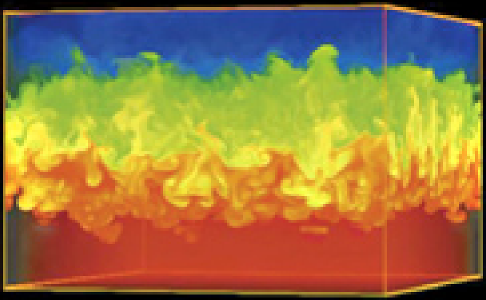
J. N. REDDY



PRINCIPLES OF CONTINUUM MECHANICS

A Study of Conservation Principles with Applications

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PRINCIPLES OF CONTINUUM MECHANICS

A Study of Conservation Principles with Applications

As most modern technologies are no longer discipline-specific but involve multidisciplinary approaches, undergraduate engineering students should be introduced to the principles of mechanics so that they have a strong background in the basic principles common to all disciplines and are able to work at the interface of science and engineering disciplines. This textbook is designed for a first course on principles of mechanics and provides an introduction to the basic concepts of stress and strain and conservation principles. It prepares engineers and scientists for advanced courses in traditional as well as emerging fields such as biotechnology, nanotechnology, energy systems, and computational mechanics. This simple book presents the subjects of mechanics of materials, fluid mechanics, and heat transfer in a unified form using the conservation principles of mechanics.

J. N. Reddy is a Distinguished Professor and holder of the Oscar S. Wyatt Endowed Chair in the Department of Mechanical Engineering at Texas A&M University (http://www.tamu.edu/acml).

Dr. Reddy is a renowned researcher and educator in the broad fields of mechanics, applied mathematics, and computational engineering science. Dr. Reddy's research areas include theory and finite element analysis of problems in structural mechanics (composite plates and shells), fluid dynamics, and heat transfer; theoretical modeling of stress and deformation of biological cells and soft tissues; nanocomposites; and development of robust computational technology (including the *K*-version finite element models based on the least-squares method in collaboration with Professor Karan Surana of the University of Kansas). He is the author of more than 400 journal papers and 16 books on these subjects. His books include *Mechanics of Laminated Plates and Shells: Theory and Analysis*, Second Edition, 2004; *An Introduction to Nonlinear Finite Element Analysis*, 2004; *Introduction to the Finite Element Method*, 2006; and *An Introduction to Continuum Mechanics*, 2008.

Dr. Reddy's outstanding research credentials have earned him wide international acclaim in the form of numerous professional awards; citations; fellowship in all major professional societies including AAM, AIAA, ASC, ASCE, ASME, IACM, and USACM; membership on two dozen archival journals; and numerous keynote and plenary lecture invitations at international conferences. Dr. Reddy is the editor-in-chief of *Applied Mechanics Reviews*, *Mechanics of Advanced Materials and Structures*, *International Journal of Computational Methods in Engineering Science and Mechanics*, and *International Journal of Structural Stability and Dynamics*.

The extent of Dr. Reddy's original and sustained contributions to education, research, and professional service is substantial. As a result of his extensive publications of archival journal papers and books on a wide range of topics in applied sciences and engineering, Dr. Reddy is one of the few researchers in engineering around the world who is recognized by *ISI Highly Cited Researchers*, with more than 10,000 citations with an H-index greater than 46. In February 2009 he was awarded a *Honoris Causa* (Honorary Doctorate) by the Technical University of Lisbon.

PRINCIPLES OF CONTINUUM MECHANICS

A Study of Conservation Principles with Applications

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When even the brightest mind in our world has been trained up from childhood in a superstition of any kind, it will never be possible for that mind, in its maturity, to examine sincerely, dispassionately, and conscientiously any evidence or any circumstance which shall seem to cast a doubt upon the validity of that superstition.

Mark Twain

The fact that an opinion has been widely held is no evidence whatever that it is not utterly absurd; indeed in view of the silliness of the majority of mankind, a widespread belief is more likely to be foolish than sensible.

Bertrand Russell

Desire for approval and recognition is a healthy motive, but the desire to be acknowledged as better, stronger, or more intelligent than a fellow being or fellow scholar easily leads to an excessively egoistic psychological adjustment, which may become injurious for the individual and for the community.

Albert Einstein

Contents

Preface

You cannot teach a man anything, you can only help him find it within himself.
Galileo Galilei

This book is a simplified version of the author's book, *An Introduction to Continuum Mechanics with Applications,* published by Cambridge University Press (New York, 2008), intended for use as an undergraduate textbook. As most modern technologies are no longer discipline-specific but involve multidisciplinary approaches, undergraduate engineering students should be educated to think and work in such environments. Therefore, it is necessary to introduce the subject of *principles of mechanics* (i.e., laws of physics applied to science and engineering systems) to undergraduate students so that they have a strong background in the basic principles common to all disciplines and are able to work at the interface of science and engineering disciplines. A first course on principles of mechanics provides an introduction to the basic concepts of stress and strain and conservation principles and prepares engineers and scientists for advanced courses in traditional as well as emerging fields such as biotechnology, nanotechnology, energy systems, and computational mechanics. Undergraduate students with such a background may seek advanced degrees in traditional (e.g., aerospace, civil, electrical or mechanical engineering; physics; applied mathematics) as well as interdisciplinary (e.g., bioengineering, engineering physics, nanoscience and engineering, biomolecular engineering) degree programs.

There are not many books on principles of mechanics that are written that keep the undergraduate engineering or science student in mind. A vast majority of books on the subject are written for graduate students of engineering and tend to be more mathematical and too advanced to be of use for third-year or senior undergraduate students. This book presents the subjects of mechanics of materials, fluid mechanics, and heat transfer in unified form using the conservation principles of mechanics. It is hoped that the book, which is simple, will facilitate in presenting the main concepts of the previous three courses under a unified framework.

With a brief discussion of the concept of a continuum in Chapter 1, a review of vectors and tensors is presented in Chapter 2. Because the analytical language of applied sciences and engineering is mathematics, it is necessary for all students

of this course to familiarize themselves with the notation and operations of vectors, matrices, and tensors that are used in the mathematical description of physical phenomena. Readers who are familiar with the topics of this chapter may refresh or skip and go to the next chapter. The subject of kinematics, which deals with geometric changes without regard to the forces causing the deformation, is discussed in Chapter 3. Measures of engineering normal and shear strains and definitions of mathematical strains are introduced here. Both simple one-dimensional systems as well as two-dimensional continua are used to illustrate the strain and strain-rate measures introduced. In Chapter 4, the concept of stress vector and stress tensor are introduced. It is here that the readers are presented with entities that require two directions – namely, the plane on which they are measured and the direction in which they act – to specify them. Transformation equations among components of stress tensor referred to two different orthogonal coordinate systems are derived, and principal values and principal planes (i.e., eigenvalue problems associated with the stress tensor) are also discussed.

Chapter 5 is dedicated to the derivation of the governing equations of mechanics using the conservation principles of continuum mechanics (or laws of physics). The principles of conservation of mass, linear momentum, angular momentum, and energy are presented using one-dimensional systems as well as general three-dimensional systems. The derivations are presented in invariant (i.e., independent of a coordinate system) as well as in component form. The equations resulting from these principles are those governing stress and deformation of solid bodies, stress and rate of deformation of fluid elements, and transfer of heat through solid media. Thus, this chapter forms the heart of the course. Constitutive relations that connect the kinematic variables (e.g., density, temperature, deformation) to the kinetic variables (e.g., internal energy, heat flux, stresses) are discussed in Chapter 6 for elastic materials, viscous fluids, and heat transfer in solids.

Chapter 7 is devoted to the application of the field equations derived in Chapter 5 and constitutive models presented in Chapter 6 to problems of heat conduction in solids, fluid mechanics (inviscid flows as well as viscous incompressible flows), diffusion, and solid mechanics (e.g., bars, beams, and plane elasticity). Simple boundary-value problems are formulated and their solutions are discussed. The material presented in this chapter illustrates how physical problems are analytically formulated with the aid of the equations resulting from the conservation principles.

As stated previously, the present book is an undergraduate version of the author's book *An Introduction to Continuum Mechanics* (Cambridge University Press, New York, 2008). The presentation herein is limited in scope when compared to the author's graduate-level textbook. The major benefit of a course based on this book is to present the governing equations of diverse physical phenomena from a unified point of view, namely, from the conservation principles (or laws of physics), so that students of applied science and engineering see the physical principles as well as the mathematical structure common to diverse fields.

Readers interested in advanced topics may consult the author's continuum mechanics book or other titles listed in references therein.

The author is pleased to acknowledge the fact that the manuscript was tested with the undergraduate students in the College of Engineering at Texas A&M University as well as in the Engineering Science Programme at the National University of Singapore. The students, in general, have liked the contents and the simplicity with which the concepts are introduced and explained. They also expressed the feeling that the subject is more challenging than most at the undergraduate level but a useful prerequisite to graduate courses in engineering.

The author wishes to thank Drs. Vinu and Ginu Unnikrishnan and Ms. Feifei Cheng for their help with the proofreading of the manuscript of this book during the course of its preparation and production. The book contains so many mathematical expressions that it is hardly possible not to have typographical and other kinds of errors. The author wishes to thank in advance those who are willing to draw the author's attention to typos and errors, using the e-mail address jn_reddy@yahoo.com.

1 Introduction

One thing I have learned in a long life: that all our science, measured against reality, is primitive and childlike—and yet it is the most precious thing we have.

Albert Einstein

1.1 Continuum mechanics

Matter is composed of discrete molecules, which in turn are made up of atoms. An atom consists of electrons, positively charged protons, and neutrons. Electrons form chemical bonds. An example of mechanical (i.e., has no living cells) matter is a carbon nanotube (CNT), which consists of carbon molecules in a certain geometric pattern in equilibrium with each other, as shown in Figure 1.1.1.

Another example of matter is a biological cell, which is a fundamental unit of any living organism. There are two types of cells: prokaryotic and eukaryotic cells. Eukaryotic cells are generally found in multicellular organs and have a true nucleus, distinct from a prokaryotic cell. Structurally, cells are composed of a large number of macromolecules, or large molecules. These macromolecules consist of large numbers of atoms and form specific structures, like chromosomes and plasma membranes in a cell. Macromolecules occur as four major types: carbohydrates, proteins, lipids, and nucleic acids. To highlight the hierarchical nature of the structures formed by the macromolecule in a cell, let us analyze a chromosome.

Chromosomes, which are carriers of hereditary traits in an individual, are found inside the nucleus of all eukaryotes. Each chromosome consists of a single nucleic acid macromolecule called deoxyribonucleic acid (DNA), each 2.2–2.4 nanometers wide. These nucleic acids are in turn formed from the specific arrangement of monomers called mononucleotides, each 0.3–0.33 nanometers wide. The fundamental units of nucleotides are formed again by a combination of a specific arrangement of a phosphate radical, nitrogenous base, and a carbohydrate sugar. The hierarchical nature of the chromosome is shown in Figure 1.1.2(a). Similar to the chromosomes, all the structures in a cell are formed from a combination of the macromolecules.

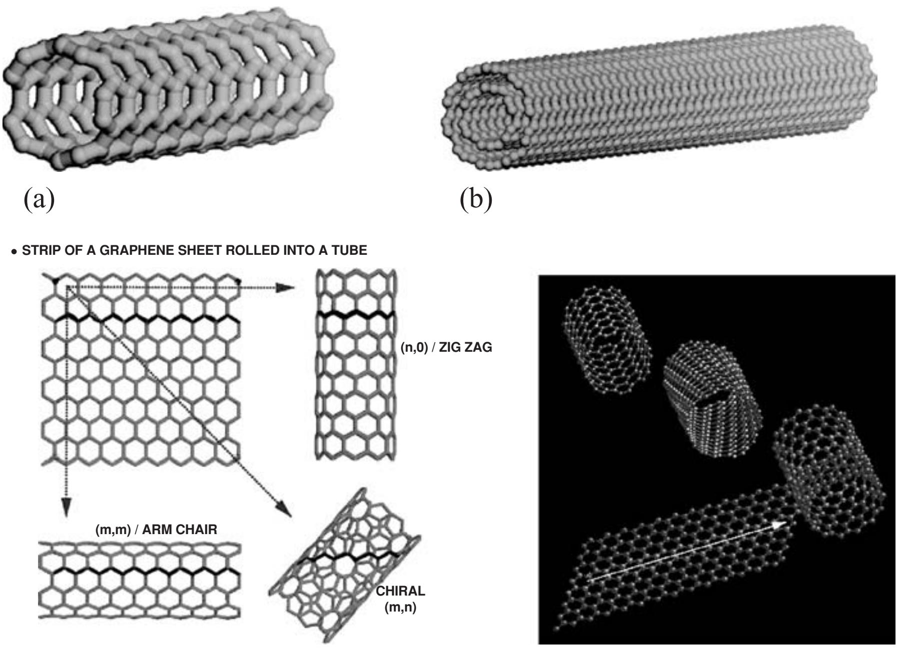


Figure 1.1.1 Carbon nanotubes (CNTs) with different chiralities.

At the macroscopic scale, eukaryotic cells can be divided into three distinct regions: nucleus, plasma membrane, and a cytoplasm having a host of other structures, as shown in Figure 1.1.2(b). The nucleus consists of the chromosomes and other protein structures and is the control center of the cell determining how the cell functions. The plasma membrane encloses the cell and separates the material outside the cell from inside. It is responsible for maintaining the integrity of the cell and also acts as channels for the transport of molecules to and from the cell. The cell membrane is made up of a double layer of phospholipid molecules (macromolecules) having embedded transmembrane proteins. The region between the cell membrane and the nucleus is the cytoplasm, which consists of a gel-like fluid called cytosol, the cytoskeleton, and other macromolecules. The cytoskeleton forms the biomechanical framework of the cell and consists of three primary protein macromolecule structures of actin filaments, intermediate filaments, and microtubules. Growth, cell expansion, and replication are all carried out in the cytoplasm.

The interactions between the different components of the cell are responsible for maintaining the structural integrity of the cell. The analysis of these interactions to obtain the response of the cell when subjected to an external stimulus (mechanical, electrical, or chemical) is studied systematically under cell mechanics. The structural framework of primary macromolecular structures in a cell is shown in Figure 1.1.2(c).

The study of matter at molecular or atomistic levels is very useful for understanding a variety of phenomena, but studies at these scales are not useful to solve common engineering problems. The understanding gained at the molecular

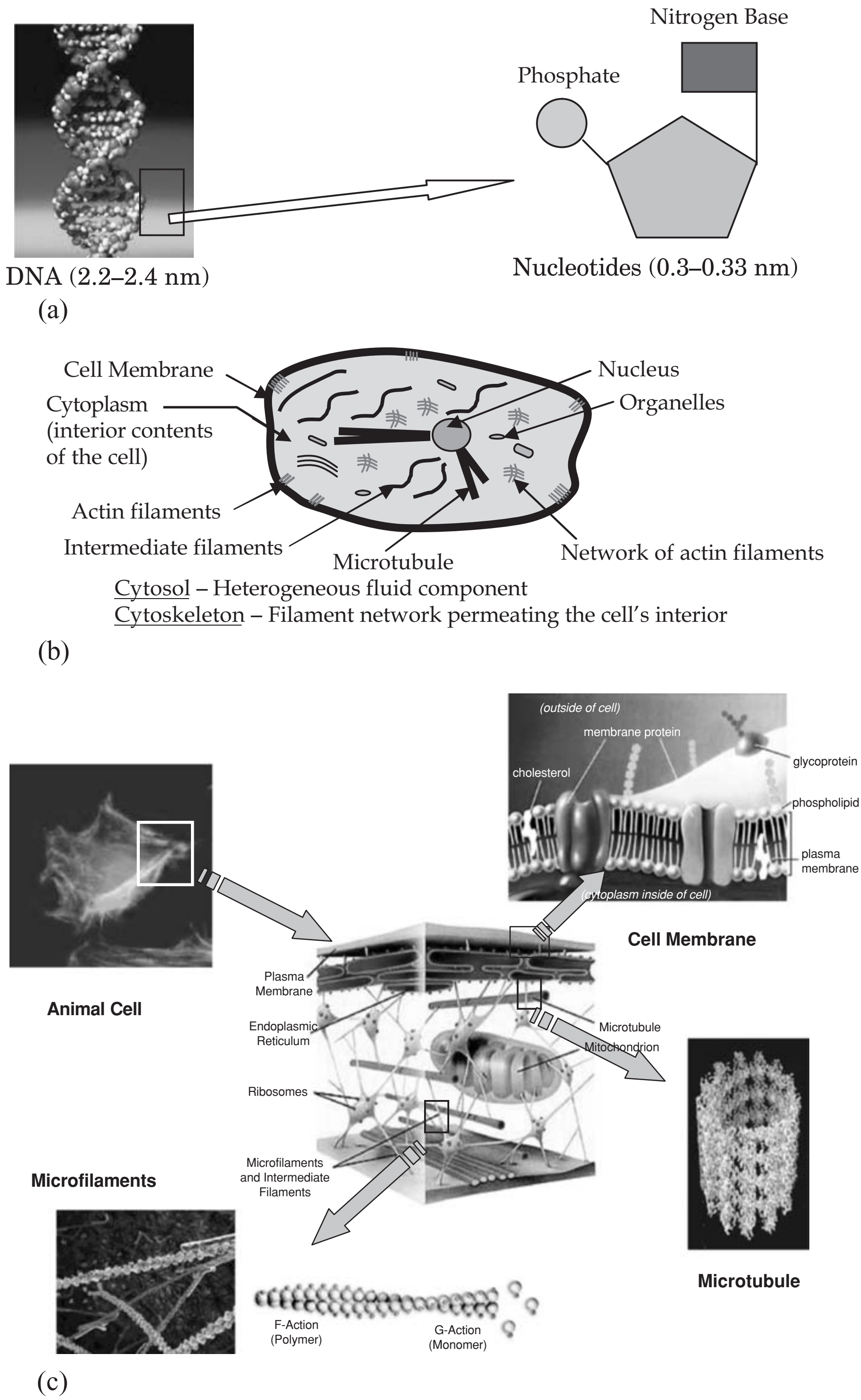


Figure 1.1.2 (a) Hierarchical nature of a chromosome. (b) Structure of a generalized cell. (c) Macromolecular structure in a cell.

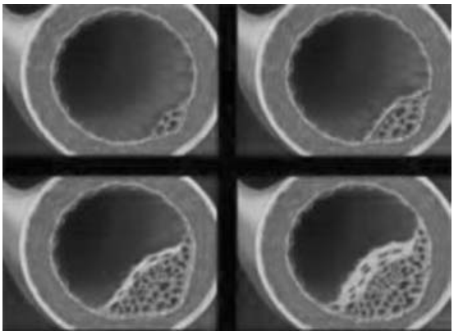


Figure 1.1.3 Progressive damage of artery due to deposition of particles in the arterial wall.

level must be taken to the macrosopic scale (i.e., a scale that a human eye can see) to be able to study its behavior. Central to this study is the assumption that the discrete nature of matter can be overlooked, provided the length scales of interest are large compared to the length scales of discrete molecular structure. Thus, matter at sufficiently large length scales can be treated as a *continuum* in which all physical quantities of interest, including density, are continuously differentiable.

The subject of *mechanics* deals with the study of motion and forces in solids, liquids, and gases and the deformation or flow of these materials. In such a study, we make the simplifying assumption, for analytic purposes, that the matter is distributed continuously, without gaps or empty spaces (i.e., we disregard the molecular structure of matter). Such a hypothetical continuous matter is termed a *continuum*. In essence, in a continuum all quantities such as the density, displacements, velocities, stresses, and so on vary continuously so that their spatial derivatives exist and are continuous. The continuum assumption allows us to shrink an arbitrary volume of material to a point, in much the same way as we take the limit in defining a derivative, so that we can define quantities of interest at a point. For example, the density (mass per unit volume) of a material at a point is defined as the ratio of the mass Δm of the material to a small volume ΔV surrounding the point in the limit that ΔV becomes a value ϵ^3, where ϵ is small compared with the mean distance between molecules,

$$\rho = \lim_{\Delta V \to \epsilon^3} \frac{\Delta m}{\Delta V}\,. \tag{1.1.1}$$

In fact, we take the limit $\epsilon \to 0$. A mathematical study of mechanics of such an idealized continuum is called *continuum mechanics*.

Engineers and scientists undertake the study of continuous systems to understand their behavior under "working conditions," so that the systems can be designed to function properly and produced economically. For example, if we were to repair or replace a damaged artery in a human body, we must understand the function of the original artery and the conditions that lead to its damage. An artery carries blood from the heart to different parts of the body. Conditions like high blood pressure and increases in cholesterol content in the blood may lead to deposition of particles in the arterial wall, as shown in Figure 1.1.3. With time, accumulation of these particles in the arterial wall hardens and constricts the passage, leading to cardiovascular diseases. A possible remedy for such diseases is to repair or replace the damaged portion of the artery. This in turn requires an understanding of the deformation and stresses caused in the arterial wall by the

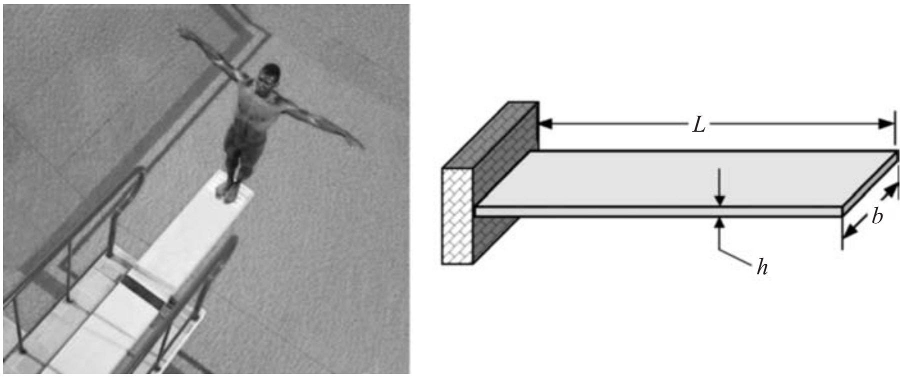


Figure 1.1.4 A diving board fixed at the left end and free at the right end.

flow of blood. The understanding is then used to design the vascular prosthesis (i.e., an artificial artery).

The primary objectives of this book are (1) to study the conservation principles in mechanics of continua and formulate the equations that describe the motion and mechanical behavior of materials, and (2) to present the applications of these equations to simple problems associated with flows of fluids, conduction of heat, and deformation of solid bodies. Although the first of these objectives is an important topic, the reason for the formulation of the equations is to gain a quantitative understanding of the behavior of an engineering system. This quantitative understanding is useful in the design and manufacture of better products. Typical examples of engineering problems sufficiently simple to cover in this course are described in the following. At this stage of discussion, it is sufficient to rely on the reader's intuitive understanding of concepts.

PROBLEM 1 (MECHANICAL STRUCTURE)

We wish to design a diving board that must enable the swimmer to gain enough momentum for the swimming exercise. The diving board is fixed at one end and free at the other end (see Figure 1.1.4). The board is initially straight and horizontal, and of length L and uniform cross section $A = bh$.

The design process consists of selecting the material with Young's modulus E and cross-sectional dimensions b and h such that the board carries the weight W of the swimmer. The design criteria are that the stresses developed do not exceed the allowable stress and the deflection of the free end does not exceed a pre-specified value δ. A preliminary design of such systems is often based on mechanics of materials equations. The final design involves the use of more sophisticated equations, such as the three-dimensional elasticity equations. The equations of elementary beam theory may be used to find a relation between the deflection δ of the free end in terms of the length L, cross-sectional dimensions b and h, Young's modulus E, and weight W:

$$\delta = \frac{4WL^3}{Ebh^3}. \tag{1.1.2}$$

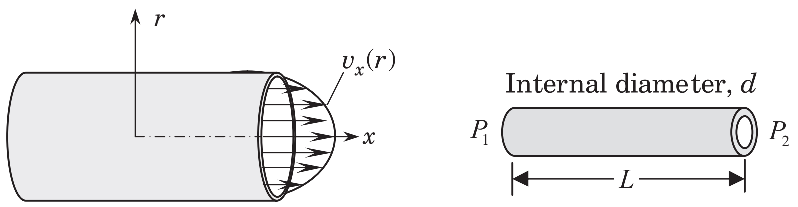


Figure 1.1.5 Measurement of viscosity of a fluid using a capillary tube.

Given δ (allowable deflection) and load W (maximum possible weight of a swimmer), one can select the material (Young's modulus, E) and dimensions L, b, and h (which must be restricted to the standard sizes fabricated by a manufacturer). In addition to the deflection criterion, one must also check if the board develops stresses that exceed the allowable stresses of the material selected. Analysis of pertinent equations provide the designer with alternatives to select the material and dimensions of the board so as to have a cost-effective but functionally reliable structure.

PROBLEM 2 (FLOW OF FLUIDS)

We wish to measure the viscosity μ of a lubricating oil used in rotating machinery to prevent the damage of the parts in contact. Viscosity, like Young's modulus of solid materials, is a material property that is useful in the calculation of shear stresses developed between a fluid and a solid body.

A capillary tube is used to determine the viscosity of a fluid via the formula

$$\mu = \frac{\pi d^4}{128L} \frac{P_1 - P_2}{Q}, \tag{1.1.3}$$

where d is the internal diameter and L is the length of the capillary tube, P_1 and P_2 are the pressures at the two ends of the tube (oil flows from one end to the other, as shown in Figure 1.1.5), and Q is the volume rate of flow at which the oil is discharged from the tube. As we shall see later in this course, Eq. (1.1.3) is derived using the principles of continuum mechanics.

PROBLEM 3 (TRANSFER OF HEAT IN SOLIDS)

We wish to determine the heat loss through the wall of a furnace. The wall typically consists of layers of brick, cement mortar, and cinder block (see Figure 1.1.6). Each of these materials provides varying degrees of thermal resistance. The Fourier heat conduction law,

$$q = -k\frac{dT}{dx}, \tag{1.1.4}$$

provides a relation between the heat flux q (heat flow per unit area) and gradient of temperature T. Here k denotes thermal conductivity ($1/k$ is the thermal resistance) of the material. The negative sign in Eq. (1.1.4) indicates that heat flows from a high-temperature region to a low-temperature region. Using the continuum mechanics equations, one can determine the heat loss when the

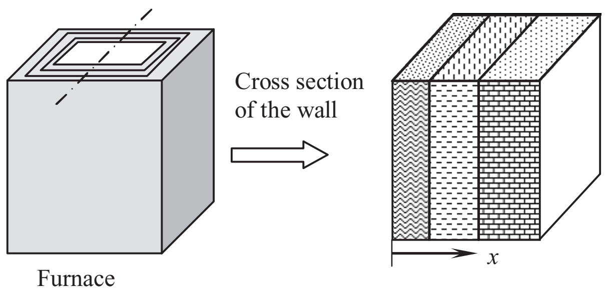


Figure 1.1.6 Heat transfer through a composite wall of a furnace.

temperatures inside and outside of the building are known. A building designer can select the materials as well as thicknesses of various components of the wall to reduce the heat loss while ensuring necessary structural strength – a structural analysis aspect.

The previous three examples provide some indication of the need for studying the response of materials under the influence of external loads. The response of a material is consistent with the laws of physics and the constitutive behavior of the material. This book has the objective of describing the physical principles and deriving the equations governing the stress and deformation of continuous materials, and then solving some simple problems from various branches of engineering to illustrate the applications of the principles discussed and equations derived.

1.2 Objective of the study

The primary objective of this book, as already stated, is twofold: (1) use the physical principles to derive the equations that govern the motion and thermomechanical response of materials and systems, and (2) apply these equations for the solution of specific problems of engineering and applied science (e.g., linearized elasticity, heat transfer, and fluid mechanics). The governing equations for the study of deformation and stress of a continuous material are nothing but an analytical representation of the global laws of conservation of mass, momenta, and energy, and the constitutive response of the continuum. They are applicable to all materials that are treated as a continuum. Tailoring these equations to particular problems and solving them constitute the bulk of engineering analysis and design.

The study of motion and deformation of a continuum (or a "body" consisting of continuously distributed material) can be broadly classified into four basic categories:

(1) Kinematics
(2) Kinetics (conservation of linear and angular momentum)
(3) Thermodynamics (first and second laws of thermodynamics)
(4) Constitutive equations

Table 1.2.1. Four major topics of the present study, principles of mechanics used, resulting governing equations, and variables involved.

Topic of study	Physical principle	Resulting equations	Variables involved
1. Kinematics	Based on geometric changes	Strain-displacement relations	Displacements and strains
		Strain rate-velocity relations	Velocities and strain rates
2. Kinetics	Conservation of linear momentum	Equations of motion	Stresses, velocities, and body forces
	Conservation of angular momentum	Symmetry of stress tensor	Stresses
3. Thermodynamics	First law	Energy equation	Temperature, heat flux, stresses, heat generation, and velocities
	Second law	Clausius–Duhem inequality	Temperature, heat flux, and entropy
4. Constitutive equations (not all relations are listed)	Constitutive axioms	Hooke's law	Stresses, strains, heat flux, and temperature
		Newtonian fluids	Stresses, pressure, velocities
		Fourier's law	Heat flux and temperature
		Equations of state	Density, pressure, temperature
5. Boundary conditions	All of the previous principles and axioms	Relations between kinematic and kinetic variables	All of the previous variables

Kinematics is the study of the geometric changes or deformation in a continuum, without the consideration of forces causing the deformation. *Kinetics* is the study of the static or dynamic equilibrium of forces and moments acting on a continuum, using the principles of conservation of linear and angular momentum. This study leads to equations of motion as well as the symmetry of stress tensors in the absence of body couples. *Thermodynamic principles* are concerned with the conservation of energy and relations among heat, mechanical work, and thermodynamic properties of the continuum. *Constitutive equations* describe the thermomechanical behavior of the material of the continuum and relate the dependent variables introduced in the kinetic description to those introduced in the kinematic and thermodynamic descriptions. Table 1.2.1 provides a brief summary of the relationship between physical principles and governing equations, and physical entities involved in the equations. To the equations derived from physical principles, one must add *boundary conditions* of the system (and *initial conditions* if the phenomenon is time-dependent) to complete the analytical description.

1.3 Summary

In this chapter, the concept of a continuous medium is discussed with the major objectives of the present study, namely, to use the principles of mechanics to

derive the equations governing a continuous medium and to present applications of the equations in the solution of specific problems arising in engineering. The study of principles of mechanics is broadly divided into four topics, as outlined in Table 1.2.1. These four topics form the subject of Chapters 3 through 6, respectively. Mathematical formulation of the governing equations of a continuous medium (that is, the development of a mathematical model of the physical phenomenon) necessarily requires the use of vectors, matrices, and tensors – mathematical tools that facilitate analytical formulation of the natural laws. Therefore, it is useful to first gain certain operational knowledge of vectors, matrices, and tensors. Chapter 2 is dedicated to this purpose.

Many of the concepts presented herein are the same as those most likely introduced in undergraduate courses on mechanics of materials, heat transfer, fluid mechanics, and material science. The present course brings together these courses under a common mathematical framework and, thus, may require mathematical tools as well as concepts not seen previously. The readers must motivate and challenge themselves to learn the new mathematical concepts introduced here, as the language of engineers is mathematics. This subject also serves as a prelude to many graduate courses in engineering and applied sciences.

Although this book is self-contained for an introduction to principles of continuum mechanics, there are several books that may provide an advanced treatment of the subject. The graduate-level textbook by the author, *An Introduction to Continuum Mechanics with Applications* (Cambridge University Press, New York, 2008), provides additional material. Interested readers may consult other titles listed in "References and Additional Readings," at the end of this book.

When a distinguished but elderly scientist states that something is possible, he is almost certainly right. When he states that something is impossible, he is very probably wrong.

Arthur C. Clarke

2 Vectors and Tensors

No great discovery was ever made without a bold guess.
Isaac Newton

2.1 Motivation

In the mathematical description of equations governing a continuous medium, we derive relations between various quantities that describe the response of the continuum by means of the laws of nature, such as Newton's laws. As a means of expressing a natural law, a coordinate system in a chosen frame of reference is often introduced. The mathematical form of the law thus depends upon the chosen coordinate system and may appear different in another coordinate system. However, the laws of nature should be independent of the choice of coordinate system, and we may seek to represent the law in a manner independent of a particular coordinate system.[1] A way of doing this is provided by objects called *vectors* and *tensors*. When vector and tensor notation is used, a particular coordinate system need not be introduced. Consequently, the use of vector and tensor notation in formulating natural laws leaves them *invariant*, and we may express them in any chosen coordinate system. A study of physical phenomena by means of vectors and tensors can lead to a deeper understanding of the problem, in addition to bringing simplicity and versatility to the analysis. This chapter is dedicated to the algebra and calculus of physical vectors and tensors, as needed in the subsequent study.

2.2 Definition of a vector

The quantities encountered in the analytical description of physical phenomena can be classified into the following groups according to the information needed to specify them completely:

[1] We always return to a particular coordinate system of our choice to solve the equations resulting from the physical law.

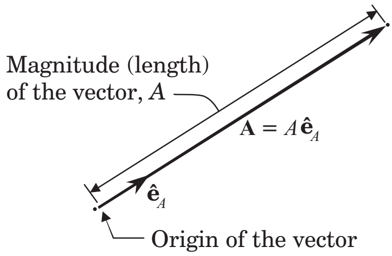


Figure 2.2.1 Geometric representation of a physical vector.

Scalars	Nonscalars
Mass	Force
Temperature	Moment
Time	Stress
Volume	Acceleration
Length	Displacement

The scalars are given by a single number. Nonscalars not only have a magnitude specified but also have additional information, such as direction. Nonscalars that obey certain rules (such as the parallelogram law of addition) are called vectors. Not all nonscalar quantities are vectors, unless they obey certain rules as discussed in the sequel.

A physical vector is often shown as a directed line segment with an arrow head at the end of the line, as shown in Figure 2.2.1. The length of the line represents the magnitude of the vector and the arrow indicates the direction. In written or typed material, it is customary to place an arrow over the letter denoting the vector, such as $\vec{A}$. In printed material, the letter used for the vector is commonly denoted by a boldface letter, $\mathbf{A}$, such as used in this study. The magnitude of the vector $\mathbf{A}$ is denoted by $|\mathbf{A}|$, $\|\mathbf{A}\|$, or A. The magnitude of a vector is a scalar.

2.3 Vector algebra

In this section, we discuss various operations with vectors and interpret them physically. First, we introduce the notion of unit and zero vectors.

2.3.1 Unit vector

A vector of unit length is called a *unit vector*. The unit vector along $\mathbf{A}$ can be defined as follows:

$$\hat{\mathbf{e}}_A = \frac{\mathbf{A}}{A}\,. \tag{2.3.1}$$

We can now write

$$\mathbf{A} = A\,\hat{\mathbf{e}}_A\,. \tag{2.3.2}$$

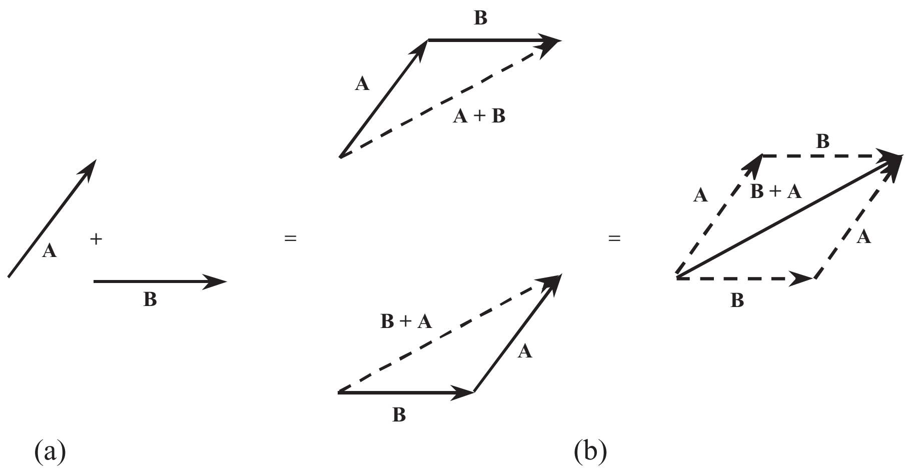


Figure 2.3.1 (a) Addition of vectors. (b) Parallelogram law of addition.

Thus, any vector can be represented as a product of its magnitude and a unit vector. A unit vector is used to designate direction. It does not have any physical dimensions. We denote a unit vector by a "hat" (caret) above the boldface letter, $\hat{\mathbf{e}}$.

2.3.2 Zero vector

A vector of zero magnitude is called a *zero vector* or a *null vector*. All null vectors are considered equal to each other without consideration as to their direction. Note that a lightface zero, 0, is a scalar and a boldface zero, $\mathbf{0}$, is the zero vector.

2.3.3 Vector addition

Let **A**, **B**, and **C** be any vectors. Then there exists a vector $\mathbf{A}+\mathbf{B}$, called the sum of **A** and **B**, such that:

(1) $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$ (commutative property).

(2) $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$ (associative property).

(3) There exists a unique vector, $\mathbf{0}$, independent of **A** such that
$\mathbf{A}+\mathbf{0}=\mathbf{A}$ (existence of zero vector). (2.3.3)

(4) To every vector **A**, there exists a unique vector $-\mathbf{A}$ (that depends on **A**) such that
$\mathbf{A}+(-\mathbf{A})=\mathbf{0}$ (existence of negative vector).

The addition of two vectors is shown in Figure 2.3.1(a). Note that the commutative property is essential for a nonscalar to qualify as a vector. The combination of the two diagrams in Figure 2.3.1(a) gives the parallelogram shown in Figure 2.3.1(b), and it characterizes the commutativity. Thus, we say that vectors add

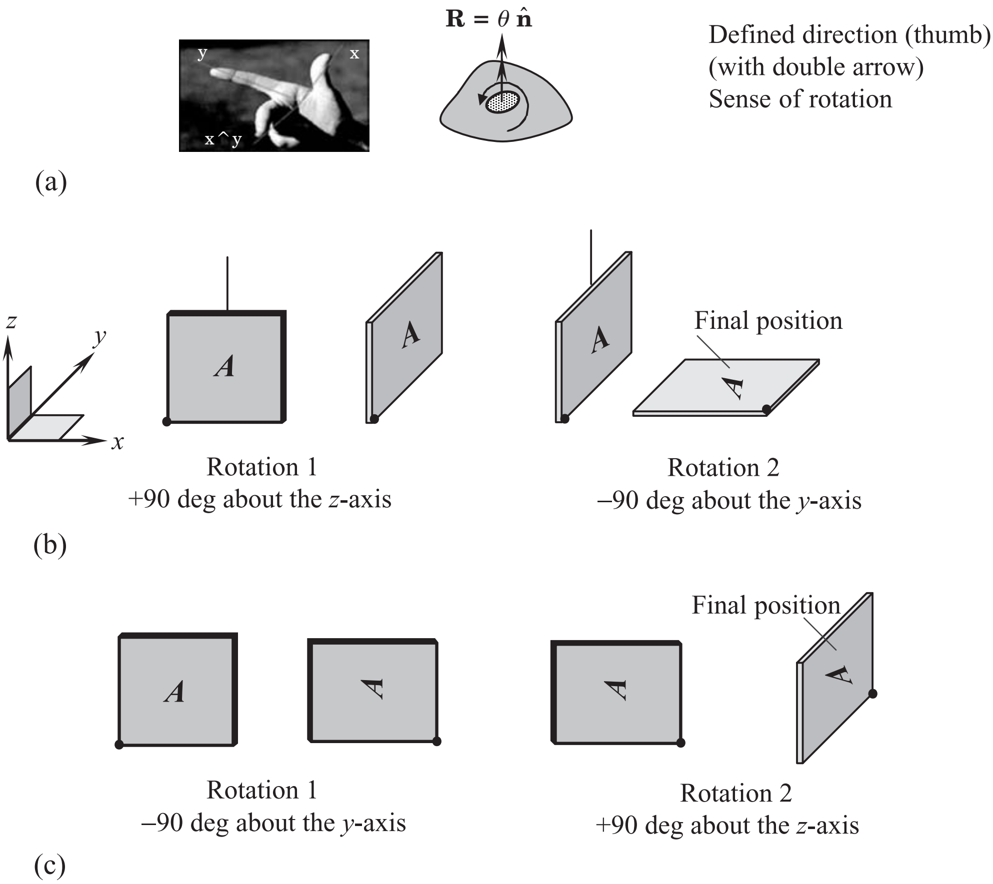


Figure 2.3.2 (a) Preferred sense of rotation. (b) Rotation θ_z followed by rotation θ_y. (c) Rotation θ_y followed by rotation θ_z.

according to the *parallelogram law* of addition. The negative vector $-\mathbf{A}$ has the same magnitude as $\mathbf{A}$ but the opposite *sense*. Subtraction of vectors is carried out along the same lines. To form the difference $\mathbf{A} - \mathbf{B}$, we write

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) \tag{2.3.4}$$

and subtraction reduces to the operation of addition.

As an example of a nonscalar that has magnitude and direction but does not obey commutativity, consider finite rotation. Finite rotation has a magnitude θ and a preferred direction as that in which a right-handed screw would advance when turned in the direction of rotation, as indicated in Figure 2.3.2(a). Now consider two different rotations of a rectangular block, in a certain order. The first rotation is about the z-axis by $\theta_z = +90^\circ$, followed by rotation $\theta_y = -90^\circ$ about the y-axis. This sequence of rotations results in the final position indicated in Figure 2.3.2(b). We may represent this pair of rotations as $\mathbf{R}_1 + \mathbf{R}_2$, as shown in Figure 2.3.2(c). Reversing the order of rotations, that is, θ_y first and θ_z next, we obtain $\mathbf{R}_2 + \mathbf{R}_1$, which is not the same as the position achieved by $\mathbf{R}_1 + \mathbf{R}_2$. Thus, a finite rotation is not a vector, even though it has a direction and magnitude.

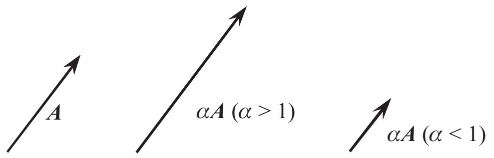


Figure 2.3.3 A typical vector **A** and its scalar multiple.

2.3.4 Multiplication of a vector by a scalar

Let **A** and **B** be vectors and α and β be real numbers (scalars). To every vector **A** and every real number α, there correspond a unique vector $\alpha\mathbf{A}$ such that:

$$\begin{aligned}
&(\mathbf{1})\ \ \alpha(\beta\mathbf{A}) = (\alpha\beta)\mathbf{A} \text{ (associative property).}\\
&(\mathbf{2})\ \ (\alpha+\beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A} \text{ (distributive scalar addition).}\\
&(\mathbf{3})\ \ \alpha(\mathbf{A}+\mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B} \text{ (distributive vector addition).}\\
&(\mathbf{4})\ \ 1\cdot\mathbf{A} = \mathbf{A}\cdot 1 = \mathbf{A},\quad 0\cdot\mathbf{A} = \mathbf{0}.
\end{aligned} \tag{2.3.5}$$

Figure 2.3.3 contains a vector **A** and its scalar multiple $\alpha\mathbf{A}$ for $\alpha > 1$ and $\alpha < 1$.

Two vectors **A** and **B** are equal if their magnitudes are equal, $|\mathbf{A}| = |\mathbf{B}|$, and if their directions are equal. Consequently, a vector is not changed if it is moved parallel to itself. This means that the position of a vector in space – that is, the point from which the line segment is drawn (or the end without the arrowhead) – may be chosen arbitrarily. However, in certain applications the actual point of the location of a vector may be important, for instance, for a moment or a force acting on a body. A vector associated with a given point is known as a *localized* or *bound vector*.

Two vectors **A** and **B** are said to be linearly dependent if they are scalar multiples of each other, that is, $c_1\mathbf{A} + c_2\mathbf{B} = \mathbf{0}$ for some nonzero scalars c_1 and c_2. If two vectors are linearly dependent, then they are *collinear*. If three vectors **A**, **B**, and **C** are linearly dependent, then they are *coplanar*. A set of n vectors $\{\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n\}$ is said to be *linearly dependent* if a set of n numbers $c_1, c_2, \ldots, c_n$ can be found such that $c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + \cdots + c_n\mathbf{A}_n = \mathbf{0}$, where $c_1, c_2, \ldots, c_n$ cannot all be zero. If this expression cannot be satisfied (that is, all c_i are zero), the set of vectors $\{\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n\}$ is said to be *linearly independent*.

2.3.5 Scalar product of vectors

When a force **F** acts on a point mass and moves through a displacement vector **d**, as shown in Figure 2.3.4(a), the work done by the force vector is defined by the *projection* of the force in the direction of the displacement, as shown in Figure 2.3.4(b), multiplied by the magnitude of the displacement. Such an operation can be defined for any two vectors. Because the result of the product is a scalar, it is called the *scalar product*. We denote this product as $\mathbf{F}\cdot\mathbf{d} \equiv (\mathbf{F},\mathbf{d})$, and it is defined as follows:

$$\mathbf{F}\cdot\mathbf{d} \equiv (\mathbf{F},\mathbf{d}) = Fd\,\cos\theta, \qquad 0 \le \theta \le \pi. \tag{2.3.6}$$

The scalar product is also known as the *dot product* or *inner product*.

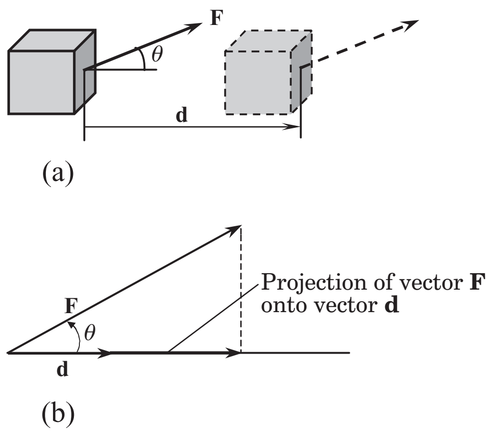


Figure 2.3.4 (a) Representation of work done. (b) Projection of a vector.

A few simple results follow from the definition in Eq. (2.3.6), and they are listed next.

(1) Because $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$, the scalar product is commutative.

(2) If the vectors $\mathbf{A}$ and $\mathbf{B}$ are perpendicular to each other, then $\mathbf{A} \cdot \mathbf{B} = AB\cos(\pi/2) = 0$. Conversely, if $\mathbf{A} \cdot \mathbf{B} = 0$, then either $\mathbf{A}$ or $\mathbf{B}$ is zero or $\mathbf{A}$ is perpendicular, or *orthogonal*, to $\mathbf{B}$.

(3) If two vectors $\mathbf{A}$ and $\mathbf{B}$ are parallel and in the same direction, then $\mathbf{A} \cdot \mathbf{B} = AB\cos 0 = AB$, because $\cos 0 = 1$. Thus, the scalar product of a vector with itself is equal to the square of its magnitude:

$$\mathbf{A} \cdot \mathbf{A} = AA = A^2. \tag{2.3.7}$$

(4) The orthogonal projection of a vector $\mathbf{A}$ in any direction $\hat{\mathbf{e}}$ is given by $(\mathbf{A} \cdot \hat{\mathbf{e}})\hat{\mathbf{e}}$.

(5) The scalar product follows the distributive law:

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \cdot \mathbf{B}) + (\mathbf{A} \cdot \mathbf{C}). \tag{2.3.8}$$

2.3.6 Vector product

Consider the concept of moment due to a force. Let us describe the *moment* about a point O of a force $\mathbf{F}$ acting at a point P, such as that shown in Figure 2.3.5(a).

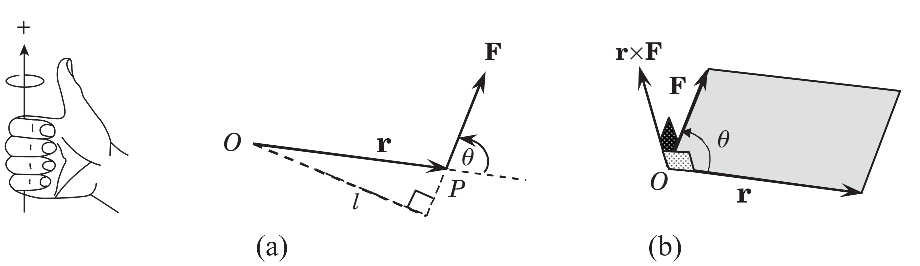


Figure 2.3.5 (a) Representation of a moment. (b) Direction of rotation.

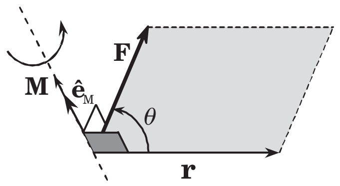


Figure 2.3.6 Axis of rotation.

By definition, the magnitude of the moment is given by

$$M = F\ell, \quad F = |\mathbf{F}|, \tag{2.3.9}$$

where ℓ is the perpendicular distance from the point O to the force $\mathbf{F}$ (called the *lever arm*). If $\mathbf{r}$ denotes the vector $\mathbf{OP}$ and θ the angle between $\mathbf{r}$ and $\mathbf{F}$, as shown in Figure 2.3.5(a), such that $0 \le \theta \le \pi$, we have $\ell = r \sin\theta$, and thus

$$M = Fr \sin\theta. \tag{2.3.10}$$

A direction can now be assigned to the moment. Drawing the vectors $\mathbf{F}$ and $\mathbf{r}$ from the common origin O, we note that the rotation due to $\mathbf{F}$ tends to bring $\mathbf{r}$ into $\mathbf{F}$, as can be seen from Figure 2.3.5(b). We now set up an axis of rotation perpendicular to the plane formed by $\mathbf{F}$ and $\mathbf{r}$. Along this axis of rotation, we set up a preferred direction as that in which a right-handed screw would advance when turned in the direction of rotation due to the moment, as shown in Figure 2.3.6. Along this axis of rotation, we draw a unit vector $\hat{\mathbf{e}}_M$ and agree that it represents the direction of the moment $\mathbf{M}$. Thus, we have

$$\mathbf{M} = M\hat{\mathbf{e}}_M = Fr \ \sin\theta \ \hat{\mathbf{e}}_M \tag{2.3.11}$$

$$= \mathbf{r} \times \mathbf{F}. \tag{2.3.12}$$

According to this expression, $\mathbf{M}$ can be looked upon as resulting from a special operation between the two vectors $\mathbf{F}$ and $\mathbf{r}$. Thus, it is the basis for defining a product between any two vectors. Because the result of such a product is a vector, it is called the *vector product*.

The product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is a vector $\mathbf{C}$ whose magnitude is equal to the product of the magnitudes of vectors $\mathbf{A}$ and $\mathbf{B}$ multiplied by the sine of the angle measured from $\mathbf{A}$ to $\mathbf{B}$ such that $0 \le \theta \le \pi$. The direction of vector $\mathbf{C}$ is specified by the condition that $\mathbf{C}$ be perpendicular to the plane of the vectors $\mathbf{A}$ and $\mathbf{B}$, and point in the direction in which a right-handed screw advances when turned so as to bring $\mathbf{A}$ into $\mathbf{B}$, as shown in Figure 2.3.7.

The vector product is usually denoted by

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = AB \ \sin(\mathbf{A}, \mathbf{B})\, \hat{\mathbf{e}} = AB \ \sin\theta \ \hat{\mathbf{e}}, \tag{2.3.13}$$

where sin(**A,B**) denotes the sine of the angle between vectors $\mathbf{A}$ and $\mathbf{B}$. This product is called the *cross product* or *vector product*. When $\mathbf{A} = a\,\hat{\mathbf{e}}_A$ and $\mathbf{B} = b\,\hat{\mathbf{e}}_B$ are the vectors representing the sides of a parallelogram, with a and b denoting the lengths of the sides, then the magnitude of the vector product $\mathbf{A} \times \mathbf{B}$

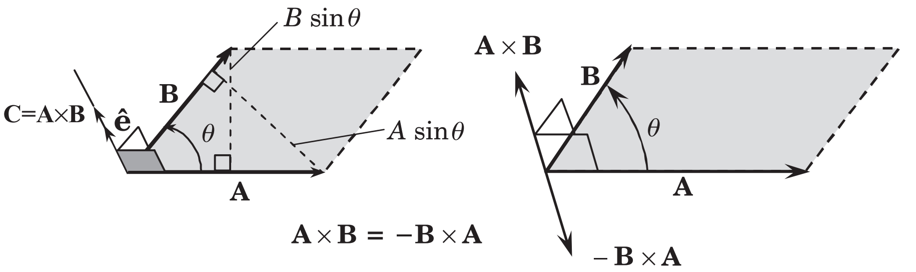


Figure 2.3.7 Representation of the vector product.

represents the area of the parallelogram, $ab\ \sin\theta$. The unit vector $\hat{\mathbf{e}} = \hat{\mathbf{e}}_A \times \hat{\mathbf{e}}_B$ denotes the normal to the plane area. Thus, an area can be represented as a vector (see Section 2.3.8 for additional discussion).

The description of the velocity of a point of a rotating rigid body is an important example of the geometric and physical application of vectors. Suppose a rigid body is rotating with an angular velocity ω about an axis and we wish to describe the velocity of some point P of the body, as shown in Figure 2.3.8(a).

Let $\mathbf{v}$ denote the velocity at the point P. Each point of the body describes a circle that lies in a plane perpendicular to the axis with its center on the axis. The radius of the circle, a, is the perpendicular distance from the axis to the point of interest. The magnitude of the velocity is equal to ωa. The direction of $\mathbf{v}$ is perpendicular both to a and to the axis of rotation. We denote the direction of the velocity by the unit vector $\hat{\mathbf{e}}$. Thus, we can write

$$\mathbf{v} = \omega a\ \hat{\mathbf{e}}. \tag{2.3.14}$$

Let O be a reference point on the axis of revolution, and let $\mathbf{OP} = \mathbf{r}$. We then have $a = r\sin\theta$, so that

$$\mathbf{v} = \omega r\ \sin\theta\ \hat{\mathbf{e}}. \tag{2.3.15}$$

The angular velocity is a vector because it has an assigned direction and magnitude, and obeys the parallelogram law of addition. We denote it by ω and represent its direction in the sense of a right-handed screw, as shown in Figure 2.3.8(b). If we further let $\hat{\mathbf{e}}_r$ be a unit vector in the direction of $\mathbf{r}$, we see that

$$\hat{\mathbf{e}}_\omega \times \hat{\mathbf{e}}_r = \hat{\mathbf{e}}\ \sin\theta. \tag{2.3.16}$$

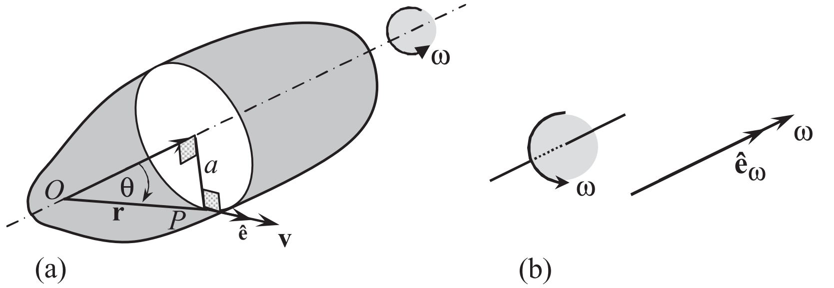


Figure 2.3.8 (a) Velocity at a point in a rotating rigid body. (b) Angular velocity as a vector.

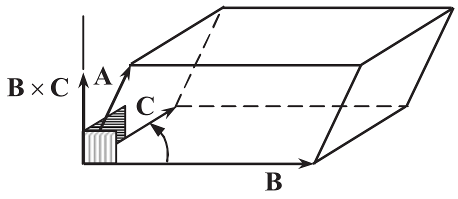


Figure 2.3.9 Scalar triple product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ as the volume of a parallelepiped.

With these relations, we have

$$\mathbf{v} = \omega \times \mathbf{r}. \tag{2.3.17}$$

Thus, the velocity of a point of a rigid body rotating about an axis is given by the vector product of ω and a position vector $\mathbf{r}$ drawn from any reference point on the axis of revolution.

From the definition of the vector (cross) product, the following few simple results follow:

(1) The products $\mathbf{A} \times \mathbf{B}$ and $\mathbf{B} \times \mathbf{A}$ are not equal. In fact, we have

$$\mathbf{A} \times \mathbf{B} \equiv -\mathbf{B} \times \mathbf{A}. \tag{2.3.18}$$

Thus, the vector product does not commute. We must therefore preserve the order of the vectors when vector products are involved.

(2) If two vectors $\mathbf{A}$ and $\mathbf{B}$ are parallel to each other, then $\theta = \pi$ or 0 and $\sin\theta = 0$. Thus,

$$\mathbf{A} \times \mathbf{B} = \mathbf{0}.$$

Conversely, if $\mathbf{A} \times \mathbf{B} = \mathbf{0}$, then either $\mathbf{A}$ or $\mathbf{B}$ is zero or they are parallel vectors. It follows that the vector product of a vector with itself is zero, that is, $\mathbf{A} \times \mathbf{A} = \mathbf{0}$.

(3) The distributive law still holds but the order of the factors must be maintained:

$$(\mathbf{A} + \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \times \mathbf{C}) + (\mathbf{B} \times \mathbf{C}). \tag{2.3.19}$$

2.3.7 Triple products of vectors

Now consider the various products of three vectors:

$$\mathbf{A}(\mathbf{B} \cdot \mathbf{C}), \quad \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}), \quad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}). \tag{2.3.20}$$

The product $\mathbf{A}(\mathbf{B} \cdot \mathbf{C})$ is merely a multiplication of the vector $\mathbf{A}$ by the scalar $\mathbf{B} \cdot \mathbf{C}$. On the other hand, the product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is a scalar and it is termed the *scalar triple product*. It can be seen that the product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$, except for the algebraic sign, is the volume of the parallelepiped formed by the vectors **A, B**, and **C**, as shown in Figure 2.3.9.

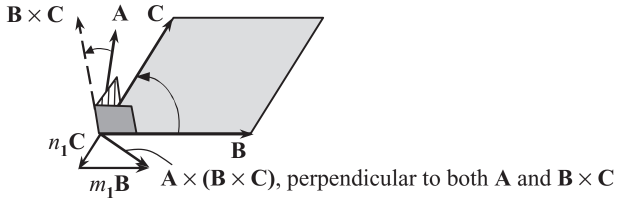


Figure 2.3.10 The vector triple product.

We also note the following properties:

(1) The dot and cross operations can be interchanged without changing the value:

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}\equiv[\mathbf{ABC}]. \tag{2.3.21}$$

(2) A cyclical permutation of the order of the vectors leaves the result unchanged:

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=\mathbf{C}\cdot\mathbf{A}\times\mathbf{B}=\mathbf{B}\cdot\mathbf{C}\times\mathbf{A}\equiv[\mathbf{ABC}]. \tag{2.3.22}$$

(3) If the cyclic order is changed, the sign changes:

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=-\mathbf{A}\cdot\mathbf{C}\times\mathbf{B}=-\mathbf{C}\cdot\mathbf{B}\times\mathbf{A}=-\mathbf{B}\cdot\mathbf{A}\times\mathbf{C}. \tag{2.3.23}$$

(4) A necessary and sufficient condition for any three vectors, **A,B,C**, to be coplanar is that $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=0$. Note also that the scalar triple product is zero when any two vectors are the same.

The *vector triple product* $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ is a vector normal to the plane formed by **A** and $(\mathbf{B}\times\mathbf{C})$. However, the vector $(\mathbf{B}\times\mathbf{C})$ is perpendicular to the plane formed by **B** and **C**. This means that $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ lies in the plane formed by **B** and **C** and is perpendicular to **A**, as shown in Figure 2.3.10. Thus, $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ can be expressed as a linear combination of **B** and **C**:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=c_1\mathbf{B}+c_2\mathbf{C}. \tag{2.3.24}$$

Likewise, we would find that

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C}=d_1\mathbf{A}+d_2\mathbf{B}. \tag{2.3.25}$$

Thus, the parentheses cannot be interchanged or removed. It can be shown that

$$c_1=\mathbf{A}\cdot\mathbf{C},\qquad c_2=-\mathbf{A}\cdot\mathbf{B},$$

and hence that

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}. \tag{2.3.26}$$

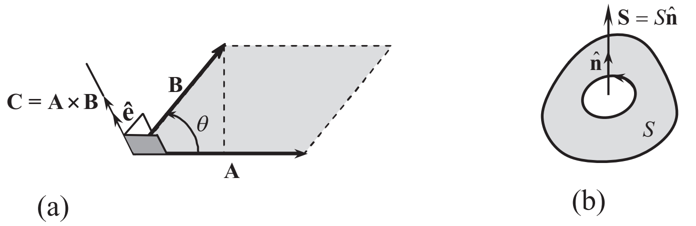


Figure 2.3.11 (a) Plane area as a vector. (b) Unit normal vector and sense of travel.

Example 2.3.1:

Let **A** and **B** be any two vectors in space. Express vector **A** in terms of its components along (that is, parallel) and perpendicular to vector **B**.

Solution: The component of **A** along **B** is given by $(\mathbf{A}\cdot\hat{\mathbf{e}}_B)$, where $\hat{\mathbf{e}}_B = \mathbf{B}/B$ is the unit vector in the direction of **B**. The component of **A** perpendicular to **B** and in the plane of **A** and **B** is given by the vector triple product $\hat{\mathbf{e}}_B \times (\mathbf{A}\times\hat{\mathbf{e}}_B)$. Thus,

$$\mathbf{A} = (\mathbf{A}\cdot\hat{\mathbf{e}}_B)\hat{\mathbf{e}}_B + \hat{\mathbf{e}}_B \times (\mathbf{A}\times\hat{\mathbf{e}}_B).$$

Alternatively, using Eq. (2.3.26) with $\mathbf{A} = \mathbf{C} = \hat{\mathbf{e}}_B$ and $\mathbf{B} = \mathbf{A}$, we obtain

$$\hat{\mathbf{e}}_B \times (\mathbf{A}\times\hat{\mathbf{e}}_B) = \mathbf{A} - (\hat{\mathbf{e}}_B\cdot\mathbf{A})\hat{\mathbf{e}}_B$$

or

$$\mathbf{A} = (\mathbf{A}\cdot\hat{\mathbf{e}}_B)\hat{\mathbf{e}}_B + \hat{\mathbf{e}}_B \times (\mathbf{A}\times\hat{\mathbf{e}}_B). \tag{2.3.27}$$

2.3.8 Plane area as a vector

As indicated previously, the magnitude of the vector $\mathbf{C} = \mathbf{A}\times\mathbf{B}$ is equal to the area of the parallelogram formed by the vectors **A** and **B**, as shown in Figure 2.3.11(a). In fact, the vector **C** may be considered to represent both the magnitude and the direction of the product **A** and **B**. Thus, a plane area may be looked upon as possessing a direction in addition to a magnitude, the directional character rising out of the need to specify an orientation of the plane in space.

It is customary to denote the direction of a plane area by means of a unit vector drawn normal to that plane. To fix the direction of the normal, we assign a sense of travel along the contour of the boundary of the plane area in question. The direction of the normal is taken by convention as that in which a right-handed screw advances as it is rotated according to the sense of travel along the boundary curve or contour, as shown in Figure 2.3.11(b). Let the unit normal vector be given by $\hat{\mathbf{n}}$; then the area can be denoted by $\mathbf{S} = S\hat{\mathbf{n}}$.

The representation of a plane as a vector has many uses. The vector can be used to determine the area of an inclined plane in terms of its projected area, as illustrated in the next example.

Example 2.3.2:

(**1**) Determine the plane area of the surface obtained by cutting a cylinder of cross-sectional area S_0 with an inclined plane whose normal is $\hat{\mathbf{n}}$, as shown in Figure 2.3.12(a).

(**2**) Express the areas of the sides of the tetrahedron obtained from a cube (or a prism) cut by an inclined plane whose normal is $\hat{\mathbf{n}}$, as shown in Figure 2.3.12(b), in terms of the area S of the inclined surface.

Solution:

(**1**) Let the plane area of the inclined surface be S, as shown in Figure 2.3.12(a). First, we express the areas as vectors,

$$\mathbf{S}_0 = S_0\,\hat{\mathbf{n}}_0 \quad \text{and} \quad \mathbf{S} = S\,\hat{\mathbf{n}}.$$

Because S_0 is the projection of $\mathbf{S}$ along $\hat{\mathbf{n}}_0$ (if the angle between $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}_0$ is acute; otherwise the negative of it),

$$S_0 = \mathbf{S} \cdot \hat{\mathbf{n}}_0 = S\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0. \tag{2.3.28}$$

The scalar product $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0$ is the cosine of the angle between the two unit normal vectors.

(**2**) For reference purposes, we label the sides of the cube by 1, 2, and 3 and the normals and surface areas by $(\hat{\mathbf{n}}_1, S_1)$, $(\hat{\mathbf{n}}_2, S_2)$, and$(\hat{\mathbf{n}}_3, S_3)$, respectively (i.e., S_i is the surface area of the plane perpendicular to the ith line or $\hat{\mathbf{n}}_i$ vector), as shown in Figure 2.3.12(b). Then we have

$$\hat{\mathbf{n}}_1 = -\hat{\mathbf{e}}_1, \quad \hat{\mathbf{n}}_2 = -\hat{\mathbf{e}}_2, \quad \hat{\mathbf{n}}_3 = -\hat{\mathbf{e}}_3 \tag{2.3.29}$$

$$S_1 = S\,\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_1 = Sn_1, \quad S_2 = S\,\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_2 = Sn_2, \quad S_3 = S\,\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_3 = Sn_3 \tag{2.3.30}$$

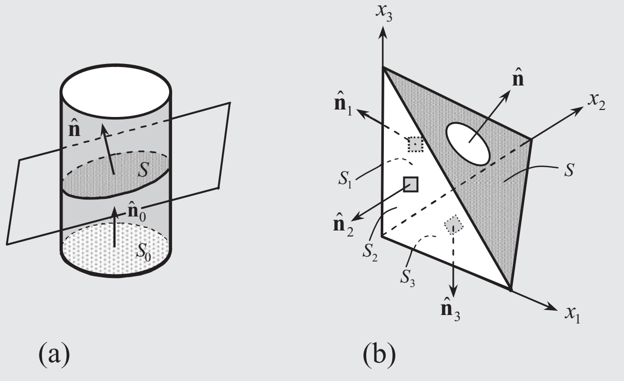


Figure 2.3.12 Vector representation of an inclined plane area.

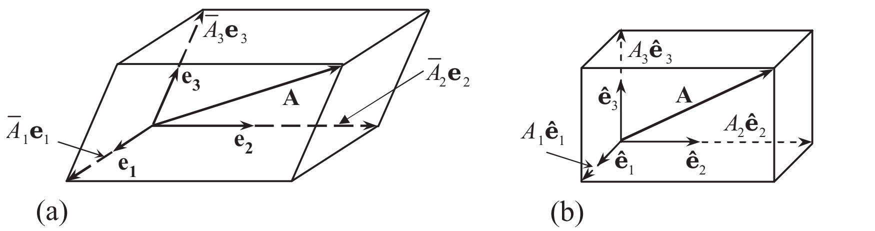


Figure 2.3.13 Components of a vector in (a) the general coordinate system that is oblique, and (b) the rectangular Cartesian system.

2.3.9 Components of a vector

So far, we have considered a geometric description of a vector as a directed line segment. We now embark on an analytic description of a vector and some of the operations associated with this description. The analytic description of vectors is useful in expressing, for example, the laws of physics in analytic form. The analytic description of a vector is based on the notion of its components.

In a three-dimensional space, a set of no more than three linearly independent vectors can be found. Let us choose any set and denote it as follows:

$$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3. \tag{2.3.31}$$

This set is called a *basis* (or a base system). A basis is called *orthonormal* if they are mutually orthogonal and have unit magnitudes. To distinguish the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ that is not orthonormal from one that is orthonormal, we denote the orthonormal basis by $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$, where

$$\begin{aligned} &\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2 = 0, \quad \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_3 = 0, \quad \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_1 = 0, \\ &\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1 = 1, \quad \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_2 = 1, \quad \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_3 = 1. \end{aligned} \tag{2.3.32}$$

In some books, the notation $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ or $(\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z)$ is used in place of $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$. In view of the previous discussion of the cross product of vectors, we note the following relations resulting from the cross products of the basis vectors:

$$\begin{aligned} &\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_1 = \mathbf{0}, &&\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3, &&\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_3 = -\hat{\mathbf{e}}_2, \\ &\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_1 = -\hat{\mathbf{e}}_3, &&\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_2 = \mathbf{0}, &&\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1, \\ &\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2, &&\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_2 = -\hat{\mathbf{e}}_1, &&\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_3 = \mathbf{0}. \end{aligned} \tag{2.3.33}$$

It is clear from the concept of linear dependence that we can represent any vector in three-dimensional space as a linear combination of the basis vectors:

$$\mathbf{A} = A_x\hat{\mathbf{e}}_x + A_y\hat{\mathbf{e}}_y + A_z\hat{\mathbf{e}}_z = A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3. \tag{2.3.34}$$

The vectors $A_1\hat{\mathbf{e}}_1$, $A_2\hat{\mathbf{e}}_2$, and $A_3\hat{\mathbf{e}}_3$ are called the *vector components* of $\mathbf{A}$, and A_1, A_2, and A_3 are called *scalar components* of $\mathbf{A}$ associated with the basis $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$, as indicated in Figure 2.3.13.

When the basis is orthonormal, A_1, A_2, and A_3 are the *physical components* of the vector $\mathbf{A}$, that is, the components have the same physical dimensions or

units as the vector. A scalar multiple of a vector is the same as the vector whose components are the scalar multiples:

$$\alpha \mathbf{A} = (\alpha A_1)\hat{\mathbf{e}}_1 + (\alpha A_2)\hat{\mathbf{e}}_2 + (\alpha A_3)\hat{\mathbf{e}}_3. \tag{2.3.35}$$

Two vectors are equal if and only if their respective components are equal. That is, $\mathbf{A} = \mathbf{B}$ implies that $A_1 = B_1$, $A_2 = B_2$, and $A_3 = B_3$.

The operations of vector addition, scalar product, and vector product of vectors can now be expressed in terms of the rectangular Cartesian components, as given in the following.

Addition of Vectors. The sum of vectors **A** and **B** is the vector **C** whose components are the sum of the respective components of vectors **A** and **B**:

$$\begin{aligned}
\mathbf{A} + \mathbf{B} &= \left(A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3\right) + \left(B_1\hat{\mathbf{e}}_1 + B_2\hat{\mathbf{e}}_2 + B_3\hat{\mathbf{e}}_3\right) \\
&= (A_1 + B_1)\hat{\mathbf{e}}_1 + (A_2 + B_2)\hat{\mathbf{e}}_2 + (A_3 + B_3)\hat{\mathbf{e}}_3 \\
&\equiv C_1\hat{\mathbf{e}}_1 + C_2\hat{\mathbf{e}}_2 + C_3\hat{\mathbf{e}}_3 = \mathbf{C},
\end{aligned} \tag{2.3.36}$$

with $C_1 = A_1 + B_1$, $C_2 = A_2 + B_2$, and $C_3 = A_3 + B_3$.

Scalar Product of Vectors. The scalar product of vectors **A** and **B** is the scalar

$$\begin{aligned}
\mathbf{A} \cdot \mathbf{B} &= \left(A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3\right) \cdot \left(B_1\hat{\mathbf{e}}_1 + B_2\hat{\mathbf{e}}_2 + B_3\hat{\mathbf{e}}_3\right) \\
&= A_1B_1 + A_2B_2 + A_3B_3,
\end{aligned} \tag{2.3.37}$$

where the orthonormal property, Eq. (2.3.32), of the basis vectors is used in arriving at this last expression.

Vector Product of Vectors. The vector product of two vectors **A** and **B** is the vector

$$\begin{aligned}
\mathbf{A} \times \mathbf{B} &= \left(A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3\right) \times \left(B_1\hat{\mathbf{e}}_1 + B_2\hat{\mathbf{e}}_2 + B_3\hat{\mathbf{e}}_3\right) \\
&= (A_2B_3 - A_3B_2)\hat{\mathbf{e}}_1 + (A_3B_1 - A_1B_3)\hat{\mathbf{e}}_2 + (A_1B_2 - A_2B_1)\hat{\mathbf{e}}_3,
\end{aligned} \tag{2.3.38}$$

where the relations in Eq. (2.3.33) are used in arriving at the final expression.

Example 2.3.3:

The velocity at a point in a flow field is $\mathbf{v} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ (m/s). Determine (**1**) the velocity vector $\mathbf{v}_n$ normal to the plane $\mathbf{n} = 3\hat{\mathbf{i}} - 4\hat{\mathbf{k}}$ passing through the point, (**2**) the angle between $\mathbf{v}$ and $\mathbf{v}_n$, (**3**) the tangential velocity vector $\mathbf{v}_t$ on the plane, and (**4**) the mass flow rate across the plane through an area $A = 0.15$ m^2 if the density of the fluid (water) is $\rho = 10^3$ kg/m^3 and the flow is uniform.

Solution: (**1**) The magnitude of the velocity normal to the given plane is given by the projection of the velocity along the normal to the plane. The unit vector normal to the plane is given by

$$\hat{\mathbf{n}} = \frac{1}{5}\left(3\hat{\mathbf{i}} - 4\hat{\mathbf{k}}\right).$$

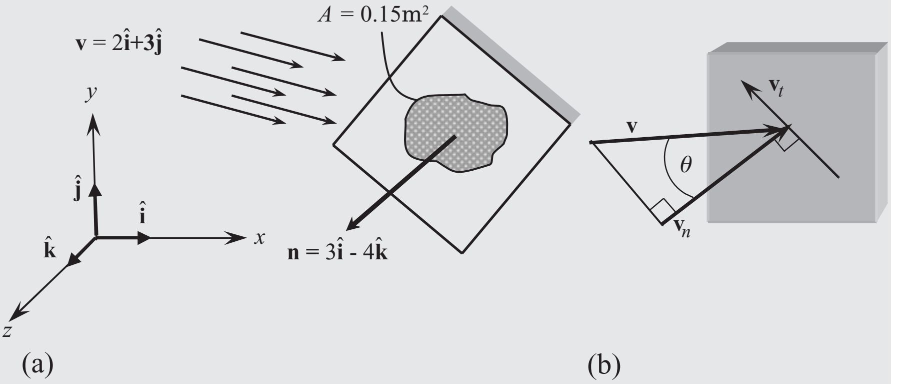


Figure 2.3.14 Flow across a plane.

Then the velocity vector normal to the plane is [see Figure 2.3.14(a)]

$$\mathbf{v}_n = (\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} = \frac{6}{5}\hat{\mathbf{n}} = 0.24\left(3\hat{\mathbf{i}} - 4\hat{\mathbf{k}}\right) \text{ m/s.}$$

(**2**) The angle between $\mathbf{v}$ and $\mathbf{v}_n$ is given by

$$\theta = \cos^{-1}\left(\frac{\mathbf{v} \cdot \mathbf{v}_n}{|\mathbf{v}||\mathbf{v}_n|}\right) = \cos^{-1}\left(\frac{v_n}{v}\right) = \cos^{-1}\left(\frac{1.2}{\sqrt{13}}\right) = 70.6^\circ.$$

(**3**) The tangential velocity vector on the plane is given by

$$\mathbf{v}_t = \mathbf{v} - \mathbf{v}_n = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 0.24\left(3\hat{\mathbf{i}} - 4\hat{\mathbf{k}}\right) = -5.2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 9.6\hat{\mathbf{k}}.$$

(**4**) The mass flow rate is given by

$$Q = \rho v_n A = 10^3 \times 1.2 \times 0.15 = 180 \text{ kg/s.}$$

Various vectors are depicted in Figure 2.3.14(b).

2.4 Index notation and summation convention

2.4.1 Summation convention

The use of index notation facilitates writing long expressions in a succinct form. For example, consider the component form of vector **A**,

$$\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3, \tag{2.4.1}$$

which can be abbreviated as

$$\mathbf{A} = \sum_{i=1}^{3} A_i\mathbf{e}_i \quad \text{or} \quad \mathbf{A} = \sum_{j=1}^{3} A_j\mathbf{e}_j.$$

If we had chosen the notation

$$\mathbf{A} = A_x\mathbf{e}_x + A_y\mathbf{e}_y + A_z\mathbf{e}_z,$$

where (A_x, A_y, A_z) are the same components as (A_1, A_2, A_3) and the basis $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ is the same as $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, it is not possible to write this expression

with the summation convention. The summation index i or j is arbitrary as long as the same index is used for both A and $\hat{\mathbf{e}}$. The expression can be further shortened by omitting the summation sign and having the understanding that a repeated index means summation over all values of that index. Thus, the three-term expression $A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$ can simply be written as

$$\mathbf{A} = A_i\mathbf{e}_i. \tag{2.4.2}$$

This notation is called the *summation convention.* The summation convention allows us to write several expressions or equations in a single statement. As we shall see shortly, the six relations in Eq. (2.3.32) can be written as a single equation with the help of index notation and certain symbols that we are about to introduce in Section 2.4.4.

2.4.2 Dummy index

The repeated index is called a *dummy index* because it can be replaced by any other symbol that has not already been used in that expression. Thus, the expression in Eq. (2.4.2) can also be written as

$$\mathbf{A} = A_i\mathbf{e}_i = A_j\mathbf{e}_j = A_m\mathbf{e}_m, \tag{2.4.3}$$

and so on. As a rule, no index must appear more than twice in an expression. For example, $A_iB_iC_i$ is not a valid expression because the index i appears more than twice. Other examples of dummy indices are

$$F_i = A_iB_jC_j, \quad G_k = H_k(2 - 3A_iB_i) + P_jQ_jF_k.$$

Each of these equations expresses three equations when the range of i and j is 1 to 3. For example, the first equation is equal to the following three equations:

$$\begin{aligned}
F_1 &= A_1(B_1C_1 + B_2C_2 + B_3C_3),\\
F_2 &= A_2(B_1C_1 + B_2C_2 + B_3C_3),\\
F_3 &= A_3(B_1C_1 + B_2C_2 + B_3C_3).
\end{aligned}$$

This amply illustrates the usefulness of the summation convention in shortening long and multiple expressions into a single expression.

2.4.3 Free index

A *free index* is one that appears in every expression of an equation except for expressions that contain real numbers (scalars) only. The index i in the equation $F_i = A_iB_jC_j$ and k in the equation $G_k = H_k(2 - 3A_iB_i) + P_jQ_jF_k$ are free indices. Another example is

$$A_i = 2 + B_i + C_i + D_i + (F_jG_j - H_jP_j)E_i.$$

This expression contains three equations ($i = 1, 2, 3$). The expressions $A_i = B_jC_k$, $A_i = B_j$, and $F_k = A_iB_jC_k$ do not make sense and should not arise because the indices on the two sides of the equal sign do not match.

