ZEROS OF THE KELVIN FUNCTIONS

The solutions $x_1, x_2, x_3, \ldots$ of the equation: $\mathrm{ber}(x_k) = 0$ are called the *zeros* of the ber function. Because the series for the Kelvin functions involve both positive and negative algebraic signs, we find that there is an infinite set of zeros for all four Kelvin functions. The first five zeros for the Kelvin functions and their derivatives are given in Table E-3.

TABLE E-3. First Five zeros of the Kelvin Functions and Their First Derivatives

ZERO	$\mathrm{ber}(x_k)$	$\mathrm{bei}(x_k)$	$\mathrm{ker}(x_k)$	$\mathrm{kei}(x_k)$
1st	2.84892	5.02622	1.71854	3.91467
2nd	7.23883	9.45541	6.12728	8.34422
3rd	11.67396	13.89349	10.56294	12.78256
4th	16.11356	18.33398	15.00269	17.22314
5th	20.55463	22.77544	19.44381	21.66464
	$\mathrm{ber}'(x_k)$	$\mathrm{bei}'(x_k)$	$\mathrm{ker}'(x_k)$	$\mathrm{kei}'(x_k)$
1st	6.03871	3.77320	2.66584	4.93181
2nd	10.51364	8.28099	7.17212	9.40405
3rd	14.96844	12.74215	11.63218	13.85827
4th	19.41758	17.19343	16.08312	18.30717
5th	23.86430	21.64114	20.53068	22.75379

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APPENDIX F

MATRICES AND DETERMINANTS

F.1 DETERMINANTS

F.1.1 Definition of a Determinant

Determinants are important elements in matrix analysis, so we first need to consider some of the main characteristics of determinants. A *determinant* is an $n \times n$ square array of numbers or functions:

$$D_n = \begin{vmatrix} a_1 & b_1 & c_1 & \cdots & \cdots \\ a_2 & b_2 & c_2 & \cdots & \cdots \\ a_3 & b_3 & c_3 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & c_n & \cdots & \cdots \end{vmatrix} \tag{F-1}$$

The value of the determinant D_n is

$$D_n = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\cdots}^{n} \varepsilon_{ijk\,\ldots}\, a_i b_j c_k \cdots \tag{F-2}$$

where the quantity $\varepsilon_{ijk\ldots}$ is defined as

$$\varepsilon_{ijk\cdots} = \begin{cases} +1 & \text{for an even permutation of } ijk\cdots \\ -1 & \text{for an odd permutation of } ijk\cdots \\ \;\;0 & \text{if any pair of indices are equal} \end{cases}$$

An *even permutation* is defined as any arrangement of the indices $ijk\cdots$ that can be brought back to the 1, 2, 3, ... order by an *even* number of interchanges of pairs of indices. For example, for a 3 × 3 determinant, the arrangement (3, 1, 2) is an even permutation, because it can be brought to the (1, 2, 3) order by two exchanges, and $\varepsilon_{312} = +1$:

$$(3, 1, 2) \rightarrow (\mathbf{2}, \mathbf{1}, \mathbf{3})\text{(1st exchange)} \rightarrow (\mathbf{1}, \mathbf{2}, 3)\text{(2nd exchange)}$$

On the other hand, the arrangement (1, 3, 2) is an odd permutation, because it can be brought to the (1, 2, 3) order by one exchange, so $\varepsilon_{132} = -1$:

$$(1,\ 3,\ 2) \rightarrow (1, \mathbf{2}, \mathbf{3})\ \text{(1st exchange)}$$

The quantity $\varepsilon_{112} = 0$ because a pair of subscripts is repeated.

