Sheri D. Sheppard • Thalia Anagnos • Sarah L. Billington

ENGINEERING MECHANICS STATICS

Modeling and Analyzing Systems in Equilibrium

WILEY

ENGINEERING MECHANICS: STATICS

MODELING AND ANALYZING SYSTEMS IN EQUILIBRIUM

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WILEY

From: Sarah, Sheri, and Thalia
This book is dedicated to all those who inspire us, including our partners (Ed, Jeff, and Peter), our children (Alexei, Anna, Bram, Chloe, and Portia), our teachers (past and present), and the many students we have had the privilege to teach.

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In *Engineering Mechanics: Statics*, our aim is to equip students with the knowledge, tools and good habits for solving mechanics problems in realistic contexts. Mechanics courses have historically presented engineering students with a precise, mathematical treatment of the material. This approach has appeal in that it presents mechanics as a relatively uncluttered "science." On the other hand, this material is generally of idealized cases, and students, when confronted with more realistic systems, are often at a loss as to how to proceed.

From the outset in Chapter 1 we focus on developing good problem solving habits that include being systematic about the analysis process, understanding the modeling assumptions, and developing intuition for how loads are transferred through structures and machines. This introduction of the material provides a motivational framework for the more mathematical presentation of statics found in Chapters 2–11.

Throughout this text, our emphasis is to present and illustrate:

a. The *physical principles* and concepts that describe non-accelerating objects. These principles and concepts are grounded in the reader's own experiences to motivate and provide a context for formal mathematical representations.

b. *An analytical problem-solving methodology* for describing and assessing physical systems, so that the reader is able to apply the principles in a systematic manner in evaluating engineered systems. Throughout the text, the methodology and its application are framed within the context of broader engineering practice.

c. A *wide variety of problems from daily life and engineering practice*. Through our "Watch-It" videos and multiple styles of artwork we demonstrate how messy-looking problems can be simplified for engineering analysis.

This online course has been written and developed explicitly with the students in mind—those in the class who are trying to get their minds around the material for the first time. Mechanics can sometimes be counterintuitive, and it can be a major frustration to those students who do not immediately relate to the logic behind the material (and this includes many of them!). Thus the presentation is a personalized one—one in which the students feel that they are having a one-on-one discussion with the authors. We do not skimp on rigor but do try to make the material accessible and, as far as we can, make it fun to learn.

Features

The goals outlined above are supported by a number of unique features in this online course:

Emphasis on sketching: We emphasize the importance of communicating solutions through graphics both to enhance learning and to prepare the reader for engineering practice. Most engineering students are visual learners.[1] In Chapter 1 we introduce the importance of visualizing and sketching skills for the successful implementation of structured analyses, and provide guidelines for sketching objects. We further reinforce the importance of drawing through:

a. A full chapter (Chapter 4) devoted to the skill of drawing free-body diagrams, including drawings on engineering graph paper background that have a hand-sketched look to provide examples to students of how to document solutions. An ideal response from a reader regarding a graphical element of the text would be, "The sketch in Figure 2.3.5 made the concept more understandable AND I can create a similar drawing to illustrate the concept to someone else."

b. A **Draw** step included in every worked example. To reinforce the drawing concept we use "hand-drawn" figures on graph paper.

Structured problem solving procedures: We introduce a structured analysis procedure early in the text and use it consistently in all worked examples. These steps include explicitly listing the **Assumptions** made and the importance of the **Draw** and **Check** steps as part of a complete solution.

[1] Felder, Richard, "Reaching the Second Tier: Learning and Teaching Styles in College Science Education." *J. College Science Teaching*, 23(5), 286–290 (1993).

Multiple paths for students to learn: Different students find they learn better in different ways and having variety is both motivating and helps deepen understanding of new concepts. We provide text to read, videos to watch, and many problems for students to tackle.

Feedback for students and faculty: Getting feedback is a key tool in effective learning for students and effective teaching for instructors. Online resources in *WileyPLUS* give students rapid feedback on their level of preparation, whether they understand a new concept, and on their ability to carry out more detailed calculations. At the same time, the instructor has a window into how her students are doing by getting individualized and class-average scores to these online problems.

Scaffolding in learning: Statics concepts often look easy, but they can be surprisingly subtle. A strong grasp of the fundamental concepts is needed to use statics successfully to analyze systems. To develop this grasp of concepts we break them up for students into individual pieces, providing multiple opportunities to explore and master new concepts before moving on. The "Are You Ready" problems at the beginning of each chapter let students assess if they have a good understanding of the math and previously covered mechanics topics they need in order to be ready to learn the next chapter material.

Multiple study tools: To facilitate speedy access to key content, we have included review and study tools, such as **Learning Objectives** at the start of each chapter, and a **Just the Facts** section at the end of each chapter giving an overview of terms, equations, and concepts from each chapter. To the greatest extent possible, all in-text figures include *descriptive figure captions* that show at a glance what is being illustrated. *Key equations* are highlighted in yellow, and *key terms* are in bold blue type when they first appear.

Instructor Resources

The following resources are available to faculty using this text in their courses:

WileyPLUS:
The *Engineering Mechanics: Statics WileyPLUS* course is a new-generation online learning system designed to address the key learning and teaching issues in today's engineering mechanics course. It includes powerful and customizable content, tools, and resources to facilitate mastery of introductory statics for students of a wide range of abilities and preparation. The system uses scaffolded practice and feedback as a means to build student competency, confidence, and commitment. The system also improves productivity and assessment of learning progress for any class size and across many sections so that instructors can focus on teaching.

Each individual element of the online experience has been crafted to become part of a larger, cohesive learning experience, one that leverages the unique capabilities available in a digital setting.

To deliver on student learning and mastery challenges, *Engineering Mechanics*: *Statics* implements:

- *Diagnostic assessment before each new chapter*—Students are able to gauge their readiness for each new chapter—and what they may need to review further—with a brief diagnostic quiz.
- *A consistent instructional cadence: tell, show, do*—For each new major concept within a chapter, students will read or watch a passage that develops it, then see solved examples that apply it, and finally have an opportunity to master it through progressive, interactive exercises.
- *Scaffolded learning*—Practice exercises and a selection of homework problems use techniques such as hints, partial solutions, feedback on common mistakes, and progressive complexity to build student confidence and reinforce skills.
- *Optional pathways and resources*—The system facilitates differences in students' ideal learning styles. For example, they are able to choose a preferred pathway through the conceptual and example content, leveraging both video and textual content to reinforce their understanding of the material presented. All practice exercises are available to students for self-study, even if they are not formally assigned by instructors for assessment.

Solutions Manual: Fully worked solutions to all exercises in the text, using the same solution procedure as the worked examples.

Electronic figures: All figures from the text are available electronically, for use in creating your own lectures.

Student Resources

The following resources are available to students:

Answers to selected exercises: The text companion site, www.wiley.com/college/sheppard, includes answers to selected exercises from the text, to help students check that they have solved the exercises correctly.

Commitment to Accuracy

From the beginning we have committed to providing accurate and error-free coverage of the material. In this mission we have benefited from the help of many, many people.

While writing solutions, each solution was solved and checked at least twice, by a combination of authors, accuracy checkers, and graduate students.

All text and art were reviewed line by line by a developmental editor. A proofreader compared all corrections to final pages to confirm that any and all corrections were made. Finally, and certainly not least, the authors themselves spent countless hours checking all elements of the project at every step of the way to guarantee accuracy.

Despite our best efforts, it is possible that some errors still remain. Should anyone find anything they question, please contact the authors and we will see that any necessary corrections are made.

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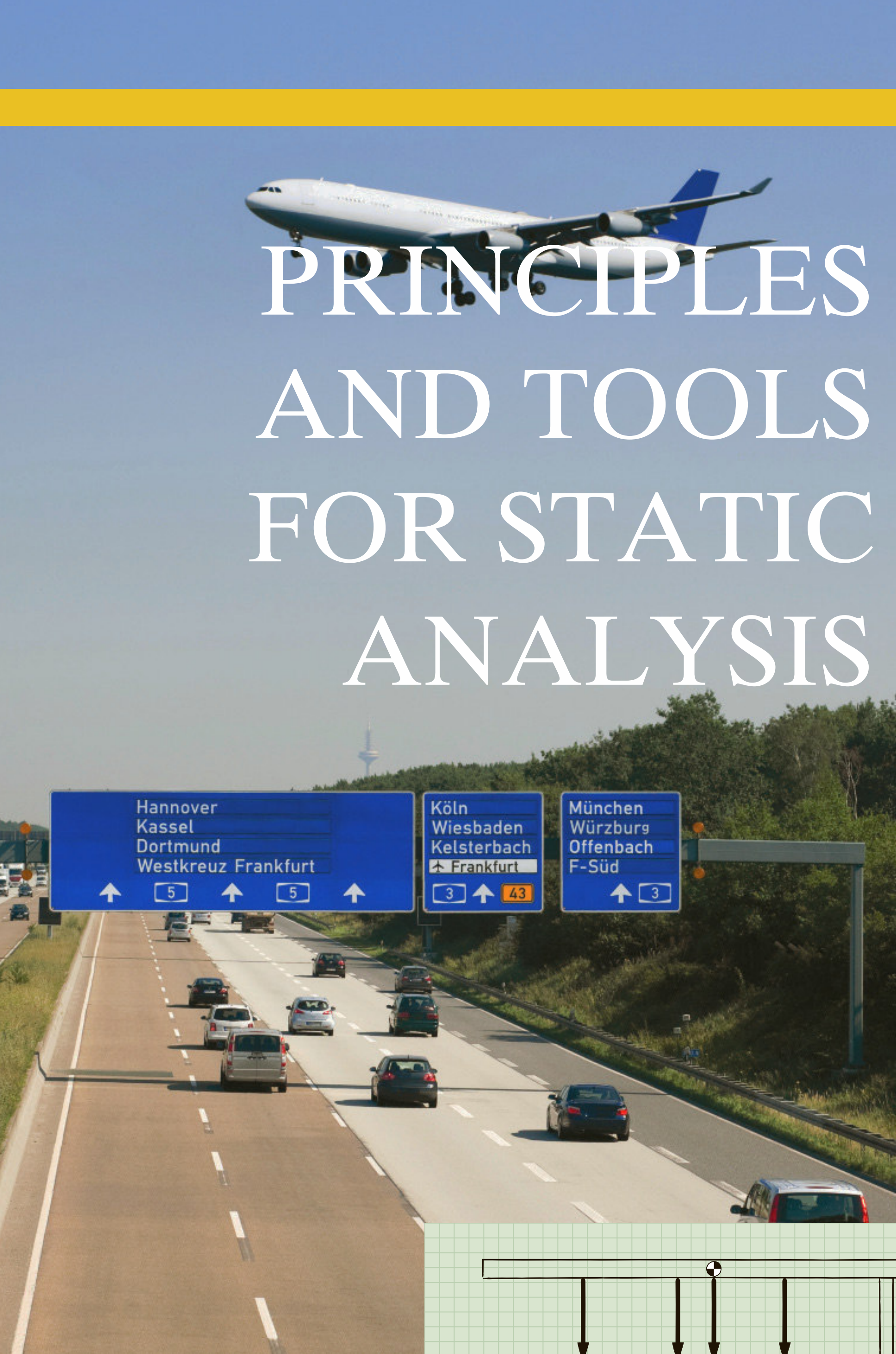
CHAPTER 1