2.4.4 Kronecker delta and permutation symbols

It is convenient to introduce the *Kronecker delta* δ_{ij} and alternating symbol e_{ijk} because they allow easy representation of the dot product (or scalar product) and cross product, respectively, of orthonormal vectors in a right-handed basis system. We define the dot product $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j$ as

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}, \tag{2.4.4}$$

where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases} \tag{2.4.5}$$

Thus, the single expression in Eq. (2.4.4) is the same as the six relations in Eq. (2.3.32). Due to its definition, the Kronecker delta δ_{ij} modifies (or contracts) the subscripts in the coefficients of an expression in which it appears:

$$A_i\delta_{ij} = A_j, \quad A_iB_j\delta_{ij} = A_iB_i = A_jB_j, \quad \delta_{ij}\delta_{ik} = \delta_{jk}.$$

As we shall see shortly, δ_{ij} denotes the Cartesian components of a second-order unit tensor, $\mathbf{I} = \delta_{ij}\,\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j = \hat{\mathbf{e}}_i\hat{\mathbf{e}}_i$.

We define the cross product $\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j$ as

$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j \equiv e_{ijk}\hat{\mathbf{e}}_k \;\text{ or }\; e_{ijk} = \hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_k = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j \times \hat{\mathbf{e}}_k, \tag{2.4.6}$$

where

$$e_{ijk} = \begin{cases} 1, & \text{if } i, j, k \text{ are in cyclic order} \\ & \text{and not repeated } (i \neq j \neq k), \\ -1, & \text{if } i, j, k \text{ are not in cyclic order} \\ & \text{and not repeated } (i \neq j \neq k), \\ 0, & \text{if any of } i, j, k \text{ are repeated.} \end{cases} \tag{2.4.7}$$

The symbol e_{ijk} is called the *alternating symbol* or *permutation symbol*. By definition, the subscripts of the permutation symbol can be permuted without changing its value; an interchange of any two subscripts will change the sign. Hence, the interchange of two subscripts twice keeps the value unchanged:

$$e_{ijk} = e_{kij} = e_{jki}, \quad e_{ijk} = -e_{jik} = e_{jki} = -e_{kji}.$$

In an orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, the scalar and vector products can be expressed with the index notation using the Kronecker delta and the alternating symbol:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= (A_i\hat{\mathbf{e}}_i) \cdot (B_j\hat{\mathbf{e}}_j) = A_iB_j\delta_{ij} = A_iB_i, \\ \mathbf{A} \times \mathbf{B} &= (A_i\hat{\mathbf{e}}_i) \times (B_j\hat{\mathbf{e}}_j) = A_iB_je_{ijk}\hat{\mathbf{e}}_k. \end{aligned} \tag{2.4.8}$$

Note that the components of a vector in an orthonormal coordinate system can be expressed as

$$A_i = \mathbf{A} \cdot \hat{\mathbf{e}}_i, \tag{2.4.9}$$

and therefore we can express vector **A** as

$$\mathbf{A} = A_i \hat{\mathbf{e}}_i = (\mathbf{A} \cdot \hat{\mathbf{e}}_i)\hat{\mathbf{e}}_i. \tag{2.4.10}$$

Further, the Kronecker delta and the permutation symbol are related by an identity known as the *e-δ identity* [see Problem 2.14],

$$e_{ijk} e_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \tag{2.4.11}$$

The permutation symbol and the Kronecker delta prove to be very useful in proving vector identities. Because a vector form of any identity is invariant (i.e., valid in any coordinate system), it suffices to prove it in only one coordinate system. In particular, an orthonormal system is very convenient because we can use the index notation, the permutation symbol, and the Kronecker delta. The following examples contain several cases of incorrect and correct use of index notation, and illustrate some of the uses of δ_{ij} and e_{ijk}.

Example 2.4.1:

Discuss the validity of the following expressions:

(1) $a_m b_s = c_m(d_r - f_r)$
(2) $a_m b_s = c_m(d_s - f_s)$
(3) $a_i = b_j c_i d_i$
(4) $x_i x_i = r^2$
(5) $a_i = 3$

Solution:

(1) This is not a valid expression because the free indices r and s do not match.
(2) Valid; both m and s are free indices. There are nine equations ($m, s = 1, 2, 3$).
(3) This is not a valid expression because the free index j is not matched on both sides of the equality, and index i is a dummy index in one expression and a free index in the other. The index i cannot be used both as a free and dummy index in the same equation. The equation would be valid if i on the left side of the equation is replaced with j; then there will be three equations).
(4) A valid expression, containing one equation: $x_1^2 + x_2^2 + x_3^2 = r^2$.
(5) This is a valid expression in some branches of mathematics but it is not a valid expression in continuum mechanics because it violates form-invariance (material frame indifference) under a basis transformation (every component of a vector cannot be the same in all bases).

Example 2.4.2:

Simplify the following expressions:

(1) $\delta_{ij}\delta_{jk}\delta_{kp}\delta_{pi}$
(2) $\varepsilon_{mjk}\varepsilon_{njk}$
(3) $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})$

Solution:

(1) Successive contraction of subscripts yield the result of

$$\delta_{ij}\delta_{jk}\delta_{kp}\delta_{pi} = \delta_{ij}\delta_{jk}\delta_{ki} = \delta_{ij}\delta_{ji} = \delta_{ii} = 3.$$

(2) Expand this expression using the e-δ identity:

$$\varepsilon_{mjk}\varepsilon_{njk} = \delta_{mn}\delta_{jj} - \delta_{mj}\delta_{nj} = 3\delta_{mn} - \delta_{mn} = 2\delta_{mn}.$$

In particular, the expression $\varepsilon_{ijk}\varepsilon_{ijk}$ is equivalent to $2\delta_{ii} = 6$.

(3) Expanding the expression using index notation, we obtain

$$\begin{aligned}
(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) &= (A_iB_je_{ijk}\hat{\mathbf{e}}_k)\cdot(C_mD_ne_{mnp}\hat{\mathbf{e}}_p)\\
&= A_iB_jC_mD_ne_{ijk}e_{mnp}\delta_{kp}\\
&= A_iB_jC_mD_ne_{ijk}e_{mnk}\\
&= A_iB_jC_mD_n(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\\
&= A_iB_jC_mD_n\delta_{im}\delta_{jn} - A_iB_jC_mD_n\delta_{in}\delta_{jm}\\
&= A_iB_jC_iD_j - A_iB_jC_jD_i\\
&= A_iC_iB_jD_j - A_iD_iB_jC_j\\
&= (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}),
\end{aligned}$$

where we have used the e-δ identity, Eq. (2.4.11). Although the previous vector identity is established in an orthonormal coordinate system, it holds in a general coordinate system. That is, the vector identity here is invariant.

Example 2.4.3:

Rewrite the expression $e_{mni}A_iB_jC_mD_n\hat{\mathbf{e}}_j$ in vector form.

Solution: We note that $B_j\hat{\mathbf{e}}_j = \mathbf{B}$. Examining the indices in the permutation symbol and the remaining coefficients, it is clear that vectors $\mathbf{C}$ and $\mathbf{D}$ must have a cross product between them and the resulting vector must have a dot product with vector $\mathbf{A}$. Thus, we have

$$e_{mni}A_iB_jC_mD_n\hat{\mathbf{e}}_j = [(\mathbf{C}\times\mathbf{D})\cdot\mathbf{A}]\mathbf{B} = (\mathbf{C}\times\mathbf{D}\cdot\mathbf{A})\,\mathbf{B}.$$

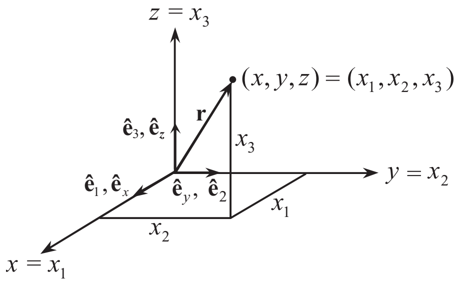


Figure 2.4.1 Rectangular Cartesian coordinates.

2.4.5 Transformation law for different bases

When the basis vectors are constant, that is, with fixed lengths (with the same units) and directions, the basis is called *Cartesian*. The general Cartesian system is oblique. When the basis vectors are unit and orthogonal (orthonormal), the basis system is called *rectangular Cartesian*, or simply *Cartesian*. In much of our study, we shall deal with Cartesian bases.

Let us denote an orthonormal Cartesian basis by

$$\{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z\} \qquad \text{or} \qquad \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}.$$

The Cartesian coordinates are denoted by (x, y, z) or (x_1, x_2, x_3). The familiar rectangular Cartesian coordinate system is shown in Figure 2.4.1. We shall always use right-handed coordinate systems.

A position vector to an arbitrary point (x, y, z) or (x_1, x_2, x_3), measured from the origin, is given by

$$\begin{aligned}\mathbf{r} &= x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z \\ &= x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3,\end{aligned} \tag{2.4.12}$$

or in summation notation by

$$\mathbf{r} = x_j\hat{\mathbf{e}}_j, \quad \mathbf{r}\cdot\mathbf{r} = r^2 = x_i x_i. \tag{2.4.13}$$

We shall also use the symbol $\mathbf{x}$ for the position vector $\mathbf{r} = \mathbf{x}$. The length of a line element $d\mathbf{r} = d\mathbf{x}$ is given by

$$d\mathbf{r}\cdot d\mathbf{r} = (ds)^2 = dx_j dx_j = (dx)^2 + (dy)^2 + (dz)^2. \tag{2.4.14}$$

Here we discuss the relationship between the components of two different orthonormal coordinate systems. Consider the first coordinate basis

$$\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$$

and the second coordinate basis

$$\{\hat{\bar{e}}_1, \hat{\bar{e}}_2, \hat{\bar{e}}_3\}.$$

Now we can express the same vector in the coordinate system without bars (referred to as "unbarred"), and also in the coordinate system with bars (referred to as "barred"):

$$\begin{aligned} \mathbf{A} &= A_i \hat{\mathbf{e}}_i = (\mathbf{A} \cdot \hat{\mathbf{e}}_i)\hat{\mathbf{e}}_i \\ &= \bar{A}_i \hat{\bar{\mathbf{e}}}_i = (\mathbf{A} \cdot \hat{\bar{\mathbf{e}}}_i)\hat{\bar{\mathbf{e}}}_i . \end{aligned} \tag{2.4.15}$$

From Eq. (2.4.10), we have

$$\bar{A}_j = \mathbf{A} \cdot \hat{\bar{\mathbf{e}}}_j = A_i(\hat{\mathbf{e}}_i \cdot \hat{\bar{\mathbf{e}}}_j) \equiv \ell_{ji} A_i , \tag{2.4.16}$$

where

$$\ell_{ij} = \hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j . \tag{2.4.17}$$

Equation (2.4.12) gives the relationship between the components $(\bar{A}_1, \bar{A}_2, \bar{A}_3)$ and (A_1, A_2, A_3), and this is called the *transformation rule* between the barred and unbarred components in the two coordinate systems. The coefficients ℓ_{ij} can be interpreted as the directional cosines of the barred coordinate system with respect to the unbarred coordinate system:

$$\ell_{ij} = \text{cosine of the angle between } \hat{\bar{\mathbf{e}}}_i \text{ and } \hat{\mathbf{e}}_j . \tag{2.4.18}$$

Note that the first subscript of ℓ_{ij} comes from the barred coordinate system and the second subscript from the unbarred system. Obviously, ℓ_{ij} is not symmetric (i.e., $\ell_{ij} \neq \ell_{ji}$). The rectangular array of these components is called a *matrix*, which is the topic of the next section. The next example illustrates the computation of directional cosines.

Example 2.4.4:

Let $\hat{\mathbf{e}}_i$ $(i = 1, 2, 3)$ be a set of orthonormal base vectors, and define a new right-handed coordinate basis by (note that $\hat{\bar{\mathbf{e}}}_1 \cdot \hat{\bar{\mathbf{e}}}_2 = 0$)

$$\hat{\bar{\mathbf{e}}}_1 = \frac{1}{3}(2\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3), \quad \hat{\bar{\mathbf{e}}}_2 = \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2),$$

$$\hat{\bar{\mathbf{e}}}_3 = \hat{\bar{\mathbf{e}}}_1 \times \hat{\bar{\mathbf{e}}}_2 = \frac{1}{3\sqrt{2}}(\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - 4\hat{\mathbf{e}}_3).$$

The original and new coordinate systems are depicted in Figure 2.4.2. Determine the directional cosines ℓ_{ij} of the transformation.

Solution: From Eq. (2.4.17), we have

$$\begin{aligned} &\ell_{11} = \hat{\bar{\mathbf{e}}}_1 \cdot \hat{\mathbf{e}}_1 = \frac{2}{3}, && \ell_{12} = \hat{\bar{\mathbf{e}}}_1 \cdot \hat{\mathbf{e}}_2 = \frac{2}{3}, && \ell_{13} = \hat{\bar{\mathbf{e}}}_1 \cdot \hat{\mathbf{e}}_3 = \frac{1}{3}, \\ &\ell_{21} = \hat{\bar{\mathbf{e}}}_2 \cdot \hat{\mathbf{e}}_1 = \frac{1}{\sqrt{2}}, && \ell_{22} = \hat{\bar{\mathbf{e}}}_2 \cdot \hat{\mathbf{e}}_2 = -\frac{1}{\sqrt{2}}, && \ell_{23} = \hat{\bar{\mathbf{e}}}_2 \cdot \hat{\mathbf{e}}_3 = 0, \\ &\ell_{31} = \hat{\bar{\mathbf{e}}}_3 \cdot \hat{\mathbf{e}}_1 = \frac{1}{3\sqrt{2}}, && \ell_{32} = \hat{\bar{\mathbf{e}}}_3 \cdot \hat{\mathbf{e}}_2 = \frac{1}{3\sqrt{2}}, && \ell_{33} = \hat{\bar{\mathbf{e}}}_3 \cdot \hat{\mathbf{e}}_3 = -\frac{4}{3\sqrt{2}} . \end{aligned}$$

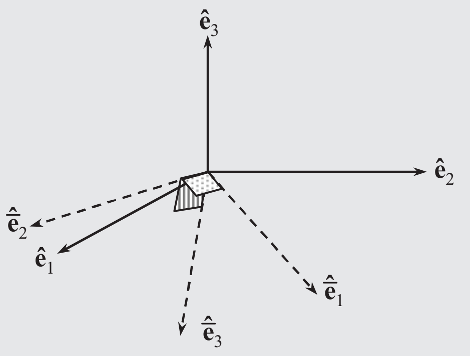


Figure 2.4.2 The original and transformed coordinate systems defined in Example 2.4.4.

2.5 Theory of matrices

2.5.1 Definition

In the preceding sections, we studied the algebra of ordinary vectors and the transformation of vector components from one coordinate system to another. For example, the transformation equation Eq. (2.4.16) relates the components of a vector in the barred coordinate system to an unbarred coordinate system. Writing Eq. (2.4.16) in an expanded form,

$$\begin{aligned}\bar{A}_1 &= \ell_{11}A_1 + \ell_{12}A_2 + \ell_{13}A_3,\\ \bar{A}_2 &= \ell_{21}A_1 + \ell_{22}A_2 + \ell_{23}A_3,\\ \bar{A}_3 &= \ell_{31}A_1 + \ell_{32}A_2 + \ell_{33}A_3,\end{aligned} \tag{2.5.1}$$

we see that there are nine coefficients relating the components A_i to $\bar{A}_i$. The form of these linear equations suggests writing the scalars of ℓ_{ij} (jth components in a ith equation) in a rectangular array,

$$\mathbf{L} = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix}.$$

This rectangular array $\mathbf{L}$ of scalars ℓ_{ij} is called a *matrix*, and the quantities ℓ_{ij} are called the *elements* of $\mathbf{L}$.[2]

If a matrix has m rows and n columns, we will say that is an m by n ($m \times n$) matrix, the number of rows always being listed first. The element in the ith row and jth column of a matrix $\mathbf{A}$ is generally denoted by a_{ij}, and we will sometimes designate a matrix by $\mathbf{A} = [A] = [a_{ij}]$. A square matrix is one that has the same number of rows as columns, $n \times n$. An $n \times n$ matrix is said to be of *order* n. The elements of a square matrix for which the row number and the column number are

[2] The word "matrix" was first used in 1850 by James Sylvester (1814–1897), an English algebraist. However, Arthur Caley (1821–1895), professor of mathematics at Cambridge, was the first one to explore the properties of matrices. Significant contributions in the early years were made by Charles Hermite, Georg Frobenius, and Camille Jordan, among others.

the same (that is, a_{ii} for any fixed i) are called *diagonal elements*, or simply the *diagonal*. A square matrix is said to be a *diagonal matrix* if all of the off-diagonal elements are zero. An *identity matrix*, denoted by $\mathbf{I} = [I]$, is a diagonal matrix whose elements are all 1's. Examples of a diagonal and an identity matrix are given as

$$\mathbf{A} = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The sum of the diagonal elements is called the *trace* of the matrix.

If the matrix has only one row or one column, we will normally use only a single subscript to designate its elements. For example,

$$\mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}, \qquad \mathbf{Y} = \{y_1 \; y_2 \; y_3\}$$

denote a column matrix and a row matrix, respectively. Row and column matrices can be used to denote the components of a vector.

2.5.2 Matrix addition and multiplication of a matrix by a scalar

The *sum* of two matrices of the same size is defined to be a matrix of the same size obtained by simply adding the corresponding elements. If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $m \times n$ matrix, their sum is an $m \times n$ matrix, $\mathbf{C}$, with

$$c_{ij} = a_{ij} + b_{ij} \quad \text{for all} \quad i, j. \tag{2.5.2}$$

A *constant multiple* of a matrix is equal to the matrix obtained by multiplying all of the elements by the constant. That is, the multiple of a matrix $\mathbf{A}$ by a scalar α, $\alpha\mathbf{A}$, is the matrix obtained by multiplying each of its elements by α:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \alpha\mathbf{A} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \dots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \dots & \alpha a_{2n} \\ \alpha a_{m1} & \alpha a_{m2} & \dots & \alpha a_{mn} \end{bmatrix}.$$

Matrix addition has the following properties:

(1) Addition is commutative: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

(2) Addition is associative: $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$.

(3) There exists a unique matrix $\mathbf{0}$, such that $\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$. The matrix $\mathbf{0}$ is called the *zero matrix*; with all elements of it are zeros.

(4) For each matrix $\mathbf{A}$, there exists a unique matrix $-\mathbf{A}$ such that $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$.

(5) Addition is distributive with respect to scalar multiplication: $\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$.

(6) Addition is distributive with respect to matrix multiplication, which will be discussed shortly (note the order):

$$(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}.$$

Calculations of the sum and difference of matrices are illustrated through the next example.

Example 2.5.1:

Compute the sum and difference of the following two matrices:

$$\mathbf{A} = \begin{bmatrix} 5 & -2 & 12 & 21 \\ 10 & 2 & 16 & -3 \\ 20 & 14 & 13 & 8 \\ -12 & 31 & 0 & 19 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 13 & -11 & 32 & 4 \\ -6 & 32 & 25 & 7 \\ 39 & 36 & -23 & 15 \\ 14 & -15 & 31 & 18 \end{bmatrix}.$$

Solution: The sum of **A** and **B** is

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} 5 & -2 & 12 & 21 \\ 10 & 2 & 16 & -3 \\ 20 & 14 & 13 & 8 \\ -12 & 31 & 0 & 19 \end{bmatrix} + \begin{bmatrix} 13 & -11 & 32 & 4 \\ -6 & 32 & 25 & 7 \\ 39 & 36 & -23 & 15 \\ 14 & -15 & 31 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & -13 & 44 & 25 \\ 4 & 34 & 41 & 4 \\ 59 & 50 & -10 & 23 \\ 2 & 16 & 31 & 37 \end{bmatrix}.$$

The difference of **A** and **B** is

$$\mathbf{A}-\mathbf{B} = \begin{bmatrix} 5 & -2 & 12 & 21 \\ 10 & 2 & 16 & -3 \\ 20 & 14 & 13 & 8 \\ -12 & 31 & 0 & 19 \end{bmatrix} - \begin{bmatrix} 13 & -11 & 32 & 4 \\ -6 & 32 & 25 & 7 \\ 39 & 36 & -23 & 15 \\ 14 & -15 & 31 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 9 & -20 & 17 \\ 16 & -30 & -9 & -10 \\ -19 & -22 & 36 & -7 \\ -26 & 46 & -31 & 1 \end{bmatrix}.$$

2.5.3 Matrix transpose and symmetric and skew symmetric matrices

If **A** is an $m \times n$ matrix, then the $n \times m$ matrix obtained by interchanging its rows and columns is called the *transpose* of **A** and is denoted by $\mathbf{A}^{\mathrm{T}}$. For example, consider the matrices

$$\mathbf{A} = \begin{bmatrix} 5 & -2 & 1 \\ 8 & 7 & 6 \\ 2 & 4 & 3 \\ -1 & 9 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 & -1 & 2 & 4 \\ -6 & 3 & 5 & 7 \\ 9 & 6 & -2 & 1 \end{bmatrix}. \qquad (2.5.3)$$

The transpose matrices of **A** and **B** are

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 5 & 8 & 2 & -1 \\ -2 & 7 & 4 & 9 \\ 1 & 6 & 3 & 0 \end{bmatrix}, \quad \mathbf{B}^{\mathrm{T}} = \begin{bmatrix} 3 & -6 & 9 \\ -1 & 3 & 6 \\ 2 & 5 & -2 \\ 4 & 7 & 1 \end{bmatrix}.$$

The following basic properties of a transpose matrix should be noted:

(1) $(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}$.
(2) $(\mathbf{A} + \mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$.

A square matrix **A** of real numbers is said to be *symmetric* if $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$. It is said to be *skew symmetric* or *antisymmetric* if $\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$. In terms of the elements of **A**, these definitions imply that **A** is symmetric if and only if $a_{ij} = a_{ji}$, and it is skew symmetric if and only if $a_{ij} = -a_{ji}$. Note that the diagonal elements of a skew symmetric matrix are always zero because $a_{ij} = -a_{ij}$ implies $a_{ij} = 0$ for $i = j$. Examples of symmetric and skew symmetric matrices, respectively, are

$$\begin{bmatrix} 5 & -2 & 12 & 21 \\ -2 & 2 & 16 & -3 \\ 12 & 16 & 13 & 8 \\ 21 & -3 & 8 & 19 \end{bmatrix}, \qquad \begin{bmatrix} 0 & -11 & 32 & 4 \\ 11 & 0 & 25 & 7 \\ -32 & -25 & 0 & 15 \\ -4 & -7 & -15 & 0 \end{bmatrix}.$$

2.5.4 Matrix multiplication

Consider a vector $\mathbf{A} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3$ in a Cartesian system. We can represent **A** as a *product* of a row matrix and a column matrix,

$$\mathbf{A} = \{a_1\ a_2\ a_3\} \begin{Bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{Bmatrix}.$$

Note that the vector **A** is obtained by multiplying the ith element in the row matrix with the ith element in the column matrix, and then adding terms. This gives us a strong motivation for defining the product of two matrices.

Let **x** and **y** be the vectors (matrices with one column)

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{Bmatrix}, \quad \mathbf{y} = \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{Bmatrix}.$$

We define the product $\mathbf{x}^{\mathrm{T}}\mathbf{y}$ to be the scalar

$$\mathbf{x}^{\mathrm{T}}\mathbf{y} = \{x_1, x_2, \ldots, x_m\} \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{Bmatrix} = x_1y_1 + x_2y_2 + \cdots + x_my_m = \sum_{i=1}^{m} x_iy_i. \tag{2.5.4}$$

It follows from Eq. (2.5.4) that $\mathbf{x}^{\mathrm{T}}\mathbf{y} = \mathbf{y}^{\mathrm{T}}\mathbf{x}$. More generally, let $\mathbf{A} = [a_{ij}]$ be $m \times n$ and $\mathbf{B} = [b_{ij}]$ be $n \times p$ matrices. The product $\mathbf{AB}$ is defined to be the $m \times p$ matrix $\mathbf{C} = [c_{ij}]$ with

$$c_{ij} = \{i\text{th row of } [\mathbf{A}]\} \begin{Bmatrix} j\text{th} \\ \text{column} \\ \text{of } \mathbf{B} \end{Bmatrix} = \{a_{i1}, a_{i2}, \ldots, a_{in}\} \begin{Bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{Bmatrix}$$

$$= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}. \tag{2.5.5}$$

The next example illustrates the computation of the product of a square matrix with a column matrix.

Example 2.5.2:

Express transformation equations of Example 2.4.4 in matrix form, and then determine the components $\bar{A}_i$ of the vector $\mathbf{A} = 6\hat{\mathbf{e}}_1 - 2\hat{\mathbf{e}}_2 + 4\hat{\mathbf{e}}_3$ referred to the unbarred system.

Solution: By Eq. (2.4.16), we have $\bar{\mathbf{A}} = \mathbf{LA}$:

$$\begin{Bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \bar{A}_3 \end{Bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \frac{1}{3\sqrt{2}} \begin{bmatrix} 2\sqrt{2} & 2\sqrt{2} & \sqrt{2} \\ 3 & -3 & 0 \\ 1 & 1 & -4 \end{bmatrix} \begin{Bmatrix} 6 \\ 2 \\ -4 \end{Bmatrix}. \tag{2.5.6}$$

Thus, a point $(A_1, A_2, A_3) = (6, -2, 4)$ referred to the unbarred coordinate system has the following coordinates in the barred coordinate system:

$$\begin{Bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \bar{A}_3 \end{Bmatrix} = \frac{1}{3\sqrt{2}} \begin{bmatrix} 2\sqrt{2} & 2\sqrt{2} & \sqrt{2} \\ 3 & -3 & 0 \\ 1 & 1 & -4 \end{bmatrix} \begin{Bmatrix} 6 \\ 2 \\ -4 \end{Bmatrix} = \frac{1}{3\sqrt{2}} \begin{Bmatrix} 12\sqrt{2} \\ 12 \\ 24 \end{Bmatrix}.$$

The following comments are in order of the matrix multiplication, wherein $\mathbf{A}$ denotes an $m \times n$ matrix and $\mathbf{B}$ denotes a $p \times q$ matrix:

(1) The product $\mathbf{AB}$ is defined only if the number of columns n in $\mathbf{A}$ is equal to the number of rows p in $\mathbf{B}$. Similarly, the product $\mathbf{BA}$ is defined only if $q = m$.

(2) If $\mathbf{AB}$ is defined, $\mathbf{BA}$ may or may not be defined. If both $\mathbf{AB}$ and $\mathbf{BA}$ are defined, it is not necessary that they be of the same size.

(3) The products $\mathbf{AB}$ and $\mathbf{BA}$ are of the same size if and only if both $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size.

(4) Generally, the products $\mathbf{AB}$ and $\mathbf{BA}$ are not equal, $\mathbf{AB} \neq \mathbf{BA}$, even if they are of equal size; that is, the matrix multiplication is not commutative.

(5) For any real square matrix $\mathbf{A}$, $\mathbf{A}$ is said to be *normal* if $\mathbf{AA}^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\mathbf{A}$; $\mathbf{A}$ is said to be *orthogonal* if $\mathbf{AA}^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}$.

(6) If $\mathbf{A}$ is a square matrix, the powers of $\mathbf{A}$ are defined as $\mathbf{A}^2 = \mathbf{AA}$, $\mathbf{A}^3 = \mathbf{AA}^2 = \mathbf{A}^2\mathbf{A}$, and so on.
(7) Matrix multiplication is associative: $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.
(8) The product of any square matrix with the identity matrix is the original matrix itself.
(9) The transpose of the product is $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ (note the order).

The next example verifies the previous Property 9.

Example 2.5.3:

Verify Property 9 using the matrices $[A]$ and $[B]$ in Eq. (2.5.3). The product of matrix $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{AB} = \begin{bmatrix} 5 & -2 & 1 \\ 8 & 7 & 6 \\ 2 & 4 & 3 \\ -1 & 9 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 & 4 \\ -6 & 3 & 5 & 7 \\ 9 & 6 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 36 & -5 & -2 & 7 \\ 36 & 49 & 39 & 87 \\ 9 & 28 & 18 & 39 \\ -57 & 28 & 43 & 59 \end{bmatrix}$$

and

$$(\mathbf{AB})^{\mathrm{T}} = \begin{bmatrix} 36 & 36 & 9 & -57 \\ -5 & 49 & 28 & 28 \\ -2 & 39 & 18 & 43 \\ 7 & 87 & 39 & 59 \end{bmatrix}.$$

Now compute the product:

$$\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 3 & -6 & 9 \\ -1 & 3 & 6 \\ 2 & 5 & -2 \\ 4 & 7 & 1 \end{bmatrix} \begin{bmatrix} 5 & 8 & 2 & -1 \\ -2 & 7 & 4 & 9 \\ 1 & 6 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 36 & 36 & 9 & -57 \\ -5 & 49 & 28 & 28 \\ -2 & 39 & 18 & 43 \\ 7 & 87 & 39 & 59 \end{bmatrix}.$$

Thus, $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ is verified.

2.5.5 Inverse and determinant of a matrix

If $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{B}$ is any $n \times n$ matrix such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, then $\mathbf{B}$ is called the *inverse* of $\mathbf{A}$. If it exists, the inverse of a matrix is unique (a consequence of the associative property). If both $\mathbf{B}$ and $\mathbf{C}$ are inverses for $\mathbf{A}$, then by definition

$$\mathbf{AB} = \mathbf{BA} = \mathbf{AC} = \mathbf{CA} = \mathbf{I}.$$

Because matrix multiplication is associative, we have

$$\begin{aligned} \mathbf{BAC} &= (\mathbf{BA})\mathbf{C} = \mathbf{IC} = \mathbf{C} \\ &= \mathbf{B}(\mathbf{AC}) = \mathbf{BI} = \mathbf{B}. \end{aligned}$$

This relationship shows that $\mathbf{B} = \mathbf{C}$, and the inverse is unique. The inverse of $\mathbf{A}$ is denoted by $\mathbf{A}^{-1}$. A matrix is said to be *singular* if it does not have an inverse.

If $\mathbf{A}$ is nonsingular, then the transpose of the inverse is equal to the inverse of the transpose: $(\mathbf{A}^{-1})^{\mathrm{T}} = (\mathbf{A}^{\mathrm{T}})^{-1}$. The actual computation of the inverse of a matrix requires evaluation of its determinant, which is discussed next.

Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ matrix. We wish to associate a scalar with $\mathbf{A}$ that in some sense measures the "size" of $\mathbf{A}$ and indicates whether or not $\mathbf{A}$ is nonsingular. The *determinant* of the matrix $\mathbf{A} = [a_{ij}]$ is defined to be the scalar $\det\mathbf{A} = |A|$ computed according to the rule

$$\det\mathbf{A} = |a_{ij}| = \sum_{i=1}^{n}(-1)^{i+1} a_{i1}|A_{i1}|,$$

where $|A_{ij}|$ is the determinant of the $(n-1) \times (n-1)$ matrix that remains on deleting out the ith row and the first column of $\mathbf{A}$. For convenience, we define the determinant of a zeroth-order matrix to be unity. For 1×1 matrices, the determinant is defined according to $|a_{11}| = a_{11}$. For a 2×2 matrix $\mathbf{A}$, the determinant is defined by

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

In the definition here, special attention is given to the first column of the matrix $\mathbf{A}$. We call it the *expansion* of $|A|$ according to the first column of $\mathbf{A}$. One can expand $|A|$ according to any column or row:

$$|A| = \sum_{i=1}^{n}(-1)^{i+j} a_{ij}|A_{ij}|, \tag{2.5.7}$$

where $|A_{i1}|$ is the determinant of the matrix obtained by deleting the ith row and jth column of matrix $\mathbf{A}$.

A numerical example of the calculation of determinant is presented next.

Example 2.5.4:

Compute the determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 5 & -1 \\ 1 & 4 & 3 \\ 2 & -3 & 5 \end{bmatrix}.$$

Solution: Using the definition of Eq. (2.5.7) and expanding by the first column, we have

$$\begin{aligned}
|A| &= \sum_{i=1}^{3}(-1)^{i+1} a_{i1}|A_{i1}| \\
&= (-1)^2 a_{11}\begin{vmatrix} 4 & 3 \\ -3 & 5 \end{vmatrix} + (-1)^3 a_{21}\begin{vmatrix} 5 & -1 \\ -3 & 5 \end{vmatrix} + (-1)^4 a_{31}\begin{vmatrix} 5 & -1 \\ 4 & 3 \end{vmatrix} \\
&= 2\big[(4)(5) - (3)(-3)\big] + (-1)\big[(5)(5) - (-1)(-3)\big] + 2\big[(5)(3) - (-1)(4)\big] \\
&= 2(20+9) - (25-3) + 2(15+4) = 74.
\end{aligned}$$

The cross product of two vectors **A** and **B** can be expressed as the value of the determinant,

$$\mathbf{A}\times\mathbf{B} \equiv \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \hat{A}_1 & \hat{A}_2 & \hat{A}_3 \\ \hat{B}_1 & \hat{B}_2 & \hat{B}_3 \end{vmatrix}$$

$$= \left(A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3\right) \times \left(B_1\hat{\mathbf{e}}_1 + B_2\hat{\mathbf{e}}_2 + B_3\hat{\mathbf{e}}_3\right)$$

$$= (A_2B_3 - A_3B_2)\hat{\mathbf{e}}_1 + (A_3B_1 - A_1B_3)\hat{\mathbf{e}}_2 + (A_1B_2 - A_2B_1)\hat{\mathbf{e}}_3, \quad (2.5.8)$$

and the scalar triple product can be expressed as the value of a determinant,

$$\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) \equiv \begin{vmatrix} \hat{C}_1 & \hat{C}_2 & \hat{C}_3 \\ \hat{A}_1 & \hat{A}_2 & \hat{A}_3 \\ \hat{B}_1 & \hat{B}_2 & \hat{B}_3 \end{vmatrix}$$

$$=(A_2B_3 - A_3B_2)C_1 + (A_3B_1 - A_1B_3)C_2 + (A_1B_2 - A_2B_1)C_3. \quad (2.5.9)$$

In general, the determinant of a 3×3 matrix **A** can be expressed in the form

$$|\mathbf{A}| = e_{ijk}a_{1i}a_{2j}a_{3k}, \quad (2.5.10)$$

where a_{ij} is the element occupying the ith row and the jth column of the matrix. The verification of these results is left as an exercise for the reader.

We note the following properties of determinants:

(1) $\det(\mathbf{AB}) = \det\mathbf{A}\cdot\det\mathbf{B}$.
(2) $\det\mathbf{A}^{\mathrm{T}} = \det\mathbf{A}$.
(3) $\det(\alpha\,\mathbf{A}) = \alpha^n\det\mathbf{A}$, where α is a scalar and n is the order of **A**.
(4) If $\mathbf{A}'$ is a matrix obtained from **A** by multiplying a row (or column) of **A** by a scalar α, then $\det\mathbf{A}' = \alpha\det\mathbf{A}$.
(5) If $\mathbf{A}'$ is the matrix obtained from **A** by interchanging any two rows (or columns) of **A**, then $\det\mathbf{A}' = -\det\mathbf{A}$.
(6) If **A** has two rows (or columns), one of which is a scalar multiple of another (i.e., linearly dependent), $\det\mathbf{A} = 0$.
(7) If $\mathbf{A}'$ is the matrix obtained from **A** by adding a multiple of one row (or column) to another, then $\det\mathbf{A}' = \det\mathbf{A}$.

We define (in fact, the definition given earlier is an indirect definition) singular matrices in terms of their determinants. A matrix is said to be *singular* if and only if its determinants are zero. By Property 6, the determinant of a matrix is zero if it has linearly dependent rows (or columns).

For an $n\times n$ matrix **A**, the determinant of the $(n-1)\times(n-1)$ submatrix of **A** obtained by deleting row i and column j of **A** is called the *minor* of a_{ij} and is denoted by $M_{ij}(\mathbf{A})$. The quantity $\mathrm{cof}_{ij}(\mathbf{A}) \equiv (-1)^{i+j}M_{ij}(\mathbf{A})$ is called the *cofactor* of a_{ij}. The determinant of **A** can be cast in terms of the minor and cofactor of a_{ij} as

$$\det\mathbf{A} = \sum_{i=1}^{n} a_{ij}\,\mathrm{cof}_{ij}(\mathbf{A}) \quad (2.5.11)$$

for any value of j. The *adjunct* (also called *adjoint*) of a matrix $\mathbf{A}$ is the transpose of the matrix obtained from $\mathbf{A}$ by replacing each element by its cofactor. The adjunct of $\mathbf{A}$ is denoted by Adj$\mathbf{A}$.

Now we have the essential tools to compute the inverse of a matrix. If $\mathbf{A}$ is non-singular (that is, det $\mathbf{A} \neq 0$), the inverse $\mathbf{A}^{-1}$ of $\mathbf{A}$ can be computed according to

$$\mathbf{A}^{-1} = \frac{1}{\mathrm{det}\mathbf{A}}\mathrm{Adj}\mathbf{A}. \tag{2.5.12}$$

The next example illustrates the computation of an inverse of a matrix.

Example 2.5.5:

Determine the inverse of the matrix of Example 2.5.4.

Solution: For example, we have

$$M_{11}(\mathbf{A}) = \begin{vmatrix} 4 & 3 \\ -3 & 5 \end{vmatrix}, \qquad M_{12}(\mathbf{A}) = \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix}, \qquad M_{13}(\mathbf{A}) = \begin{vmatrix} 1 & 4 \\ 2 & -3 \end{vmatrix},$$

$$\mathrm{cof}_{11}(\mathbf{A}) = (-1)^2 M_{11}(\mathbf{A}) = 4 \times 5 - (-3)3 = 29,$$
$$\mathrm{cof}_{12}(\mathbf{A}) = (-1)^3 M_{12}(\mathbf{A}) = -(1 \times 5 - 3 \times 2) = 1,$$
$$\mathrm{cof}_{13}(\mathbf{A}) = (-1)^4 M_{13}(\mathbf{A}) = 1 \times (-3) - 2 \times 4 = -11.$$

The Adj($\mathbf{A}$) is given by

$$\mathrm{Adj}(\mathbf{A}) = \begin{bmatrix} \mathrm{cof}_{11}(\mathbf{A}) & \mathrm{cof}_{12}(\mathbf{A}) & \mathrm{cof}_{13}(\mathbf{A}) \\ \mathrm{cof}_{21}(\mathbf{A}) & \mathrm{cof}_{22}(\mathbf{A}) & \mathrm{cof}_{23}(\mathbf{A}) \\ \mathrm{cof}_{31}(\mathbf{A}) & \mathrm{cof}_{32}(\mathbf{A}) & \mathrm{cof}_{33}(\mathbf{A}) \end{bmatrix}^{\mathrm{T}}$$
$$= \begin{bmatrix} 29 & -22 & 19 \\ 1 & 12 & 19 \\ -11 & 16 & 3 \end{bmatrix}.$$

The determinant is given by (expanding by the first row)

$$|A| = 2(29) + 5(1) + (-1)(-11) = 74.$$

The inverse of $\mathbf{A}$ can be now computed using Eq. (2.5.12),

$$\mathbf{A}^{-1} = \frac{1}{74}\begin{bmatrix} 29 & -22 & 19 \\ 1 & 12 & -7 \\ -11 & 16 & 3 \end{bmatrix}.$$

It can be easily verified that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

2.6 Vector calculus

2.6.1 The del operator

The basic notions of vector and scalar calculus, especially with regard to physical applications, are closely related to the rate of change of a scalar field (such

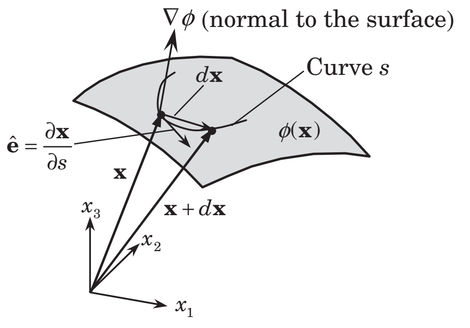


Figure 2.6.1 Directional derivative of a scalar function.

as the temperature of a continuous body) with distance. Let us denote a scalar field by $\phi = \phi(\mathbf{x})$, $\mathbf{x}$ being the position vector in a rectangular Cartesian system (x_1, x_2, x_3). Let us now denote a differential element with $d\mathbf{x}$ and its magnitude by $ds \equiv |d\mathbf{x}|$. Then $\hat{\mathbf{e}} = d\mathbf{x}/ds$ is a unit vector in the direction of $d\mathbf{x}$, and we may express it as (by the use of the chain rule of differentiation)

$$\left(\frac{d\phi}{ds}\right)_{\hat{\mathbf{e}}} = \frac{d\mathbf{x}}{ds} \cdot \frac{\partial \phi}{\partial \mathbf{x}} = \hat{\mathbf{e}} \cdot \left(\hat{\mathbf{e}}_1 \frac{\partial \phi}{\partial x_1} + \hat{\mathbf{e}}_2 \frac{\partial \phi}{\partial x_2} + \hat{\mathbf{e}}_3 \frac{\partial \phi}{\partial x_3}\right). \tag{2.6.1}$$

The derivative $(d\phi/ds)_{\hat{\mathbf{e}}}$ is called the *directional derivative* of ϕ. We see that it is the rate of change of ϕ with respect to distance, and that it depends on the direction $\hat{\mathbf{e}}$ in which the distance is taken, as shown in Figure 2.6.1. The vector $\partial\phi/\partial\mathbf{x}$ is called the *gradient vector* and is denoted by grad ϕ:

$$\text{grad}\,\phi \equiv \hat{\mathbf{e}}_1 \frac{\partial \phi}{\partial x_1} + \hat{\mathbf{e}}_2 \frac{\partial \phi}{\partial x_2} + \hat{\mathbf{e}}_3 \frac{\partial \phi}{\partial x_3} = \hat{\mathbf{e}}_x \frac{\partial \phi}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial \phi}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial \phi}{\partial z}. \tag{2.6.2}$$

We interpret grad ϕ as some operator operating on function ϕ, that is, grad $\phi \equiv \nabla\phi$. This operator is denoted by

$$\nabla \equiv \hat{\mathbf{e}}_1 \frac{\partial}{\partial x_1} + \hat{\mathbf{e}}_2 \frac{\partial}{\partial x_2} + \hat{\mathbf{e}}_3 \frac{\partial}{\partial x_3} = \hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \tag{2.6.3}$$

and is called the *del operator*. The del operator is a vector differential operator. It is important to note that although the del operator has some of the properties of a vector, it does not have them all because it is an operator. For instance, $\nabla \cdot \mathbf{A}$ is a scalar, called the *divergence of* $\mathbf{A}$ $(\partial A_i/\partial x_i)$, whereas $\mathbf{A} \cdot \nabla$ is a scalar differential operator $[A_i(\partial/\partial x_i)]$. Thus, the del operator does not commute in this sense.

When the scalar function $\phi(\mathbf{x})$ is set equal to a constant, $\phi(\mathbf{x}) = c$, a family of surfaces is generated for each value of c. If the direction in which the directional derivative is taken lies within a surface, then $d\phi/ds$ is zero because ϕ is a constant on a surface. In this case, the unit vector $\hat{\mathbf{e}}$ is tangent to a level surface. Then it follows from Eq. (2.6.1) that if $d\phi/ds$ is zero, grad ϕ must be perpendicular to $\hat{\mathbf{e}}$, and hence perpendicular to the surface. Thus, if a surface is given by $\phi(\mathbf{x}) = c$, the unit normal to the surface is determined by

$$\hat{\mathbf{n}} = \pm \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|}. \tag{2.6.4}$$

The sign appears because the direction of $\hat{\mathbf{n}}$ may point in either direction away from the surface. If the surface is closed, the usual convention is to take $\hat{\mathbf{n}}$ pointing outward.

An important note is in order concerning the del operator. Two types of gradients are used in continuum mechanics: forward and backward gradients. The forward gradient is the usual gradient, and the backward gradient is the transpose of the forward gradient operator. To see the difference between the two types of gradients, consider a vector function $\mathbf{A} = A_i(\mathbf{x})\hat{\mathbf{e}}_i$. The forward and backward gradients of $\mathbf{A}$ are (both are second-order tensors, as will be discussed in Section 2.7)

$$\overrightarrow{\nabla}\mathbf{A} = \nabla\mathbf{A} = \hat{\mathbf{e}}_j \frac{\partial}{\partial x_j}\left(A_i\hat{\mathbf{e}}_i\right) = \frac{\partial A_i}{\partial x_j}\hat{\mathbf{e}}_j\hat{\mathbf{e}}_i = A_{i,j}\hat{\mathbf{e}}_j\hat{\mathbf{e}}_i, \tag{2.6.5}$$

$$\overleftarrow{\nabla}\mathbf{A} = \left(\nabla\mathbf{A}\right)^{\mathrm{T}} = \frac{\partial A_i}{\partial x_j}\left(\hat{\mathbf{e}}_j\hat{\mathbf{e}}_i\right)^{\mathrm{T}} = A_{i,j}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j, \tag{2.6.6}$$

where $A_{i,j} = \partial A_i/\partial x_j$. The backward gradient is often used (without explanation) in defining the deformation gradient tensor and velocity gradient tensor, which will be introduced in Chapter 3.

2.6.2 Divergence and curl of a vector

The dot product of a del operator with a vector is called the *divergence* of a vector and denoted by

$$\nabla \cdot \mathbf{A} \equiv \mathrm{div}\mathbf{A}. \tag{2.6.7}$$

If we take the divergence of the gradient vector, we have

$$\mathrm{div}(\mathrm{grad}\ \phi) \equiv \nabla \cdot \nabla\phi = (\nabla \cdot \nabla)\phi = \nabla^2\phi. \tag{2.6.8}$$

The notation $\nabla^2 = \nabla \cdot \nabla$ is called the *Laplacian operator*. In Cartesian systems, this reduces to the simple form

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{\partial^2\phi}{\partial x_i \partial x_i}. \tag{2.6.9}$$

The Laplacian of a scalar appears frequently in partial differential equations governing physical phenomena.

The *curl* of a vector is defined as the del operator operating on a vector by means of the cross product:

$$\mathrm{curl}\ \mathbf{A} = \nabla \times \mathbf{A} = e_{ijk}\hat{\mathbf{e}}_i \frac{\partial A_k}{\partial x_j}. \tag{2.6.10}$$

Let $\hat{\mathbf{n}}$ denote the unit vector, taken positive outward, normal to the surface Γ of a continuous medium occupying the region Ω, as shown in Figure 2.6.2. In a Cartesian coordinate system, the unit normal vector can be expressed in terms of its components as

$$\hat{\mathbf{n}} = n_1\hat{\mathbf{e}}_1 + n_2\hat{\mathbf{e}}_2 + n_3\hat{\mathbf{e}}_3 = n_x\hat{\mathbf{e}}_x + n_y\hat{\mathbf{e}}_y + n_z\hat{\mathbf{e}}_z. \tag{2.6.11}$$

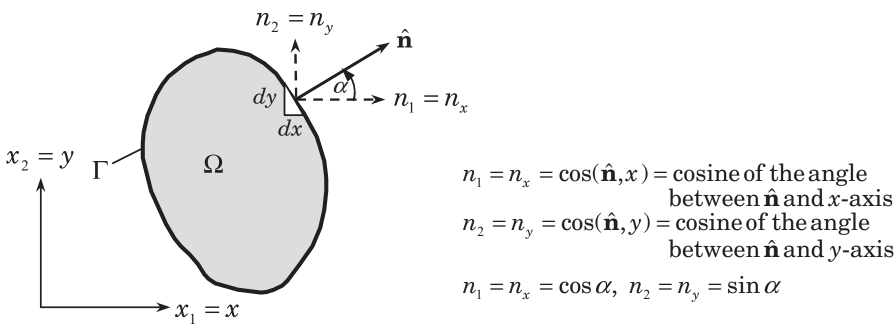


Figure 2.6.2 A unit vector normal to a surface.

The components $(n_1, n_2, n_3) = (n_x, n_y, n_z)$ are called *direction cosines* because of the fact that n_i equals the cosine of the angle between $\hat{\mathbf{n}}$ and the x_i-axis.

The quantity $\hat{\mathbf{n}} \cdot \text{grad}\,\phi$ of a function ϕ is called the *normal derivative* of ϕ and is denoted by

$$\frac{\partial \phi}{\partial n} \equiv \hat{\mathbf{n}} \cdot \text{grad}\,\phi = \hat{\mathbf{n}} \cdot \nabla\phi. \tag{2.6.12}$$

In a rectangular Cartesian coordinate system (x, y, z), $\partial\phi/\partial n$ takes the form

$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial x} n_x + \frac{\partial \phi}{\partial y} n_y + \frac{\partial \phi}{\partial z} n_z. \tag{2.6.13}$$

Next, we present several examples to illustrate the use of index notation to prove certain identities involving vector calculus.

Example 2.6.1:

Establish the following identities using the index notation:
(1) $\nabla(r) = \frac{\mathbf{r}}{r}$, $r = |\mathbf{r}|$. **(2)** $\nabla(r^p) = p\,r^{p-2}\,\mathbf{r}$, p an integer.
(3) $\nabla \times (\nabla F) = \mathbf{0}$.

Solution:

(1) Note that we use the notation $\mathbf{r} \equiv \mathbf{x}$ for a position vector, and $r = |\mathbf{r}| = \sqrt{\mathbf{r} \cdot \mathbf{r}} = \sqrt{x_j x_j}$. Consider

$$\begin{aligned}\nabla(r) &= \hat{\mathbf{e}}_i \frac{\partial r}{\partial x_i} = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} \left(x_j x_j\right)^{\frac{1}{2}} \\ &= \hat{\mathbf{e}}_i \frac{1}{2} \left(x_j x_j\right)^{\frac{1}{2}-1} 2x_i = \hat{\mathbf{e}}_i x_i \left(x_j x_j\right)^{-\frac{1}{2}} = \frac{\mathbf{r}}{r} = \frac{\mathbf{x}}{r},\end{aligned}$$

from which we note the identity

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}. \tag{2.6.14}$$

(2) In this case, we have

$$\nabla(r^p) = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}(r^p) = p\, r^{p-1}\, \hat{\mathbf{e}}_i \frac{\partial r}{\partial x_i} = p\, r^{p-2} x_i \hat{\mathbf{e}}_i = p\, r^{p-2}\, \mathbf{r}.$$

(3) Consider the expression

$$\nabla \times (\nabla F) = \left(\hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}\right) \times \left(\hat{\mathbf{e}}_j \frac{\partial F}{\partial x_j}\right) = e_{ijk} \hat{\mathbf{e}}_k \frac{\partial^2 F}{\partial x_i \partial x_j}.$$

Note that $\frac{\partial^2 F}{\partial x_i \partial x_j}$ is symmetric in i and j. Consider the kth component of the vector,

$$\begin{aligned} e_{ijk} \frac{\partial^2 F}{\partial x_i \partial x_j} &= -e_{jik} \frac{\partial^2 F}{\partial x_i \partial x_j} \quad \text{(interchanged } i \text{ and } j\text{)} \\ &= -e_{ijk} \frac{\partial^2 F}{\partial x_j \partial x_i} \quad \text{(renamed } i \text{ as } j \text{ and } j \text{ as } i\text{)} \\ &= -e_{ijk} \frac{\partial^2 F}{\partial x_i \partial x_j} \quad \text{(used the symmetry of } \tfrac{\partial^2 F}{\partial x_i \partial x_j}\text{).} \end{aligned}$$

Thus, the expression is equal to its own negative. Obviously, the only parameter that is equal to its own negative is zero. Hence, we have $\nabla \times (\nabla F) = \mathbf{0}$. It also follows that $e_{ijk} F_{ij} = 0$ whenever $F_{ij} = F_{ji}$, that is, F_{ij} is symmetric.

The examples presented previously illustrate the power of index notation in establishing vector identities. The difficult step in these proofs is recognizing vector operations from index notations. A list of vector operations in both vector notation and in Cartesian component form is presented in Table 2.6.1.

2.6.3 Cylindrical and spherical coordinate systems

Two commonly used orthogonal curvilinear coordinate systems are the *cylindrical* coordinate system and the *spherical* coordinate system. Table 2.6.2 contains a summary of the basic information for the two coordinate systems. The matrix of directional cosines between the orthogonal rectangular Cartesian system (x, y, z) and the orthogonal curvilinear systems (r, θ, z) and (R, ϕ, θ), as shown in Figure 2.6.3, are given next.

Cylindrical coordinate system

$$\begin{Bmatrix} \hat{\mathbf{e}}_r \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_z \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z \end{Bmatrix}, \tag{2.6.15}$$

$$\begin{Bmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_r \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_z \end{Bmatrix}, \tag{2.6.16}$$

Table 2.6.1. Vector expressions and their Cartesian component forms. **A**, **B**, and **C** are vector functions and U is a scalar function; $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ are the Cartesian unit vectors.

Vector Form	Component form
$\mathbf{A}$	$A_i \hat{\mathbf{e}}_i$
$\mathbf{A}\cdot\mathbf{B}$	$A_i B_i$
$\mathbf{A}\times\mathbf{B}$	$e_{ijk} A_i B_j \hat{\mathbf{e}}_k$
$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$	$e_{ijk} A_i B_j C_k$
$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$	$e_{ijk} e_{klm} A_j B_l C_m \hat{\mathbf{e}}_i$
∇U	$\frac{\partial U}{\partial x_i}\hat{\mathbf{e}}_i$
$\nabla \mathbf{A}$	$\frac{\partial A_j}{\partial x_i}\hat{\mathbf{e}}_i \hat{\mathbf{e}}_j$
$\nabla\cdot\mathbf{A}$	$\frac{\partial A_i}{\partial x_i}$
$\nabla\times\mathbf{A}$	$e_{ijk}\frac{\partial A_j}{\partial x_i}\hat{\mathbf{e}}_k$
$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$	$e_{ijk}\frac{\partial}{\partial x_i}(A_j B_k)$
$\nabla\cdot(U\mathbf{A}) = U\nabla\cdot\mathbf{A} + \nabla U\cdot\mathbf{A}$	$\frac{\partial}{\partial x_i}(UA_i)$
$\nabla\times(U\mathbf{A}) = \nabla U\times\mathbf{A} + U\nabla\times\mathbf{A}$	$e_{ijk}\frac{\partial}{\partial x_j}(UA_k)\hat{\mathbf{e}}_i$
$\nabla(U\mathbf{A}) = \nabla U\mathbf{A} + U\nabla\mathbf{A}$	$\hat{\mathbf{e}}_j\frac{\partial}{\partial x_j}(UA_k\hat{\mathbf{e}}_k)$
$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + \mathbf{B}\cdot\nabla\mathbf{A} - \mathbf{A}\cdot\nabla\mathbf{B}$	$e_{ijk} e_{mkl}\frac{\partial}{\partial x_m}(A_i B_j)\hat{\mathbf{e}}_l$
$(\nabla\times\mathbf{A})\times\mathbf{B} = \mathbf{B}\cdot[\nabla\mathbf{A} - (\nabla\mathbf{A})^{\mathrm{T}}]$	$e_{ijk} e_{klm} B_l \dfrac{\partial A_j}{\partial x_i}\hat{\mathbf{e}}_m$
$\nabla\cdot(\nabla U) = \nabla^2 U$	$\frac{\partial^2 U}{\partial x_i \partial x_i}$
$\nabla\cdot(\nabla\mathbf{A}) = \nabla^2\mathbf{A}$	$\frac{\partial^2 A_j}{\partial x_i \partial x_i}\hat{\mathbf{e}}_j$
$\nabla\times\nabla\times\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - (\nabla\cdot\nabla)\mathbf{A}$	$e_{mil} e_{jkl}\frac{\partial^2 A_k}{\partial x_i \partial x_j}\hat{\mathbf{e}}_m$
$(\mathbf{A}\cdot\nabla)\mathbf{B}$	$A_j\frac{\partial B_i}{\partial x_j}\hat{\mathbf{e}}_i$
$\mathbf{A}(\nabla\cdot\mathbf{B})$	$A_i\hat{\mathbf{e}}_i\frac{\partial B_j}{\partial x_j}$

Spherical coordinate system

$$\begin{Bmatrix} \hat{\mathbf{e}}_R \\ \hat{\mathbf{e}}_\phi \\ \hat{\mathbf{e}}_\theta \end{Bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z \end{Bmatrix}, \tag{2.6.17}$$

$$\begin{Bmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z \end{Bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_R \\ \hat{\mathbf{e}}_\phi \\ \hat{\mathbf{e}}_\theta \end{Bmatrix}. \tag{2.6.18}$$

Table 2.6.2. Base vectors and del and Laplace operators in cylindrical and spherical coordinate systems.

Cylindrical coordinate system (r, θ, z)

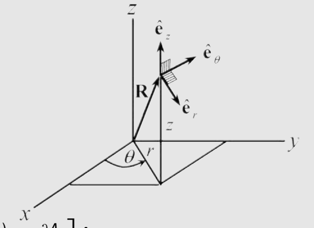
$x = r\cos\theta,\ y = r\sin\theta,\ z = z,\ \mathbf{R} = r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z$

$\mathbf{A} = A_r\hat{\mathbf{e}}_r + A_\theta\hat{\mathbf{e}}_\theta + A_z\hat{\mathbf{e}}_z$ (typical vector)

$\hat{\mathbf{e}}_r = \cos\theta\,\hat{\mathbf{e}}_x + \sin\theta\,\hat{\mathbf{e}}_y,\ \hat{\mathbf{e}}_\theta = -\sin\theta\,\hat{\mathbf{e}}_x + \cos\theta\,\hat{\mathbf{e}}_y,\ \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_z$

$\frac{\partial\hat{\mathbf{e}}_r}{\partial\theta} = -\sin\theta\,\hat{\mathbf{e}}_x + \cos\theta\,\hat{\mathbf{e}}_y = \hat{\mathbf{e}}_\theta,\ \frac{\partial\hat{\mathbf{e}}_\theta}{\partial\theta} = -\cos\theta\,\hat{\mathbf{e}}_x - \sin\theta\,\hat{\mathbf{e}}_y = -\hat{\mathbf{e}}_r$

All other derivatives of the base vectors are zero.



$$\nabla = \hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \frac{1}{r}\hat{\mathbf{e}}_\theta\frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_z\frac{\partial}{\partial z},$$

$$\nabla^2 = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial^2}{\partial\theta^2} + r\frac{\partial^2}{\partial z^2}\right]$$

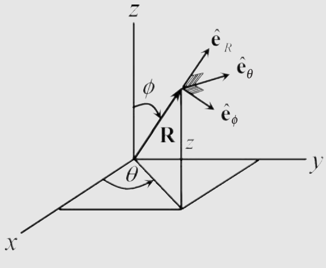
$$\nabla\cdot\mathbf{A} = \frac{1}{r}\left[\frac{\partial(rA_r)}{\partial r} + \frac{\partial A_\theta}{\partial\theta} + r\frac{\partial A_z}{\partial z}\right]$$

$$\nabla\times\mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{\mathbf{e}}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\mathbf{e}}_\theta + \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]\hat{\mathbf{e}}_z$$

Spherical coordinate system (R, ϕ, θ)

$x = R\sin\phi\cos\theta,\ \ y = R\sin\phi\sin\theta,\ \ z = R\cos\phi,\ \mathbf{R} = R\hat{\mathbf{e}}_R$

$\mathbf{A} = A_R\hat{\mathbf{e}}_R + A_\phi\hat{\mathbf{e}}_\phi + A_\theta\hat{\mathbf{e}}_\theta$ (typical vector)



$\hat{\mathbf{e}}_R = \sin\phi\cos\theta\,\hat{\mathbf{e}}_x + \sin\phi\sin\theta\,\hat{\mathbf{e}}_y + \cos\phi\,\hat{\mathbf{e}}_z$

$\hat{\mathbf{e}}_\phi = \cos\phi\cos\theta\,\hat{\mathbf{e}}_x + \cos\phi\sin\theta\,\hat{\mathbf{e}}_y - \sin\phi\,\hat{\mathbf{e}}_z$

$\hat{\mathbf{e}}_\theta = -\sin\theta\,\hat{\mathbf{e}}_x + \cos\theta\,\hat{\mathbf{e}}_y$

$\hat{\mathbf{e}}_x = \sin\phi\cos\theta\,\hat{\mathbf{e}}_R + \cos\phi\cos\theta\,\hat{\mathbf{e}}_\phi - \sin\theta\,\hat{\mathbf{e}}_\theta$

$\hat{\mathbf{e}}_y = \sin\phi\sin\theta\,\hat{\mathbf{e}}_R + \cos\phi\sin\theta\,\hat{\mathbf{e}}_\phi + \cos\theta\,\hat{\mathbf{e}}_\theta$

$\hat{\mathbf{e}}_z = \cos\phi\,\hat{\mathbf{e}}_R - \sin\phi\,\hat{\mathbf{e}}_\phi$

$\frac{\partial\hat{\mathbf{e}}_R}{\partial\phi} = \hat{\mathbf{e}}_\phi,\ \frac{\partial\hat{\mathbf{e}}_R}{\partial\theta} = \sin\phi\,\hat{\mathbf{e}}_\theta,\ \frac{\partial\hat{\mathbf{e}}_\phi}{\partial\phi} = -\hat{\mathbf{e}}_R,\ \frac{\partial\hat{\mathbf{e}}_\phi}{\partial\theta} = \cos\phi\,\hat{\mathbf{e}}_\theta,\ \frac{\partial\hat{\mathbf{e}}_\theta}{\partial\theta} = -\sin\phi\,\hat{\mathbf{e}}_R - \cos\phi\,\hat{\mathbf{e}}_\phi$

All other derivatives of the base vectors are zero.

$$\nabla = \hat{\mathbf{e}}_R\frac{\partial}{\partial R} + \frac{1}{R}\hat{\mathbf{e}}_\phi\frac{\partial}{\partial\phi} + \frac{1}{R\sin\phi}\hat{\mathbf{e}}_\theta\frac{\partial}{\partial\theta}$$

$$\nabla^2 = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right) + \frac{1}{R^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2}$$

$$\nabla\cdot\mathbf{A} = 2\frac{A_R}{R} + \frac{\partial A_R}{\partial R} + \frac{1}{R\sin\phi}\frac{\partial(A_\phi\sin\phi)}{\partial\phi} + \frac{1}{R\sin\phi}\frac{\partial A_\theta}{\partial\theta}$$

$$\nabla\times\mathbf{A} = \frac{1}{R\sin\phi}\left[\frac{\partial(\sin\phi A_\theta)}{\partial\phi} - \frac{\partial A_\phi}{\partial\theta}\right]\hat{\mathbf{e}}_R + \left[\frac{1}{R\sin\phi}\frac{\partial A_R}{\partial\theta} - \frac{1}{R}\frac{\partial(RA_\theta)}{\partial R}\right]\hat{\mathbf{e}}_\phi + \frac{1}{R}\left[\frac{\partial(RA_\phi)}{\partial R} - \frac{\partial A_R}{\partial\phi}\right]\hat{\mathbf{e}}_\theta$$

2.6.4 Gradient, divergence, and curl theorems

Integral identities involving the gradient of a vector, divergence of a vector, and curl of a vector can be established from integral relations between volume integrals and surface integrals. These identities will be useful in later chapters when we derive the equations of a continuous medium.

Let Ω denote a region in $\Re^3$ bounded by the closed surface Γ. Let ds be a differential element of surface and $\hat{\mathbf{n}}$ the unit outward normal, and let $d\mathbf{x}$ be a

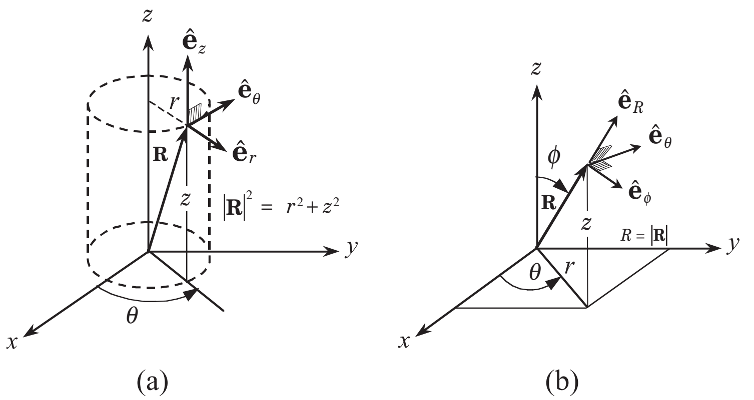


Figure 2.6.3 (a) Cylindrical coordinate system. (b) Spherical coordinate system.

differential volume element in Ω. The following relations, known from advanced calculus, hold:

$$\int_\Omega \nabla\phi \, d\mathbf{x} = \oint_\Gamma \hat{\mathbf{n}}\,\phi \, ds \quad \text{(Gradient theorem)}, \tag{2.6.19}$$

$$\int_\Omega \nabla\cdot\mathbf{A} \, d\mathbf{x} = \oint_\Gamma \hat{\mathbf{n}}\cdot\mathbf{A} \, ds \quad \text{(Divergence theorem)}, \tag{2.6.20}$$

$$\int_\Omega \nabla\times\mathbf{A} \, d\mathbf{x} = \oint_\Gamma \hat{\mathbf{n}}\times\mathbf{A} \, ds \quad \text{(Curl theorem)}. \tag{2.6.21}$$

The combination $\mathbf{A}\cdot\hat{\mathbf{n}}\,ds$ is called the *outflow* of $\mathbf{A}$ through the differential surface ds. The integral is called the total or net outflow through the surrounding surface Δs. This is easiest to see if one imagines that $\mathbf{A}$ is a velocity vector and the outflow is an amount of fluid flow. In the limit as the region shrinks to a point, the net outflow per unit volume is therefore associated with the divergence of the vector field.

2.7 Tensors

2.7.1 Dyads

Like physical vectors, *tensors* are more general objects that are endowed with a magnitude and multiple directions but satisfy the rules of "vector addition and scalar multiplication." In fact, physical vectors are often termed first-order tensors. For example, the stress vector $\mathbf{t}$, which is a measure of force per unit area, depends not only on the magnitude and direction of the force but also on the orientation of the plane on which the force acts, as shown in Figure 2.7.1:

$$\mathbf{t}(\hat{\mathbf{n}}) = \lim_{\Delta a\to 0} \frac{\Delta\mathbf{f}(\hat{\mathbf{n}})}{\Delta a}. \tag{2.7.1}$$

Thus, specification of the stress vector at a point requires two vectors, one perpendicular to the plane on which the force is acting and the other in the direction of the force. A detailed discussion of the stress vector will be presented in Chapter 4.

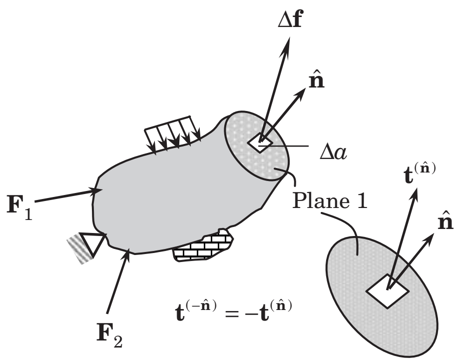


Figure 2.7.1 Definition of a stress vector acting on a plane with normal $\hat{\mathbf{n}}$.

Quantities that require two directions to be specified are known as *dyads*, or what we shall call a *second-order tensor*. Because of its utilization in physical applications, a dyad is defined as two vectors standing side by side and acting as a single unit. A linear combination of dyads is called a *dyadic*. Let $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ and $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_n$ be arbitrary vectors. Then we can represent a dyadic as

$$\boldsymbol{\Phi} = \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 + \cdots + \mathbf{A}_n\mathbf{B}_n. \tag{2.7.2}$$

The transpose of a dyadic is defined as the result obtained by the interchange of the two vectors in each of the dyads. For example, the transpose of the dyadic in Eq. (2.7.2) is

$$\boldsymbol{\Phi}^{\mathrm{T}} = \mathbf{B}_1\mathbf{A}_1 + \mathbf{B}_2\mathbf{A}_2 + \cdots + \mathbf{B}_n\mathbf{A}_n.$$

One of the properties of a dyadic is defined by the dot product with a vector, say $\mathbf{V}$:

$$\begin{aligned} \boldsymbol{\Phi}\cdot\mathbf{V} &= \mathbf{A}_1(\mathbf{B}_1\cdot\mathbf{V}) + \mathbf{A}_2(\mathbf{B}_2\cdot\mathbf{V}) + \cdots + \mathbf{A}_n(\mathbf{B}_n\cdot\mathbf{V}),\\ \mathbf{V}\cdot\boldsymbol{\Phi} &= (\mathbf{V}\cdot\mathbf{A}_1)\mathbf{B}_1 + (\mathbf{V}\cdot\mathbf{A}_2)\mathbf{B}_2 + \cdots + (\mathbf{V}\cdot\mathbf{A}_n)\mathbf{B}_n. \end{aligned} \tag{2.7.3}$$

The dot operation with a vector produces another vector. In the first case, the dyadic acts as a *prefactor* and in the second case as a *postfactor*. The two operations in general produce different vectors. The expressions in Eq. (2.7.3) can also be written in alternative form using the definition of the transpose of a dyad as

$$\mathbf{V}\cdot\boldsymbol{\Phi} = \boldsymbol{\Phi}^{\mathrm{T}}\cdot\mathbf{V}, \quad \boldsymbol{\Phi}\cdot\mathbf{V} = \mathbf{V}\cdot\boldsymbol{\Phi}^{\mathrm{T}}. \tag{2.7.4}$$

In general, one can show that the transpose of the product of tensors (of any order) follows the rule

$$(\boldsymbol{\Phi}\cdot\boldsymbol{\Psi})^{\mathrm{T}} = \boldsymbol{\Psi}^{\mathrm{T}}\cdot\boldsymbol{\Phi}^{\mathrm{T}}, \quad (\boldsymbol{\Phi}\cdot\boldsymbol{\Psi}\cdot\mathbf{V})^{\mathrm{T}} = \mathbf{V}^{\mathrm{T}}\cdot\boldsymbol{\Psi}^{\mathrm{T}}\cdot\boldsymbol{\Phi}^{\mathrm{T}}. \tag{2.7.5}$$

The dot product of a dyad with itself is a dyad, and is denoted by

$$\boldsymbol{\Phi}\cdot\boldsymbol{\Phi} = \boldsymbol{\Phi}^2.$$

In general, we have

$$\mathbf{\Phi}^n = \mathbf{\Phi}^{n-1} \cdot \mathbf{\Phi}. \tag{2.7.6}$$

2.7.2 Nonion form of a dyad

Let each of the vectors in the dyadic of Eq. (2.7.2) be represented in a given basis system. In a Cartesian system, we have

$$\mathbf{A}_i = A_{ij}\mathbf{e}_j, \quad \mathbf{B}_i = B_{ik}\mathbf{e}_k.$$

The summations on j and k are implied by the repeated indices.

We can display all of the components of a dyad $\mathbf{\Phi}$ by letting the k index run to the right and the j index run downward:

$$\begin{aligned}\mathbf{\Phi} &= \phi_{11}\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + \phi_{12}\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 + \phi_{13}\hat{\mathbf{e}}_1\hat{\mathbf{e}}_3 \\ &+ \phi_{21}\hat{\mathbf{e}}_2\hat{\mathbf{e}}_1 + \phi_{22}\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 + \phi_{23}\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3 \\ &+ \phi_{31}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_1 + \phi_{32}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_2 + \phi_{33}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_3.\end{aligned} \tag{2.7.7}$$

This form is called the *nonion* form of a dyad. Equation (2.7.7) illustrates that a dyad in three-dimensional space has nine independent components in general, each component associated with a certain dyad pair. The components are thus said to be ordered. When the ordering is understood, such as suggested by the nonion form of Eq. (2.7.7), the explicit writing of the dyads can be suppressed and the dyad is written as an array:

$$[\mathbf{\Phi}] = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix} \quad \text{and} \quad \mathbf{\Phi} = \begin{Bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{Bmatrix}^{\mathrm{T}} [\mathbf{\Phi}] \begin{Bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{Bmatrix}. \tag{2.7.8}$$

This representation is simpler than Eq. (2.7.7) but it is taken to mean the same.

The unit dyad is defined as

$$\mathbf{I} = \hat{\mathbf{e}}_i\hat{\mathbf{e}}_i. \tag{2.7.9}$$

It is clear that the second-order unit tensor is symmetric. With the help of the Kronecker delta δ_{ij}, the unit dyadic in an orthogonal Cartesian coordinate system can be written alternatively as

$$\mathbf{I} = \delta_{ij}\, \hat{\mathbf{e}}_i\hat{\mathbf{e}}_j, \quad \mathbf{I} = \begin{Bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{Bmatrix}^{\mathrm{T}} [I] \begin{Bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{Bmatrix}, \quad [\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.7.10}$$

The permutation symbol e_{ijk} can be viewed as the Cartesian components of a third-order tensor of a special kind.

The "double-dot product" between two dyads, $(\mathbf{AB})$ and $(\mathbf{CD})$, is defined as the scalar

$$(\mathbf{AB}) : (\mathbf{CD}) \equiv (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \cdot \mathbf{D}). \tag{2.7.11}$$

By this definition, the double-dot product is commutative. The double-dot product between two dyadics in a rectangular Cartesian system is given by

$$\begin{aligned}\boldsymbol{\Phi} : \boldsymbol{\Psi} &= (\phi_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j) : (\psi_{mn}\hat{\mathbf{e}}_m\hat{\mathbf{e}}_n)\\ &= \phi_{ij}\psi_{mn}(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_n)(\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_m)\\ &= \phi_{ij}\psi_{mn}\delta_{in}\delta_{jm}\\ &= \phi_{ij}\psi_{ji}. \end{aligned} \tag{2.7.12}$$

In the general scheme that is developed, scalars are zeroth-order tensors, vectors are first-order tensors, and dyads are second-order tensors. Third-order tensors can be viewed as those derived from *triadics*, or three vectors standing side by side.

2.7.3 Transformation of components of a dyad

A second-order Cartesian tensor $\boldsymbol{\Phi}$ can be represented in barred and unbarred coordinate systems as

$$\begin{aligned}\boldsymbol{\Phi} &= \phi_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\\ &= \bar{\phi}_{kl}\hat{\bar{\mathbf{e}}}_k\hat{\bar{\mathbf{e}}}_l. \end{aligned} \tag{2.7.13}$$

The unit base vectors in the barred and unbarred systems are related by

$$\hat{\mathbf{e}}_i = \ell_{ji}\hat{\bar{\mathbf{e}}}_j \quad \text{or} \quad \hat{\bar{\mathbf{e}}}_i = \ell_{ij}\hat{\mathbf{e}}_j, \qquad \text{where} \quad \ell_{ij} = \hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j, \tag{2.7.14}$$

where ℓ_{ij} denotes the directional cosines between barred and unbarred systems; see Eqs. (2.4.15)–(2.4.17). Thus, the components of a second-order tensor transform according to

$$\bar{\phi}_{k\ell} = \ell_{ki}\ell_{\ell j}\phi_{ij} \quad \text{or} \quad \bar{\phi} = \mathbf{L}\phi\mathbf{L}^{\mathrm{T}}. \tag{2.7.15}$$

In some books, a second-order tensor is defined as one whose components transform according to Eq. (2.7.15). In orthogonal coordinate systems, the determinant of the matrix of directional cosines is unity and its inverse is equal to the transpose:

$$\mathbf{L}^{-1} = \mathbf{L}^{\mathrm{T}} \quad \text{or} \quad \mathbf{L}\mathbf{L}^{\mathrm{T}} = \mathbf{I}. \tag{2.7.16}$$

Tensors $\mathbf{L}$ that satisfy the property of Eq. (2.7.16) are called *orthogonal tensors*.

2.7.4 Tensor calculus

Operations involving the del operator and second- and higher-order tensors follow the same rules as first-order tensors (i.e., vectors), except that the order of the base vectors is kept intact, that is, not switched from the order in which they appear. First, consider the gradient of a vector $\mathbf{A}$:

$$\text{grad}\,\mathbf{A} = \nabla\mathbf{A} = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}(A_j\hat{\mathbf{e}}_j) = \frac{\partial A_j}{\partial x_i}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j. \tag{2.7.17}$$

We note that the gradient of a vector function **A** is a second-order tensor **S**:

$$\nabla \mathbf{A} \equiv \mathbf{S} = S_{ij}\, \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j, \quad S_{ij} = \frac{\partial A_j}{\partial x_i} = A_{j,i}. \tag{2.7.18}$$

Of course, in general $\mathbf{S} = \nabla \mathbf{A}$ is not a symmetric tensor. The gradient $\nabla \mathbf{A}$ and its transpose can be expressed as the sum of symmetric and antisymmetric parts by adding and subtracting $(1/2)(\partial A_i/\partial x_j)$:

$$\begin{aligned}
\nabla \mathbf{A} &= \frac{1}{2}\left(\frac{\partial A_j}{\partial x_i} + \frac{\partial A_i}{\partial x_j}\right)\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j + \frac{1}{2}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j \\
&= \frac{1}{2}\left[(\nabla \mathbf{A}) + (\nabla \mathbf{A})^{\mathrm{T}}\right] + \frac{1}{2}\left[(\nabla \mathbf{A}) - (\nabla \mathbf{A})^{\mathrm{T}}\right] = \mathbf{V} - \mathbf{W}, \qquad (2.7.19) \\
\left(\nabla \mathbf{A}\right)^{\mathrm{T}} &= \frac{1}{2}\left(\frac{\partial A_j}{\partial x_i} + \frac{\partial A_i}{\partial x_j}\right)\hat{\mathbf{e}}_j\hat{\mathbf{e}}_i + \frac{1}{2}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)\hat{\mathbf{e}}_j\hat{\mathbf{e}}_i \\
&= \frac{1}{2}\left[(\nabla \mathbf{A})^{\mathrm{T}} + (\nabla \mathbf{A})\right] + \frac{1}{2}\left[(\nabla \mathbf{A})^{\mathrm{T}} - (\nabla \mathbf{A})\right] = \mathbf{V} + \mathbf{W}, \qquad (2.7.20)
\end{aligned}$$

where **V** and **W** are symmetric and antisymmetric second-order tensors, respectively,

$$\mathbf{V} = \frac{1}{2}\left[(\nabla \mathbf{A})^{\mathrm{T}} + (\nabla \mathbf{A})\right] = \mathbf{V}^{\mathrm{T}}, \quad \mathbf{W} = \frac{1}{2}\left[(\nabla \mathbf{A})^{\mathrm{T}} - (\nabla \mathbf{A})\right] = -\mathbf{W}^{\mathrm{T}}. \tag{2.7.21}$$

Analogously to the divergence of a vector, the divergence of a second-order tensor function **S** is a vector **A**:

$$\begin{aligned}
\nabla \cdot \mathbf{S} &= \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} \cdot (S_{jk}\hat{\mathbf{e}}_j\hat{\mathbf{e}}_k) = \frac{\partial S_{jk}}{\partial x_i}(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j)\hat{\mathbf{e}}_k = \frac{\partial S_{jk}}{\partial x_j}\hat{\mathbf{e}}_k, \\
\nabla \cdot \mathbf{S} &\equiv \mathbf{A} = A_k\, \hat{\mathbf{e}}_k, \quad A_k = \frac{\partial S_{jk}}{\partial x_j} = S_{jk,j}.
\end{aligned} \tag{2.7.22}$$

Thus, the gradient of a tensor $\mathbf{\Phi}$ increases its order by one whereas the divergence of $\mathbf{\Phi}$ decreases its order by one. The gradient and divergence of a tensor $\mathbf{\Phi}$ in other coordinate systems can be readily obtained by writing ∇ and $\mathbf{\Phi}$ in that coordinate system and accounting for the derivatives of the bases vectors (see Problems 2.21–2.24).