The expressions for the values of the 2 × 2 and 3 × 3 determinants are relatively simple:

$$D_2 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \tag{F-3}$$

$$D_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1 \tag{F-4}$$

F.1.2 Expansion of a Determinant by Cofactors

For larger determinants, the expanded expression for the value of the determinant becomes more cumbersome. Note that for a 3 × 3 determinant, eq. (F-4) may be written in the following form:

$$D_3 = a_1(b_2 c_3 - b_3 c_2) - a_2(b_1 c_3 - b_3 c_1) + a_3(b_1 c_2 - b_2 c_1)$$

or

$$D_3 = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \tag{F-5}$$

The determinants in eq. (F-5) are called the *minors* of the element a_1, a_2, and a_3, respectively. The minor M_{ij} of the element in the ith row and jth column is obtained by omitting the corresponding row and column in the original determinant. The *cofactor* Cof(i, j) of the element in the ith row and jth column is the corresponding minor multiplied by the algebraic sign given by $(-1)^{i+j}$:

$$\text{Cof}\ (i,\ j) = (-1)^{i+j} M_{ij} \tag{F-6}$$

For example, eq. (F-5) for the 3 × 3 determinant may be written as

$$D_3 = \sum_{i=1}^{3} a_i\ \text{Cof}\,(i,\ 1) = \sum_{i=1}^{3} (-1)^{i+1} a_i M_{i1} \tag{F-7}$$

Equation (F-7) can be extended to determinants of any size (for example, $n \times n$) by increasing the limit to which the summation is carried (for example, to n). In evaluating the determinant, eq. (F-7) would need to be applied successively in evaluating the resulting $(n-1) \times (n-1)$ determinants, the $(n-2) \times (n-2)$ determinants, etc. Following is an example of the evaluation of a larger determinant.

Example F-1 Determine the value of the following 4×4 determinant.

$$D_4 = \begin{vmatrix} 4 & 2 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 4 \end{vmatrix}$$

The determinant may be evaluated by expanding along the first column:

$$D_4 = (4)\begin{vmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 2 & 4 \end{vmatrix} - (1)\begin{vmatrix} 2 & 0 & 0 \\ 1 & 4 & 1 \\ 0 & 2 & 4 \end{vmatrix} + (0)\begin{vmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 2 & 4 \end{vmatrix} - (0)\begin{vmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 4 & 1 \end{vmatrix}$$

Each of the resulting 3×3 determinants may be evaluated by a similar procedure:

$$\begin{aligned} D_4 = (4)&\left[(4)\begin{vmatrix} 4 & 1 \\ 2 & 4 \end{vmatrix} - (1)\begin{vmatrix} 1 & 0 \\ 2 & 4 \end{vmatrix} + (0)\begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix}\right] \\ &- \left[(2)\begin{vmatrix} 4 & 1 \\ 2 & 4 \end{vmatrix} - (1)\begin{vmatrix} 0 & 0 \\ 2 & 4 \end{vmatrix} + (0)\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}\right] + 0 \end{aligned}$$

The resulting 2×2 determinants may be evaluated from eq. (F-3):

$$D_4 = (4)[(4)(16-2) - (1)(4-0) + 0] - [(2)(16-2) - (1)(0) + 0]$$

$$D_4 = (4)[56-4] - [28] = 208 - 28 = 180$$

F.1.3 Properties of Determinants

There are several properties of determinants that are helpful in efficient evaluation of a determinant.

(a) If each element in a row or each element in a column is zero, then the value of the determinant is zero. It is obvious from eq. (F-5) for a 3×3 determinant that, if $a_1 = a_2 = a_3 = 0$, then $D_3 = 0$.

(b) If any two rows or any two columns are equal, then the value of the determinant is zero.

(c) If each element in a row or each element in a column is multiplied by a constant (k, for example), then the value of the determinant is multiplied by the same constant.

(d) If two rows or two columns are interchanged in a determinant, the algebraic sign of the value of the determinant is changed.

(e) The value of the determinant is not changed if a multiple of one row is added (column by column) to another row or if a multiple of one column is added (row by row) to another column. For example, for a 3×3 determinant, if we multiply the 2nd column by a constant k and add this result to the 1st column, then expand the resulting determinant by cofactors, we obtain

$$\begin{vmatrix} a_1 + kb_1 & b_1 & c_1 \\ a_2 + kb_2 & b_2 & c_2 \\ a_3 + kb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}$$

However, the 2nd determinant on the right side has two identical columns; therefore, the value of this determinant is zero.

(f) If two rows are proportional or if two columns are proportional, the value of the determinant is zero. This result may be demonstrated by using properties (b), (c), and (e).

These properties may be utilized in developing a computer-aided algorithm for evaluating large determinants.