PRINCIPLES AND TOOLS FOR STATIC ANALYSIS



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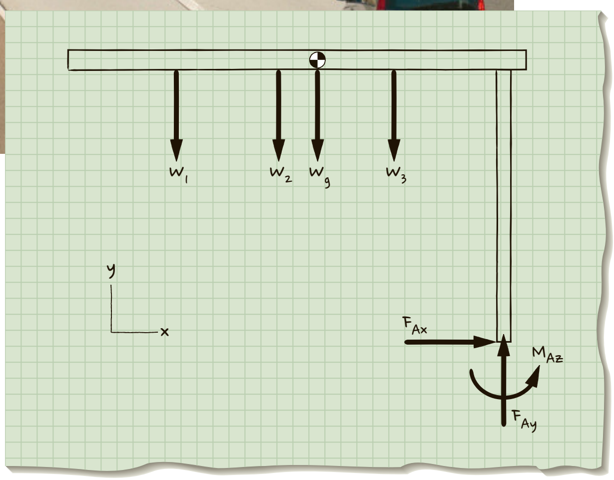
This text is about how to describe the forces that act on structures in equilibrium. Newton's laws of motion are used to establish mathematical relationships between the various quantities involved. These relationships enable us to predict how the quantities affect one another. After studying the material in this text, you should be able to use **static analysis**, which involves

1. looking at a structure and seeing how it resists loads,
2. creating a model of the structure,
3. evaluating the loads on the structure that keep it in equilibrium, and
4. postulating and answering "what if" questions about the structure.

This sequence of events is illustrated in **Figure 1.1.1**.

Static analysis is one example of **engineering analysis**. More generally, engineering analysis involves performing the calculations needed to assess the behavior of a system. The basis for these calculations is often physical principles from chemistry and physics. This chapter presents background material for static analysis.



LEARNING OBJECTIVES

On completion of this chapter, you will be able to:

- **Summarize the steps of the product realization process and an engineering analysis procedure.** (1.1)
- **State Newton's three laws of motion.** (1.2)
- **Convert between SI and USCS units.** (1.3)
- **Represent vectors.** (1.4)
- **Recognize the different types of drawings used in engineering analysis and basic guidelines for creating them.** (1.5)
- **Describe good problem-solving habits.** (1.6)
- **State the overall goal of this text.** (1.7)

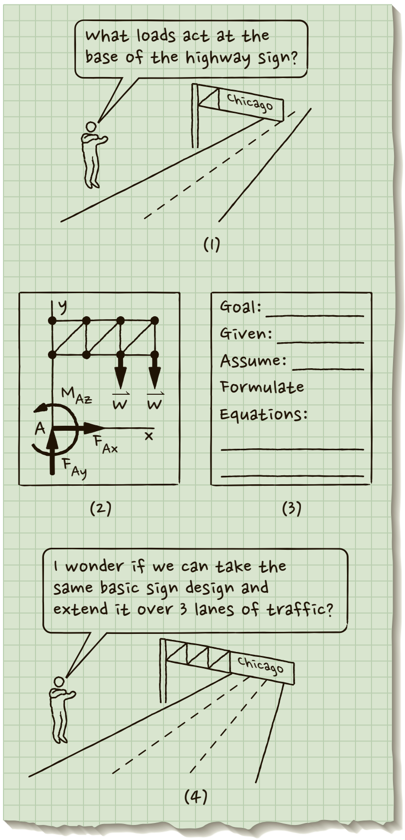


Figure 1.1.1 Engineer using analysis to answer a question.

1.1 HOW DOES ENGINEERING ANALYSIS FIT INTO ENGINEERING PRACTICE?

Learning Objective: Summarize the steps of the product realization process and an engineering analysis procedure.

There are some 1.5 million practicing engineers in the United States; this is less than 1% of the U.S. population. Engineers create the products and systems that we interact with daily. They create products that improve our quality of life (surgical devices, air-scrubbers in smoke stacks), entertain us (roller blades, roller coasters, electric trains, bikes), and educate us (LCD projection systems, computers). Engineers also create the systems that extend our reach from our planet's surface to the bottom of the ocean and to distant planets.

The process by which engineers design and manufacture these products and systems is referred to as the **product realization process** and may extend over months (less than six months for disk drives), years (for automobiles or bridges), or even decades (as in the case of the space station).

Any product or system begins with someone identifying an initial *client need* (the design problem). This need may arise from the market, the development of new technology, the demand for more sophisticated engineered systems or simply the President of the United States stating, "We will go to the moon before the end of the decade." Engineers design a product or system to solve some problem. Identification of a problem includes development of a list of design requirements. These design requirements are benchmarks used to evaluate progress toward a design solution, as well as the effectiveness of the final design solution. They may have to do with, for example, the final design's performance, appearance, time-to-market, cost, ease of manufacture, safety, impact on the environment, or ability to meet national or international standards.

Listing of design requirements is followed by generation of ideas on how to address the need or problem. These early ideas are referred to

as *design concepts*, and this phase of the product realization process is known as *conceptual design*.

Conceptual design is followed by *preliminary design*, where some of the concepts are developed further and some are discarded. Often the decision to continue with or discard a concept is based on an evaluation of how well the concept meets the design requirements. Evaluation may involve calculations and/or building prototypes (physical or virtual) of the concept. Typically, preliminary design ends with the selection of a single concept that will be detailed and refined in the next phase of design (called detail design).

Decisions made during *detail design* about specific configurations of components, types of materials, size of connections, methods of manufacturing, and so on, are often based on analysis to confirm that design decisions and choices continue to meet the design requirements. The analysis may involve numerical modeling and simulation. Building and testing of prototypes may also be involved.

Detail design results in a *comprehensive description* of the product or system. This description consists of drawings, complete fabrication specifications, and supporting documentation that describes the design decisions. It should also include analysis details and test results that support these decisions.

Detail design is followed by production, in which the product or system is constructed or manufactured. Here engineers oversee the process to verify that the final product meets the design requirements. Analysis may be used in this verification.

The product realization process that we have described may sound like a linear, sequential process, with one phase connecting to the start of another phase. In reality the process is a continuous loop, as suggested in **Figure 1.1.2**. For example, new design requirements may be generated later in the process as additional details of the design are being worked out. Also the real problem being solved may not be identified until well into the conceptual phase of design, or two competing concepts may be carried into detail design before a decision is made as to which one will be produced.

Regardless of where in the product realization process flowchart an engineer is working, he or she is likely to be involved in verifying and justifying decisions about the product. Engineering analysis is one of the main tools the engineer will use. The major steps in engineering analysis are summarized as an **engineering analysis procedure** (see **Box 1.1**).

In carrying out engineering analysis it is critical to simultaneously consider the physical situation and the mathematical model of the physical situation. The mathematical model allows us to understand and predict performance of the physical situation. At the same time, any model is only an approximation of the physical situation, and so is an estimate of real performance. One of the challenges in undertaking engineering analysis is learning to appropriately model a physical situation to obtain insights into its approximate performance.

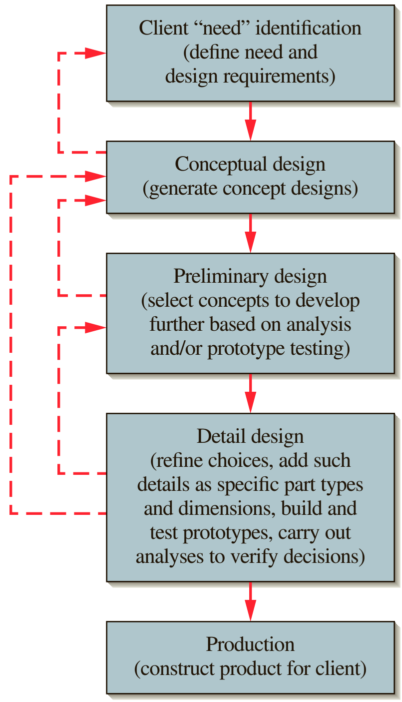
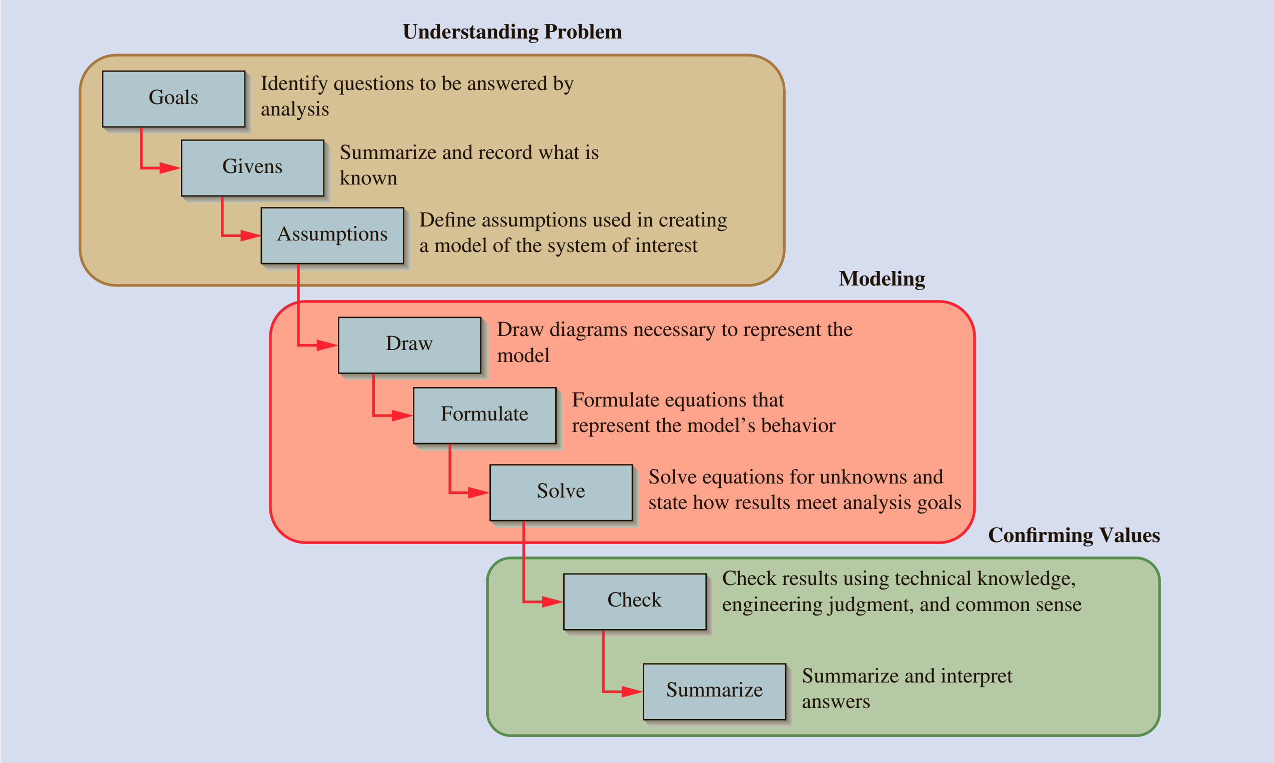


Figure 1.1.2 Product realization process flowchart.