The integral theorems of vectors presented in Section 2.6.4 are also valid for tensors (second-order and higher) but it is important that the order of the operations be observed:

$$\int_\Omega \operatorname{grad} \mathbf{A}\, d\mathbf{x} = \oint_\Gamma \hat{\mathbf{n}}\mathbf{A}\, ds, \tag{2.7.23}$$

$$\int_\Omega \operatorname{div} \mathbf{\Phi}\, d\mathbf{x} = \oint_\Gamma \hat{\mathbf{n}} \cdot \mathbf{\Phi}\, ds, \tag{2.7.24}$$

$$\int_\Omega \operatorname{curl} \mathbf{\Phi}\, d\mathbf{x} = \oint_\Gamma \hat{\mathbf{n}} \times \mathbf{\Phi}\, ds. \tag{2.7.25}$$

2.8 Summary

In this chapter, the mathematical preliminaries necessary for the study of the principles of mechanics are reviewed. In particular, the notion of geometric vectors, algebra and calculus of vectors, matrix theory, and tensors and tensor calculus are presented. Components of vectors and tensors in rectangular Cartesian, cylindrical, and spherical coordinate systems are discussed. Index notation is introduced and its utility in the proof of vector identities and the representation of tensors is illustrated. Transformations of vector and tensor components are also presented. A number of examples are presented at appropriate places to apply the ideas introduced.

PROBLEMS

2.1. Find the equation of a line (or a set of lines) passing through the terminal point of a vector **A** and in the direction of vector **B**.

2.2. Find the equation of a plane connecting the terminal points of vectors **A**, **B**, and **C**. Assume that all three vectors refer to a common origin.

2.3. In a rectangular Cartesian coordinate system, find the length and directional cosines of a vector **A** that extends from the point $(1, -1, 3)$ to the midpoint of the line segment from the origin to the point $(6, -6, 4)$.

2.4. The vectors **A** and **B** are defined as follows:

$$\mathbf{A} = 3\hat{\mathbf{i}} - 4\hat{\mathbf{k}}, \quad \mathbf{B} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}},$$

where $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ is an orthonormal basis. Find
(a) the orthogonal projection of **A** in the direction of **B**, and
(b) the angle between the positive directions of the vectors.

2.5. In Example 2.3.3, take $\mathbf{n} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{k}}$ and $\mathbf{v} = 5\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ to determine the quantities required in (a)–(d).

2.6. In Example 2.3.3, take $\mathbf{n} = -\hat{\mathbf{i}} + 2\hat{\mathbf{k}}$ and $\mathbf{v} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{k}}$ to determine the quantities required in (a)–(d).

2.7. Determine whether the following set of vectors is linearly independent:

$$\mathbf{A} = 2\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3, \quad \mathbf{B} = -\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3, \quad \mathbf{C} = -\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2.$$

Here $\hat{\mathbf{e}}_i$ are orthonormal unit base vectors in $\Re^3$.

2.8. Using index notation, prove the identities

$$\text{(a)} \quad (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A}) = (\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}))^2,$$

$$\text{(b)} \quad (\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = [\mathbf{A} \cdot (\mathbf{C} \times \mathbf{D})]\mathbf{B} - [\mathbf{B} \cdot (\mathbf{C} \times \mathbf{D}]\mathbf{A}.$$

2.9. Given the following matrix components

$$\mathbf{A} = \begin{Bmatrix} 2 \\ -1 \\ 4 \end{Bmatrix}, \quad \mathbf{S} = \begin{bmatrix} -1 & 0 & 5 \\ 3 & 7 & 4 \\ 9 & 8 & 6 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} 8 & -1 & 6 \\ 5 & 4 & 9 \\ -7 & 8 & -2 \end{bmatrix},$$

determine

(a) $\mathrm{tr}(\mathbf{S})$ (b) $\mathbf{S}:\mathbf{S}$ (c) $\mathbf{S}:\mathbf{S}^{\mathrm{T}}$

(d) $\mathbf{A}\cdot\mathbf{S}$ (e) $\mathbf{S}\cdot\mathbf{A}$ (f) $\mathbf{S}\cdot\mathbf{T}\cdot\mathbf{A}$

2.10. Consider two rectangular Cartesian coordinate systems that are translated and rotated with respect to each other. The transformation between the two coordinate systems is given by

$$\bar{\mathbf{x}} = \mathbf{c} + \mathbf{L}\mathbf{x},$$

where $\mathbf{c}$ is a constant vector and $\mathbf{L} = [\ell_{ij}]$ is the matrix of directional cosines,

$$\ell_{ij} \equiv \hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j.$$

Deduce that the following orthogonality conditions hold:

$$\mathbf{L}\cdot\mathbf{L}^{\mathrm{T}} = \mathbf{I},$$

that is, $\mathbf{L}$ is an orthogonal matrix.

2.11. Determine the transformation matrix relating the orthonormal basis vectors $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ and $(\hat{\mathbf{e}}_1', \hat{\mathbf{e}}_2', \hat{\mathbf{e}}_3')$, when $\hat{\mathbf{e}}_i'$ are given by:

(a) $\hat{\mathbf{e}}_1'$ is along the vector $\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$ and $\hat{\mathbf{e}}_2'$ is perpendicular to the plane $2x_1 + 3x_2 + x_3 - 5 = 0$.

(b) $\hat{\mathbf{e}}_1'$ is along the line segment connecting point $(1, -1, 3)$ to $(2, -2, 4)$ and $\hat{\mathbf{e}}_3' = (-\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + 2\hat{\mathbf{e}}_3)/\sqrt{6}$.

2.12. The angles between the barred and unbarred coordinate lines are given by

	$\hat{\mathbf{e}}_1$	$\hat{\mathbf{e}}_2$	$\hat{\mathbf{e}}_3$
$\hat{\bar{\mathbf{e}}}_1$	60°	30°	90°
$\hat{\bar{\mathbf{e}}}_2$	150°	60°	90°
$\hat{\bar{\mathbf{e}}}_3$	90°	90°	0°

Determine the directional cosines of the transformation.

2.13. The angles between the barred and unbarred coordinate lines are given by

	x_1	x_2	x_3
$\bar{x}_1$	45°	90°	45°
$\bar{x}_2$	60°	45°	120°
$\bar{x}_3$	120°	45°	60°

Determine the transformation matrix.

2.14. Establish the following identities for a second-order tensor $\mathbf{A}$:

(a) $|A| = e_{ijk}A_{1i}A_{2j}A_{3k}$. (b) $|A| = \dfrac{1}{6}A_{ir}A_{js}A_{kt}e_{rst}e_{ijk}$.

(c) $e_{lmn}|A| = e_{ijk}A_{il}A_{jm}A_{kn}$. (d) $e_{ijk}e_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$.

(e) $e_{ijk} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix}$. (f) $e_{ijk}e_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix}$.

2.15. Show that the following expressions for the components of an arbitrary second-order tensor $\mathbf{S} = [s_{ij}]$ are invariant: (a) s_{ii}, (b) $s_{ij}s_{ij}$, and (c) $s_{ij}s_{jk}s_{ki}$.

2.16. Let $\mathbf{r}$ denote a position vector $\mathbf{r} = \mathbf{x} = x_i\, \hat{\mathbf{e}}_i$ $(r^2 = x_i x_i)$ and $\mathbf{A}$ an arbitrary constant vector. Show that:

(a) $\nabla^2(r^p) = p(p+1)r^{p-2}$, (b) $\text{grad}\,(\mathbf{r} \cdot \mathbf{A}) = \mathbf{A}$,

(c) $\text{div}\,(\mathbf{r} \times \mathbf{A}) = 0$, (d) $\text{curl}(\mathbf{r} \times \mathbf{A}) = -2\mathbf{A}$,

(e) $\text{div}\,(r\mathbf{A}) = \dfrac{1}{r}(\mathbf{r} \cdot \mathbf{A})$, (f) $\text{curl}(r\mathbf{A}) = \dfrac{1}{r}(\mathbf{r} \times \mathbf{A})$.

2.17. Let $\mathbf{A}$ and $\mathbf{B}$ be continuous vector functions of the position vector $\mathbf{x}$ with continuous first derivatives, and let F and G be continuous scalar functions of position $\mathbf{x}$ with continuous first and second derivatives. Show that:

(a) $\text{div}(\text{curl}\,\mathbf{A}) = 0$,
(b) $\text{div}(\text{grad}\,F \times \text{grad}\,G) = 0$,
(c) $\text{grad}(\mathbf{A} \cdot \mathbf{x}) = \mathbf{A} + \text{grad}\,\mathbf{A} \cdot \mathbf{x}$,
(d) $\text{div}(F\mathbf{A}) = \mathbf{A} \cdot \text{grad}\,F + F\text{div}\mathbf{A}$,
(e) $\text{curl}(F\mathbf{A}) = F\,\text{curl}\,\mathbf{A} - \mathbf{A} \times \text{grad}\,F$,
(f) $\text{div}\,(\mathbf{A} \times \mathbf{B}) = \text{curl}\,\mathbf{A} \cdot \mathbf{B} - \text{curl}\,\mathbf{B} \cdot \mathbf{A}$,
(g) $\text{curl}\,(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla\mathbf{A} - \mathbf{A} \cdot \nabla\mathbf{B} + \mathbf{A}\text{div}\mathbf{B} - \mathbf{B}\text{div}\mathbf{A}$,
(h) $\mathbf{A} \cdot \text{grad}\,\mathbf{A} = \text{grad}\left(\frac{1}{2}\mathbf{A} \cdot \mathbf{A}\right) - \mathbf{A} \times \text{curl}\,\mathbf{A}$.

2.18. Show that the *vector area* of a closed surface Γ is zero, that is,

$$\oint_\Gamma \hat{\mathbf{n}}\, ds = \mathbf{0}.$$

2.19. Let $\phi(\mathbf{r})$ be a scalar field. Show that

$$\int_\Omega \nabla^2\phi\, d\mathbf{x} = \oint_\Gamma \frac{\partial\phi}{\partial n}\, ds.$$

2.20. If $\mathbf{A}$ is an arbitrary vector and $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are arbitrary dyadics, verify that: (a) $(\mathbf{I} \times \mathbf{A}) \cdot \mathbf{\Phi} = \mathbf{A} \times \mathbf{\Phi}$ (b) $(\mathbf{A} \times \mathbf{I}) \cdot \mathbf{\Phi} = \mathbf{A} \times \mathbf{\Phi}$ (c) $(\mathbf{\Phi} \times \mathbf{A})^{\mathrm{T}} = -\mathbf{A} \times \mathbf{\Phi}^{\mathrm{T}}$ (d) $(\mathbf{\Phi} \cdot \mathbf{\Psi})^{\mathrm{T}} = \mathbf{\Psi}^{\mathrm{T}} \cdot \mathbf{\Phi}^{\mathrm{T}}$

2.21. For an arbitrary second-order tensor $\mathbf{S}$, show that $\nabla \cdot \mathbf{S}$ in the cylindrical coordinate system is given by

$$\begin{aligned}
\nabla \cdot \mathbf{S} = &\left[\frac{\partial S_{rr}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta r}}{\partial \theta} + \frac{\partial S_{zr}}{\partial z} + \frac{1}{r}\left(S_{rr} - S_{\theta\theta}\right)\right]\hat{\mathbf{e}}_r \\
&+ \left[\frac{\partial S_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta\theta}}{\partial \theta} + \frac{\partial S_{z\theta}}{\partial z} + \frac{1}{r}\left(S_{r\theta} + S_{\theta r}\right)\right]\hat{\mathbf{e}}_\theta \\
&+ \left[\frac{\partial S_{rz}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta z}}{\partial \theta} + \frac{\partial S_{zz}}{\partial z} + \frac{1}{r}S_{rz}\right]\hat{\mathbf{e}}_z.
\end{aligned}$$

2.22. For an arbitrary second-order tensor $\mathbf{S}$, show that $\nabla \times \mathbf{S}$ in the cylindrical coordinate system is given by

$$\begin{aligned}\nabla \times \mathbf{S} &= \hat{\mathbf{e}}_r \hat{\mathbf{e}}_r \left(\frac{1}{r}\frac{\partial S_{zr}}{\partial \theta} - \frac{\partial S_{\theta r}}{\partial z} - \frac{1}{r} S_{z\theta} \right) + \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_\theta \left(\frac{\partial S_{r\theta}}{\partial z} - \frac{\partial S_{z\theta}}{\partial r} \right) \\
&+ \hat{\mathbf{e}}_z \hat{\mathbf{e}}_z \left(\frac{1}{r} S_{\theta z} - \frac{1}{r}\frac{\partial S_{rz}}{\partial \theta} + \frac{\partial S_{\theta z}}{\partial r} \right) + \hat{\mathbf{e}}_r \hat{\mathbf{e}}_\theta \left(\frac{1}{r}\frac{\partial S_{z\theta}}{\partial \theta} - \frac{\partial S_{\theta\theta}}{\partial z} + \frac{1}{r} S_{zr} \right) \\
&+ \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_r \left(\frac{\partial S_{rr}}{\partial z} - \frac{\partial S_{zr}}{\partial r} \right) + \hat{\mathbf{e}}_r \hat{\mathbf{e}}_z \left(\frac{1}{r}\frac{\partial S_{zz}}{\partial \theta} - \frac{\partial S_{\theta z}}{\partial z} \right) \\
&+ \hat{\mathbf{e}}_z \hat{\mathbf{e}}_r \left[\frac{\partial S_{\theta r}}{\partial r} - \frac{1}{r}\frac{\partial S_{rr}}{\partial \theta} + \frac{1}{r}(S_{r\theta} + S_{\theta r}) \right] + \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_z \left(\frac{\partial S_{rz}}{\partial z} - \frac{\partial S_{zz}}{\partial r} \right) \\
&+ \hat{\mathbf{e}}_z \hat{\mathbf{e}}_\theta \left[\frac{\partial S_{\theta\theta}}{\partial r} + \frac{1}{r}(S_{\theta\theta} - S_{rr}) - \frac{1}{r}\frac{\partial S_{r\theta}}{\partial \theta} \right].\end{aligned}$$

2.23. For an arbitrary second-order tensor $\mathbf{S}$, show that $\nabla \cdot \mathbf{S}$ in the spherical coordinate system is given by

$$\begin{aligned}\nabla \cdot \mathbf{S} = &\left\{ \frac{\partial S_{RR}}{\partial R} + \frac{1}{R}\frac{\partial S_{\phi R}}{\partial \phi} + \frac{1}{R \sin\phi}\frac{\partial S_{\theta R}}{\partial \theta} \right. \\
&\left. + \frac{1}{R}\left[2S_{RR} - S_{\phi\phi} - S_{\theta\theta} + S_{\phi R}\cot\phi \right] \right\} \hat{\mathbf{e}}_R \\
&+ \left\{ \frac{\partial S_{R\phi}}{\partial R} + \frac{1}{R}\frac{\partial S_{\phi\phi}}{\partial \phi} + \frac{1}{R \sin\phi}\frac{\partial S_{\theta\phi}}{\partial \theta} \right. \\
&\left. + \frac{1}{R}\left[(S_{\phi\phi} - S_{\theta\theta})\cot\phi + S_{\phi R} + 2S_{R\phi} \right] \right\} \hat{\mathbf{e}}_\phi \\
&+ \left\{ \frac{\partial S_{R\theta}}{\partial R} + \frac{1}{R}\frac{\partial S_{\phi\theta}}{\partial \phi} + \frac{1}{R \sin\phi}\frac{\partial S_{\theta\theta}}{\partial \theta} \right. \\
&\left. + \frac{1}{R}\left[(S_{\phi\theta} + S_{\theta\phi})\cot\phi + 2S_{R\theta} + S_{\theta R} \right] \right\} \hat{\mathbf{e}}_\theta .\end{aligned}$$

2.24. Show that $\nabla \mathbf{u}$ in the spherical coordinate system is given by

$$\begin{aligned}\nabla \mathbf{u} = &\frac{\partial u_R}{\partial R} \hat{\mathbf{e}}_R \hat{\mathbf{e}}_R + \frac{\partial u_\phi}{\partial R} \hat{\mathbf{e}}_R \hat{\mathbf{e}}_\phi + \frac{\partial u_\theta}{\partial R} \hat{\mathbf{e}}_R \hat{\mathbf{e}}_\theta \\
&+ \frac{1}{R}\left(\frac{\partial u_R}{\partial \phi} - u_\phi \right) \hat{\mathbf{e}}_\phi \hat{\mathbf{e}}_R + \frac{1}{R}\left(\frac{\partial u_\phi}{\partial \phi} + u_R \right) \hat{\mathbf{e}}_\phi \hat{\mathbf{e}}_\phi + \frac{1}{R}\frac{\partial u_\theta}{\partial \phi} \hat{\mathbf{e}}_\phi \hat{\mathbf{e}}_\theta \\
&+ \frac{1}{R \sin\phi}\left[\left(\frac{\partial u_R}{\partial \theta} - u_\theta \sin\phi \right) \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_R + \left(\frac{\partial u_\phi}{\partial \theta} - u_\theta \cos\phi \right) \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_\phi \right. \\
&\left. + \left(\frac{\partial u_\theta}{\partial \theta} + u_R \sin\phi + u_\phi \cos\phi \right) \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_\theta \right].\end{aligned}$$

3 Kinematics of a Continuum

Science advances, not by the accumulation of new facts, but by the continuous development of new concepts.

James Bryant Conant

3.1 Deformation and configuration

The present chapter is devoted to the study of geometric changes in a continuous medium that is in static or dynamic equilibrium under the action of some stimuli, such as mechanical, thermal, or other types of forces. The change of geometry or rate of change of geometry of a continuous medium can be used as a measure of so-called strains or strain rates, which are responsible for inducing stresses in the continuum. In the subsequent chapters, we will study stresses and physical principles that govern the mechanical response of a continuous medium. The study of geometric (or rate of geometric) changes in a continuum without regard to the stimuli (forces) causing the changes is known as *kinematics*.

Consider a continuous body of known geometry, material constitution, and loading in a three-dimensional space $\Re^3$; the body may be viewed as a set of particles, each particle representing a large collection of molecules, having a continuous distribution of matter in space and time. Examples of such a body are provided by a diving board, the artery in a human body, a can of soda, and so on. Suppose that the body is subjected to a set of forces that tend to change the shape of the body. For a given geometry and forces, the body will undergo macroscopic geometric changes, which are referred to as *deformation*. The geometric changes may be small enough not to be noticed by the human eye, like the changes in lengths of connecting members in a bicycle, or they may be large enough to be noticeable, like the rubber tire in a bicycle. The forces causing the deformation can be mechanical, thermal, or otherwise. If the applied forces are time-dependent, the deformation of the body will be a function of time, that is, the geometry of the body will change continuously with time. The region occupied by the continuum at a given time t is termed a *configuration* and denoted by $\mathcal{C}$.

In general, the deformation throughout a body is not uniform. That is, a body may experience a small deformation in one part and a large deformation in another

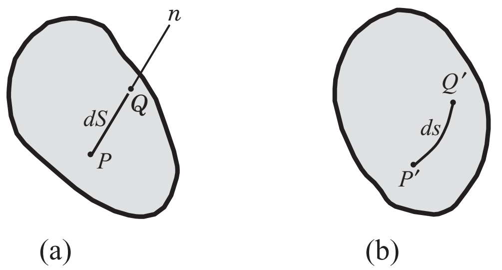


Figure 3.2.1 (a) Line segment in the undeformed body. (b) Deformed line segment – curve – in the deformed body.

part. The deformation varies from point to point. Deformation is measured in a number of different ways. The change in the length of a line segment per unit undeformed length is taken as a measure of an *extensional* or *normal* strain. The change of the angle between two line segments that were mutually perpendicular to each other in the reference configuration is taken as a measure of a *shear* strain. The point-wise strains are measured by taking line segments that are infinitesimally small at the point. Also, the strains at a point depend on the orientation of the line segment, that is, at the same point a line segment oriented in one direction may experience elongation and another line segment oriented in a different direction may contract.

In this chapter, we first present the concepts of infinitesimal (i.e., small) normal and shear strains. Then a more formal discussion of the subject of kinematics is presented. In Chapter 6, the strains are related to force per unit area (i.e., stress) induced in the continuum. Sections with an asterisk (*) may be skipped without a loss of continuity.

3.2 Engineering strains

3.2.1 Normal strain

The change in the length (elongation or contraction) of a line segment per unit length is termed the *normal strain*. To give a mathematical definition of normal strain, we consider the line segment connecting material points P and Q, separated by a distance dS in the undeformed body, as shown in Figure 3.2.1(a). After deformation (due to externally applied forces), the material points P and Q are displaced to P' and Q', respectively; the material points on line PQ are displaced such that $P'Q'$ becomes a curve of length ds, as shown in Figure 3.2.1(b). This deformation does not include any rigid body displacements the body may have undergone. Then the *average normal strain* in the line segment PQ is defined by

$$\varepsilon_n^{\text{avg}} = \frac{ds - dS}{dS}, \tag{3.2.1}$$

where the subscript n denotes the direction in which the original line segment is oriented.

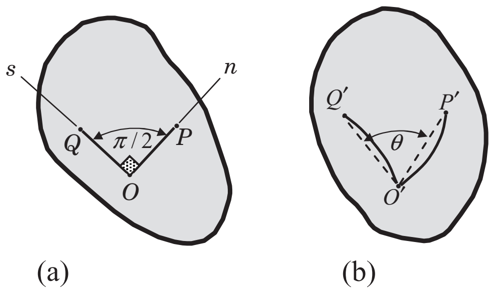


Figure 3.2.2 (a) Two line segments perpendicular to each other in the undeformed body. (b) Deformed line segments – curves – in the deformed body.

To define the normal strain at point P in the direction of n, we choose point Q closer and closer to P, such that $dS \to 0$. Because we are dealing with a continuum, this causes point Q' to approach P', as $ds \to 0$. Thus, in the limit the normal strain at point P in the direction of n is

$$\varepsilon_n = \lim_{dS \to 0} \frac{ds - dS}{dS} . \tag{3.2.2}$$

The normal strain is a dimensionless quantity because it is the ratio of two lengths. However, to indicate the fact it is a ratio of lengths, the units of normal strain are often stated as m/m (meters per meter) or μm/m (micrometers per meter, μ denoting 10^{-6}).

3.2.2 Shear strain

As stated previously, the change in the angle between two line segments that were perpendicular to each other in the undeformed body is termed *shear strain*. To write a mathematical expression of the shear strain, consider line segments OP and OQ in the undeformed body that were perpendicular to each other, as shown in Figure 3.2.2(a). After deformation, the line segments OP and OQ become curves $O'P'$ and $O'Q'$, with included angle θ.

The shear strain at point O is then defined as

$$\gamma_{ns} = \frac{\pi}{2} - \lim_{P,Q \to O} \theta . \tag{3.2.3}$$

The units of shear strain are in radians (rad).

The next couple of examples are concerned with the computation of strain components using the definitions in Eqs. (3.2.1) and (3.2.3).

Example 3.2.1:

Consider the two-member truss shown in Figure 3.2.3. The deformed configuration of the truss is shown by dashed lines, where point O has moved vertically down by v and horizontally by u. Determine the axial strains in each member in terms of the displacements u and v.

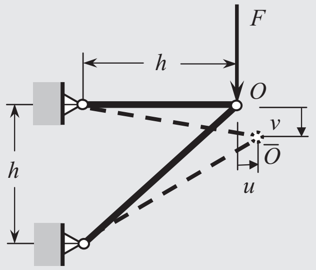


Figure 3.2.3 Undeformed and deformed configurations of a truss.

Solution: The axial strains can be computed in terms of the displacements u and v of point O as follows:

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{h}\left(\sqrt{(h+u)^2+v^2}-h\right) = \sqrt{1+\frac{u^2+v^2+2hu}{h^2}}-1\\
&\approx \frac{u^2+v^2+2hu}{2h^2},\\
\varepsilon_2 &= \frac{1}{\sqrt{2}h}\left(\sqrt{(h+u)^2+(h-v)^2}-\sqrt{2}h\right) = \sqrt{1+\frac{u^2+v^2+2h(u-v)}{2h^2}}-1\\
&\approx \frac{u^2+v^2+2h(u-v)}{4h^2},
\end{aligned}$$

where we have used the approximation

$$(1+x)^n \approx 1+nx+(nx)^2+\cdots.$$

If the displacements are small compared to unity, we may neglect the products and squares of u and v and obtain the following linearized strains:

$$\varepsilon_1 = \frac{u^2+v^2+2hu}{2h^2} \approx \frac{u}{h}, \qquad \varepsilon_2 = \frac{u^2+v^2+2h(u-v)}{4h^2} \approx \frac{u-v}{2h}.$$

Example 3.2.2:

Consider a rectangular block $ABCD$ of dimensions $a \times b \times h$ where h is the thickness, which is very small compared to a and b. Suppose that the block is deformed into the diamond shape $\bar{A}\bar{B}\bar{C}\bar{D}$ shown in Figure 3.2.4. Determine the normal and shear strains with respect to the coordinate system given.

Solution: A line element AB in the undeformed configuration of the body moves to position $\bar{A}\bar{B}$, as shown in Figure 3.2.4. Then the normal strain in the line AB is given by

$$\begin{aligned}
\varepsilon_{11} = \varepsilon_{AB} &= \frac{\bar{A}\bar{B}-AB}{AB} = \frac{1}{a}\sqrt{a^2+e_2^2}-1 = \sqrt{1+\left(\frac{e_2}{a}\right)^2}-1\\
&= \left[1+\frac{1}{2}\left(\frac{e_2}{a}\right)^2+\cdots\right]-1 \approx \frac{1}{2}\left(\frac{e_2}{a}\right)^2 = \frac{1}{2}k_2^2.
\end{aligned}$$

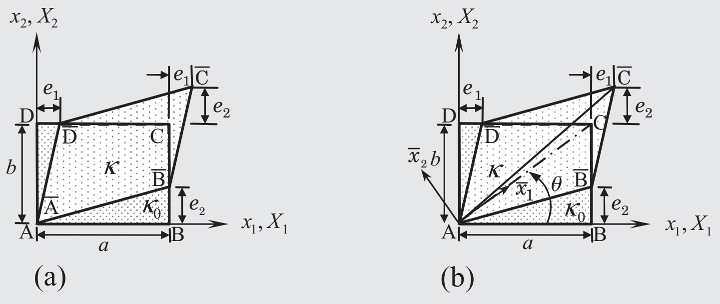


Figure 3.2.4 Undeformed and deformed configurations of a rectangular block.

Similarly,

$$\varepsilon_{22} = \left[1 + \frac{1}{2}\left(\frac{e_1}{b}\right)^2 + \cdots\right] - 1 \approx \frac{1}{2}\left(\frac{e_1}{b}\right)^2 = \frac{1}{2}k_1^2.$$

The shear strain γ_{12} is equal to the change in the angle between two line elements that were originally at 90°, that is, the change in the angle DAB. The change is clearly equal to

$$\gamma_{12} = \angle DAB - \angle \bar{D}\bar{A}\bar{B} = \frac{e_1}{b} + \frac{e_2}{a} = k_1 + k_2.$$

The axial strain in line element AC is ($\bar{A} = A$):

$$\begin{aligned}
\varepsilon_{AC} &= \frac{\bar{A}\bar{C} - AC}{AC} = \frac{1}{\sqrt{a^2+b^2}}\sqrt{(a+e_1)^2 + (b+e_2)^2} - 1 \\
&= \frac{1}{\sqrt{a^2+b^2}}\sqrt{a^2+b^2+e_1^2+e_2^2+2ae_1+2be_2} - 1 \\
&= \left[1 + \frac{e_1^2+e_2^2+2ae_1+2be_2}{a^2+b^2}\right]^{\frac{1}{2}} - 1 \approx \frac{1}{2}\frac{e_1^2+e_2^2+2ae_1+2be_2}{a^2+b^2} \\
&= \frac{1}{2(a^2+b^2)}\left[a^2k_2^2 + 2ab(k_1+k_2) + b^2k_1^2\right].
\end{aligned}$$

The linearized normal and shear strains are given by

$$\varepsilon_{11} = 0, \quad \varepsilon_{22} = 0, \quad \gamma_{12} = k_1 + k_2, \quad \varepsilon_{AC} = \frac{ab}{(a^2+b^2)}(k_1+k_2).$$

Example 3.2.3:

Consider a thin strip of material of length L, height h, and thickness t in the undeformed configuration. If the strip is bent into an arc of a circle, as shown in Figure 3.2.5, determine the axial strain $\varepsilon_{11} = \varepsilon_{xx}$ induced in the strip. Assume that the changes in the height h and thickness t are negligible.

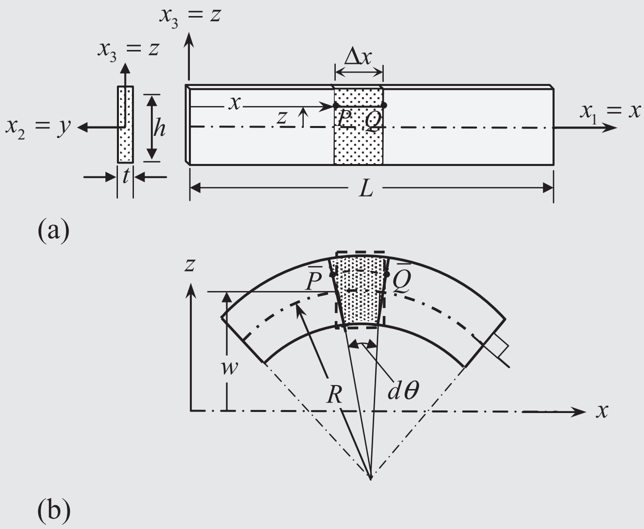


Figure 3.2.5 **Undeformed and deformed configurations of a strip.**

Solution: From the deformed configuration, it is clear that the lines parallel to the length of the strip have deformed proportionately, with the top line elongating and the bottom line shortening. Thus, the centerline of the strip neither elongated nor shortened. All lines parallel to the y-axis and z-axis have remained the same in length, and they remained perpendicular to the axis of the strip at every point along the length, implying that all shear strains are zero. Hence, the only nonzero strain is ε_{xx}.

To compute the infinitesimal strain ε_{xx}, we use the definition of change in length divided by the original length. We have

$$\varepsilon_{xx} = \frac{\bar{P}\bar{Q} - PQ}{PQ} = \frac{(R+z)\theta - R\theta}{R\theta} = \frac{z}{R}.$$

From a course on geometry, the *radius of curvature* R ($1/R$ is called *curvature*) is related to the transverse displacement w of the center line of the strip by

$$\frac{1}{R} = -\frac{\frac{d^2w}{dx^2}}{\left[1 + \left(\frac{dw}{dx}\right)^2\right]^{3/2}} \approx -\frac{d^2w}{dx^2}.$$

Thus, the infinitesimal strain is

$$\varepsilon_{xx} = -z\frac{d^2w}{dx^2}. \tag{3.2.4}$$

The kinematics described here correspond to pure bending of beams, and the theory is known as the *Euler–Bernoulli beam theory*. A slender solid body, such as the one shown in Figure 3.2.5, is called a *bar* if it is subjected to forces that tend to stretch it along its length; it is termed a *beam* if it is subjected to forces

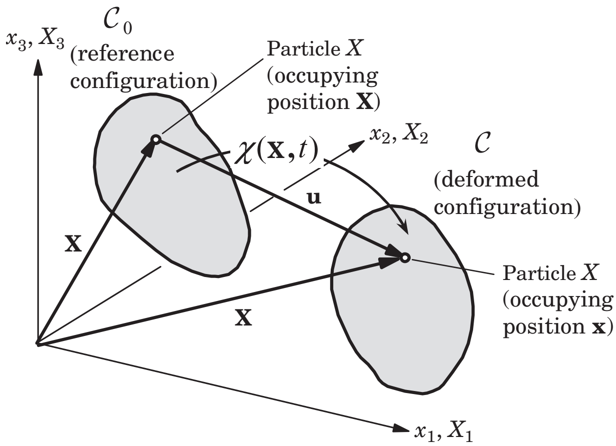


Figure 3.3.1 Reference and deformed configurations of a body.

that tend to bend it about an axis perpendicular to its length. We will return to stretching of bars and bending of beams in Chapters 5 and 6.

3.3 General kinematics of a solid continuum

3.3.1 Configurations of a continuous medium

Suppose that the continuum initially occupies a configuration $\mathcal{C}_0$, in which a typical particle X occupies the position $\mathbf{X}$, referred to in a rectangular Cartesian system as (X_1, X_2, X_3). Note that X (lightface letter) is the name of the particle that occupies the location $\mathbf{X}$ (boldface letter) in configuration $\mathcal{C}_0$, and therefore (X_1, X_2, X_3) are called the *material coordinates*. After the application of the forces, the continuum changes its geometric shape and thus assumes a new configuration $\mathcal{C}$, called the *current* or *deformed configuration*. The particle X now occupies the position $\mathbf{x}$ in configuration $\mathcal{C}$, as shown in Figure 3.3.1. The mapping of $\mathbf{X}$ to $\mathbf{x}$ at any time t can be represented by

$$\mathbf{x} = \chi(\mathbf{X}, t), \quad \chi(\mathbf{X}, 0) = \mathbf{X}. \tag{3.3.1}$$

The mapping χ is called the *deformation mapping* in the study of solid continua. It gives the position $\mathbf{x}$ in the deformed configuration of every material particle X that was occupying position $\mathbf{X}$. The inverse mapping is given by

$$\mathbf{X} = \chi^{-1}(\mathbf{x}, t), \tag{3.3.2}$$

which is not always possible to determine.

A coordinate system is chosen, explicitly or implicitly, to describe the deformation. We shall use the same coordinate system for reference and current configurations. For example, the components X_i and x_i of vectors $\mathbf{X} = X_i\,\hat{\mathbf{E}}_i$ and $\mathbf{x} = x_i\,\hat{\mathbf{e}}_i$, referred to in rectangular Cartesian coordinate systems as (X_1, X_2, X_3) and (x_1, x_2, x_3), respectively, are the same at $t = 0$, and $\mathbf{E}_i = \mathbf{e}_i$ for all t.

3.3.2 Material and spatial descriptions

To understand the meaning of the mapping in Eq. (3.3.1), consider a property ϕ (such as the temperature or density) of the body. For a fixed value of $\mathbf{X}$ in $\mathcal{C}_0$, $\phi(\mathbf{X}, t)$ gives the value of ϕ at time t associated with the fixed material point X whose position in the reference configuration is $\mathbf{X}$. Thus, a change in time t, say $t = \tau$, implies that the same material particle X occupying position $\mathbf{X}$ in $\mathcal{C}_0$ has a different value $\phi(\mathbf{X}, \tau)$. Thus, the attention is focused on the material particles X of the continuum. Such a description – that is, a description in which attention is focused on material points instead of their locations – is called a *material description* or *Lagrangian description*.

In the spatial description, the motion is referred to by the current configuration $\mathcal{C}$ occupied by the body, and a typical property ϕ is described with respect to the current position $\mathbf{x}$ in $\mathcal{C}$ in space, currently occupied by material particle X:

$$\phi = \phi(\mathbf{x}, t), \quad \mathbf{X} = \mathbf{X}(\mathbf{x}, t) = \chi^{-1}(\mathbf{x}, t). \tag{3.3.3}$$

The coordinates ($\mathbf{x}$) are termed the *spatial coordinates*. For a fixed value of $\mathbf{x}$ in $\mathcal{C}$, $\phi(\mathbf{x}, t)$ gives the value of ϕ associated with a fixed point $\mathbf{x}$ in space, which will be the value of ϕ associated with different material points at different times, because different material points occupy the position $\mathbf{x}$ in $\mathcal{C}$ at different times. Thus, a change in time t implies that a different value ϕ is observed at the same spatial location $\mathbf{x}$ in $\mathcal{C}$, now probably occupied by a different material particle X. Hence, attention is focused on a spatial position $\mathbf{x}$ in $\mathcal{C}$. This description is also known as the *Eulerian description*.

The material description is commonly used to study the stress deformation of solid bodies, as one is interested in the body irrespective of what spatial location it occupies. On the other hand, spatial description is adopted in studying fluid motions, where one is interested in the conditions of flow (e.g., density, temperature, pressure, and so on) at a fixed spatial location, rather than in the material particles that happen to occupy the fixed spatial location. A simple analogy of the descriptions is provided by the traffic policeman. When the policeman sits at a traffic light and observes the vehicles (which may be seen as material particles) moving through the junction, it is like a spatial description. In contrast, if the policeman follows a fixed set of vehicles along a expressway, it is like a material description. In both cases, the traffic policeman is looking for violators of the traffic rules.

In the study of solid bodies, the Eulerian description is less useful because the configuration $\mathcal{C}$ is unknown. On the other hand, it is the preferred description for the study of the motion of fluids because the configuration is known and remains unchanged, and we wish to determine the changes in the fluid velocities, pressure, density, and so on. Thus, in the Eulerian description attention is focused on a given region of space instead of a given body of matter.

When ϕ is known in the material description, $\phi = \phi(\mathbf{X}, t)$, its time derivative is simply the partial derivative with respect to time because the material coordinates

$\mathbf{X}$ do not change with time:

$$\frac{d}{dt}[\phi(\mathbf{X},t)] = \frac{d}{dt}[\phi(\mathbf{X},t)]\bigg|_{\mathbf{X}=\text{fixed}} = \frac{\partial\phi}{\partial t}\,. \tag{3.3.4}$$

However, when ϕ is known in the spatial description, $\phi = \phi(\mathbf{x}, t)$, its time derivative, known as the *material derivative*[1], consists of two parts: the local change at point $\mathbf{x}$ and the change brought in by the new particle. This is expressed, using the chain rule of differentiation, as

$$\begin{aligned}\frac{d}{dt}[\phi(\mathbf{x},t)] &= \frac{\partial}{\partial t}[\phi(\mathbf{x},t)] + \frac{dx_i}{dt}\frac{\partial}{\partial x_i}[\phi(\mathbf{x},t)]\\ &= \frac{\partial\phi}{\partial t} + v_i\frac{\partial\phi}{\partial x_i} = \frac{\partial\phi}{\partial t} + v_x\frac{\partial\phi}{\partial x} + v_y\frac{\partial\phi}{\partial y} + v_z\frac{\partial\phi}{\partial z}\\ &= \frac{\partial\phi}{\partial t} + \mathbf{v}\cdot\nabla\phi,\end{aligned} \tag{3.3.5}$$

where $\mathbf{v}$ is the velocity vector $\mathbf{v} = d\mathbf{x}/dt = \dot{\mathbf{x}}$ when $\mathbf{X}$ is fixed. Thus, the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ [let $\phi \;\rightarrow\; \mathbf{v}$ in Eq. (3.3.5)] of a particle are defined by

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{x}}{dt} = \frac{d}{dt}[\mathbf{x}(\mathbf{X},t)]\bigg|_{\mathbf{X}=\text{fixed}},\\ \mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v},\quad \left(a_i = \frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right).\end{aligned} \tag{3.3.6}$$

Thus, the time derivative of (Eq. 3.3.5) in the spatial description consists of two parts: The first part is the instantaneous change of the property ϕ with time at the spatial location $\mathbf{x}$. The second part is the additional change in the property at $\mathbf{x}$ brought about by the material particle that happens to occupy the position $\mathbf{x}$ at that instant. The second term is sometimes referred to as the *convective part* of the material time derivative, and it is a source of nonlinearity in fluid mechanics.

The next example illustrates the determination of the inverse of a given mapping and computation of the material time derivative of a given function.

Example 3.3.1:

Suppose that the motion of a continuous body is described by the mapping

$$\mathbf{x} = \chi(\mathbf{X},t) = (X_1 + AtX_2)\hat{\mathbf{e}}_1 + (X_2 - AtX_1)\hat{\mathbf{e}}_2 + X_3\,\hat{\mathbf{e}}_3,$$

and that the temperature θ in the continuum in the spatial description is given by

$$\theta(\mathbf{x},t) = x_1 + tx_2.$$

Determine (a) the inverse of the mapping, (b) the velocity and acceleration components, and (c) the time derivatives of θ in the two descriptions.

[1] Stokes' notation for material derivative d/dt is D/Dt.

Solution: The mapping implies that a unit square is mapped into a rectangle that is rotated in the clockwise direction, as shown in Figure 3.3.2.

(a) First, note that x_i are related to X_i by the simultaneous equations

$$x_1 = X_1 + AtX_2, \quad x_2 = X_2 - AtX_1, \quad x_3 = X_3.$$

Solving for (X_1, X_2, X_3) in terms of (x_1, x_2, x_3), we obtain

$$X_1 = \frac{x_1 - Atx_2}{1 + A^2t^2}, \quad X_2 = \frac{x_2 + Atx_1}{1 + A^2t^2}, \quad X_3 = x_3.$$

Hence, the inverse mapping is given by $\chi^{-1} : \kappa \to \kappa_0$:

$$\chi^{-1}(\mathbf{x}, t) = \left(\frac{x_1 - Atx_2}{1 + A^2t^2}\right)\hat{\mathbf{E}}_1 + \left(\frac{x_2 + Atx_1}{1 + A^2t^2}\right)\hat{\mathbf{E}}_2 + x_3\,\hat{\mathbf{E}}_3,$$

where $(\hat{\mathbf{E}}_1, \hat{\mathbf{E}}_2, \hat{\mathbf{E}}_3)$ is the basis in the reference coordinate system (X_1, X_2, X_3).

(b) The velocity vector in the material coordinate system is given by

$$\mathbf{v} = v_1(X_1, X_2, X_3, t)\hat{\mathbf{E}}_1 + v_2(X_1, X_2, X_3, t)\hat{\mathbf{E}}_2 + v_3(X_1, X_2, X_3, t)\hat{\mathbf{E}}_3$$

with

$$v_1 = \frac{\partial x_1}{\partial t} = AX_2, \quad v_2 = \frac{\partial x_2}{\partial t} = -AX_1, \quad v_3 = \frac{\partial x_3}{\partial t} = 0.$$

The same vector expressed in the spatial coordinate system is given by

$$\mathbf{v} = v_1(x_1, x_2, x_3, t)\hat{\mathbf{e}}_1 + v_2(x_1, x_2, x_3, t)\hat{\mathbf{e}}_2 + v_3(x_1, x_2, x_3, t)\hat{\mathbf{e}}_3$$

with

$$v_1 = AX_2 = A\left(\frac{x_2 + Atx_1}{1 + A^2t^2}\right), \quad v_2 = -AX_1 = -A\left(\frac{x_1 - Atx_2}{1 + A^2t^2}\right), \quad v_3 = 0.$$

The acceleration vector $\mathbf{a} = a_1\hat{\mathbf{E}}_1 + a_2\hat{\mathbf{E}}_2 + a_3\hat{\mathbf{E}}_3$ is a zero vector because

$$a_1 = \frac{\partial v_1(\mathbf{X}, t)}{\partial t} = 0, \quad a_2 = \frac{\partial v_2(\mathbf{X}, t)}{\partial t} = 0, \quad a_3 = \frac{\partial v_3(\mathbf{X}, t)}{\partial t} = 0.$$

Alternatively, a_1 can also be computed (but algebraically more involved) using the definition of the material time derivative as

$$\begin{aligned}
a_1 &= \frac{dv_1(\mathbf{x}, t)}{dt} = \frac{\partial v_1(\mathbf{x}, t)}{\partial t} + v_1\frac{\partial v_1}{\partial x_1} + v_2\frac{\partial v_1}{\partial x_2} + v_3\frac{\partial v_1}{\partial x_3} \\
&= \left[A\left(\frac{Ax_1}{1 + A^2t^2}\right) - 2A^3t\left(\frac{x_2 + Atx_1}{(1 + A^2t^2)^2}\right)\right] \\
&\quad + \left[v_1\left(\frac{A^2t}{1 + A^2t^2}\right) + v_2\left(\frac{A}{1 + A^2t^2}\right) + v_3 \cdot 0\right] \\
&= \frac{1}{(1 + A^2t^2)^2}[(A^2x_1 - A^4x_1t^2 - 2A^3tx_2) \\
&\quad + (2A^3tx_2 + A^4x_1t^2 - A^2x_1)] = 0.
\end{aligned}$$

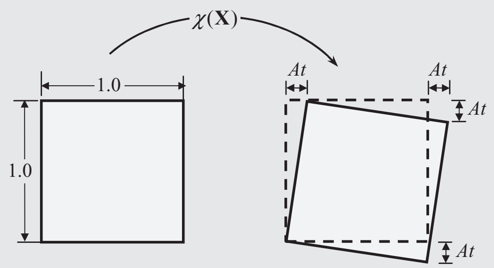


Figure 3.3.2 A sketch of the mapping as applied to a unit square.

(c) The time rate of change of temperature of a material particle in the body is simply

$$\frac{d}{dt}[\theta(\mathbf{X}, t)] = \frac{\partial}{\partial t}[\theta(\mathbf{X}, t)]\Big|_{\mathbf{X}=\text{fixed}} = -2AtX_1 + (1 + A)X_2.$$

On the other hand, the time rate of change of temperature at point $\mathbf{x}$, which is now occupied by particle X, is

$$\frac{d}{dt}[\theta(\mathbf{x}, t)] = \frac{\partial \theta}{\partial t} + v_i \frac{\partial \theta}{\partial x_i} = x_2 + v_1 \cdot 1 + v_2 \cdot t$$
$$= -2AtX_1 + (1 + A)X_2.$$

3.3.3 Displacement field

The phrase "deformation of a continuum" refers to relative displacements and changes in the geometry experienced by the continuum under the influence of a force system. The displacement of the particle X is given, as can be seen from Figure 3.3.3, by

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{x} - \mathbf{X}(\mathbf{x}, t). \tag{3.3.7}$$

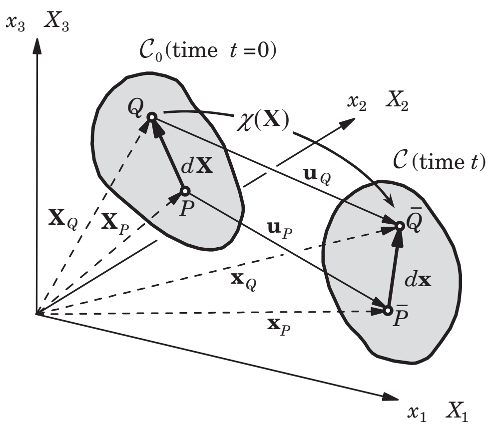


Figure 3.3.3 Points P and Q separated by a distance $d\mathbf{X}$ in the undeformed configuration $\mathcal{C}_0$ take up positions $\bar{P}$ and $\bar{Q}$, respectively, in the deformed configuration $\mathcal{C}$, where they are separated by distance $d\mathbf{x}$.

Displacements are of no interest in spatial descriptions but are used in fluid mechanics as the gradients of the time rate of change displacements (i.e., velocity gradients).

In the Lagrangian description used in solid mechanics, the displacements are expressed in terms of the material coordinates $\mathbf{X}$ and time t,

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X}. \tag{3.3.8}$$

For example, the displacement vector of the deformation given in Example 3.3.1 in the material coordinates X_i is given by

$$\begin{aligned}\mathbf{u}(\mathbf{X}, t) &= \mathbf{x} - \mathbf{X}\\ &= (X_1 + AtX_2)\hat{\mathbf{e}}_1 + (X_2 - AtX_1)\hat{\mathbf{e}}_2 + X_3\,\hat{\mathbf{e}}_3 - \big(X_1\,\hat{\mathbf{e}}_1 + X_2\,\hat{\mathbf{e}}_2 + X_3\,\hat{\mathbf{e}}_3\big)\\ &= AtX_2\hat{\mathbf{e}}_1 - AtX_1\hat{\mathbf{e}}_2.\end{aligned}$$

If the displacement of every particle in the body is known, we can construct the current configuration $\mathcal{C}$ from the reference configuration κ_0, $\mathbf{x} = \chi(\mathbf{X}, t) = \mathbf{X} + \mathbf{u}(\mathbf{X}, t)$.

The velocity and acceleration vectors $\mathbf{v}$ and $\mathbf{a}$ in a spatial description are given by Eq. (3.3.6). In the material description, the velocity and acceleration vectors are simply given by [note that $\mathbf{u} = \mathbf{u}(\mathbf{X}, t)$]

$$\mathbf{v}(\mathbf{X}, t) = \frac{\partial \mathbf{u}}{\partial t}, \quad \mathbf{a}(\mathbf{X}, t) = \frac{\partial \mathbf{v}}{\partial t} = \frac{\partial^2 \mathbf{u}}{\partial t^2}. \tag{3.3.9}$$

A *rigid body* is one in which all material particles of the continuum undergo the same linear and angular displacements. On the other hand, a *deformable body* is one in which the material particles can move relative to each other. The deformation of a continuum can be determined only by considering the change of distance between any two arbitrary but infinitesimally close points of the continuum.

3.4 Analysis of deformation

3.4.1* Deformation gradient tensor

The relationship of a material line $d\mathbf{X}$ before deformation to the line $d\mathbf{x}$ (consisting of the same material as $d\mathbf{X}$) after deformation is provided by *deformation gradient* tensor $\mathbf{F}$. It is one of the most important quantities in the kinematic description, and hence discussed next.

Because $x_1 = x_1(X_1, X_2, X_3), x_2 = x_2(X_1, X_2, X_3)$, and $x_3 = x_3(X_1, X_2, X_3)$, we have

$$\begin{aligned} dx_1 &= \frac{\partial x_1}{\partial X_1} dX_1 + \frac{\partial x_1}{\partial X_2} dX_2 + \frac{\partial x_1}{\partial X_3} dX_3, \\ dx_2 &= \frac{\partial x_2}{\partial X_1} dX_1 + \frac{\partial x_2}{\partial X_2} dX_2 + \frac{\partial x_2}{\partial X_3} dX_3, \\ dx_3 &= \frac{\partial x_3}{\partial X_1} dX_1 + \frac{\partial x_3}{\partial X_2} dX_2 + \frac{\partial x_3}{\partial X_3} dX_3. \end{aligned} \tag{3.4.1}$$

These three relations can be written in matrix notation as

$$\begin{Bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{Bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \begin{Bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{Bmatrix} \equiv [F] \begin{Bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{Bmatrix}, \tag{3.4.2}$$

where $[F]$ is the matrix associated with the deformation gradient tensor $\mathbf{F}$,

$$[F] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix}. \tag{3.4.3}$$

Equations (3.4.1) through (3.4.3) can be expressed succinctly using index notation or vector and tensor notation. In index notation, Eqs. (3.4.) through (3.4.3) take the form

$$dx_i = \frac{\partial x_i}{\partial X_J} dX_J \equiv F_{iJ}\, dX_J, \qquad F_{iJ} = \frac{\partial x_i}{\partial X_J} = x_{i,J}. \tag{3.4.4}$$

In vector and tensor notation, we have [recall the backward gradient defined in Eq. (2.6.6)]

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} = d\mathbf{X} \cdot \mathbf{F}^{\mathrm{T}}, \ \ \mathbf{F} = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right)^{\mathrm{T}} \equiv (\nabla_0 \mathbf{x})^{\mathrm{T}}, \tag{3.4.5}$$

where ∇_0 is the gradient operator with respect to $\mathbf{X}$,

$$\nabla_0 = \hat{\mathbf{E}}_i \frac{\partial}{\partial X_i} = \hat{\mathbf{E}}_1 \frac{\partial}{\partial X_1} + \hat{\mathbf{E}}_2 \frac{\partial}{\partial X_2} + \hat{\mathbf{E}}_3 \frac{\partial}{\partial X_3}. \tag{3.4.6}$$

By definition, $\mathbf{F}$ is a second-order tensor. The definition of $\mathbf{F}$ given in Eq. (3.4.5) may appear as the transpose of that defined in other books but these books use the backward gradient operator without explicitly stating such. We note that $\mathbf{F} \equiv (\nabla_0 \mathbf{x})^{\mathrm{T}} = F_{iJ}\, \hat{\mathbf{e}}_i\, \hat{\mathbf{E}}_J$, where $\nabla_0 \mathbf{x} = \hat{\mathbf{E}}_J \frac{\partial x_i}{\partial X_J} \hat{\mathbf{e}}_i = F_{iJ}\, \hat{\mathbf{E}}_J\, \hat{\mathbf{e}}_i = \mathbf{F}^{\mathrm{T}}$, and the definition of F_{iJ} used here and in other books is the same.

The equation $\mathbf{F} \cdot d\mathbf{X} = 0$ for $d\mathbf{X} \neq 0$ implies that a material line in the reference configuration is reduced to zero by the deformation. As this is physically not realistic, we conclude that $\mathbf{F} \cdot d\mathbf{X} \neq 0$ for $d\mathbf{X} \neq 0$, that is, $\mathbf{F}$ is a nonsingular tensor, $J \neq 0$. Hence, $\mathbf{F}$ has an inverse $\mathbf{F}^{-1}$. The deformation gradient can be expressed in terms of the displacement vector as

$$\mathbf{F} = (\nabla_0 \mathbf{x})^{\mathrm{T}} = (\nabla_0 \mathbf{u} + \mathbf{I})^{\mathrm{T}} \quad \text{or} \quad \mathbf{F}^{-1} = (\nabla \mathbf{X})^{\mathrm{T}} = (\mathbf{I} - \nabla \mathbf{u})^{\mathrm{T}}. \tag{3.4.7}$$

In general, the deformation gradient $\mathbf{F}$ is a function of $\mathbf{X}$. If $\mathbf{F}$=$\mathbf{I}$ everywhere in the body, then the body is not rotated and is undeformed. If $\mathbf{F}$ has the same value at every material point in a body (i.e., $\mathbf{F}$ is independent of $\mathbf{X}$), then the mapping $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ is said to be a *homogeneous* motion of the body and the deformation is said to be homogeneous. The determinant of $\mathbf{F}$ is called the *Jacobian* of the motion, and it is denoted by $J = \det \mathbf{F}$.

The next example illustrates the computation of the components of the deformation gradient tensor and displacement vector from known deformation mapping.

Example 3.4.1:

Consider the uniform deformation of a square block with sides of two units and initially centered at $\mathbf{X} = (0, 0)$. The deformation is defined by the mapping (not time-dependent)

$$\chi(\mathbf{X}) = \left(3.5 + X_1 + 0.5X_2\right) \hat{\mathbf{e}}_1 + \left(4 + X_2\right) \hat{\mathbf{e}}_2 + X_3\, \hat{\mathbf{e}}_3.$$

Determine the deformation gradient tensor $\mathbf{F}$, sketch the deformation, and compute the displacements.

Solution: From the given mapping, we have

$$x_1 = 3.5 + X_1 + 0.5X_2, \quad x_2 = 4 + X_2, \quad x_3 = X_3. \tag{a}$$

The displacement vector is given by

$$\mathbf{u} = \mathbf{x} - \mathbf{X} = \left(3.5 + 0.5X_2\right)\hat{\mathbf{e}}_1 + 4\, \hat{\mathbf{e}}_2 \; (u_1 = 3.5 + 0.5X_2, \; u_2 = 4, \; u_3 = 0). \tag{b}$$

The relations in (a) can be inverted to obtain

$$X_1 = -1.5 + x_1 - 0.5x_2, \quad X_2 = -4 + x_2, \quad X_3 = x_3. \tag{c}$$

Hence, the inverse mapping is given by

$$\mathbf{X} = \chi^{-1}(\mathbf{x}) = \left(-1.5 + x_1 - 0.5x_2\right) \hat{\mathbf{E}}_1 + \left(-4 + x_2\right) \hat{\mathbf{E}}_2 + x_3\, \hat{\mathbf{E}}_3, \tag{d}$$

which produces the deformed shape shown in Figure 3.4.1. This type of deformation is known as *simple shear*, in which there exists a set of line elements (in the present case, lines parallel to the X_1-axis) whose orientation is such that they are unchanged in length and orientation by the deformation.

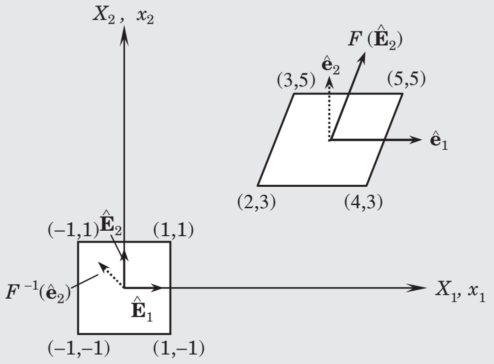


Figure 3.4.1 **Uniform deformation of a square.**

The components of the deformation gradient tensor and its inverse can be expressed in matrix form as

$$[F] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1.0 & 0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}, \quad [F]^{-1} = \begin{bmatrix} 1.0 & -0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}. \tag{e}$$

The unit vectors $\hat{\mathbf{E}}_1$ and $\hat{\mathbf{E}}_2$ in the initial configuration deform to the vectors

$$\begin{bmatrix} 1.0 & 0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \quad \begin{bmatrix} 1.0 & 0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.5 \\ 1.0 \\ 0.0 \end{Bmatrix}.$$

The unit vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ in the current configuration are deformed from the vectors

$$\begin{bmatrix} 1.0 & -0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \quad \begin{bmatrix} 1.0 & -0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -0.5 \\ 1.0 \\ 0.0 \end{Bmatrix}.$$

3.4.2* Various types of deformations

A mapping $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ is a homogeneous motion if and only if it can be expressed as

$$\mathbf{x} = \mathbf{A} \cdot \mathbf{X} + \mathbf{c}, \tag{3.4.8}$$

where the second-order tensor $\mathbf{A}$ and vector $\mathbf{c}$ are constants, and $\mathbf{c}$ represents a rigid-body translation. Note that for a homogeneous motion, we have $\mathbf{F} = \mathbf{A}$. If the Jacobian is unity, $J = 1$ (i.e., $\mathbf{A} = \mathbf{I}$), then the deformation is either a rigid rotation or the current and reference configurations coincide ($\mathbf{x} = \mathbf{X}$). If volume

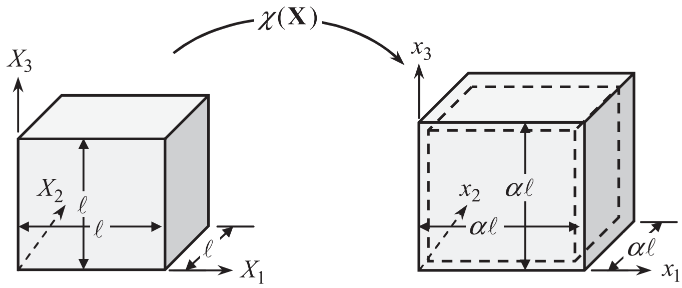


Figure 3.4.2

A deformation mapping of pure dilatation.

does not change at point $\mathbf{X}$ during the deformation, the deformation is called *isochoric* at $\mathbf{X}$. If $J = 1$ everywhere in the body, then the deformation of the body is said to be isochoric.

Several simple forms of homogeneous deformations are outlined next.

3.4.2.1 PURE DILATATION

If a cube of material has edges of length L and ℓ in the reference and current configurations, respectively, then the deformation mapping has the form

$$\chi(\mathbf{X}) = \lambda X_1\, \hat{\mathbf{e}}_1 + \lambda X_2\, \hat{\mathbf{e}}_2 + \lambda X_3\, \hat{\mathbf{e}}_3, \quad \lambda = \frac{L}{\ell}, \tag{3.4.9}$$

and $\mathbf{F}$ has the following matrix representation with respect to rectangular Cartesian coordinates:

$$[F] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}. \tag{3.4.10}$$

This deformation is known as *pure dilatation* or *pure stretch*, and it is isochoric if and only if $\lambda = 1$ (λ is called the principal stretch), as shown in Figure 3.4.2.

3.4.2.2 SIMPLE EXTENSION

An example of homogeneous extension in the X_1-direction is shown in Figure 3.4.3. The deformation mapping for this case is given by

$$\chi(\mathbf{X}) = (1+\alpha)X_1\, \hat{\mathbf{e}}_1 + X_2\, \hat{\mathbf{e}}_2 + X_3\, \hat{\mathbf{e}}_3. \tag{3.4.11}$$

The components of the deformation gradient tensor are given by

$$[F] = \begin{bmatrix} 1+\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.4.12}$$

For example, a line $X_2 = a + bX_1$ in the undeformed configuration transforms under the mapping to

$$x_2 = a + \frac{b}{1+\alpha} x_1$$

because $x_1 = (1+\alpha)X_1$, $x_2 = X_2$, and $x_3 = X_3$.

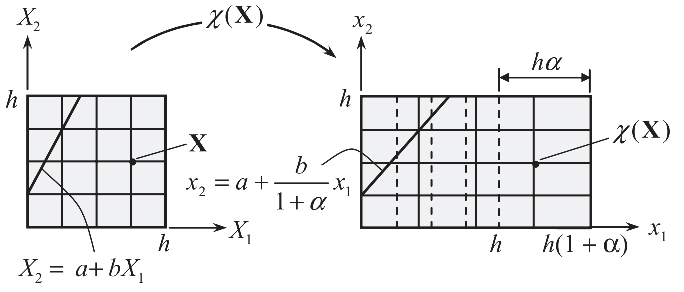


Figure 3.4.3 A deformation mapping of simple extension.

3.4.2.3 SIMPLE SHEAR

As discussed in Example 3.4.1, this deformation is defined to be one in which there exists a set of line elements whose lengths and orientations are unchanged, as shown in Figure 3.4.4. The deformation mapping in this case is

$$\chi(\mathbf{X}) = (X_1 + \gamma X_2)\hat{\mathbf{e}}_1 + X_2\,\hat{\mathbf{e}}_2 + X_3\,\hat{\mathbf{e}}_3. \tag{3.4.13}$$

The matrix representation of the deformation gradient tensor is

$$[F] = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.4.14}$$

where γ denotes the amount of shear.

3.4.2.4 NONHOMOGENEOUS DEFORMATION

As discussed previously, a nonhomogeneous deformation is one in which the deformation gradient $\mathbf{F}$ is a function of $\mathbf{X}$. An example of nonhomogeneous deformation mapping is provided, as shown in Figure 3.4.5, by

$$\chi(\mathbf{X}) = X_1(1 + \gamma_1 X_2)\hat{\mathbf{e}}_1 + X_2(1 + \gamma_2 X_1)\hat{\mathbf{e}}_2 + X_3\,\hat{\mathbf{e}}_3. \tag{3.4.15}$$

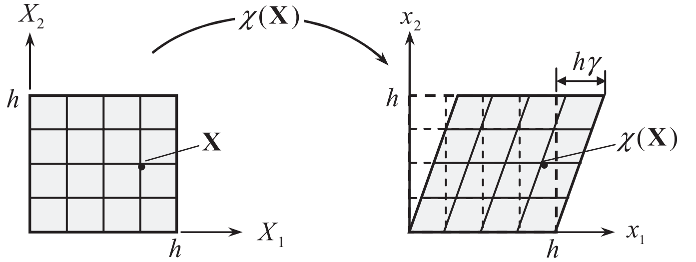


Figure 3.4.4 A deformation mapping of simple shear.

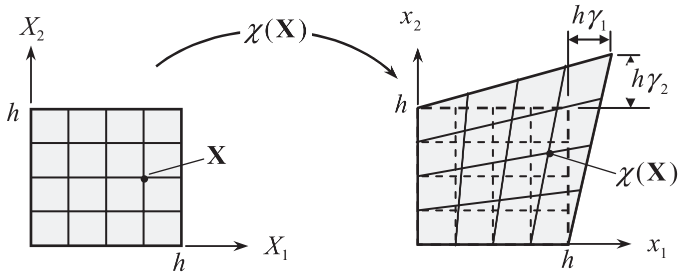


Figure 3.4.5 A deformation mapping of combined shearing and extension.

The matrix representation of the deformation gradient tensor is

$$[F] = \begin{bmatrix} 1+\gamma_1 X_2 & \gamma_1 X_1 & 0 \\ \gamma_2 X_2 & 1+\gamma_2 X_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.4.16}$$

3.4.3 Green strain tensor

The geometric changes that a solid continuum experiences can be measured in a number of ways. Here we discuss a general measure of deformation of a solid body. Consider two material particles P and Q in the neighborhood of each other, separated by $d\mathbf{X}$ in the reference configuration, as shown in Figure 3.4.6. In the current (deformed) configuration, the material points P and Q occupy positions $\bar{P}$ and $\bar{Q}$ and are separated by $d\mathbf{x}$. We wish to determine the change in the distance $d\mathbf{X}$ between the material points P and Q as the body deforms and the material points move to the new locations $\bar{P}$ and $\bar{Q}$.

The distances between points P and Q and points $\bar{P}$ and $\bar{Q}$ are given, respectively, by

$$\begin{aligned} (dS)^2 &= d\mathbf{X}\cdot d\mathbf{X} = dX_i dX_i = (dX_1)^2 + (dX_2)^2 + (dX_3)^2, \\ (ds)^2 &= d\mathbf{x}\cdot d\mathbf{x} = dx_i dx_i = (dx_1)^2 + (dx_2)^2 + (dx_3)^2. \end{aligned} \tag{3.4.17}$$

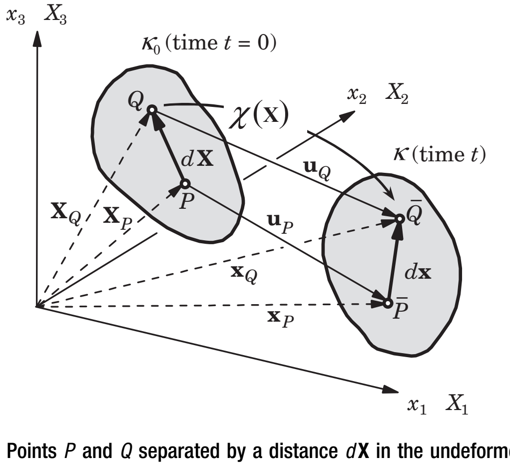


Figure 3.4.6 Points P and Q separated by a distance $d\mathbf{X}$ in the undeformed configuration κ_0 take up positions $\bar{P}$ and $\bar{Q}$, respectively, in the deformed configuration κ, where they are separated by distance $d\mathbf{x}$.

The change in the squared lengths that occurs as a body deforms from the reference configuration to the current configuration can be expressed relative to the original length as

$$
\begin{aligned}
(ds)^2 - (dS)^2 &= d\mathbf{x} \cdot d\mathbf{x} - d\mathbf{X} \cdot d\mathbf{X} = d\mathbf{X} \cdot (\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}) \cdot d\mathbf{X} - d\mathbf{X} \cdot d\mathbf{X} \\
&\equiv 2\, d\mathbf{X} \cdot \mathbf{E} \cdot d\mathbf{X},
\end{aligned}
\tag{3.4.18}
$$

where Eq. (3.4.5) is used in arriving at the last expression of the first line in Eq. (3.4.18), and

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} - \mathbf{I}\right) = \frac{1}{2}\left[(\mathbf{I} + \nabla_0 \mathbf{u}) \cdot (\mathbf{I} + \nabla_0 \mathbf{u})^{\mathrm{T}} - \mathbf{I}\right] \\
&= \frac{1}{2}\left[\nabla_0 \mathbf{u} + (\nabla_0 \mathbf{u})^{\mathrm{T}} + (\nabla_0 \mathbf{u}) \cdot (\nabla_0 \mathbf{u})^{\mathrm{T}}\right],
\end{aligned}
\tag{3.4.19}
$$

where Eq. (3.4.7) is used to express $\mathbf{E}$ in terms of the displacement gradient $\nabla_0 \mathbf{u}$. By definition, $\mathbf{E}$ is a symmetric second-order tensor[2] called the *Green-St. Venant (Lagrangian) strain tensor*, or simply the *Green strain tensor*. We note that the change in the squared lengths is zero if and only if $\mathbf{E} = \mathbf{0}$. Also, $\mathbf{E}$ is a nonlinear function of the displacement gradient because of the third term in Eq. (3.4.19).

In index notation, the rectangular Cartesian components of $\mathbf{E}$ are given by

$$
E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right),
\tag{3.4.20}
$$

or more explicitly, we have

$$
\begin{aligned}
E_{11} &= \frac{\partial u_1}{\partial X_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_1}\right)^2 + \left(\frac{\partial u_2}{\partial X_1}\right)^2 + \left(\frac{\partial u_3}{\partial X_1}\right)^2\right], \\
E_{22} &= \frac{\partial u_2}{\partial X_2} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_2}\right)^2 + \left(\frac{\partial u_2}{\partial X_2}\right)^2 + \left(\frac{\partial u_3}{\partial X_2}\right)^2\right], \\
E_{33} &= \frac{\partial u_3}{\partial X_3} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_3}\right)^2 + \left(\frac{\partial u_2}{\partial X_3}\right)^2 + \left(\frac{\partial u_3}{\partial X_3}\right)^2\right], \\
E_{12} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_2}\right), \\
E_{13} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_3}\right), \\
E_{23} &= \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} + \frac{\partial u_1}{\partial X_2}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_2}\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\frac{\partial u_3}{\partial X_3}\right).
\end{aligned}
\tag{3.4.21}
$$

The components E_{11}, E_{22}, and E_{33} are called normal strains and E_{12}, E_{23}, and E_{13} are called shear strains. The components of $\mathbf{E}$ in other coordinate systems can be obtained using the definition of Eq. (3.4.19) and writing ∇_0 and $\mathbf{u}$ in that

[2] The reader should not confuse the symbol $\mathbf{E}$ used for the Lagrangian strain tensor and $\mathbf{E}_i$ used for the basis vectors in the reference configuration. One should always pay attention to different typefaces and subscripts used.

coordinate system. The Green strain components associated with the deformations discussed in Sections 3.4.2.1 through 3.4.2.4 are as follows:

Pure dilatation:

$$[E] = \frac{1}{2}\begin{bmatrix} \lambda^2 - 1 & 0 & 0 \\ 0 & \lambda^2 - 1 & 0 \\ 0 & 0 & \lambda^2 - 1 \end{bmatrix}. \tag{3.4.22}$$

Simple extension:

$$[E] = \frac{1}{2}\begin{bmatrix} 2\alpha + \alpha^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.4.23}$$

Simple shear:

$$[E] = \frac{1}{2}\begin{bmatrix} 0 & \gamma & 0 \\ \gamma & \gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.4.24}$$

Nonhomogeneous deformation:

$$[E]=\frac{1}{2}\begin{bmatrix} 2\gamma_1 X_2 + (\gamma_1^2 + \gamma_2^2)X_2^2 & \gamma_1 X_1 + \gamma_2 X_2 + (\gamma_1^2 + \gamma_2^2)X_1 X_2 & 0 \\ \gamma_1 X_1 + \gamma_2 X_2 + (\gamma_1^2 + \gamma_2^2)X_1 X_2 & 2\gamma_2 X_1 + (\gamma_1^2 + \gamma_2^2)X_1^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.4.25}$$

The next two examples illustrate the calculation of the Green strain tensor from a known mapping of the motion.

Example 3.4.2:

Consider the uniform deformation of a square block with sides of length 2 units and initially centered at $\mathbf{X} = (0, 0)$, as shown in Figure 3.4.7. If the deformation is defined by the mapping

$$\chi(\mathbf{X}) = \frac{1}{4}(18 + 4X_1 + 6X_2)\hat{\mathbf{e}}_1 + \frac{1}{4}(14 + 6X_2)\hat{\mathbf{e}}_2 + X_3\hat{\mathbf{e}}_3,$$

(a) sketch the deformed configuration κ of the body, and

(b) compute the Green strain tensor components (display them in matrix form).

Solution:

(a) Sketch of the deformed configuration of the body is shown in Figure 3.4.7.

(b) The displacement vector is given by ($X_3 = x_3$):

$$\mathbf{u} = \mathbf{x} - \mathbf{X} = \frac{1}{2}(9 + 3X_2)\hat{\mathbf{e}}_1 + \frac{1}{2}(7 + X_2)\hat{\mathbf{e}}_2$$

$$\left[u_1 = \frac{1}{2}(9 + 3X_2),\ u_2 = \frac{1}{2}(7 + X_2),\ u_3 = 0\right].$$

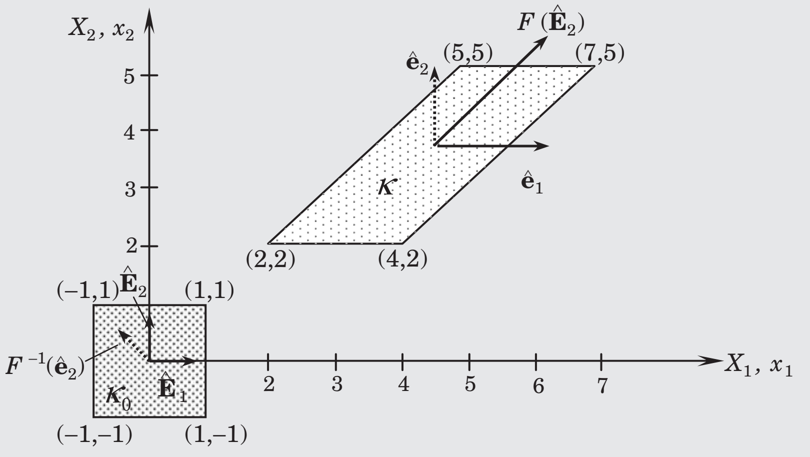


Figure 3.4.7 Undeformed and deformed configurations of a square block.

The only nonzero derivatives of the displacement components are

$$\frac{\partial u_1}{\partial X_2} = \frac{3}{2}, \quad \frac{\partial u_2}{\partial X_2} = \frac{1}{2}.$$

Substituting into the definitions of Eq. (3.4.21), we obtain

$$[E] = \frac{1}{4}\begin{bmatrix} 0 & 3 & 0 \\ 3 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Example 3.4.3:

We revisit the problem of deformation of the rectangular block of Example 3.2.2, where the strains were determined using the definition of engineering strains. Here we shall compute the strains using the mathematical definition of strains. Suppose that the rectangular block $ABCD$ is deformed into the diamond shape $\bar{A}\bar{B}\bar{C}\bar{D}$, as shown in Figure 3.4.8. Determine the deformation, displacements, and strains in the body.

Solution: By inspection, the deformation is clearly a homogeneous deformation. The geometry of the deformed body, which is a quadrilateral, can be determined as follows: Let (X_1, X_2, X_3) denote the coordinates of a material point in the undeformed configuration. The X_3-axis is taken out of the plane of the page and not shown in the figure.

The deformation is defined by the mapping $\chi(\mathbf{x}) = x_1\,\hat{\mathbf{e}}_1 + x_2\,\hat{\mathbf{e}}_2 + x_3\,\hat{\mathbf{e}}_3$, where

$$x_1 = A_0 + A_1X_1 + A_2X_2 + A_{12}X_1X_2,$$
$$x_2 = B_0 + B_1X_1 + B_2X_2 + B_{12}X_1X_2,$$
$$x_3 = X_3,$$

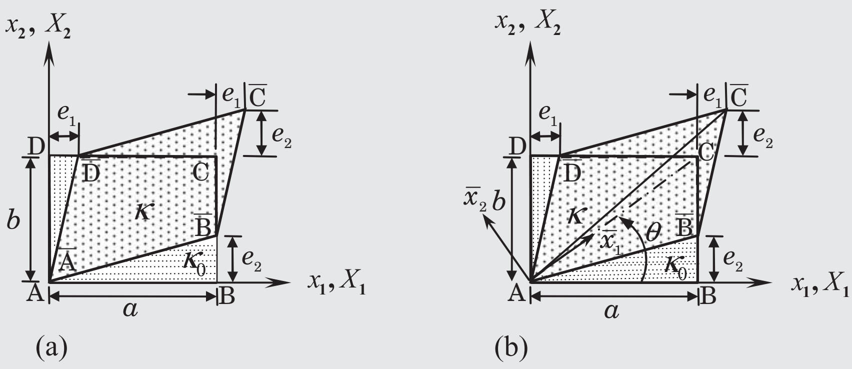


Figure 3.4.8 Undeformed and deformed configurations of a rectangular block.

and A_i and B_i are constants, which can be determined using the deformed configuration.

Using the known coordinates of the four vertex points of the quadrilateral, we can determine A_0, A_1, A_2, A_{12}, B_0, B_1, B_2, and B_{12} as follows:

$$
\begin{aligned}
&(X_1, X_2) = (0, 0), \quad (x_1, x_2) = (0, 0) \;\rightarrow\; A_0 = 0, \quad B_0 = 0,\\
&(X_1, X_2) = (a, 0), \quad (x_1, x_2) = (a, e_2) \;\rightarrow\; A_1 = 1, \quad B_1 = \frac{e_2}{a},\\
&(X_1, X_2) = (0, b), \quad (x_1, x_2) = (e_1, b) \;\rightarrow\; A_2 = \frac{e_1}{b}, \quad B_2 = 1,\\
&(X_1, X_2) = (a, b), \quad (x_1, x_2) = (a + e_1, b + e_2) \;\rightarrow\; A_{12} = 0, \quad B_{12} = 0.
\end{aligned}
$$

Thus, the deformation is defined by the transformation

$$
\chi(\mathbf{x}) = \big(X_1 + k_1 X_2\big)\hat{\mathbf{e}}_1 + \big(X_2 + k_2 X_1\big)\hat{\mathbf{e}}_2 + X_3\,\hat{\mathbf{e}}_3,
$$

where $k_1 = e_1/b$ and $k_2 = e_2/a$. Thus, the displacement vector of a material point in the Lagrangian description is

$$
\mathbf{u} = k_1 X_2\,\hat{\mathbf{e}}_1 + k_2 X_1\,\hat{\mathbf{e}}_2 \quad (u_1 = k_1 X_2, \;\; u_2 = k_2 X_1, \;\; u_3 = 0).
$$

The only nonzero Green strain tensor components are given by

$$
\begin{aligned}
E_{11} &= \frac{\partial u_1}{\partial X_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_1}\right)^2 + \left(\frac{\partial u_2}{\partial X_1}\right)^2 + \left(\frac{\partial u_3}{\partial X_1}\right)^2\right] = \frac{1}{2}k_2^2,\\
E_{12} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_2}\right) = \frac{k_1 + k_2}{2},\\
E_{22} &= \frac{\partial u_2}{\partial X_2} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_2}\right)^2 + \left(\frac{\partial u_2}{\partial X_2}\right)^2 + \left(\frac{\partial u_3}{\partial X_2}\right)^2\right] = \frac{1}{2}k_1^2.
\end{aligned}
$$

Note that the contribution of the linear terms to E_{11} and E_{22} is zero, whereas the contribution of the nonlinear terms to E_{12} is also zero.