F.1.4 Computer-Aided Evaluation of Determinants

Although the procedure of evaluating a determinant using cofactors is an exact mathematical method, it is not computationally convenient for evaluation of large determinants, such as a 100×100 determinant.

A more computer-friendly method for evaluation of a determinant involves "diagonalization" of the determinant. For example, suppose we have a determinant of the form given by eq. (F-1). We may factor out a_1 from each element of the first row to obtain

$$D_n = a_1 \begin{vmatrix} 1 & (b_1/a_1) & (c_1/a_1) & \cdots & \cdots \\ a_2 & b_2 & c_2 & \cdots & \cdots \\ a_3 & b_3 & c_3 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & c_n & \cdots & \cdots \end{vmatrix} \qquad \text{(F-8)}$$

By multiplying the first row by a_2 and subtracting the first row from the second row, we obtain a new second row with a zero as the first element. By repeating the process for the remaining rows, the following determinant is obtained:

$$D_n = a_1 \begin{vmatrix} 1 & (b_1/a_1) & (c_1/a_1) & \cdots & \cdots \\ 0 & b_2 - (b_1/a_1)a_2 & c_2 - (c_1/a_1)a_2 & \cdots & \cdots \\ 0 & b_3 - (b_1/a_1)a_3 & c_3 - (c_1/a_1)a_3 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & b_n - (b_1/a_1)a_n & c_n - (c_1/a_1)a_n & \cdots & \cdots \end{vmatrix} \qquad \text{(F-9)}$$

This process may be repeated for the second row, then the resulting third row, etc., until the diagonal elements of the determinant are all equal to one and the elements below the diagonal are all equal to zero. During this process, round-off error may be reduced by exchange of adjacent rows, if needed, to make the value of the diagonal element (before factorization) large. The resulting determinant has a sign change for each exchange. The value of the determinant is then found from the product of the quantities factored from the determinant. This process is best illustrated by an example.

Example F-2 Let us evaluate the 4×4 matrix given in Example F-1 by the diagonalization procedure. First, let us factor out 4 from the first row:

$$D_4 = (4)\begin{vmatrix} 1 & 0.5000 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 4 \end{vmatrix}$$

The element in the first column and second row is already one, so we may subtract the first row from the second row, element by element, to obtain

$$D_4 = (4)\begin{vmatrix} 1 & 0.5000 & 0 & 0 \\ 0 & 3.5000 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 4 \end{vmatrix}$$

Next, factoring out 3.5000 from the second row, the following is obtained:

$$D_4 = (4)(3.5000)\begin{vmatrix} 1 & 0.5000 & 0 & 0 \\ 0 & 1 & 0.2857 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 4 \end{vmatrix}$$

Subtracting the element in the second row from those in the third row, we obtain

$$D_4 = (4)(3.5000)\begin{vmatrix} 1 & 0.5000 & 0 & 0 \\ 0 & 1 & 0.2857 & 0 \\ 0 & 0 & 3.7143 & 1 \\ 0 & 0 & 2 & 4 \end{vmatrix}$$

Continuing, we may factor out 3.7143 from the third row:

$$D_4 = (4)(3.500)(3.7143)\begin{vmatrix} 1 & 0.5000 & 0 & 0 \\ 0 & 1 & 0.2857 & 0 \\ 0 & 0 & 1 & 0.2692 \\ 0 & 0 & 2 & 4 \end{vmatrix}$$

Factoring out 2 from the elements of the fourth row and subtracting the third row from the elements in the fourth row, we get

$$D_4 = (4)(3.5000)(3.7143)(2) \begin{vmatrix} 1 & 0.5000 & 0 & 0 \\ 0 & 1 & 0.2857 & 0 \\ 0 & 0 & 1 & 0.2692 \\ 0 & 0 & 0 & 1.7308 \end{vmatrix}$$

Finally, we may factor 1.7308 from the fourth row:

$$D_4 = (4)(3.5000)(3.7143)(2)(1.7308) \begin{vmatrix} 1 & 0.5000 & 0 & 0 \\ 0 & 1 & 0.2857 & 0 \\ 0 & 0 & 1 & 0.2692 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

The value of the last determinant is 1; therefore the value of the original determinant is

$$D_4 = (4)(3.5000)(3.7143)(2)(1.7308)(1) = 180.00$$

F.2 MATRICES

F.2.1 Definition of a Matrix

A matrix is defined as a rectangular (or square) array of numbers or functions that obey certain rules. The matrix is *not* a determinant. It is not a single number; instead, it is an ordered arrangement of numbers or functions.