Box 1.1: Overview of Engineering Analysis Procedure



1.2 PHYSICS PRINCIPLES: NEWTON'S LAWS REVIEWED

Learning Objective: State Newton's three laws of motion.

The physical principles that underlie engineering analysis in this text are Newton's three laws of motion:[1]

First Law: An object will remain at rest (if originally at rest) or will move with constant speed in a straight line (if originally in motion) if the resultant force acting on the object is zero. Another way of stating the same law is that an object originally at rest, or moving in a straight line with constant velocity, will remain in this state provided the object is acted on by balanced forces.

Second Law: If the resultant force acting on an object is not zero, the object will have an acceleration proportional to the magnitude of the resultant force and in the direction of this resultant force.

[1]The man most immediately responsible for what you'll be learning in this book is Sir Isaac Newton. Even among geniuses, Newton stands out. He needed a new mathematical approach to handle his investigations and so he invented calculus. That same year, he revolutionized optics by realizing that white light is made up of a spectrum of colors. And, to top it all off, he laid down his three laws of motion. Even more amazing, he did all of this when he was in his early twenties while taking a short break from London in order to avoid the plague.

Third Law: The forces exerted by two objects on each other are equal in magnitude and opposite in direction.

In this text we use the first and third laws extensively to describe situations where objects are at rest or are moving at constant velocity as a result of being acted on by balanced forces. We call these situations "static." This text is about static analysis, which is often referred to simply as **statics**. Statics can be used to design and describe a wide array of engineered systems, from the propulsion of bicycles (as described in Appendix D) to the tension in the cables in a suspension bridge (as described in Appendix E).

Closely related to statics is **dynamics**, the area of engineering that also embodies analysis based on Newton's laws except that the object is moving at a nonconstant velocity, an acceleration, as described by Newton's second law. In mathematical terms, the second law says that if an object is acted upon by an unbalanced force $\boldsymbol{F}$, the object experiences acceleration $\boldsymbol{a}$ in the same direction as the force. The acceleration is proportional to the force (and the proportionality factor is the mass m of the object):

$$\boldsymbol{F} = m\boldsymbol{a} \tag{1.1}$$

The bold italic notations $\boldsymbol{F}$ and $\boldsymbol{a}$ denote that these are vector quantities. Dynamics is usually covered in a separate course apart from statics.

Together statics and dynamics make up the study of "rigid body mechanics." A **rigid body** is a combination of a large number of particles in which all the particles remain at a fixed distance from one another before, during, and after a force is applied to the object. As a result, the material properties of any object that is assumed to be rigid will not be considered when analyzing the forces acting on the object. In most cases, the actual deformations occurring in structures, machines, mechanisms, and the like are relatively small, and the rigid-body assumption is suitable for analysis or preliminary design. Detail design requires full investigation of the deformations.

1.3 PROPERTIES AND UNITS IN ENGINEERING ANALYSIS

Learning Objective: Convert between SI and USCS units.

Static analysis involves quantifying, manipulating, and measuring properties of objects. The properties we are concerned with are length, time, mass, and force:

Length is a description of distance.

Time is conceived as a succession of events. Although the principles of statics are time-independent, this quantity does play an important role in the study of dynamics.

Mass is a property of matter by which the action of one object can be compared with the action of another. This property manifests itself as a gravitational attraction between two bodies and provides a quantitative measure of the resistance of matter to a change in velocity.

Force is considered as a push or pull exerted by one object on another.

Table 1.1 Standard Measures

Name	Standard Unit of Length	Standard Unit of Time	Standard Unit of Mass	Standard Unit of Force
International System of Units (SI)	meter (m)	second (s)	kilogram (kg)	newton (N)*
U.S. Customary System of Units (USCS)	foot (ft)	second (s)	slug**	pound (lb)

*derived quantity, based on meter, second, and kilogram, as discussed below $\left(\mathrm{N} = \frac{\mathrm{kg \cdot m}}{\mathrm{s^2}}\right)$

**derived quantity, based on foot, second, and pound, as discussed below $\left(\mathrm{slug} = \frac{\mathrm{lb \cdot s^2}}{\mathrm{ft}}\right)$

In working with these quantities we need consistent and standard measures—these are provided by the **International System of Units** (abbreviated SI after the French Le Système International d'Unités) and the **U.S. Customary System of Units** (USCS), as summarized in **Table 1.1**. The SI system is the accepted national standard of measurement in all countries except Myanmar, Liberia, and the United States.

SI Units

As shown in **Table 1.1**, the standard measure of length in the SI system is the **meter**, which is roughly the length from an adult's nose to his or her extended finger tips. Often engineers deal with lengths that are much larger (e.g., Earth's radius) or smaller (e.g., the thickness of a sheet of paper) than a meter; therefore, it may be more appropriate to deal with multiples or submultiples of the meter. We denote these multiples or submultiples with the prefixes listed in **Table 1.2**. For example, the

Table 1.2 SI Prefixes*

Factor	Prefix	Symbol
10^{18}	exa-	E
10^{15}	peta-	P
$1\ 000\ 000\ 000\ 000 = 10^{12}$	tera-	T
$1\ 000\ 000\ 000 = 10^{9}$	giga-	G
$1\ 000\ 000 = 10^{6}$	mega-	M
$1\ 000 = 10^{3}$	**kilo-**	**k**
$100 = 10^{2}$	hecto-	h
$10 = 10^{1}$	deka-	da
$0.1 = 10^{-1}$	deci-	d
$0.01 = 10^{-2}$	centi-	c
$0.001 = 10^{-3}$	**milli-**	**m**
$0.000\ 001 = 10^{-6}$	micro-	μ
$0.000\ 000\ 001 = 10^{-9}$	nano-	n
$0.000\ 000\ 000\ 001 = 10^{-12}$	pico-	p
$0.000\ 000\ 000\ 000\ 001 = 10^{-15}$	femto-	f
$0.000\ 000\ 000\ 000\ 000\ 001 = 10^{-18}$	atto-	a

*Prefixes commonly used in this text are shown in boldface type.

mean radius of Earth is 6.37×10^6 m or 6370 km, and a sheet of paper is 1×10^{-4} m or 0.1 mm thick.

The standard measure of mass in the SI system is the **kilogram** (kg), defined as the mass of a particular platinum-iridium cylinder kept at the International Bureau of Weights and Measures near Paris. From **Table 1.2**, we see that the prefix of "kilo" means that this standard has a mass of 1000 grams. Engineers work with a range of mass sizes, from the very large (mass of a Boeing 787) to the very small (mass of a white blood cell).

The standard measure of time is the **second** (s).

The standard unit of force in the SI system is the **newton** (N). One newton is equal to the force required to give 1 kilogram of mass an acceleration of 1 m/s^2. We will have a lot more to say about forces in Chapter 2.

In the SI system, length, mass, and time are the fundamental properties, and force is a derived quantity from Newton's second law. By Newton's second law (1.1), one newton (1 N) of force equals $[1\ \text{kg}][1\ \frac{\text{m}}{\text{s}^2}] = [\frac{\text{kg}\cdot\text{m}}{\text{s}^2}]$. Guidelines for working with SI prefixes and units are given in **Box 1.2**.

U.S. Customary Units

The standard measure of length in this system is the **foot**, as shown in **Table 1.1**. The standard measures for time and force are the **second** and **pound**, respectively.

In the U.S. Customary system, the fundamental properties are length, force, and time. The standard unit of mass in the U.S. Customary system is called the **slug** and is derived from the foot, second, and pound using Newton's second law. One slug is equal to the amount of matter that is accelerated at 1 ft/s^2 when acted upon by a force of 1 pound (1 slug = 1 lb $\cdot$ s^2/ft).

No matter which system of units you are working with, it is imperative that you *use consistent units*. For example, if you are using kilometers as the measure of length, make sure that you use kilometers consistently for all measures of length in the problem. Do not mix with feet or miles. Sometimes you may need to convert quantities from one measurement system to another; **Table 1.3** lists some conversion factors for going between U.S. Customary units and SI units.

Table 1.3 Conversion Factors

Converting from U.S. Customary to SI		
Quantity	**U.S. Customary**	**To SI multiply by**
Force	lb	4.4482 N/lb
Mass	slug	14.5938 kg/slug
Length	ft	0.3048 m/ft

Converting from SI to U.S. Customary		
Quantity	**SI**	**To U.S. Customary multiply by**
Force	N	0.2248 lb/N
Mass	kg	0.06852 slug/kg
Length	m	3.2808 ft/m

Box 1.2: Guidelines for Working with SI Prefixes and Units

1. Unit symbols are always written in lowercase letters, with the following exceptions: symbols for some prefixes and symbols named after an individual are capitalized (e.g., N for newton).
2. Unit symbols are never written with a plural "s" because this may be confused with the unit for second (s).
3. Compound prefixes should not be used. For example, k μm (kilo-micrometer) should be expressed as mm (millimeter) since $1(10^3)(10^{-6})$ m = $1(10^{-3})$ m = 1 mm.
4. The exponential power given for a unit having a prefix refers to both the unit and its prefix (e.g., $\text{mm}^2 = (\text{mm})^2 = \text{mm} \cdot \text{mm}$).
5. In engineering notation, exponents are generally displayed in multiples of three. This convention facilitates conversion to the appropriate prefix. For example, $4.0(10^3)$ N can be rewritten as 4.0 kN.
6. Quantities defined by several units that are multiples of one another are separated by a dot to avoid confusion with prefix notion (e.g., $\text{N} = \text{kg} \cdot \text{m/s}^2 = \text{kg} \cdot \text{m} \cdot \text{s}^{-2}$). The dot notation differentiates m·s (meter-second) from ms (millisecond).
7. Avoid prefixes in the denominator of composite units. For example, write kN/m rather than N/mm. The exception to this rule is the kilogram (kg); since it is the base unit of mass, it is fine to use it in the denominator (e.g., write Mm/kg rather than km/g).
8. When calculating, convert all prefixes to powers of 10. For example, (100 kN)(200 μm) = $[100(10^3)\,\text{N}][200(10^{-6})\text{m}] = 20{,}000(10^{-3})\,\text{N} \cdot \text{m}$. Then express the final result using a single prefix combined with a numerical value between 0.1 and 1000: $20{,}000(10^{-3})$ N·m becomes 20 N·m.
9. Minutes, hours, days, and so forth are used for multiples of the second. Plane angular measurement is made using radians (rad) or degrees (°).