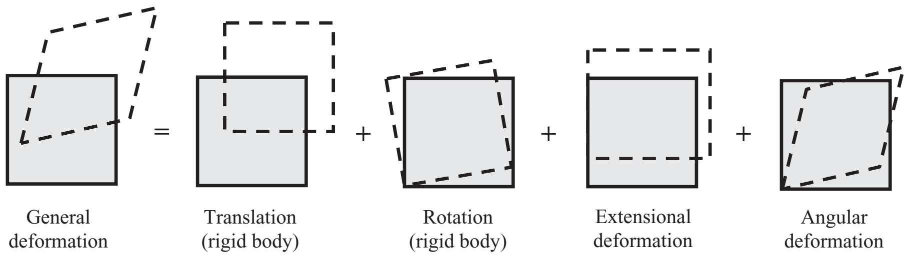


Figure 3.4.9 **Various types of motion and deformation of a continuum element.**

3.4.4 Infinitesimal strain tensor

When all displacement gradients are small (or infinitesimal), that is, $|\nabla \mathbf{u}| \ll 1$, we can neglect the nonlinear terms in the definition of the Green strain tensor defined in Eq. (3.4.20). In the case of infinitesimal strains, no distinction is made between the material coordinates $\mathbf{X}$ and the spatial coordinates $\mathbf{x}$ because the changes in the geometry are very small.

An infinitesimal element in the shape of a cube at a position in the undeformed continuum will move as the continuum moves and deforms. In general, the element will translate, rotate, and undergo a change in shape (i.e., the element deforms). The deformation itself involves extensional and rotational changes of line elements in the cube. Although these movements and deformation occur simultaneously, we can superpose individual changes to obtain the deformed shape in the case of infinitesimal motion and deformation of the cube, as depicted in Figure 3.4.9.

The *infinitesimal strain tensor* is denoted by ε and is defined by the linear part of the Green strain tensor $\mathbf{E}$,

$$\varepsilon = \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}}\right]. \tag{3.4.26}$$

The rectangular Cartesian components of the infinitesimal strain tensor are given by (x_i and X_i are interchangeable for the infinitesimal strains)

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) \tag{3.4.27}$$

or in expanded form,

$$\begin{aligned}
\varepsilon_{11} &= \frac{\partial u_1}{\partial x_1}, & \varepsilon_{22} &= \frac{\partial u_2}{\partial x_2},\\
\varepsilon_{33} &= \frac{\partial u_3}{\partial x_3}, & \varepsilon_{12} &= \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right),\\
\varepsilon_{13} &= \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right), & \varepsilon_{23} &= \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right).
\end{aligned} \tag{3.4.28}$$

The strain components ε_{11}, ε_{22}, and ε_{33} are the infinitesimal normal strains and ε_{12}, ε_{13}, and ε_{23} are the infinitesimal shear strains. The shear strains $\gamma_{12} = 2\varepsilon_{12}$,

$\gamma_{13} = 2\varepsilon_{13}$, and $\gamma_{23} = 2\varepsilon_{23}$ are called the *engineering shear strains*. The physical meaning of the strains was discussed in Section 3.2.

If the coordinate system chosen is labeled as (x, y, z) and the respective components of the displacement vector are denoted as (u_x, u_y, u_z), the infinitesimal strains in Eq. (3.4.28) take the form

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u_x}{\partial x}, & \varepsilon_{yy} &= \frac{\partial u_y}{\partial y},\\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z}, & \varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right),\\
\varepsilon_{xz} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right), & \varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right).
\end{aligned} \tag{3.4.29}$$

In the cylindrical coordinate system, the displacement vector and ∇ are expressed as

$$\begin{aligned}
\mathbf{u} &= u_r \hat{\mathbf{e}}_r + u_\theta \hat{\mathbf{e}}_\theta + u_z \hat{\mathbf{e}}_z,\\
\nabla &= \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{1}{r}\hat{\mathbf{e}}_\theta \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z},\\
\frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} &= \hat{\mathbf{e}}_\theta, \quad \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} = -\hat{\mathbf{e}}_r\,.
\end{aligned} \tag{3.4.30}$$

Use of Eq. (3.4.30) in Eq. (3.4.26) yields the following infinitesimal strain tensor components in the cylindrical coordinate system:

$$\begin{aligned}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r}, & \varepsilon_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right),\\
\varepsilon_{rz} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right), & \varepsilon_{\theta\theta} &= \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta},\\
\varepsilon_{z\theta} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right), & \varepsilon_{zz} &= \frac{\partial u_z}{\partial z}\,.
\end{aligned} \tag{3.4.31}$$

Example 3.4.4:

Consider an open hollow circular cylinder of internal radius a and outside radius b. Assume that the cylinder is made of homogeneous and isotropic material.[3] If the cylinder is subjected at $r = b$ to a uniform load of p_b, determine the strains in terms of the displacement field experienced by the cylinder.

Solution: It is convenient to use the cylindrical coordinate system (r, θ, z) with the origin at the center of the cylinder. Due to the axisymmetric (about the z-axis) geometry, loads applied, and material, the cylinder experiences axisymmetric deformation. Therefore, the displacement field is of the form (i.e., the cylinder experiences only radial displacement independent of θ and z)

$$u_r = u_r(r), \quad u_\theta = u_z = 0. \tag{3.4.32}$$

[3] The words "homogeneous" and "isotropic" will be explained in Chapter 6, but for the moment assume that the material properties do not vary with position or direction.

The strains associated with the displacement field of Eq. (3.4.32) can be obtained using Eq. (3.4.31) and noting that all derivatives with respect to θ and z are zero:

$$\varepsilon_{rr} = \frac{du_r}{dr}, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r}, \quad \varepsilon_{zz} = 0, \quad \varepsilon_{r\theta} = 0, \quad \varepsilon_{z\theta} = 0, \quad \varepsilon_{rz} = 0. \qquad (3.4.33)$$

3.4.5 Principal values and principal planes of strains

It is of interest to inquire whether there are certain vectors $\mathbf{x}$ that have only their lengths, and not their orientation, changed when operated upon by a given *tensor* $\mathbf{A}$ – that is, to seek vectors that are transformed into multiples of themselves. If such vectors exist, they must satisfy the equation

$$\mathbf{A} \cdot \mathbf{x} = \lambda \mathbf{x}. \qquad (3.4.34)$$

Such vectors $\mathbf{x}$ are called *characteristic vectors*, or *eigenvectors*, associated with $\mathbf{A}$. The parameter $\boldsymbol{\lambda}$ is called the *characteristic value*, or *eigenvalue*, and it characterizes the change in length of the eigenvector $\mathbf{x}$ after it has been operated upon by $\mathbf{A}$.

Because $\mathbf{x}$ can be expressed as $\mathbf{x} = \mathbf{I} \cdot \mathbf{x}$, Eq. (3.4.34) can also be written as

$$(\mathbf{A} - \lambda \mathbf{I}) \cdot \mathbf{x} = \mathbf{0}. \qquad (3.4.35)$$

Because this is a homogeneous set of equations for $\mathbf{x}$, a nontrivial solution (that is, a vector with at least one component of $\mathbf{x}$ that is nonzero) will not exist unless the determinant of the matrix $[\mathbf{A} - \lambda \mathbf{I}]$ vanishes,

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0. \qquad (3.4.36)$$

The vanishing of this determinant yields an algebraic equation of degree n, called the *characteristic equation*, for λ when $\mathbf{A}$ is an $n \times n$ matrix.

For the strain tensor, which is a second-order tensor, finding its eigenvectors amounts to finding the planes on which the normal strains are the maximum (and the shear strains are zero). Because the strain tensor is represented by a 3×3 matrix, the characteristic equation yields three eigenvalues, λ_1, λ_2, and λ_3. The eigenvalues of a strain tensor are called the *principal strains*, and the corresponding eigenvectors are called the *principal planes*. The following example illustrates the calculation of the principal strains and principal planes.

Example 3.4.5:

The state of strain at a point in an elastic body is given by

$$\varepsilon = \begin{bmatrix} 4 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} (10^{-3} \text{ in./in.}).$$

Determine the principal strains and principal directions of the strain.

Solution: Setting $|[\varepsilon] - \lambda[I]| = 0$, we obtain

$$(4-\lambda)[(-\lambda)(3-\lambda)-0] + 4[-4(3-\lambda)] = 0 \rightarrow [(4-\lambda)\lambda + 16](3-\lambda) = 0.$$

We see that $\lambda_1 = 3$ is an eigenvalue of the matrix. The remaining two eigenvalues are obtained from $\lambda^2 - 4\lambda - 16 = 0$. Thus, the principal strains are (10^{-3} in./in.)

$$\lambda_1 = 3, \quad \lambda_2 = 2(1+\sqrt{5}), \quad \lambda_3 = 2(1-\sqrt{5}).$$

The eigenvector components x_i associated with $\varepsilon_1 = \lambda_1 = 3$ are calculated from

$$\begin{bmatrix} 4-3 & -4 & 0 \\ -4 & 0-3 & 0 \\ 0 & 0 & 3-3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$

which gives $x_1 - 4x_2 = 0$ and $-4x_1 - 3x_2 = 0$, or $x_1 = x_2 = 0$. Using the normalization $x_1^2 + x_2^2 + x_3^2 = 1$, we obtain $x_3 = 1$. Thus, the principal direction associated with the principal strain $\varepsilon_1 = 3$ is $\hat{\mathbf{x}}^{(1)} = \pm(0, 0, 1)$.

The eigenvector components associated with principal strain $\varepsilon_2 = \lambda_2 = 2(1+\sqrt{5})$ are calculated from

$$\begin{bmatrix} 4-\lambda_2 & -4 & 0 \\ -4 & 0-\lambda_2 & 0 \\ 0 & 0 & 3-\lambda_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$

which gives

$$x_1 = -\frac{2+2\sqrt{5}}{4} x_2 = -1.618 x_2, \quad x_3 = 0, \rightarrow \hat{\mathbf{x}}^{(2)} = \pm(-0.851, 0.526, 0).$$

Similarly, the eigenvector components associated with principal strain $\varepsilon_3 = \lambda_3 = 2(1-\sqrt{5})$ are obtained as

$$x_1 = \frac{2+2\sqrt{5}}{4} x_2 = 1.618 x_2, \quad x_3 = 0, \rightarrow \hat{\mathbf{x}}^{(3)} = \pm(0.526, 0.851, 0).$$

The principal planes of strain are shown in Figure 3.4.10.

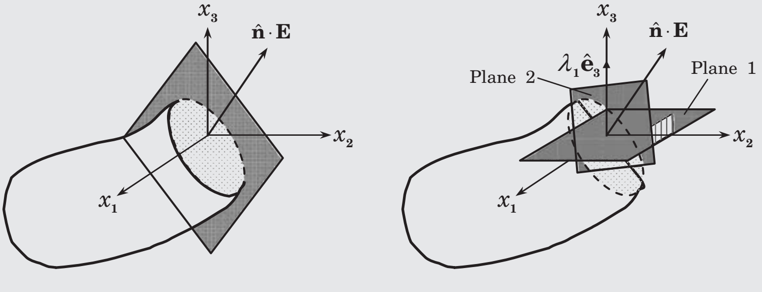


Figure 3.4.10 Principal planes 1 and 2 of strain.

3.5 Rate of deformation and vorticity tensors

3.5.1 Velocity gradient tensor

In fluid mechanics, velocity vector $\mathbf{v}(\mathbf{x}, t)$ is the variable of interest, as opposed to the displacement vector $\mathbf{u}$ in solid mechanics. This is because fluids respond to the time rate of deformation. In solid bodies, the symmetric part of the displacement gradient $\nabla\mathbf{u}$ determines the infinitesimal strains; see Eq. (3.4.26). On the other hand, in fluid mechanics the symmetric part of the velocity gradient tensor $\mathbf{L} \equiv (\nabla\mathbf{v})^{\mathrm{T}}$ determines the strain rates. The *velocity gradient* tensor $\mathbf{L}$ can be expressed as the sum of symmetric and antisymmetric parts [see Section 2.7.4 and Eq. (2.7.20)]:

$$\mathbf{L} = (\nabla\mathbf{v})^{\mathrm{T}} = \frac{1}{2}\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\right] - \frac{1}{2}\left[(\nabla\mathbf{v}) - (\nabla\mathbf{v})^{\mathrm{T}}\right] \equiv \mathbf{D} + \boldsymbol{\Omega}. \tag{3.5.1}$$

The symmetric part of the velocity gradient, denoted by $\mathbf{D}$, is called the *rate of deformation tensor* and the antisymmetric part, denoted by $\boldsymbol{\Omega}$, is called the *vorticity tensor*:

$$\mathbf{D} = \frac{1}{2}\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\right], \quad \boldsymbol{\Omega} = \frac{1}{2}\left[(\nabla\mathbf{v})^{\mathrm{T}} - \nabla\mathbf{v}\right]. \tag{3.5.2}$$

3.5.2 Rate of deformation tensor

The rectangular Cartesian components of the rate of deformation tensor $\mathbf{D}$ are given by

$$\begin{aligned}
\mathbf{D} &= \frac{1}{2}\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\right] = D_{ij}\,\hat{\mathbf{e}}_i\,\hat{\mathbf{e}}_j, & D_{ij} &= \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right), \\
D_{11} &= \frac{\partial v_1}{\partial x_1}, & D_{22} &= \frac{\partial v_2}{\partial x_2}, \\
D_{33} &= \frac{\partial v_3}{\partial x_3}, & D_{12} &= \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right), \\
D_{13} &= \frac{1}{2}\left(\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3}\right), & D_{23} &= \frac{1}{2}\left(\frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3}\right).
\end{aligned} \tag{3.5.3}$$

The components D_{11}, D_{22}, and D_{33} are the *normal strain rates* and D_{12}, D_{13}, and D_{23} are the *shear strain rates*. Expressed in terms of the velocity components in the (x, y, z) coordinates, Eq. (3.5.3) takes the form

$$\begin{aligned}
D_{xx} &= \frac{\partial v_x}{\partial x}, & D_{yy} &= \frac{\partial v_y}{\partial y}, \\
D_{zz} &= \frac{\partial v_z}{\partial z}, & D_{xy} &= \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right), \\
D_{xz} &= \frac{1}{2}\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right), & D_{yz} &= \frac{1}{2}\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right).
\end{aligned} \tag{3.5.4}$$

In the cylindrical coordinate system, we have

$$
\begin{aligned}
D_{rr} &= \frac{\partial v_r}{\partial r}, & D_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right),\\
D_{rz} &= \frac{1}{2}\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right), & D_{\theta\theta} &= \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta},\\
D_{z\theta} &= \frac{1}{2}\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right), & D_{zz} &= \frac{\partial v_z}{\partial z}.
\end{aligned}
\tag{3.5.5}
$$

A comparison of Eq. (3.5.3) with Eq. (3.4.26) through (3.4.28) shows that they are similar except that displacement vector **u** is the variable in solid mechanics and velocity vector **v** is the variable in fluid mechanics.

3.5.3 Vorticity tensor and vorticity vector

The vorticity tensor $\boldsymbol{\Omega}$ is the antisymmetric (or skew-symmetric) part of the velocity gradient tensor, also known as the *spin tensor*.

$$
\boldsymbol{\Omega} = \frac{1}{2}\left[(\nabla\mathbf{v})^{\mathrm{T}} - \nabla\mathbf{v}\right], \quad \Omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right). \tag{3.5.6}
$$

In expanded rectangular Cartesian component form, the vorticity components are

$$
\begin{gathered}
\Omega_{11} = 0, \quad \Omega_{22} = 0, \quad \Omega_{33} = 0,\\
\Omega_{12} = \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}\right), \quad \Omega_{13} = \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right), \quad \Omega_{23} = \frac{1}{2}\left(\frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2}\right).
\end{gathered}
\tag{3.5.7}
$$

Because $\boldsymbol{\Omega}$ is skew-symmetric (i.e., $\boldsymbol{\Omega}^{\mathrm{T}} = -\boldsymbol{\Omega}$), it has only three independent components,

$$
\boldsymbol{\Omega} = \begin{bmatrix} 0 & \Omega_{12} & \Omega_{13} \\ -\Omega_{12} & 0 & \Omega_{23} \\ -\Omega_{13} & -\Omega_{23} & 0 \end{bmatrix} \equiv \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}, \tag{3.5.8}
$$

where

$$
\begin{aligned}
\omega_1 &\equiv -\Omega_{23} = \Omega_{32} = \frac{1}{2}\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right),\\
\omega_2 &\equiv -\Omega_{31} = \Omega_{13} = \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right),\\
\omega_3 &\equiv -\Omega_{12} = \Omega_{21} = \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right).
\end{aligned}
\tag{3.5.9}
$$

These components can be used to define a vector $\boldsymbol{\omega}$, called the *axial vector* of $\boldsymbol{\Omega}$, as follows:

$$
\boldsymbol{\omega} = \frac{1}{2}\,\mathrm{curl}\,\mathbf{v} = \frac{1}{2}\nabla\times\mathbf{v}, \quad \omega_i = -\frac{1}{2}e_{ijk}\Omega_{jk} = \frac{1}{2}e_{ijk}\frac{\partial v_k}{\partial x_j}, \quad \text{and} \quad \Omega_{ij} = -e_{ijk}\,\omega_k. \tag{3.5.10}
$$

The vector $\boldsymbol{\omega}$ is also called the *rotation* vector, and twice the rotation vector, $\boldsymbol{\zeta} = 2\omega$, is called the *vorticity* vector. The flow is said to be *irrotational* if the vorticity vector is zero, $\boldsymbol{\zeta} = 0$.

Equations (3.5.9) and (3.5.10) can be expressed in terms of the velocity components in the (x, y, z) coordinates as

$$
\begin{aligned}
\boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{v} &= \frac{1}{2}\begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \\
&= \frac{1}{2}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{e}}_x + \frac{1}{2}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{e}}_y + \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{e}}_z \\
&\equiv \omega_x\hat{\mathbf{e}}_x + \omega_y\hat{\mathbf{e}}_y + \omega_z\hat{\mathbf{e}}_z,
\end{aligned} \tag{3.5.11}
$$

where

$$
\begin{aligned}
\omega_x &\equiv -\Omega_{yz} = \Omega_{zy} = \frac{1}{2}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right), \\
\omega_y &\equiv -\Omega_{zx} = \Omega_{xz} = \frac{1}{2}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right), \\
\omega_z &\equiv -\Omega_{xy} = \Omega_{yx} = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right).
\end{aligned} \tag{3.5.12}
$$

For a two-dimensional flow field, say in the xy-plane, we have $v_x = v_x(x, y)$, $v_y = v_y(x, y)$, and $v_z = 0$; consequently, ω_x and ω_y will always be zero.

In the cylindrical coordinate system, Eq. (3.5.12) takes the form (see Table 2.6.2 for curl of a vector in the cylindrical coordinate system)

$$
\omega = \omega_r\hat{\mathbf{e}}_r + \omega_\theta\hat{\mathbf{e}}_\theta + \omega_z\hat{\mathbf{e}}_z, \quad \mathbf{v} = v_r\hat{\mathbf{e}}_r + v_\theta\hat{\mathbf{e}}_\theta + v_z\hat{\mathbf{e}}_z \tag{3.5.13}
$$

$$
\omega_r = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right), \quad \omega_\theta = \frac{1}{2}\left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right), \quad \omega_z = \frac{1}{2r}\left[\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right]. \tag{3.5.14}
$$

Example 3.5.1:

Derive the strain rate-velocity relations and vorticity-velocity relations in two dimensions by considering a moving rectangular fluid element of dimensions dx and dy at time t that experiences deformation and becomes distorted at time $t + \Delta t$, as shown in Figure 3.5.1.

Solution: Referring to Figure 3.5.1, the translation of point A is given by $v_x\,dt$ and $v_y\,dt$ in the x- and y-directions, respectively. Corresponding rates of translations are v_x and v_y.

The extensional strain-rate in the x direction can be calculated using the definition, "the rate of increase in length per unit length of a line element parallel to the x-axis." We then have

$$
D_{xx}\,dt = \frac{[dx + (\partial v_x/\partial x)dx\,dt] - dx}{dx} = \frac{\partial v_x}{\partial x}\,dt \quad \text{or} \quad D_{xx} = \frac{\partial v_x}{\partial x}.
$$

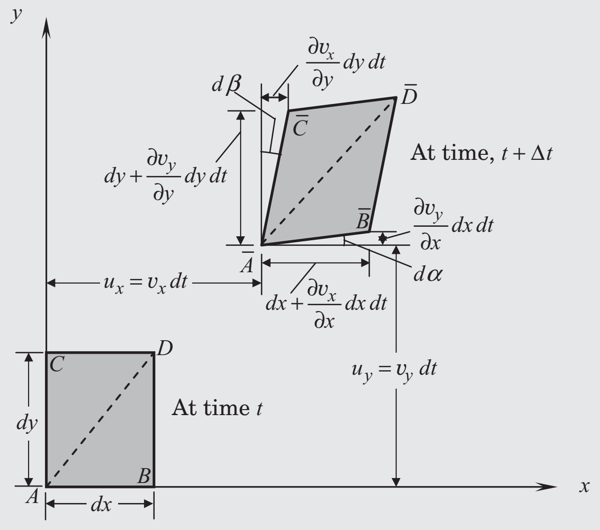


Figure 3.5.1 Strain rates in a moving fluid element.

Similarly, the extensional strain-rate in the y-direction is

$$D_{yy}\,dt = \frac{[dy + (\partial v_y/\partial y)dy\,dt] - dy}{dy} = \frac{\partial v_y}{\partial y}\,dt \quad \text{or} \quad D_{yy} = \frac{\partial v_y}{\partial y}.$$

The engineering shear strain rate can be calculated as the rate of change of the angle between line elements that were parallel to the x- and y-axes at time t. From Figure 3.5.1, we have

$$2D_{xy} = \frac{d\alpha}{dt} + \frac{d\beta}{dt} = \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y},$$

where

$$d\alpha = \tan^{-1}\left[\frac{(\partial v_y/\partial x)dx\,dt}{dx + (\partial v_x/\partial x)dx\,dt}\right] \approx \frac{\partial v_y}{\partial x}\,dt,$$

$$d\beta = \tan^{-1}\left[\frac{(\partial v_x/\partial y)dy\,dt}{dx + (\partial v_y/\partial y)dy\,dt}\right] \approx \frac{\partial v_x}{\partial y}\,dt.$$

Finally, the rotation of line AD about the z-axis is given by

$$\omega_z = -\Omega_{xy} = \frac{1}{2}\left(\frac{d\alpha}{dt} - \frac{d\beta}{dt}\right) = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right).$$

3.6 Compatibility equations

The task of computing strains (infinitesimal or finite) from a given displacement field is a straightforward exercise. However, sometimes we face the problem of finding the displacements from a given strain field. This is not as

straightforward due to the fact that there are six independent partial differential equations (i.e., strain-displacement relations) for only three unknown displacements, which would in general over-determine the solution. We will find some conditions, known as *St. Venant's compatibility equations*, that will ensure the computation of a unique displacement field from a given strain field. The derivation is presented here for infinitesimal strains.

To understand the meaning of strain compatibility, imagine that a material body is cut up into pieces before it is strained, and then each piece is given a certain strain. The strained pieces cannot be fitted back into a single continuous body without further deformation. On the other hand, if the strain in each piece is related to or compatible with the strains in the neighboring pieces, then they can be fitted together to form a continuous body. Mathematically, the six relations that connect six strain components to the three displacement components should be consistent.

To further explain this point, consider the two-dimensional state of deformation. We have three strain-displacement relations in two displacements:

$$\frac{\partial u_1}{\partial x_1} = \varepsilon_{11}, \tag{3.6.1}$$

$$\frac{\partial u_2}{\partial x_2} = \varepsilon_{22}, \tag{3.6.2}$$

$$\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 2\varepsilon_{12}. \tag{3.6.3}$$

If the given data $(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12})$ are compatible (or consistent), any two of the three equations should yield the same displacement components. The compatibility of the data can be established as follows. Differentiating the first equation with respect to x_2 twice, the second equation with respect to x_1 twice, and the third equation with respect to x_1 and x_2 each, we obtain

$$\frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} = \frac{\partial^2 \varepsilon_{11}}{\partial x_2^2}, \tag{3.6.1$'$}$$

$$\frac{\partial^3 u_2}{\partial x_2 \partial x_1^2} = \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2}, \tag{3.6.2$'$}$$

$$\frac{\partial^3 u_1}{\partial x_2^2 \partial x_1} + \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}. \tag{3.6.3$'$}$$

Using Eqs. (3.6.1$'$) and (3.6.2$'$) in (3.6.3$'$), we arrive at the following relation between the three strains:

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}. \tag{3.6.4}$$

Equation (3.6.4) is called the *strain compatibility condition* among the three strains $(\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{22})$ for a two-dimensional elasticity problem. For a three-dimensional problem, there are six compatibility conditions. They can be written compactly using index notation as

$$\frac{\partial^2 \varepsilon_{mn}}{\partial x_i \partial x_j} + \frac{\partial^2 \varepsilon_{ij}}{\partial x_m \partial x_n} = \frac{\partial^2 \varepsilon_{im}}{\partial x_j \partial x_n} + \frac{\partial^2 \varepsilon_{jn}}{\partial x_i \partial x_m}. \tag{3.6.5}$$

Equation (3.6.5) contains $(3)^4 = 81$ equations, but only six of them involving six strain components $(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23})$ are linearly independent. These conditions are both necessary and sufficient to determine a single-valued displacement field. Similar compatibility conditions hold for the rate of deformation tensor **D**. Note that the issue of compatibility does not arise when the displacements or velocities are given. It arises only when six strains are given in terms of three displacement components.

The next example illustrates how to determine if a given two-dimensional strain field is compatible.

Example 3.6.1:

Given the following two-dimensional, infinitesimal strain field:

$$\varepsilon_{11} = c_1 x_1 \left(x_1^2 + x_2^2\right), \quad \varepsilon_{22} = \frac{1}{3} c_2 x_1^3, \quad \varepsilon_{12} = c_3 x_1^2 x_2,$$

where c_1, c_2, and c_3 are constants, determine if the strain field is compatible.

Solution: Using Eq. (3.6.4), we obtain

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 2c_1 x_1 + 2c_2 x_1 - 4c_3 x_1.$$

Thus, the strain field is not compatible unless $c_1 + c_2 - 2c_3 = 0$.

3.7 Summary

In this chapter, the kinematics of motion and deformation are presented. Two descriptions of motion, namely the spatial (Eulerian) and material (Lagrange) descriptions, are discussed, and the deformation gradient tensor, the Green–Lagrange strain tensor, and several forms of homogeneous deformations are presented. Engineering and mathematical definitions of the normal and shear strain components are presented. The infinitesimal strain components and components of the deformation rate tensor and the vorticity vector are also introduced. Compatibility conditions on strains to ensure a unique determination of displacements from a given strain field are discussed.

PROBLEMS

3.1. A steel wire is connected to an immovable fixture at point A and wrapped around a lever arm at point B, as depicted in Figure P3.1. The lever arm is rotated clockwise by an angle of 6.5°. Determine the normal strain developed in the steel wire AB.

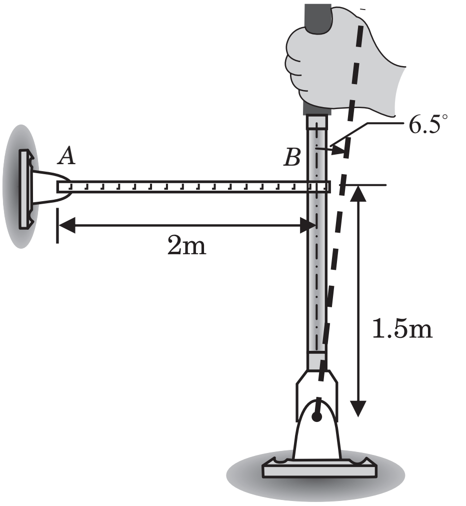


Figure P3.1

3.2. Wires BE and CF support a rigid bar $ABCD$, as shown in Figure P3.2. The rigid bar $ABCD$ is rotated clockwise by a force F through an angle of 5°. Determine the normal strains developed in wires BE and CF.

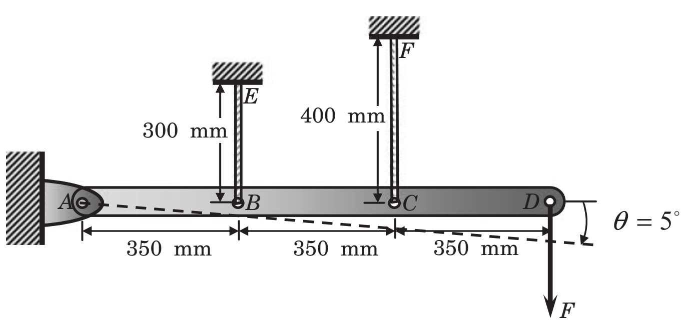


Figure P3.2

3.3. A steel wire is connected to an immovable fixture at point A and a lever arm at point C, as depicted in Figure P3.3. The lever arm is rotated clockwise by a force F through an angle of 5.75°. Determine the normal strain developed in the steel wire AC.

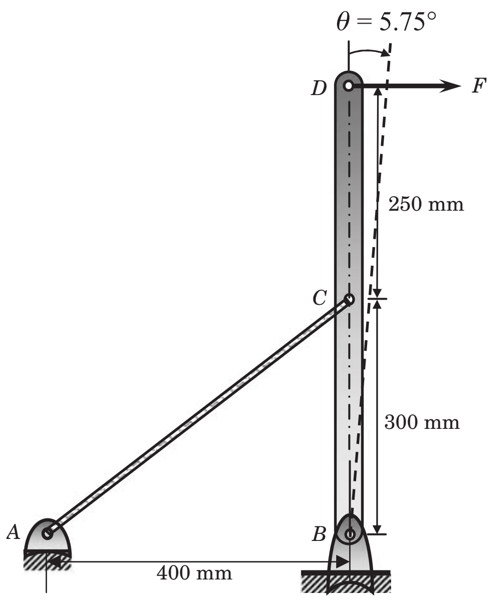


Figure P3.3

3.4. A rectangular block $ABCD$ is deformed into the shape shown in Figure P3.4 (dashed lines). Determine the normal strains in (a) the line segment parallel to side AB, (b) the line segment parallel to side BC, (c) the diagonal segment AC, and (d) the diagonal segment BD. What is the shear strain at point A?

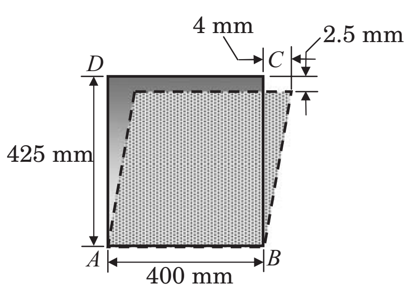


Figure P3.4

3.5. Suppose that the motion is described by the mapping

$$\mathbf{x} = (1 + t)\mathbf{X}.$$

Determine (a) the velocity and accelerations in the spatial and material descriptions, and (b) the time derivative of a function $\phi(X, t) = Xt^2$ in material description.

3.6. For a one-dimensional flow through a converging nozzle, the velocity distribution is given by

$$v_x(x) = U_0\left(1 + 2\frac{x}{L}\right),$$

where U_0 is the velocity at the entry and L is the length of the nozzle. Determine the acceleration dv_x/dt of the flow.

3.7. The motion of a body is described by the mapping

$$\chi(\mathbf{X}) = (X_1 + t^2 X_2)\,\hat{\mathbf{e}}_1 + (X_2 + t^2 X_1)\,\hat{\mathbf{e}}_2 + X_3\,\mathbf{e}_3,$$

where t denotes time. Determine

(a) the components of the deformation gradient tensor **F**, and

(b) the position (X_1, X_2, X_3) of the particle in an undeformed configuration that occupies the position $(x_1, x_2, x_3) = (9, 6, 1)$ at time $t = 2$ sec in the deformed configuration.

3.8. *Homogeneous stretch.* Consider a body with deformation mapping of the form

$$\chi(\mathbf{X}) = k_1 X_1\,\hat{\mathbf{e}}_1 + k_2 X_2\,\hat{\mathbf{e}}_2 + k_3 X_3\,\hat{\mathbf{e}}_3,$$

where k_i are constants. Determine the components of the deformation gradient tensor **F** and the Green–Lagrange strain tensor **E**.

3.9. *Homogeneous stretch followed by simple shear.* Consider a body with deformation mapping of the form

$$\chi(\mathbf{X}) = (k_1 X_1 + e_0 k_2 X_2)\,\hat{\mathbf{e}}_1 + k_2 X_2\,\hat{\mathbf{e}}_2 + k_3 X_3\,\hat{\mathbf{e}}_3,$$

where k_i and e_0 are constants. Determine the components of the deformation gradient tensor **F** and the Green–Lagrange strain tensor **E**.

3.10. Suppose that the motion of a continuous medium is given by

$$x_1 = X_1 \cos At + X_2 \sin At,$$
$$x_2 = -X_1 \sin At + X_2 \cos At,$$
$$x_3 = (1 + Bt)X_3,$$

where A and B are constants. Determine the components of

(a) the displacement vector in the material description,

(b) the displacement vector in the spatial description, and

(c) the Green–Lagrange strain tensor.

3.11. If the deformation mapping of a body is given by

$$\chi(\mathbf{X}) = (X_1 + AX_2)\,\hat{\mathbf{e}}_1 + (X_2 + BX_1)\,\hat{\mathbf{e}}_2 + X_3\,\hat{\mathbf{e}}_3,$$

where A and B are constants, determine

(a) the displacement components in the material description,

(b) the displacement components in the spatial description, and

(c) the components of the Green–Lagrange strain tensor.

3.12. The motion of a continuous medium is given by

$$x_1 = \frac{1}{2}(X_1 + X_2)e^t + \frac{1}{2}(X_1 - X_2)e^{-t},$$
$$x_2 = \frac{1}{2}(X_1 + X_2)e^t - \frac{1}{2}(X_1 - X_2)e^{-t},$$
$$x_3 = X_3.$$

Determine

(a) the velocity components in the material description,

(b) the velocity components in the spatial description, and

(c) the components of the rate of deformation and vorticity tensors.

3.13. Consider a square block of material of thickness h, as shown in Figure P3.13. If the material is subjected to the deformation mapping given in Eq. (3.5.8) with $\gamma_1 = 1$ and $\gamma_2 = 3$,

$$\chi(\mathbf{X}) = X_1(1 + X_2)\hat{\mathbf{e}}_1 + X_2(1 + 3X_1)\mathbf{e}_2 + X_3\,\hat{\mathbf{e}}_3,$$

(a) compute the components of the Green strain tensor $\mathbf{E}$ at the point $\mathbf{X} = (1, 1, 0)$, and

(b) the principal strains and directions associated with $\mathbf{E}$ at $\mathbf{X} = (1, 1, 0)$.

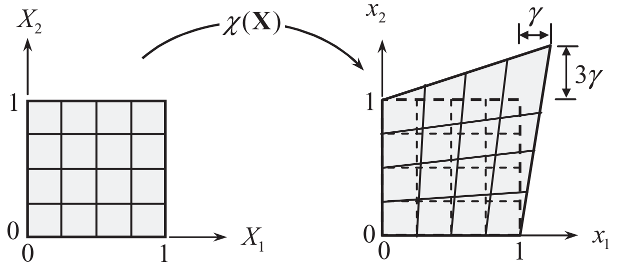


Figure P3.13

3.14. Determine the displacements and Green–Lagrange strain tensor components for the deformed configuration shown in Figure P3.14. The undeformed configuration is shown by the dashed lines.

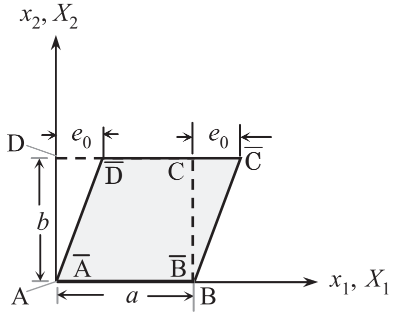


Figure P3.14

3.15. Determine the displacements and Green–Lagrange strain components for the deformed configuration shown in Figure P3.15. The undeformed configuration is shown by the dashed lines.

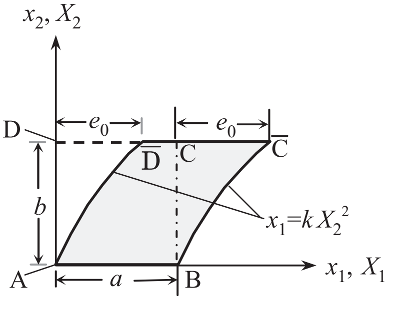


Figure P3.15

3.16. Determine the displacements and Green–Lagrange strains in the (x_1, x_2, x_3) system for the deformed configuration shown in Figure P3.16. The undeformed configuration is shown by the dashed lines.

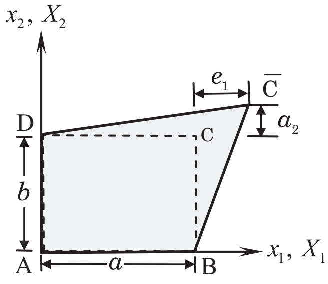
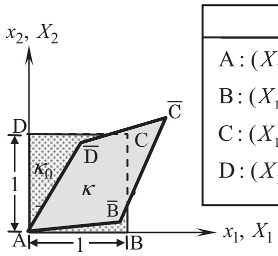


Figure P3.16

3.17. Determine the displacements and Green–Lagrange strains for the deformed configuration shown in Figure P3.17. The undeformed configuration is shown by the dashed lines.



κ_0	κ
A: $(X_1, X_2) = (0,0)$;	$\bar{A}$: $(x_1, x_2) = (0,0)$;
B: $(X_1, X_2) = (1,0)$;	$\bar{B}$: $(x_1, x_2) = (0.8, 0.2)$;
C: $(X_1, X_2) = (1,1)$;	$\bar{C}$: $(x_1, x_2) = (1.3, 1.2)$;
D: $(X_1, X_2) = (0,1)$.	$\bar{D}$: $(x_1, x_2) = (0.5, 0.9)$.

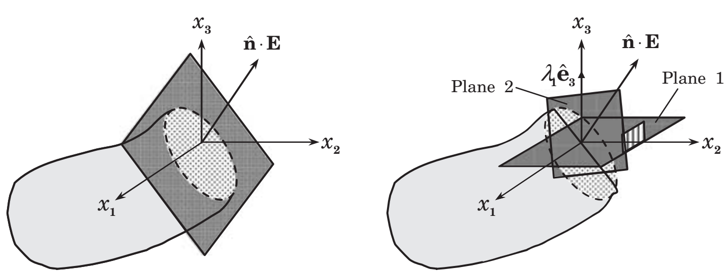


Figure P3.17

3.18. The two-dimensional displacement field in a body is given by

$$u_1 = x_1\left[x_1^2 x_2 + c_1\left(2c_2^3 + 3c_2^2 x_2 - x_2^3\right)\right],$$

$$u_2 = -x_2\left(2c_2^3 + \frac{3}{2}c_2^2 x_2 - \frac{1}{4}x_2^3 + \frac{3}{2}c_1 x_1^2 x_2\right),$$

where c_1 and c_2 are constants. Find the linear and nonlinear Green–Lagrange strains.

3.19. Determine whether the following strain fields are possible in a continuous body:

$$\text{(a) } [\varepsilon] = \begin{bmatrix} (x_1^2 + x_2^2) & x_1 x_2 \\ x_1 x_2 & x_2^2 \end{bmatrix}, \quad \text{(b) } [\varepsilon] = \begin{bmatrix} x_3(x_1^2 + x_2^2) & 2x_1x_2x_3 & x_3 \\ 2x_1x_2x_3 & x_2^2 & x_1 \\ x_3 & x_1 & x_3^2 \end{bmatrix}.$$

3.20. Determine the transformation relations between the components of strain ε_{ij} referred to in the rectangular Cartesian system (x_1, x_2, x_3) and components $\bar{\varepsilon}_{ij}$ referred to in another rectangular Cartesian system $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$. The two systems are such that the "barred" system is obtained by rotation of the unbarred system by θ about the x_3-axis (i.e., $\bar{x}_3 = x_3$) in the counterclockwise direction. *Hint:* Use the transformation in Eq. (2.7.15) for a second-order tensor to obtain the required relations. The transformation matrix **L** is given by

$$\mathbf{L} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3.21. Find the axial strain in the diagonal element of Problem 3.16 using (a) the engineering definition of normal strain, and (b) the strain transformation equations derived in Problem 3.20 to a point at $(X_1, X_2) = (a/2, b/2)$.

3.22. The biaxial state of strain at a point is given by $\varepsilon_{11} = 800 \times 10^{-6}$ in./in., $\varepsilon_{22} = 200 \times 10^{-6}$ in./in., and $\varepsilon_{12} = 400 \times 10^{-6}$ in./in. Find the principal strains and their directions, that is, determine the eigenvalues and eigenvectors associated with the strain tensor.

3.23. Given the following velocity field in a certain flow, determine the components of **D**, $\boldsymbol{\Omega}$, and the rotation vector $\boldsymbol{\omega}$:

$$v_x = x^2 + y^2 + z^2, \quad v_y = xy + yz + z^2, \quad v_z = 4 - 3xz - \frac{z^2}{2}.$$

3.24. Consider the following infinitesimal strain field:

$$\varepsilon_{11} = c_1 X_2^2, \quad \varepsilon_{22} = c_1 X_1^2, \quad 2\varepsilon_{12} = c_2 X_1 X_2,$$
$$\varepsilon_{31} = \varepsilon_{32} = \varepsilon_{33} = 0,$$

where c_1 and c_2 are constants. Determine c_1 and c_2 such that there exists a continuous, single-valued displacement field that corresponds to this strain field.

3.25. Given the infinitesimal strain components

$$\varepsilon_{11} = f(X_2, X_3), \quad \varepsilon_{22} = \varepsilon_{33} = -\nu f(X_2, X_3), \quad \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0,$$

determine the form of $f(X_2, X_3)$ so that the strain field is compatible. Here, ν denotes a constant (Poisson's ratio).

4 Stress Vector and Stress Tensor

A man may imagine things that are false, but he can only understand things that are true, for if the things be false, the apprehension of them is not understanding.
Isaac Newton

4.1 Introduction

In the beginning of Chapter 3, we briefly discussed the need to study deformation in materials that we may design for engineering applications. All materials have a certain threshold to withstand forces, beyond which they "fail" to perform their intended function. The force per unit area, called *stress*, is a measure of the capacity of the material to carry loads, and all designs are based on the criterion that the materials used have the capacity to carry the working loads of the system. Thus, it is necessary to determine the state of stress in materials that are used in a system.

In the present chapter, we study the concept of stress and its various measures. For instance, stress can be measured as a force (that occurs inside a deformed body) per unit deformed area or undeformed area. Stress at a point on the surface and at a point inside a three-dimensional continuum are measured using different entities. The stress at a point on the surface is measured in terms of force per unit area and depends on (magnitude and direction) the force vector as well as the plane on which the force is acting. Therefore, the stress defined at a point on the surface is a vector. As we shall see shortly, the stress at a point inside the body can be measured in terms of nine quantities, three per plane, on three mutually perpendicular planes at the point. These nine quantities may be viewed as the components of a second-order tensor, called a *stress tensor*. One may suspect that the stress vector defined at a point on the surface of a continuum is related to the stress tensor defined inside the continuum at the point. Such a relation, known as the Cauchy's formula, is derived here. Coordinate transformations and principal values associated with the stress tensor are also discussed in this chapter.

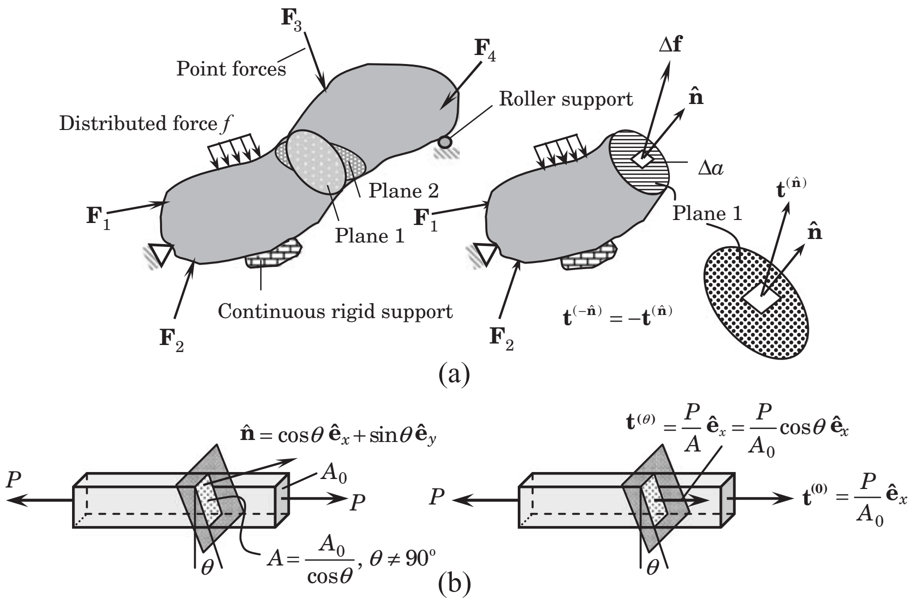


Figure 4.2.1 (a) A material body supported at various points on the surface and subjected to a number forces; shown are cuts through a point by planes of different orientation. (b) Stress vectors on a plane normal to the x-axis and on a plane with unit normal $\hat{\mathbf{n}}$ for an uniaxially loaded member.

4.2 Stress vector, stress tensor, and Cauchy's formula

First we introduce the true stress, that is, the force in the deformed configuration that is measured per unit area of the deformed configuration. The surface force acting on an element of area in a continuous medium depends not only on the magnitude of the area but also upon the orientation of the area. It is customary to denote the direction of a plane area of magnitude A by means of a unit vector drawn normal to that plane, as discussed in Section 2.3.8. The direction of the normal is taken by convention as that in which a right-handed screw advances as it is rotated according to the sense of travel along the boundary curve or contour. Let the unit normal vector be denoted by $\hat{\mathbf{n}}$; then the area is expressed as $\mathbf{A} = A\,\hat{\mathbf{n}}$.

If we denote by $\Delta\mathbf{f}(\hat{\mathbf{n}})$ the force on a small area Δa located at the position $\mathbf{x}$, the stress vector can be defined, shown graphically in Figure 4.2.1(a), as

$$\mathbf{t}(\hat{\mathbf{n}}) = \lim_{\Delta a \to 0} \frac{\Delta\mathbf{f}(\hat{\mathbf{n}})}{\Delta a}. \tag{4.2.1}$$

We see that the stress vector is a point function of the unit normal $\hat{\mathbf{n}}$, which denotes the orientation of the slant surface. Because of Newton's third law for action and reaction, we see that $\mathbf{t}(-\hat{\mathbf{n}}) = -\mathbf{t}(\hat{\mathbf{n}})$. The dependence of the stress vector $\mathbf{t}$ on the orientation of the plane on which it acts is further illustrated in Figure 4.2.1(b) with an uniaxiallyloaded member.

Example 4.2.1:

With reference to a rectangular Cartesian system (x_1, x_2, x_3), write stress vectors associated with the force system shown in Figure 4.2.2.

Solution: The stress (that is, the force per unit area) distribution on the top face of the block in Figure 4.2.2(a) is given by

$$S_1(x, y, z) = c_1 + c_2 x + c_3 y + c_4 xy,$$

where the constants c_1, c_2, c_3, and c_4 can be evaluated in terms of a, b, c, and d. Hence, the stress vector on the face is given by $\mathbf{t} = S_1\hat{\mathbf{e}}_y$. The stress on the right face of the block in Figure 4.2.2(a) is

$$S_2(x, y, z) = \sigma\left(1 - \frac{y}{h_2}\right).$$

Hence, the stress vector has the form $\mathbf{t} = S_2\hat{\mathbf{e}}_x$.

The stress on the slant face of the block in Figure 4.2.2(b) is given by

$$S(x, y, z) = a\frac{y}{h} = a\left(1 - \frac{x}{h}\right),$$

which is acting on the slant face at an angle of θ from the y-axis. Hence, the stress vector is given by

$$\mathbf{t} = S\big(\sin\theta\,\hat{\mathbf{e}}_x + \cos\theta\,\hat{\mathbf{e}}_y\big).$$

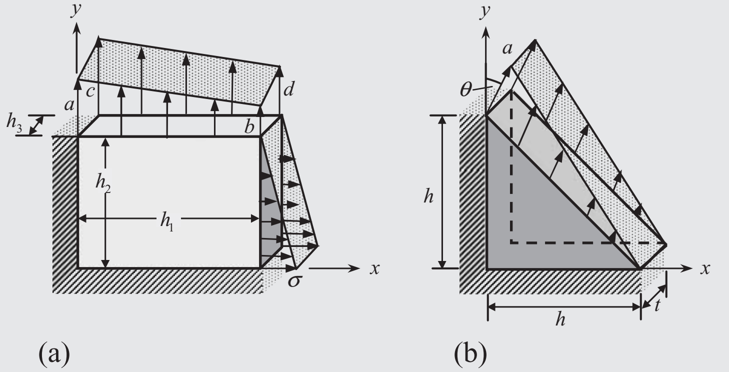


Figure 4.2.2 Stress vectors associated with a given force system.

Example 4.2.2:

Given the structure shown in Figure 4.2.3(a), determine the normal (i.e., axial) stresses in each member. What are the normal and shear stresses on the horizontal section through the inclined member?

Solution: First, we find the axial forces in each member. The members carry only axial forces because of the pin joints at points A, B, and C, as shown in Figure 4.2.3(b). The equilibrium of forces at joint B can be used to determine

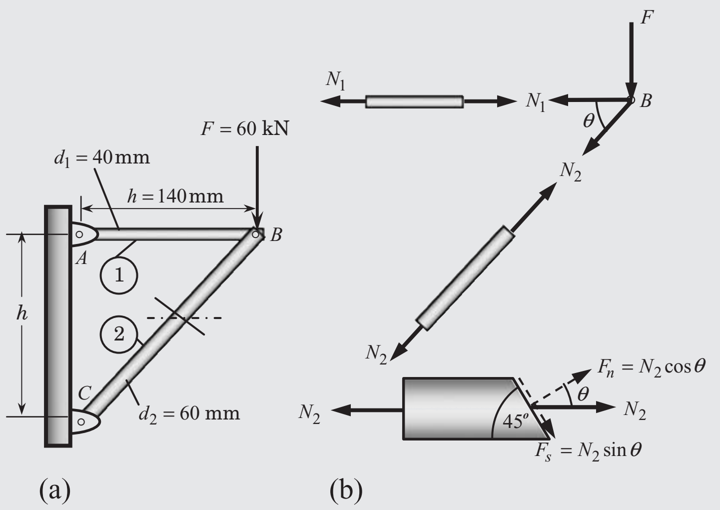


Figure 4.2.3 A structure with a pin joint at *B*.

the member forces. Summing forces horizontally and vertically, we obtain (for $\theta = 45^\circ$) the following axial forces:

$$N_1 + N_2 \cos\theta = 0, \quad F + N_2 \sin\theta = 0 \rightarrow \quad N_2 = -\sqrt{2}F = -84.85 \text{ kN},$$
$$N_1 = F = 60 \text{ kN}.$$

The negative sign indicates that the direction shown in the Figure for the forces must be reversed. The axial stresses in the two members are

$$\sigma_1 = \frac{N_1}{A_1} = \frac{60 \times 10^3 \text{ N}}{\pi(0.020 \text{ m})^2} = 47.746 \text{ MPa},$$
$$\sigma_2 = \frac{N_2}{A_2} = -\frac{84.85 \times 10^3 \text{ N}}{\pi(0.030 \text{ m})^2} = -30.01 \text{ MPa}.$$

The normal and shear forces on the horizontal cross section of member 2 are

$$F_n = N_2 \cos\theta = -60 \text{ kN}, \quad F_s = N_2 \sin\theta = -60 \text{ kN}.$$

The corresponding stresses are computed by dividing the forces with the cross-sectional area on which they are acting. The inclined cross-sectional area of member 2 is $A_n = A_0/\cos\theta$, where A_0 denotes the cross-sectional area perpendicular to the axis of the member. Hence, the stresses are

$$\sigma_2^{(n)} = \frac{F_n \cos\theta}{A_0} = -\frac{60 \times 10^3 \text{ N}}{\sqrt{2}\pi(0.030 \text{ m})^2} = -15.005 \text{ MPa},$$
$$\sigma_2^{(s)} = \frac{F_s \cos\theta}{A_0} = -\frac{60 \times 10^3 \text{ N}}{\sqrt{2}\pi(0.030 \text{ m})^2} = -15.005 \text{ MPa}.$$

For the purpose of labeling the stress components at an interior point A of the continuum, we enclose the point A inside a rectangular box whose sides

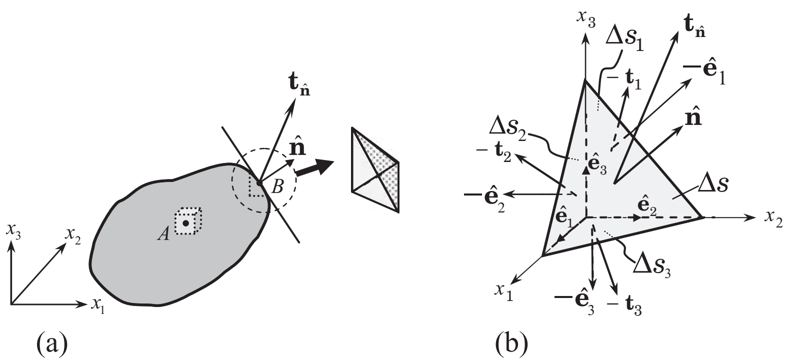


Figure 4.2.4 A tetrahedral element with stress vectors on all its four faces.

are parallel to the coordinate axes, as shown in Figure 4.2.4(a). The box has no dimensions (i.e., a point box). A boundary point B can only be enclosed inside a tetrahedron, as shown in Figure 4.2.4(a). To establish the relationship between the (external) stress vector $\mathbf{t}$ at point B and the state of stress (inside the continuum) at the point, we set up an infinitesimal tetrahedron in Cartesian coordinates, as shown in Figure 4.2.4(b). Let $-\mathbf{t}_1$, $-\mathbf{t}_2$, $-\mathbf{t}_3$, and $\mathbf{t}$ denote the stress vectors in the outward directions on the faces (of the infinitesimal tetrahedron) whose normals are $-\hat{\mathbf{e}}_1$, $-\hat{\mathbf{e}}_2$, $-\hat{\mathbf{e}}_3$, and $\hat{\mathbf{n}}$, respectively, and whose areas are Δs_1, Δs_2, Δs_3, and Δs, respectively. By Newton's second law of motion for the mass inside the tetrahedron ($\mathbf{F} = m\mathbf{a}$), we can write

$$\mathbf{t}\Delta s - \mathbf{t}_1\Delta s_1 - \mathbf{t}_2\Delta s_2 - \mathbf{t}_3\Delta s_3 + \rho\,\Delta v\,\mathbf{f} = \rho\,\Delta v\,\mathbf{a}, \tag{4.2.2}$$

where Δv is the volume of the tetrahedron, ρ the density, $\mathbf{f}$ the body force per unit mass, and $\mathbf{a}$ the acceleration. Examples of a body force are provided by electromagnetic force and gravitational force. Because the total vector area of a closed surface is zero,[1]

$$0 = \oint_s \hat{\mathbf{n}}\, ds = \sum_i \Delta s_i\, \hat{\mathbf{n}}_i,$$

where Δs_i denotes the area of the ith face of the tetrahedron and $\hat{\mathbf{n}}_i$ is the unit vector normal to the ith face, we can write

$$\Delta s\,\hat{\mathbf{n}} - \Delta s_1\hat{\mathbf{e}}_1 - \Delta s_2\hat{\mathbf{e}}_2 - \Delta s_3\hat{\mathbf{e}}_3 = \mathbf{0}. \tag{4.2.3}$$

It follows that

$$\Delta s_1 = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_1)\Delta s, \quad \Delta s_2 = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_2)\Delta s, \quad \Delta s_3 = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_3)\Delta s. \tag{4.2.4}$$

The volume of the element Δv can be expressed in terms of the slant surface area Δs and the perpendicular distance Δh from the origin to the slant face as

$$\Delta v = \frac{\Delta h}{3}\Delta s. \tag{4.2.5}$$

[1] Use the gradient theorem in Eq. (2.6.19) with $\phi = 1$ to obtain the result.

Substitution of Eqs. (4.2.3) through (4.2.5) into Eq. (4.2.2) and dividing throughout by Δs yields

$$\mathbf{t} = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_1)\mathbf{t}_1 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_2)\mathbf{t}_2 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_3)\mathbf{t}_3 + \rho\frac{\Delta h}{3}(\mathbf{a} - \mathbf{f}). \tag{4.2.6}$$

In the limit when the tetrahedron shrinks to a point, that is, as $\Delta h \to 0$, we are left with

$$\mathbf{t} = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_1)\mathbf{t}_1 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_2)\mathbf{t}_2 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_3)\mathbf{t}_3. \tag{4.2.7}$$

It is now convenient to display the previous equation as

$$\mathbf{t} = \hat{\mathbf{n}} \cdot (\hat{\mathbf{e}}_1\mathbf{t}_1 + \hat{\mathbf{e}}_2\mathbf{t}_2 + \hat{\mathbf{e}}_3\mathbf{t}_3). \tag{4.2.8}$$

The terms in the parentheses are to be treated as a dyadic, called the *stress dyadic* or *stress tensor* σ:

$$\sigma \equiv \hat{\mathbf{e}}_1\mathbf{t}_1 + \hat{\mathbf{e}}_2\mathbf{t}_2 + \hat{\mathbf{e}}_3\mathbf{t}_3. \tag{4.2.9}$$

The stress tensor is a property of the medium that is independent of the $\hat{\mathbf{n}}$. Thus, from Eqs. (4.2.8) and (4.2.9) we have

$$\mathbf{t}(\hat{\mathbf{n}}) = \hat{\mathbf{n}} \cdot \sigma = \sigma^{\mathrm{T}} \cdot \hat{\mathbf{n}}, \tag{4.2.10}$$

and the dependence of $\mathbf{t}$ on $\hat{\mathbf{n}}$ has been explicitly displayed. Equation (4.2.10) is known as the *Cauchy stress formula*, and σ is termed the *Cauchy stress tensor*. Thus, the Cauchy stress tensor σ is defined to be the current force per unit deformed area, $d\mathbf{f} = \mathbf{t}da = \sigma \cdot d\mathbf{a}$, where Cauchy's formula, $\mathbf{t} = \hat{\mathbf{n}} \cdot \sigma = \sigma^{\mathrm{T}} \cdot \hat{\mathbf{n}}$, is used.

In Cartesian component form, the Cauchy formula in Eq. (4.2.10) can be written as $t_i = n_j\sigma_{ji}$. The matrix form of Cauchy's formula (for computing purposes) in a rectangular Cartesian system is given by

$$\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}. \tag{4.2.11}$$

It is useful to resolve the stress vectors $\mathbf{t}_1$, $\mathbf{t}_2$, and $\mathbf{t}_3$ into their orthogonal components in a rectangular Cartesian system,

$$\mathbf{t}_i = \sigma_{i1}\,\hat{\mathbf{e}}_1 + \sigma_{i2}\,\hat{\mathbf{e}}_2 + \sigma_{i3}\,\hat{\mathbf{e}}_3 = \sigma_{ij}\,\hat{\mathbf{e}}_j \tag{4.2.12}$$

for $i = 1, 2, 3$. Hence, the stress tensor can be expressed in the Cartesian component form as

$$\begin{aligned} \sigma &= \hat{\mathbf{e}}_i\mathbf{t}_i = \sigma_{ij}\,\hat{\mathbf{e}}_i\,\hat{\mathbf{e}}_j \\ &= \sigma_{11}\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + \sigma_{12}\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 + \sigma_{13}\hat{\mathbf{e}}_1\hat{\mathbf{e}}_3 \\ &\quad + \sigma_{21}\hat{\mathbf{e}}_2\hat{\mathbf{e}}_1 + \sigma_{22}\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 + \sigma_{23}\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3 \\ &\quad + \sigma_{31}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_1 + \sigma_{32}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_2 + \sigma_{33}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_3. \end{aligned} \tag{4.2.13}$$

The component σ_{ij} represents the stress (force per unit area) on a plane perpendicular to the x_i coordinate and in the x_j-coordinate direction, as shown in

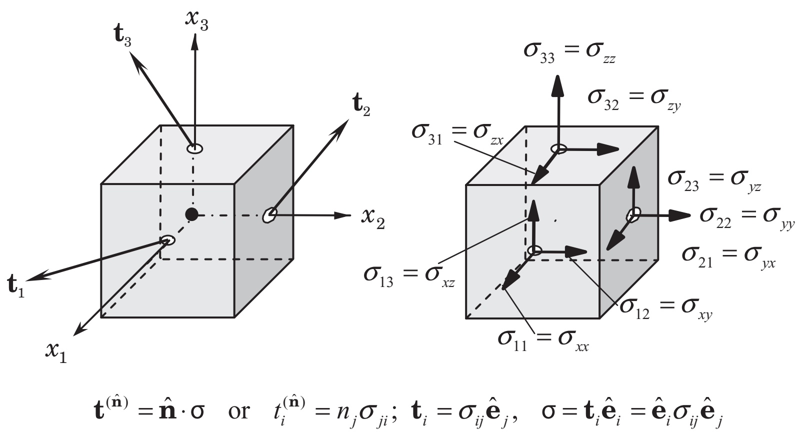


Figure 4.2.5 Display of stress vectors and stress components in Cartesian rectangular coordinates. The same quantities occur on the opposite faces with directions reversed (not shown in the figure).

Figure 4.2.5. Thus, the first subscript of σ_{ij} refers to the plane on which the stress component is acting, whereas the second subscript denotes the direction of the stress component.

The component of $\mathbf{t}$ that is in the direction of $\hat{\mathbf{n}}$ is called the *normal stress*. The component of $\mathbf{t}$ that is normal to $\hat{\mathbf{n}}$ is called the *shear* stress. The stress vector $\mathbf{t}$ can be represented as the sum of vectors along and perpendicular to the unit normal vector $\hat{\mathbf{n}}$,

$$\mathbf{t} = (\mathbf{t} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + \hat{\mathbf{n}} \times (\mathbf{t} \times \hat{\mathbf{n}}). \tag{4.2.14}$$

The magnitudes of the component of the stress vector $\mathbf{t}$ normal to the plane are given by

$$t_{nn} = \mathbf{t} \cdot \hat{\mathbf{n}} = t_i \, n_i = n_j \, \sigma_{ji} \, n_i, \tag{4.2.15}$$

and the component of $\mathbf{t}$ perpendicular to $\hat{\mathbf{n}}$, as depicted in Figure 4.2.6, is

$$t_{ns} = \sqrt{|\mathbf{t}|^2 - t_{nn}^2}. \tag{4.2.16}$$

The tangential component lies in the $\hat{\mathbf{n}}$-$\mathbf{t}$ plane but perpendicular to $\hat{\mathbf{n}}$.

The next example illustrates the ideas presented here.

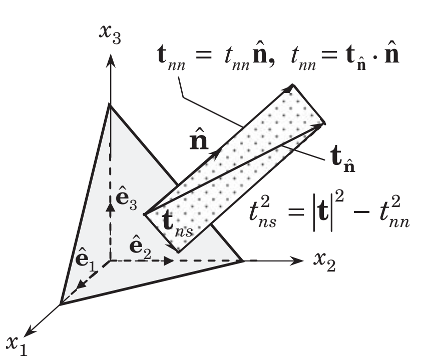


Figure 4.2.6 Tetrahedral element with stress vector and its normal and shear components.

Example 4.2.3:

With reference to a rectangular Cartesian system (x_1, x_2, x_3), the components of the stress dyadic at a certain point of a continuous medium are given by

$$[\sigma] = \begin{bmatrix} 200 & 400 & 300 \\ 400 & 0 & 0 \\ 300 & 0 & -100 \end{bmatrix} \text{ MPa.}$$

Determine the stress vector $\mathbf{t}$ and its normal and tangential components at the point on the plane, $\phi(x_1, x_2) \equiv x_1 + 2x_2 + 2x_3 =$ constant, passing through the point.

Solution: First, we should find the unit normal to the plane on which we are required to find the stress vector. The unit normal to the plane defined by $\phi(x_1, x_2, x_3) =$ constant is given by

$$\hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{1}{3}(\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 2\hat{\mathbf{e}}_3).$$

The components of the stress vector are

$$\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{bmatrix} 200 & 400 & 300 \\ 400 & 0 & 0 \\ 300 & 0 & -100 \end{bmatrix} \frac{1}{3} \begin{Bmatrix} 1 \\ 2 \\ 2 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} 1600 \\ 400 \\ 100 \end{Bmatrix} \text{ MPa,}$$

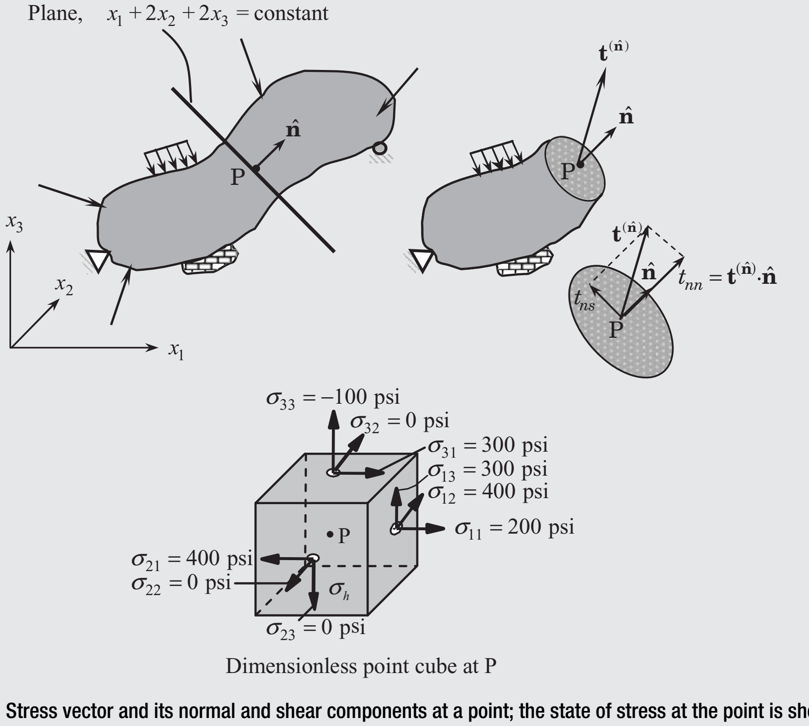


Figure 4.2.7 Stress vector and its normal and shear components at a point; the state of stress at the point is shown on the point cube.

or

$$\mathbf{t}(\hat{\mathbf{n}}) = \frac{1}{3}(1600\hat{\mathbf{e}}_1 + 400\hat{\mathbf{e}}_2 + 100\hat{\mathbf{e}}_3)\ \text{MPa}.$$

The normal component t_{nn} of the stress vector $\mathbf{t}$ on the plane is given by

$$t_{nn} = \mathbf{t}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{n}} = \frac{2600}{9}\ \text{MPa},$$

and the tangential component is given by (using the Pythagorean theorem)

$$t_{ns} = \sqrt{|\mathbf{t}|^2 - t_{nn}^2} = \frac{10^2}{9}\sqrt{(256+16+1)9 - 26 \times 26} = 468.9\ \text{MPa}.$$

The stress components, stress vector, and its normal and shear components are shown in Figure 4.2.7.

Example 4.2.4:

Consider a thin cylindrical pressure vessel of length L, inner diameter D, and thickness h. The ends of the cylindrical vessel are closed and the internal (gauge) pressure is p, as shown in Figure 4.2.8(a). Determine the stresses at a typical point away from the ends in the pressure vessel and display them on a rectangular element with sides parallel to the longitudinal and circumferential directions.

Solution: The internal pressure causes the cylindrical pressure to experience longitudinal stress $\sigma_{xx} = \sigma_L$ and circumferential stress $\sigma_{\theta\theta} = \sigma_h$, also known as the *hoop stress*. The remaining stresses $\sigma_{rr}, \sigma_{x\theta}, \sigma_{xr}$, and $\sigma_{r\theta}$ are either negligible or zero by symmetry (the radial stress σ_{rr} has a maximum value of p at the interior surface and decreases through the wall to zero at the exterior surface).

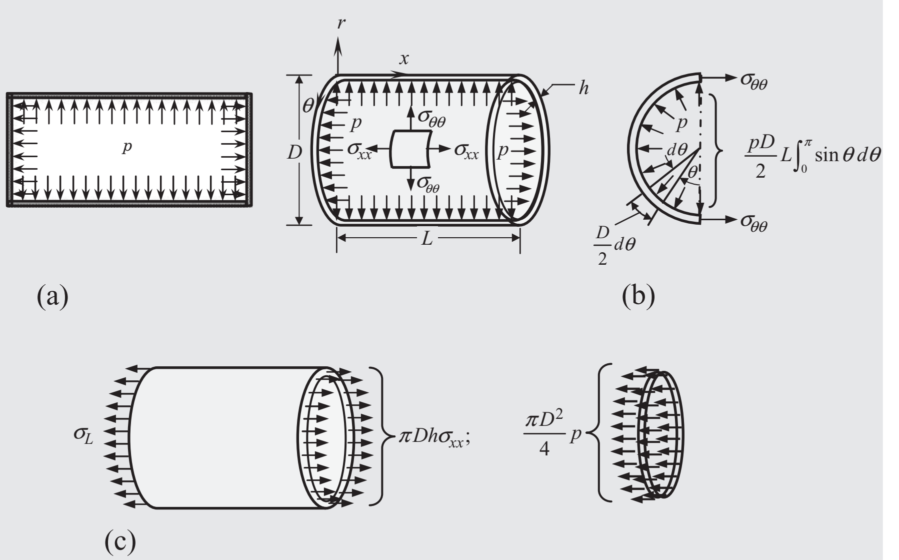


Figure 4.2.8 Stresses in a closed-end, thin-walled pressure vessel.

From Figure 4.2.8(b), we obtain

$$\frac{pD}{2}L\int_0^{\pi}\sin\theta\, d\theta = 2hL\sigma_{\theta\theta} \quad\rightarrow\quad \sigma_{\theta\theta} = \frac{pD}{2h}. \tag{4.2.17}$$

To calculate σ_{xx}, we consider a cut of the cylinder along the length and balance the forces, as shown in Figure 4.2.7(c). We obtain

$$\pi Dh\sigma_{xx} = \frac{\pi D^2}{4}p \quad\rightarrow\quad \sigma_{xx} = \frac{pD}{4h}. \tag{4.2.18}$$

Note that the maximum stress occurs in the circumferential direction ($\sigma_{\theta\theta} = 2\sigma_{xx}$). The stresses on an element of the lateral surface are shown in Figure 4.2.8(a).

4.3 Transformations of stress components and principal stresses

4.3.1 Transformation of stress components

The components of a stress tensor σ in one rectangular Cartesian coordinate system can be related to the components of the same tensor in another rectangular Cartesian system. In the next two examples, we show how the stress transformation equations can be derived for a two-dimensional problem and illustrate the use of the derived transformation equations in the calculation of stresses in a different coordinate system.

Example 4.3.1:

Use equilibrium of forces to derive the relations between the normal and shear stresses (σ_n, σ_s) on an inclined edge with unit normal $\hat{\mathbf{n}} = \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2$ and the stress components ($\sigma_{11}, \sigma_{22}, \sigma_{12} = \sigma_{21}$) on the edges perpendicular to the base vector $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$, as shown in Figure 4.3.1.

Solution: Identify the forces associated with the stresses on various planes (assume the thickness of the triangle to be t and its diagonal length to be dL).

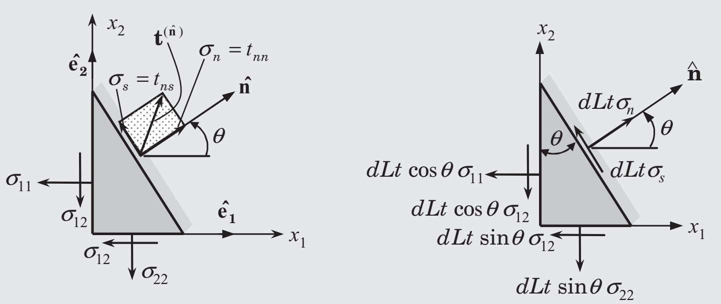


Figure 4.3.1 Stress components on the three edges of a wedge; the inclined edge is denoted with the unit normal vector $\hat{\mathbf{n}}$.

The base is of length $dL\sin\theta$ and the height is of length $dL\cos\theta$. Summing the forces along the normal to the inclined plane we obtain

$$\sigma_n\, t\, dL - (\sigma_{11}\, t\, dL\cos\theta)\cos\theta - (\sigma_{12}\, t\, dL\cos\theta)\sin\theta - (\sigma_{22}\, t\, dL\sin\theta)\sin\theta - (\sigma_{12}\, t\, dL\sin\theta)\cos\theta = 0.$$

Dividing throughout by $t\, dL$, we obtain

$$\sigma_n = \sigma_{11}\cos^2\theta + 2\sigma_{12}\cos\theta\sin\theta + \sigma_{22}\sin^2\theta. \tag{4.3.1}$$

Similarly, summing the forces along the tangent to the inclined plane we obtain

$$\sigma_s\, t\, dL + (\sigma_{11}\, t\, dL\cos\theta)\sin\theta - (\sigma_{12}\, t\, dL\cos\theta)\cos\theta - (\sigma_{22}\, t\, dL\sin\theta)\cos\theta + (\sigma_{12}\, t\, dL\sin\theta)\sin\theta = 0.$$

Dividing throughout by $t\, dL$, we obtain

$$\sigma_s = (\sigma_{22} - \sigma_{11})\cos\theta\sin\theta + \sigma_{12}\left(\cos^2\theta - \sin^2\theta\right). \tag{4.3.2}$$

The relations can be expressed in terms of the double angle 2θ as

$$\begin{aligned}\sigma_n &= \frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta + \sigma_{12}\sin 2\theta,\\ \sigma_s &= -\frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta + \sigma_{12}\cos 2\theta.\end{aligned} \tag{4.3.3}$$

Example 4.3.2:

Consider a thin, closed, filament-wound cylindrical pressure vessel as shown in Figure 4.3.2. The vessel is 63.5 cm (25 in.) in internal diameter, and it is pressurized to 1.379 MPa (200 psi). If the filament winding angle is $\theta = 53.125^\circ$ from the longitudinal axis of the pressure vessel, determine the shear and normal forces per unit length of filament winding. Assume that the material

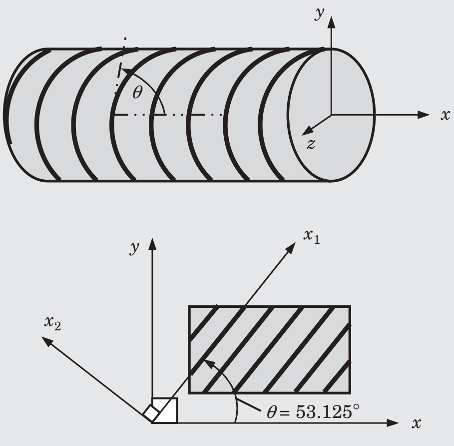


Figure 4.3.2 A filament-wound cylindrical pressure vessel.

used is graphite-epoxy with the following material properties [see Reddy (2004)]:

$$E_1 = 140 \text{ MPa } (20.3 \times 10^6 \text{ psi}), \quad E_2 = 10 \text{ MPa } (1.45 \times 10^6 \text{ psi}),$$
$$G_{12} = 7 \text{ MPa } (1.02 \times 10^6 \text{ psi}), \quad \nu_{12} = 0.3,$$

where MPa denotes mega (10^6) Pascal (Pa) and Pa $=$N/m^2 (1 psi $=$6,894.76 Pa).

Solution: First, we compute the stresses in the pressure vessel using the formulas developed earlier in Example 4.2.4. The longitudinal (σ_{xx}) and circumferential ($\sigma_{\theta\theta} = \sigma_{yy}$) stresses are given by Eqs. (4.2.17) and (4.2.18),

$$\sigma_{xx} = \frac{pD}{4h}, \quad \sigma_{\theta\theta} = \sigma_{yy} = \frac{pD}{2h}, \tag{4.3.4}$$

where p is the internal pressure, D is the internal diameter, and h is the thickness of the pressure vessel. Note that the stresses are independent of the material properties and only depend on the geometry and load (pressure). Using the values of various parameters, we calculate the stresses as

$$\sigma_{xx} = \frac{1.379 \times 0.635}{4h} = \frac{0.2189}{h} \text{ MPa}, \quad \sigma_{yy} = \frac{1.379 \times 0.635}{2h} = \frac{0.4378}{h} \text{ MPa}.$$

The shear stress σ_{xy} is zero. Next, we determine the shear stress σ_{12} along the fiber and the normal stress σ_{11} in the fiber using the transformation equations, Eq. (4.3.3):

$$\sigma_{11} = \frac{0.2189}{h}(0.6)^2 + \frac{0.4378}{h}(0.8)^2 = \frac{0.3590}{h} \text{ MPa},$$
$$\sigma_{22} = \frac{0.2189}{h}(0.8)^2 + \frac{0.4378}{h}(0.6)^2 = \frac{0.2977}{h} \text{ MPa},$$
$$\sigma_{12} = \left(\frac{0.4378}{h} - \frac{0.2189}{h}\right) \times 0.6 \times 0.8 = \frac{0.1051}{h} \text{ MPa}.$$

Thus, the normal and shear forces per unit length along the fiber-matrix interface are $F_{22} = 0.2977$ kN and $F_{12} = 0.1051$ kN, whereas the force per unit length in the fiber direction is $F_{11} = 0.359$ kN.

4.3.2 Principal stresses and principal planes

For a given state of stress, the determination of maximum normal stresses and shear stresses at a point is of considerable interest in the design of structures because failures occur when the magnitudes of stresses exceed the allowable (normal or shear) stress values, called *strengths*, of the material. In this regard, it is of interest to determine the values and the planes on which the stresses are the maximum. The maximum values are called *principal values* and the planes on which they occur are called *principal planes*. The principal values are nothing but the eigenvalues and the principal planes are characterized by the eigenvectors

associated with the stress tensor. Thus, the determination of eigenvalues and eigenvectors of a stress tensor at a point is important. Alternatively, we can consider a two-dimensional state of stress and use the stress transformation equations in Eq. (4.3.3) to derive the expressions for the maximum normal and shear stresses. The latter approach is illustrated here.