Several notations have been used for matrices. For example, a rectangular $m \times n$ matrix may be written by enclosing the elements in rectangular brackets, as follows. Note that the first subscript (m) always refers to the *row* of the element and the second subscript (n) refers to the *column* of the element:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \tag{F-8}$$

Some authors represent the matrix by one of its general elements, a_{ij}:

$$\mathbf{A} = [a_{ij}]$$

where $i = 1, 2, \ldots, m$; and $j = 1, 2, \ldots, n$

The $1 \times n$ and $m \times 1$ matrices are called *row* matrices and *column* matrices, respectively. These quantities are also called *vectors* in the numerical analysis vocabulary. A common notation for these matrices is

$$\mathbf{R} = [\, a_{11} \quad a_{12} \quad a_{13} \quad \cdots \quad a_{1n} \quad \text{(Row matrix or row vector)}\,]$$

$$\mathbf{C} = \begin{Bmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{Bmatrix} \quad \text{(Column matrix or column vector)}$$

F.2.2 Special Matrices

There are several special matrices that are used in matrix analysis. A *square matrix* is a matrix for which the number of rows and number of columns are the same. A *diagonal* matrix is a square matrix for which only the elements along the diagonal a_{ii} are nonzero:

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \tag{F-9}$$

Two matrices are of particular interest in matrix algebra: the *null* matrix $\boldsymbol{\emptyset}$ and the *unit* matrix **I**. The null matrix is analogous to "zero" in numerical analysis and is defined as a matrix for which all elements are zero, as follows.

$$\emptyset = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{F-10}$$

Similarly, the *unit matrix* (sometimes called the *identity matrix*) is analogous to the number "1" in numerical analysis and is defined as a matrix for which the values of each element is given by $a_{ij} = \delta_{ij} =$ *Kronecker delta*, as follows:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{F-11}$$

The null matrix and the unit matrix are usually square matrices.

F.2.3 Matrix Addition

The addition of two matrices $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ to obtain the sum matrix $\mathbf{C} = [c_{ij}]$ is defined by

$$c_{ij} = a_{ij} + b_{ij} \qquad \text{for all values of } i \text{ and } j \tag{F-12}$$

To be able to add two matrices, the number of rows must be the same for both matrices and the number of columns must be the same for both matrices. In other words, it is impossible to add the following matrices:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

As mentioned in the infamous line in an old B-rated science-fiction movie, "There are some things that mankind is not meant to do." One of these things is to add matrices of unequal number of rows and/or columns.

Based on the definition of the operation of addition of matrices, it is noted that the *commutation principle* is valid for addition of matrices:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{F-13}$$

The *associative principle* is also valid:

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \tag{F-14}$$

It is observed that adding the null matrix $\boldsymbol{\emptyset}$ to any matrix $\mathbf{A}$ yields the same matrix $\mathbf{A}$, which is similar to the fact that adding zero to any number yields the number:

$$\mathbf{A} + \emptyset = \mathbf{A} \tag{F-15}$$

F.2.4 Matrix Multiplication

The multiplication of an $(m \times n)$ matrix $\mathbf{A}$ by a constant (scalar) c is defined as follows:

$$c\mathbf{A} = [ca_{ij}] = \begin{bmatrix} ca_{11} & ca_{12} & ca_{13} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & ca_{23} & \cdots & ca_{2n} \\ ca_{31} & ca_{32} & ca_{33} & \cdots & ca_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ ca_{m1} & ca_{m2} & ca_{m3} & \cdots & ca_{mn} \end{bmatrix} \tag{F-16}$$