EXERCISES 1.3

1.3.1. [*] Derive conversion factors for changing the following U.S. Customary units to their SI equivalents:

a. Pressure, lb/in.^{2}
b. Force, kip
c. Volume, ft^3
d. Area, in.^{2}

1.3.2. [*] Derive conversion factors for changing the following SI units to their U.S. Customary equivalents:

a. Pressure, N/m^2 (pascal)
b. Pressure, MPa (Megapascal)
c. Volume, m^3
d. Area, mm^2

1.3.3. [*] Jamaican sprinter Asafa Powell set the world record for the 100-meter dash on May 27, 2010. His time was 9.07 seconds. Calculate his average speed in m/s, ft/s, and mph.

1.3.4. [*] Calculate the percent difference between the mile and the metric mile (1500 meters).

1.3.5. [*] The world best performance in the women's marathon is 2:17:42, set by Paula Radcliffe of the United Kingdom on April 17, 2005 in the London Marathon. On average, how long did it take her to run each mile? What was her average speed in m/s? A previous best performance was 2:18:47, turned in by Catherine Ndereba from Kenya. (The race was run in Chicago on October 7, 2001.) How much faster did Paula Radcliffe run each mile of the race?

1.3.6. [*] In the heavyweight division, Russian Aleksey Lovchev holds the world record for the clean and jerk. He lifted a mass of 264 kg. Calculate the mass in slugs. What is the corresponding weight in newtons and pounds? How many people would it take to clean and jerk a Porsche 911 if they were all as strong as Aleksey Lovchev? (Make sure to document your source for weight data.)

1.3.7. [*] When a certain linear spring has a length of 180 mm, the tension in it is 170 N. For a length of 160 mm, the compressive force in the spring is 120 N.

a. What is the stiffness of the spring in SI units? In U.S. Customary units?

b. What is its unstretched length in SI units? In U.S. Customary units?

1.3.8. [*] Complete the following two tables:

MEN'S World Records for Selected Field Events

Event	Meters	Centimeters	Inches	Feet	Miles
High jump	2.45		96.46		
Pole vault	6.16			20.21	
Long jump			352.36		
Triple jump			720.08	60.01	1.14E-02
Shot put	23.12			75.85	
Discus throw		7408			
Hammer throw		8674	3414.96	284.58	
Javelin throw	98.48			323.10	

WOMEN'S World Records for Selected Field Events

Event	Meters	Centimeters	Inches	Feet	Miles
High jump	2.09			6.86	
Pole vault		506	199.2		
Long jump				24.67	4.67E-03
Triple jump	15.50		610.2		9.63E-03
Shot put		2263		74.25	
Discus throw			3023.6	251.97	
Hammer throw			3192.1	266.01	
Javelin throw				237.14	

1.4 COORDINATE SYSTEMS AND VECTORS

Learning Objective: Represent vectors.

Coordinate Systems

In working with physical objects it is useful to specify information about them relative to a **Cartesian coordinate system**, which uses three axes that are orthogonal to one another, as shown in **Figure 1.4.1*a***. In addition, the system is **right-handed**. In a right-handed system, if you point the fingers of your right hand in the direction of the positive x axis and bend them (as in preparing to make a fist) toward the positive y axis, your thumb will point in the direction of the positive z axis, as shown in **Figure 1.4.1*b***.

The assignment of coordinate axes is often a matter of convenience, and the choice is frequently up to the engineer. The logical choice is usually indicated by the geometry of the situation. For example, when the principal dimensions of a system or structure are given in the horizontal and vertical directions, the assignment of coordinate axes in these directions is generally convenient (**Figure 1.4.2*a***). If the structure and/or the forces are not aligned with the horizontal and vertical directions, alternative orientations of the coordinate axes may be appropriate, as shown in **Figure 1.4.2*b***.

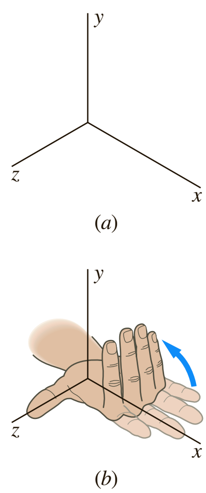


Figure 1.4.1 xyz coordinates arranged in right-handed manner.

Scalar and Vector Quantities

Static analysis deals with two kinds of quantities—scalars and vectors. **Scalar quantities** can be completely described with a magnitude (number

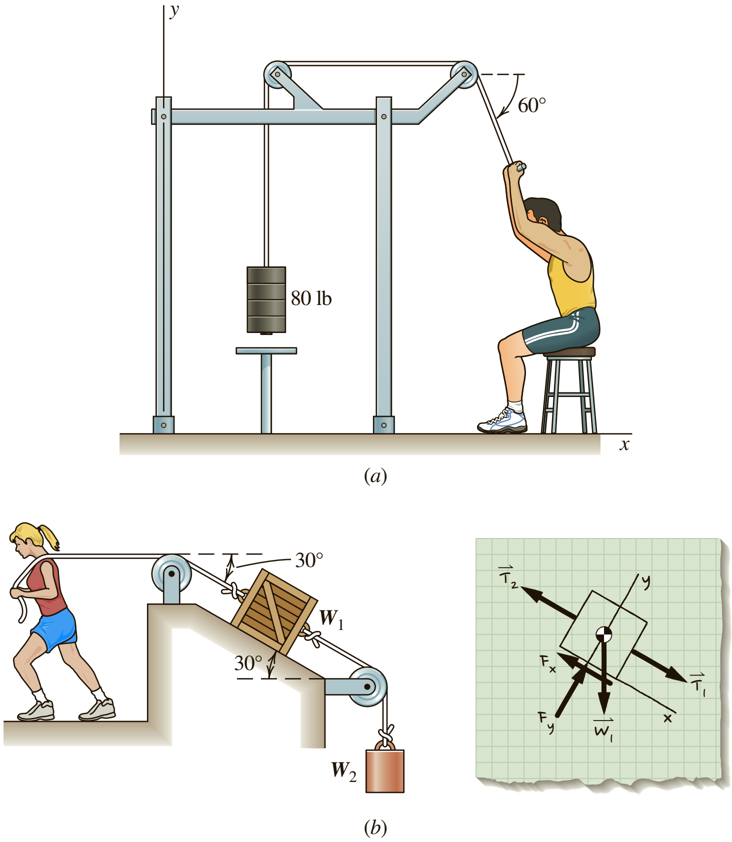


Figure 1.4.2 Various orientations of coordinate axes.

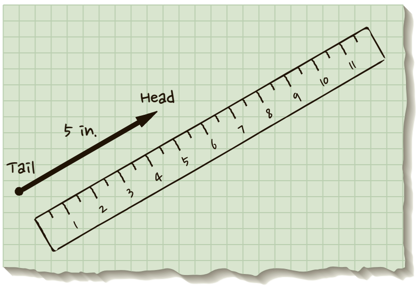


Figure 1.4.3 A position vector.

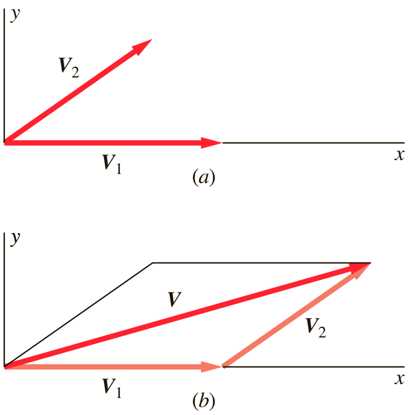


Figure 1.4.4 (*a*) Two vectors to be added; (*b*) vector addition using the parallelogram law.

only) and associated units. Examples of scalar quantities are mass, density, length, area, volume, speed, energy, time, and temperature. In mathematical operations, scalars follow the rules of elementary algebra. Scalars in this text are represented with italic type (V).

In contrast to scalars, **vector quantities** have both magnitude (with units) and direction, and obey the parallelogram law of addition, as described below. Examples of vector quantities are velocity, acceleration, momentum, force, moment, and position.

A vector is typically represented in drawings by an arrow with a head and a tail (Figure 1.4.3). The direction from the tail to the head of the arrow represents the direction of the vector, and the length of the arrow is often drawn proportional to the magnitude of the vector. The magnitude of the vector is generally written next to the arrow.

In this text, vector quantities are distinguished from scalar quantities through the use of boldface italic type ($\boldsymbol{V}$). In longhand writing, a vector may be denoted by drawing a "half arrow" above the letter, $\overline{V}$. Euclidean norm bars surrounding the vector symbol are used to denote the *magnitude* of a vector. Thus, the magnitude of the vector $\boldsymbol{V}$ is denoted by $\|\boldsymbol{V}\|$, or $\|\overline{V}\|$ (in longhand).

As mentioned above, vectors obey the **parallelogram law of addition**. This means that the two vectors $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ in Figure 1.4.4*a* can be replaced by an equivalent vector $\boldsymbol{V}$ that is the diagonal of a parallelogram

having $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ as two of its sides (**Figure 1.4.4*b***). This combination, or *vector sum*, is represented by the vector equation

$$\boldsymbol{V} = \boldsymbol{V}_1 + \boldsymbol{V}_2$$

where the plus sign used in conjunction with the vector quantities (boldface italic type) means vector and not scalar addition. It is important to note that the vector sum of $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ is not equal to the scalar sum (which is $V_1 + V_2$).

Vectors are generally specified relative to a Cartesian coordinate system. The specification can be in terms of a magnitude plus angle representation or in terms of the vector components. The **magnitude–angle representation** involves specifying the vector's magnitude accompanied by angles indicating the vector's direction relative to right-handed coordinate axes. For example, the **space angles** θ_x, θ_y, and θ_z, as depicted in **Figure 1.4.5**, can be used to specify the direction of $\boldsymbol{V}$. In Chapter 2, we discuss in detail how to use angles to specify the direction of a vector.