The expressions for σ_n and σ_s in Eq. (4.3.3) can be viewed as functions of θ (which represents the orientation of the plane). Hence, their maximum values with respect to θ can be determined by setting the first derivatives of σ_n and σ_s with respect to θ to zero, a necessary condition for an extremum. We begin with the normal stress σ_n,

$$\frac{d\sigma_n}{d\theta} = -\left(\sigma_{11} - \sigma_{22}\right)\sin 2\theta_n + 2\sigma_{12}\cos 2\theta_n = 0,$$

and determine the angle θ_n for which the normal stress is the maximum or minimum,

$$\tan 2\theta_n = \pm\left[\frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}}\right]. \tag{4.3.5}$$

There are two values of θ_n, differing by 90°, corresponding to the two principal stresses. Substituting for $\cos 2\theta_n$ and $\sin 2\theta_n$ from

$$\cos 2\theta_n = \pm\left[\frac{\frac{\sigma_{11}-\sigma_{22}}{2}}{\sqrt{(\sigma_{12})^2 + \left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2}}\right], \quad \sin 2\theta_n = \pm\left[\frac{\sigma_{12}}{\sqrt{(\sigma_{12})^2 + \left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2}}\right] \tag{4.3.6}$$

into Eq. (4.3.3), we obtain the following principal (normal) stresses:

$$\begin{aligned}\sigma_{p1} &= \frac{\sigma_{11} + \sigma_{22}}{2} + \sqrt{(\sigma_{12})^2 + \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2}\\ \sigma_{p2} &= \frac{\sigma_{11} + \sigma_{22}}{2} - \sqrt{(\sigma_{12})^2 + \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2}.\end{aligned} \tag{4.3.7}$$

To derive the maximum shear stress, set the derivative of σ_s with respect to θ to zero,

$$\frac{d\sigma_s}{d\theta} = -\left(\sigma_{11} - \sigma_{22}\right)\cos 2\theta - 2\sigma_{12}\sin 2\theta = 0,$$

and determine the angle θ_s for which the shear stress is maximum or minimum,

$$\tan 2\theta_s = \pm\left[\frac{\sigma_{11} - \sigma_{22}}{2\sigma_{12}}\right]. \tag{4.3.8}$$

Again, there are two values of θ_s, differing by 90°, corresponding to the two shear stresses, which only differ in sign and not in magnitude.

Substituting for $\sin 2\theta_s$ and $\cos 2\theta_s$ from

$$\sin 2\theta_s = \pm\left[\frac{\frac{\sigma_{11}-\sigma_{22}}{2}}{\sqrt{(\sigma_{12})^2 + \left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2}}\right], \quad \cos 2\theta_s = \pm\left[\frac{\sigma_{12}}{\sqrt{(\sigma_{12})^2 + \left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2}}\right], \tag{4.3.9}$$

into Eq. (4.3.1), we obtain the following maximum and minimum shear stresses:

$$\sigma_{s1} = \sqrt{(\sigma_{12})^2 + \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2}, \quad \sigma_{s2} = -\sqrt{(\sigma_{12})^2 + \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2}. \tag{4.3.10}$$

The next example illustrates an application of Eqs. (4.3.6) through (4.3.10).

Example 4.3.3:

Find the maximum and minimum normal stresses and the orientations of the principal planes for the state of stress shown in Figure 4.3.3.

Solution: From Eq. (4.3.7), we have (with $\sigma_{11} = 40$ MPa, $\sigma_{22} = 0$ MPa, and $\sigma_{12} = -100$ MPa)

$$\begin{aligned} \sigma_{p1} &= \frac{\sigma_{11} + \sigma_{22}}{2} + \sqrt{(\sigma_{12})^2 + \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2} \\ &= 20 + \sqrt{10000 + 400} = 121.98 \text{ MPa} \\ \sigma_{p2} &= \frac{\sigma_{11} + \sigma_{22}}{2} - \sqrt{(\sigma_{12})^2 + \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2}, \\ &= 20 - \sqrt{10000 + 400} = -81.98 \text{ MPa}. \end{aligned}$$

The principal plane is given by

$$\theta_n = \pm\frac{1}{2}\tan_n^{-1}\left[\frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}}\right] = \pm\frac{1}{2}\tan^{-1}\left(\frac{200}{40}\right),$$

or $\theta_{n1} = 39.35°$ and $\theta_{n2} = 90 + 39.35 = 129.35°$. The maximum shear stress is given by

$$\sigma_s = \sqrt{(\sigma_{12})^2 + \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2} = \sqrt{10000 + 400} = 101.98 \text{ MPa}$$

and its plane is

$$\theta_s = \pm\frac{1}{2}\tan^{-1}\left[\frac{\sigma_{11} - \sigma_{22}}{2\sigma_{12}}\right] = \pm\frac{1}{2}\tan_s^{-1}\left(\frac{40}{200}\right),$$

or $\theta_{s1} = 84.35°$ and $\theta_{s2} = 90 + 84.35 = 174.35°$.

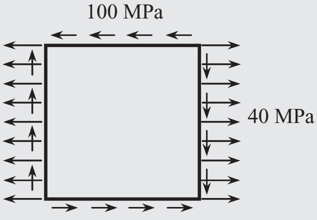


Figure 4.3.3 A two-dimensional state of stress at a point.

4.4 Summary

In this chapter, concepts of stress vector and stress tensor are introduced and the Cauchy formula that relates the stress tensor to the stress vector at a point on the boundary is derived. The transformation relations among stress components from two different rectangular coordinate systems are derived. The principal values and principal directions of a stress tensor are also discussed.

PROBLEMS

4.1. Suppose that $\mathbf{t}^{\hat{\mathbf{n}}_1}$ and $\mathbf{t}^{\hat{\mathbf{n}}_2}$ are stress vectors acting on planes with unit normals $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$, respectively, and passing through a point with the stress state σ. Show that the component of $\mathbf{t}^{\hat{\mathbf{n}}_1}$ along $\hat{\mathbf{n}}_2$ is equal to the component of $\mathbf{t}^{\hat{\mathbf{n}}_2}$ along the normal $\mathbf{t}^{\hat{\mathbf{n}}_1}$.

4.2. Write the stress vectors on each boundary surface in terms of the given values and base vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ for the system shown in Figure P4.2.

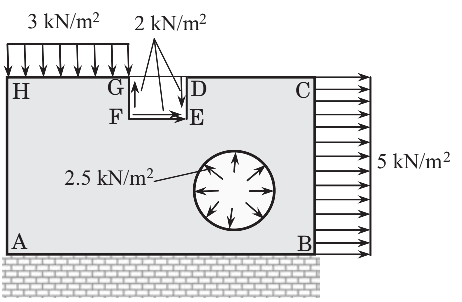


Figure P4.2

4.3. The components of a stress dyadic at a point, referred to as the (x_1, x_2, x_3) system, are (in ksi $= 1000$ psi):

$$\text{(i)} \begin{bmatrix} 12 & 9 & 0 \\ 9 & -12 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} 9 & 0 & 12 \\ 0 & -25 & 0 \\ 12 & 0 & 16 \end{bmatrix}, \quad \text{(iii)} \begin{bmatrix} 1 & -3 & \sqrt{2} \\ -3 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 4 \end{bmatrix}.$$

Find the following:

(a) The stress vector acting on a plane perpendicular to the vector $2\hat{\mathbf{e}}_1 - 2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$.

(b) The magnitude of the stress vector and the angle between the stress vector and the normal to the plane.

(c) The magnitudes of the normal and tangential components of the stress vector.

4.4. Consider a kinematically infinitesimal stress field whose matrix of scalar components in the vector basis $\{\hat{\mathbf{e}}_i\}$ is

$$\begin{bmatrix} 1 & 0 & 2X_2 \\ 0 & 1 & 4X_1 \\ 2X_2 & 4X_1 & 1 \end{bmatrix} \times 10^3 \text{ (psi)},$$

where the Cartesian coordinate variables X_i are in inches (in.) and the units of stress are pounds per square inch (psi).

(a) Determine the traction vector acting at point $\mathbf{X} = \hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 3\hat{\mathbf{e}}_3$ on the plane $X_1 + X_2 + X_3 = 6$.

(b) Determine the normal and projected shear tractions acting at this point on this plane.

(c) Determine the principal stresses and principal directions of stress at this point.

(d) Determine the maximum shear stress at this point.

4.5. The three-dimensional state of stress at a point $(1, 1, -2)$ within a body relative to the coordinate system (x_1, x_2, x_3) is

$$\begin{bmatrix} 2.0 & 3.5 & 2.5 \\ 3.5 & 0.0 & -1.5 \\ 2.5 & -1.5 & 1.0 \end{bmatrix} \times 10^6 \text{ (Pa)}.$$

Determine the normal and shear stresses at the point and on the surface of an internal sphere whose equation is $x_1^2 + (x_2 - 2)^2 + x_3^2 = 6$.

4.6. For the state of stress given in Problem 4.5, determine the normal and shear stresses on a plane intersecting the point where the plane is defined by the points $(0, 0, 0)$, $(2, -1, 3)$, and $(-2, 0, 1)$.

4.7. Determine the normal and shear stress components on the plane indicated in Figure P4.7.

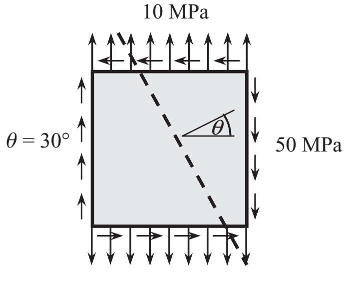


Figure P4.7

4.8. Determine the normal and shear stress components on the plane indicated in Figure P4.8.

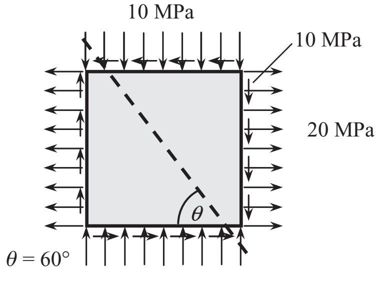


Figure P4.8

4.9. Determine the normal and shear stress components on the plane indicated in Figure P4.9.

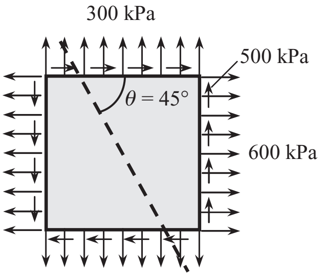


Figure P4.9

4.10. Determine the normal and shear stress components on the plane indicated in Figure P4.10.

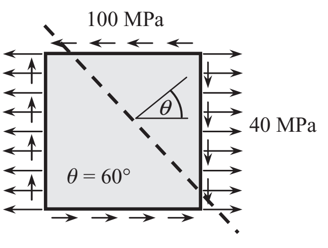


Figure P4.10

4.11. Find the values of σ_s and σ_{22} for the state of stress shown in Figure P4.11.

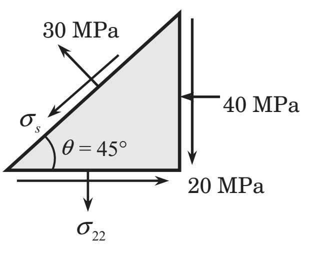


Figure P4.11

4.12. Find the maximum and minimum normal stresses and the orientations of the principal planes for the state of stress shown in Figure P4.9. What is the maximum shear stress at this point?

4.13. Find the maximum and minimum normal stresses and the orientations of the principal planes for the state of stress shown in Figure **??**. What is the maximum shear stress at this point?

4.14. A wine barrel is made of wood planks bound together using steel straps, as shown in Figure **??**. If the cross-sectional area of the steel strap is 150 mm^2 and the allowable stress is $\sigma_{\text{allow}} = 85$ MPa, determine the maximum spacing d along the length of the vessel so that it can resist an internal pressure of $p = 0.032$ MPa.

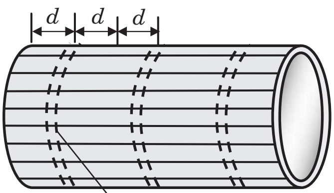


Figure P4.14

4.15. Derive the expression for the hoop stress in a spherical pressure vessel of inner diameter D, thickness t, and subjected to internal pressure p, as shown in Figure **??**.

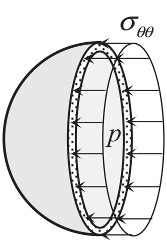


Figure P4.15

4.16. Given the following state of stress ($\sigma_{ij} = \sigma_{ji}$),

$$\sigma_{11} = -2x_1^2, \quad \sigma_{12} = -7 + 4x_1x_2 + x_3, \quad \sigma_{13} = 1 + x_1 - 3x_2,$$

$$\sigma_{22} = 3x_1^2 - 2x_2^2 + 5x_3, \quad \sigma_{23} = 0, \quad \sigma_{33} = -5 + x_1 + 3x_2 + 3x_3,$$

determine (a) the stress vector at point (x_1, x_2, x_3) on the plane $x_1 + x_2 + x_3 =$ constant, (b) the normal and shearing components of the stress vector at point $(1, 1, 3)$, and (c) the principal stresses and their orientation at point $(1, 2, 1)$.

4.17. The components of a stress dyadic at a point, referred to as the (x_1, x_2, x_3) system, are

$$\begin{bmatrix} 25 & 0 & 0 \\ 0 & -30 & -60 \\ 0 & -60 & 5 \end{bmatrix} \text{ MPa.}$$

Determine (a) the stress vector acting on a plane perpendicular to the vector $2\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + 2\hat{\mathbf{e}}_3$, and (b) the magnitude of the normal and tangential components of the stress vector.

4.18. The components of a stress dyadic at a point, referred to as the (x_1, x_2, x_3) system, are

$$\begin{bmatrix} 57 & 0 & 24 \\ 0 & 50 & 0 \\ 24 & 0 & 43 \end{bmatrix} \text{ MPa.}$$

Determine the principal stresses and principal directions at this point. What is the maximum shear stress at this point?

5 Conservation of Mass, Momentum, and Energy

It is the mark of an educated mind to be able to entertain a thought without accepting it.

Aristotle

5.1 Introduction

Virtually every phenomenon in nature, whether mechanical, biological, chemical, geological, or geophysical, can be described in terms of mathematical relations among various quantities of interest. Such relationships are called *mathematical models* and are based on fundamental scientific laws of physics that are extracted from centuries of research on the behavior of mechanical systems subjected to the action of external stimuli. What is most exciting is that the laws of physics also govern biological systems because of mass and energy transports. However, biological systems may require additional laws, yet to be discovered, from biology and chemistry to complete their description.

This chapter is devoted to the study of the fundamental laws of physics as applied to mechanical systems. The laws of physics are expressed in analytical form with the aid of the concepts and quantities introduced in previous chapters. The laws or principles of physics that we study here are the principle of conservation of mass, the principle of conservation of linear momentum, the principle of conservation of angular momentum, and the principle of conservation of energy. These laws allow us to write mathematical relationships – algebraic, differential, or integral – of physical quantities such as displacements, velocities, temperatures, stresses, and strains in mechanical systems. The solution of these equations represents the response of the system, which aids the design and manufacturing of the system. The equations developed here not only will be used in the later chapters of this book, but they are also useful in other courses in engineering and applied science. Thus, the present chapter is the heart and soul of a course on continuum mechanics.

As discussed in Chapter 3 [see Eqs. (3.3.4) and (3.3.5)], the partial time derivative with the material coordinates $\mathbf{X}$ held constant should be distinguished from the partial time derivative with spatial coordinates $\mathbf{x}$ held constant due to the

difference in the descriptions of motion. The *material time derivative*, denoted here by D/Dt, is the time derivative with the material coordinates held constant. The time derivative of a function ϕ in material description, $\phi = \phi(\mathbf{X}, t)$, with $\mathbf{X}$ held constant is nothing but the partial derivative with respect to time [see Eq. (3.3.4)],

$$\frac{D\phi}{Dt} \equiv \left(\frac{\partial \phi}{\partial t}\right)_{\mathbf{X}=\text{const}} = \frac{\partial \phi}{\partial t}. \tag{5.1.1}$$

The material time derivative of a function of spatial coordinates, $\phi = \phi(\mathbf{x}, t)$, is computed using the chain rule of differentiation:

$$\begin{aligned}\frac{D\phi}{Dt} \equiv \left(\frac{\partial \phi}{\partial t}\right)_{\mathbf{X}=\text{const}} &= \left(\frac{\partial \phi}{\partial t}\right)_{\mathbf{x}=\text{const}} + \left(\frac{\partial x_i}{\partial t}\right)_{\mathbf{X}=\text{const}} \frac{\partial \phi}{\partial x_i} \\ &= \left(\frac{\partial \phi}{\partial t}\right)_{\mathbf{x}=\text{const}} + v_i \frac{\partial \phi}{\partial x_i} = \left(\frac{\partial \phi}{\partial t}\right)_{\mathbf{x}=\text{const}} + \mathbf{v}\cdot\nabla\phi.\end{aligned} \tag{5.1.2}$$

Thus, the material derivative operator is given by

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_{\mathbf{x}=\text{const}} + \mathbf{v}\cdot\nabla. \tag{5.1.3}$$

5.2 Conservation of mass

5.2.1 Preliminary discussion

It is common knowledge that the mass of a given system cannot be created or destroyed. For example, the mass flow of the blood entering a section of an artery is equal to the mass flow leaving the artery, provided that no blood is added or lost through the artery walls. Thus, the mass of the blood is conserved even when the artery cross section changes along the length.

The principle of conservation of mass states that the total mass of any part of a body does not change in any motion. The mathematical form of this principle is different in different descriptions of motion. The equation resulting from the principle of conservation of mass is also known as the *continuity equation*.

5.2.2 Conservation of mass in spatial description

First, we derive the one-dimensional version of the equation resulting from the principle of conservation of mass by considering the flow of a material along the x-axis (see Figure 5.2.1). The amount of mass entering (i.e., mass flow) per unit time at the left section of the elemental volume is the density $\times$ cross-sectional area $\times$ velocity of the flow $= (\rho A v_x)_x$. The mass leaving at the right section of the elemental volume is $(\rho A v_x)_{x+\Delta x}$, where v_x is the velocity along the x-direction. The subscript of this form denotes the distance at which the enclosed quantity is

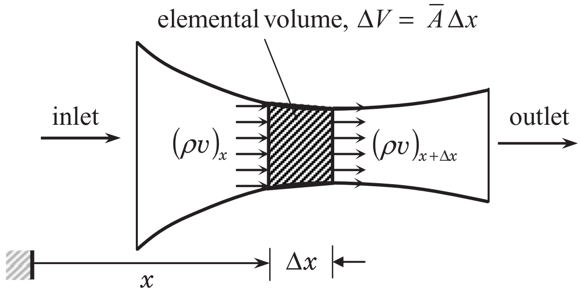


Figure 5.2.1 Derivation of the local form of the continuity equation in one dimension.

evaluated. It is assumed that the cross-sectional area A is a function of position x but not of time t. The net mass inflow into the elemental volume is

$$(A\rho v_x)_x - (A\rho v_x)_{x+\Delta x}.$$

The time rate of increase of the total mass inside the elemental volume is

$$\bar{A}\Delta x \frac{(\bar{\rho})_{t+\Delta t} - (\bar{\rho})_t}{\Delta t},$$

where $\bar{\rho}$ and $\bar{A}$ are the average values of the density and cross-sectional area, respectively, inside the elemental volume.

If no mass is created or destroyed inside the elemental volume, the rate of increase of mass should be equal to the mass inflow:

$$\bar{A}\Delta x \frac{(\bar{\rho})_{t+\Delta t} - (\bar{\rho})_t}{\Delta t} = (A\rho v_x)_x - (A\rho v_x)_{x+\Delta x}.$$

Dividing throughout by Δx and taking the limits $\Delta t \to 0$ and $\Delta x \to 0$, we obtain

$$\lim_{\Delta t,\, \Delta x \to 0} \bar{A}\frac{(\rho)_{t+\Delta t} - (\rho)_t}{\Delta t} + \frac{(A\rho v_x)_{x+\Delta x} - (A\rho v_x)_x}{\Delta x} = 0,$$

or ($\bar{\rho} \to \rho$ and $\bar{A} \to A$ as $\Delta x \to 0$)

$$A\frac{\partial \rho}{\partial t} + \frac{\partial (A\rho v_x)}{\partial x} = 0. \tag{5.2.1}$$

For the steady-state case, which is when the flow is not dependent on time, Eq. (5.2.1) reduces to

$$\frac{\partial (A\rho v_x)}{\partial x} = 0 \;\to\; A\rho v_x = \text{constant} \Rightarrow (A\rho v_x)_1 = (A\rho v_x)_2 = \cdots = (A\rho v_x)_i, \tag{5.2.2}$$

where the subscript i refers to ith section along the direction of the (one-dimensional) flow. The quantity $Q = Av_x$ is called the *flow*, whereas $\rho A v_x$ is called the *mass flow*.

Next, we derive the general form of Eq. (5.2.1) that is applicable to three-dimensional flows. Let an arbitrary region in a continuous medium be denoted by Ω, and let the bounding closed surface of this region be continuous and denoted by Γ. Let each point on the bounding surface move with the velocity $\mathbf{v}_s$. It can

be shown that the time derivative of the volume integral of a continuous function $\phi(\mathbf{x}, t)$ is given by

$$\begin{aligned}\frac{d}{dt}\int_{\Omega}\phi(\mathbf{x}, t)\, d\mathbf{x} &\equiv \frac{\partial}{\partial t}\int_{\Omega}\phi\, d\mathbf{x} + \oint_{\Gamma}\phi\mathbf{v}_s\cdot\hat{\mathbf{n}}\, ds \\ &= \int_{\Omega}\frac{\partial\phi}{\partial t}\, d\mathbf{x} + \oint_{\Gamma}\phi\mathbf{v}_s\cdot\hat{\mathbf{n}}\, ds.\end{aligned} \tag{5.2.3}$$

Thus, the total time derivative of the integral of ϕ over the region Ω is the sum of the integral of the change of the function ϕ with time and the integral of the outward flow of the quantity through the surface Γ. This expression for the differentiation of a volume integral with variable limits is sometimes known as the three-dimensional *Leibnitz rule*.

Let each element of mass in the medium move with the velocity $\mathbf{v}(\mathbf{x}, t)$, and consider a special region Ω such that the bounding surface Γ is attached to a fixed set of material elements. Then each point of this surface moves itself with the material velocity, that is, $\mathbf{v}_s = \mathbf{v}$, and the region Ω thus contains a fixed mass because no mass crosses the boundary surface Γ. To distinguish the time rate of change of an integral over this material region, we replace d/dt by D/Dt and write

$$\frac{D}{Dt}\int_{\Omega}\phi(\mathbf{x}, t)\, d\mathbf{x} \equiv \int_{\Omega}\frac{\partial\phi}{\partial t}\, d\mathbf{x} + \oint_{\Gamma}\phi\mathbf{v}\cdot\hat{\mathbf{n}}\, ds, \tag{5.2.4}$$

which holds for a material region, that is, a region of fixed total mass. Then the relation between the time derivative following an arbitrary region and the time derivative following a material region (fixed total mass) is

$$\frac{d}{dt}\int_{\Omega}\phi(\mathbf{x}, t)\, d\mathbf{x} \equiv \frac{D}{Dt}\int_{\Omega}\phi(\mathbf{x}, t)\, d\mathbf{x} + \oint_{\Gamma}\phi(\mathbf{v}_s - \mathbf{v})\cdot\hat{\mathbf{n}}\, ds. \tag{5.2.5}$$

The velocity difference $\mathbf{v} - \mathbf{v}_s$ is the velocity of the material measured relative to the velocity of the surface. The surface integral

$$\oint_{\Gamma}\phi(\mathbf{v}_s - \mathbf{v})\cdot\hat{\mathbf{n}}\, ds$$

thus measures the total outflow of the property ϕ from the region Ω.

Let $\rho(\mathbf{x}, t)$ denote the mass density of a continuous region Ω with closed boundary Γ. Here Ω denotes the *control volume* (cv) and Γ the *control surface* (cs) enclosing Ω. Then the principle of conservation of mass for a fixed material region requires that ($\mathbf{v}_s = 0$):

$$\begin{matrix}\text{time rate of} & & \text{time rate of change} & & \text{rate of flow of} & \\ \text{change of the mass} & \equiv & \text{of the mass inside} & + & \text{mass through the} & = 0 \\ \text{of the system} & & \text{the control volume} & & \text{control surface} & \end{matrix}$$

or

$$\frac{D}{Dt}\int_{\Omega}\rho\, d\mathbf{x} \equiv \frac{d}{dt}\int_{\Omega}\rho\, d\mathbf{x} + \oint_{\Gamma}\rho\mathbf{v}\cdot\hat{\mathbf{n}}\, ds = 0. \tag{5.2.6}$$

The integrand $\mathbf{v}\cdot\hat{\mathbf{n}}\, ds$ represents the volume flow rate through the elemental area ds, whereas $\rho\mathbf{v}\cdot\hat{\mathbf{n}}\, ds$ denotes the mass flow rate (out) through ds.

Equation (5.2.6) is known as the control volume formulation of the conservation of mass principle. Using Eq. (5.2.3) with $\phi = \rho$, Eq. (5.2.6) can be expressed as

$$\int_\Omega \frac{\partial \rho}{\partial t}\, d\mathbf{x} + \oint_\Gamma \rho \mathbf{v} \cdot \hat{\mathbf{n}}\, ds = 0. \tag{5.2.7}$$

When the flow is steady, all quantities including density remain constant with time, and we have

$$\int_\Omega \frac{\partial \rho}{\partial t}\, d\mathbf{x} = 0,$$

and net the mass flow rate through the control surface is zero,

$$\oint_\Gamma \rho \mathbf{v} \cdot \hat{\mathbf{n}}\, ds = 0.$$

Converting the surface integral in Eq. (5.2.6) to a volume integral by means of the divergence theorem, Eq. (2.6.20),

$$\int_\Omega \nabla \cdot \mathbf{A}\, d\mathbf{x} = \oint_\Gamma \mathbf{A} \cdot \hat{\mathbf{n}}\, ds,$$

we obtain (with $\mathbf{A} = \rho \mathbf{v}$)

$$\int_\Omega \left[\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{v})\right] d\mathbf{x} = 0. \tag{5.2.8}$$

The previous equation also follows directly from Eq. (5.2.4). Because this integral vanishes, for a continuous medium and for any arbitrary region Ω, we deduce that this can be true only if the integrand itself vanishes identically, giving the following local (i.e., point-wise) form:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{v}) = 0. \tag{5.2.9}$$

This equation, called the *continuity equation*, expresses the local conservation of mass at any point in a continuous medium. Equation (5.2.9) can be written in an alternative form as

$$0 = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} = \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v}, \tag{5.2.10}$$

where the definition of *material time derivative*, Eq. (5.1.3), is used in arriving at the final result.

For steady state, we set the time derivative term in Eq. (5.2.9) to zero and obtain

$$\nabla \cdot (\rho \mathbf{v}) = 0. \tag{5.2.11}$$

For materials with constant density, called *incompressible materials*, we set $D\rho/Dt = 0$ and obtain

$$\rho \nabla \cdot \mathbf{v} = 0, \quad \text{or} \quad \nabla \cdot \mathbf{v} = 0. \tag{5.2.12}$$

Next, we consider examples of the application of the principle of conservation of mass in spatial description.

Example 5.2.1:

Consider a water hose with a conical-shaped nozzle at its end, as shown in Figure 5.2.2(a). **(a)** Determine the pumping capacity required so that the velocity of the water (assuming an incompressible that for the present case) exiting the nozzle is 25 m/sec. (**b**) If the hose is connected to a rotating sprinkler through its base, as shown in Figure 5.2.2(b), determine the average speed of the water leaving the sprinkler nozzle.

Solution:

(**a**) The principle of conservation of mass for steady one-dimensional flow requires

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2. \tag{5.2.13}$$

If the exit of the nozzle is taken as the section 2, we can calculate the flow at section 1 as (for an incompressible fluid, $\rho_1 = \rho_2$)

$$Q_1 = A_1 v_1 = A_2 v_2 = \frac{\pi (20 \times 10^{-3})^2}{4} 25 = 0.0025\pi \text{ m}^3/\text{sec}.$$

(**b**) The average speed of the water leaving the sprinkler nozzle can be calculated using the principle of conservation of mass for steady one-dimensional flow. We obtain

$$Q_1 = 2A_2 v_2 \quad \to \quad v_2 = \frac{2Q_1}{\pi d^2} = \frac{0.005}{(12.5 \times 10^{-3})^2} = 32 \text{ m/sec}.$$

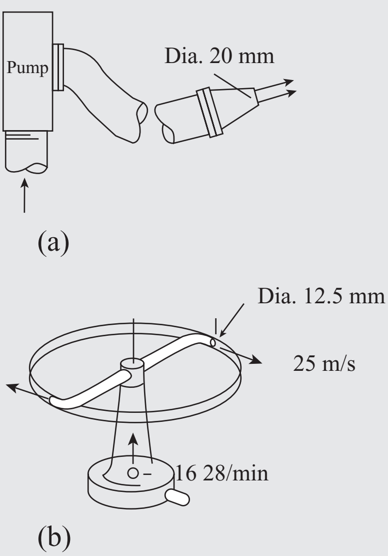


Figure 5.2.2

Example 5.2.2:

Consider a jet airplane moving at a speed of $V_p = 900$ km/hr. The intake and exhaust areas of the jet engine are $A_a = 0.75$ m^2 and $A_g = 0.51$ m^2,

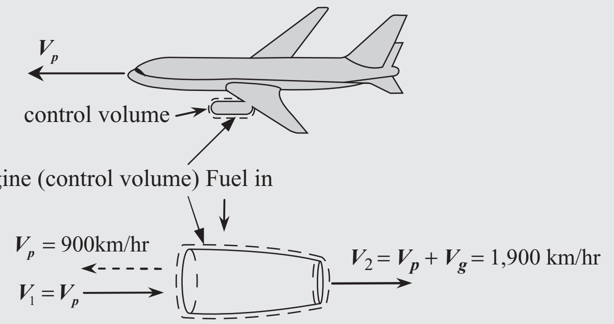


Figure 5.2.3 Estimation of mass flow rate of fuel into the engine of an airplane.

respectively, and the densities of the intake air and the exhaust gas are $\rho_a = 0.735$ kg/m^3 and $\rho_g = 0.525$ kg/m^3, respectively. If a stationary observer estimates that relative to the earth the jet engine exhaust gases move away from the engine at a speed of $V_g = 1{,}000$ km/hr, determine the mass flow rate of the fuel into the engine.

Solution: We select the control volume to be the engine, which moves with the airplane, with flows of all its fluids at an instant, as shown in Figure 5.2.3. The flow is considered to be one-dimensional and steady relative to moving control volume. Then Eq. (5.2.7) takes the form

$$0 = \oint_\Gamma \rho \mathbf{v} \cdot \hat{\mathbf{n}}\, ds = -\dot{m}_{\rm fin} - \rho_a A_a V_1 + \rho_g A_g V_2,$$

where $\dot{m}_{\rm fin}$ is the mass flow rate of fuel into the engine; V_1 is the velocity of intake air, which is the same as the speed of the airplane, $V_1 = V_p$; and V_2 is the velocity of the exhaust gases, which is the sum of the airplane velocity and the velocity of exhaust gases relative to the airplane $V_2 = V_p + V_g = 1{,}000 + 900 = 1{,}900$ km/hr. We obtain

$$\begin{aligned}\dot{m}_{\rm fin} &= \rho_g A_g V_2 - \rho_a A_a V_1 = 0.525 \times 0.51 \times 1{,}900(10^3)\\ &\quad - 0.735 \times 0.75 \times 900(10^3)\\ &= 508{,}725 - 496{,}125 = 12{,}600 \text{ kg/hr.}\end{aligned}$$

5.2.3 Conservation of mass in material description

Under the assumption that the mass is neither created nor destroyed during the motion, we require that the total mass of any material volume be the same at any instant during the motion. To express this in analytical terms, we consider a material body $\mathcal{B}$ that occupies configuration κ_0 with density ρ_0 and volume Ω_0 at time $t = 0$. The same material body occupies the configuration κ with volume Ω at time $t > 0$, and it has a density ρ. As per the principle of conservation of mass, we have

$$\int_{\Omega_0} \rho_0\, d\mathbf{X} = \int_\Omega \rho\, d\mathbf{x}. \tag{5.2.14}$$

Using the relation between $d\mathbf{X}$ and $d\mathbf{x}$, $d\mathbf{x} = J\,d\mathbf{X}$, where J is the determinant of the deformation gradient tensor $\mathbf{F}$, we arrive at

$$\int_{\Omega_0} (\rho_0 - J\rho)\,d\mathbf{X} = 0. \tag{5.2.15}$$

This is the *global form* of the continuity equation. Because the material volume Ω_0 we selected is arbitrarily small, as we shrink the volume to a point we obtain the *local form* of the continuity equation,

$$\rho_0 = J\rho. \tag{5.2.16}$$

The next example illustrates the use of the material time derivative in computing velocities and the use of the continuity equation to compute the density in the current configuration.

Example 5.2.3:

Consider the motion of a body $\mathcal{B}$ described by the mapping

$$x_1 = \frac{X_1}{1 + tX_1}, \quad x_2 = X_2, \quad x_3 = X_3.$$

Determine the material density as a function of position $\mathbf{x}$ and time t.

Solution: First, we compute the velocity components,

$$\mathbf{v} = \frac{D\mathbf{x}}{Dt} = \left(\frac{\partial \mathbf{x}}{\partial t}\right)_{\mathbf{X}=\text{fixed}}, \quad v_i = \frac{Dx_i}{Dt} = \left(\frac{\partial x_i}{\partial t}\right)_{\mathbf{X}=\text{fixed}}. \tag{5.2.17}$$

Therefore, we have

$$v_1 = -\frac{X_1^2}{(1 + tX_1)^2} = -x_1^2, \quad v_2 = 0, \quad v_3 = 0.$$

Next, we compute $D\rho/Dt$ from the continuity equation, Eq. (5.2.10):

$$\frac{D\rho}{Dt} = -\rho\,\text{div}\,\mathbf{v} = 2\rho x_1 = 2\rho\,\frac{X_1}{1 + tX_1}.$$

Integrating this equation, we obtain

$$\int \frac{1}{\rho}\,D\rho = 2\int \frac{X_1}{1 + tX_1}\,Dt \;\Rightarrow\; \ln\rho = 2\ln(1 + tX_1) + \ln c,$$

where c is the constant of integration. If $\rho = \rho_0$ at time $t = 0$, we have $\ln c = \ln\rho_0$. Thus, the material density in the current configuration is

$$\rho = \rho_0\left(1 + tX_1\right)^2 = \frac{\rho_0}{(1 - tx_1)^2}.$$

It can be verified that[1]

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + v_1\frac{\partial\rho}{\partial x_1} = \frac{2\rho_0 x_1}{(1 - tx_1)^2} = 2\rho x_1.$$

[1] Note that $\rho = \rho(x_1, t)$ and $(1 + tX_1) = (1 - tx_1)^{-1}$.

The material density in the current configuration can also be computed using the continuity equation in the material description, $\rho_0 = \rho J$. We have

$$J = \frac{dx_1}{dX_1} = \frac{1}{(1+tX_1)^2}, \quad \text{hence } \rho = \frac{1}{J}\rho_0 = \rho_0(1+tX_1)^2.$$

5.2.4 Reynolds transport theorem

The material derivative operator D/Dt corresponds to changes with respect to a fixed mass, that is, $\rho\, d\mathbf{x}$ is constant with respect to this operator. Therefore, from Eq. (5.2.4) it follows that for $\phi = \rho Q(\mathbf{x}, t)$, the result is

$$\frac{D}{Dt}\int_\Omega \rho Q(\mathbf{x}, t)\, d\mathbf{x} = \frac{\partial}{\partial t}\int_\Omega \rho Q\, d\mathbf{x} + \oint_\Gamma \rho Q \mathbf{v}\cdot\hat{\mathbf{n}}\, ds, \tag{5.2.18}$$

or

$$\begin{aligned}\frac{D}{Dt}\int_\Omega \rho Q(\mathbf{x}, t)\, d\mathbf{x} &= \int_\Omega \left[\rho\frac{\partial Q}{\partial t} + Q\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho Q\mathbf{v})\right] d\mathbf{x}\\ &= \int_\Omega \left[\rho\left(\frac{\partial Q}{\partial t} + \mathbf{v}\cdot\nabla Q\right) + Q\left(\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\mathbf{v})\right)\right] d\mathbf{x},\end{aligned} \tag{5.2.19}$$

and using the continuity equation, Eq. (5.2.9), and the definition of the material time derivative, we arrive at the result

$$\frac{D}{Dt}\int_\Omega \rho Q\, d\mathbf{x} = \int_\Omega \rho\frac{DQ}{Dt}\, d\mathbf{x}. \tag{5.2.20}$$

Equation (5.2.20) is known as the *Reynolds transport theorem*. It is useful in simplifying integral statements that appear in the sequel.

5.3 Conservation of momenta

5.3.1 Principle of conservation of linear momentum

The principle of conservation of linear momentum, or Newton's second law of motion, applied to a set of particles (or rigid body) can be stated as the time rate of change of (linear) momentum of a collection of particles equals the net force exerted on the collection. Written in vector form, the principle implies

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{F}, \tag{5.3.1}$$

where m is the total mass, $\mathbf{v}$ the velocity, and $\mathbf{F}$ the resultant force on the collection particles. For constant mass, Eq. (5.3.1) becomes

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}, \tag{5.3.2}$$

which is the familiar form of Newton's second law. Newton's second law for a control volume Ω can be expressed as

$$\mathbf{F} = \frac{\partial}{\partial t}\int_{\Omega} \rho \mathbf{v}\, d\mathbf{x} + \int_{\Gamma} \rho \mathbf{v}\mathbf{v}\cdot d\mathbf{s}, \tag{5.3.3}$$

where $\mathbf{F}$ is the resultant force and $d\mathbf{s}$ denotes the vector representing an area element of the outflow. Several simple examples that illustrate the use of Eqs. (5.3.2) or (5.3.3) are presented next.

Example 5.3.1:

Suppose that a jet of fluid with area of cross section A and mass density ρ issues from a nozzle with a velocity v and impinges against a smooth inclined flat plate, as shown in Figure 5.3.1. Assuming that there is no frictional resistance between the jet and the plate, determine the distribution of the flow and the force required to keep the plate in position.

Solution: Because there is no change in pressure or elevation before or after impact, the velocity of the fluid remains the same before and after impact. Let the amounts of flow to the left be Q_L and to the right be Q_R. Then the total flow $Q = vA$ of the jet is equal to the sum (by the continuity equation)

$$Q = Q_L + Q_R.$$

Next, we use the principle of conservation of linear momentum to relate Q_L and Q_R. Applying Eq. (5.3.3) to the tangential direction to the plate and noting that the resultant force is zero and the first term on the right-hand side is zero by virtue of the steady-state condition, we obtain

$$0 = \int_{\Gamma} \rho v_t \mathbf{v}\cdot d\mathbf{s} = \rho v(vA_L) + \rho(-v)(vA_R) + \rho v \cos\theta(-vA),$$

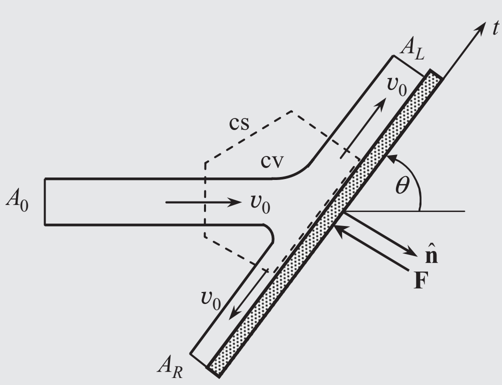


Figure 5.3.1 Jet of fluid impinging on an inclined plate.

which, with $Q_L = A_L v$, $Q_R = A_R v$, and $Q = Av$, yields

$$Q_L - Q_R = Q\cos\theta.$$

Solving the two equations for Q_L and Q_R, we obtain

$$Q_L = \frac{1}{2}(1+\cos\theta)\,Q, \quad Q_R = \frac{1}{2}(1-\cos\theta)\,Q.$$

Thus, the total flow Q is divided into the left flow of Q_L and right flow of Q_R, as given here.

The force exerted on the plate is normal to the plate. By applying the conservation of linear momentum in the normal direction, we obtain

$$-F_n = \int_\Gamma \rho v_n \mathbf{v}\cdot d\mathbf{s} = \rho(v\sin\theta)(-vA) \;\rightarrow\; F_n = \rho Q v\sin\theta.$$

Example 5.3.2:

A chain of total length L and mass ρ per unit length slides down from the edge of a smooth table with an initial overhang x_0 to initiate motion, as shown in Figure 5.3.2. Assuming that the chain is rigid, find the equation of motion governing the chain and the tension in the chain.

Solution: Let x be the amount of chain sliding down the table at any instant t. By considering the entire chain as the control volume, the linear momentum of the chain is

$$\rho(L-x)\cdot\dot{x}\,\hat{\mathbf{e}}_x - \rho x\cdot\dot{x}\,\hat{\mathbf{e}}_y.$$

The resultant force in the chain is $-\rho x g\,\hat{\mathbf{e}}_y$. The principle of linear momentum gives

$$-\rho x g\,\hat{\mathbf{e}}_y = \frac{d}{dt}\left[\rho(L-x)\dot{x}\hat{\mathbf{e}}_x - \rho x\dot{x}\,\hat{\mathbf{e}}_y\right],$$

or

$$(L-x)\ddot{x} - \dot{x}^2 = 0, \quad x\ddot{x} + \dot{x}^2 = gx.$$

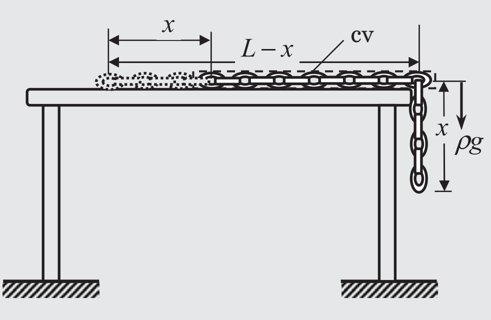


Figure 5.3.2 Chain sliding down a table.

Eliminating $\dot{x}^2$ from these two equations, we arrive at the equation of motion,

$$\ddot{x} - \frac{g}{L}x = 0. \tag{1}$$

The solution of the second-order differential equation is

$$x(t) = A\cosh\lambda t + B\sinh\lambda t, \quad \text{where} \quad \lambda = \sqrt{\frac{g}{L}}. \tag{2}$$

The constants of integration A and B are determined from the initial conditions,

$$x(0) = x_0, \quad \dot{x}(0) = 0, \tag{3}$$

where x_0 denotes the initial overhang of the chain. We obtain

$$A = x_0, \quad B = 0,$$

and the solution becomes

$$x(t) = x_0\cosh\lambda t, \quad \lambda = \sqrt{\frac{g}{L}}. \tag{4}$$

The tension in the chain can be computed by using the principle of linear momentum applied to the control volume of the chain on the table,

$$T = \frac{d}{dt}\left[\rho(L-x)\dot{x}\right] + \rho\dot{x}\dot{x} = \rho(L-x)\ddot{x} = \frac{\rho g}{L}(L-x)x,$$

where the term $\rho\dot{x}\dot{x}$ denotes the momentum flux.

Example 5.3.3:

Consider the problem of a simple pendulum. The system consists of a bob of mass m attached to one end of a rod of length l and the other end is pivoted to a fixed point O, as shown in Figure 5.3.3. Derive the governing equation of motion, that is, the equation governing the angular displacement of the bob. Assume (1) the bob as well as the rod connecting the bob to the fixed point O are rigid, (2) the mass of the rod is negligible relative to the mass of the bob, and (3) there is no friction at the pivot.

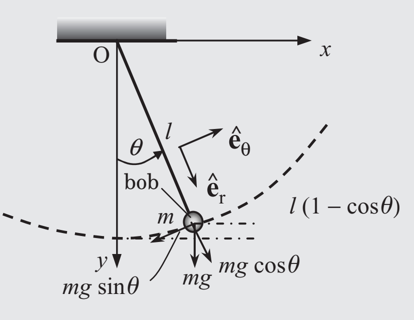


Figure 5.3.3 Simple pendulum.

Solution: The equation governing the motion can be derived using the principle of conservation of linear momentum; that is, the sum of externally applied forces on a body in any direction is equal to the time rate of change of the linear momentum of the body in the same direction. Using the principle in the x direction, we have

$$F_x = m\frac{dv}{dt}, \quad \text{where} \quad F_x = -mg\,\sin\theta, \quad v = l\frac{d\theta}{dt},$$

where m is the mass, θ is the angular displacement, v is the velocity along x, F_x is the force in the x-direction, g is the acceleration due to gravity, and t is the time. Then, the equation for motion becomes

$$-mg\,\sin\theta = ml\frac{d^2\theta}{dt^2} \qquad \text{or} \qquad \frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0.$$

Thus, the problem at hand involves solving the nonlinear differential equation,

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0, \quad 0 < t \leq T, \tag{1}$$

subjected to the initial (i.e., at time $t = 0$) conditions

$$\theta(0) = \theta_0, \qquad \frac{d\theta}{dt}(0) = v_0, \tag{2}$$

where θ_0 and v_0 are the initial values of angular displacement and velocity, respectively. Mathematically, the problem is called an *initial-value problem*. For small θ, $\sin\theta$ is approximately equal to the angle θ, and the motion is described by the linear equation

$$\frac{d^2\theta}{dt^2} + \lambda^2\theta = 0, \qquad \lambda^2 = \frac{g}{l}. \tag{3}$$

The general analytical solution to the linear equation (3) is given by

$$\theta(t) = A\sin\lambda t + B\cos\lambda t, \quad \lambda = \sqrt{\frac{g}{l}}. \tag{4}$$

where A and B are constants to be determined using the initial conditions given previously. We obtain

$$A = \frac{v_0}{\lambda}, \quad B = \theta_0,$$

and the solution to the linear problem is

$$\theta(t) = \frac{v_0}{\lambda}\sin\lambda t + \theta_0\cos\lambda t.$$

In particular, for zero initial velocity and nonzero initial position θ_0, the solution becomes

$$\theta(t) = \theta_0\cos\lambda t, \tag{5}$$

which represents a simple harmonic motion.

Example 5.3.4:

A rod of total length L and cross section $A(x)$ at distance x from its left end, as shown in Figure 5.3.4(a), is fixed at its left end and connected at its right end to a rigid support through a spring. The right end is also subjected to a force P applied at the geometric centroid of the cross section. Assuming that the cross section A is very small compared to its length and the forces applied are small enough that the strains are infinitesimal, derive the equations governing the forces inside the member and at the ends.

Solution: Because the cross-sectional dimension is very small compared to its length, it is reasonable to assume that the stress is uniform at any section and all other stresses are zero. To derive the equation governing the axial forces at any arbitrary point x, we isolate an element of length Δx, as shown in Figure 5.3.4(b), and apply Newton's second law, that is, set the sum of axial forces to zero.

Let the axial stress in the bar at x be $\sigma(x)$. Then the net force at x is $[A\sigma]_x$ acting to the left; the net force at $x + \Delta x$ is $[A\sigma]_{x+\Delta x}$ acting to the right. If the body force per unit length is $f(x)$, the net force is $f(x)\Delta x$. Then summing the forces, we obtain

$$[A\sigma]_{x+\Delta x} - [A\sigma]_x + f(x)\Delta x = 0.$$

Dividing throughout by Δx,

$$\frac{[A\sigma]_{x+\Delta x} - [A\sigma]_x}{\Delta x} + f = 0$$

and taking the limit $\Delta x \to 0$, we obtain

$$\frac{d}{dx}(A\sigma) + f = 0, \quad 0 < x < L. \tag{5.3.4}$$

The condition at $x = 0$ is $u(0) = 0$. The force equilibrium at $x = L$ gives

$$P - (A\sigma)_L - F_s = 0, \tag{5.3.5}$$

where F_s is the compressive force in the spring.

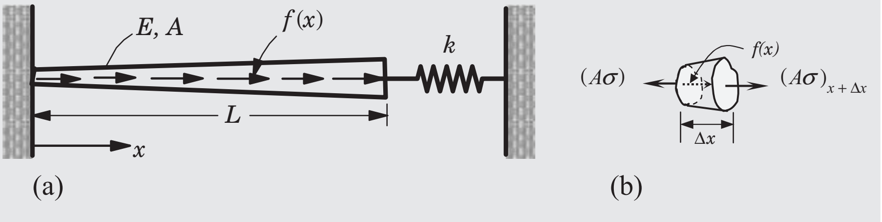


Figure 5.3.4 A rod of variable cross section and subjected to axial load.

Example 5.3.5:

Consider the bending of a straight beam according to the classical (Euler–Bernoulli) beam theory, as discussed in Example 3.2.3. Suppose that the beam is subjected to distributed axial force $f(x)$ and transverse load $q(x)$, as shown

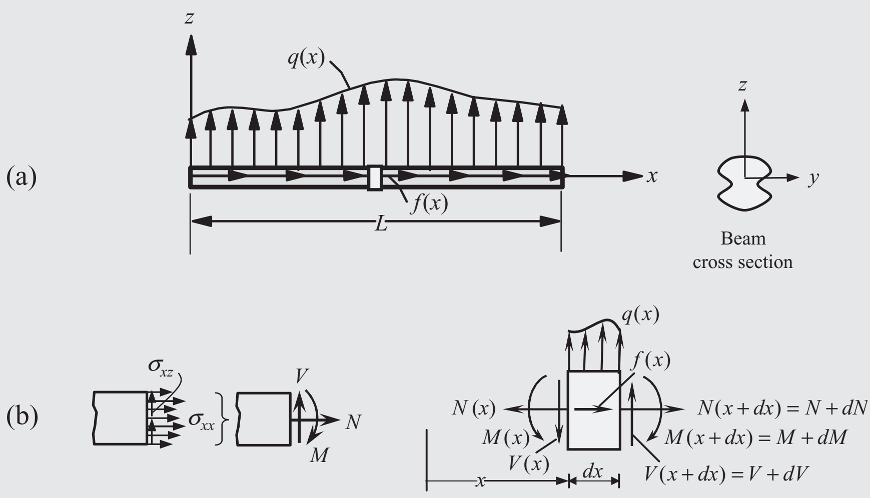


Figure 5.3.5 Bending of beams. (a) A typical beam with axial and transverse forces. (b) Equilibrium of a beam element.

in Figure 5.3.5(a). Consider a typical element of length dx from the beam acted by area-integrated forces and moments, as shown in Figure 5.3.5(b). Using the principle of linear momentum, derive the equations governing the equilibrium of the beam.

Solution: As shown in Figure 5.3.5(b), the following area-integrated forces and moment, called the *stress resultants*, are defined:

$$N(x) = \int_A \sigma_{xx}\, dA, \quad M(x) = \int_A \sigma_{xx} z\, dA, \quad V(x) = \int_A \sigma_{xz} dA. \tag{5.3.6}$$

Here A denotes the area of cross section, $N(x)$ is called the *axial force*, $V(x)$ is called the transverse *shear force*, and $M(x)$ is called the *bending moment*.

First, sum the forces acting in the x-direction on the element of the beam and obtain

$$\sum F_x = 0: \quad -N + (N + dN) + f(x)\, dx = 0.$$

Dividing throughout by dx and taking the limit $dx \to 0$ gives

$$\frac{dN}{dx} + f(x) = 0. \tag{5.3.7}$$

Next, we sum the forces in the z-direction to obtain

$$\sum F_z = 0: \quad -V + (V + dV) + q\, dx = 0.$$

Dividing throughout by dx and taking the limit $dx \to 0$, one obtains

$$\frac{dV}{dx} + q = 0. \tag{5.3.8}$$

Finally, summing the moments of all forces about the y-axis using the point on the right end of the beam, we obtain (clockwise moments are

taken as positive)

$$\sum M_y = 0: \qquad -V\,dx - M + (M + dM) - (q\,dx)(\alpha\,dx) = 0,$$

where α is a constant $0 < \alpha < 1$. Dividing throughout by dx and taking the limit $dx \to 0$, we obtain

$$-V + \frac{dM}{dx} = 0 \text{ or } V = \frac{dM}{dx}. \tag{5.3.9}$$

Equations (5.3.8) and (5.3.9) can be combined by eliminating the shear force V,

$$\frac{d^2M}{dx^2} + q = 0. \tag{5.3.10}$$

Example 5.3.6:

Given the beam with the supports and loads shown in Figure 5.3.6(a), determine the expressions for the shear force $V(x)$ and bending moment $M(x)$ as functions of x.

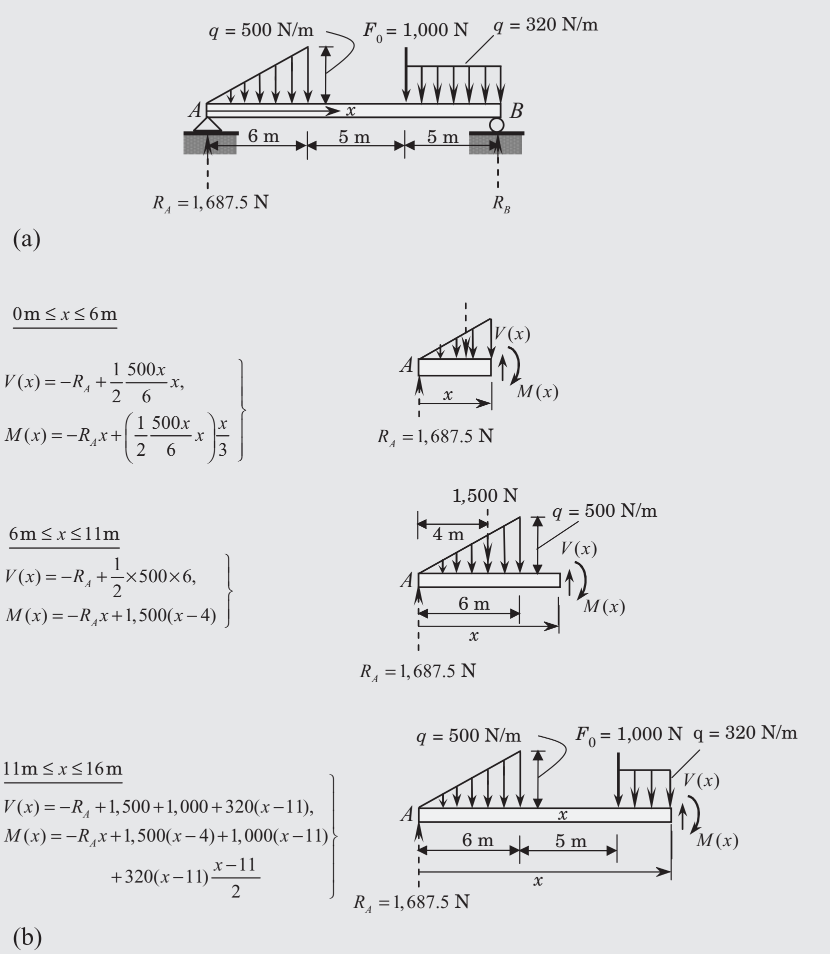


Figure 5.3.6 (a) A beam with given supports and loads. (b) Shear force and bending moment expressions.

Solution: First, we shall determine the reaction force R_A at point A. By taking the moment of all forces about the y-axis at point B, so that the unknown reaction R_B at point B does not enter the equation, we obtain (clockwise moment is taken as positive)

$$R_A \times 16 - \left(\frac{1}{2} \times 500 \times 6\right)\left(10 + \frac{1}{3} \times 6\right) - 1{,}000 \times 5 - (320 \times 5)\frac{5}{2} = 0,$$

or $R_A = 1{,}687.5$ N. Figure 5.3.6(b) contains the expressions for $V(x)$ and $M(x)$, which are derived by summing forces in the vertical direction and moments at a point that is x meter from point A.

Example 5.3.7:

When long slender members like bars or beams are subjected to compressive loads along their lengths, they may deflect laterally and cause a sudden failure. The phenomena of the onset of lateral deflection that causes the member to fail is called *buckling*. For a safe design of slender members subjected to compressive loads, called *columns*, one must ensure that the columns support the working compressive loads without buckling. Derive the equation governing the onset of buckling of an elastic column, and determine the critical buckling load of a column that is hinged at both ends. Assume a uniaxial stress-strain relationship $\sigma = E\varepsilon$, where E denotes the modulus of elasticity (or Young's modulus).

Solution: Consider a column with applied compressive load N_0, as shown in Figure 5.3.7(a). At the onset of buckling, the column deflects laterally, like a beam. Momentarily, the beam is in equilibrium with all its forces, as shown in Figure 5.3.7(b). Identify a typical element of length Δx with all its forces and moments, as shown in Figure 5.3.7(b), where N_0 is the axial compressive force, $V(x)$ is the vertical shear force,[2] and $M(x)$ is the bending moment (see Example 5.3.5). Summing the forces in the z coordinate direction and moments about the y axis, we obtain

$$\sum F_z = 0: \qquad -V + (V + \Delta V) = 0,$$

$$\sum M_y = 0: \qquad -M + (M + \Delta M) - V\Delta x - N_0 \frac{dw}{dx}\Delta x = 0.$$

Dividing throughout by Δx and taking the limit $\Delta x \rightarrow 0$, we obtain the following two equations:

$$\frac{dV}{dx} = 0,$$

$$\frac{dM}{dx} - V - N_0\frac{dw}{dx} = 0,$$

[2] Note that V is the shear force on a section perpendicular to the x-axis, and it is not equal to the shear force $Q(x)$ acting on the section perpendicular to the deformed beam. In fact, one can show that $V = Q - N_{xx}(dw_0/dx)$.

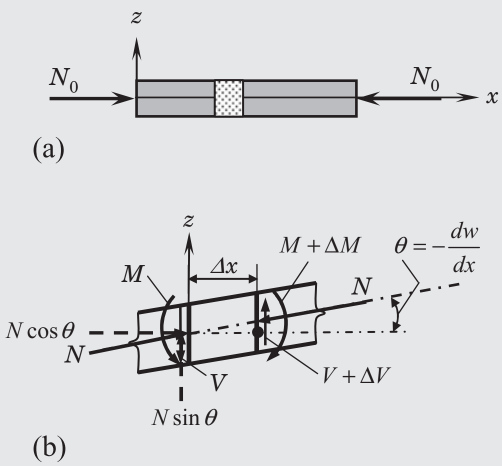


Figure 5.3.7 **(a) A column subjected to axial force. (b) Free-body diagram of an element of the column at the onset of buckling.**

which are equivalent to the single equation

$$\frac{d^2 M}{dx^2} - \frac{d}{dx}\left(N_0 \frac{dw}{dx}\right) = 0. \tag{5.3.11}$$

To complete the derivation, we must relate M to w. Using the definition of M given in Eq. (5.3.6), the strain-displacement relation Eq. (3.2.4), and the stress-strain relation $\sigma = E\varepsilon$ (see Chapter 6 for more discussion), we obtain the relation

$$M(x) = -EI\frac{d^2w}{dx^2}, \quad I = \int_A z^2\, dA. \tag{5.3.12}$$

Combining Eqs. (5.3.11) and (5.3.12), we have the equation governing the buckling of columns:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) + N_0\frac{d^2w}{dx^2} = 0. \tag{5.3.13}$$

Equation (5.3.13) must be solved, using suitable boundary conditions of the column, for the smallest value of N_0, called the *critical buckling load* and for nonzero w, called the *mode shape*. Figure 5.3.8 shows some typical boundary conditions for columns subjected to axial compressive loads.

The general solution of Eq. (5.3.13) for a constant value of EI is

$$w(x) = c_1 \sin\lambda x + c_2 \cos\lambda x + c_3 x + c_4, \quad \lambda^2 = N_0/EI, \tag{5.3.14}$$

where $\lambda^2 = N_0/EI$ and c_i are constants of integration, which are determined using the support conditions at the ends of the beam. For a column hinged at both ends ($x = 0$ and $x = L$), the support conditions can expressed as

$$w = 0, \quad M = -EI\frac{d^2w}{dx^2} = 0. \tag{5.3.15}$$

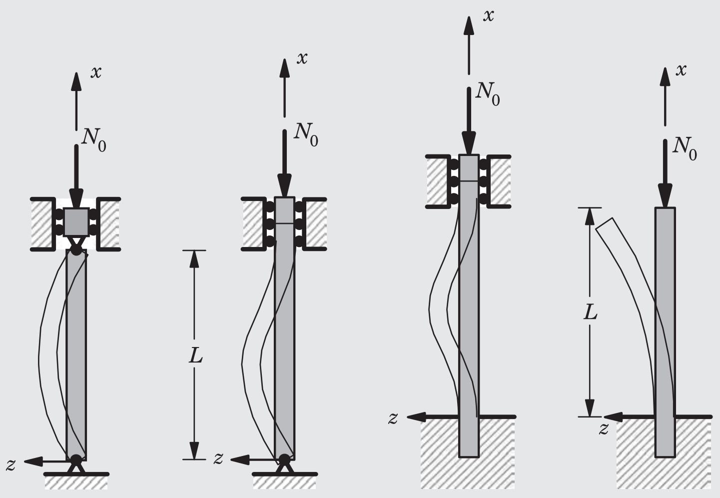


Figure 5.3.8 (a) Hinged-hinged column. (b) Hinged-clamped column. (c) Clamped-clamped column. (d) Clamped-free column.

Use of these conditions on w gives

$$\begin{aligned} w(0) = 0: &\quad c_2 + c_4 = 0 \quad \text{or} \quad c_4 = -c_2. \\ \frac{d^2w}{dx^2}\Big|_{x=0} = 0: &\quad -\lambda^2 c_2 = 0 \quad \text{or} \quad c_2 = 0. \\ w(L) = 0: &\quad c_1 \sin\lambda L + c_3 L = 0. \\ \frac{d^2w}{dx^2}\Big|_{x=L} = 0: &\quad -\lambda^2 c_1 \sin\lambda L = 0. \end{aligned} \tag{5.3.16}$$

From these equations, it follows that

$$c_1 \sin\lambda L = 0, \quad c_2 = 0, \quad c_3 = 0, \quad c_4 = 0. \tag{5.3.17}$$

The first equation implies that either $c_1 = 0$ or (and) $\sin\lambda L = 0$. If $c_1 = 0$, then the buckling deflection w is zero, implying that the column did not begin to buckle. For nonzero deflection w, we must have

$$\sin\lambda L = 0, \quad \text{which implies} \quad \lambda L = n\pi \quad \text{or} \quad N_0 = EI\left(\frac{n\pi}{L}\right)^2. \tag{5.3.18}$$

The smallest value of N_0 at which the column buckles occurs when $n = 1$,

$$N_{cr} = \frac{\pi^2 EI}{L^2} = 9.8696\frac{EI}{L^2}. \tag{5.3.19}$$

Thus, the buckling load is proportional to the column stiffness EI and inversely proportional to the length of the column (i.e., the shorter the column, the more compressive load it can carry).

To derive the equation of motion applied to an arbitrarily fixed region in space through which material flows (i.e., a control volume), we must identify the forces acting on it. Forces acting in system can be classified as *internal* and *external*.

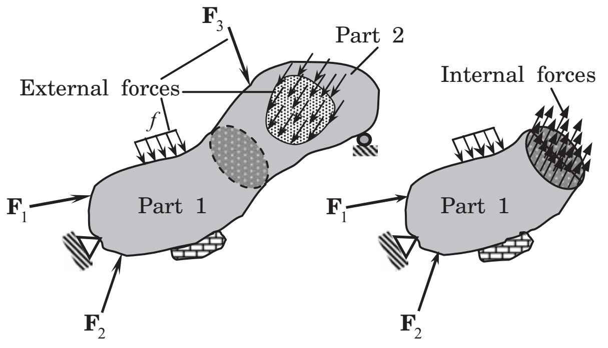


Figure 5.3.9 Internal and external forces in a continuum.

The internal forces, which are generated due to the application of external forces, resist the tendency of one part of the continuum to be separated from the other and are seen only when one part is separated from the other,[3] as shown in Figure 5.3.6. The internal force per unit area is termed stress, as defined in Eq. (4.2.1). The external forces are those applied external to the body; they are transmitted from one part of the structure to the other, as shown in Figure 5.3.9.

The external forces can be further classified as *body* (or *volume*) *forces* and *surface forces*, as shown in Figure 5.3.10(a). Body forces act on the distribution of mass inside the body. Examples of body forces are provided by the gravitational and electromagnetic forces. Body forces are usually measured per unit mass or unit volume of the body. Surface forces are contact forces acting on the boundary surface of the body. Examples of surface forces are provided in Figure 5.3.10(b). Surface forces are reckoned per unit area.

Let $\mathbf{f}$ denote the body force per unit mass. Consider an elemental volume $d\mathbf{x}$ inside Ω. The body force of the elemental volume is equal to $\rho\, d\mathbf{x}\, \mathbf{f}$. Hence, the total body force of the control volume is

$$\int_\Omega \rho\, \mathbf{f}\, d\mathbf{x}. \tag{5.3.20}$$

If $\mathbf{t}$ denotes the surface force per unit area (or surface stress vector), then the surface force on an elemental surface ds of the volume is $\mathbf{t}\, ds$. The total surface force acting on the closed surface of the region Ω is

$$\oint_\Gamma \mathbf{t}\, ds.$$

Because the stress vector $\mathbf{t}$ on the surface is related to the (internal) stress tensor σ by Cauchy's formula [see Eq. (4.2.10)],

$$\mathbf{t} = \hat{\mathbf{n}} \cdot \sigma, \tag{5.3.21}$$

[3] The shear forces and bending moments shown in Figure 5.3.6(b) are seen only because the left part of the beam is separated from the right part, and only the left part of the beam is used to write the expressions. The shear forces and bending moments are developed inside the beam.

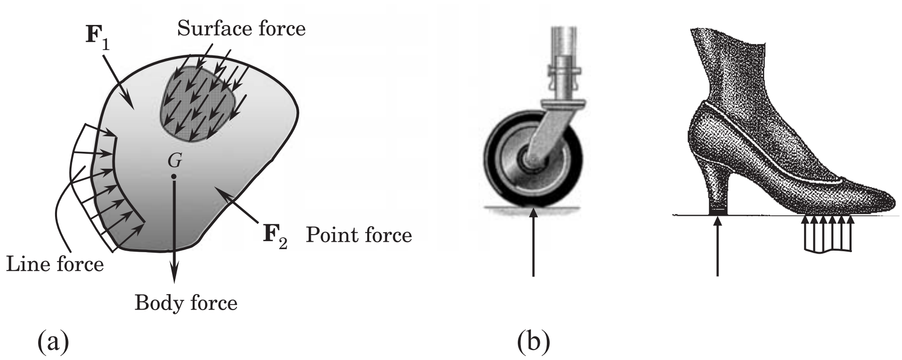


Figure 5.3.10 Body and surface forces on a continuum.

where $\hat{\mathbf{n}}$ denotes the unit normal to the surface, we can express the surface force as

$$\oint_{\Gamma} \hat{\mathbf{n}} \cdot \sigma \, ds.$$

Using the divergence theorem, we can write

$$\oint_{\Gamma} \hat{\mathbf{n}} \cdot \sigma \, ds = \int_{\Omega} \nabla \cdot \sigma \, d\mathbf{x}. \tag{5.3.22}$$

The principle of conservation of linear momentum applied to a given mass of a medium $\mathcal{B}$, instantaneously occupying a region Ω with bounding surface Γ and acted upon by external surface force $\mathbf{t}$ per unit area and body force $\mathbf{f}$ per unit mass, requires

$$\int_{\Omega} (\nabla \cdot \sigma + \rho \mathbf{f}) \, d\mathbf{x} = \frac{D}{Dt} \int_{\Omega} \rho \mathbf{v} \, d\mathbf{x}, \tag{5.3.23}$$

where $\mathbf{v}$ is the velocity vector. Using the Reynolds transport theorem, Eq. (5.2.20), we arrive at

$$0 = \int_{\Omega} \left[\nabla \cdot \sigma + \rho \mathbf{f} - \rho \frac{D\mathbf{v}}{Dt} \right] d\mathbf{x}, \tag{5.3.24}$$

which is the global form of the equation of motion. The local form is given by

$$\nabla \cdot \sigma + \rho \mathbf{f} = \rho \frac{D\mathbf{v}}{Dt} \tag{5.3.25}$$

or

$$\nabla \cdot \sigma + \rho \mathbf{f} = \rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right). \tag{5.3.26}$$

In a Cartesian rectangular system, we have

$$\frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i = \rho \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right). \tag{5.3.27}$$

In the case of steady-state conditions, Eq. (5.3.26) reduces to

$$\nabla \cdot \sigma + \rho \mathbf{f} = \rho \, \mathbf{v} \cdot \nabla \mathbf{v}, \quad \text{or} \quad \frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i = \rho \, v_j \frac{\partial v_i}{\partial x_j}. \tag{5.3.28}$$

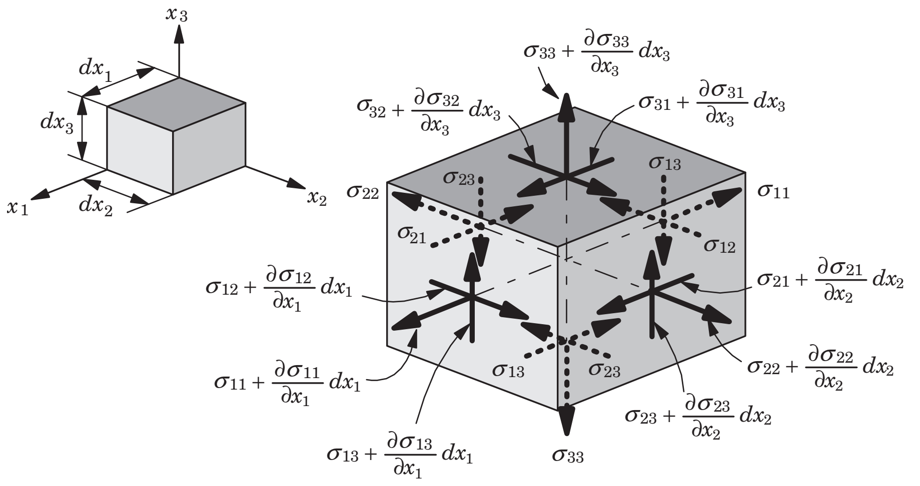


Figure 5.3.11 Stresses on a parallelepiped element.

For kinematically infinitesimal deformation (i.e., when $\mathbf{x} \sim \mathbf{X}$) of solid bodies in static equilibrium, Eq. (5.3.25) reduces to the equations of equilibrium:

$$\nabla \cdot \sigma + \rho \mathbf{f} = \mathbf{0}, \quad \text{or} \quad \frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i = 0. \tag{5.3.29}$$

When the state of stress in the medium is of the form $\sigma = -p\mathbf{I}$ (i.e., the hydrostatic state of stress), the equations of motion, Eq. (5.3.25), reduce to

$$-\nabla p + \rho \mathbf{f} = \rho \frac{D\mathbf{v}}{Dt}. \tag{5.3.30}$$

The stress equilibrium equations of Eq. (5.3.29) can be derived for a solid body directly by considering a volume element of the body, as shown in Figure 5.3.11. Consider the stresses and body forces on an infinitesimal parallelepiped element of a material body. The stresses acting on various faces of the infinitesimal parallelepiped with dimensions dx_1, dx_2, and dx_3 along coordinate lines (x_1, x_2, x_3) are shown in Figure 5.3.11. By Newton's second law of motion, the sum of the forces in the x_1-direction are zero if the body is in equilibrium.

The sum of all forces in the x_1-direction is given by

$$\begin{aligned} 0 &= \left(\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1} dx_1\right) dx_2 dx_3 - \sigma_{11} dx_2 dx_3 + \left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2} dx_2\right) dx_1 dx_3 \\ &\quad - \sigma_{21} dx_1 dx_3 + \left(\sigma_{31} + \frac{\partial \sigma_{31}}{\partial x_3} dx_3\right) dx_1 dx_2 - \sigma_{31} dx_1 dx_2 + \rho f_1 dx_1 dx_2 dx_3 \\ &= \left(\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} - \rho f_1\right) dx_1 dx_2 dx_3. \end{aligned} \tag{5.3.31}$$

Upon dividing throughout by $dx_1 dx_2 dx_3$, we obtain

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + \rho f_1 = 0. \tag{5.3.32}$$

Similarly, the application of Newton's second law in the x_2- and x_3-directions gives, respectively,

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + \rho f_2 = 0 \tag{5.3.33}$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho f_3 = 0. \tag{5.3.34}$$

In index notation, the previous three equations can be expressed as

$$\frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i = 0. \tag{5.3.35}$$

The invariant form of this equation is given by

$$\nabla \cdot \sigma + \rho \mathbf{f} = \mathbf{0}. \tag{5.3.36}$$

In terms of the (x, y, z) coordinate system, we have

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + \rho f_x &= 0, \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + \rho f_y &= 0, \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho f_z &= 0. \end{aligned} \tag{5.3.37}$$

Next, we consider a couple of the examples of the application of the stress equilibrium equations.

Example 5.3.8:

Given the following state of stress ($\sigma_{ij} = \sigma_{ji}$) in a kinematically infinitesimal deformation,

$$\begin{aligned} &\sigma_{11} = -2x_1^2, \quad \sigma_{12} = -7 + 4x_1x_2 + x_3, \quad \sigma_{13} = 1 + x_1 - 3x_2, \\ &\sigma_{22} = 3x_1^2 - 2x_2^2 + 5x_3, \quad \sigma_{23} = 0, \quad \sigma_{33} = -5 + x_1 + 3x_2 + 3x_3, \end{aligned}$$

determine the body force components for which the stress field describes a state of equilibrium.

Solution: The body force components are

$$\begin{aligned} \rho f_1 &= -\left(\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} \right) = -[(-4x_1) + (4x_1) + 0] = 0, \\ \rho f_2 &= -\left(\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} \right) = -[(4x_2) + (-4x_2) + 0] = 0, \\ \rho f_3 &= -\left(\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} \right) = -[1 + 0 + 3] = -4. \end{aligned}$$

Thus, the body is in equilibrium when the body forces are $\rho f_1 = 0$, $\rho f_2 = 0$, and $\rho f_3 = -4$.

Example 5.3.9:

Determine if the following stress field in a kinematically infinitesimal deformation satisfies the equations of equilibrium:

$$\sigma_{11} = x_2^2 + k\left(x_1^2 - x_2^2\right), \quad \sigma_{12} = -2kx_1x_2, \quad \sigma_{13} = 0,$$
$$\sigma_{22} = x_1^2 + k\left(x_2^2 - x_1^2\right), \quad \sigma_{23} = 0, \quad \sigma_{33} = k\left(x_1^2 + x_2^2\right).$$

Solution: We have

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = (2kx_1) + (-2kx_1) + 0 = 0,$$
$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = (-2kx_2) + (2kx_2) + 0 = 0,$$
$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = 0 + 0 + 0 = 0.$$

Thus, the given stress field is in equilibrium in the absence of body forces.

5.3.2 Principle of conservation of angular momentum

The principle of conservation of angular momentum states that the time-rate of change of the total moment of momentum for a continuum is equal to vector sum of the moments of external forces acting on the continuum. The principle as applied to a control volume Ω with a control surface Γ can be expressed as

$$\mathbf{r} \times \mathbf{F} = \frac{\partial}{\partial t}\int_\Omega \rho \mathbf{r} \times \mathbf{v}\, d\mathbf{x} + \int_\Gamma \rho \mathbf{r} \times \mathbf{v}\,(\mathbf{v} \cdot d\mathbf{s}). \tag{5.3.38}$$

An application of the principle is presented next.

Example 5.3.10:

Consider the top view of a sprinkler as shown in Figure 5.3.12. The sprinkler discharges water outward in a horizontal plane, which is in the plane of the paper. The sprinkler exits are oriented at an angle of θ from the tangent line to the circle formed by rotating the sprinkler about its vertical centerline. The sprinkler has a constant cross-sectional flow area of A_0 and discharges a flow rate of Q when $\omega = 0$ at time $t = 0$. Hence, the radial velocity is equal to $v_r = Q/2A_0$. Determine ω (counterclockwise) as a function of time.

Solution: Suppose that the moment of inertia of the empty sprinkler head is I_z and the resisting torque due to friction (from bearings and seals) is T (clockwise). Taking the control volume to be the cylinder of unit height, formed by the rotating sprinkler head. The inflow, being along the axis, has no moment. Thus, the time rate of moment of momentum of the sprinkler head plus the net efflux of the moment of momentum from the control surface is equal to the torque T:

$$-T\hat{\mathbf{e}}_z = \left[2\frac{d}{dt}\int_0^R A_0\rho\omega r^2\, dr + I_z\frac{d\omega}{dt} + 2R\left(\rho\frac{Q}{2}\right)(\omega R - v_r\cos\theta)\right]\hat{\mathbf{e}}_z,$$

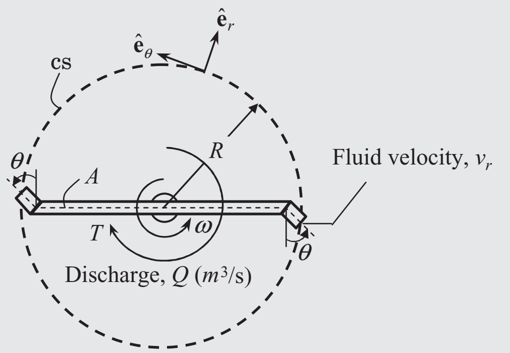


Figure 5.3.12 A rotating sprinkler system.

where the first term represents the time rate of change of the moment of momentum – the moment arm multiplied by the mass of a differential length dr multiplied by the velocity, $r \times (\rho A_0\, dr)(\omega r)$; – the second term is the time rate of change of angular momentum; and the last term represents the efflux of the moment of momentum at the control surface (i.e., the exit of the sprinkler nozzles).