Matrix multiplication is defined by

$$\mathbf{AB} = \mathbf{C} = [c_{ij}] \tag{F-17}$$

where the components of the $\mathbf{C}$ matrix are found as follows:

$$c_{ij} = \sum_k a_{ik} b_{kj} \tag{F-18}$$

The (ij) element of $\mathbf{C}$ is found by multiplying the ith row of $\mathbf{A}$ by the jth column of $\mathbf{B}$. To perform this operation, $\mathbf{A}$ must have the same number of *columns* as the number of *rows* of $\mathbf{B}$. For example, it is possible to form the product $(\mathbf{AB})$ of the following matrices:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

The resulting product is

$$\mathbf{C} = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}) & (a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}) \\ (a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}) & (a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}) \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

Except for special cases, the commutative principle does not apply for matrix multiplication. Therefore, in general, it is true that

$$\mathbf{AB} \neq \mathbf{BA} \tag{F-19}$$

On the other hand, the associative principle and distributive principle apply for matrix multiplication:

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) \tag{F-20}$$

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC} \tag{F-21}$$

F.2.5 Inverse Matrix

The *inverse matrix* is defined from the following, if $\mathbf{A}$ is a square matrix:

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \tag{F-22}$$

The mathematical procedure for finding the inverse of a matrix involves the operation of *transposing* a matrix. The transpose of matrix $\mathbf{A}$ is denoted by $\mathbf{A}^T$, and is defined by interchanging rows and columns of the original matrix:

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{n1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{n2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{n3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn} \end{bmatrix} \tag{F-23}$$

For a matrix $\mathbf{A} = [a_{ij}]$, the mathematical procedure for finding the inverse is

$$\mathbf{A}^{-1} = \frac{1}{\det(A)}\left[A_{ij}\right]^T = \frac{1}{\det(A)}\begin{bmatrix} A_{11} & A_{21} & A_{31} & \cdots & A_{n1} \\ A_{12} & A_{22} & A_{32} & \cdots & A_{n2} \\ A_{13} & A_{23} & A_{33} & \cdots & A_{n3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & A_{3n} & \cdots & A_{nn} \end{bmatrix} \tag{F-24}$$

The quantity det(A) is the determinant of the elements of the matrix $\mathbf{A}$. The elements A_{ij} are the cofactors of the matrix elements a_{ij}, as given by eq. (F-6). For example,

$$A_{12} = (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

The elements of the determinant are selected by striking out the 1st row and the 2nd column of the $\mathbf{A}$ matrix.

Example F-3 Determine the inverse $\mathbf{A}^{-1}$ of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 0 \\ 1 & 4 & 1 \\ 0 & 2 & 4 \end{bmatrix}$$

The determinant of the matrix is found as follows:

$$\det(A) = \begin{vmatrix} 4 & 2 & 0 \\ 1 & 4 & 1 \\ 0 & 2 & 4 \end{vmatrix} = (4)\begin{vmatrix} 4 & 1 \\ 2 & 4 \end{vmatrix} - (1)\begin{vmatrix} 2 & 0 \\ 2 & 4 \end{vmatrix} + (0)\begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix} = (4)(14) - 8 = 48$$

The cofactors for the matrix are found as follows:

$$A_{11} = (-1)^2 \begin{vmatrix} 4 & 1 \\ 2 & 4 \end{vmatrix} = +14$$

$$A_{12} = (-1)^3 \begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix} = -4$$

$$A_{13} = (-1)^4 \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = +2$$

Similarly,

$$A_{21} = -8; \quad A_{22} = +16; \quad A_{23} = -8$$

$$A_{31} = -2; \quad A_{32} = -4; \quad A_{33} = +14$$

The inverse matrix is

$$\mathbf{A}^{-1} = \frac{1}{48}\begin{bmatrix} 14 & -8 & -2 \\ -4 & 16 & -4 \\ 2 & -8 & 14 \end{bmatrix} = \begin{bmatrix} \frac{7}{24} & -\frac{1}{6} & -\frac{1}{24} \\ -\frac{1}{12} & \frac{1}{3} & -\frac{1}{12} \\ \frac{1}{24} & -\frac{1}{6} & \frac{7}{24} \end{bmatrix}$$

F.2.6 Computer-Aided Methods for Matrix Inversion

The method for calculating the inverse of a square matrix described in Section F.2.5 is the direct mathematical method; however, it is not the most computationally effective method for matrix inversion. Numerical calculation of the inverse of a 2×2 or a 3×3 matrix using the direct method is satisfactory; however, the calculation of the inverse of a 100×100 matrix by the direct method is extremely cumbersome. A more computationally effective method of calculating the inverse may be developed from the definition of the inverse, eq. (F-11).