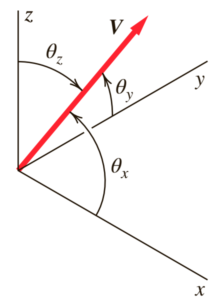


Figure 1.4.5 Space angles defined.

A **vector component representation** of a vector $\boldsymbol{V}$ involves specifying the "hike" you would take in the x direction, the y direction, and the z direction to get from the tail of $\boldsymbol{V}$ to its head (**Figure 1.4.6**). The "hike" in each direction is a **component vector**. In this text component vectors are not bold. We indicate that $\boldsymbol{V}$ is the sum of the three component vectors V_x, V_y, and V_z by the expression

$$\boldsymbol{V} = V_x + V_y + V_z \tag{1.2A}$$

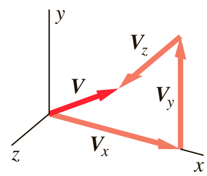


Figure 1.4.6 Vector represented in terms of its x, y, and z components.

To define the component vectors further, we introduce the concept of a **unit vector**. By definition, a unit vector has magnitude of 1. Align a unit vector with each of the axes, as shown in **Figure 1.4.7*a***; let $\boldsymbol{i}$ be a unit vector aligned with the x axis, $\boldsymbol{j}$ a unit vector aligned with the y axis, and $\boldsymbol{k}$ a unit vector aligned with the z axis.

The component vector V_x can then be written as $V_x\boldsymbol{i}$, where V_x is the distance we hike in the direction of $\boldsymbol{i}$. We will refer to V_x as the x component. If V_x is positive, we hike in the positive x direction; if V_x is negative, we hike in the negative x direction. Similarly, V_y can be written as $V_x\boldsymbol{j}$, and V_z as $V_z\boldsymbol{k}$. Therefore, $\boldsymbol{V}$ can be written in terms of unit vectors:

$$\boldsymbol{V} = V_x\boldsymbol{i} + V_y\boldsymbol{j} + V_z\boldsymbol{k} \tag{1.2B}$$

We can also interpret the vector components V_x, V_y, and V_z as scale factors of the unit vectors $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$.

Finally, if we align a unit vector with $\boldsymbol{V}$ (call this unit vector $\boldsymbol{u}$, where $\boldsymbol{u} = \cos\theta_x\boldsymbol{i} + \cos\theta_y\boldsymbol{j} + \cos\theta_z\boldsymbol{k}$; see **Figure 1.4.7*b***), we are able to rewrite $\boldsymbol{V}$ as $\|\boldsymbol{V}\|\ \boldsymbol{u}$, where $\|\boldsymbol{V}\|$ is the magnitude (size) of the vector. Therefore, (1.2B) becomes

$$\boldsymbol{V} = \|\boldsymbol{V}\|(\cos\theta_x\boldsymbol{i} + \cos\theta_y\boldsymbol{j} + \cos\theta_z\boldsymbol{k}) \tag{1.2C}$$

This expression is depicted graphically in **Figure 1.4.7*c***.

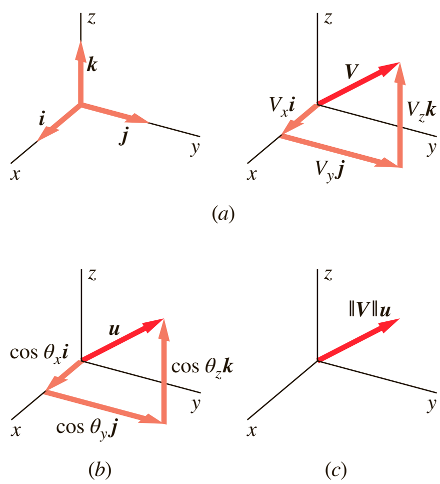
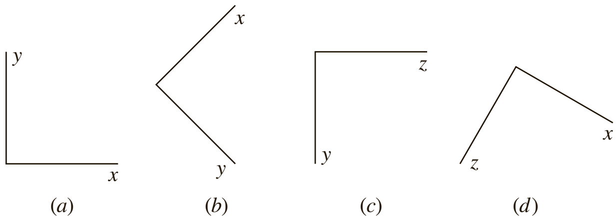


Figure 1.4.7 Unit vectors.

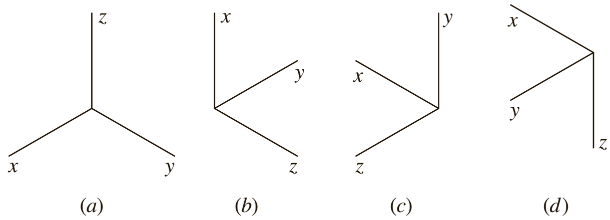
EXERCISES 1.4

1.4.1. **[*]** Determine whether the missing axis in each case is oriented into or out of the page for a right-handed coordinate system.



EX 1.4.1

1.4.2. **[*]** Which of the coordinate systems shown are right-handed?



EX 1.4.2

1.5 DRAWING

Learning Objective: Recognize the different types of drawings used in engineering analysis and basic guidelines for creating them.

Graphical representation is used to convey information that would be difficult or impossible to communicate about a product or system with words alone. For example, graphics in the form of drawings can be used to document the size and configuration of a product or how it is assembled. We refer to these types of drawings as **formal engineering drawings**. Graphics in the form of charts, schematics, and diagrams are used to convey information about the sequence of steps in a product's manufacture or how it was modeled for the purpose of analysis. We refer to these types of representations as **engineering diagrams**. Examples are shown in Figure 1.5.1.

The engineering diagram created in static analysis is the **free-body diagram** (Figure 1.5.2). This diagram describes a structure in terms of its size, the relationships between its important components, and how the rest of the world interacts with it. Many details of the real structure are simplified and represented in terms of their function. For example, a joint between two components that allows their relative

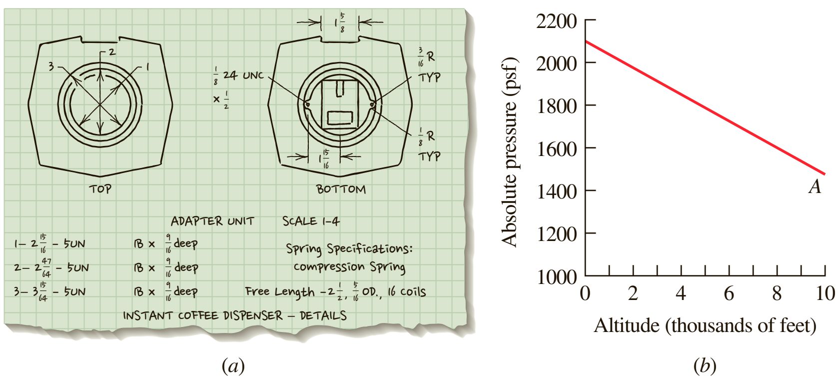


Figure 1.5.1 Examples of (*a*) formal engineering drawings and (*b*) engineering diagrams.

rotation may be represented by a circle, and a distributed load may be represented as a point force. For the purpose of static analysis, it is reasonable to assume that these free-body diagrams adequately model the real structure's behavior. Keep in mind, a free-body diagram is a model, or representation, of the real structure. More details on creating free-body diagrams are given in Chapter 4. It will be very important in our studies!

Beyond formal engineering drawings and engineering diagrams, drawings are also created by engineers as "think pieces." For example, in the conceptual stage of design, an engineer draws a thumbnail sketch of an idea as part of explaining the idea to a colleague, and this process spawns other ideas in both of their imaginations. Or the process of committing a design solution to paper may help an engineer identify and work out problems toward a solution. The value of these types of drawings is often in the act of their creation. We refer to this type as **informal engineering drawing** (Figure 1.5.3).

Informal drawing helps you to visualize, understand, and define the overall structure and relationships between components. It is drawing done as part of a visual study of the structure and in preparation for creating a free-body diagram. Informal drawing is typically self-intended or directed to a small group and is often concerned more with overall features than with details. It generally assumes that additional information will be added verbally.

The process of informal drawing starts with a "picture" of the structure that may be a photograph, the actual structure, another drawing, or even an image from your imagination. The process cycles through seeing–deciding–drawing, beginning at a large scale and then moving through progressively smaller scales:

Seeing means scanning the overall structure, noticing details, shapes and spaces, relationships, scales, sizes, patterns, proportions, and connections. It is a focused visual study. If you can actually touch, move, and/or manipulate the structure, you should do this too. Avoid applying labels as you go about a visual study. Contrary to common belief, seeing is an active art to be developed, not a passive experience to be taken for granted.

Deciding means making choices about which features are important, which can be simplified and which can be ignored, as well as what is inside the structure and what is outside. These choices should be based on your visual study. Deciding filters your seeing. It also means coming up with labels for features or components to make reasoning about them easier and planning the overall size of the drawing.

Drawing is putting down on paper what you decided was important. Continue to look at the structure as you draw but now you should be able to focus on those features that you have decided should be included. Some specific guidelines for putting line on paper are included in Box 1.3.

Many people think that the ability to create a drawing is a gift and that only talented individuals can draw. But like reading, writing, and riding a bicycle, with a few guidelines, some practice, and a little

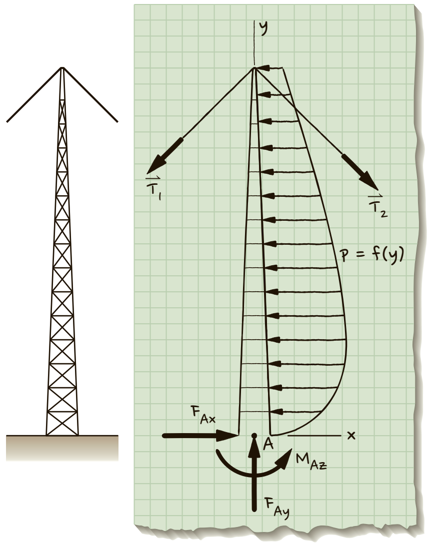


Figure 1.5.2 A system represented as a free-body diagram.