Simplifying the previous equation, we arrive at

$$\left(I_z + \frac{2}{3}\rho A_0 R^3\right) + \rho Q R^2 \omega = \rho Q R v_r \cos\theta - T.$$

This equation indicates that for rotation to start, $\rho Q R v_r \cos\theta - T > 0$. The final value of ω is obtained when the sprinkler motion reaches the steady state, that is, $d\omega/dt = 0$. Thus, at steady state we have

$$\omega_f = \frac{v_r}{R}\cos\theta - \frac{T}{\rho Q R^2}.$$

A continuum said to have no body couples (i.e., volume-dependent couples **M**) if

$$\lim_{\Delta A \to 0} \frac{\Delta \mathbf{M}}{\Delta A} = \mathbf{0} \tag{5.3.39}$$

holds. For a continuum without body couples, the stress tensor can be shown to be symmetric. Consider the moment of all forces acting on the parallelepiped about the x_3-axis, as shown in Figure 5.3.11. Using the right-handed screw rule for positive moment, we obtain

$$\left[\left(\sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_1} dx_1\right) dx_2 dx_3\right] \frac{dx_1}{2} + (\sigma_{12} dx_2 dx_3)\frac{dx_1}{2}$$
$$- \left[\left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2} dx_2\right) dx_1 dx_3\right] \frac{dx_2}{2} - (\sigma_{21} dx_1 dx_3)\frac{dx_2}{2} = 0.$$

Dividing throughout by $\frac{1}{2} dx_1 dx_2 dx_3$ and taking the limit $dx_1 \to 0$ and $dx_2 \to 0$, we obtain

$$\sigma_{12} - \sigma_{21} = 0.$$

Similar considerations of moments about the x_1-axis and x_2-axis give

$$\sigma_{23} - \sigma_{32} = 0, \quad \sigma_{13} - \sigma_{31} = 0.$$

These relations can be expressed as

$$\sigma_{ij} = \sigma_{ji} \text{ for all } i, j = 1, 2, 3. \tag{5.3.40}$$

5.4 Thermodynamic principles

5.4.1 Introduction

The first law of thermodynamics is commonly known as the principle of conservation of energy, and it can be regarded as a statement of the inter-convertibility of heat and work. The law does not place any restrictions on the direction of the process. For instance, in the study of mechanics of particles and rigid bodies, the kinetic energy and potential energy can be fully transformed from one to the other in the absence of friction and other dissipative mechanisms. From our experience, we know that mechanical energy that is converted into heat cannot be entirely converted back into mechanical energy. For example, the motion (i.e., kinetic energy) of a flywheel can be entirely converted into heat (i.e., internal energy) by means of a friction brake; if the whole system is insulated, the internal energy causes the temperature of the system to rise. Although the first law does not restrict the reversal process, namely the conversion of heat to internal energy and internal energy to motion (the flywheel), such a reversal cannot occur because the frictional dissipation is an *irreversible process*. The second law of thermodynamics provides the restriction on the inter-convertibility of energies and will not be discussed here.

5.4.2 Energy equation for one-dimensional flows

In this section, a simple form of the energy equation is derived for use with one-dimensional fluid flow problems. The first law of thermodynamics for a system occupying the domain (control volume) Ω can be written as

$$\frac{D}{Dt}\int_{\Omega} \rho\epsilon \, dV = W_{\text{net}} + H_{\text{net}}, \tag{5.4.1}$$

where ϵ is the total energy stored per unit mass, W_{net} is the net rate of work transferred into the system, and H_{net} is the net rate of heat transfer into the system. The total stored energy per unit mass ϵ consists of the internal energy per unit mass e, the kinetic energy per unit mass $v^2/2$, and the potential energy per unit mass gz (g is the gravitational acceleration and z is the vertical distance above a reference value),

$$\epsilon = e + \frac{v^2}{2} + gz. \tag{5.4.2}$$

The rate of work done in the absence of body forces is given by ($\sigma = \tau - P\mathbf{I}$)

$$W_{\text{net}} = W_{\text{shaft}} - \oint_{\Gamma} P\,\mathbf{v}\cdot\hat{\mathbf{n}}\,ds, \tag{5.4.3}$$

where P is the pressure (normal stress) and W_{shaft} is the rate of work done by the tangential force (due to shear stress).

Using the Reynolds transport theorem, Eq. (5.2.20), and Eqs. (5.4.2) and (5.4.3), we can write Eq. (5.4.1) as

$$\frac{\partial}{\partial t}\int_{\Omega} \rho\epsilon\, dV + \oint_{\Gamma}\left(e + \frac{P}{\rho} + \frac{v^2}{2} + gz\right)\rho\mathbf{v}\cdot\hat{\mathbf{n}}\,ds = W_{\text{shaft}} + H_{\text{net}}, \tag{5.4.4}$$

If only one stream of fluid (compressible or incompressible) enters the control volume, the integral over the control surface in Eq. (5.4.4) can be written as

$$\left(e + \frac{P}{\rho} + \frac{v^2}{2} + gz\right)_{\text{out}}(\rho Q)_{\text{out}} - \left(e + \frac{P}{\rho} + \frac{v^2}{2} + gz\right)_{\text{in}}(\rho Q)_{\text{in}}, \tag{5.4.5}$$

where ρQ denotes the mass flow rate. Finally, if the flow is steady, Eq. (5.4.4) can be written as

$$\left(e + \frac{P}{\rho} + \frac{v^2}{2} + gz\right)_{\text{out}}(\rho Q)_{\text{out}} - \left(e + \frac{P}{\rho} + \frac{v^2}{2} + gz\right)_{\text{in}}(\rho Q)_{\text{in}} = W_{\text{shaft}} + H_{\text{net}}. \tag{5.4.6}$$

In writing this equation, it is assumed that the flow is one-dimensional and the velocity field is uniform. If the velocity profile at sections crossing the control surface is not uniform, a correction must be made to Eq. (5.4.6). In particular, when the velocity profile is not uniform, the integral

$$\oint_{\Gamma}\frac{v^2}{2}\rho\mathbf{v}\cdot\hat{\mathbf{n}}\,ds$$

cannot be replaced with $(v^2/2)(\rho Q) = \rho A v^3/2$, where A is the cross section area of the flow because the integral of v^3 is different when v is uniform or varies across the section. If we define the ratio, called the *kinetic energy coefficient*,

$$\alpha = \frac{\oint_{\Gamma}\frac{v^2}{2}\rho\mathbf{v}\cdot\hat{\mathbf{n}}\,ds}{(\rho Q v^2/2)}, \tag{5.4.7}$$

Eq. (5.4.6) can be expressed as

$$\begin{aligned}&\left(e + \frac{P}{\rho} + \frac{\alpha v^2}{2} + gz\right)_{\text{out}}(\rho Q)_{\text{out}}\\ &\quad - \left(e + \frac{P}{\rho} + \frac{\alpha v^2}{2} + gz\right)_{\text{in}}(\rho Q)_{\text{in}} = W_{\text{shaft}} + H_{\text{net}}.\end{aligned} \tag{5.4.8}$$

An example of the application of the energy equation, Eq. (5.4.7), is presented next.

Example 5.4.1:

A pump delivers water at a steady rate of Q_0 (gal/min), as shown in Figure 5.4.1. If the left-side pipe is of diameter d_1 (in.) and the right-side pipe is of diameter d_2 (in.), and the pressures in the two pipes are P_1 and P_2 (psi), respectively, determine the horsepower (hp) required by the pump if the rise in the internal energy across the pump is e. Assume that there is no change of elevation in water level across the pump, and the pumping process is adiabatic, that is, the heat transfer rate is zero. Use the following data ($\alpha = 1$):

$$\rho = 1.94 \text{ slugs/ft}^3, \quad d_1 = 4 \text{ in.}, \quad d_2 = 1 \text{ in.},$$

$$P_1 = 20 \text{ psi}, \quad P_2 = 50 \text{ psi}, \quad Q_0 = 350 \text{ gal/min}, \quad e = 3300 \text{ lb-ft/slug}.$$

Solution: We take the control volume between the entrance and exit sections of the pump, as shown in dashed lines in Figure 5.4.1. The mass flow rate entering and exiting the pump is the same (conservation of mass) and equal to

$$\rho Q_0 = \frac{1.94 \times 350}{7.48 \times 60} = 1.513 \text{ slugs/sec}.$$

The velocities at Sections 1 and 2 are (converting all quantities to proper units)

$$v_1 = \frac{Q_0}{A_1} = \frac{350}{7.48 \times 60} \frac{4 \times 144}{16\pi} = 8.94 \text{ ft/sec},$$
$$v_2 = \frac{Q_0}{A_2} = \frac{350}{7.48 \times 60} \frac{4 \times 144}{\pi} = 143 \text{ ft/sec}.$$

For adiabatic flow $H_{\text{net}} = 0$, the potential energy term is zero because of no elevation difference between the entrance and exits, and $e = e_2 - e_1 = 3300$ ft-lb/slug. Thus, we have

$$\begin{aligned} W_{\text{shaft}} &= \rho Q_0 \left[\left(e + \frac{P}{\rho} + \frac{v^2}{2} \right)_2 - \left(e + \frac{P}{\rho} + \frac{v^2}{2} \right)_1 \right] \\ &= (1.513) \left[3300 + \frac{(50-20) \times 144}{1.94} + \frac{(143)^2 - (8.94)^2}{2} \right] \frac{1}{550} \\ &= 43.22 \text{ hp}. \end{aligned}$$

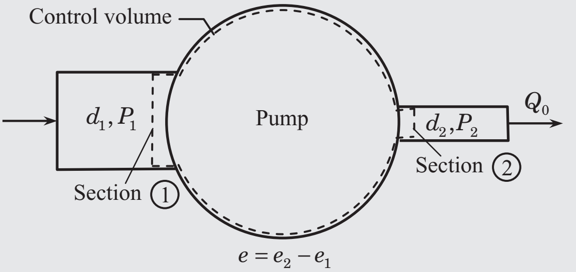


Figure 5.4.1 The pump considered in Example 5.4.1.

Example 5.4.2:

A rod of total length L and cross section $A(x)$ is maintained at temperature T_0 at the left end, and the right end is exposed to an ambient temperature, as shown in Figure 5.4.2(a). Assuming that the cross section A is very small compared to its length, derive the equation governing steady-state heat conduction along the length of the rod.

Solution: Let the coordinate x be taken along the length of the rod. When the temperature varies in the x direction, then there is a heat flux q along the x direction. Suppose that the surface of the rod is insulated so that there is no heat transfer across the surface of the rod and that there is a source within the rod generating energy at a rate of $f = f(x)$ (W/m^3). In practice, such an energy source can be due to nuclear fission or chemical reactions taking place within the rod, or due to the passage of electric current passing through the medium. Because the cross-sectional dimension is very small compared to its length, it is reasonable to assume that the heat flux is uniform at any section.

We isolate an element of length Δx, as shown in Figure 5.4.2(b), and apply the conservation of energy to the element. Let the axial flux in the bar at x be $q(x)$. Then the net heat flow in at x is $[Aq]_x$; the net heat flow out at $x + \Delta x$ is $[Aq]_{x+\Delta x}$. The net internal heat generation is $f(x)\Delta x$. Then the balance of energy gives

$$-[Aq]_{x+\Delta x} + [Aq]_x + f\Delta x = 0.$$

Dividing throughout by Δx,

$$-\frac{[Aq]_{x+\Delta x} - [Aq]_x}{\Delta x} + f = 0,$$

and taking the limit $\Delta x \to 0$, we obtain

$$-\frac{d}{dx}(Aq) + f = 0, \quad 0 < x < L. \tag{5.4.9}$$

For heat transfer across a plane wall, we take $A = 1$.

The condition at $x = 0$ is $T(0) = T_0$. The heat balance at $x = L$ gives

$$-(Aq)_L - F_c = 0, \tag{5.4.10}$$

where F_c is the heat flow into the rod due to convection at $x = L$.

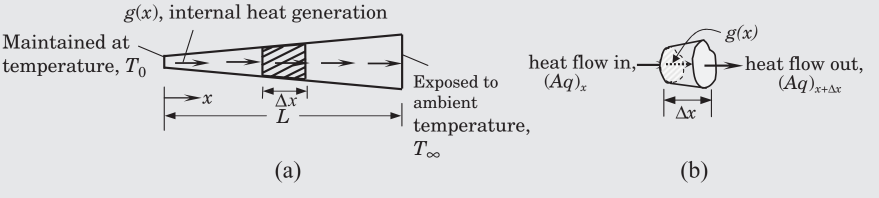


Figure 5.4.2 Steady-state heat transfer in a rod of variable cross section.

5.4.3 Energy equation for a three-dimensional continuum

The first law of thermodynamics states that the time rate of change of the total energy is equal to the sum of the rate of work done by the external forces and the change of heat content per unit time. The total energy is the sum of the kinetic energy and the internal energy. The principle of conservation of energy can be expressed as

$$\frac{D}{Dt}(K+U) = W + H. \tag{5.4.11}$$

Here K denotes the kinetic energy, U is the internal energy, W is the power input, and H is the heat input to the system.

The kinetic energy of the system is given by

$$K = \frac{1}{2}\int_\Omega \rho \mathbf{v}\cdot\mathbf{v}\, d\mathbf{x}, \tag{5.4.12}$$

where $\mathbf{v}$ is the velocity vector. If e is the energy per unit mass (or *specific internal energy*), the total internal energy of the system is given by

$$U = \int_\Omega \rho e\, d\mathbf{x}. \tag{5.4.13}$$

The kinetic energy (K) of a system is the energy associated with the macroscopically observable velocity of the continuum. The kinetic energy associated with the (microscopic) motions of molecules of the continuum is a part of the internal energy; the elastic strain energy and other forms of energy are also parts of internal energy, U.

In the nonpolar case (i.e., without body couples), the power input consists of the rate of work done by external surface tractions $\mathbf{t}$ per unit area and body forces $\mathbf{f}$ per unit volume of the region Ω bounded by Γ:

$$\begin{aligned} W &= \oint_\Gamma \mathbf{t}\cdot\mathbf{v}\, ds + \int_\Omega \rho\mathbf{f}\cdot\mathbf{v}\, d\mathbf{x} \\ &= \oint_\Gamma (\hat{\mathbf{n}}\cdot\sigma)\cdot\mathbf{v}\, ds + \int_\Omega \rho\mathbf{f}\cdot\mathbf{v}\, d\mathbf{x} \\ &= \int_\Omega [\nabla\cdot(\sigma\cdot\mathbf{v}) + \rho\mathbf{f}\cdot\mathbf{v}]\, d\mathbf{x} \\ &= \int_\Omega [(\nabla\cdot\sigma + \rho\mathbf{f})\cdot\mathbf{v} + \sigma : \nabla\mathbf{v}]\, d\mathbf{x} \\ &= \int_\Omega \left(\rho\frac{D\mathbf{v}}{Dt}\cdot\mathbf{v} + \sigma : \nabla\mathbf{v}\right) d\mathbf{x}, \end{aligned} \tag{5.4.14}$$

where the ":" symbol denotes the "double-dot product," $\mathbf{\Phi} : \mathbf{\Psi} = \Phi_{ij}\Psi_{ji}$. Note that the Cauchy formula, the symmetry of the stress tensor, and the equation of motion Eq. (5.3.25) are used in arriving at the last line. Using the symmetry of σ,

we can write $\sigma\colon \nabla\mathbf{v} = \sigma\colon \mathbf{D}$. Hence, we can write

$$W = \frac{1}{2}\int_\Omega \rho \frac{D}{Dt}(\mathbf{v}\cdot\mathbf{v})\,d\mathbf{x} + \int_\Omega \sigma\colon \mathbf{D}\,d\mathbf{x}$$
$$= \frac{1}{2}\frac{D}{Dt}\int_\Omega \rho\,\mathbf{v}\cdot\mathbf{v}\,d\mathbf{x} + \int_\Omega \sigma\colon \mathbf{D}\,d\mathbf{x}, \tag{5.4.15}$$

where $\mathbf{D}$ is the rate of deformation tensor [see Eq. (3.5.2)],

$$\mathbf{D} = \frac{1}{2}\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\right],$$

and the Reynolds theorem is used to write the final expression.

The rate of heat input consists of conduction through the surface s and heat generation inside the region Ω (possibly from a radiation field or transmission of electric current). Let $\mathbf{q}$ be the heat flux vector and $\mathcal{E}$ be the internal heat generation per unit mass. Then the heat inflow across the surface element ds is $-\mathbf{q}\cdot\hat{\mathbf{n}}\,ds$, and internal heat generation in the volume element $d\mathbf{x}$ is $\rho\mathcal{E}d\mathbf{x}$. Hence, the total heat input is

$$H = -\oint_\Gamma \mathbf{q}\cdot\hat{\mathbf{n}}\,ds + \int_\Omega \rho\mathcal{E}\,d\mathbf{x} = \int_\Omega (-\nabla\cdot\mathbf{q} + \rho\mathcal{E})\,d\mathbf{x}. \tag{5.4.16}$$

Substituting expressions for K, U, W, and H from Eqs. (5.4.12), (5.4.13), (5.4.15), and (5.4.16) into Eq. (5.4.11), we obtain

$$\frac{D}{Dt}\int_\Omega \rho\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v} + e\right)d\mathbf{x} = \frac{1}{2}\frac{D}{Dt}\int_\Omega \rho\,\mathbf{v}\cdot\mathbf{v}\,d\mathbf{x} + \int_\Omega (\sigma\colon \mathbf{D} - \nabla\cdot\mathbf{q} + \rho\mathcal{E})\,d\mathbf{x},$$

or

$$0 = \int_\Omega \left(\rho\frac{De}{Dt} - \sigma\colon \mathbf{D} + \nabla\cdot\mathbf{q} - \rho\mathcal{E}\right)d\mathbf{x}, \tag{5.4.17}$$

which is the global form of the energy equation. The local form of the energy equation is given by

$$\rho\frac{De}{Dt} = \sigma\colon \mathbf{D} - \nabla\cdot\mathbf{q} + \rho\mathcal{E}, \tag{5.4.18}$$

which is known as the *thermodynamic form* of the energy equation for a continuum. The term $\sigma\colon \mathbf{D}$ is known as the *stress power*, which can be regarded as the internal production of energy.

In the case of viscous fluids, the total stress tensor σ is decomposed into a viscous part and a pressure part,

$$\sigma = \tau - p\,\mathbf{I}, \tag{5.4.19}$$

where p is the pressure and τ is the viscous stress tensor. Then Eq. (5.4.18) can be written as (note that $\mathbf{I} : \mathbf{D} = \nabla\cdot\mathbf{v}$)

$$\rho\frac{De}{Dt} = \Phi - p\,\nabla\cdot\mathbf{v} - \nabla\cdot\mathbf{q} + \rho\mathcal{E}, \tag{5.4.20}$$

where Φ is called the *viscous dissipation* function,

$$\Phi = \tau\colon \mathbf{D}. \tag{5.4.21}$$

For incompressible materials (i.e., div $\mathbf{v} = 0$), Eq. (5.4.20) reduces to

$$\rho \frac{De}{Dt} = \Phi - \nabla \cdot \mathbf{q} + \rho \mathcal{E}. \tag{5.4.22}$$

The internal energy e can be related to heat capacity c_v or c_p, pressure P, specific volume $v = 1/\rho$, or temperature T. The heat capacity at constant volume, c_v, may be defined as

$$c_v \equiv \left(\frac{\partial e}{\partial T}\right)_{\text{fixed volume}}. \tag{5.4.23}$$

Then for incompressible materials, we have

$$\frac{De}{Dt} = \frac{\partial e}{\partial T}\frac{DT}{Dt} = c_v \frac{DT}{Dt}. \tag{5.4.24}$$

Consequently, for incompressible continua, the conservation of energy equation, Eq. (5.4.22), becomes

$$\rho\, c_v \frac{DT}{Dt} = \Phi - \nabla \cdot \mathbf{q} + \rho \mathcal{E}. \tag{5.4.25}$$

For a solid material, we have $\mathbf{v} = \mathbf{0}$ and $\Phi = 0$, and Eq. (5.4.25) becomes

$$\rho c_v \frac{\partial T}{\partial t} = -\nabla \cdot \mathbf{q} + \rho \mathcal{E}. \tag{5.4.26}$$

For steady one-dimensional heat flow ($\mathbf{q} = q\,\hat{\mathbf{i}}$) through a plane wall, Eq. (5.4.26) reduces to Eq. (5.4.9) with $f = \rho \mathcal{E}$.

Example 5.4.3:

The heat flow through a solid is $\mathbf{q} = q_x\,\hat{\mathbf{i}} + q_y\,\hat{\mathbf{j}} + q_z\,\hat{\mathbf{k}}$ (J/(s-m^2)). Determine the heat flux across planes whose normals are (a) $\hat{\mathbf{i}}$, (b) $\hat{\mathbf{j}}$, and (c) $\hat{\mathbf{n}} = 0.6\hat{\mathbf{i}} + 0.8\hat{\mathbf{j}}$.

Solution: The heat flux through any plane with normal $\hat{\mathbf{n}}$ is equal to $q_n = \hat{\mathbf{n}} \cdot \mathbf{q}$. Hence, the heat fluxes on the three planes given in the question are (a) $q = q_x$, (b) $q = q_y$, and (c) $q_n = 0.6\,q_x + 0.8\,q_y$, respectively.

5.5 Summary

This chapter was devoted to the derivation of the field equations governing a continuous medium using the principles of conservation of mass, momentum, and energy, and therefore constitutes the heart of the book. The equations are derived in invariant (i.e., vector and tensor) form so that they can be expressed in any chosen coordinate system (e.g., rectangular, cylindrical, spherical, or even curvilinear system). The principle of conservation of mass results in the continuity equation; the principle of conservation of linear momentum, which is equivalent to Newton's second law of motion, leads to the equations of motion in terms of the Cauchy stress tensor; the principle of conservation of angular momentum, in

the absence of body couples, yields the symmetry of the Cauchy stress tensor; and the first law of thermodynamics gives rise to the energy equation. Numerous examples taken from fluid mechanics, solid mechanics, and heat transfer are presented to illustrate the utility of the conservation principles in the solutions of engineering problems.

The second law of thermodynamics (not derived here) places restrictions on thermodynamic processes. Often, the constitutive relations developed are required to be consistent with the second law of thermodynamics.

The subject of mechanics is primarily concerned with the determination of the behavior (that is, ρ, $\mathbf{v}$, θ, and so on) of a body under externally applied causes (for example, $\mathbf{f}$, r, and so on). After introducing suitable constitutive relations for σ, e, and $\mathbf{q}$, to be discussed in the next chapter, this task involves solving governing differential equations under specified initial and boundary conditions.

PROBLEMS

5.1. Derive the continuity equation in the cylindrical coordinate system by considering a differential volume element, shown in Figure P5.1.

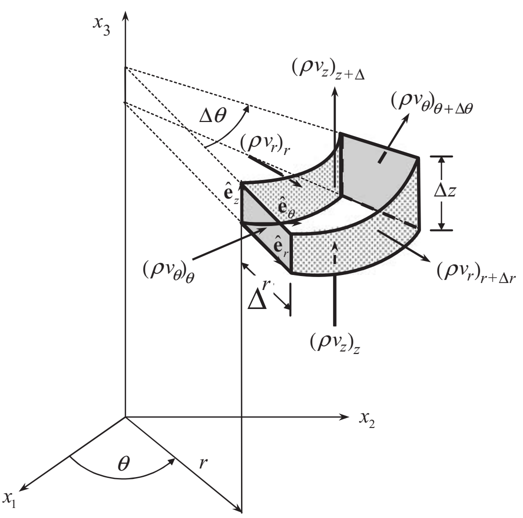


Figure P5.1

5.2. Determine if the following velocity fields for an incompressible flow satisfy the continuity equation:

(a) $v_1(x_1, x_2) = -\frac{x_1}{r^2}, \quad v_2(x_1, x_2) = -\frac{x_2}{r^2}$ where $r^2 = x_1^2 + x_2^2$.

(b) $v_r = 0, \ v_\theta = 0, \ v_z = c\left(1 - \frac{r^2}{R^2}\right)$ where c and R are constants.

5.3. The velocity distribution between two parallel plates separated by a distance b is

$$v_x(y) = \frac{y}{b}v_0 - c\frac{y}{b}\left(1 - \frac{y}{b}\right), \quad v_y = 0, \quad v_z = 0, \quad 0 < y < b,$$

where y is measured from and normal to the bottom plate, x is taken along the plates (as shown in Figure **??**), v_x is the velocity component parallel to the plates, v_0 is the velocity of the top plate in the x direction, and c is a constant. Determine if the velocity field satisfies the continuity equation and find the volume rate of flow and the average velocity.

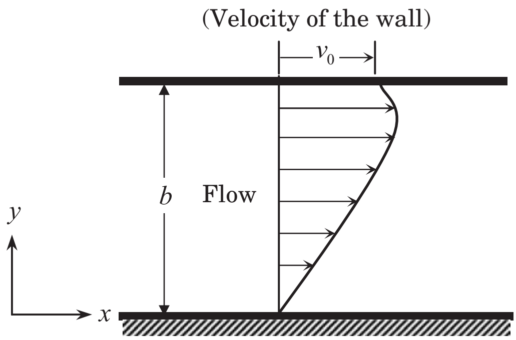


Figure P5.3

5.4. A jet of air ($\rho = 1.206$ kg/m^3) impinges on a smooth vane with a velocity $v = 50$ m/sec at the rate of $Q = 0.4$ m^3/sec. Determine the force required to hold the plate in position for the two different vane configurations shown in Figure P5.4. Assume that the vane splits the jet into two equal streams, and neglect any energy loss in the streams.

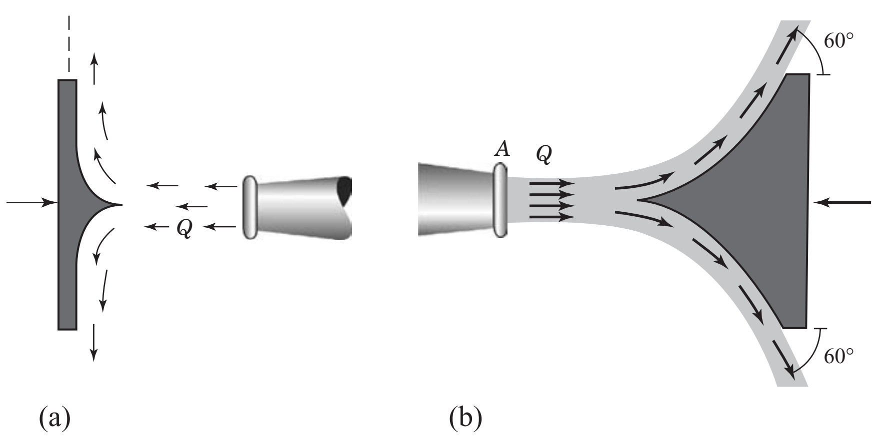


Figure P5.4

5.5. *Bernoulli's Equations.* Consider a flow with hydrostatic pressure, $\sigma = -P\mathbf{I}$, and conservative body force, $\mathbf{f} = -\text{grad}\,\phi$.

(a) For steady flow, show that

$$\mathbf{v} \cdot \text{grad}\left(\frac{v^2}{2} + \phi\right) + \frac{1}{\rho}\mathbf{v} \cdot \text{grad}\, P = 0.$$

(b) For steady and irrotational (i.e., curl $\mathbf{v} = \mathbf{0}$) flow, show that

$$\text{grad}\left(\frac{v^2}{2} + \phi\right) + \frac{1}{\rho}\text{grad}\, P = 0.$$

5.6. Use the Bernoulli's equation (which is valid for steady, frictionless, incompressible flow) derived in Problem 5.5 to determine the velocity and discharge of the fluid at the exit of the nozzle in the wall of the reservoir shown in Figure P5.6.

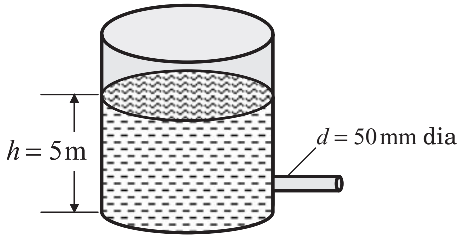


Figure P5.6

5.7. Determine the reactions at the fixed end A of the structure shown in Figure P5.7.

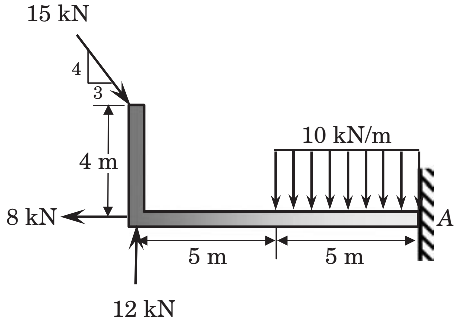


Figure P5.7

5.8.–5.11. For the beam problems shown in Figures P5.8 through P5.11, determine the expressions for shear force $V(x)$ and $M(x)$.

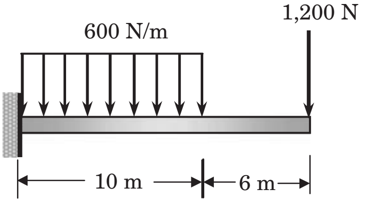


Figure P5.8

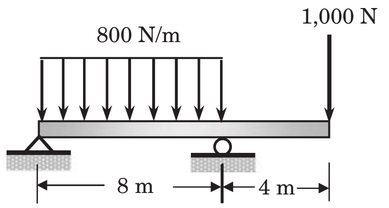


Figure P5.9

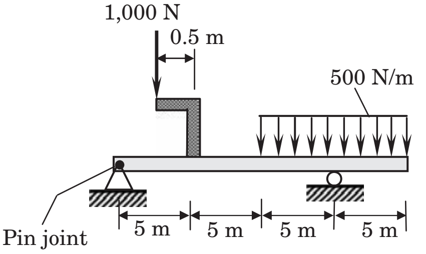


Figure P5.10

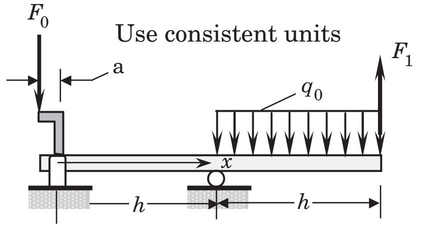


Figure P5.11

5.12. If the stress field in a body has the following components in a rectangular Cartesian coordinate system

$$[\sigma] = a\begin{bmatrix} x_1^2x_2 & (b^2 - x_2^2)x_1 & 0 \\ (b^2 - x_2^2)x_1 & \frac{1}{3}(x_2^2 - 3b^2)x_2 & 0 \\ 0 & 0 & 2bx_3^2 \end{bmatrix},$$

where a and b constants, determine the body force components necessary for the body to be in equilibrium.

5.13. A two-dimensional state of stress exists in a body with no body forces. The following components of stress are given:

$$\sigma_{11} = c_1x_2^3 + c_2x_1^2x_2 - c_3x_1, \quad \sigma_{22} = c_4x_2^3 - c_5,$$
$$\sigma_{12} = c_6x_1x_2^2 + c_7x_1^2x_2 - c_8,$$

where c_i are constants. Determine the conditions on the constants so that the stress field is in equilibrium.

5.14. For a cantilevered beam bent by a point load at the free end, as shown in Figure **??**, the bending moment M about the y-axis is given by $M = -Px$. The axial stress σ_{xx} is given by

$$\sigma_{xx} = \frac{Mz}{I} = -\frac{Pxz}{I},$$

where I is the moment of inertia of the cross section about the y-axis. Starting with this equation, use the two-dimensional equilibrium equations to determine stresses σ_{zz} and σ_{xz} as functions of x and z.

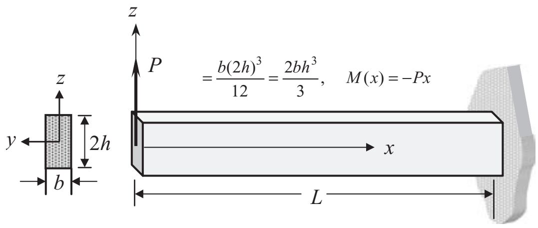


Figure P5.14

5.15. Derive the stress equilibrium equations in cylindrical coordinates by considering the equilibrium of a typical volume element, shown in Figure P5.15.

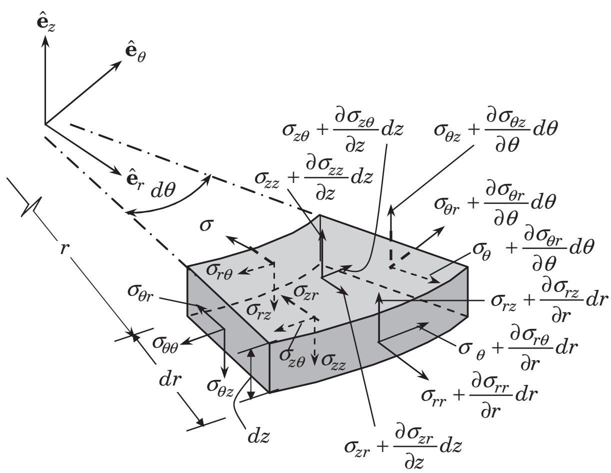


Figure P5.15

5.16. A sprinkler with four nozzles, each nozzle having a length $r = 0.1$ m and an exit area of $A = 0.25$ cm^2, rotates at a constant angular velocity of $\omega =$ 20 rad/sec and distributes water ($\rho = 10^3$ kg/m^3) at the rate of $Q = 0.5$ L/sec (see Figure P5.16). Determine (a) the torque T required on the shaft of the sprinkler to maintain the given motion, and (b) the angular velocity ω_0 at which the sprinkler rotates when no external torque is applied.

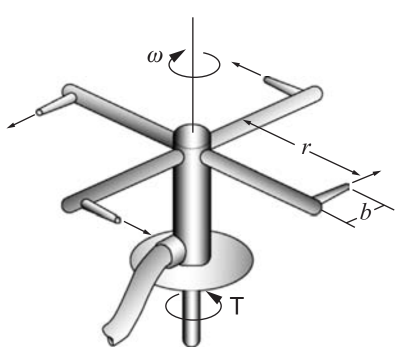


Figure P5.16

5.17. *Finite difference method*. The nonlinear equation (1) in Example 5.3.3 can be solved using a numerical method, such as the finite difference method. As an example, consider the first-order equation

$$\frac{du}{dt} = f(t, u), \tag{1}$$

where f is a known function of t and u. Equation (1) is subjected to an initial condition on u, that is, $u(0) = u_0$. In the finite difference method, one seeks the values of u for discrete values of t. Suppose that the range of the coordinate t (usually denoting time), $0 \le t \le T$, is divided into a finite number of subintervals called time steps, Δt_1, $\Delta t_2, \ldots$, and let t_s denote the value $t_s = \sum_{i=1}^{s} \Delta t_i$. Suppose that u is known for values of $t = 0, t_1, t_2, \ldots, t_s$, and we wish to find its value for $t = t_{s+1}$. The derivative du/dt at $t = t_s$ can be approximated by

$$\left(\frac{du}{dt}\right)\bigg|_{t=t_s} \approx \frac{u(t_{s+1}) - u(t_s)}{t_{s+1} - t_s} = f(u_s, t_s) \tag{2}$$

or

$$u_{s+1} = u_s + \Delta t_s\ f(u_s, t_s), \tag{3}$$

where $u_s = u(t_s)$ and $\Delta t_s = t_{s+1} - t_s$. Equation (3) can be solved, starting from the known value u_0 of $u(0)$ at $t = 0$, for $u_1 = u(t_1) = u(\Delta t_1)$. This process can be repeated to determine the values of u at times $t = \Delta t_1, \Delta t_1 + \Delta t_2, \ldots$. This is known as *Euler's explicit method*, also known as the *forward difference scheme*. Note that we are able to convert the ordinary differential equation, Eq. (1), to an algebraic equation, Eq. (3), which must be evaluated at different times to construct the time history of $u(t)$. Euler's explicit method can be applied to the nonlinear second-order equation in Example 5.3.3. First, rewrite Eq. (1) of Example 5.3.3 as a pair of first-order equations,

$$\frac{d\theta}{dt} = v, \qquad \frac{dv}{dt} = -\lambda^2 \sin\theta, \tag{4}$$

which are coupled (i.e., one cannot be solved without the other). Then, apply the scheme of Eq. (3) to the two equations at hand and obtain

$$\theta_{s+1} = \theta_s + \Delta t_s\ v_s; \quad v_{s+1} = v_s - \Delta t\ \lambda^2 \sin\theta_s. \tag{5}$$

The expressions for θ_{s+1} and v_{s+1} in Eq. (5) must be evaluated repeatedly using the known solution (θ_s, v_s) from the previous step. At time $t = 0$, use the known initial values (θ_0, v_0). Write a computer program to determine θ_{s+1} and v_{s+1} for uniform step size $\Delta t_1 = \Delta_2 = \ldots = \Delta t_s$. Obtain numerical solutions for two different step sizes, $\Delta t = 0.05$ and $\Delta t = 0.025$, and compare these with the exact linear solution (with $v_0 = 0$ and $\theta_0 = \pi/4$).

6 Constitutive Equations

The truth is, the science of Nature has been already too long made only a work of the brain and the fancy. It is now high time that it should return to the plainness and soundness of observations on material and obvious things.

Robert Hooke

6.1 Introduction

The kinematic relations developed in Chapter 3, and the principles of conservation of mass and momenta and thermodynamic principles discussed in Chapter 5, are applicable to any continuum irrespective of its physical constitution. The kinematic variables such as the strains and temperature gradient, and kinetic variables such as the stresses and heat flux were introduced independently of each other. *Constitutive equations* are those relations that connect the primary field variables (e.g., ρ, T, $\mathbf{x}$, and $\mathbf{u}$ or $\mathbf{v}$) to the secondary field variables (e.g., e, $\mathbf{q}$, and σ). In essence, constitutive equations are mathematical models of the behavior of materials that are validated against experimental results. The differences between theoretical predictions and experimental findings are often attributed to inaccurate representation of the constitutive behavior.

A material body is said to be *homogeneous* if the material properties are the same throughout the body (i.e., independent of position). In a *heterogeneous* body, the material properties are a function of position. An *anisotropic* body is one that has different values of a material property in different directions at a point, that is, material properties are direction-dependent. An *isotropic* material is one for which every material property is the same in all directions at a point. An isotropic or anisotropic material can be nonhomogeneous or homogeneous.

Materials for which the constitutive behavior is only a function of the current state of deformation are known as *elastic*. If the constitutive behavior is only a function of the current state of rate of deformation, such materials are termed *viscous*. In this study, we shall be concerned with the Hookean solids and Newtonian fluids for which the constitutive relations are linear. A study of these "theoretical" materials is important because these materials provide good mathematical models for the behavior of "real" materials. There exist other materials

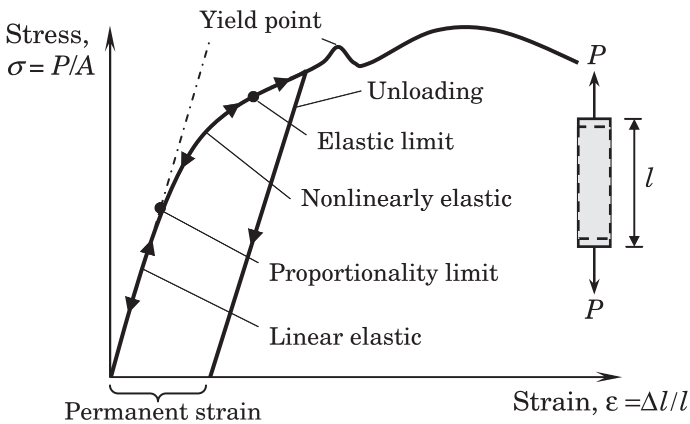


Figure 6.2.1 A nonlinear elastic stress-strain curve.

(e.g., polymers and elastomers) whose constitutive relations cannot be adequately described by those of a Hookean solid or Newtonian fluid.

Constitutive equations are often postulated directly from experimental observations. The approach typically involves assuming the form of the constitutive equation and then restricting the form to a specific one by appealing to certain physical requirements, including invariance of the equations and material frame indifference. The mathematical model involves undetermined parameters that characterize certain responses of the material being studied. Then the parameters of the constitutive model are determined by validating the mathematical model against experimentally determined response. This chapter is primarily focused on Hookean solids and Newtonian fluids (which include biological materials), and they are restricted to linear material response.

6.2 Elastic solids

6.2.1 Introduction

The constitutive equations to be developed here for the stress tensor σ do not include creep at constant stress and stress relaxation at constant strain. Thus, the material coefficients that specify the constitutive relationship between the stress and strain components are assumed to be constant during the deformation. This does not automatically imply that we neglect temperature effects on deformation. We account for the thermal expansion of the material, which can produce strains or stresses as large as those produced by the applied mechanical forces.

Materials for which the constitutive behavior is only a function of the current state of deformation are known as *elastic*. For an elastic material, all of the deformation is recoverable on removal of loads causing the deformation and there is no loss of energy, that is, loading and unloading is along the same stress-strain path, as shown in Figure 6.2.1. A *linearly elastic* material is one for which

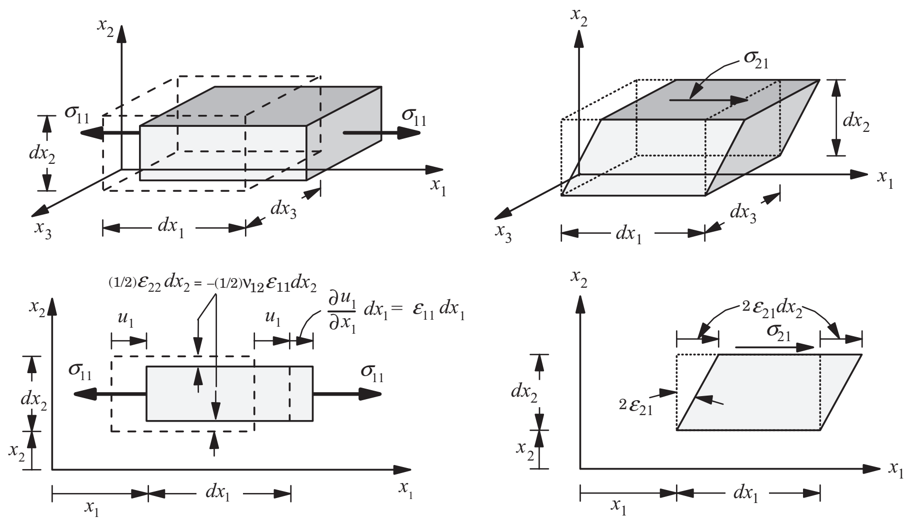


Figure 6.2.2

Strains produced by stresses in a cube of material.

the relationship between the stress and strain is linear. A *nonlinearly elastic* material is one that has a nonlinear relationship between stress and strain.

6.2.2 Generalized Hooke's law for orthotropic materials

Here, we discuss the constitutive equations of Hookean solids, that is, where relations between stress and strain are linear, for the case of infinitesimal deformation (i.e., $|\nabla \mathbf{u}| \ll 1$). Hence, we will not distinguish between $\mathbf{x}$ and $\mathbf{X}$, and use σ for the stress tensor and ε for strain tensor in the material description. The linear constitutive model for infinitesimal deformations is referred to as the *generalized Hooke's law.*

Most often, the material properties are determined in a laboratory in terms of the engineering constants such as Young's modulus, shear modulus, and so on. These constants are measured using simple tests like the uniaxial tension test or the pure shear test. Because of their direct and obvious physical meaning, engineering constants are used in deriving the stress-strain relations.

One of the consequences of linearity (both kinematic and material linearizations) is that the principle of superposition applies. That is, if the applied loads and geometric constraints are independent of deformation, the sum of the displacements (and hence strains) produced by two sets of loads is equal to the displacements (and strains) produced by the sum of the two sets of loads. In particular, the strains of the same type produced by the application of individual stress components can be superposed. For example, the extensional strain $\varepsilon_{11}^{(1)}$ in the material coordinate direction x_1 due to the stress σ_{11} in the same direction is σ_{11}/E_1, where E_1 denotes Young's modulus of the material in the x_1 direction, as shown in Figure 6.2.2. The extensional strain $\varepsilon_{11}^{(2)}$ due to the stress σ_{22} applied in the x_2 direction is (a result of the Poisson effect) $-\nu_{21}\,(\sigma_{22}/E_2)$, where ν_{21} is

Poisson's ratio (note that the first subscript in ν_{ij}, $i \neq j$, corresponds to the load direction and the second subscript refers to the directions of the strain):

$$\nu_{21} = -\frac{\varepsilon_{11}}{\varepsilon_{22}}, \tag{6.2.1}$$

and E_2 is Young's modulus of the material in the x_2 direction. Similarly, σ_{33} produces a strain $\varepsilon_{11}^{(3)}$ equal to $-\nu_{31}(\sigma_{33}/E_3)$. Hence, the total strain ε_{11} due to the simultaneous application of all three normal stress components is

$$\varepsilon_{11} = \varepsilon_{11}^{(1)} + \varepsilon_{11}^{(2)} + \varepsilon_{11}^{(3)} = \frac{\sigma_{11}}{E_1} - \nu_{21}\frac{\sigma_{22}}{E_2} - \nu_{31}\frac{\sigma_{33}}{E_3}, \tag{6.2.2}$$

where the direction of loading is denoted by the superscript. Similarly, we can write

$$\begin{aligned}
\varepsilon_{22} &= -\nu_{12}\frac{\sigma_{11}}{E_1} + \frac{\sigma_{22}}{E_2} - \nu_{32}\frac{\sigma_{33}}{E_3}, \\
\varepsilon_{33} &= -\nu_{13}\frac{\sigma_{11}}{E_1} - \nu_{23}\frac{\sigma_{22}}{E_2} + \frac{\sigma_{33}}{E_3}.
\end{aligned} \tag{6.2.3}$$

The simple shear tests with an orthotropic material give the results

$$2\varepsilon_{12} = \frac{\sigma_{12}}{G_{12}}, \quad 2\varepsilon_{13} = \frac{\sigma_{13}}{G_{13}}, \quad 2\varepsilon_{23} = \frac{\sigma_{23}}{G_{23}}, \tag{6.2.4}$$

where G_{ij} $(i \neq j)$ are the shear moduli in the x_i-x_j plane.

Writing Eqs. (6.2.2) through (6.2.4) in matrix form, we obtain

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}, \tag{6.2.5}$$

where E_1, E_2, and E_3 are Young's moduli in 1, 2, and 3 material directions, respectively, ν_{ij} is Poisson's ratio, defined as the ratio of transverse strain in the jth direction to the axial strain in the ith direction when stressed in the i-direction, and G_{23}, G_{13}, and G_{12} are shear moduli in the 2-3, 1-3, and 1-2 planes, respectively. Because the matrix of elastic coefficients should be symmetric (due to the symmetry of the stress and strain tensors and the existence of strain energy density), the following reciprocal relations hold, that is, compare the off-diagonal terms in Eq. (6.2.5):

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}; \quad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1}; \quad \frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2} \quad \rightarrow \quad \frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \tag{6.2.6}$$

for $i, j = 1, 2, 3$. The nine independent material coefficients for an orthotropic material are

$$E_1,\ E_2,\ E_3,\ G_{23},\ G_{13},\ G_{12},\ \nu_{12},\ \nu_{13},\ \nu_{23}. \tag{6.2.7}$$

The strain-stress relations in Eq. (6.2.5) can be inverted to obtain the stress-strain relations

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}, \tag{6.2.8}$$

where $C_{ij} = C_{ji}$ are the stiffness coefficients, which can be expressed in terms of the engineering constants as

$$\begin{aligned} C_{11} &= \frac{1-\nu_{23}\nu_{23}}{E_2E_3\Delta}, \quad C_{12} = \frac{\nu_{21}+\nu_{31}\nu_{23}}{E_2E_3\Delta} = \frac{\nu_{12}+\nu_{32}\nu_{13}}{E_1E_3\Delta}, \\ C_{13} &= \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2E_3\Delta} = \frac{\nu_{13}+\nu_{12}\nu_{23}}{E_1E_2\Delta}, \quad C_{22} = \frac{1-\nu_{13}\nu_{31}}{E_1E_3\Delta}, \\ C_{23} &= \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1E_3\Delta} = \frac{\nu_{23}+\nu_{21}\nu_{13}}{E_1E_3\Delta}, \quad C_{33} = \frac{1-\nu_{12}\nu_{21}}{E_1E_2\Delta}, \\ &C_{44} = G_{23}, \quad C_{55} = G_{31}, \quad C_{66} = G_{12}, \\ \Delta &= \frac{1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-2\nu_{21}\nu_{32}\nu_{13}}{E_1E_2E_3}. \end{aligned} \tag{6.2.9}$$

6.2.3 Generalized Hooke's law for isotropic materials

Isotropic materials are those for which the material properties are independent of the direction, and we have

$$E_1 = E_2 = E_3 = E, \quad G_{12} = G_{13} = G_{23} = G = \frac{E}{2(1+\nu)}, \quad \nu_{12} = \nu_{23} = \nu_{13} = \nu.$$

The stiffness coefficients in Eq. (6.2.9) become ([$\Delta = (1+\nu)^2(1-2\nu)$]

$$\begin{aligned} C_{11} = C_{22} = C_{33} &= \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}, \\ C_{12} = C_{13} = C_{23} &= \frac{\nu E}{(1+\nu)(1-2\nu)}, \\ C_{44} = C_{55} = C_{66} &= G\,. \end{aligned} \tag{6.2.10}$$

The stress-strain relations in Eq. (6.2.8), with C_{ij} defined in Eq. (6.2.10), can be written in the index-notation form (sum on repeated indices is implied)

$$\sigma_{ij} = \frac{E}{1+\nu}\varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij}. \tag{6.2.11}$$

The inverse relations are

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}. \tag{6.2.12}$$

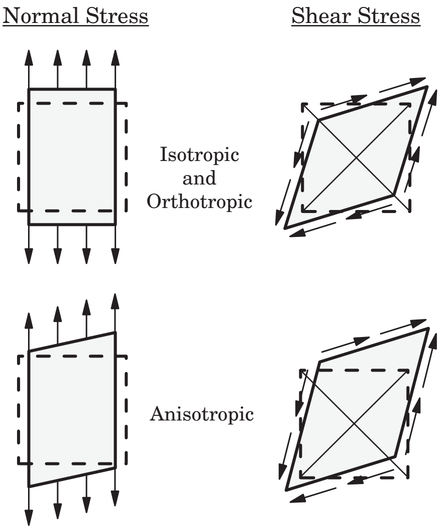


Figure 6.2.3 Deformation of orthotropic and anisotropic rectangular block under uniaxial tension.

For a one-dimensional state of stress and strain, the Hooke's law reduces to

$$\sigma_{xx} = E\varepsilon_{xx}\,, \quad \sigma_{xy} = 2G\varepsilon_{xy}\,. \tag{6.2.13}$$

For orthotropic and isotropic materials, the normal stress-strain relations are decoupled from shear stress-strain relations. Thus, application of a normal stress to a rectangular block of an isotropic or orthotropic material leads to only extension in the direction of the applied stress and contraction perpendicular to it, as shown in Figure 6.2.3. Normal stress applied to an orthotropic material at an angle to its principal material directions causes it to behave like an anisotropic material, which when subjected to shearing stress causes shearing strain as well as normal strains.

We revisit Example 4.3.2 to determine the strains.

Example 6.2.1:

Consider a thin, filament-wound, closed cylindrical pressure vessel shown in Figure 6.2.4. The vessel is 63.5 cm (25 in.) in internal diameter, 2 cm thick (0.7874 in.), and pressurized to 1.379 MPa (200 psi); note that MPa means mega (10^6) Pascal (Pa), Pa $=$ N/m^2, and 1 psi $=$ 6,894.76 Pa. A giga Pascal (GPa) is 1,000 MPa. Determine

(a) stresses σ_{xx}, σ_{yy}, and σ_{xy} in the vessel,

(b) stresses σ_{11}, σ_{22}, and σ_{12} in the material coordinates (x_1, x_2, x_3) with x_1 being along the filament direction,

(c) strains ε_{11}, ε_{22}, and $2\varepsilon_{12}$ in the material coordinates, and

(d) strains ε_{xx}, ε_{yy}, and γ_{xy} in the vessel. Assume a filament winding angle of $\theta = 53.125^\circ$ from the longitudinal axis of the pressure vessel, and use the following material properties, typical of graphite-epoxy material:

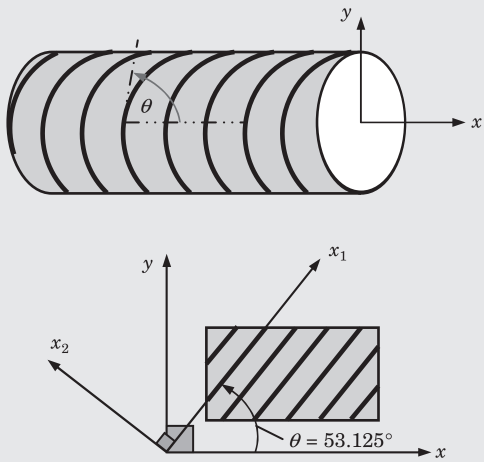


Figure 6.2.4 A filament-wound cylindrical pressure vessel.

$E_1 = 140$ GPa (20.3 Msi), $E_2 = 10$ GPa (1.45 Msi), $G_{12} = 7$ GPa (1.02 Msi), and $\nu_{12} = 0.3$.

Solution: **(a)** The equations of equilibrium of forces in a structure do not depend on the material properties. Hence, equations derived for the longitudinal (σ_{xx}) and circumferential (σ_{yy}) stresses in a thin-walled cylindrical pressure vessel are valid here:

$$\sigma_{xx} = \frac{pD_i}{4h}\,, \quad \sigma_{yy} = \frac{pD_i}{2h}\,,$$

where p is the internal pressure, D_i is the internal diameter, and h is the thickness of the pressure vessel. We obtain (the shear stress σ_{xy} is zero)

$$\sigma_{xx} = \frac{1.379 \times 0.635}{4h} = \frac{0.2189}{h}\text{ MPa}\,, \quad \sigma_{yy} = \frac{1.379 \times 0.635}{2h} = \frac{0.4378}{h}\text{ MPa.}$$

(b) Next, we determine the shear stress along the fiber and the normal stress in the fiber using the transformation equations, the inverse of Eqs. (4.3.3):

$$\begin{aligned}
\sigma_{11} &= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\cos\theta\sin\theta,\\
\sigma_{22} &= \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\sigma_{xy}\cos\theta\sin\theta,\\
\sigma_{12} &= -\sigma_{xx}\sin\theta\cos\theta + \sigma_{yy}\cos\theta\sin\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta).
\end{aligned} \tag{6.2.14}$$

We obtain

$$\begin{aligned}
\sigma_{11} &= \frac{0.2189}{h}(0.6)^2 + \frac{0.4378}{h}(0.8)^2 = \frac{0.3590}{h}\text{ MPa},\\
\sigma_{22} &= \frac{0.2189}{h}(0.8)^2 + \frac{0.4378}{h}(0.6)^2 = \frac{0.2977}{h}\text{ MPa},\\
\sigma_{12} &= \left(\frac{0.4378}{h} - \frac{0.2189}{h}\right) \times 0.6 \times 0.8 = \frac{0.1051}{h}\text{ MPa}.
\end{aligned}$$

Thus, the normal and shear forces per unit length along the fiber-matrix interface are $F_{22} = 297.7$ kN and $F_{12} = 105.1$ kN, whereas the force per unit

length in the fiber direction is $F_{11} = 359$ kN. For $h = 2$ cm, the stress field in the material coordinates becomes

$$\sigma_{11} = 17.95 \text{ MPa}, \; \sigma_{22} = 14.885 \text{ MPa}, \; \sigma_{12} = 5.255 \text{ MPa}.$$

(c) The strains in the material coordinates can be calculated using the strain-stress relations of Eq. (6.2.12). We have ($\nu_{21}/E_2 = \nu_{12}/E_1$)

$$\varepsilon_{11} = \frac{\sigma_{11}}{E_1} - \frac{\sigma_{22}\nu_{12}}{E_1} = \frac{17.95}{140 \times 10^3} - \frac{14.885 \times 0.3}{140 \times 10^3} = 0.0963 \times 10^{-3} \text{ m/m},$$

$$\varepsilon_{22} = -\frac{\sigma_{11}\nu_{12}}{E_1} + \frac{\sigma_{22}}{E_2} = -\frac{17.95 \times 0.3}{140 \times 10^3} + \frac{14.885}{10 \times 10^3} = 1.45 \times 10^{-3} \text{ m/m},$$

$$\varepsilon_{12} = \frac{\sigma_{12}}{2G_{12}} = \frac{5.255}{2 \times 7} = 0.3757 \times 10^{-3}.$$

(d) The strains in the (x, y) coordinates can be computed using

$$\begin{aligned}
\varepsilon_{xx} &= \varepsilon_{11}\cos^2\theta + \varepsilon_{22}\sin^2\theta - 2\varepsilon_{12}\cos\theta\sin\theta, \\
\varepsilon_{yy} &= \varepsilon_{11}\sin^2\theta + \varepsilon_{22}\cos^2\theta + 2\varepsilon_{12}\cos\theta\sin\theta, \\
\varepsilon_{xy} &= (\varepsilon_{11} - \varepsilon_{22})\cos\theta\sin\theta + \varepsilon_{12}\left(\cos^2\theta - \sin^2\theta\right),
\end{aligned} \tag{6.2.15}$$

or

$$\begin{aligned}
\varepsilon_{xx} &= 10^{-3}\left[0.0963 \times (0.6)^2 + 1.45 \times (0.8)^2 - 0.3757 \times 0.6 \times 0.8\right] \\
&= 0.782 \times 10^{-3} \text{ m/m}, \\
\varepsilon_{yy} &= 10^{-3}\left[0.0963 \times (0.8)^2 + 1.45 \times (0.6)^2 + 0.3757 \times 0.6 \times 0.8\right] \\
&= 0.764 \times 10^{-3} \text{ m/m}, \\
\varepsilon_{xy} &= 10^{-3}\left\{2(0.0963 - 1.45) \times (0.6) \times 0.8 + 0.3757[(0.6)^2 - (0.8)^2]\right\} \\
&= -1.405 \times 10^{-3}.
\end{aligned}$$

6.3 Constitutive equations for fluids

6.3.1 Introduction

All bulk matter in nature exists in one of two forms: solid or fluid. A solid body is characterized by relative immobility of its molecules, whereas a fluid state is characterized by their relative mobility. Fluids can exist either as gases or liquids.

The stress in a fluid is proportional to the time rate of strain (i.e., time rate of deformation). The proportionality parameter is known as the *viscosity*. It is a measure of the intermolecular forces exerted as layers of fluid attempt to slide past one another. In general, the viscosity of a fluid is a function of the thermodynamic state of the fluid, and in some cases the strain rate. A *Newtonian fluid* is one for which the stresses are linearly proportional to the velocity gradients.

If the constitutive equation for stress tensor is nonlinear, the fluid is said to *non-Newtonian*.

6.3.2 Ideal fluids

A fluid is said to be *incompressible* if the volume change is zero,

$$\nabla \cdot \mathbf{v} = 0, \tag{6.3.1}$$

where $\mathbf{v}$ is the velocity vector. A fluid is termed *inviscid* if the viscosity is zero, $\mu = 0$. An *ideal fluid* is one that has zero viscosity and is incompressible.

The simplest constitutive equations are those for an ideal fluid. The most general constitutive equations for an ideal fluid are of the form

$$\sigma = -p(\rho, T)\mathbf{I}, \tag{6.3.2}$$

where p is the pressure and T is the absolute temperature. The dependence of p on ρ and T has been experimentally verified many times during several centuries. The thermomechanical properties of an ideal fluid are the same in all directions, that is, the material is isotropic.

An explicit functional form of $p(\rho, T)$ valid for gases over a wide range of temperature and density is

$$p = R\rho T/m, \tag{6.3.3}$$

where R is the universal gas constant and m is the mean molecular weight of the gas. Equation (6.3.3) is known to define a "perfect" gas. When p is only a function of the density, the fluid is said to be *barotropic*, and the barotropic constitutive model is applicable under isothermal conditions. If p is independent of both ρ and θ ($\rho = \rho_0 =$ constant), p is determined from the equations of motion.

6.3.3 Viscous incompressible fluids

The constitutive equation for stress tensor in a fluid motion is assumed to be of the general form

$$\sigma = \mathcal{F}(\mathbf{D}) - p\,\mathbf{I}, \tag{6.3.4}$$

where $\mathcal{F}$ is a tensor-valued function of the rate of deformation $\mathbf{D}$ and p is the thermodynamic pressure. The viscous stress τ is equal to the total stress σ minus the equilibrium stress $-p\mathbf{I}$

$$\sigma = \tau - p\,\mathbf{I}, \quad \tau = \mathcal{F}(\mathbf{D}). \tag{6.3.5}$$

For a Newtonian fluid, $\mathcal{F}$ is assumed to be a linear function of $\mathbf{D}$,

$$\tau = \mathbf{C} : \mathbf{D} \quad \text{or} \quad \tau_{ij} = C_{ijkl} D_{kl}, \tag{6.3.6}$$

where $\mathbf{C}$ is the fourth-order tensor of viscosities of the fluid. For an isotropic viscous fluid, Eq. (6.3.6) reduces to

$$\tau = 2\mu\mathbf{D} + \lambda(\mathrm{tr}\,\mathbf{D})\mathbf{I} \quad \text{or} \quad \tau_{ij} = 2\mu D_{ij} + \lambda D_{kk}\delta_{ij}, \tag{6.3.7}$$

where μ and λ are the Lamé constants. Equation (6.3.5) takes the form

$$\sigma = 2\mu\mathbf{D} + \lambda(\mathrm{tr}\,\mathbf{D})\mathbf{I} - p\,\mathbf{I}, \quad \sigma_{ij} = 2\mu D_{ij} + (\lambda D_{kk} - p)\,\delta_{ij}. \tag{6.3.8}$$

The constitutive equation for a viscous, isotropic, incompressible fluid (i.e., $\nabla\cdot\mathbf{v}$) reduces to

$$\sigma = -p\,\mathbf{I} + 2\mu\mathbf{D}, \quad (\sigma_{ij} = -p\,\delta_{ij} + 2\mu D_{ij}). \tag{6.3.9}$$

For inviscid fluids, the constitutive equation for the stress tensor has the form

$$\sigma = -p\,\mathbf{I} \quad (\sigma_{ij} = -p\,\delta_{ij}). \tag{6.3.10}$$

and p in this case represents the mean normal stress or *hydrostatic pressure.*

6.4 Heat transfer

6.4.1 General introduction

Heat transfer is a branch of engineering that deals with the transfer of thermal energy within a medium or from one medium to another due to a temperature difference. Heat transfer may take place in one or more of the three basic forms: conduction, convection, and radiation (see Reddy and Gartling, 2001). The transfer of heat within a medium due to diffusion process is called *conduction* heat transfer. Fourier's law states that the heat flow is proportional to the temperature gradient. The constant of proportionality depends on, among other things, a material parameter known as the *thermal conductivity* of the material. For heat conduction to occur, there must be temperature differences between neighboring points.

6.4.2 Fourier's heat conduction law

Convection heat transfer is the energy transport effected by the motion of a fluid. The convection heat transfer between two dissimilar media is governed by Newton's law of cooling. It states that the heat flow is proportional to the difference of the temperatures of the two media. For heat convection to occur, there must be a fluid that is free to move and transport energy with it.

The Fourier heat conduction law states that the heat flow $\mathbf{q}$ is related to the temperature gradient by the relation

$$\mathbf{q} = -\mathbf{k}\cdot\nabla T, \tag{6.4.1}$$

where $\mathbf{k}$ is the thermal conductivity tensor of order two. The negative sign in Eq. (6.4.1) indicates that heat flows downhill on the temperature scale.

For an isotropic material, Eq. (6.4.1) can be written as

$$\mathbf{q} = -k\nabla T, \quad q_i = -k\frac{\partial T}{\partial x_i}. \tag{6.4.2}$$

The balance of energy of Eq. (5.3.25) requires that

$$\rho c_v \frac{DT}{Dt} = \Phi - \nabla \cdot \mathbf{q} + \rho\mathcal{E}, \quad \Phi = \tau : \mathbf{D}, \tag{6.4.3}$$

which, in view of Eq. (6.4.2), becomes

$$\rho c_v \frac{DT}{Dt} = \Phi + \nabla \cdot (k\nabla T) + \rho\mathcal{E}, \tag{6.4.4}$$

where $\rho\mathcal{E}$ is the heat energy generated per unit volume, ρ is the density, and c_v is the specific heat at constant volume.

For heat transfer in a solid medium, Eq. (6.4.4) reduces to

$$\rho c_v \frac{\partial T}{\partial t} = \nabla \cdot (k\nabla T) + \rho\mathcal{E}, \tag{6.4.5}$$

which forms the subject of the field of conduction heat transfer. For a fluid medium, Eq. (6.4.4) becomes

$$\rho c_v \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T\right) = \Phi + \nabla \cdot (k\nabla T) + \rho\mathcal{E}, \tag{6.4.6}$$

where $\mathbf{v}$ is the velocity field and Φ is the viscous dissipation function.

6.4.3 Newton's law of cooling

At a solid-fluid interface, the heat flux is related to the difference between the temperature at the interface and that in the fluid,

$$q_n \equiv \hat{\mathbf{n}} \cdot \mathbf{q} = h\left(T - T_{\text{fluid}}\right), \tag{6.4.7}$$

where $\hat{\mathbf{n}}$ is the unit normal to the surface of the body and h is known as the *heat transfer coefficient* or *film conductance*. This relation is known as Newton's law of cooling, which also defines h. Clearly, Eq. (6.4.7) defines a boundary condition on the bounding surface of a conducting medium.

6.4.4 Stefan–Boltzmann law

Radiation is a mechanism that is different from the three transport processes discussed so far: (1) momentum transport in Newtonian fluids that is proportional to the velocity gradient, (2) energy transport by conduction that is proportional to the negative of the temperature gradient, and (3) energy transport by convection that is proportional to the difference in temperatures of the body and the moving fluid in contact with the body. Thermal radiation is an electromagnetic mechanism that allows energy transport with the speed of light through regions of space that are devoid of any matter. Radiant energy exchange between surfaces or between a region and its surroundings is described by the *Stefan–Boltzmann law*.

The heat flow from surface 1 to surface 2 by radiation is governed by the Stefan–Boltzman law:

$$q_n = \sigma \left(T_1^4 - T_2^4 \right), \tag{6.4.8}$$

where T_1 and T_2 are the temperatures of surfaces 1 and 2, respectively, and σ is the Stefan–Boltzman constant. Again, Eq. (6.4.8) defines a boundary condition on the surface 1 of a body.

6.5 Summary

This chapter was dedicated to a discussion of the constitutive equations, that is, relations between the primary variables such as the displacements, velocities, and temperatures to the secondary variables such as the stresses, pressures, and heat flux of continua. Although there are no physical principles to derive these mathematical relations, there are rules or guidelines that help to develop mathematical models of the constitutive behavior that must be, ultimately, validated against actual response characteristics observed in physical experiments. In general, the constitutive relations can be algebraic, differential, or integral relations, depending on the nature of the material behavior being modelled.

In this chapter, the generalized Hooke's law governing linear elastic solids, Newtonian relations for viscous fluids, and the Fourier heat conduction equation for heat transfer in solids are presented. These equations are used in Chapter 7 to analyze problems of solid mechanics, fluid mechanics, and heat transfer.

PROBLEMS

6.1. Establish the following relations between the Lamé constants μ and λ and engineering constants E, ν, and K:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = G = \frac{E}{2(1+\nu)}, \quad K = \frac{E}{3(1-2\nu)}.$$

6.2. Determine the stress tensor components at a point in a 7075-T6 aluminum alloy body ($E = 72$ GPa, and $G = 27$ GPa) if the strain tensor at the point has the following components with respect to the Cartesian basis vectors $\hat{\mathbf{e}}_i$:

$$[\varepsilon] = \begin{bmatrix} 200 & 100 & 0 \\ 100 & 300 & 400 \\ 0 & 400 & 0 \end{bmatrix} \times 10^{-6}.$$

6.3. The components of strain at a point in a body made of structural steel are

$$[\varepsilon] = \begin{bmatrix} 36 & 12 & 30 \\ 12 & 40 & 0 \\ 30 & 0 & 25 \end{bmatrix} \times 10^{-6}.$$

Assuming that the Lamé constants for the structural steel are $\lambda = 207$ GPa (30×10^6 psi) and $\mu = 79.6$ GPa (11.54×10^6 psi), determine the stresses.

6.4. The components of stress at a point in a body are

$$[\sigma] = \begin{bmatrix} 42 & 12 & 30 \\ 12 & 15 & 0 \\ 30 & 0 & -5 \end{bmatrix} \text{ MPa.}$$

Assuming that the Lamé constants for are $\lambda = 207$ GPa (30×10^6 psi) and $\mu = 79.6$ GPa (11.54×10^6 psi), determine the strains.

6.5. Using the stress-strain relations in Eq. (6.2.13), express the stress resultants N, V, and M of Eq. (5.3.6) in terms of the axial displacement u and transverse displacement w. *Hint:* The axial strain ε_{xx} is given by

$$\varepsilon_{xx} = \frac{du}{dx} - z\frac{d^2w}{dx^2}\,.$$

6.6. Given the following motion of an isotropic continuum,

$$\chi(\mathbf{X}) = (X_1 + kt^2X_2^2)\,\hat{\mathbf{e}}_1 + (X_2 + ktX_2)\,\hat{\mathbf{e}}_2 + X_3\,\hat{\mathbf{e}}_3,$$

determine the components of the viscous stress tensor as a function of position and time.

6.7. Write the Navier–Stokes equations, Eq. (5.3.27), and the continuity equation, Eq. (5.2.12), governing two-dimensional flows of an incompressible viscous fluid in terms of the velocity components (v_x, v_y) and pressure P.

6.8. Determine the conditions under which the Navier–Stokes equations and continuity equation are satisfied by the flow field of Problem 5.3 (flow between parallel plates).

6.9. Use the Fourier heat conduction law, $q = -k(dT/dx)$, to rewrite the steady heat conduction equation, Eq. (5.4.9), governing one-dimensional heat flow in terms of the temperature.

6.10. Use the Fourier heat conduction law, $\mathbf{q} = -k\nabla T$, to rewrite the steady heat conduction equation, Eq. (5.4.26), governing three-dimensional heat flow in terms of the rectangular Cartesian coordinates (x, y, z) and temperature T.

7 Applications in Heat Transfer, Fluid Mechanics, and Solid Mechanics

It is really quite amazing by what margins competent but conservative scientists and engineers can miss the mark, when they start with the preconceived idea that what they are investigating is impossible. When this happens, the most well-informed men become blinded by their prejudices and are unable to see what lies directly ahead of them.

Arthur C. Clarke

7.1 Introduction

This chapter is dedicated to the application of the conservation principles to the solution of some simple problems of solid mechanics, fluid mechanics, and heat transfer. In the solid mechanics applications, we assume that stresses and strains are small so that linear strain-displacement relations and Hooke's law are valid, and we use appropriate governing equations derived in the previous chapters. In fluid mechanics applications, finding exact solutions of the Navier–Stokes equations is an impossible task. The principal reason is the nonlinearity of the equations, and consequently, the principle of superposition is not valid. We shall find exact solutions for certain flow problems for which the convective terms (i.e., $\mathbf{v} \cdot \nabla \mathbf{v}$) vanish and problems become linear. Of course, even for linear problems flow geometry must be simple to be able to determine the exact solution. The solution of problems of heat transfer in solid bodies is largely an exercise of solving Poisson's equation in one, two, and three dimensions. We limit our discussion largely to one-dimensional problems. In all cases, mathematically speaking, we seek solutions to differential equations subject to specified boundary conditions.

7.2 Heat transfer

7.2.1 Governing equations

Recall from Eqs. (5.4.26) and (6.4.2) that the principle of conservation of energy applied to a solid medium reduces to

$$\rho c_v \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + Q, \tag{7.2.1}$$

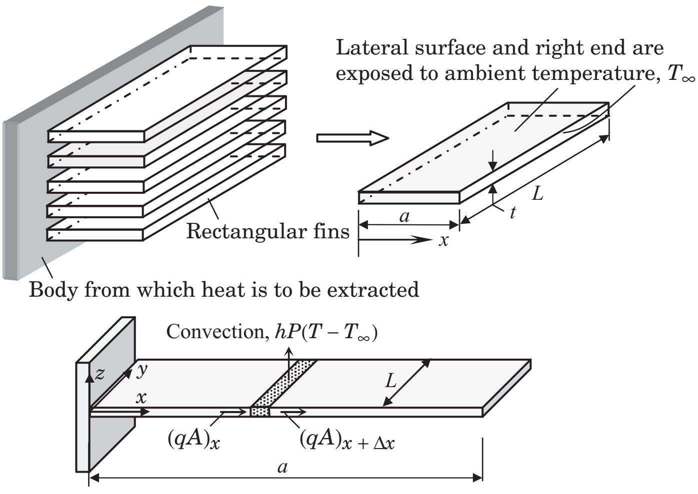


Figure 7.2.1 **Heat transfer in a cooling fin.**

where $Q\ (= \rho\mathcal{E})$ is the rate of internal heat generation per unit volume, k is the conductivity of the (isotropic) solid, ρ is the density, and c_v is the specific heat at a constant volume. All material parameters are assumed to be constant.

The expanded form of Eq. (7.2.1) in a rectangular Cartesian system is given by

$$\rho c_v \frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + Q. \tag{7.2.2}$$

The second-order equation in Eq. (7.2.2) is to be solved subjected to suitable boundary conditions. The boundary conditions involve specifying either the value of the temperature T on the boundary or balancing the heat flux normal to the boundary $q_n = \hat{\mathbf{n}} \cdot \mathbf{q}$. In the following sections, we consider application of Eq. (7.2.2) to some one- and two-dimensional problems.

Heat transfer from a surface to the surrounding fluid medium can be increased by attaching thin strips, called *fins*, of conducting material to the surface (see Figure 7.2.1). We assume that the fins are very long in the y-direction, and heat conducts only along the x-direction and convects through the lateral surface, that is, $T = T(x, t)$. This assumption reduces the three-dimensional problem to a one-dimensional problem. By noting that $T = T(x, t)$, we obtain from Eq. (7.2.2) the result

$$\rho c_v \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + Q. \tag{7.2.3}$$

This equation does not account for the cross-sectional area of the fin and convective heat transfer through the surface. Therefore, we must derive the governing equation for the fin from the first principles (as illustrated in Example 5.4.2).

Consider an element of length Δx at a distance x in the fin, as shown in Figure 7.2.2. The balance of energy in the element requires that the net heat added

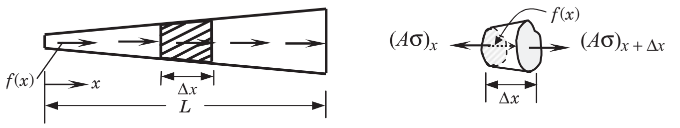


Figure 7.2.2 Heat transfer in a bar of variable cross section.

to the element is equal to the time rate of change of internal energy:

$$(qA)_x - (qA)_{x+\Delta x} - hP\Delta x(T_\infty - T) + Q\left(\frac{A_x + A_{x+\Delta x}}{2}\right)\Delta x = \rho \Delta x c_v \frac{\partial T}{\partial t}, \tag{7.2.4}$$

where q is the heat flux, A is the area of cross section (which can be a function of x), P is the perimeter, h is the film conductance, and Q is the internal heat generation per unit mass (which is zero in the case of fins). Here $(qA)_x$ denotes the heat input at the left end, whereas $(qA)_{x+\Delta x}$ denotes the heat output at the right end of the element; the quantity $hP\Delta x(T_\infty - T)$ denotes the heat leaving the surface of the element due to convection; $A\Delta x Q$ denotes the internal heat generation; and $\rho A \Delta x c_v(\partial T/\partial t)$ is the time rate of change of internal energy. Dividing throughout Eq. (7.2.4) by Δx and taking the limit $\Delta x \to 0$, we obtain

$$-\frac{\partial}{\partial x}(qA) + Ph(T - T_\infty) + QA = \rho A c_v \frac{\partial T}{\partial t}, \quad 0 < x < a, \quad t > 0. \tag{7.2.5}$$

Fourier's law of heat conduction provides a relation between heat flux and gradient of temperature, $q = -k(dT/dx)$, where k is thermal conductivity of the fin. The negative sign indicates that heat flows from the high temperature region to the low temperature region. Substituting Fourier's hear conduction law into Eq. (7.2.5), we obtain

$$\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + Ph(T - T_\infty) + QA = \rho A c_v \frac{\partial T}{\partial t}, \quad 0 < x < a, \quad t > 0. \tag{7.2.6}$$

Equation (7.2.6) must be solved subject to suitable boundary and initial conditions. Because Eq. (7.2.6) is a second-order differential equation in space and first-order in time, two boundary conditions and one initial condition are required. The boundary conditions involve specifying one of the following variables at a boundary point: temperature T, heat qA (or heat flux q), or balance of energy transfer between the body and the surrounding medium when the end is exposed to another medium with temperature T_∞, called the *ambient temperature*:

$$T = \hat{T}, \quad kA\frac{\partial T}{\partial x} = \hat{H}, \quad \text{or} \quad n_x kA\frac{\partial T}{\partial x} + h(T - T_\infty) = \hat{H}, \tag{7.2.7}$$

where $\hat{T}$ and $\hat{H}$ denote specified temperature and heat, respectively, n_x is the unit outward normal (takes a value of $+1$ or -1, depending on if the boundary point is at $x = 0$ or $x = L$, respectively), and h is the *film conductance* or *convective heat transfer coefficient*.

Equations (7.2.6) and (7.2.7) are valid for heat transfer in a fin of arbitrary cross-sectional area $A = A(x)$. If the fin is insulated along its length, the second

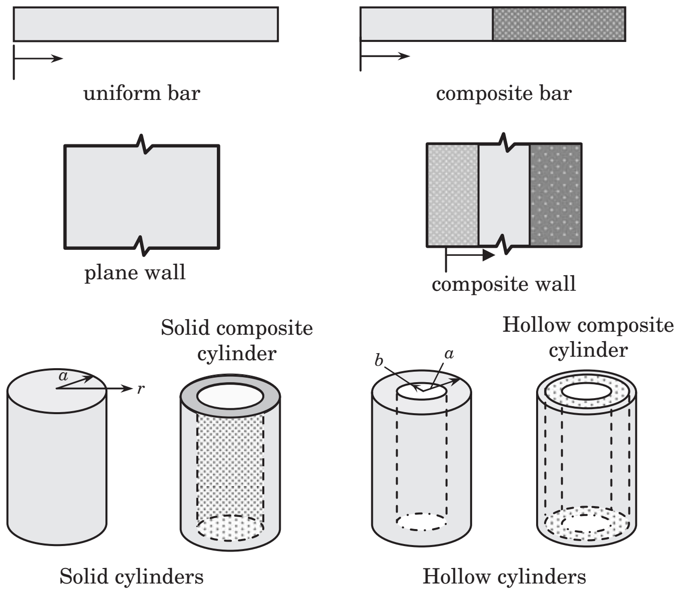


Figure 7.2.3 One-dimensional heat transfer in different geometries.

term on the left side of the equality in Eq. (7.2.6) is set to zero, and we obtain the governing equation for the case as

$$\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + QA = \rho A c_v \frac{\partial T}{\partial t}. \tag{7.2.8}$$

Equation (7.2.6) is also valid for a plane wall, where A is set to unity and the convective heat transfer term is set to zero, and the governing equation takes the form

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + Q = \rho c_v \frac{\partial T}{\partial t}. \tag{7.2.9}$$

The governing equation of one-dimensional, axisymmetric heat transfer in cylindrical geometries is given by (Problem 7.1 provides a physical background for such cases)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + Q = \rho c_v \frac{\partial T}{\partial t}, \quad b < r < a, \quad t > 0, \tag{7.2.10}$$

where a and b denote the outside and inside radii of the cylinder. The equations listed here are also valid in each portion of a composite body made of different materials or geometries (i.e., cross sections). Various geometries of one-dimensional heat transfer are shown in Figure 7.2.3.