Suppose we let the inverse be written as

$$\mathbf{A}^{-1} = [b_{ij}]$$

For a 3×3 matrix, the definition of the inverse may be written as

$$\mathbf{AA}^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I} \quad \text{(F-25)}$$

The numerical values of the a_{ij}'s are known quantities, and the values of the b_{ij}'s are to be determined.

Based on the principles of matrix multiplication, eq. (F-18), the definition of the matrix inverse may be written as a set of three equations:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \tag{F-26}$$
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

These relationships are equivalent to three sets of three simultaneous equations in each set:

$$\begin{cases} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = 1 \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} = 0 \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} = 0 \end{cases}$$
$$\begin{cases} a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} = 0 \\ a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} = 1 \\ a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} = 0 \end{cases} \tag{F-27}$$
$$\begin{cases} a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} = 0 \\ a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} = 0 \\ a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} = 1 \end{cases}$$

There are many computer programs available for the solution of simultaneous equations [Yakowitz and Szidarovszky, 1989; Griffiths and Smith, 2006]. Generally, for up to about 200 simultaneous equations, the Gauss-Jordan algorithm can be used efficiently. For more than about 200 simultaneous equations, the Jacobi iterative algorithm or the Gauss-Seidel algorithm is used to avoid round-off error problems.

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INDEX

A

Airy, Sir George Biddell, 256
Airy stress function, 217, 243, 247, 253, 440
Axial stress, 322, 340, 378, 451–453

B

Bars, 27
  with expansion gaps, 28, 35
  with nonuniform temperature, 43–48
  in parallel, 32–35
  with spring elements, 29, 39–43
Bathysphere, 414
Beam
  bending, 65, 257, 264, 270, 311, 422
  cantilever, 69
  elastic support, 87, 274, 350
  simply-supported, 74, 114, 274
  statically indeterminate, 80
Beam-column, *see* Columns
Beltrami, Eugenio, 213, 215–217
Beltrami-Michell equations, 213
Bending
  axisymmetric, 268, 287, 291
  of plates, 264–310
  of shells, 319–371
  two-dimensional, 305
Bernoulli, Daniel, 109
Bernoulli, Jacob, 109, 469
Bessel, Friedrich W., 470
Bessel functions, 481 (table)
  asymptotic relations for, 476, 480
  differential equation for, 474
  modified, 234, 477, 482 (table)
  noninteger order, 470–472
  recurrence relations for, 475, 479–480
  series for, 391, 470
  zeroes for, 483
Biharmonic operator, 243, 245, 272, 432
Bifurcation, 423
Billiard ball, 414
Biot number, 390, 403
Bowing, thermal, 97
Buckling, 416–454
  of circular plates, 432–437

Buckling (*continued*)
of columns, 416–426
critical temperature, 418
lateral, 426–431
mechanical load, 417, 444
of rectangular plates, 437–450
of shells, 451–454
thermal load, 445
Bulk modulus, 9, 24, 208, 214, 381

C

Castigliano, A., 185
Castigliano's theorem, 118, 186
Centroid axis, 46, 111
Circular plate, 287–310, 432–437
Circumferential stress, 232, 259, 322, 376
Coefficient of thermal expansion, 7–11, 214, 381, 462 (table)
Columns
buckling of, 416–426
critical temperature difference, 418
lateral buckling of, 426–431
postbuckling behavior, 423–426
Compatibility equations, 201, 217, 254
Complementary energy, 120
Complementary energy method, 119
Conduction, 45, 98, 208, 233, 384, 397
Constraints, 1
external, 129
internal, 132
partial, 85
reduction of, 4, 28–29
Convection, 233, 390, 403
Cooldown of vessels, 329–334
Coulomb, C. A., 109
Cryogenic transfer line, 42, 53
Curvature, 60–65, 269, 354
Cylinder
hollow, 384
long shell, 325–327
short shell, 344–350
solid, 388
steady-state conduction in, 384
thick-walled, 378–384
transient conduction in, 388–393