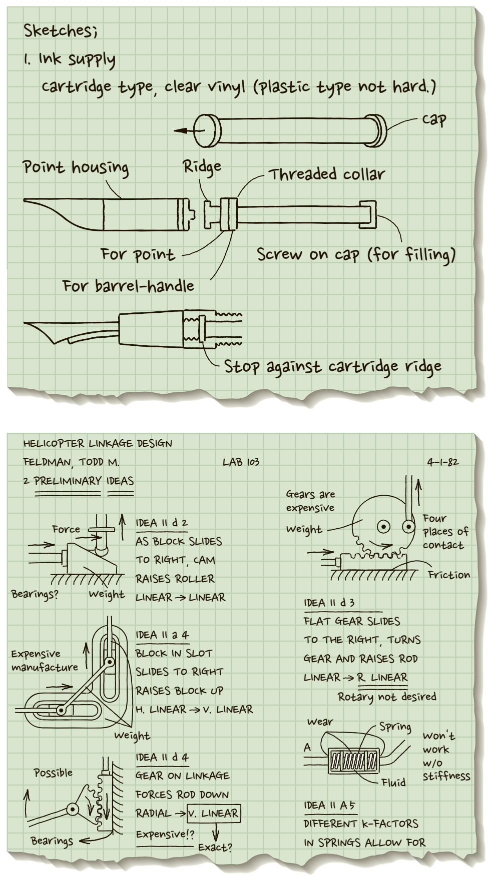
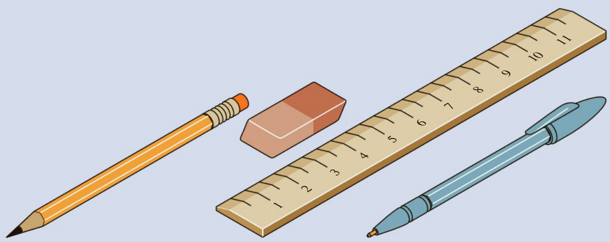


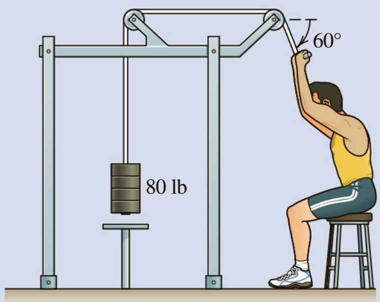
Figure 1.5.3 Examples of informal engineering drawings.

Box 1.3: Drawing Guidelines

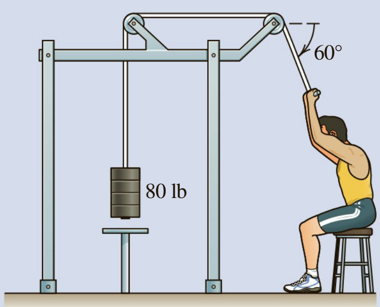
1. The **tools** needed for drawing are commonplace, so don't feel that you need to make a trip to an art supply store before you can begin drawing. The tools include pencils, pens, an eraser, a straightedge or ruler with a metal edge, unlined paper, and engineering grid paper.



2. Create drawings with a sense of **proportion**. This means depicting the right relative size of components and features. This can be accomplished by taking a few relative measurements of features of the device either with a ruler or other aid (pencil or fingers). These relative measurements then become guidemarks on the paper. Here, for instance, the person is drawn too large in (*a*) relative to the size of the equipment; the properly scaled drawing of (*b*) is much more useful.



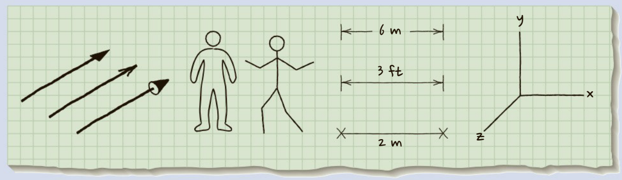
(*a*)



(*b*)

3. Create drawings with a sense of **scale**. This doesn't mean including all dimensions, but it does mean including some way of knowing how big the object is. This can be accomplished by including:
   - Something of known size in the drawing (e.g., a person)
   - A textual note that states the scale (e.g., the drawing is 3 times actual size)
   - A background grid with scale noted
   - A few dimensions

4. **Symbols** are useful for conveying standard information compactly. Along with words and numeric labels in drawings, you will probably use arrows, people, circles, ellipses, and boxes. It is worth practicing these symbols so that they become second nature when drawing.



5. **Plan** your drawing by considering overall size relative to your paper. If you like to create large drawings that do not fit well into your homework solution, consider drawing large, then using a scanner to **shrink** your final drawing for the purpose of your homework solution. Sometimes you may need **multiple views** of an object to convey it. As you go through the see–decide–draw cycle, use tracing paper over prior drawings or photographs to **trace** key lines. Outlining in pen will help you to see those lines through the tracing paper.

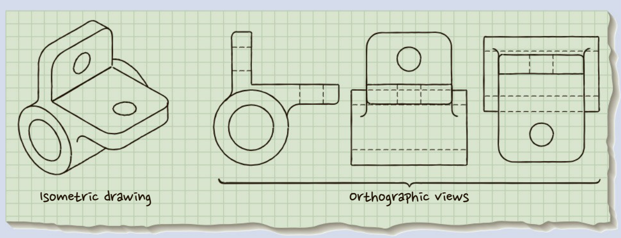


Figure 1.5.4 Examples of how a little practice (and instruction) in drawing can go a long way. (Left image is before practice and instruction, right image is after.)

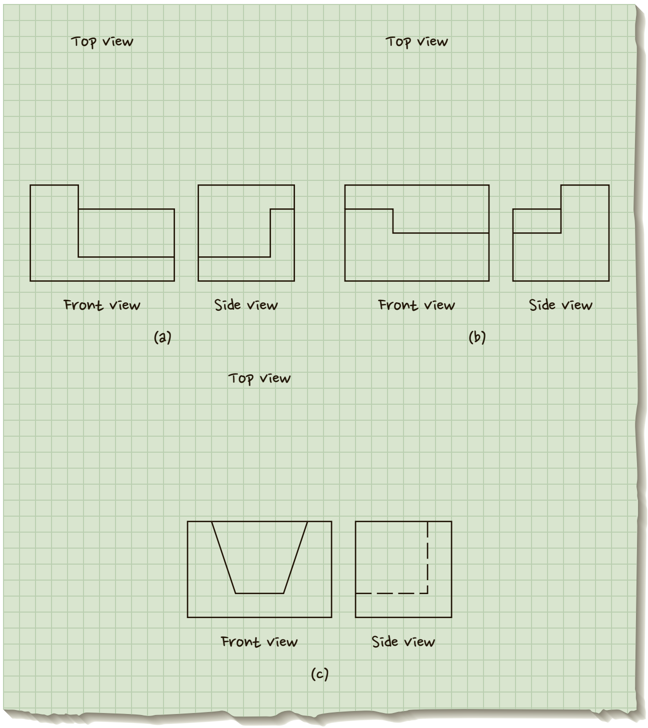
encouragement, everyone can create drawings that embody what things look like. As evidence, consider the examples in **Figure 1.5.4** from Betty Edwards' book *Drawing on the Right Side of the Brain*.

EXERCISES 1.5

1.5.1. [*] Identify three devices that involve a conversion of human input (i.e., forces and movements) into some other force or movement. Create sketches of each device that show how you think it works and what forces are involved. Examples to get your thinking jump-started are piano, typewriter, house window crank, foot-actuated garbage can, and bicycle pump. Do not draw those.

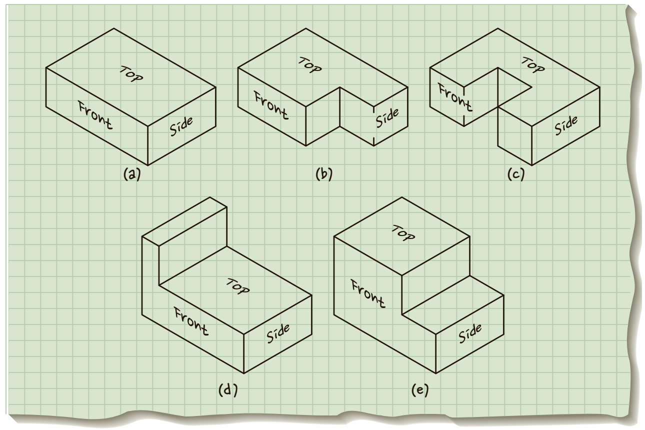
1.5.2. [*] Find an interesting artifact in your kitchen, garage, or dorm room. Create a storyboard (cartoon-like description) of how the artifact works or how it is operated. Examples of artifacts include hand mixer, hole punch, and nail clipper. Do not use these examples.

1.5.3. [*] Given the front and side views in the three multi-view drawings shown, sketch the missing top view in each case.



EX 1.5.3

1.5.4. [*] For each of the five objects shown, create a multi-view drawing showing separate front, side, and top views.



EX 1.5.4

1.5.5. To get more experience inspecting and drawing systems, complete exercise SA D.1 (1, 2, and 4) in Appendix D on how a bicycle works.

1.5.6. To get more experience inspecting and drawing systems, complete exercise SA E.2 in Appendix E on how a beam bridge works.

1.5.7. To get more experience inspecting and drawing systems, complete exercise SA E.3 in Appendix E on how an arch bridge works.

1.6 PROBLEM SOLVING

Learning Objective: Describe good problem-solving habits.

Good analysis habits do not guarantee that your solutions will be consistent and correct, but they do make it more likely! Neat and well-compiled analyses are also a means for engineers to communicate with one another about design decisions. It is not unusual for an analysis to be reexamined when there is a design change or a problem in the field.

Static analysis is generally carried out on paper using the steps outlined in **Box 1.1**. Make sure that your written work is neat and complete. A reader should be able to follow your work from the drawings, equations, and words that are on the paper and should not need to go to another reference. Lay out steps clearly, so another engineer can easily follow your thinking. This means that there should be a neatly drawn free-body diagram (with multiple views if needed to convey the structure and loads) and a list of assumptions. The algebra that is part of the *formulate equations and solve* steps should be included as part of the solution. The values of the unknowns that result from the algebra should be boxed so that they are easy to find. It is also useful to draw the correct direction of a load next to its boxed value. Finally, a check (qualitative and/or quantitative) is carried out to confirm the answers.

Good analysis habits also consist of ensuring that the analysis (1) maintains dimensional consistency, (2) solves any algebraic equation for the desired variable before plugging in known numeric values, (3) maintains consistent numerical accuracy and the correct number of significant figures, and (4) is based on reasonable assumptions. Each of these elements is discussed below.