7.2.2 Analytical solutions of one-dimensional heat transfer

7.2.2.1 STEADY-STATE HEAT TRANSFER IN A COOLING FIN

First, we consider the specific problem of a fin of length a and constant cross-sectional area A, and with zero internal heat generation ($Q = 0$). Suppose that

the fin is maintained at temperature T_0 at its left end and exposed to ambient temperature T_∞ at the right end:

$$T(0) = T_0, \qquad \left[kA\frac{dT}{dx} + hA(T - T_\infty)\right]_{x=a} = 0. \tag{7.2.11}$$

The second boundary condition is a statement of the balance of energy (conductive and convective) at $x = a$. We wish to determine the steady-state temperature field in the fin. The governing equation is provided by

$$\frac{d}{dx}\left(kA\frac{dT}{dx}\right) + Ph(T - T_\infty) + QA = 0, \quad 0 < x < a. \tag{7.2.12}$$

We introduce the following non-dimensional quantities for the convenience of solving the governing equations:

$$\theta = \frac{T - T_\infty}{T_0 - T_\infty}, \quad \xi = \frac{x}{a}, \quad m^2 = \frac{hPa^2}{kA}, \quad N = \frac{ha}{k}. \tag{7.2.13}$$

Then Eqs. (7.2.12) and (7.2.11) take the form

$$\frac{d^2\theta}{d\xi^2} - m^2\theta = 0, \quad \theta(0) = 1, \quad \left[\frac{d\theta}{d\xi} + N\theta\right]_{\xi=1} = 0. \tag{7.2.14}$$

The general solution to the differential equation in Eq. (7.2.14) is

$$\theta(\xi) = C_1 \cosh m\xi + C_2 \sinh m\xi, \quad 0 < \xi < a, \tag{7.2.15}$$

where the constants C_1 and C_2 are determined using the boundary conditions in Eq. (7.2.14). The condition $\theta = 1$ at $\xi = 0$ gives $C_1 = 1$. To use the second boundary condition from Eq. (7.2.14), we first compute the derivative of θ using Eq. (7.2.15):

$$\frac{d\theta}{d\xi} = m\left(C_1 \sinh m\xi + C_2 \cosh m\xi\right), \quad 0 < \xi < a. \tag{7.2.16}$$

Then we have

$$\frac{d\theta}{d\xi} + N\theta = m\left(C_1 \sinh m\xi + C_2 \cosh m\xi\right) + N\left(C_1 \cosh m\xi + C_2 \sinh m\xi\right). \tag{7.2.17}$$

The second boundary condition in Eq. (7.2.14) yields, with $C_1 = 1$, the result

$$C_2 = -\left(\frac{m \sinh m + N \cosh m}{m \cosh m + N \sinh m}\right). \tag{7.2.18}$$

Then the solution in Eq. (7.2.15) becomes

$$\begin{aligned}\theta(\xi) &= \frac{\cosh m\xi\,(m \cosh m + N \sinh m) - (m \sinh m + N \cosh m) \sinh m\xi}{m \cosh m + N \sinh m} \\ &= \frac{m \cosh m(1-\xi) + N \sinh m(1-\xi)}{m \cosh m + N \sinh m}.\end{aligned} \tag{7.2.19}$$

The *effectiveness* of a fin is defined by (omitting the end effects)

$$
\begin{aligned}
E &= \frac{\text{Actual heat convected by the fin surface}}{\text{Heat that would be convected if the fin surface were held at } T_0} \\
&= \frac{\int_0^L \int_0^a h(T - T_\infty)dxdy}{\int_0^L \int_0^a h(T_0 - T_\infty)dxdy} = \int_0^1 \theta(\xi)\, d\xi \\
&= \int_0^1 \frac{m \cosh m(1-\xi) + N \sinh m(1-\xi)}{m \cosh m + N \sinh m}\, d\xi \\
&= \frac{1}{m} \frac{m \sinh m + N(\cosh m - 1)}{m \cosh m + N \sinh m}.
\end{aligned}
\tag{7.2.20}
$$

7.2.2.2 STEADY-STATE HEAT TRANSFER IN A SURFACE-INSULATED ROD

The governing equation for one-dimensional heat transfer in a surface-insulated bar of length a and constant cross section A is

$$
k\frac{d^2T}{dx^2} + Q = 0, \quad 0 < x < a, \tag{7.2.21}
$$

whose general solution is

$$
T(x) = -\frac{1}{k}\int\left(\int Q(x)dx\right)dx + C_1 x + C_2. \tag{7.2.22}
$$

The constants of integration C_1 and C_2 are determined using any pair of the following boundary conditions:

$$
\begin{aligned}
&\text{Case 1:} \quad T(0) = T_0, \quad T(a) = T_a, \\
&\text{Case 2:} \quad T(0) = T_0, \quad \left[kA\frac{dT}{dx} + h_a A(T - T_\infty^a)\right]_{x=a} = H_a, \\
&\text{Case 3:} \quad \left[-kA\frac{dT}{dx} + h_0 A(T - T_\infty^0)\right]_{x=0} = H_0, \\
&\qquad\qquad \left[kA\frac{dT}{dx} + h_a A(T - T_\infty^a)\right]_{x=a} = H_a,
\end{aligned}
\tag{7.2.23}
$$

where superscripts and subscripts 0 and a on variables T_∞, T, h, and H denote their specified values at $x = 0$ and $x = a$, respectively; H denotes the heat input.

As an example, we consider uniform internal heat generation Q and the boundary conditions in Case 3 of Eq. (7.2.23). Then $T(x)$ and its derivative are

$$
T(x) = -\frac{Q}{2k}x^2 + C_1 x + C_2, \quad \frac{dT}{dx} = -\frac{Q}{k}x + C_1. \tag{7.2.24}
$$

We have

$$\left[-kA\frac{dT}{dx}+h_0A(T-T_\infty^0)\right]_{x=0}=H_0 \quad\rightarrow\quad -kAC_1+h_0A(C_2-T_\infty^0)=H_0,$$
$$\left[kA\frac{dT}{dx}+h_aA(T-T_\infty^a)\right]_{x=a}=H_a,$$

yields

$$-aAQ+kAC_1+h_aA\left(-\frac{Q}{2k}a^2+C_1a+C_2-T_\infty^a\right)=H_a,$$

or

$$\begin{bmatrix}-\beta_0 & 1\\ \beta_a+a & 1\end{bmatrix}\begin{Bmatrix}C_1\\ C_2\end{Bmatrix}=\begin{Bmatrix}\frac{H_0}{h_0A}+T_\infty^0\\ \frac{H_a}{h_aA}+T_\infty^a+\frac{a}{h_a}Q(1+a/2\beta_a)\end{Bmatrix},\quad \beta_0=\frac{k}{h_0},\quad \beta_a=\frac{k}{h_a}. \tag{7.2.25}$$

Using Cramer's rule, we obtain

$$C_1=\frac{1}{(a+\beta_0+\beta_a)}\left[T_\infty^a-T_\infty^0+\frac{1}{A}\left(\frac{H_a}{h_a}-\frac{H_0}{h_0}\right)+\frac{a}{h_a}Q(1+a/2\beta_a)\right],$$
$$C_2=\frac{1}{(a+\beta_0+\beta_a)}\left[\beta_0T_\infty^a+\beta_aT_\infty^0+\frac{1}{A}\left(\frac{\beta_0H_a}{h_a}+\frac{\beta_aH_0}{h_0}\right)+aT_\infty^0+\frac{a}{h_0A}H_0\right.$$
$$\left.+\frac{a}{h_a}\beta_0Q(1+a/2\beta_a)\right]. \tag{7.2.26}$$

Thus, the solution of Eq. (7.2.21) subject to the boundary conditions in Case 3 of Eq. (7.2.23) is given by Eq. (7.2.24), with C_1 and C_2 as given in Eq. (7.2.26).

As special cases, one can obtain solutions of Eq. (7.2.21) for other boundary conditions. For example, for the boundary conditions in Case 1 we take the limit $h\rightarrow\infty$ (no convection at $x=0$ and $x=a$), or $\alpha,\beta\rightarrow 0$, and set $T_\infty^0=T_0$ and $T_\infty^a=T_a$. This gives $C_1=(T_a-T_0)/a$ and $C_2=T_0$. Hence, the temperature distribution in a bar with constant internal heat generation Q per unit mass and with Case 1 boundary conditions is

$$T(x)=-\frac{\rho a^2Q}{2k}\frac{x^2}{a^2}+(T_a-T_0)\frac{x}{a}+T_0\,. \tag{7.2.27}$$

Note that when the internal heat generation is zero, $Q=0$, the temperature varies linearly from T_0 at $x=0$ to T_a at $x=a$.

Similarly, for the boundary conditions in Case 2 we take the limit $h_0\rightarrow\infty$ or $\alpha_0,\beta_0\rightarrow 0$, and set $T_\infty^0=T_0$. This gives

$$C_1=\frac{1}{(a+\beta_a)}\left[T_\infty^a-T_0+\frac{H_a}{Ah_a}+\frac{a}{h_a}Q(1+a/2\beta_a)\right],\quad C_2=T_0. \tag{7.2.28}$$

Then the temperature distribution in a bar with constant internal heat generation Q per unit mass and with Case 2 boundary conditions is

$$T(x)=-\frac{\rho a^2Q}{2k}\frac{x^2}{a^2}+\frac{a}{(a+\beta_a)}\left[T_\infty^a-T_0+\frac{H_a}{Ah_a}+\frac{a}{h_a}Q(1+a/2\beta_a)\right]\frac{x}{a}+T_0\,. \tag{7.2.29}$$

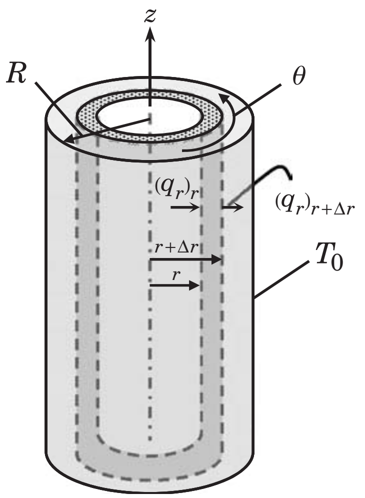


Figure 7.2.4 Heat conduction in a circular cylinder.

7.2.3 Axisymmetric heat conduction in a circular cylinder

Here we consider heat transfer in a long circular cylinder (see Figure 7.2.4). If the boundary conditions and material of the cylinder are axisymmetric, that is, independent of the circumferential coordinate θ, it is sufficient to consider a typical rz-plane, where r is the radial coordinate and z is the axial coordinate. Further, if the cylinder is very long, say 10 diameters in length, then the heat transfer along a typical radial line is all we need to determine; thus, the problem is reduced to a one-dimensional one.

The governing equation for this one-dimensional problem is

$$\rho c_p \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \rho Q(r), \tag{7.2.30}$$

where ρQ is internal heat generation (per unit volume). For example, in the case of an electric wire of circular cross section and electrical conductivity k_e (1/Ohm/m) heat is produced at the rate of

$$\rho Q = \frac{I^2}{k_e}, \tag{7.2.31}$$

where I is electric current density (amps/m^2) passing through the wire. Equation (7.2.30) is to be solved subjected to appropriate initial condition and boundary conditions at $r = 0$ and $r = R$, where R is the radius of the cylinder.

Here we consider a steady heat transfer when there is an internal heat generation of $\rho Q = g$ and the surface of the cylinder is subjected to a temperature $T(R) = T_0$. Then the problem becomes one of solving the equation

$$k\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + g = 0, \quad (rq_r)_{r=0} = \left[-kr\frac{dT}{dr}\right]_{r=0} = 0, \quad T(R) = T_0. \tag{7.2.32}$$

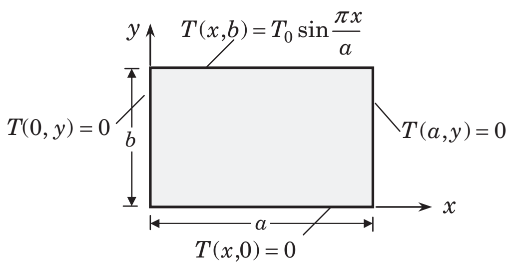


Figure 7.2.5 **Heat conduction in a rectangular plate.**

The general solution is given by

$$T(r) = -\frac{gr^2}{4k} + A\log r + B. \tag{7.2.33}$$

The constants A and B are determined using the boundary conditions

$$(rq_r)|_{r=0} = 0 \;\rightarrow\; A = 0; \quad T(R) = T_0 \;\rightarrow\; B = T_0 + \frac{gR^2}{4k}.$$

The final solution is given by

$$T(r) = T_0 + \frac{gR^2}{4k}\left[1 - \left(\frac{r}{R}\right)^2\right], \tag{7.2.34}$$

which is a parabolic function of the distance r. The heat flux is given by

$$q(r) = -k\frac{dT}{dr} = \frac{gr}{2}, \tag{7.2.35}$$

and the total heat flow at the surface is

$$Q = 2\pi RLq(R) = \pi R^2 Lg. \tag{7.2.36}$$

7.2.4 Two-dimensional heat transfer

Here we consider steady heat conduction in a rectangular plate with sinusoidal temperature distribution on one edge, as shown in Figure 7.2.5, and with zero internal heat generation. The governing equation is a special case of Eq. (7.2.2). Taking $T = T(x, y)$ and setting the time derivative term and velocity components to zero, we obtain

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = 0. \tag{7.2.37}$$

The boundary conditions are

$$T(x, 0) = 0, \quad T(0, y) = 0, \quad T(a, y) = 0, \quad T(x, b) = T_0\sin\frac{\pi x}{a}. \tag{7.2.38}$$

Once the temperature $T(x, y)$ is known, we can determine the components of heat flux, q_x and q_y, from Fourier's law:

$$q_x = -k\frac{\partial T}{\partial x}, \quad q_y = -k\frac{\partial T}{\partial y}. \tag{7.2.39}$$

The classical approach to an analytical solution of the Laplace or Poisson equation over a regular (i.e., rectangular or circular) domain is the separation-of-variables technique. In this technique, we assume the temperature $T(x, y)$ to be of the form

$$T(x, y) = X(x)Y(y), \tag{7.2.40}$$

where X is a function of x alone and Y is a function of y alone. Substituting Eq. (7.2.40) into Eq. (7.2.37) and rearranging the terms, we obtain

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2}. \tag{7.2.41}$$

Because the left side is a function of x alone and the right side is a function of y alone, it follows that both sides must be equal to a constant, which we choose to be $-\lambda^2$ (because the solution must be periodic in x so as to satisfy the boundary condition on the edge $y = b$). Thus, we have

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0, \quad \frac{d^2Y}{dy^2} - \lambda^2 Y = 0, \tag{7.2.42}$$

whose general solutions are

$$X(x) = C_1 \cos\lambda x + C_2 \sin\lambda x, \quad Y(y) = C_3 e^{-\lambda y} + C_4 e^{\lambda y}. \tag{7.2.43}$$

The solution $T(x, y)$ is given by

$$T(x, y) = (C_1 \cos\lambda x + C_2 \sin\lambda x)\left(C_3 e^{-\lambda y} + C_4 e^{\lambda y}\right). \tag{7.2.44}$$

The constants C_i $(i = 1, 2, 3, 4)$ are determined using the boundary conditions in Eq. (7.2.38). We obtain

$$\begin{aligned}
&T(x, 0) = 0 \;\rightarrow\; (C_1 \cos\lambda x + C_2 \sin\lambda x)(C_3 + C_4) = 0 \;\rightarrow\; C_3 = -C_4,\\
&T(0, y) = 0 \;\rightarrow\; C_1\left(C_3 e^{-\lambda y} + C_4 e^{\lambda y}\right) = 0 \;\rightarrow\; C_1 = 0,\\
&T(a, y) = 0 \;\rightarrow\; C_2 \sin\lambda a\left(C_3 e^{-\lambda y} + C_4 e^{\lambda y}\right) = 0 \;\rightarrow\; \sin\lambda a = 0.
\end{aligned}$$

The last conclusion is reached because $C_2 = 0$ will make the whole solution trivial. We have

$$\sin\lambda a = 0 \;\rightarrow\; \lambda a = n\pi \;\text{ or }\; \lambda_n = \frac{n\pi}{a}. \tag{7.2.45}$$

The solution in Eq. (7.2.44) now can be expressed as

$$T(x, y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}. \tag{7.2.46}$$

The constants A_n, $n = 1, 2, \ldots$ are determined using the remaining boundary condition. We have

$$T(x, b) = T_0 \sin \frac{\pi x}{a} = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi b}{a}.$$

Multiplying both sides with $\sin(m\pi x/a)$ and integrating from 0 to a, and using the orthogonality of the sine functions,

$$\int_0^a \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a}\, dx = \begin{cases} 0, & m \neq n, \\ \frac{a}{2}, & m = n, \end{cases} \tag{7.2.47}$$

we obtain

$$A_1 = \frac{T_0}{\sinh \frac{n\pi b}{a}}, \qquad A_n = 0 \text{ for } n \neq 1.$$

Hence, the final solution is

$$T(x, y) = T_0 \frac{\sinh \frac{\pi y}{a}}{\sinh \frac{\pi b}{a}} \sin\left(\frac{\pi x}{a}\right). \tag{7.2.48}$$

When the boundary condition at $y = b$ is replaced with $T(x, b) = f(x)$, then the solution is given by

$$T(x, y) = \sum_{n=1}^{\infty} A_n \frac{\sinh \frac{n\pi y}{a}}{\sinh \frac{n\pi b}{a}} \sin\left(\frac{n\pi x}{a}\right), \tag{7.2.49}$$

with A_n given by

$$A_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx. \tag{7.2.50}$$

7.3 Fluid mechanics

7.3.1 Preliminary comments

Matter exists only in two states: solid and fluid. The difference between the two is that a solid can resist shear force in static deformation whereas a fluid cannot. Shear force acting on a fluid causes it to deform continuously. Thus, a fluid at rest can only take hydrostatic pressure and no shear stress.

Fluid mechanics is a branch of mechanics that deals with the effects of fluids at rest (statics) or in motion (dynamics) on surfaces they contact. Fluids do not have the so-called natural state to which they return upon removal of forces causing deformation. Therefore, we use a spatial (or Eulerian) description to write the governing equations. Pertinent equations are summarized here for an isotropic, incompressible, Newtonian fluid. Viscous dissipation Φ is also omitted.

7.3.2 Summary of equations

The number of equations N_{eq} and number of new dependent variables N_{var} for three-dimensional problems are listed in parentheses. *Continuity equation* ($N_{\text{eq}} = 1$, $N_{\text{var}} = 3$):

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0. \tag{7.3.1}$$

Equations of motion ($N_{\text{eq}} = 3$, $N_{\text{var}} = 6$):

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho f_x &= \rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right),\\
\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho f_y &= \rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right),\\
\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho f_z &= \rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right).
\end{aligned} \tag{7.3.2}$$

Energy equation ($N_{\text{eq}} = 1$, $N_{\text{var}} = 1$):

$$\rho c_v\left(\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \rho Q. \tag{7.3.3}$$

Constitutive equation ($N_{\text{eq}} = 6$, $N_{\text{var}} = 1$):

$$\begin{gathered}
\sigma_{xx} = 2\mu\frac{\partial v_x}{\partial x} - P, \quad \sigma_{yy} = 2\mu\frac{\partial v_y}{\partial y} - P, \quad \sigma_{zz} = 2\mu\frac{\partial v_z}{\partial z} - P,\\
\sigma_{xy} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right), \quad \sigma_{xz} = \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right), \quad \sigma_{yz} = \mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right).
\end{gathered} \tag{7.3.4}$$

Thus, there are 11 equations and 11 variables. Equations (7.3.2) and (7.3.4) together are known as the *Navier–Stokes equations.* When the stresses in Eq. (7.3.2) are expressed in terms of the velocities and pressure with the help of Eq. (7.3.4), the resulting equations together with Eq. (7.3.1) contain four equations in four unknowns (v_1, v_2, v_3, P). The expanded forms of these four equations for the steady-state case, in the absence of body forces, are

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0, \tag{7.3.5}$$

$$\begin{aligned}
&\mu\left[2\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)\right] - \frac{\partial P}{\partial x}\\
&\quad = \rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right),
\end{aligned} \tag{7.3.6}$$

$$\begin{aligned}
&\mu\left[2\frac{\partial^2 v_y}{\partial y^2} + \frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)\right] - \frac{\partial P}{\partial y}\\
&\quad = \rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right),
\end{aligned} \tag{7.3.7}$$

$$\mu\left[2\frac{\partial^2 v_z}{\partial z^2}+\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}\right)\right]-\frac{\partial P}{\partial z}$$
$$=\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right), \tag{7.3.8}$$

In general, finding exact solutions of the Navier–Stokes equations is an impossible task. The principal reason is the nonlinearity of the equations and, consequently, the principle of superposition is not valid. In the following sections, we shall find exact solutions of Eqs. (7.3.5) through (7.3.8) for certain flow problems for which the convective term (i.e., $\mathbf{v}\cdot\nabla\mathbf{v}$) vanishes and problems become linear. Of course, even for linear problems flow geometry must be simple to be able to determine the exact solution.

7.3.3 Inviscid fluid statics

For inviscid fluids (i.e., fluids with zero viscosity), the constitutive equation for stress is

$$\sigma_{xx}=\sigma_{yy}=\sigma_{zz}=-P, \tag{7.3.9}$$

where P is the hydrostatic pressure. The body force in hydrostatics problem often represents the gravitational force, $\rho\mathbf{f}=-\rho g\,\hat{\mathbf{k}}$, where the z-axis is taken positive upward. Consequently, the equations of motion reduce to

$$-\frac{\partial P}{\partial x}=\rho a_x,\quad -\frac{\partial P}{\partial y}=\rho a_y,\quad -\frac{\partial P}{\partial z}=\rho g+\rho a_z, \tag{7.3.10}$$

where (a_x, a_y, a_z) are the components of acceleration vector $\mathbf{a}=D\mathbf{v}/Dt$. For steady flows with a constant velocity field, the equations in Eq. (7.3.10) simplify to

$$-\frac{\partial P}{\partial x}=0,\quad -\frac{\partial P}{\partial y}=0,\quad -\frac{\partial P}{\partial z}=\rho g. \tag{7.3.11}$$

The first two equations in Eq. (7.3.11) imply that $P=P(z)$. Integrating the third equation with respect to z, we obtain

$$P(z)=-\rho g z+c_1,$$

where c_1 is the constant of integration, which can be evaluated using the pressure boundary condition at $z=H$, and H is the height of the column of liquid; see Figure 7.3.1(a). On the free surface we have $P=P_0$, where P_0 is the atmospheric pressure. Then the constant of integration is $c_1=P_0+\rho g H$, and we have

$$P(z)=\rho g(H-z)+P_0. \tag{7.3.12}$$

For the unsteady case in which the fluid (i.e., a rectangular container with the fluid) moves at a constant acceleration a_x in the x-direction, the equations of

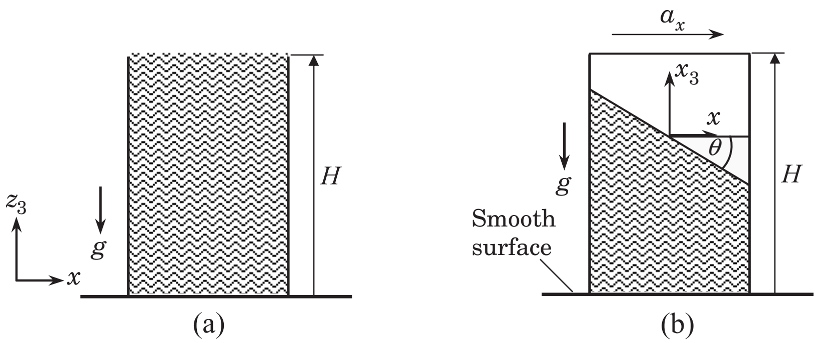


Figure 7.3.1 (a) Column of liquid of height H. (b) A container of fluid moving with a constant acceleration, $\mathbf{a} = a_x \hat{\mathbf{i}}$.

motion in Eq. (7.3.10) become

$$-\frac{\partial P}{\partial x} = \rho a_x, \quad -\frac{\partial P}{\partial y} = 0, \quad -\frac{\partial P}{\partial z} = \rho g, \tag{7.3.13}$$

From the second equation, it follows that $P = P(x, z)$. Integrating the first equation with respect to x, we obtain

$$P(x, z) = -\rho a_x x + f(z),$$

where $f(z)$ is a function of z alone. Substituting the equation for P into the third equation in Eq. (7.3.13) and integrating with respect to z, we arrive at

$$f(z) = -\rho g z + c_2, \quad P(x, z) = -\rho a_x x - \rho g z + c_2,$$

where c_2 is a constant of integration. If $z = 0$ is taken on the free surface of the fluid in the container, then $P = P_0$ at $x = z = 0$, giving $c_2 = P_0$. Thus,

$$P(x, z) = P_0 - \rho a_x x - \rho g z. \tag{7.3.14}$$

Equation (7.3.14) suggests that the free surface (which is a plane), where $P = P_0$, is given by the equation $a_x x = -gz$. The orientation of the plane is given by the angle θ, where

$$\tan\theta = \frac{dz}{dx} = -\frac{a_x}{g}. \tag{7.3.15}$$

7.3.4 Parallel flow (Navier–Stokes equations)

A flow is called *parallel* if only one velocity component is nonzero, that is, all fluid particles are moving in the same direction. Suppose that $v_y = v_z = 0$ and that the body forces are negligible. Then from the continuity equation, Eq. (7.3.5), it follows that

$$\frac{\partial v_x}{\partial x} = 0 \quad \rightarrow \quad v_x = v_x(y, z, t). \tag{7.3.16}$$

Thus, for a parallel flow we have

$$v_x = v_x(y, z, t), \quad v_y = v_z = 0. \tag{7.3.17}$$

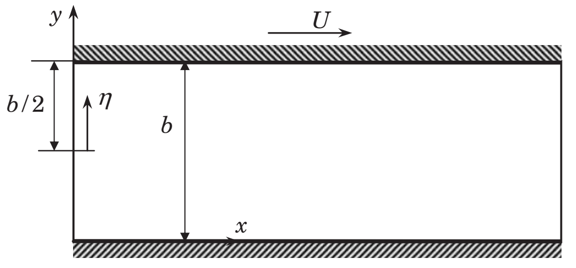


Figure 7.3.2 Flow of viscous fluid between parallel plates.

Consequently, the three equations of motion in Eqs. (7.3.6) through (7.3.8) simplify to the following linear differential equations:

$$-\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) = \rho\frac{\partial v_x}{\partial t}, \quad \frac{\partial P}{\partial y} = 0, \quad \frac{\partial P}{\partial z} = 0. \tag{7.3.18}$$

The last two equations in Eq. (7.3.18) imply that P is only a function of x. Thus, given the pressure gradient dP/dx, the first equation in (7.3.18) can be used to determine v_x.

7.3.4.1 STEADY FLOW OF VISCOUS INCOMPRESSIBLE FLUID BETWEEN PARALLEL PLATES

Consider a steady flow (that is, $\partial v_x/\partial t = 0$) in a channel with two parallel flat walls (see Figure 7.3.2). Let the distance between the two walls be b. Equation (7.3.18) reduces to the boundary value problem:

$$\begin{gathered}\mu\frac{d^2 v_x}{dy^2} = \frac{dP}{dx}, \quad 0 < y < b, \\ v_x(0) = 0, \quad v_x(b) = U.\end{gathered} \tag{7.3.19}$$

When $U \neq 0$, the problem is known as the *Couette flow*. The solution of Eq. (7.3.19) is given by

$$v_x(y) = \frac{y}{b}U - \frac{b^2}{2\mu}\frac{dP}{dx}\frac{y}{b}\left(1 - \frac{y}{b}\right), \quad 0 < y < b, \tag{7.3.20}$$

$$\bar{v}_x(\bar{y}) = \bar{y} + f\bar{y}\,(1-\bar{y}), \quad \bar{v}_x = \frac{v_x}{U}, \quad \bar{y} = \frac{y}{b}, \quad f = -\frac{b^2}{2\mu U}\frac{dP}{dx}. \tag{7.3.21}$$

When $U = 0$, the flow is known as the *Poiseuille flow*. In this case, the solution in Eq. (7.3.21) reduces to

$$v_x(y) = -\frac{b^2}{2\mu}\frac{dP}{dx}\frac{y}{b}\left(1 - \frac{y}{b}\right), \quad 0 < y < b, \tag{7.3.22}$$

$$v_x(\eta) = -\frac{1}{2\mu}\frac{dP}{dx}\left(\frac{b^2}{4} - \eta^2\right), \quad \eta = y - \frac{b}{2}, \quad -\frac{b}{2} < \eta < \frac{b}{2}. \tag{7.3.23}$$

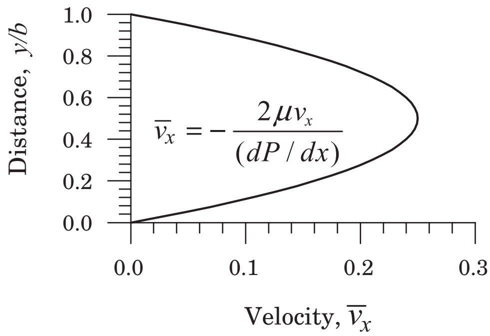


Figure 7.3.3 Velocity distributions for Poiseuille flow.

Figures 7.3.3 and 7.3.4 show the velocity distributions for cases $U = 0$ and $U \neq 0$ (Couette flow).

7.3.4.2 STEADY FLOW OF VISCOUS INCOMPRESSIBLE FLUID THROUGH A PIPE

The steady flow through a long, straight, horizontal circular pipe is another problem that admits exact solution to the Navier–Stokes equations. We use the cylindrical coordinate system with r being the radial coordinate and the z-coordinate is taken along the axis of the pipe, as shown in Figure 7.3.5. The continuity equation is given by

$$\frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0. \tag{7.3.24}$$

The momentum equations are

$$\begin{aligned} &\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\left(\frac{\partial^2 v_r}{\partial \theta^2} - 2\frac{\partial v_\theta}{\partial \theta}\right) + \frac{\partial^2 v_r}{\partial z^2}\right] - \frac{\partial P}{\partial r} + \rho f_r \\ &= \rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_z}{\partial z}\right), \end{aligned} \tag{7.3.25}$$

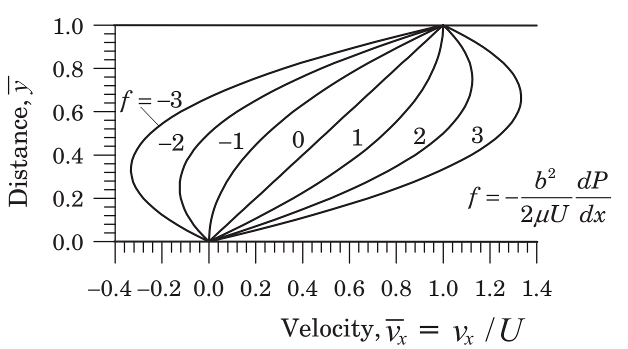


Figure 7.3.4 Velocity distributions for the Couette flow.

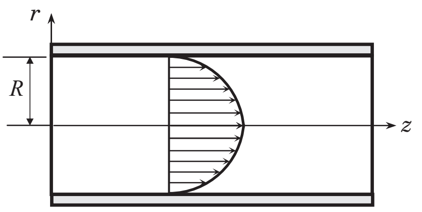


Figure 7.3.5 Steady flow of a viscous incompressible fluid through a pipe.

$$\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)+\frac{1}{r^2}\left(\frac{\partial^2 v_\theta}{\partial \theta^2}+2\frac{\partial v_r}{\partial \theta}\right)+\frac{\partial^2 v_\theta}{\partial z^2}\right]-\frac{\partial P}{\partial \theta}+\rho f_\theta$$
$$=\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right), \tag{7.3.26}$$

$$\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]-\frac{\partial P}{\partial z}+\rho f_z$$
$$=\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right). \tag{7.3.27}$$

For the problem at hand, the velocity components v_r and v_θ in the radial and tangential directions, respectively, are both zero. Then the continuity equation, Eq. (7.3.24), coupled with the axisymmetric flow situation (i.e., the flow field is independent of θ) implies that the velocity component parallel to the axis of the pipe, v_z, is only a function of r. Equations (7.3.25) and (7.3.26), in the absence of any body forces, yield $(\partial P/\partial r) = 0$ and $(\partial P/\partial \theta) = 0$, implying that P is only a function of z (or P is a constant in every cross section). The momentum equation in the z-coordinate direction, Eq. (7.3.27), simplifies to

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)=\frac{dP}{dz}, \tag{7.3.28}$$

whose solution is given by

$$v_z(r)=\frac{r^2}{4\mu}\frac{dP}{dz}+A\log r+B. \tag{7.3.29}$$

The constants of integration, A and B, are determined using the boundary conditions

$$r\tau_{rz}\equiv r\mu\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial r}\right)=0 \text{ at } r=0, \quad \text{and} \quad v_z=0 \text{ at } r=R, \tag{7.3.30}$$

where R is the radius of the pipe. We find that

$$A=0, \quad B=-\frac{R^2}{4\mu}\frac{dP}{dz},$$

and the solution becomes

$$v_z(r)=-\frac{1}{4\mu}\frac{dP}{dz}\left(R^2-r^2\right). \tag{7.3.31}$$

Thus, the velocity over the cross section of the pipe varies as a paraboloid of revolution. The maximum velocity occurs along the axis of the pipe and is equal to

$$(v_z)_{\max} = v_z(0) = -\frac{R^2}{4\mu}\frac{dP}{dz}, \tag{7.3.32}$$

The volume rate of flow through the pipe is

$$Q = \int_0^{2\pi}\int_0^R v_z(r)\,r\,dr\,d\theta = \frac{\pi R^4}{8\mu}\left(-\frac{dP}{dz}\right). \tag{7.3.33}$$

The wall shear stress is

$$\tau_w = -\mu\left(\frac{dv_z}{dr}\right)_{r=R} = \frac{R}{2}\frac{dP}{dz}. \tag{7.3.34}$$

7.3.5 Diffusion processes

Diffusion is the process in which there is movement of the particles (or molecules) of a substance from an area of high concentration of that substance to an area of lower concentration. The flow of fluid particles through a (porous) solid medium and the flow of heat from high-temperature region to a low-temperature region in a solid or fluid medium provide examples of diffusion. The governing equation of such a process has already been discussed in the case of heat transfer [see Eq. (7.2.1)]; depending on the process, the meaning of the variables appearing in Eq. (7.2.1) will vary. In the following, we present a discussion of the diffusion equation and its applications.

The transport of solutes, for cellular metabolism and energy generation, is a fundamental requirement for the sustainment of life in an organism. At the cellular level, the transfer of molecules is primarily a diffusion process, which assists in the delivery of nutrients and metabolites from blood capillaries to cells. Convection and electrical conduction methods are required for the transport of molecules at distances greater than 100 to 200 microns. The flow of molecules and cells within the blood, the lymphatic system, and organs like kidneys are the most common examples of diffusion of mass transfer in bio-systems.

The transfer of molecules is governed by the Fick's first law (similar to Fourier's heat conduction law), which relates the diffusion flux of a solute to the concentration gradient in a dilute solution given as

$$\mathbf{J} = -D\nabla C, \tag{7.3.35}$$

where C denotes concentration, D the diffusion coefficient, and J the diffusion flux. The negative sign on the right side of the equality indicates that the solute moves from the region of higher concentration to the region of lower concentration. Mass conservation of the solutes in a tissue without any chemical reaction leads to the following equation:

$$\frac{\partial C}{\partial t} - \nabla\cdot(D\nabla C) = 0. \tag{7.3.36}$$

Example 7.3.1: (Diffusion Through Artery Wall)

Most of the biological tissues consist of multiple layers having different material properties, a prominent example being the artery wall. The diffusion of low density lipoprotein (LDL) from the blood in the lumen through the artery wall causes diseases like atherosclerosis. For analysis purposes, we consider the artery wall to be composed of the intima near the blood flow region called the lumen and the media, as shown in Figure 7.3.6; the intermediate region between the intima and the media, called the internal elastic lamina (IEL), the adventitia, and the region between the media and the adventitia, called the external elastic layer (EEL), are neglected. Assuming that the artery is a long two-layer cylindrical shell and the diffusion of LDL (cholesterol) is axisymmetric, one can adopt a one-dimensional model. Let the lumen radius be a, the radius of the media be b, and let the radial distance to the interface between the intima and the media be R. Assuming that the diffusion coefficients for LDL in the intima and the media are D_1 and D_2, respectively, and the concentration at the intima surface is C_a and at the outside of the media is C_b, determine the LDL concentration as a function of the radial distance from the center of the artery.

Solution: The steady-state axisymmetric diffusion in a cylindrical geometry is governed by

$$\frac{d}{dr}\left(r D_i \frac{dC_i}{dr}\right) = 0, \quad i = 1, 2 \tag{7.3.37}$$

in intima (layer 1) and media (layer 2). Solving the equations, we obtain

$$C_1(r) = A_1 \log r + B_1 \text{ for } a < r < R;\ C_2(r) = A_2 \log r + B_2 \text{ for } R < r < b. \tag{7.3.38}$$

The four conditions required to determine the four constant of integration are

$$\begin{aligned} r = a: &\quad C_1 = C_a, &\quad r = b: &\quad C_2 = C_b,\\ r = R: &\quad C_1 = C_2, &\quad r = R: &\quad r D_1 \frac{dC_1}{dr} = r D_2 \frac{dC_2}{dr}. \end{aligned} \tag{7.3.39}$$

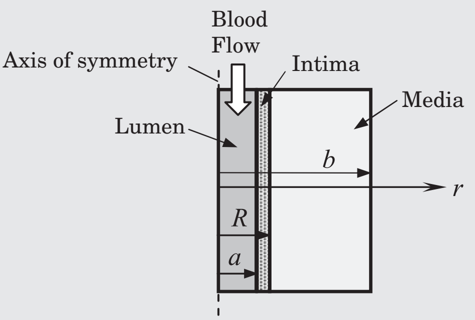


Figure 7.3.6 Diffusion of low density lipoprotein through two-layer artery wall.

These conditions yield

$$A_2 = \beta A_1, \quad A_1 = \frac{C_b - C_a}{\log \frac{R}{a} - \beta \log \frac{R}{b}}, \quad \beta = \frac{D_1}{D_2}, \tag{7.3.40}$$
$$B_1 = C_a - A_1 \log R, \quad B_2 = C_b - \beta A_1 \log R,$$

and the solution is given by Eq. (7.3.38).

Example 7.3.2: (Intratumor Drug Injection)

Tumors can be considered as porous elastic materials with blood, and lymph vessels can be considered as sources of injection and fluid absorption. Drugs can be directly injected into the solid tumors thereby enhancing convective transport, which is critical to the delivery of macromolecules and nanoparticles in tumors.

Consider a spherical solid tumor of radius R and hydraulic conductivity of K, stripped of blood and lymph circulation, and having zero pressure at its surface. A drug is infused into the tumor using a needle tip at the center of the tumor, which creates a small fluid cavity around the tip with a radius of a. Assuming that the tumor is homogenous, isotropic, and consists of chemically inert solid and fluid phases, determine the steady-state pressure distribution in the tumor for a constant infusion rate of Q (see Figure 7.3.7).

Solution: The momentum balance in a porous media is Darcy's law, which states that the fluid velocity through the porous material is proportional to the pressure gradient. The proportionality constant is defined as hydraulic conductivity of the material. Darcy's law (similar to Fick's law or Fourier's law) can be expressed as

$$\mathbf{v} = -K\nabla p, \tag{7.3.41}$$

where $\mathbf{v}$ denotes the velocity and p the pressure. The principle of conservation of mass for an incompressible fluid without any source or sink is given by

$$\nabla \cdot \mathbf{v} = 0. \tag{7.3.42}$$

Combining Eqs. (7.3.41) and (7.3.42), we obtain

$$\nabla \cdot (K\nabla p) = 0. \tag{7.3.43}$$

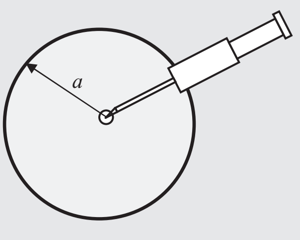


Figure 7.3.7 Intratumor drug infusion.

For the problem at hand, it is convenient to use the spherical coordinate system and express Eq. (7.3.43) in spherical coordinates (see Table 2.6.2 and Figure 2.6.3). Because of axisymmetry, p is only a function of r, the radial coordinate. Equation (7.3.43) takes the form

$$\frac{K}{r^2}\frac{d}{dr}\left(r^2\frac{dp}{dr}\right) = 0, \tag{7.3.44}$$

whose solution and the velocity are given by

$$p(r) = -\frac{C_1}{r} + C_2, \quad v_r(r) = K\frac{C_1}{r^2}, \tag{7.3.45}$$

where C_1 and C_2 are constants of integration, which are to be determined using the boundary conditions. One boundary condition is that $p = 0$ at $r = a$; this yields $C_2 = C_1/a$. The other boundary condition is provided by the fact that the infusion rate must be equal to the rate of flow across any spherical surface (a requirement of the principle of conservation of mass),

$$Q = \int_0^{2\pi}\left(\int_0^{\pi} v_r\, r^2 \sin\phi\, d\phi\right)d\theta.$$

Substituting for v_r from Eq. (7.3.45), we obtain

$$Q = -4\pi K C_1 \ \text{ or } \ C_1 = -\frac{Q}{4\pi K} \quad \to \quad C_2 = -\frac{Q}{4\pi K a}.$$

Thus, the pressure and velocity distributions in the solid tumor are given by

$$p(r) = \frac{Q}{4\pi K}\left(\frac{1}{r} - \frac{1}{a}\right), \quad v_r(r) = \frac{Q}{4\pi r^2}. \tag{7.3.46}$$

7.4 Solid mechanics

7.4.1 Governing equations

It is useful to summarize the equations of linearized elasticity for use in the remainder of the chapter. For the moment, we consider isothermal elasticity and study only equilibrium, that is, static problems. The governing equations of a three-dimensional elastic body involve: (1) six strain-displacement relations among nine variables, six strain components and three displacements; (2) three equilibrium equations among six components of stress, assuming symmetry of the stress tensor; and (3) six stress-strain equations among the six stress and six strain components that are already counted. Thus, there are a total of 15 coupled equations among 15 scalar fields. These equations are listed here in Cartesian component forms for an isotropic, linearized elastic body in static equilibrium.

Strain-displacement equations:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z},$$

$$2\varepsilon_{xy} \equiv \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad 2\varepsilon_{xz} \equiv \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad 2\varepsilon_{yz} \equiv \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}. \tag{7.4.1}$$

Equilibrium equations:

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho f_x &= 0,\\
\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho f_y &= 0,\\
\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho f_z &= 0.
\end{aligned} \tag{7.4.2}$$

Stress-strain relations:

$$\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu\left(\varepsilon_{yy} + \varepsilon_{zz}\right)\right],\\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{yy} + \nu\left(\varepsilon_{xx} + \varepsilon_{zz}\right)\right],\\
\sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{zz} + \nu\left(\varepsilon_{xx} + \varepsilon_{yy}\right)\right],\\
\sigma_{xy} &= G\gamma_{xy}, \quad \sigma_{xz} = G\gamma_{xz}, \quad \sigma_{yz} = G\gamma_{yz}.
\end{aligned} \tag{7.4.3}$$

These equations are valid for all problems of linearized elasticity; different problems differ from each other only in the geometry of the domain, boundary conditions, and material constitution. The general form of the boundary condition is given as

$$\begin{aligned}
t_x &\equiv n_x\sigma_{xx} + n_y\sigma_{xy} + n_z\sigma_{xz} = \hat{t}_x,\\
t_y &\equiv n_x\sigma_{xy} + n_y\sigma_{yy} + n_z\sigma_{yz} = \hat{t}_y,\\
t_z &\equiv n_x\sigma_{xz} + n_y\sigma_{yz} + n_z\sigma_{zz} = \hat{t}_z,
\end{aligned} \tag{7.4.4}$$

$$u = \hat{u}, \quad v = \hat{v}, \quad w = \hat{w}. \tag{7.4.5}$$

Only one element of each of the following pairs may be specified at a point:

$$(u, t_x), \quad (v, t_y), \quad (w, t_z).$$

The continuity equation resulting from the principle of conservation of mass is not used in linearized elasticity because there we deal with a solid body with a fixed mass, and material does not flow out of the body during the infinitesimal deformations. Thus, the continuity equation is trivially satisfied. The energy equation is not used here here because of the assumption of isothermal conditions. When temperatures are not much different from the room temperature or

temperature of the stress-free state of the body, the material constants such as Young's modulus are assumed to be independent of the temperature. Then strains due to thermal expansion of the continuum are accounted as $\varepsilon = \alpha \Delta T$, where α is the coefficient of thermal expansion and ΔT is the temperature change from a reference state.

In most formulations of boundary value problems of elasticity, one does not use the 15 equations in 15 unknowns. Most often, the 15 equations are reduced to either three equations in terms of the three displacement components (u, v, w) or six equations in terms of the six stress components $(\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz})$. When all governing equations are expressed in terms of the displacements, there is no need to use the compatibility conditions. On the other hand, when the previous equations are expressed in terms of stresses (or strains), one must satisfy the six compatibility conditions among six components of stress (or strain). The compatibility conditions are required only when the strain or stress field is given and the displacement field is to be determined.

In solid and structural mechanics, not all problems are solved by resorting to the equations of elasticity. Due to the nature of the stress and stress fields experienced, certain problems can be reduced to two- or one-dimensional problems by certain simplifying assumptions. Examples of one-dimensional problems were formulated in Examples 3.2.3, 5.3.4, and 5.3.5 for a slender body subjected to forces along or transverse to its length. Such structural members are classified as *bars* and *beams* depending on the nature of the deformation, that is, whether the members experience extensional (bars) or bending (beams) deformation. The basic assumption in formulating the problem was that the state of stress is one-dimensional; that is, the only nonzero stress in the body is $\sigma_{xx} = \sigma$, where x is taken along the length of the member. Many of the concepts may already be familiar to the reader through a course on elementary mechanics of materials. In this study, we will primarily study the problems of bars, beams, and some simple two-dimensional elasticity problems. We begin with the analysis of bars.

7.4.2 Analysis of bars

The equation governing the equilibrium of a bar is [see Eq. (5.3.4) or Eq. (5.3.7)]

$$\frac{dN}{dx} + f = 0, \quad 0 < x < L, \tag{7.4.6}$$

where f is the body force per unit length, as shown in Figure 7.4.1, and N is the internal axial force. The axial stress is related to the axial strain by $\sigma_{xx} = E\varepsilon_{xx}$, and the strain-displacement relation is $\varepsilon_{xx} = du/dx$. Substituting these relations into the definition of N [see Eq. (5.3.6)] gives

$$N(x) \equiv \int_A \sigma_{xx}\, dA = \int_A E\varepsilon_{xx}\, dA = \int_A E\frac{du}{dx}\, dA = EA\frac{du}{dx}. \tag{7.4.7}$$

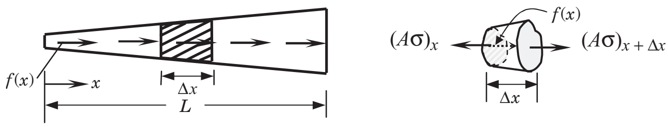


Figure 7.4.1 A bar of variable cross section and subjected to axial loads.

Then, Eq. (7.4.6) reduces to

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + f = 0, \quad 0 < x < L. \tag{7.4.8}$$

The second-order differential equation of Eq. (7.4.8) in u requires two boundary conditions to completely determine the solution $u(x)$. The boundary conditions are provided by the known geometric constraints and forces applied to the bar. The general solution of Eq. (7.4.8) is obtained by successive integrations:[1]

$$EA\frac{du}{dx} = -\left(\int f(x)\,dx + C_1\right), \tag{7.4.9}$$

$$u(x) = -\left\{\int\left[\frac{1}{EA}\left(\int f(x)\,dx + C_1\right)\right]dx + C_2\right\}, \tag{7.4.10}$$

where C_1 and C_2 are constants of integration that are to be evaluated using known conditions on u and $EA(du/dx)$. Such conditions are called *boundary conditions*. From Eqs. (7.4.9) and (7.4.10), it should be clear that the solution can be obtained only if the following integrals can be evaluated analytically:

$$\int f(x)\,dx, \quad \int\left[\frac{1}{EA}\left(\int f(x)\,dx + C_1\right)\right]dx.$$

Next, we consider couple of examples of axially loaded bars.

Example 7.4.1:

A bridge is supported by several concrete piers, and the geometry and loads of a typical pier are shown in Figure 7.4.2. The load 20×10^3 N/m^2 represents the weight of the bridge and an assumed distribution of the traffic on the bridge. The concrete weighs approximately $\gamma = 25 \times 10^3$ N/m^3 and its modulus is $E = 28 \times 10^9$ N/m^2. Determine the axial displacement, strain, and stress in the pier using a one-dimensional model.

Solution: The cross-sectional area $A(x)$ is

$$A(x) = (0.5 + 0.5x)0.5 = \frac{1}{4}(1 + x)\ (\text{m}^2).$$

We represent the distributed force at the top of the pier as a point force of magnitude

$$F = (0.5 \times 0.5)20 \times 10^3 = 5 \times 10^3\ (\text{N}).$$

[1] The symbols C_1, C_2, and so on are used to denote constants of integration, and they have no relation to the same symbols appearing in different examples.

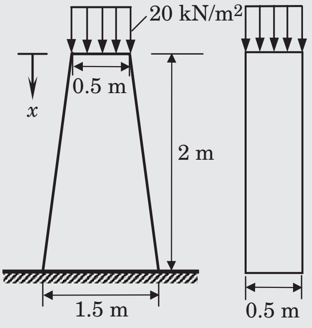


Figure 7.4.2 The geometry and loading in the concrete pier.

The weight of the concrete is represented as the body force per unit length. The total force at any distance x is equal to the weight of the concrete above that point. The weight at a distance x is equal to the product of the volume of the body above x and the specific weight $\rho g = \gamma$ of the concrete:

$$W(x) = 0.5\frac{0.5 + (0.5 + 0.5x)}{2}x \times 25 \times 10^3 = 6.25 \times 10^3(1 + 0.5x)x \text{ (N)}.$$

The body force per unit length is computed from

$$f = \gamma A(x) = \frac{dW}{dx} = 6.25(1 + x)10^3 \text{ (N/m)}.$$

The governing differential equation, Eq. (7.4.8), for the problem becomes

$$-\frac{d}{dx}\left[\frac{1}{4}E(1 + x)\frac{du}{dx}\right] = f(x), \tag{7.4.11}$$

subject to the boundary conditions

$$\left.\left[\frac{1}{4}E(1 + x)\frac{du}{dx}\right]\right|_{x=0} = -5 \times 10^3, \quad u(2) = 0. \tag{7.4.12}$$

The integrals can be evaluated as

$$\int f(x)\, dx = 3.125(1 + x)^2 10^3$$

$$\begin{aligned}
\int\left[\frac{1}{EA}\left(\int f(x)\, dx + C_1\right)\right]dx
&= \frac{12.5 \times 10^3}{E}\int \frac{(1 + x)^2}{(1 + x)}\, dx + \frac{4C_1}{E}\int \frac{1}{1 + x}\, dx \\
&= \frac{6.25 \times 10^3}{E}(1 + x)^2 + \frac{4C_1}{E}\ln(1 + x).
\end{aligned}$$

Hence, we have

$$EA\frac{du}{dx} = -\left[3.125(1 + x)^2 10^3 + C_1\right],$$

$$u(x) = -\frac{1}{E}\left\{\left[6.25 \times 10^3(1 + x)^2 + 4C_1\ln(1 + x)\right] + C_2\right\}.$$

Use of the boundary conditions in Eq. (7.4.12) yields

$$C_1 = 1.875 \times 10^3, \quad C_2 = -56.25 \times 10^3 - 4C_1 \ln 3.$$

The displacement, strain, and stress in the bar are

$$\begin{aligned} u &= \frac{10^3}{E}\left\{56.25 - 6.25(1+x)^2 + 7.5\left[\ln(1+x) - \ln 3\right]\right\} \text{ (m)} \\ \varepsilon &= \frac{du}{dx} = \frac{10^3}{E}\left[-12.5(1+x) + \frac{7.5}{1+x}\right] \text{ (m/m)} \\ \sigma &= E\varepsilon = 10^3\left[-12.5(1+x) + \frac{7.5}{1+x}\right] \text{ (N/m}^2\text{)}. \end{aligned} \tag{7.4.13}$$

Example 7.4.2:

Consider the composite bar consisting of a tapered steel bar fastened to an aluminum rod of uniform cross section and subjected loads, as shown in Figure 7.4.3. Determine the displacement field in the composite bar. Take the following values of the data:

$$E_1 = 200 \text{ GPa}, \; A_1 = \left(1.5 - \frac{x}{1.92}\right)^2 \times 10^{-4} \text{ m}^2, \; E_2 = 73 \text{ GPa},$$

$$A_2 = 10^{-4} \text{ m}^2, \; a = 0.96 \text{ m}, \; L = 2.16 \text{ m}, \; P_1 = 2,000 \text{ N}, \; P_2 = 1,000 \text{ N}.$$

Solution: The governing equations in each part of the composite bar are given by

$$\begin{aligned} \frac{d}{dx}\left(E_1 A_1 \frac{du_1}{dx}\right) &= 0, \quad 0 < x < a, \\ \frac{d}{dx}\left(E_2 A_2 \frac{du_2}{dx}\right) &= 0, \quad a < x < L, \end{aligned} \tag{7.4.14}$$

where the subscript 1 refers to steel and 2 to aluminum. The solutions of the previous two equations are

$$\begin{aligned} u_1(x) &= \frac{C_1}{(2.88 - x)} + C_2, \quad 0 < x < a, \\ u_2(x) &= C_3 x + C_4, \quad a < x < L, \end{aligned} \tag{7.4.15}$$

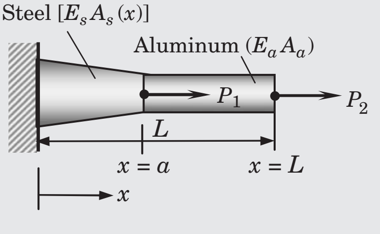


Figure 7.4.3 Axial deformation of a composite member.

where C_1, C_2, C_3, and C_4 are constants of integration, which are to be determined using the boundary and interface conditions. The boundary conditions are

$$u_1(0) = 0, \quad \left(E_2 A_2 \frac{du_2}{dx}\right)_{x=2.16} = P_2. \tag{7.4.16}$$

The interface conditions include the continuity of the displacement at $x = a$,

$$u_1(a) = u_2(a), \tag{7.4.17}$$

and the balance of forces at $x = a$,

$$\left[\left(E_1 A_1 \frac{du_1}{dx}\right)_{x=0.96^+} - \left(E_2 A_2 \frac{du_2}{dx}\right)_{x=0.96^-}\right] = P_1. \tag{7.4.18}$$

The condition $u_1(0) = 0$ gives $C_1 = -2.88C_2$, and the second condition in Eq. (7.4.16) results in

$$C_3 = \frac{P_2}{E_2 A_2} = \frac{10^{-2}}{73}.$$

The continuity condition of Eq. (7.4.17) leads to the result

$$\frac{C_1}{1.92} + C_2 = 0.96C_3 + C_4 \quad \text{or} \quad \left(1 - \frac{2.88}{1.92}\right)C_2 = 0.96\frac{P_2}{E_2 A_2} + C_4.$$

The condition in Eq. (7.4.18) gives

$$E_1 C_1 \frac{1}{1.92} - E_2 C_3 = 10^4 P_1 \quad \text{or} \quad -2.88 E_1 C_2 \frac{10^{-4}}{(1.92)^2} = P_1 + P_2.$$

Solving for the constants, we obtain

$$C_1 = 5.5296 \times 10^{-4}, \quad C_2 = -1.92 \times 10^{-4}, \quad C_3 = 1.36986 \times 10^{-4},$$
$$C_4 = -0.94685 \times 10^{-4}.$$

The displacement, strain, and stress fields in the composite bar are given by

$$\begin{aligned}
u(x) &= \begin{cases} \left(\frac{1.92x}{2.88-x}\right) 10^{-4}\ \text{m}, & 0 \le x \le 0.96, \\ (1.36986x - 0.94685)10^{-4}\ \text{m}, & 0.96 \le x \le 2.16, \end{cases} \\
\varepsilon(x) &= \frac{du}{dx} = \begin{cases} \frac{5.5296}{(2.88-x)^2} 10^{-4}\ \text{m/m}, & 0 \le x \le 0.96, \\ 1.36986 \times 10^{-4}\ \text{m/m}, & 0.96 \le x \le 2.16, \end{cases} \\
\sigma(x) &= E\varepsilon = \begin{cases} \frac{11.0592}{(2.88-x)^2} 10^7\ \text{N/m}^2, & 0 \le x \le 0.96, \\ 10^7\ \text{N/m}^2, & 0.96 \le x \le 2.16. \end{cases}
\end{aligned} \tag{7.4.19}$$

It can be verified that the force at $x = L$ is equal to $P_1 = 10^3$ N and that at $x = 0$ is equal to $P_1 + P_2 = 3 \times 10^3$ N.

7.4.3 Analysis of beams

Another elasticity problem whose governing equation cannot be deduced directly from the equations of elasticity is a beam (see Example 5.3.5). A *beam* is a

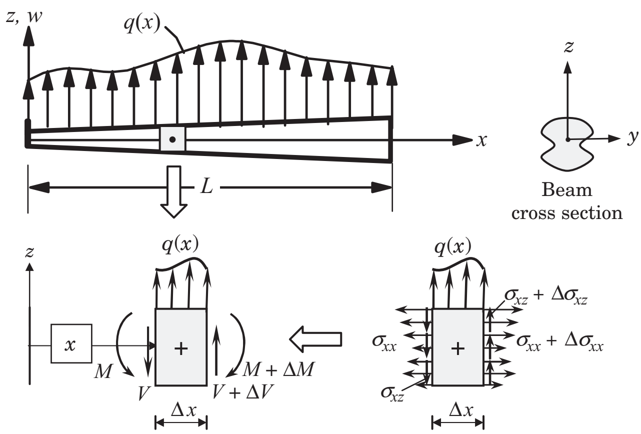


Figure 7.4.4 A beam of variable cross section subjected to transverse loads.

solid whose cross-sectional dimensions are much smaller than the longitudinal dimension (as in the case of bars) and subjected to loads that are transverse to the length, and hence tend to bend the solid about an axis (y or z) perpendicular to the beam axis (x). Thus, a beam differs from a bar only in the way forces are applied. In the case of a bar, the forces applied are axial and tend to elongate or shorten the bar. On the other hand, forces applied on a beam are in a direction transverse to the length and tend to bend the beam.

To derive the equations governing a beam, we consider a beam with loads applied transversely in the xz-plane passing through the geometric centroid (such that there is no twisting of the beam about the x-axis), as shown in Figure 7.4.4. Application of Newton's second law to the beam element shown in Figure 7.4.4 gives the following equation [see Example 5.3.5 and Eqs. (5.3.8) and (5.3.9)]:

$$\frac{d^2 M}{dx^2} + q = 0, \quad 0 < x < L. \tag{7.4.20}$$

The kinematics of deformation of the beam must be studied to determine the strain-displacement relations. Central to the development of the strain-displacement relations of a beam is the following three-part hypothesis:

(1) Plane sections before deformation remain plane after deformation, that is, straight lines perpendicular to the axis of the beam before deformation remain straight after deformation.
(2) Plane sections rotate about the y-axis as rigid disks such that they always remain perpendicular to the centroidal axis, which is now bent into an arc of a circle; that is, straight lines perpendicular to the centroidal axis of the beam before deformation remain normal after deformation.
(3) Plane sections do not change their geometric dimensions, that is, straight lines perpendicular to the axis of the beam remain inextensible during deformation. This amounts to neglecting the Poisson effect.

This set of assumptions is known as the *Euler–Bernoulli hypothesis*, and the resulting beam equations are known as the *Euler–Bernoulli beam theory*. The consequence of Assumptions 1 and 2 is that the transverse shear strain ε_{xz} is zero. Assumption 3 implies that the transverse normal strain is zero, $\varepsilon_{zz} = 0$. Strains ε_{yy}, ε_{xy}, and ε_{yz} are zero because the Poisson effect is neglected and the bending is in the xz-plane. The only nonzero strain is ε_{xx}, which was shown in Example 3.2.3 to be equal to

$$\varepsilon_{xx} = -z\frac{d^2 w}{dx^2}, \tag{7.4.21}$$

where w is the transverse deflection of the beam. The stresses vary from one section to the other along the length, and they vary linearly along the height. Because the stresses are not uniform over the cross section, the net moment due to the stress $\sigma_{xx} = E\varepsilon_{xx}$ is defined by

$$M = \int_A z\sigma_{xx}\, dA = \int_A z(E\varepsilon_{xx})\, dA = -E\frac{d^2 w}{dx^2}\left(\int_A z^2\, dA\right) = -EI\frac{d^2 w}{dx^2}, \tag{7.4.22}$$

where I is the *second moment of area*. The shear force is given by [see Eq. (5.3.9)]

$$V = \frac{dM}{dx} = -\frac{d}{dx}\left(EI\frac{d^2 w}{dx^2}\right). \tag{7.4.23}$$

Combining Eqs. (7.4.21) and (7.4.22), we arrive at the following fourth-order equation governing the transverse deflection w:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2 w}{dx^2}\right) = q(x), \quad 0 < x < L. \tag{7.4.24}$$

Because Eq. (7.4.24) is a fourth-order equation, four boundary conditions are required to solve it.

Next, we consider solution of Eqs. (7.4.20) and (7.4.24) for some representative problems of straight beams. First, the bending moment M can be determined from Eq. (7.4.20) for any given load and independent of the material and cross-sectional area as

$$M(x) = K_1 + K_2 x - \int\left(\int q(x)\, dx\right)dx, \tag{7.4.25}$$

where K_1 and K_2 constants of integration, which are determined using the boundary conditions on M and $dM/dx = V$.

As an example, consider a beam of length L, hinged at both ends (called *simply supported*), and subjected to uniformly distributed transverse load of intensity q_0, as shown in Figure 7.4.5. For this case, the bending moment becomes

$$M(x) = K_1 + K_2 x - \frac{q_0 x^2}{2}.$$

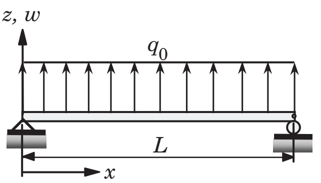


Figure 7.4.5 A simply supported beam under uniformly distributed load.

The boundary conditions on M are $M(0) = 0$ and $M(L) = 0$. Hence, we obtain $K_1 = 0$ and $K_2 = q_0L/2$, and the bending moment at any point x is

$$M(x) = \frac{q_0}{2}x(L - x), \quad 0 < x < L.$$

The maximum bending moment occurs at the center of the beam:

$$M_{\max} = M(L/2) = \frac{q_0L^2}{8}.$$

Next, the transverse displacement w is determined using Eq. (7.4.24). Integrating Eq. (7.4.24) successively four times, we obtain

$$\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) = \int q(x)\,dx + C_1, \tag{7.4.26}$$

$$EI\frac{d^2w}{dx^2} = \int\left(\int q(x)\,dx\right)dx + C_1x + C_2, \tag{7.4.27}$$

$$\begin{aligned}\frac{dw}{dx} &= \int \frac{1}{EI}\left[\int\left(\int q(x)\,dx\right)dx\right]dx \\ &\quad + C_1\int\frac{x}{EI}\,dx + C_2\int\frac{1}{EI}\,dx + C_3,\end{aligned} \tag{7.4.28}$$

$$\begin{aligned}w &= \int\left\{\int\frac{1}{EI}\left[\int\left(\int q(x)\,dx\right)dx\right]dx\right\}dx \\ &\quad + C_1\int\left(\int\frac{x}{EI}\,dx\right)dx + C_2\int\left(\int\frac{1}{EI}\,dx\right)dx + C_3x + C_4.\end{aligned} \tag{7.4.29}$$

The constants of integration, C_i $(i = 1, 2, 3, 4)$, are determined using the boundary conditions on the displacement w, the slope dw/dx, the bending moment $M = -EI(d^2w/dx^2)$, and the shear force $V = dM/dx$. Only four conditions are required, and one will have only four boundary conditions for any beam problem to be solved.

For the simply supported beam under uniform transverse load shown in Figure 7.4.5, the boundary conditions are

$$w(0) = 0, \quad w(L) = 0, \quad M(0) = 0, \quad M(L) = 0.$$

The boundary conditions on the bending moment give (from the previous discussion) $C_2 = 0$, $C_1 = -q_0L/2$, and

$$M(x) = -EI\frac{d^2w}{dx^2} = \frac{q_0}{2}x(L - x), \quad 0 < x < L.$$

If EI is a constant, Eq. (7.4.29) gives

$$w(x) = \frac{q_0 x^4}{24EI} - \frac{q_0 L x^3}{12EI} + C_3 x + C_4.$$

Using $w(0) = 0$ and $w(L) = 0$, we find $C_4 = 0$ and $C_3 = (q_0 L^3/24EI)$. The displacement w becomes

$$w(x) = \frac{q_0 x^4}{24EI} - \frac{q_0 L x^3}{12EI} + \frac{q_0 x L^3}{24EI} = \frac{q_0}{24EI}(x^4 - 2x^3 L + L^3 x).$$

The maximum displacement occurs at the center of the beam and is equal to

$$w_{\max} = w(L/2) = \frac{5 q_0 L^4}{384EI}.$$

Once the deflection w is known, we can compute the strain, stress, bending moment, and shear force from the relations

$$\begin{aligned} \varepsilon_{xx} &= -z\frac{d^2 w}{dx^2}, \quad \sigma_{xx} = \frac{Mz}{I} = -E\frac{d^2 w}{dx^2}, \\ M &= -EI\frac{d^2 w}{dx^2}, \quad V = -\frac{d}{dx}\left(EI\frac{d^2 w}{dx^2}\right). \end{aligned} \tag{7.4.30}$$

Although a simply supported beam under uniform load is used to illustrate the use of Eqs. (7.4.26) through (7.4.29), they are valid for beams with any boundary conditions. In the next example, we consider a beam with a spring support.

Example 7.4.3:

Consider a beam (with constant EI) fixed at the left end, spring-supported at the right end, and subjected to uniform load of intensity q_0 distributed over the entire span and a point load F_0 at the right end, as shown in Figure 7.4.6. Assuming that the spring is linearly elastic with a spring constant k, determine the elongation of the spring.

Solution: The displacement $w(x)$ can be determined from Eq. (7.4.29):

$$w(x) = \frac{q_0 x^4}{24EI} + C_1 \frac{x^3}{6EI} + C_2 \frac{x^2}{2EI} + C_3 x + C_4. \tag{7.4.31}$$

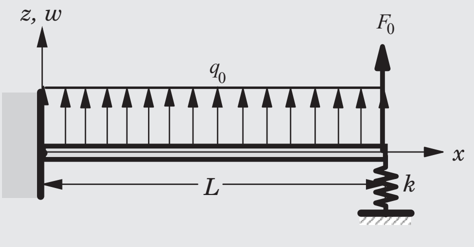


Figure 7.4.6 A beam fixed at the left end and spring-supported at the right end.

The boundary conditions for this problem are

$$\text{at } x = 0 : w = 0, \quad \frac{dw}{dx} = 0; \quad \text{at } x = L : M = 0, \quad V = F_0 - F_s; \tag{7.4.32}$$

where F_s is the force in the spring, $F_s = k\delta$, and δ is the elongation of the spring [which at the moment, is not known: $\delta = w(L)$], as shown in Figure 7.4.6. Use of the first two boundary conditions of Eq. (7.4.32) in Eq. (7.4.31) yields $C_3 = C_4 = 0$. Using the last two conditions of (7.4.32) in Eqs. (7.4.26) and (7.4.27), we obtain $C_1 = -q_0 L - F_0 + F_s$ and $C_2 = q_0 L^2/2 + F_0 L - F_s L$. Substitution of the values of C_1 through C_4 into Eq. (7.4.31) yields the result

$$w(x) = \frac{q_0 x^4}{24EI} - (q_0 L + F_0 - F_s)\frac{x^3}{6EI} + \left(\frac{q_0 L^2}{2} + F_0 L - F_s L\right)\frac{x^2}{2EI}. \tag{7.4.33}$$

Evaluating the previous expression at $x = L$ and noting that $F_s = kw(L)$, we obtain

$$\left(1 + \frac{kL^3}{3EI}\right) w(L) = \frac{q_0 L^4}{8EI} + \frac{F_0 L^3}{3EI}.$$

Thus, the elongation in the spring is

$$\delta = w(L) = \left(\frac{q_0 L^4}{8EI} + \frac{F_0 L^3}{3EI}\right)\left(1 + \frac{kL^3}{3EI}\right)^{-1}. \tag{7.4.34}$$

The displacement $w(x)$ can be determined from Eq. (7.4.33) by replacing F_s with $k\delta$.

The problem discussed in Example 7.4.3 has several special cases, as given here.

Case 1: $k = 0$ corresponds to a cantilever beam with uniformly distributed load q_0 and a point load F_0 at its right end, as shown in Figure 7.4.7(a). The displacement $w(x)$ and tip deflection $w(L)$ are given by

$$w(x) = \frac{q_0}{24EI}(x^4 - 4Lx^3 + 6L^2x^2) + \frac{F_0}{6EI}(3Lx^2 - x^3), \tag{7.4.35}$$

$$w(L) = \frac{q_0 L^4}{8EI} + \frac{F_0 L^3}{3EI}. \tag{7.4.36}$$

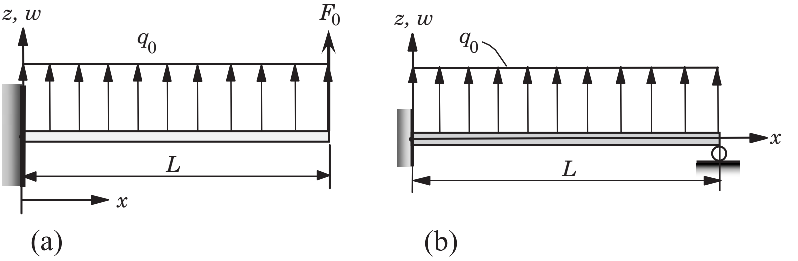
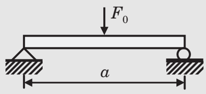
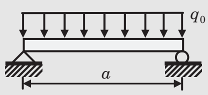
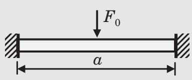
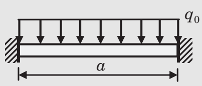
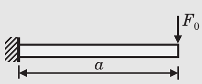
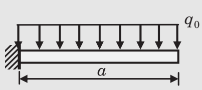


Figure 7.4.7 (a) Cantilever beam under uniformly distributed load q_0 and point load F_0. (b) Clamped-hinged beam under uniformly distributed load q_0.

Table 7.4.1. Transverse deflections of beams with various boundary conditions and subjected to point load or uniformly distributed load

Boundary conditions	Deflection, $w_0(x)$	w_{max} and M_{max}
• *Hinged-Hinged*		
Central point load 	$\frac{c_1}{48}\left[3\left(\frac{x}{a}\right) - 4\left(\frac{x}{a}\right)^3\right]$	$w^c_{max} = \frac{1}{48}c_1$ $M^c_{max} = -\frac{1}{4}c_3$
Uniform load 	$\frac{c_2}{24}\left[\left(\frac{x}{a}\right) - 2\left(\frac{x}{a}\right)^3 + \left(\frac{x}{a}\right)^4\right]$	$w^c_{max} = \frac{5}{384}c_2$ $M^c_{max} = -\frac{1}{8}c_4$
• *Fixed-Fixed*		
Central point load 	$\frac{c_1}{48}\left[3\left(\frac{x}{a}\right)^2 - 4\left(\frac{x}{a}\right)^3\right]$	$w^c_{max} = \frac{1}{192}c_1$ $M^0_{max} = \frac{1}{8}c_3$
Uniform load 	$\frac{c_2}{24}\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)\right]^2$	$w^c_{max} = \frac{1}{384}c_2$ $M^0_{max} = \frac{1}{12}c_4$
• *Fixed-Free*		
Point load at free 	$\frac{c_1}{6}\left[3\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^3\right]$	$w^a_{max} = \frac{1}{3}c_1$ $M^0_{max} = c_3$
Uniform load 	$\frac{c_2}{24}\left[6\left(\frac{x}{a}\right)^2 - 4\left(\frac{x}{a}\right)^3 + \left(\frac{x}{a}\right)^4\right]$	$w^a_{max} = \frac{1}{8}c_2$ $M^0_{max} = \frac{1}{2}c_4$

Case 2: $k \to \infty$ corresponds to a beam clamped at the left end and hinged at the right end and subjected to uniformly distributed load q_0 (the point load F_0 has no effect on the beam displacement other than to add to the reaction force at $x = L$), as shown in Figure 7.4.7(b). The boundary condition on V is now replaced by $w(L) = 0$. Then the constants C_1 and C_2 are given by

$$C_1 L + C_2 + \frac{q_0 L^2}{2} = 0, \quad \frac{q_0 L^4}{24} + \frac{C_1 L^3}{6} + \frac{C_2 L^2}{2} = 0,$$

which give $C_1 = -(5q_0 L/8)$ and $C_2 = (q_0 L^2/8)$. The displacement $w(x)$ becomes

$$w(x) = \frac{q_0}{48EI}\left(2x^4 - 5Lx^3 + 3L^2x^2\right). \tag{7.4.37}$$

The point x_0 where the maximum displacement occurs can be determined by setting $dw/dx = 0$. Table 7.4.1 contains transverse deflections as functions of x, and maximum deflections and bending moments for three different boundary conditions.

7.4.3.1 PRINCIPLE OF SUPERPOSITION

For linear boundary value problems, the principle of superposition holds. The *principle of superposition* is said to hold for a solid body if the displacements obtained under two sets of boundary conditions and forces are equal to the sum of the displacements that would be obtained by applying each set of boundary conditions and forces separately. To fix the ideas, consider the following two sets of boundary conditions and forces:

$$\textbf{Set1}: \quad \mathbf{u} = \mathbf{u}^{(1)} \text{ on } \Gamma_u; \quad \mathbf{t} = \mathbf{t}^{(1)} \text{ on } \Gamma_\sigma; \quad \mathbf{f} = \mathbf{f}^{(1)} \text{ in } \Omega. \tag{7.4.38}$$

$$\textbf{Set2}: \quad \mathbf{u} = \mathbf{u}^{(2)} \text{ on } \Gamma_u; \quad \mathbf{t} = \mathbf{t}^{(2)} \text{ on } \Gamma_\sigma; \quad \mathbf{f} = \mathbf{f}^{(2)} \text{ in } \Omega. \tag{7.4.39}$$

The superscript c refers to the center (at $x = a/2$), a refers to the end $x = a$, and 0 refers to $x = 0$. The constants c_i in the expressions for the deflections and bending moments are defined as

$$c_1 = \frac{F_0 ba^3}{EI}, \quad c_2 = \frac{q_0 ba^4}{EI}, \quad c_3 = -F_0 ba, \quad c_4 = -q_0 ba^2,$$

where the specified data ($\mathbf{u}^{(1)}$, $\mathbf{t}^{(1)}$, $\mathbf{f}^{(1)}$) and ($\mathbf{u}^{(2)}$, $\mathbf{t}^{(2)}$, $\mathbf{f}^{(2)}$) is independent of the deformation. Suppose that the solution to the two problems is $\mathbf{u}(\mathbf{x})^{(1)}$ and $\mathbf{u}(\mathbf{x})^{(2)}$, respectively. The superposition of the two sets of boundary conditions is

$$\mathbf{u} = \mathbf{u}^{(1)} + \mathbf{u}^{(2)} \text{ on } \Gamma_u; \quad \mathbf{t} = \mathbf{t}^{(1)} + \mathbf{t}^{(1)} \text{ on } \Gamma_\sigma; \quad \mathbf{f} = \mathbf{f}^{(1)} + \mathbf{f}^{(2)} \text{ in } \Omega. \tag{7.4.40}$$

Because of the linearity of the elasticity equations, the solution of the boundary value problem with the superposed data is $\mathbf{u}(\mathbf{x}) = \mathbf{u}^{(1)}(\mathbf{x}) + \mathbf{u}^{(2)}(\mathbf{x})$ in Ω. This is known as the *superposition principle.* The principle of superposition can be used to represent a linear problem with complicated boundary conditions or loads as a combination of linear problems that are equivalent to the original problem. The next example illustrates this point.

Example 7.4.4:

Consider the indeterminate beam shown in Figure 7.4.9. Determine the deflection of point A using the principle of superposition.

Solution: The solution to the problem can be viewed as the superposition of the solutions of the two beam problems shown there. Within the restrictions of the linear Euler–Bernoulli beam theory, the deflections are linear functions of the loads. Therefore, the principle of superposition holds. In particular, the deflection w_A at point A is equal to the sum of w_A^q and w_A^s due to the distributed load q_0 and spring force F_s, respectively, at point A:

$$w_A = w_A^q + w_A^s = \frac{q_0 L^4}{8EI} - \frac{F_s L^3}{3EI}.$$

Because the spring force F_s is equal to kw_A, we can calculate w_A from

$$w^A = \frac{q_0 L^4}{8EI\left(1 + \frac{kL^3}{3EI}\right)}.$$

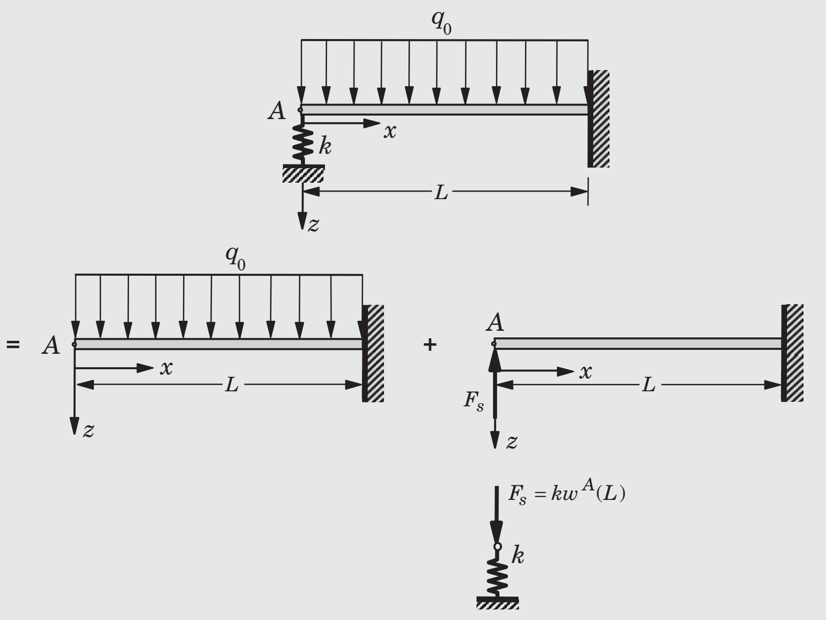


Figure 7.4.8 Solution of an indeterminate beam using the superposition principle.