D

Deflection, 43, 102, 121–123, 137, 350, 424
Density, 462 (table)
Determinants, 67, 118, 125, 131, 136, 140, 443–447, 494
Dilatation, 208, 211, 214
Direction cosine, 126, 140
Displacement
axial, 321
circumferential, 200, 245
formulation, 226, 322, 381
transverse, 77, 82, 322, 338
Displacement potential, 219, 242, 247
Distortion, 26, 59, 199, 207, 219
Domes, spherical, 317, 357–361, 376
Duhamel, J. M., 22
Dummy load, 123

E

Eigenvalue, 390, 404
Elastic energy method, 118–123
Elastic limit, 27, 59, 222
Elongation, 13, 21, 29, 39, 119
Equilibrium equations
in cylindrical coordinates, 205, 225, 290, 380
in rectangular coordinates, 205, 241, 270
in spherical coordinates, 206, 398
Euler, Leonard, 109, 417, 454, 469
Euler buckling load, 417
Euler's constant, 486
Extensional rigidity, 26, 158, 198–202, 241, 268, 353

F

Factor of safety, *see* Safety factor
Failure criteria, 4–7, 108, 216, 413
Finite-element method, 142–146, 195–196
Flexibility factor, 168
Flexural rigidity, 270, 321
Flügge, Wilhelm, 372–373
Formulation
  displacement, 226, 242, 322, 381
  stress, 230, 243, 323, 383
Foundation modulus, 87, 92
Fourier, Jean Baptiste Joseph, 389, 411, 469
Fourier
  conduction rate equation, 98, 208, 233, 412
  number, 391
  series, 279, 285, 442
Frames, 153

G

Galileo, Galilei, 108, 117
Gamma function, 471
Gaps, expansion, 28, 35, 417
Goodier, J. N., 48, 58, 117, 180, 196, 257
Goodier displacement function, 242, 247, 262
Grüneisen constant, 10 (table), 24–25, 211
Grüneisen relationship, 9
Gyration, radius of, 68

H

Heat
  conduction, 45, 98, 208, 233, 384, 397
  exchanger, 55, 57, 172
  generation, 286, 413
  transfer coefficient, 233, 330, 390, 403
Hooke, Robert, 19
Hooke's law, 20, 27, 197, 289, 380–383
Hot spot, 224–239

I

Inertia, area moment of, 31, 62, 100, 427
Initial condition, 330, 389, 403
Internal heat generation, 210, 413
Inverse matrix, 502
Isotropic body, 8, 13–15, 181, 197, 206

K

Kelvin functions, 362, 364, 368, 485–493, 488 (table)
Kirchhoff, Gustave Robert, 310
Kirchhoff-Love hypothesis, 288, 310, 364, 368
Kroneker delta, 207, 214, 500

L

Lamé, Gabriel, 15, 21, 207, 214, 381
Lamé constant, 15, 22, 207, 381
Laplace operator, 212, 242, 272, 432
Love, Augustus Edward Hough, 371

M

Mariotte, M., 109
Material properties, 62, 318, 379, 462–464
Matrices, 146–151, 499
Maxwell, James Clerk, 186
Maxwell-Mohr method, 119, 186
Metric (SI) units, 461
Michell, John Henry, 213, 215, 217
Modulus of elasticity
  bulk, 15, 24, 208, 214, 381
  linear, 11, 462 (table)
  in shear, 15, 179

Moment
bending, 62
sign convention for, 63
thermal, 62, 81
twisting, 265
Moment of inertia, 31, 62, 75, 100, 156, 183, 427

N

Navier, C.L.M.H., 21, 109, 215, 316, 411
Navier thermoelasticity equations, 214
Neumann, Franz, 22, 310
Neutral axis, *see* Centroid axis