Dimensional Consistency

In setting up and solving equations, make sure that all terms in an equation are in the same units. This is called **dimensional consistency**. For example, the value of the magnitude of the total force $\boldsymbol{F}_{\text{total}}$ acting vertically on the overhang in **Figure 1.6.1** is

$$\|\boldsymbol{F}_{\text{total}}\| = \underbrace{pA}_{\substack{\textit{force from} \\ \textit{snow pressure}}} + \underbrace{\frac{Gm_1m_2}{r^2}}_{\substack{\textit{gravitational} \\ \textit{force}}} \tag{1.3}$$

where
- p is the pressure of snow pressing down on the structure
- A is the area of the upper surface of the structure
- G is the universal gravitational constant
- m_1 is the mass of the structure
- m_2 is the mass of the earth
- r is the distance between the centers of mass of the structure and earth

The SI units in the various quantities are $\|\boldsymbol{F}_{\text{total}}\|$ in newtons, p in newtons per meter squared, A in meters squared, m_1 and m_2 in kilograms, r in meters, and G in $[\text{m}^3/(\text{kg}\cdot\text{s}^2)]$. Thus in (1.3), each term is expressed in newtons:

$$\|\boldsymbol{F}_{total}\| \quad : \quad \text{N}$$

$$\underbrace{pA}_{\substack{\textit{force from} \\ \textit{snow pressure}}} \quad : \quad \frac{\text{N}}{\text{m}^2}\,\text{m}^2 = \text{N}$$

$$\underbrace{\frac{Gm_1m_2}{r^2}}_{\substack{\textit{gravitional} \\ \textit{force}}} \quad : \quad \frac{\dfrac{\text{m}^3}{\text{kg}\cdot\text{s}^2}(\text{kg})(\text{kg})}{\text{m}^2} = \frac{\text{kg}\cdot\text{m}}{\text{s}^2} = \text{N}$$

Each term maintains consistency. *Because static analysis involves the solution of equations that must remain dimensionally consistent, the fact that all terms of an equation you might formulate are represented by a consistent set of units can be used as a partial check.*

Numerical versus Symbolic Solution

In setting up and solving equations, you can either insert known numerical values right from the beginning or solve the equation for the desired variable before inserting any numbers. For example, let's say you want to calculate the acceleration of an object at time $t = 20$ s that is currently at position s of 10 m. Furthermore, its current speed v is 15 m/s. You

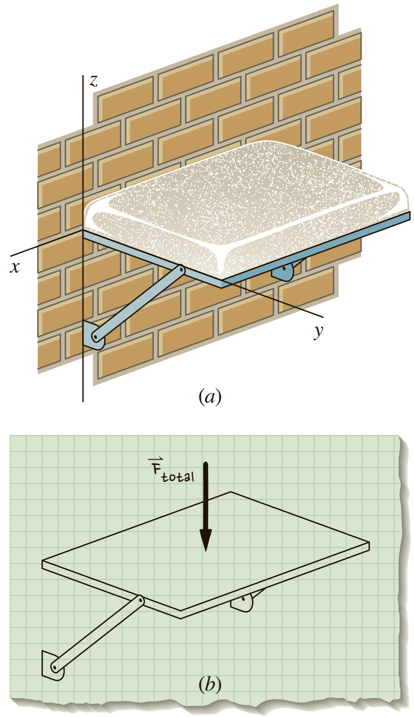


Figure 1.6.1 (*a*) Snow on top of an overhang; (*b*) forces acting on the overhang.

could substitute these numerical values directly into the expression that relates distance traveled to velocity, time, and acceleration—namely,

$$s = vt + \frac{1}{2}at^2$$
$$10 \text{ m} = (15 \text{ m/s})20 \text{ s} + \frac{1}{2}a(20 \text{ s})^2$$

which you can solve for a,

$$a = \frac{2(10 \text{ m} - (15 \text{ m/s})20 \text{ s})}{(20 \text{ s})^2} = -1.45 \text{ m/s}^2$$

With this approach, the magnitude of each quantity expressed in its particular units is evident at each stage of the calculation. The main advantage of the approach is that the practical significance of the magnitude of each term can be assessed.

Alternatively, you could first solve the equation for the acceleration to obtain a **symbolic solution** and then make numerical substitutions:

$$s = vt + \frac{1}{2}at^2$$
$$a = 2\frac{s - vt}{t^2}$$
$$a = \frac{2(10 \text{ m} - (15 \text{ m/s})20 \text{ s})}{(20 \text{ s})^2} = -1.45 \text{ m/s}^2$$

The symbolic solution allows you to consider the relationships between physical variables and performance. For example, in the second expression above, we see that an increase in distance will increase acceleration, whereas an increase in time decreases acceleration. The symbolic solution also allows you to make dimensional checks as expressions are manipulated. And finally, the symbolic solution is amenable to being manipulated in a spreadsheet so that a range of values of variables can be considered.

Numerical Accuracy and Significant Figures

In calculations, you should consider the *accuracy* of the numbers that you are working with. The accuracy of the solution depends on the accuracy of the given data and of the computations performed. For example, if the load on a bridge is known to be 120,000 N with a possible error of 240 N higher or lower, the **relative error** that measures the degree of accuracy of the data is

$$\frac{240 \text{ N}}{120{,}000 \text{ N}} = 0.002 = 0.2\%$$

In engineering problems, the data are seldom known with an accuracy greater than 0.2%, so writing the answers with any greater accuracy

generally is not justified. For the bridge example, if we compute the force acting on the bridge at one of its supports to be 43,625 N, the force is actually somewhere between 43,537 N ($= 0.998 \times 43{,}625$ N) and 43,712 N ($= 1.002 \times 43{,}625$ N). What answer do we give if we want it to reflect an accuracy of 0.2%?

To answer this question consider that the accuracy of a number is reflected in its number of **significant figures**. A significant figure is any digit in a number that we know with certainty (including zero, provided it is not used to specify the location of the decimal point for the number) plus one uncertain digit. The rule for counting the number of significant figures is to count digits from the left and ignore leading zeros, and keep all digits up to and including the first doubtful one. For example, $x = 3$ m has only one significant figure, and expressing this value as $x = 0.003$ km does not change the number of significant figures. If we instead wrote $x = 3.0$ m (or, equivalently, $x = 0.0030$ km), we would imply that we know the value of x to two significant figures.

Be careful of ambiguity; $x = 300$ m does not indicate whether there are one, two, or three significant figures; we don't know whether the zeros are carrying information or merely serving as place holders. Instead, we should write $x = 3(10^2)$, $x = 3.0(10^2)$, or $x = 3.00(10^2)$, to indicate one, two, or three significant figures, respectively.

Let's return to the question of recording the bridge reaction force of 43,625 N. How many significant figures should be retained to reflect the 0.2% accuracy? Following the rule above, the first doubtful digit from the left is the 6 (in the hundreds slot), since it might be a 5 or a 7. Therefore, recording any digits to the right of the 6 is meaningless. We should record the answer as $43.6(10^3)$ N, which is three significant figures. Even though your calculator display may show 9 or 10 digits, you are not justified by the accuracy of the input data to record more than three significant figures.

A practical rule in engineering calculations is to use four figures to record numbers with a leading "1" and three figures in all other cases in presenting your final answer. Intermediate calculation steps should retain more significant figures. With this rule, a force of 40 is 40.0 N, and a force of 15 is 15.00 N. Numbers are generally **rounded** in reporting values to the correct number of significant figures. For example, 29.694 N would be written with three significant figures as 29.7 N with rounding (and not as 29.6 N, which is what we would get if we truncated the answer).

Making Approximations

As we develop a mathematical model we need to make approximations. It is not reasonable to include all of the physical detail in the mathematical model. What matters is knowing which details are important for the analysis at hand. Some of these approximations may be mathematical. For instance, we may ignore one term in an expression if that term is small relative to others in the expression, or we may approximate a trigonometric function for small angles, such as $\tan\theta = \theta$, for very small θ. Some approximations may be physical; for instance, it is often necessary to neglect distances, angles, weights, or forces that are much

smaller than other distances, angles, weights, or forces. Being able to make justifiable and reasonable assumptions is one of the hallmarks of a competent engineer. A major goal of this text is to provide you with many chances to exercise this ability.

EXERCISES 1.6

1.6.1. [*] Round off the numbers listed below to three significant figures.

a. 0.015362
b. 0.837482
c. 1.839462
d. 26.39473
e. 374.9371
f. 6471.907

1.6.2. [*] When an object moves through a fluid, the magnitude of the drag force F_{drag} acting on the object is given by $\frac{1}{2}C_D \rho V^2 A$, where ρ is the density of the fluid, V is the velocity of the object relative to the fluid, and A is the cross-sectional area of the object. What are the dimensions of the drag coefficient C_D?

1.6.3. [*] The pressure within objects subjected to forces is called stress and is given the symbol σ. The equation for stress in an eccentrically loaded short column is

$$\sigma = -\frac{P}{A} - \frac{Pey}{I}$$

where P is force, A is area, and e and y are lengths. What are the dimensions of the stress σ and the second moment of area I?

1.6.4. [*] In the expressions that follow, c_1 and c_2 are constants, θ is an angle, x is distance, v is velocity, and a is acceleration. Determine the dimensions of these constants if the formula is to be dimensionally correct.

a. $a = c_1 \dfrac{v^2}{x}$
b. $\frac{1}{2}mv^2 = c_1 x^2$
c. $x = c_1 v + c_2 a^2$
d. θ (degrees) $= c_1\theta$ (radians)

1.6.5. [*] The ability to make good educated guesses (often called engineering estimation or intuition) is an important engineering skill that can be practiced. In this problem you'll practice your skill in estimating how far an average individual would have to run or jog in order to burn off the calories found in a typical candy bar. Table 1.6.5 contains information about some of the more popular brands.

The "Nutritional Facts" that the FDA requires on food packages always provide the number of Calories in each serving. It is worth noting that Calories with a capital C is an abbreviation for kilocalories (kcal), where the calorie is

Table 1.6.5 Nutritional Content of Assorted Candy Bars

Maker	Candy Bar	Calories	Fat Grams	Protein (g)
Nestle	Crunch	230	12	2
Nestle	100 Grand	190	8	1
Nestle	Butterfinger	270	11	3
Nestle	Kit-Kat	220	11	3
M&M/Mars	3 Musketeers	260	8	2
M&M/Mars	Twix	280	14	3
M&M/Mars	Snickers	280	14	4
M&M/Mars	Milky Way	270	10	2
M&M/Mars	Milky Way—Lite	170	5	2
M&M/Mars	Milky Way—Midnight	220	8	1
	Average	**239**	**10.1**	**2.3**
	Standard Deviation	**39**	**2.9**	**0.9**

the standard unit of heat that you study in chemistry and biology. Typically we do not make value judgments on the unit that is used to measure a given quantity, but it seems inherently better to think of a Snickers bar as having 280 Calories rather than 280,000 calories.

Choose one of the candy bars in the table and estimate how far an average individual would have to run in order to burn off its calories. Next, solve the problem assuming that a typical runner or jogger burns 100 kcal for every mile he or she travels. In fact, this is a pretty good approximation and is not strongly affected by how fast the person moves. Calculate the distance in meters or miles and compare with your initial estimate. What if the person decides not to run that day and all of the calories in the candy bar are converted to fat (9.4 kcal yields 1 g fat)? Calculate the weight gain in newtons or pounds. Is it significant? Did this analysis affect your appetite in any way?