7.4.4 Analysis of plane elasticity problems

7.4.4.1 PLANE STRAIN AND PLANE STRESS PROBLEMS

A class of problems in elasticity, due to geometry, boundary conditions, and external applied loads, have their solutions (i.e., displacements and stresses) not dependent on one of the coordinates. Such problems are called *plane elasticity* problems. The plane elasticity problems considered here are grouped into *plane strain* and *plane stress* problems. Both classes of problems are described by a set of two coupled partial differential equations expressed in terms of two dependent variables that represent the two components of the displacement vector. The governing equations of plane strain problems differ from those of the plane stress problems only in the coefficients of the differential equations. The discussion here is limited to isotropic materials.

7.4.4.2 PLANE STRAIN PROBLEMS

Plane strain problems are characterized by the displacement field,

$$u = u(x, y), \quad v = v(x, y), \quad w = 0, \tag{7.4.41}$$

where (u, v, w) denote the components of the displacement vector $\mathbf{u}$ in the (x, y, z) coordinate system. An example of a plane strain problem is provided by the long cylindrical member under external loads that are independent of z, as shown in Figure 7.4.9. For cross sections sufficiently far from the ends, it is clear that the displacement u_z is zero and that u and v are independent of z, that is, a state of plane strain exists.

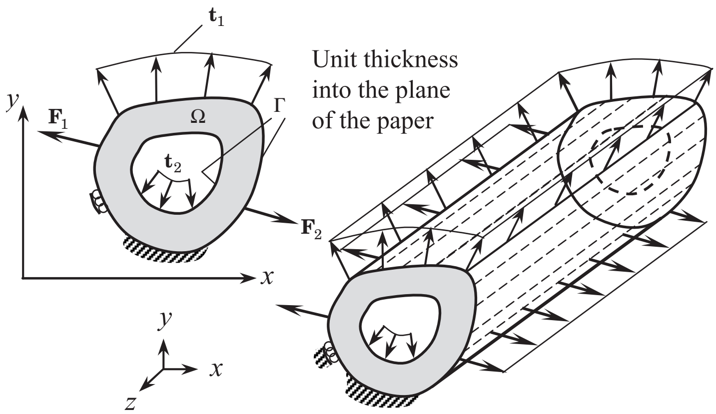


Figure 7.4.9 Examples of plane strain problems.

The displacement field of Eq. (7.4.41) results in the following strain field:

$$\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zz} = 0,$$
$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}. \tag{7.4.42}$$

Clearly, the body is in a state of plane strain.

For an isotropic material, the stress components are given by

$$\sigma_{xz} = \sigma_{yz} = 0, \quad \sigma_{zz} = \nu\,(\sigma_{xx} + \sigma_{yy}), \tag{7.4.43}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}. \tag{7.4.44}$$

The equations of equilibrium of three-dimensional linear elasticity, with the body force components

$$f_z = 0, \quad f_x = f_x(x, y), \quad f_y = f_y(x, y), \tag{7.4.45}$$

reduce to the following two plane strain equations:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \rho f_x = 0, \tag{7.4.46}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \rho f_y = 0. \tag{7.4.47}$$

The boundary conditions are either the stress type,

$$\left.\begin{aligned} t_x &\equiv \sigma_{xx} n_x + \sigma_{xy} n_y = \hat{t}_x \\ t_y &\equiv \sigma_{xy} n_x + \sigma_{yy} n_y = \hat{t}_y \end{aligned}\right\} \quad \text{on } \Gamma_\sigma, \tag{7.4.48}$$

or the displacement type,

$$u = \hat{u}, \quad v = \hat{v}, \quad \text{on } \Gamma_u. \tag{7.4.49}$$

Here (n_x, n_y) denote the components (or directional cosines) of the unit normal vector on the boundary Γ, Γ_σ and Γ_u are disjoint portions of the boundary Γ,

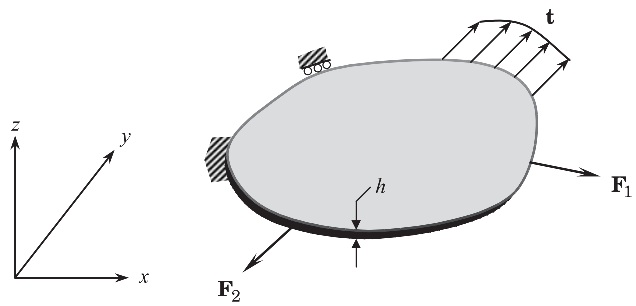


Figure 7.4.10 A thin plate in a state of plane stress.

$\hat{t}_x$ and $\hat{t}_y$ are the components of the specified traction vector, and $\hat{u}$ and $\hat{v}$ are the components of specified displacement vector. Only one element of each pair, (u, t_x) and (v, t_y), may be specified at a boundary point.

7.4.4.3 PLANE STRESS PROBLEMS

A state of *plane stress* is defined as one in which the following stress field exists:

$$\begin{aligned} \sigma_{xz} &= \sigma_{yz} = \sigma_{zz} = 0, \\ \sigma_{xx} = \sigma_{xx}(x, y), \quad \sigma_{xy} &= \sigma_{xy}(x, y), \quad \sigma_{yy} = \sigma_{yy}(x, y). \end{aligned} \tag{7.4.50}$$

An example of a plane stress problem is provided by a thin plate under external loads applied in the xy-plane (or parallel to it) that are independent of z, as shown in Figure 7.4.11. The top and bottom surfaces of the plate are assumed to be traction-free, and the specified boundary forces are in the xy-plane so that $f_z = 0$ and $u_z = 0$.

The stress-strain relations of a plane stress state are

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1+\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}. \tag{7.4.51}$$

The equations of equilibrium as well as boundary conditions of a plane stress problem are the same as those listed in Eqs. (7.4.46) through (7.4.49). The equilibrium equations of Eqs. (7.4.46) and (7.4.47) can be written in index notation as

$$\sigma_{\beta\alpha,\beta} + \rho f_\alpha = 0, \tag{7.4.52}$$

where α and β take the values of 1 and 2. The governing equations of plane stress and plane strain differ from each other only because of the difference in the constitutive equations for the two cases. To unify the formulation for plane strain and plane stress, we introduce the parameter s,

$$s = \begin{cases} \frac{1}{1-\nu} & \text{for plane strain,} \\ 1+\nu & \text{for plane stress.} \end{cases} \tag{7.4.53}$$

Then the constitutive equations of plane stress as well as plane strain can be expressed as

$$\sigma_{\alpha\beta} = 2\mu \left[\varepsilon_{\alpha\beta} + \left(\frac{s-1}{2-s} \right) \varepsilon_{\gamma\gamma}\delta_{\alpha\beta} \right], \tag{7.4.54}$$

$$\varepsilon_{\alpha\beta} = \frac{1}{2\mu} \left[\sigma_{\alpha\beta} - \left(\frac{s-1}{s} \right) \sigma_{\gamma\gamma}\delta_{\alpha\beta} \right], \tag{7.4.55}$$

where α, β, and γ take values of 1 and 2. The compatibility equations, Eq. (3.6.5), for plane stress and plane strain now take the form

$$\nabla^2 \sigma_{\alpha\alpha} = -s\, \rho f_{\alpha,\alpha}. \tag{7.4.56}$$

7.4.4.4 SOLUTION METHODS

The *analytical solution* of a problem is one that satisfies the governing differential equation at every point of the domain as well as the boundary conditions exactly. In general, finding analytical solutions of elasticity problems is not simple due to complicated geometries and boundary conditions. The *approximate solution* is one that satisfies governing differential equations as well as the boundary conditions approximately. *Numerical solutions* are approximate solutions that are developed using a numerical method, such as finite difference methods, the finite element method, the boundary element method, and so on. Often one seeks approximate solutions of practical problems using numerical methods. In this section, we discuss methods for finding solutions, exact as well as approximate.

The solutions of elasticity problems are developed using one of the following methods:

1. The *inverse method* is one in which one finds the solution for displacement, strain,andstress fields that satisfy the governing equations of elasticity and then tries to find a problem with boundary conditions to which the fields correspond.
2. The *semi-inverse method* is one in which the solution form in terms of unknown functions is arrived with the help of a qualitative understanding of the problem characteristics, and then the unknown functions are determined to satisfy the governing equations.
3. The *method of potentials* is one in which some of the governing equations are trivially satisfied by the choice of potential functions from which stresses or displacements are derived. The undetermined parametersin the potential functions are determined by finding solutions to remaining equations.
4. The *variational methods* are those that make use of extremum (i.e., minimum or maximum) and stationary principles. The principles are often cast in terms of energies of the system.
5. *Computational methods* are those that make use of numerical methods, such as the finite difference and finite element methods, and computers to determine solutions in an approximate sense.

Next, we consider an example of application of the semi-inverse method to solve an elasticity problem.

Example 7.4.5:

Consider an isotropic, hollow circular cylinder of internal radius a and outside radius b. The cylinder is pressurized at $r = a$ and at $r = b$, and rotating with a uniform speed of ω about its axis (z-axis). Determine the stresses developed in the cylinder under these applied loads.

Solution: Define a cylindrical coordinate system (r, θ, z), as shown in Figure 7.4.11. We assume that the body force vector is $\mathbf{f} = \rho\omega^2 r\,\hat{\mathbf{e}}_r$.

For this problem, we have only stress boundary conditions. We have

$$\text{At } r = a: \ \hat{\mathbf{n}} = -\hat{\mathbf{e}}_r, \quad \mathbf{t} = p_a\,\hat{\mathbf{e}}_r, \quad \text{or } \sigma_{rr} = -p_a, \quad \sigma_{r\theta} = 0, \tag{7.4.57}$$

$$\text{At } r = b: \ \hat{\mathbf{n}} = \hat{\mathbf{e}}_r, \quad \mathbf{t} = -p_b\,\hat{\mathbf{e}}_r, \quad \text{or } \sigma_{rr} = -p_b, \quad \sigma_{r\theta} = 0. \tag{7.4.58}$$

We wish to determine the displacements, strains, and stresses in the cylinder using the semi-inverse method. Due to the symmetry about the z-axis, we assume the displacement field to be of the form

$$u_r = U(r), \quad u_\theta = u_z = 0, \tag{7.4.59}$$

where $U(r)$ is an unknown function to be determined such that the equations of elasticity and boundary conditions are satisfied. If we cannot find $U(r)$ that satisfies the governing equations, then we must abandon the assumption of Eq. (7.4.59).

The strains associated with the displacement field, Eq. (7.4.59), are

$$\begin{aligned} \varepsilon_{rr} &= \frac{dU}{dr}, \quad \varepsilon_{\theta\theta} = \frac{U}{r}, \quad \varepsilon_{zz} = 0, \\ \varepsilon_{r\theta} &= 0, \quad \varepsilon_{z\theta} = 0, \quad \varepsilon_{rz} = 0. \end{aligned} \tag{7.4.60}$$

The stresses are given by

$$\begin{aligned} \sigma_{rr} &= 2\mu\varepsilon_{rr} + \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta}) = (2\mu + \lambda)\frac{dU}{dr} + \lambda\frac{U}{r}, \\ \sigma_{\theta\theta} &= 2\mu\varepsilon_{\theta\theta} + \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta}) = (2\mu + \lambda)\frac{U}{r} + \lambda\frac{dU}{dr}, \\ \sigma_{zz} &= 2\mu\varepsilon_{zz} + \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta}) = \lambda\left(\frac{dU}{dr} + \frac{U}{r}\right), \\ \sigma_{r\theta} &= 0, \quad \sigma_{rz} = 0, \quad \sigma_{\theta z} = 0. \end{aligned} \tag{7.4.61}$$

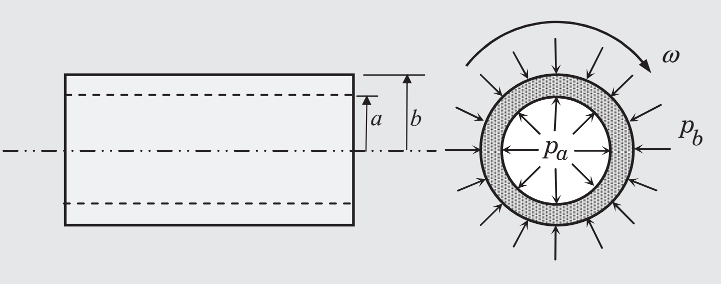


Figure 7.4.11 Rotating cylindrical pressure vessel.

Substituting the stresses from Eq. (7.4.61) into the equations of equilibrium (see the answer to Problem 5.15), we note that the last two equations are trivially satisfied, and the first equation reduces to

$$\begin{aligned}\frac{d\sigma_{rr}}{dr} + \frac{1}{r}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) &= -\rho\omega^2 r,\\ (2\mu+\lambda)\frac{d^2U}{dr^2} + \lambda\frac{d}{dr}\left(\frac{U}{r}\right) + \frac{2\mu}{r}\left(\frac{dU}{dr} - \frac{U}{r}\right) &= -\rho\omega^2 r.\end{aligned} \tag{7.4.62}$$

Simplifying the expression, we obtain

$$r^2\frac{d^2U}{dr^2} + r\frac{dU}{dr} - U = -\alpha r^3, \quad \alpha = \frac{\rho\omega^2}{2\mu+\lambda}. \tag{7.4.63}$$

The ordinary differential equation, Eq. (7.4.63), can be transformed to one with constant coefficients by a change of independent variable, $r = e^{\xi}$ (or $\xi = \ln r$). Using the chain rule of differentiation, we obtain

$$\frac{dU}{dr} = \frac{dU}{d\xi}\frac{d\xi}{dr} = \frac{1}{r}\frac{dU}{d\xi}, \quad \frac{d^2U}{dr^2} = \frac{d}{dr}\left(\frac{1}{r}\frac{dU}{d\xi}\right) = \frac{1}{r^2}\left(-\frac{dU}{d\xi} + \frac{d^2U}{d\xi^2}\right). \tag{7.4.64}$$

Substituting these expressions into Eq. (7.4.63), we obtain

$$\frac{d^2U}{d\xi^2} - U = -\alpha e^{3\xi}. \tag{7.4.65}$$

Seeking a solution in the form, $U(\xi) = e^{m\xi}$, we obtain the following general solution to the problem:

$$U_h(\xi) = c_1 e^{\xi} + c_2 e^{-\xi} - \frac{\alpha}{8}e^{3\xi}. \tag{7.4.66}$$

Changing back to the original independent variable r, we have

$$U(r) = c_1 r + \frac{c_2}{r} - \frac{\alpha}{8}r^3. \tag{7.4.67}$$

The stress σ_{rr} is given by

$$\begin{aligned}\sigma_{rr} &= (2\mu+\lambda)\left(c_1 - \frac{c_2}{r^2} - \frac{3\alpha}{8}r^2\right) + \lambda\left(c_1 + \frac{c_2}{r^2} - \frac{\alpha}{8}r^2\right)\\ &= 2(\mu+\lambda)c_1 - 2\mu\frac{c_2}{r^2} - \frac{(3\mu+2\lambda)\alpha}{4}r^2.\end{aligned} \tag{7.4.68}$$

Applying the stress boundary conditions in Eqs. (7.4.57) and (7.4.58), we obtain

$$\begin{aligned}2(\mu+\lambda)c_1 - 2\mu\frac{c_2}{a^2} - \frac{(3\mu+2\lambda)\alpha}{4}a^2 &= -p_a,\\ 2(\mu+\lambda)c_1 - 2\mu\frac{c_2}{b^2} - \frac{(3\mu+2\lambda)\alpha}{4}b^2 &= -p_b.\end{aligned} \tag{7.4.69}$$

Solving for the constants c_1 and c_2,

$$\begin{aligned} c_1 &= \frac{1}{2(\mu+\lambda)}\left[\left(\frac{p_a a^2 - p_b b^2}{b^2-a^2}\right) + (b^2+a^2)\frac{(3\mu+2\lambda)}{(2\mu+\lambda)}\frac{\rho\omega^2}{4}\right], \\ c_2 &= \frac{a^2b^2}{2\mu}\left[\left(\frac{p_a - p_b}{b^2-a^2}\right) + \frac{(3\mu+2\lambda)}{(2\mu+\lambda)}\frac{\rho\omega^2}{4}\right]. \end{aligned} \quad (7.4.70)$$

Finally, the displacement u_r and stress σ_{rr} in the cylinder are given by

$$\begin{aligned} u_r &= \frac{1}{2(\mu+\lambda)}\left[\left(\frac{p_a a^2 - p_b b^2}{b^2-a^2}\right) + (b^2+a^2)\frac{(3\mu+2\lambda)}{(2\mu+\lambda)}\frac{\rho\omega^2}{4}\right] r \\ &\quad + \frac{a^2b^2}{2\mu}\left[\left(\frac{p_a - p_b}{b^2-a^2}\right) + \frac{(3\mu+2\lambda)}{(2\mu+\lambda)}\frac{\rho\omega^2}{4}\right]\frac{1}{r} - \frac{\rho\omega^2}{8(2\mu+\lambda)}r^3, \end{aligned} \quad (7.4.71)$$

$$\begin{aligned} \sigma_{rr} &= \left[\left(\frac{p_a a^2 - p_b b^2}{b^2-a^2}\right) + (b^2+a^2)\frac{(3\mu+2\lambda)}{(2\mu+\lambda)}\frac{\rho\omega^2}{4}\right] \\ &\quad - \frac{a^2b^2}{r^2}\left[\left(\frac{p_a - p_b}{b^2-a^2}\right) + \frac{(3\mu+2\lambda)}{(2\mu+\lambda)}\frac{\rho\omega^2}{4}\right] - \frac{(3\mu+2\lambda)\alpha}{4}r^2. \end{aligned} \quad (7.4.72)$$

7.4.4.5 AIRY STRESS FUNCTION

The *Airy stress function* is a potential function introduced to identically satisfy the equations of equilibrium, Eqs. (7.4.46) and (7.4.47). First, we assume that the body force vector **f** is derivable from a scalar potential V such that

$$\rho\mathbf{f} = -\nabla V \quad \text{or} \quad \rho f_x = -\frac{\partial V}{\partial x}, \quad \rho f_y = -\frac{\partial V}{\partial y}. \quad (7.4.73)$$

This amounts to assuming that body forces are conservative. Next, we introduce the Airy stress function $\Phi(x, y)$ such that

$$\sigma_{xx} = \frac{\partial^2\Phi}{\partial y^2} + V, \quad \sigma_{yy} = \frac{\partial^2\Phi}{\partial x^2} + V, \quad \sigma_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y}. \quad (7.4.74)$$

This definition of $\Phi(x, y)$ automatically satisfies the equations of equilibrium, Eqs. (7.4.46) and (7.4.47).

The stresses derived from Eq. (7.4.74) are subject to the compatibility conditions of Eq. (7.4.56). Substituting for $\sigma_{\alpha\beta}$ in terms of Φ from Eq. (7.4.74) into Eq. (7.4.56), we obtain

$$\nabla^4\Phi + (2-s)\nabla^2 V = 0, \quad (7.4.75)$$

where $\nabla^4 = \nabla^2\nabla^2$ is the *biharmonic operator*, which in two dimensions has the form

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4}. \quad (7.4.76)$$

If the body forces are zero, we have $V = 0$ and Eq. (7.4.75) reduces to the *biharmonic equation*,

$$\nabla^4 \Phi = 0. \tag{7.4.77}$$

In summary, the solution to a plane elastic problem using the Airy stress function involves finding the solution to Eq. (7.4.75) and satisfying the boundary conditions of the problem. The most difficult part is finding a solution to the fourth-order equation, Eq. (7.4.75), over a given domain. Often the form of the Airy stress function is obtained by either the inverse method or the semi-inverse method. Next, we consider some examples of the Airy stress function approach.

Example 7.4.6:

1. Suppose that the Airy stress function is a second-order polynomial (which is the lowest order that gives a nonzero stress field) of the form

$$\Phi(x, y) = c_1 xy + c_2 x^2 + c_3 y^2. \tag{7.4.78}$$

Determine the constants c_1, c_2, and c_3 such that Φ satisfies the biharmonic equation $\nabla^4 \Phi = 0$ (the body force field is zero) and corresponds to a possible state of stress for some boundary value problem.

Solution: Clearly, the biharmonic equation is satisfied by Φ in Eq. (7.4.78). The corresponding stress field is

$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2} = 2c_3, \quad \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2} = 2c_2, \quad \sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} = -c_1. \tag{7.4.79}$$

The state of stress is uniform (i.e., constant) throughout the body, and it is independent of the geometry. Thus, there are an infinite number of problems for which the stress field is a solution. In particular, the rectangular domain shown in Figure 7.4.13 is one such problem.

2. Take the Airy stress function to be a third-order polynomial of the form

$$\Phi(x, y) = c_1 xy + c_2 x^2 + c_3 y^2 + c_4 x^2 y + c_5 xy^2 + c_6 x^3 + c_7 y^3. \tag{7.4.80}$$

Determine the stress field and identify various possible boundary-value problems.

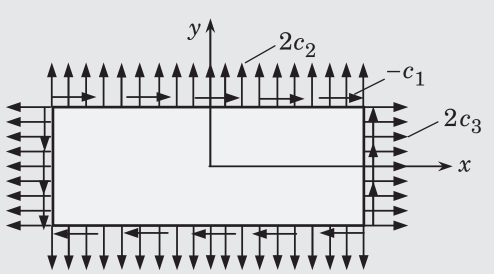


Figure 7.4.12 A problem with uniform stress field.

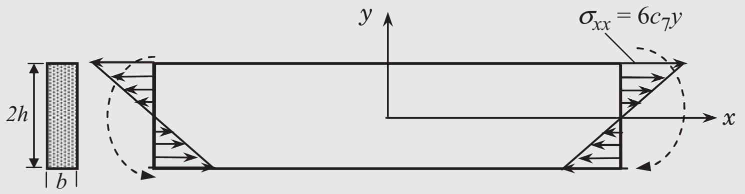


Figure 7.4.13 A thin beam in pure bending.

Solution: We note that $\nabla^4\Phi = 0$ for any c_i. The corresponding stress field is

$$\begin{aligned}\sigma_{xx} &= 2c_3 + 2c_5x + 6c_7y, \quad \sigma_{yy} = 2c_2 + 2c_4y + 6c_6y,\\ \sigma_{xy} &= -c_1 - 2c_4x - 2c_5y.\end{aligned} \tag{7.4.81}$$

Again, there are an infinite number of problems for which the stress field is a solution. In particular, for $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0$ the solution corresponds to a thin beam in pure bending (see Figure 7.4.13).

3. Lastly, take the Airy stress function to be a fourth-order polynomial of the form (omit terms that were already considered in the last two cases)

$$\Phi(x, y) = c_8x^2y^2 + c_9x^3y + c_{10}xy^3 + c_{11}x^4 + c_{12}y^4. \tag{7.4.82}$$

Determine the stress field and associated boundary value problems.

Solution: Computing $\nabla^4\Phi$ and equating it to zero (the body force field is zero), we find that

$$c_8 + 3(c_{11} + c_{12}) = 0.$$

Thus, out of five constants only four are independent. The corresponding stress field is

$$\begin{aligned}\sigma_{xx} &= 2c_8x^2 + 6c_{10}xy + 12c_{12}y^2 = -6c_{11}x^2 + 6c_{10}xy + 6c_{12}(2y^2 - x^2),\\ \sigma_{yy} &= 2c_8y^2 + 6c_9xy + 12c_{11}x^2 = 6c_9xy + 2c_{11}(2x^2 - y^2) - 6c_{12}y^2,\\ \sigma_{xy} &= -4c_8xy - 3c_9x^2 - 3c_{10}y^2 = 12c_{11}xy + 12c_{12}xy - 3c_9x^2 - 3c_{10}y^2.\end{aligned} \tag{7.4.83}$$

By suitable adjustment of the constants, we can obtain various loads on rectangular plates. For instance, taking all coefficients except c_{10} equal to zero, we obtain

$$\sigma_{xx} = 6c_{10}xy, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = -3c_{10}y^2.$$

7.5 Summary

In this chapter, some applications of problems from heat transfer, fluid mechanics, and solid mechanics are presented. Beginning with a summary of the governing equations for each field, a number of examples are presented to illustrate how the equations derived in the previous chapters are useful in solving the problems of

science and engineering. In particular, heat transfer in one-dimensional geometries (e.g., fins, plane wall, and axisymmetric geometries), flows of viscous fluids through channels and pipes, and deflection and stress analysis of problems of bars, beams, and plane elasticity problems are presented. There exist a number of books dedicated to each of the three topic areas, and interested readers may consult them for additional study.

PROBLEMS

7.1. Consider a long electric wire of length L and cross section with radius R and electrical conductivity k_e [1/(Ohm·m)]. An electric current with current density I (amps/m^2) is passing through the wire. The transmission of an electric current is an irreversible process in which some electrical energy is converted into thermal energy (heat). The rate of heat production per unit volume is given by

$$\rho Q_e = \frac{I^2}{k_e}.$$

Assuming that the temperature rise in the cylinder is so small as not to affect the thermal or electrical conductivities and that the transfer of heat is one-dimensional along the radius of the cylinder, derive the governing equation using the balance of energy.

7.2. Solve the equation derived in Problem 7.1 using the boundary conditions

$$q(0) = \text{finite}, \quad T(R) = T_0.$$

7.3. A slab of length L is initially at temperature $f(x)$. For times $t > 0$, the boundaries at $x = 0$ and $x = L$ are kept at temperatures T_0 and T_L, respectively. Obtain the temperature distribution in the slab as a function of position x and time t.

7.4. Obtain the steady-state temperature distribution $T(x, y)$ in a rectangular region, $0 \leq x \leq a$, $0 \leq y \leq b$ for the boundary conditions

$$q_x(0, y) = 0, \quad q_y(x, b) = 0, \quad q_x(a, y) + hT(a, y) = 0, \quad T(x, 0) = f(x).$$

7.5. Consider the steady flow through a long, straight, horizontal circular pipe. The velocity field is given by

$$v_r = 0, \quad v_\theta = 0, \quad v_z(r) = -\frac{R^2}{4\mu}\frac{dP}{dr}\left(1 - \frac{r^2}{R^2}\right). \tag{1}$$

If the pipe is maintained at a temperature T_0 on the surface, determine the steady-state temperature distribution in the fluid.

7.6. Consider the free convection problem of flow between two parallel plates of different temperature. A fluid with density ρ and viscosity μ is placed between two vertical plates a distance $2a$ apart, as shown in Figure P7.6. Suppose that the plate at $x = a$ is maintained at a temperature T_1 and the plate at $x = -a$ is maintained at a temperature T_2, with $T_2 > T_1$. Assuming that the plates are very long in the y-direction and hence the temperature and velocity fields are only a function of x, determine the temperature

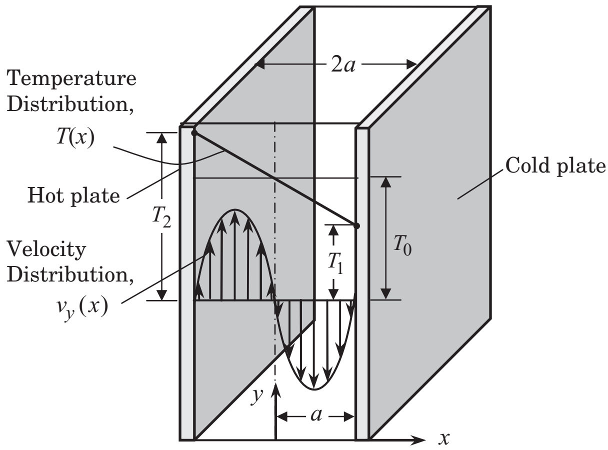


Figure P7.6

$T(x)$ and velocity $v_y(x)$. Assume that the volume rate of flow in the upward moving stream is the same as that in the downward moving stream, and the pressure gradient is solely due to the weight of the fluid.

7.7. An engineer is to design a sea lab 4 m high, 5 m wide, and 10 m long to withstand submersion to 120 m, measured from the surface of the sea to the top of the sea lab. Determine (a) the pressure on the top and (b) the pressure variation on the side of the cubic structure. Assume the density of salt water to be $\rho = 1,020$ kg/m^3.

7.8. Compute the pressure and density at an elevation of 1,600 m for isothermal conditions. Assume $P_0 = 10^2$ kPa, $\rho_0 = 1.24$ kg/m^3 at sea level.

7.9. For the steady, two-dimensional flow between parallel plates of Problem 5.3, determine c such that the velocity field satisfies the principle of conservation of linear momentum. Assume that the axial and shear stresses are related to the velocities by the relations

$$\sigma_{xy} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right), \quad \sigma_{xx} = 2\mu\frac{\partial v_x}{\partial x} - P,$$

where μ is a constant, called the *viscosity* of the fluid, and P is the pressure.

7.10. Consider the steady flow of a viscous incompressible Newtonian fluid down an inclined surface of slope α under the action of gravity (see Figure P7.10).

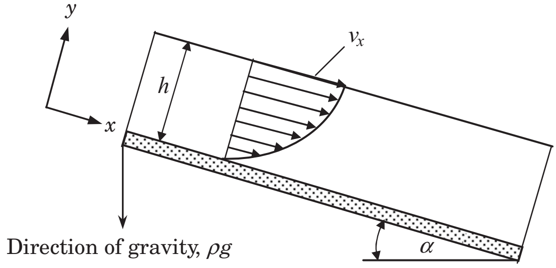


Figure P7.10

The thickness of the fluid perpendicular to the plane is h and the pressure on the free surface is p_0, a constant. Use the semi-inverse method (i.e., assume the form of the velocity field) to determine the pressure and velocity field.

7.11. Two immiscible fluids are flowing in the x-direction in a horizontal channel of length L and width $2b$ under the influence of a fixed pressure gradient. The fluid rates are adjusted such that the channel is half filled with Fluid I (denser phase) and half filled with Fluid II (less dense phase). Assuming that the gravity of the fluids is negligible, determine the velocity field. Use the geometry and coordinate system shown in Figure P7.11.

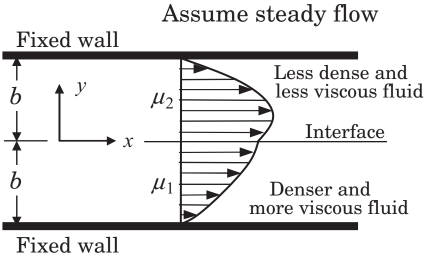


Figure P7.11

7.12. Consider the flow of a viscous incompressible fluid through a circular pipe (see Section 7.3.4.2). Reformulate the problem when the weight of the fluid is taken into account.

7.13. Consider a steady, isothermal, incompressible fluid flowing between two vertical concentric long circular cylinders with radii $r_1 = R$ and $r_2 = \alpha\, R$, as shown in Figure P7.13. If the outer one rotates with an angular velocity Ω, show that the Navier–Stokes equations reduce to the following equations governing the circumferential velocity $v_\theta = v(r)$ and pressure P:

$$\rho\frac{v^2}{r} = \frac{\partial P}{\partial r}, \quad \mu\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv)\right) = 0, \quad 0 = -\frac{\partial P}{\partial z} + \rho g.$$

Determine the velocity v and shear stress $\tau_{r\theta}$ distributions.

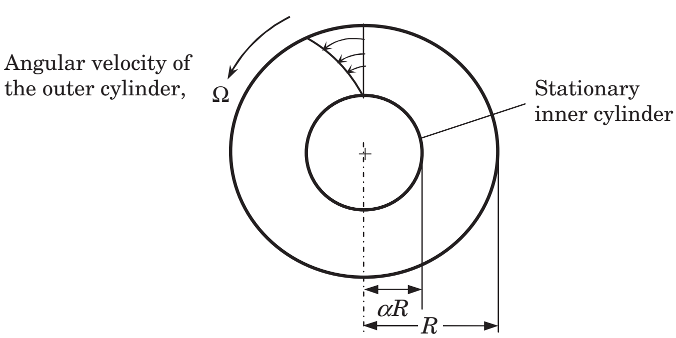


Figure P7.13

7.14. An isotropic body ($E = 210$ GPa and $\nu = 0.3$) with a two-dimensional state of stress experiences the following displacement field (in mm):

$$u_1 = 3x_1^2 - x_1^3 x_2 + 2x_2^3, \quad u_2 = x_1^3 + 2x_1 x_2,$$

where x_i are in meters. Determine the stresses and rotation of the body at point $(x_1, x_2) = (0.05, 0.02)$ m. Is the displacement field compatible?

7.15. A two-dimensional state of stress exists in a body with the following components of stress:

$$\sigma_{11} = c_1 x_2^3 + c_2 x_1^2 x_2 - c_3 x_1, \quad \sigma_{22} = c_4 x_2^3 - c_5,$$

$$\sigma_{12} = c_6 x_1 x_2^2 + c_7 x_1^2 x_2 - c_8,$$

where c_i are constants. Assuming that the body forces are zero, determine the conditions on the constants so that the stress field is in equilibrium and satisfies the compatibility equations.

7.16.–7.18. For the truss structures shown in Figures P7.16–P7.18, determine the member stresses and strains. Assume linear elastic behavior, and let A_i be the area of cross section and E_i be the modulus of the ith member.

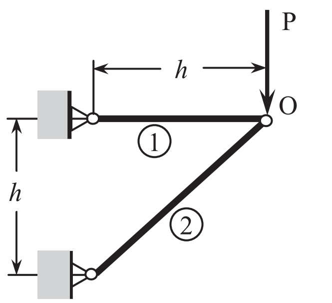


Figure P7.16

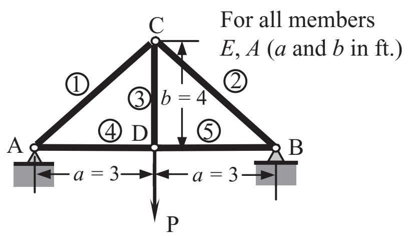


Figure P7.17

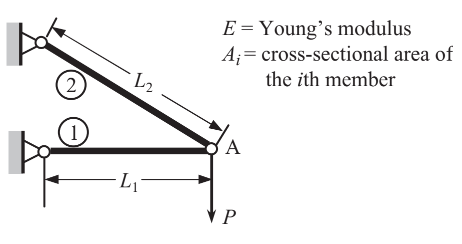


Figure P7.18

7.19.–7.20. For the straight beam structures shown in Figures P7.19–P7.20, determine the transverse deflection as a function of position along the length of the beam. Assume linear elastic behavior with constant EI.

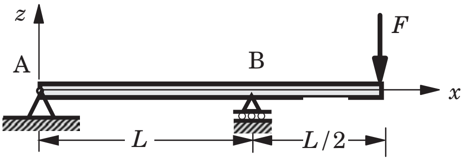


Figure P7.19

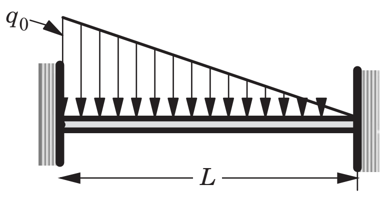


Figure P7.20

7.21. Consider a simply supported beam of length L under point loads F_0 at $x = L/4$ and $x = 3L/4$ (the so-called *four-point bending*). Use the symmetry about $x = L/2$ to determine the transverse deflection $w(x)$.

7.22. Consider a semicircular curved beam of mean radius R. The beam is fixed at $\theta = \pi$ and subjected to a vertical upward load of P at $\theta = 0$, as shown in Figure P7.22. Determine the normal and shear forces (N, V) and bending moment M at any section between $\theta = 0$ and $\theta = \pi$.

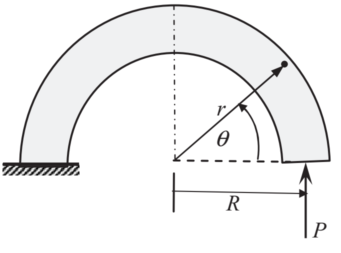


Figure P7.22

7.23. Derive the equilibrium equations governing the deformation of a homogeneous isotropic circular plate under the action of axisymmetric (about the z-axis) radial (f) and transverse (q) forces. The free-body diagram of an element of the plate with all relevant forces is shown in Figure P7.23. Note that the shear stresses are zero due to the symmetry. The stress resultants are defined by

$$N_{rr} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{rr}\, dz, \quad N_{\theta\theta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\theta\theta}\, dz,$$

$$M_{rr} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{rr} z\, dz, \quad M_{\theta\theta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{rr} z\, dz.$$

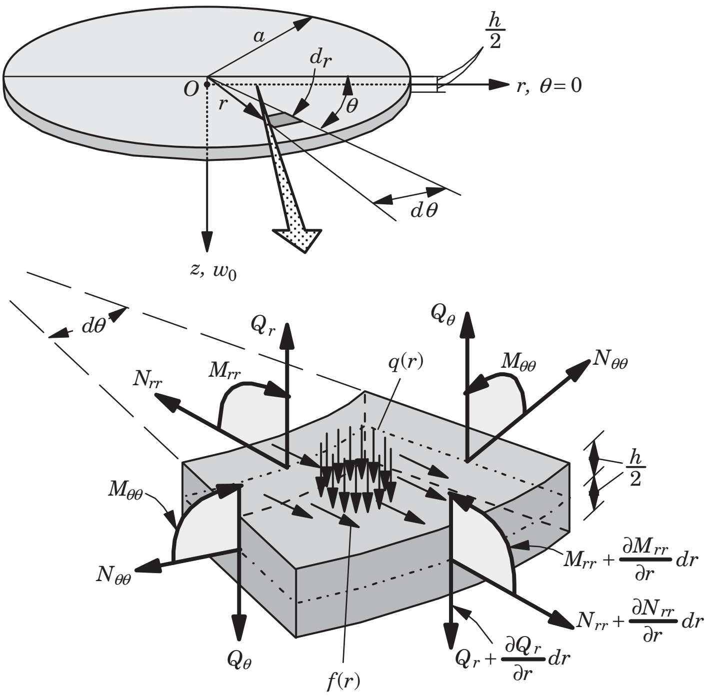


Figure P7.23

7.24. Use the following displacement field for axisymmetric deformation of a circular plate and Hooke's law to express the stress resultants of Problem 7.23 in terms of the displacements u_0 and w_0:

$$u_r(r, z) = u_0(r) - z\frac{dw_0}{dr}, \quad u_\theta(r, z) = 0 \quad u_z(r, z) = w_0(r).$$

7.25. Consider an isotropic, solid circular plate of radius a and constant thickness h, spinning about the z-axis at an angular velocity of ω. Show that the governing equation is of the form

$$-\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru_0)\right] = \frac{1-\nu^2}{E}\rho\omega^2 r,$$

and its general solution is

$$u_0(r) = C_1 r + \frac{C_2}{r} - \frac{1-\nu^2}{E}\rho\omega^2\frac{r^3}{8},$$

where C_1 and C_2 are constants of integration. Use suitable boundary conditions to determine the constants of integration, C_1 and C_2, and evaluate the stresses σ_{rr} and $\sigma_{\theta\theta}$.

7.26. Consider an isotropic, solid circular plate of radius a and constant thickness h, fixed at $r = a$ and subjected to uniformly distributed transverse load of intensity q_0. Show that the governing equation is of the form

$$D\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2 w_0}{dr^2} + \frac{1}{r}\frac{dw_0}{dr}\right) = q_0,$$

where $D = Eh^3/[12(1-\nu^2)]$. The general solution is

$$w_0(r) = C_1 + C_2 \log r + C_3 r^2 + C_4 r^2 \log r + \frac{q_0 r^4}{64D}.$$

Use suitable boundary conditions to determine the constants of integration, C_i, $i = 1, 2, 3, 4$, and determine the deflection w_0 and bending moments M_{rr} and $M_{\theta\theta}$ as functions of the radial coordinate r.

7.27. The lateral surface of a homogeneous, isotropic, solid circular cylinder of radius a, length L, and mass density ρ is bonded to a rigid surface. Assuming that the ends of the cylinder at $z = 0$ and $z = L$ are traction-free (see Figure P7.27), determine the displacement and stress fields in the cylinder due to its own weight.

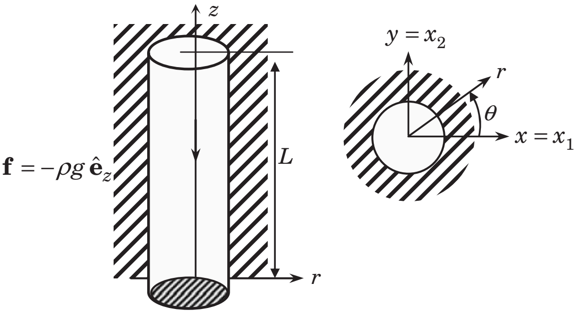


Figure P7.27

7.28. A solid circular cylindrical body of radius a and height h is placed between two rigid plates, as shown in Figure **??**. The plate at B is held stationary and the plate at A is subjected to a downward displacement of δ. Using a suitable coordinate system, write the boundary conditions for the following two cases: (a) when the cylindrical object is bonded to the plates at A and B; (b) when the plates at A and B are frictionless.

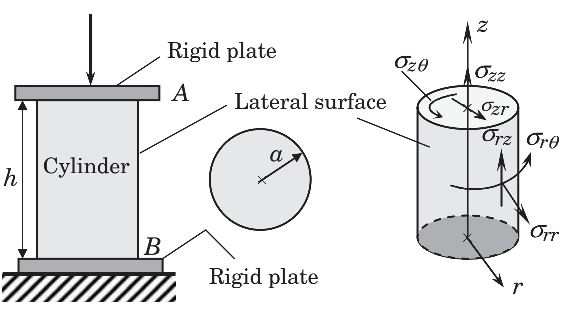


Figure P7.28

7.29. An external hydrostatic pressure of magnitude p is applied to the surface of a spherical body of radius b with a concentric rigid spherical inclusion of radius a, as shown in Figure P7.29. Determine the displacement and stress fields in the spherical body.

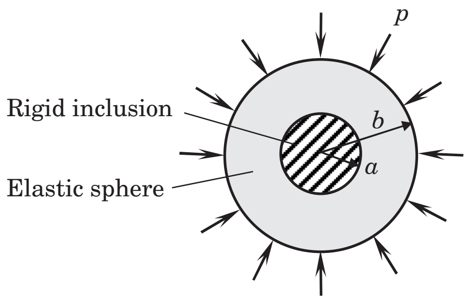


Figure P7.29

7.30. Reconsider the concentric spheres of Problem 7.29. As opposed to the rigid core in Problem 7.29, suppose that the core is elastic and the outer shell is subjected to external pressure p (both are linearly elastic). Assuming Lamé constants of μ_1 and λ_1 for the core and μ_2 and λ_2 for the outer shell (see Figure **??**), and that the interface is perfectly bonded at $r = a$, determine the displacements of the core as well as the shell.

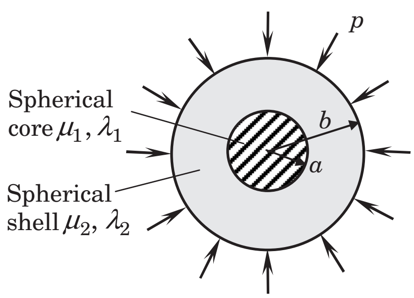


Figure P7.30

7.31. Consider a long hollow circular shaft with a rigid internal core (a cross section of the shaft is shown in Figure P7.31). Assuming that the inner surface of the shaft at $r = a$ is perfectly bonded to the rigid core and the outer boundary at $r = b$ is subjected to a uniform shearing traction of magnitude τ_0, find the displacement and stress fields in the problem.

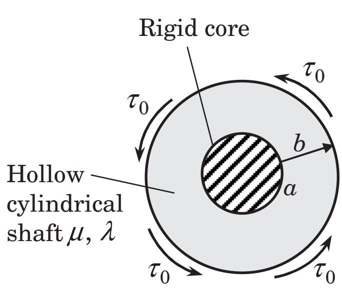


Figure P7.31

7.32. Interpret the following stress field obtained in case 3 of Example 7.4.6 using Figure 7.4.13:

$$\sigma_{xx} = 6c_{10}xy, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = -3c_{10}y^2.$$

Assume that c_{10} is a positive constant.

7.33. Compute the stress field associated with the Airy stress function

$$\Phi(x, y) = Ax^5 + Bx^4y + Cx^3y^2 + Dx^2y^3 + Exy^4 + Fy^5.$$

Interpret the stress field for the case in which A, B, and C are zero. Use Figure 7.4.13 to sketch the stress field.

7.34. Investigate what problem is solved by the Airy stress function

$$\Phi = \frac{3A}{4b}\left(xy - \frac{xy^3}{3b^2}\right) + \frac{B}{4b}y^2.$$

7.35. Show that the Airy stress function

$$\Phi(x, y) = \frac{q_0}{8b^3}\left[x^2\left(y^3 - 3b^2y + 2b^3\right) - \frac{1}{5}y^3\left(y^2 - 2b^2\right)\right]$$

satisfies the compatibility condition. Determine the stress field and find what problem it corresponds to when applied to the region $-b \leq y \leq b$ and $x = 0, a$ (see Figure **??**).

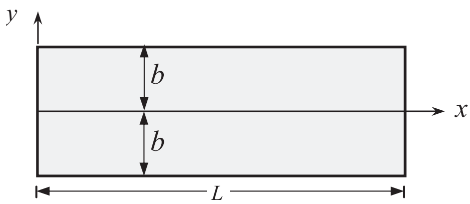


Figure P7.35

7.36. Determine the Airy stress function for the stress field of the domain shown in Figure **??** and evaluate the stress field.

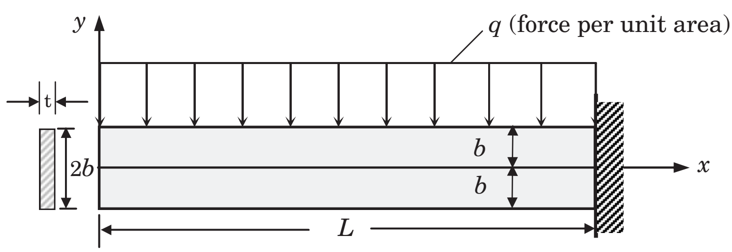


Figure P7.36

7.37. The thin cantilever beam shown in Figure **??** is subjected to a uniform shearing traction of magnitude τ_0 along its upper surface. Determine if the Airy stress function

$$\Phi(x, y) = \frac{\tau_0}{4}\left(xy - \frac{xy^2}{b} - \frac{xy^3}{b^2} + \frac{ay^2}{b} + \frac{ay^3}{b^2}\right)$$

satisfies the compatibility condition and stress boundary conditions of the problem.

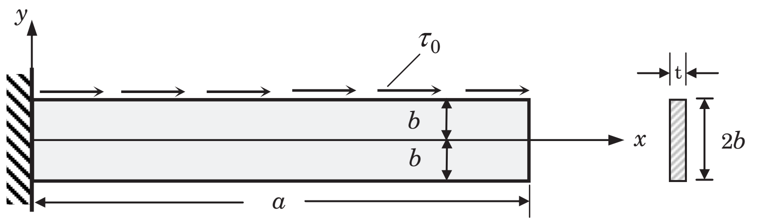


Figure P7.37

7.38. The curved beam shown in Figure **??** is curved along a circular arc. The beam is fixed at the upper end and is subjected at the lower end to a distribution of tractions statically equivalent to a force per unit thickness $\mathbf{P} = -P\hat{\mathbf{e}}_1$. Assume that the beam is in a state of plane strain/stress. Show that an Airy stress function of the form

$$\Phi(r) = \left(Ar^3 + \frac{B}{r} + C\,r\log r\right)\sin\theta$$

provides an approximate solution to this problem and solve for the values of the constants A, B, and C.

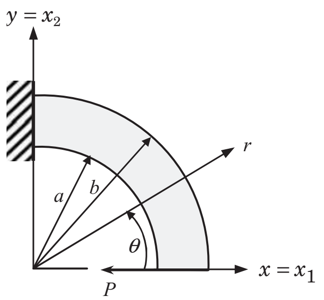


Figure P7.38

The only solid piece of scientific truth about which I feel totally confident is that we are profoundly ignorant about nature. It is this sudden confrontation with the depth and scope of ignorance that represents the most significant contribution of twentieth-century science to the human intellect.

Lewis Thomas

Answers to Selected Problems

Chapter 2

2.1 $\mathbf{C} \cdot \left[\mathbf{A} - (\mathbf{A} \cdot \hat{\mathbf{e}}_B)\, \hat{\mathbf{e}}_B\right] = 0$ is the equation of the required line (or any multiple of it) .

2.2 $(\mathbf{A} - \mathbf{B}) \times (\mathbf{B} - \mathbf{C}) \cdot (\mathbf{A} - \mathbf{C}) = 0.$

2.3 $\mathbf{A} = (2\hat{\mathbf{e}}_1 - 2\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3).$

2.4 $\mathbf{A}_p = \frac{2}{9}(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$ and $\theta = 82.34^\circ.$

2.5 $v_n = \frac{15}{\sqrt{13}},\ \theta = 75.08^\circ,\ Q_m = 624.04$ (kg/sec).

2.6 $v_n = -\frac{8}{\sqrt{5}},\ \theta = 166^\circ,\ Q_m = -536.656$ (kg/sec).

2.7 The vectors are linearly dependent.

2.9 $S_{ii} = 12,\ S_{ij}S_{ji} = 240,\ S_{ij}S_{ij} = 281,$

$$\{R\} = \begin{Bmatrix} 31 \\ 25 \\ 30 \end{Bmatrix}, \quad \{C\} = \begin{Bmatrix} 18 \\ 15 \\ 34 \end{Bmatrix}, \quad \{B\} = \begin{Bmatrix} -191 \\ 297 \\ 525 \end{Bmatrix}.$$

2.11

$$[L] = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} \\ \frac{-4}{\sqrt{42}} & \frac{1}{\sqrt{42}} & \frac{5}{\sqrt{42}} \end{bmatrix}, \quad [L] = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}.$$

2.12

$$[L] = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2.13

$$[L] = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}.$$

2.14 (a) Let $\mathbf{A}_2 = A_{2i}\hat{\mathbf{e}}_i,\ \ \mathbf{A}_3 = A_{3i}\hat{\mathbf{e}}_i$. Then the determinant form of the vector product is given by

$$\mathbf{A}_2 \times \mathbf{A}_3 = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix},$$

and the determinant form of the scalar triple product is

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \mathbf{A}_1 \cdot \mathbf{A}_2 \times \mathbf{A}_3 = e_{ijk} A_{1i} A_{2j} A_{3k}. \tag{1}$$

(**b**) From Eq. (1) it follows that

$$\mathbf{A}_i \cdot (\mathbf{A}_j \times \mathbf{A}_k) = e_{rst} A_{ir} A_{js} A_{kt}. \tag{2}$$

Multiplying both sides by e_{ijk},

$$e_{ijk}[\mathbf{A}_i \cdot (\mathbf{A}_j \times \mathbf{A}_k)] = e_{ijk} e_{rst} A_{ir} A_{js} A_{kt}.$$

By expanding the left side, we obtain

$$\begin{aligned} e_{ijk}\mathbf{A}_i \cdot (\mathbf{A}_j \times \mathbf{A}_k) &= e_{1jk}\mathbf{A}_1 \cdot (\mathbf{A}_j \times \mathbf{A}_k) + e_{2jk}\mathbf{A}_2 \cdot (\mathbf{A}_j \times \mathbf{A}_k) \\ &\quad + e_{3jk}\mathbf{A}_3 \cdot (\mathbf{A}_j \times \mathbf{A}_k), \\ &= 6\mathbf{A}_1 \cdot (\mathbf{A}_2 \times \mathbf{A}_3) = 6 \det(A). \end{aligned}$$

(**f**) Note that det $([A][B]) =$ det $[A]\cdot$ det $[B]$ and det $[A]^{\mathrm{T}}=$det $[A]$. Then

$$e_{ijk} e_{pqr} = \left| \begin{bmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{bmatrix} \begin{bmatrix} \delta_{p1} & \delta_{q1} & \delta_{r1} \\ \delta_{p2} & \delta_{q2} & \delta_{r2} \\ \delta_{p3} & \delta_{q3} & \delta_{r3} \end{bmatrix} \right| = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix}.$$

2.15 (a) $\bar{S}_{ii} = \ell_{im}\ell_{in} S_{mn} = \delta_{mn} S_{mn} = S_{mm} = S_{ii}$.

2.16 First, establish the following identity:

$$\mathrm{grad}(r) = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} \left(x_j x_j\right)^{\frac{1}{2}} = \hat{\mathbf{e}}_i \frac{1}{2} \left(x_j x_j\right)^{\frac{1}{2}-1} 2x_i = \hat{\mathbf{e}}_i x_i \left(x_j x_j\right)^{-\frac{1}{2}} = \frac{\mathbf{r}}{r}.$$

(a) We have

$$\begin{aligned} \nabla^2(r^n) &= \frac{\partial^2}{\partial x_i \partial x_i}(r^n) = \frac{\partial}{\partial x_i}\left(n r^{n-2} x_i\right) \\ &= n(n-2) r^{n-3} \frac{\partial r}{\partial x_i} x_i + n r^{n-2} \delta_{ii} = n(n-2) r^{n-3} \frac{x_i}{r} x_i + 3n r^{n-2} \\ &= [n(n-2) + 3n] r^{n-2} = n(n+1) r^{n-2}. \end{aligned}$$

(c) Carrying out the indicated operation, we obtain

$$\begin{aligned} \mathrm{div}(\mathbf{r} \times \mathbf{A}) &= \hat{\mathbf{e}}_i \cdot \frac{\partial}{\partial x_i} \left(e_{jk\ell} x_j A_k \hat{\mathbf{e}}_\ell\right) = e_{jk\ell}\delta_{i\ell} \left(\frac{\partial x_j}{\partial x_i} A_k + x_j \frac{\partial A_k}{\partial x_i}\right) \\ &= (0+0) = 0. \end{aligned}$$

2.17 (a) Using index notation, we write

$$\begin{aligned} \mathrm{div}(\mathrm{curl}\,\mathbf{A}) &= \hat{\mathbf{e}}_i \cdot \frac{\partial}{\partial x_i} \left(e_{jk\ell} \frac{\partial A_k}{\partial x_j} \hat{\mathbf{e}}_\ell\right) = e_{jk\ell}\delta_{i\ell} \frac{\partial^2 A_k}{\partial x_i \partial x_j} \\ &= e_{ijk} \frac{\partial^2 A_k}{\partial x_i \partial x_j} = 0 \end{aligned}$$

because of the symmetry of $A_{k,ij}$ in i and j. Note that because $\frac{\partial^2 F}{\partial x_i \partial x_j}$ is symmetric in i and j,

$$\nabla \times (\nabla F) = \left(\hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}\right) \times \left(\hat{\mathbf{e}}_j \frac{\partial F}{\partial x_j}\right) = e_{ijk} \hat{\mathbf{e}}_k \frac{\partial^2 F}{\partial x_i \partial x_j} = 0.$$

(c) Because $\mathbf{A} = \mathbf{A}(\mathbf{x})$, we obtain

$$\begin{aligned}\text{grad}(\mathbf{A}\cdot\mathbf{x}) &= \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}\left(A_j x_j\right) = \hat{\mathbf{e}}_i\left(\frac{\partial A_j}{\partial x_i} x_j + A_j \frac{\partial x_j}{\partial x_i}\right)\\ &= \nabla\mathbf{A}\cdot\mathbf{x} + \mathbf{A},\end{aligned}$$

where $A_j \frac{\partial x_j}{\partial x_i} = A_j \delta_{ij} = A_i$ is used in arriving at the final expression.

(f) Because $\mathbf{A} = \mathbf{A}(\mathbf{x})$ and $\mathbf{B} = \mathbf{B}(\mathbf{x})$, we obtain

$$\begin{aligned}\text{div}\,(\mathbf{A}\times\mathbf{B}) &= \hat{\mathbf{e}}_i \cdot \frac{\partial}{\partial x_i}\left(e_{jk\ell} A_j B_k \hat{\mathbf{e}}_\ell\right) = e_{ijk}\left(\frac{\partial A_j}{\partial x_i} B_k + A_j \frac{\partial B_k}{\partial x_i}\right)\\ &= \text{curl}\mathbf{A}\cdot\mathbf{B} - \text{curl}\mathbf{B}\cdot\mathbf{A}.\end{aligned}$$

(h) Begin with the right-hand side of the identity:

$$\begin{aligned}\nabla\left(\frac{A^2}{2}\right) - \mathbf{A}\times\nabla\times\mathbf{A} &= \frac{1}{2}\left[\hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}\left(A_j A_j\right)\right] - (A_i\hat{\mathbf{e}}_i)\times\left(e_{rst}\frac{\partial A_s}{\partial x_r}\hat{\mathbf{e}}_t\right)\\ &= \hat{\mathbf{e}}_i A_j \frac{\partial A_j}{\partial x_i} - e_{rst} e_{kit}\hat{\mathbf{e}}_k A_i \frac{\partial A_s}{\partial x_r}\\ &= \hat{\mathbf{e}}_i A_j \frac{\partial A_j}{\partial x_i} - \hat{\mathbf{e}}_k\left(\delta_{rk}\delta_{si} - \delta_{ri}\delta_{sk}\right) A_i \frac{\partial A_s}{\partial x_r}\\ &= \hat{\mathbf{e}}_i A_j \frac{\partial A_j}{\partial x_i} - \hat{\mathbf{e}}_k\left(A_s \frac{\partial A_s}{\partial x_k} - A_i \frac{\partial A_k}{\partial x_i}\right)\\ &= \hat{\mathbf{e}}_k A_i \frac{\partial A_k}{\partial x_i} = A_i \frac{\partial}{\partial x_i}(A_k\hat{\mathbf{e}}_k) = \mathbf{A}\cdot\text{grad}\,\mathbf{A}.\end{aligned}$$

2.18 Use the Gradient Theorem of Eq. (2.6.19) with $\phi = 1$ to obtain the required result.

2.19 Using the Divergence Theorem of Eq. (2.6.20), we can write

$$\int_\Omega \nabla\cdot\nabla\phi\, d\mathbf{x} = \oint_\Gamma \hat{\mathbf{n}}\cdot\nabla\phi\, dS = \oint_\Gamma \frac{\partial\phi}{\partial n}\, ds.$$

2.20 (a) $(\mathbf{I}\times\mathbf{A})\cdot\mathbf{\Phi} = [\delta_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j \times (A_k\hat{\mathbf{e}}_k)]\cdot\phi_{mn}\hat{\mathbf{e}}_m\hat{\mathbf{e}}_n = A_k\delta_{ij}e_{jk\ell}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_\ell\cdot\phi_{mn}\hat{\mathbf{e}}_m\hat{\mathbf{e}}_n$
$= A_k\delta_{ij}e_{jk\ell}\delta_{\ell m}\phi_{mn}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_n = \hat{\mathbf{e}}_j e_{jkm}A_k\phi_{mn}\hat{\mathbf{e}}_n = \mathbf{A}\times\mathbf{\Phi}.$

(d)
$$\begin{aligned}(\mathbf{\Phi}\cdot\mathbf{\Psi})^{\mathrm{T}} &= \left(\phi_{ij}\psi_{jk}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_k\right)^{\mathrm{T}}\\ &= \phi_{ij}\psi_{jk}\hat{\mathbf{e}}_k\hat{\mathbf{e}}_i \quad \text{(interchange the base vectors)}\\ &= \left(\hat{\mathbf{e}}_k\psi_{kj}\right)^{\mathrm{T}}\left(\phi_{ji}\hat{\mathbf{e}}_i\right)^{\mathrm{T}} \quad \text{(interchange the subscripts)}\\ &= \mathbf{\Psi}^{\mathrm{T}}\cdot\mathbf{\Phi}^{\mathrm{T}}.\end{aligned}$$

Chapter 3

3.1 $\varepsilon_{AB} = 0.085$ m/m. It is assumed that $\cos\theta \approx 1$ and $\sin\theta \approx \theta$.

3.2 $\varepsilon_{BE} = 0.102$ m/m and $\varepsilon_{CF} = 0.153$ m/m.

3.3 $\varepsilon_{AC} = 0.048$ m/m.

3.4 $\varepsilon_{BC} = -0.0058$ m/m, $\varepsilon_{AC} = -0.00257$, and $\gamma_A = 0.0095$.

3.5 $\mathbf{v} = \frac{\mathbf{x}}{1+t}$ and $\mathbf{a} = \mathbf{0}$.

3.6 $\frac{dv_x}{dt} = \frac{2}{L}\left(1 + 2\frac{x}{L}\right)U_0^2$.

3.7 $\mathbf{X} = \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix}.$

3.8 $2[E] = \begin{bmatrix} k_1^2 - 1 & 0 & 0 \\ 0 & k_2^2 - 1 & 0 \\ 0 & 0 & k_3^2 - 1 \end{bmatrix}.$

3.9 $2[E] = \begin{bmatrix} k_1^2 - 1 & e_0 k_1 k_2 & 0 \\ e_0 k_1 k_2 & (1 + e_0^2) k_2^2 - 1 & 0 \\ 0 & 0 & k_3^2 - 1 \end{bmatrix}.$

3.10 $2[E] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (2 + Bt)Bt \end{bmatrix}.$

3.11 $2[E] = \begin{bmatrix} B^2 & A + B & 0 \\ A + B & A^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

3.12 The velocity components in the material description are $v_1 = X_1 \sinh t + X_2 \cosh t$, $v_2 = X_1 \cosh t + X_2 \sinh t$, $v_3 = 0$. The components of the velocity gradient tensor are

$$[L] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

3.13 The components of the Green–Lagrange strain tensor are

$$[E] = \begin{bmatrix} 6 & 7 & 0 \\ 7 & 8 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The principal strains are $\lambda_1 = -0.071$ and $\lambda_2 = 14.071$. The eigenvectors are given by

$$\hat{\mathbf{n}}_1 = -0.7554\,\hat{\mathbf{e}}_1 + 0.6552\,\hat{\mathbf{e}}_2, \quad \hat{\mathbf{n}}_2 = 0.6552\,\hat{\mathbf{e}}_1 + 0.7554\,\hat{\mathbf{e}}_2.$$

3.14 The strain components are $E_{11} = 0, \quad E_{12} = \frac{e_0}{2b}, \quad E_{22} = \frac{1}{2}\left(\frac{e_0}{b}\right)^2.$

3.15 The Lagrange–Green strain tensor components are

$$E_{11} = 0, \quad E_{12} = \frac{e_0}{b^2} X_2, \quad E_{22} = \frac{1}{2}\left(2X_2 \frac{e_0}{b^2}\right)^2.$$

3.17 The Lagrange–Green strain tensor components are given by

$$E_{11} = -0.2 + 0.5[(-0.2)^2 + (0.2 + 0.1X_2)^2],$$
$$2E_{12} = 0.5 + (0.2 + 0.1X_2) + (-0.2)(0.5) + (0.2 + 0.1X_2)(-0.1 + 0.1X_1),$$
$$E_{22} = -0.1 + 0.1X_1 + 0.5[(0.5)^2 + (-0.1 + 0.1X_1)^2].$$

3.18 The linear components are given by

$$\varepsilon_{11} = 3x_1^2x_2 + c_1\left(2c_2^3 + 3c_2^2x_2 - x_2^3\right),$$
$$\varepsilon_{22} = -\left(2c_2^3 + 3c_2^2x_2 - x_2^3 + 3c_1x_1^2x_2\right),$$
$$2\varepsilon_{12} = x_1\left[x_1^2 + c_1\left(3c_2^2 - 3x_2^2\right)\right] - 3c_1x_1x_2^2.$$

3.19 The strain field is not compatible (why?).

3.21 $E_{AC} = \frac{1}{2}\frac{e_1^2+e_2^2+2(ae_1+be_2)}{a^2+b^2}$.

3.22 $\lambda_2 = 1,000\ \mu$m; $\hat{\mathbf{n}}^{(2)} = (2\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2)/\sqrt{3}$.

3.25 The function $f(X_2, X_3)$ is of the form

$$f(X_2, X_3) = A + BX_2 + CX_3,$$

where A, B, and C are arbitrary constants.

Chapter 4

4.1 Use Cauchy's formula and symmetry of σ to establish the required equality.

4.2 On BC: $\mathbf{t} = 5\hat{\mathbf{i}}$, and on CD: $\mathbf{t} = \mathbf{0}$.

4.3 (i) $\mathbf{t}^{\hat{n}} = 2(\hat{\mathbf{e}}_1 + 7\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3)$; $|\mathbf{t}^{\hat{n}}| = \sqrt{204} = 14.28$ MPa; $\sigma_n = -7.33$ MPa, $\sigma_s = 12.26$ MPa.

4.4 $\mathbf{t}^{\hat{n}} = \frac{1}{\sqrt{3}}(5\hat{\mathbf{e}}_1 + 5\hat{\mathbf{e}}_2 + 9\hat{\mathbf{e}}_3)10^3$ psi; $\sigma_n = 6,333.33$ psi, and $\sigma_s = 1,885.62$ psi.

4.5 The normal and shear components of the stress vector are $\sigma_n = -2.833$ MPa and $\sigma_s = 8.67$ MPa.

4.6 $\mathbf{t}^{\hat{n}} = \frac{1}{\sqrt{69}}(35\hat{\mathbf{e}}_1 + 0.5\hat{\mathbf{e}}_2 - 7.5\hat{\mathbf{e}}_3)$ MPa. The normal and shear components of the stress vector are $\sigma_n = 0.3478$ MPa and $\sigma_s = 4.2955$ MPa.

4.7 $\sigma_n = -40.80$ MPa and $\sigma_s = -20.67$ MPa.

4.8 $\sigma_n = 3.84$ MPa and $\sigma_s = -17.99$ MPa.

4.9 $\sigma_n = 950$ kPa and $\sigma_s = -150$ kPa.

4.10 $\sigma_n = -76.60$ MPa and $\sigma_s = 32.68$ MPa.

4.11 $\sigma_{22} = 140$ MPa and $\sigma_s = 90$ MPa.

4.12 $\sigma_{p1} = 972.015$ kPa and $\sigma_{p2} = -72.015$ kPa. The principal planes are given by $\theta_{p1} = 36.65°$ and $\theta_{p2} = 126.65°$.

4.13 $\sigma_{p1} = 121.98$ MPa and $\sigma_{p2} = -81.98$ MPa. The principal planes are given by $\theta_{p1} = 39.35°$ and $\theta_{p2} = 129.35°$.

4.15 $\sigma_{\theta\theta} = \frac{pD}{4t}$

4.16 $\lambda_1 = 6.856$, $\lambda_2 = -10.533$, and $\lambda_3 = -3.323$. The eigenvector associated with $\lambda_1 = 6.856$ is (only components are displayed; it is sufficient to find A_i; it is not necessary to normalize them) $\hat{\mathbf{A}}^{(1)} = \pm(1, 0.11865, -2.155)\frac{1}{\text{magnitude}} = \pm(0.42, 0.0498, -0.906)$.

4.17 (**b**) The normal and shear components of the stress vector are $t_n = -16.67$ MPa and $t_s = 52.7$ MPa.

4.18 The principal stresses are $\sigma_1 = 25$ MPa, $\sigma_2 = 50$ MPa, $\sigma_3 = 75$ MPa. The principal planes are $\hat{\mathbf{n}}^{(1)} = \pm\left(\frac{3}{5}\hat{\mathbf{e}}_1 - \frac{4}{5}\hat{\mathbf{e}}_3\right)$, $\hat{\mathbf{n}}^{(2)} = \pm\hat{\mathbf{e}}_2$.

Chapter 5

5.1 $-\frac{\partial}{\partial r}(r\rho v_r) - \frac{\partial}{\partial \theta}(\rho v_\theta) - r\frac{\partial}{\partial z}(\rho v_z) = r\frac{\partial \rho}{\partial t}$.

5.2 (b) Satisfies the continuity equation.

5.3 The average velocity is given by $v_{\text{avg}} = \frac{Q}{b} = \frac{1}{6}(3v_0 - c)$ m/s.

5.4 (a) $F = 24.12$ N. (b) $F = 12.06$ N.

5.6 $v_2 = 9.9$ m/sec; $Q = 19.45$ liters/sec.

5.7 $\mathbf{R}_A = -\hat{\mathbf{e}}_1 + 50\,\hat{\mathbf{e}}_2$ (kN) and $\mathbf{M}_A = 89\,\hat{\mathbf{e}}_3$ (kN-m).

5.10 $V(x) = -4,700 + 500x$ and $M(x) = 20,500 - 4,700x + 250x^2$ in $10 < x < 15$; $V(x) = -10{,}000 + 500x$ and $M(x) = 100{,}000 - 1{,}000x + 250x^2$ in $15 < x < 20$. The x-coordinate is taken from the left end of the beam. Units used are meters and newtons.

5.12 The equilibrium equations are satisfied only if the body forces are $\rho f_1 = 0, \quad \rho f_2 = a\left(b^2 + 2x_1x_2 - x_2^2\right), \quad \rho f_3 = -4abx_3.$

5.13 The equilibrium equations are satisfied only if $c_2 = -c_6 = 3c_4$.

5.14 $\sigma_{xz} = -\frac{P}{2I}\left(h^2 - z^2\right)$ and $\sigma_{zz} = 0$.

5.15 The equilibrium equations are

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \rho f_r = 0,$$
$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + \rho f_\theta = 0,$$
$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + \rho f_z = 0.$$

5.16 (a) $T = 0.15$ N-m. (b) When $T = 0$, the angular velocity is $\omega_0 = 477.5$ rpm.

Chapter 6

6.2 $$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = 10^6 \begin{Bmatrix} 37.8 \\ 43.2 \\ 27.0 \\ 21.6 \\ 0.0 \\ 5.4 \end{Bmatrix} \text{ Pa.}$$

6.6 The velocity vector is

$$\mathbf{v} = \left(\frac{d\mathbf{x}}{dt}\right)_{X=\mathit{fixed}} = 2ktX_2^2\,\hat{\mathbf{e}}_1 + kX_2\,\hat{\mathbf{e}}_2$$
$$= 2kt\left(\frac{x_2}{1+kt}\right)^2\hat{\mathbf{e}}_1 + \frac{kx_2}{1+kt}\,\hat{\mathbf{e}}_2.$$

Then the components of the strain rate tensor are given by

$$\varepsilon_{11} = \frac{\partial v_1}{\partial x_1} = 0,\quad 2\varepsilon_{12} = \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} = \frac{4ktx_2}{(1+kt)^2},\quad \varepsilon_{22} = \frac{\partial v_2}{\partial x_2} = \frac{k}{1+kt}.$$

The viscous stresses are given by

$$\tau_{11} = 0,\ \tau_{22} = \frac{2\mu k}{1+kt},\ \tau_{12} = \frac{4\mu tkx_2}{(1+kt)^2}.$$

6.7 The continuity equation is

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0.$$

The momentum equations take the form

$$\frac{\partial}{\partial x}\left(2\mu\frac{\partial v_x}{\partial x} - P\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] + \rho f_x$$
$$= \rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right),$$
$$\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left(2\mu\frac{\partial v_y}{\partial y} - P\right) + \rho f_y$$
$$= \rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y}\right).$$

6.8 The three equations reduce to

$$-\frac{\partial P}{\partial x} + \mu\frac{d^2 v_x}{dy^2} = 0,\quad -\frac{\partial P}{\partial y} = 0,\quad -\frac{\partial P}{\partial z} = 0.$$

The previous equations imply that $P = P(x)$ and

$$-\frac{dP}{dx} + \frac{2\mu c}{b^2} = 0 \;\rightarrow\; c = \frac{b^2}{2\mu}\frac{dP}{dx}.$$

6.10 The energy equation for heat transfer in a three-dimensional solid medium is

$$\rho c_v\frac{\partial T}{\partial t} - \frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right) - \frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right) - \frac{\partial}{\partial z}\left(k_z\frac{\partial T}{\partial z}\right) = \rho\mathcal{E}.$$

Chapter 7

7.1 $-\frac{1}{r}\frac{d}{dr}(rq_r) + \rho Q_e = 0.$

7.2 $T(r) = T_0 + \frac{\rho Q_e R^2}{4k}\left[1 - \left(\frac{r}{R}\right)^2\right].$

7.3 $\theta(x,t) = \sum_{n=1}^{\infty} B_n \sin\lambda_n x\, e^{-\alpha\lambda_n^2 t}$, $B_n = \frac{2}{L}\int_0^L f(x)\sin\lambda_n x\,dx$, where $\theta \equiv \frac{T-T_0}{T_L-T_0}$.

7.7 (**a**) The pressure at the top of the sea lab (i.e., at a depth $h = 120$ m) is $P = \gamma h = 1.2$ kN/m^2. (**b**) The pressure variation, measured from the top of the lab downward, is $P(4) = 1.24$ MPa.

7.8 $P = 82.314$ kPa abs and $\rho = \rho_0 = 1.02$ kg/m^3.

7.10 $U(y) = \frac{\rho g h^2 \sin\alpha}{2\mu}\left(2\frac{y}{h} - \frac{y^2}{h^2}\right)$.

7.11 Assume $v_x = U(x, y)$, $v_y = 0$, and $v_z = 0$. The velocity fields in the two portions of the channel are given by

$$U_1(y) = \frac{dP}{dx}\frac{y^2}{2\mu_1} + \frac{A_1}{\mu_1}y + B_1,$$

$$U_2(y) = \frac{dP}{dx}\frac{y^2}{2\mu_2} + \frac{A_2}{\mu_2}y + B_2,$$

where the constants A_1, A_2, B_1, and B_2 are determined using boundary conditions.

7.14 $\sigma_{11} = 96.88$ MPa, $\sigma_{22} = 64.597$ MPa, $\sigma_{33} = 48.443$ MPa, $\sigma_{12} = 4.02$ MPa, $\sigma_{13} = 0$ MPa, $\sigma_{23} = 0$.

When a displacement field is given, there is no question of compatibility.

7.16 The member forces are $F_2 = -\sqrt{2}\,P$ and $F_1 = P$. Then the stresses and strains are computed using $\sigma^{(i)} = \frac{F_i}{A_i}$ and $\varepsilon^{(i)} = \frac{\sigma^{(i)}}{E_i}$.

7.19 The deflections in the beam are

$$w_1(x) = \frac{FL^3}{12EI}\left(1 - \frac{x^2}{L^2}\right)\frac{x}{L}, \qquad 0 \le x \le L,$$

$$w_2(x) = -\frac{FL^3}{12EI}\left(3 - 10\frac{x}{L} + 9\frac{x^2}{L^2} - 2\frac{x^3}{L^3}\right), \quad L \le x \le 1.5L.$$

7.20 The deflection is

$$w(x) = \frac{q_0 L^4}{120EI}\frac{x^2}{L^2}\left(3 - 7\frac{x}{L} + 5\frac{x^2}{L^2} - \frac{x^3}{L^3}\right).$$

7.21 The maximum deflection is $w_{max} = 11F_0L^3/384EI$.

7.22 $N = -P\cos\theta$, $V = -P\sin\theta$, $M = -PR(1 - \cos\theta)$.

7.23 The governing equations are

$$-\frac{1}{r}\left[\frac{d}{dr}(rN_{rr}) - N_{\theta\theta}\right] - f = 0, \quad -\frac{1}{r}\left[\frac{d^2}{dr^2}(rM_{rr}) - \frac{dM_{\theta\theta}}{dr}\right] - q = 0.$$

7.24 The plate constitutive equations are

$$N_{rr} = A\left(\frac{du_0}{dr} + \nu\frac{u_0}{r}\right), \quad N_{\theta\theta} = A\left(\nu\frac{du_0}{dr} + \frac{u_0}{r}\right),$$

$$M_{rr} = -D\left(\frac{d^2w_0}{dr^2} + \nu\frac{1}{r}\frac{dw_0}{dr}\right), \quad M_{\theta\theta} = -D\left(\nu\frac{d^2w_0}{dr^2} + \frac{1}{r}\frac{dw_0}{dr}\right),$$

where $A = Eh/(1 - \nu^2)$ and $D = Eh^3/[12(1 - \nu^2)]$.

7.27 $u_z(r) = -\frac{\rho g a^2}{4\mu}\left(1 - \frac{r^2}{a^2}\right)$, $\sigma_{\theta z} = 0$, $\sigma_{zr} = \frac{\rho g}{2}r$.

7.29 $\sigma_{rr} = -\left(1 + \frac{4\mu}{3K}\right)p$, $\sigma_{\theta\theta} = \sigma_{\phi\phi} = -\left(1 - \frac{2\mu}{3K}\right)p$.

7.31 $u_\theta(r) = \frac{\tau_0 b^2}{2\mu a}\left(\frac{r}{a} - \frac{a}{r}\right)$, $\sigma_{r\theta} = \frac{b^2\tau_0}{r^2}$.

7.33 $\sigma_{xx} = 2D\left(3x^2y - 2y^3\right)$, $\sigma_{yy} = 2Dy^3$, $\sigma_{xy} = -6Dxy^2$.

7.35

$$\sigma_{xx} = \frac{3q_0}{10}\left(\frac{y}{b} + \frac{5a^2}{2b^2}\frac{x^2}{a^2}\frac{y}{b} - \frac{5}{3}\frac{y^3}{b^3}\right),$$

$$\sigma_{yy} = \frac{q_0}{4}\left(-2 - 3\frac{y}{b} + \frac{y^3}{b^3}\right),$$

$$\sigma_{xy} = \frac{3q_0 a}{4b}\frac{x}{a}\left(1 - \frac{y^2}{b^2}\right).$$

7.37

$$\sigma_{xx} = \frac{\partial^2\Phi}{\partial y^2} = \frac{\tau_0}{4}\left(-\frac{2x}{b} - \frac{6xy}{b^2} + \frac{2a}{b} + \frac{6ay}{b^2}\right), \quad \sigma_{yy} = \frac{\partial^2\Phi}{\partial x^2} = 0,$$

$$\sigma_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y} = -\frac{\tau_0}{4}\left(1 - \frac{2y}{b} - \frac{3y^2}{b^2}\right).$$

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