P

Pipe
bends, 172
bowing, 97
elbows, 168
expansion loops, 158–168
fittings, 168
smoker's, 413
Plane strain, 378–384
Plane stress, 221–256
Airy stress function, 217
biharmonic equation, 245
definition, 221
Plate
annular, 300
bending, 264–310
boundary conditions, 273–276, 291
circular, 287–310, 432–437
composite, 312
rectangular, 265–287, 437–450
with hole, 248, 300
Poisson, S.D., 21
Poisson's ratio, 13, 59, 462 (table)
Preliminary design, 3
Product of inertia, 64, 426
Properties of materials, 462–464

R

Radius of curvature, 60–65, 269
Radius of gyration, 68
Rectangular plate, 267, 437–450, 483
Redundant member, 132
Resultant force, 48, 53, 276
Rigidity
extensional, 241, 353
flexural, 270, 321
Ring, stiffening, 330–333, 375
Rotation, 60, 156, 267, 323, 353, 438

S

Safety factor, 4–7, 69, 283, 447
Saint-Venant, Barré, 48, 52, 58, 109
Saint-Venant's principle, 48, 52, 109, 202
Separation of variables, 248, 278, 389, 403
Shallow shells, 361–371
Shear modulus, 15, 24, 179, 214
Shear stress resultant, 265, 275, 290, 323
Shells
buckling of, 451–454
cylindrical, 319–350
long, 325–327
of revolution, 317
shallow, 361–371
short, 344–350
spherical, 350–356, 397–400
thick-walled, 378–384
Specific heat, 9, 16, 330, 375
Sphere
hollow, 397–400
solid, 402–406
steady-state conduction in, 397–400
thick-walled, 397–400
transient conduction in, 402–406
Spring constant, 29, 39, 53, 119, 417
Stiffness matrix, 146, 150

Strain
  axial, 378
  azimuth, 352
  bending, 64
  compatibility relations, 201
  direct, 26
  energy, 120, 216
  extensional, 198
  thermal, 27
  large values of, 202
  shearing, 26, 178, 199
  in spherical coordinates, 200
  thermal, 27, 206
Strength
  ultimate, 5, 462 (table)
  yield, 5, 462 (table)
Strength of materials, 59
Stress
  allowable, 7
  axial, 322, 340, 380, 451–453
  bending, 320, 322, 340
  circumferential, 322, 327, 341
  direct, 26, 69
  discontinuity, 319
  formulation, 230, 243, 323, 383
  intensity factor, 168, 175
  membrane, 320, 322, 327
  shearing, 26, 67
  in spherical coordinates, 206
  −strain relations, 27, 206, 222, 240, 267, 288
  thermal, 1, 206
Stress function, Airy, 217, 243, 440
Stress resultant, 222–224
  bending, 265, 321
  shear, 265, 271, 322
  thermal, 222, 266, 321, 433
Stress−strain relationship, 27, 44, 206, 222, 267, 288, 352

T

Temperature
  change, 27
  critical difference, 418
  gradient, 209
  initial, 389, 403
  nonuniform, 43
  stress-free, 27
Thermal
  buckling, 416–454
  conductivity, 209, 412
  diffusivity, 16, 389, 462 (table)
  expansion coefficient, 7–11, 208, 214, 381, 462 (table)
  expansion loops, 158, 171
  force, 47, 62, 81
  moment, 62, 81
  shear force, 68
  strain, 27, 206
  strain parameter, 10, 462 (table)
  stress ratio, 18
  stress resultant, 222, 266, 321, 433
Thermal shock
  definition, 17
  parameters, 17
  tables, 18
Thermoelastic instability, *see* Buckling
Thermoelasticity, equations of, 197–215
Time constant, 330, 335
Timoshenko, Stephen P., 257, 373
Torsion, 178, 265
Transient heat conduction equation, 210, 212, 389
Trusses, 118–153
  external constraints in, 129
  internal constraints in, 132
  pin-connected, 121, 138
  Warren, 187
Twist, 269, 441

U

Unit-load method
  beams, 153–156
  torsion, 178
  trusses, 123, 186
Units (SI), 461

V

Vessel cooldown, 329–334
Volume change, 8, 210
Von Mises-Hencky theory, 217

Y

Yield strength, 5, 239, 342, 431, 462 (table)
Young, Thomas, 20
Young's modulus, 11, 14, 462 (table)