1.6.6. Read Sections D.1 and D.2 in Appendix D on how a bicycle works. Complete the following tasks:

- Draw a free-body diagram of the bicycle and cyclist, traveling at constant speed. Make sure to state any assumptions.
- Present your answer to the question "What is your estimate of maximum velocity on a bicycle if the coefficient of drag (C_d) is reduced by 15%?" Include your supporting analysis, per the steps outlined in **Box 1.1**.
- List at least two suggestions for how the aerodynamic drag might be reduced.

1.6.7. Read Sections E.1 and E.2 in Appendix E on how the Golden Gate Bridge works. Complete the following tasks:

- Draw a free-body diagram of the Golden Gate Bridge. Make sure to state any assumptions.
- Present your answer to the question "What is your estimate of minimum required anchorage weight if the coefficient of friction is increased by 15%?" Include your supporting analysis, per the steps outlined in **Box 1.1**.
- List at least two suggestions for how an increase in the coefficient of friction could be achieved.

1.7 A MAP OF THIS TEXT

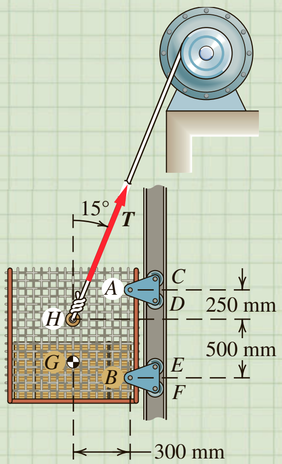
Learning Objective: State the overall goal of this text.

Figure 1.7.1 shows an example of a well-laid-out static analysis. The solution follows the analysis steps outlined in **Box 1.1**. This example also illustrates the application of the physical principles that form the basis for all static analysis—namely, Newton's laws of motion. This is the sort of problem that you probably handled in physics class.

At this point you may be wondering why this text has ten more chapters on performing static analysis if we have been able to review the physical principles and key skills in the first chapter. The answer is that you probably need more experience in applying Newton's laws to real engineering situations such as those described in Appendix D (with the bicycle) and Appendix E (with the bridge). With experience you will be able to look at a structure and identify its significant features and forces, study the loads "within" a structure, be comfortable with the vocabulary engineers use to describe engineering structures, and be able to reason how changes to a structure will affect its performance. That's what the remaining chapters are about.

The key concepts underlying static analysis are presented in this text through the following:

1. *Concrete experience.* This may entail recalling some prior experience (e.g., see-sawing as a child, removing a lug nut from a wheel assembly), and/or actually carrying out a simple physical or thought experiment (e.g., building a simple truss structure out of straws and paperclips). This concrete experience helps give a context for the

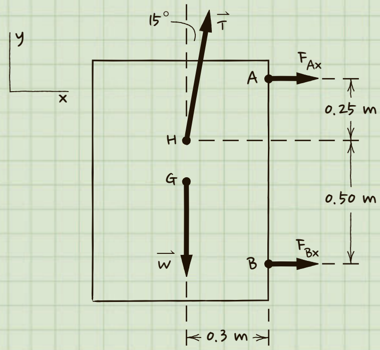


Goal Find the forces at A and B that act on the hopper shown on the left.

Given Combined mass of hopper and its contents is 4000 kg and its center of mass, location where cable is applied and its orientation angle relative to the vertical (15°).

Assume This is a planar system. A and B are rollers in a vertical track. Hopper is in equilibrium.

Draw Free body diagram of hopper.



Formulate Equations Based on equilibrium conditions (Newton's first law) and the free-body diagram, we write:

$$\sum F_x = T \sin 15° + F_{Ax} + F_{Bx} = 0 \qquad \text{(A)}$$

$$\sum F_y = T \cos 15° - W = 0, \qquad \text{(B)}$$

where $W = (4000 \text{ kg})(9.8 \text{ m/s}^2) = 39.2 \text{ kN}$

With moment center at H, we write:

$$\sum M_{z@H} = (-F_{Ax})(0.25 \text{ m}) + (F_{Bx})(0.5 \text{ m}) = 0 \qquad \text{(C)}$$

Solve

From (B), $T \cos 15° = 39.2 \text{ kN} \Rightarrow T = 40.58 \text{ kN}$ (D)

From (C), $F_{Ax} = 2F_{Bx}$ (E)

Substitute (D) and (E) into (A):

$$40.58 \text{ kN}(\sin 15°) + 2F_{Bx} + F_{Bx} = 0 \Rightarrow F_{Bx} = -3.50 \text{ kN}$$

Substitute this into (E) to find:

$$F_{Ax} = -7.00 \text{ kN}$$

or in vector notation,

$$\vec{F}_{Bx} = -3.50 \text{ kN}\,\vec{i}\,;\, F_{Ax} = -7.00 \text{ kN}\,\vec{i}$$

Answers

$$\vec{F}_B = -3.50 \text{ kN}\,\vec{i}$$

$$\vec{F}_A = -7.00 \text{ kN}\,\vec{i}$$

Check It makes sense that A bears more of the load than B since it is closer to the point of application of the force.

Figure 1.7.1 A well-laid-out analysis follows a systematic procedure and includes appropriate drawings and assumptions.

concept and motivates you to read a more formal representation of the concept.

2. *Mathematics and physics.* The experience exercise is immediately reinforced by application to solving straightforward examples. Structured problem-solving approaches are introduced at this point.
3. *The application* to a real engineered component and/or system. This includes showing you the assumptions and idealizations that engineers must make in applying a structured problem-solving approach.

1.8 JUST THE FACTS

How Does Engineering Analysis Fit into Engineering Practice?

Engineering analysis is a critical tool in an engineer's process of designing and producing any product. **Box 1.1** summarizes the major steps in an engineering analysis, and we will follow these steps throughout this text as we solve statics problems.

Physics Principles: Newton's Laws Reviewed

Newton's three laws of motion are the foundation for the engineering analysis we will undertake in this text. In particular, we will apply the first and third laws in our study of statics.

Properties and Units in Engineering Analysis

Length, mass, time, and force are the properties we will be most concerned with in analyzing statics problems. **Table 1.1** summarizes these standard measures, for both SI and USCS units. Statics problems can be solved with either SI or USCS units and we will use both throughout this text. The important thing is to be *consistent* with which system of units we use throughout an analysis.

Coordinate Systems and Vectors

We will use the Cartesian coordinate system to specify the position of physical objects for static analysis. This system is a right-handed set of three axes (x, y, z), as shown in **Figure 1.4.1**. We will also use both scalar and vector quantities. **Scalar quantities** can be completely described with a magnitude (number only) and associated units. **Vector quantities** have both magnitude (with units) and direction, and obey the parallelogram law of addition. **Figure 1.4.4** summarizes the parallelogram law of addition visually.

Drawing

Drawing plays a key role in engineering analysis and design. For statics, the key type of drawing we will use in solving virtually every problem is a free-body diagram. A **free-body diagram** describes a structure in terms

of its size, the relationships between its important components, and how the rest of the world interacts with it. **Figure 1.5.2** shows an example of a free-body diagram, and we consider them in much more detail in Chapter 4.

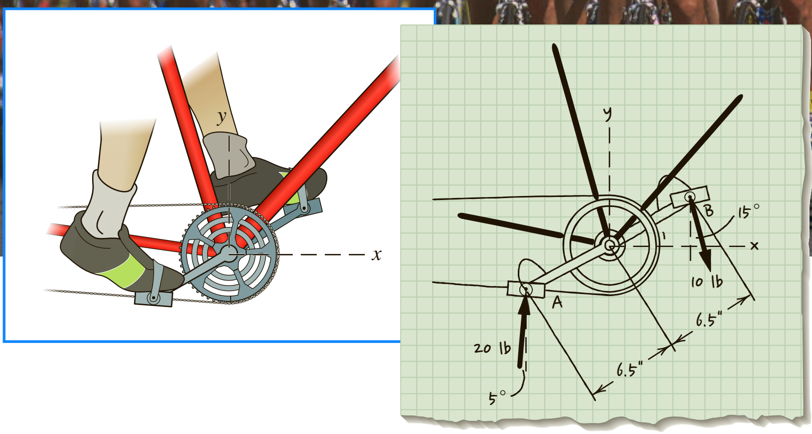
Problem Solving

Developing good and consistent problem solving habits increases the likelihood that you will obtain correct solutions—or be able to figure out where you went wrong when an answer is incorrect. Following the engineering analysis steps outlined in **Box 1.1** is a good approach to honing your problem-solving skills. Good analysis habits also include:

- Maintaining dimensional consistency throughout the solution
- Solving any algebraic equation for the desired variable before plugging in known numeric values
- Maintaining consistent numerical accuracy and the correct number of significant digits
- Making reasonable assumptions

CHAPTER 2

FORCES



Doug Pensinger/Allsport/Getty Images News and Sport Services

This chapter looks at forces—what they are, how we categorize them, and how we represent them. Learning how to categorize and represent forces is the first step in developing the analytic skills you must have in order to ask the right questions about any structure or machine you are working on and come up with the right answers to those questions.

Analyzing the forces that structures and systems apply and resist is central to the responsibilities of many engineers. For example, the gravitational force exerted by Earth must be considered every time a civil engineer[1] designs a building or a crane hoists a 2-ton beam high over a city street. And of course it's a matter of life and death that the mechanical engineer[2] have an intimate knowledge of the forces keeping the crane's cable attached both to the crane and to the beam it is hoisting into place.

To learn more about how engineers consider forces in design, review the case studies of the bicycle and the Golden Gate Bridge, which are just two examples of systems that withstand or apply forces. In the case of the bicycle an engineer might consider how the pedal force you apply while cycling relates to maximum velocity. In the case of the Golden Gate Bridge engineers might consider how the weight of a vehicle or the weight of the deck gets transferred to the ground.

[1]Civil engineers are responsible for planning, designing, constructing, and maintaining the infrastructure of our civilization—buildings, bridges, power plants, transportation systems, water systems, and much more. The civil engineer is called on to apply physical (and, in some cases, chemical and biological) principles, assess social and environmental impact, and evaluate the costs and benefits of infrastructure projects.

[2]Mechanical engineers work in a variety of industries, including transportation, product manufacturing, energy generation, consumer products, and applied research. Their work involves the design, manufacture, and maintenance of products or systems to meet human needs. The mechanical engineer is called on to use knowledge of physical principles, understanding of existing products, and imagination of what products might be.

LEARNING OBJECTIVES

On completion of this chapter, you will be able to:

- **Define a force and recognize the proper notation for force quantities. (2.1)**
- **Calculate gravitational forces. (2.2)**
- **Identify the different types of contact forces. (2.3)**
- **Isolate a system from the rest of the world and identify the types of external forces acting on it. (2.4)**
- **Represent a force mathematically and be able to convert between different representations. (2.5)**
- **Determine the resultant of forces using vector addition. (2.6)**
- **Use the dot product to find the components of a force vector or the angle between two forces. (2.7)**

2.1 WHAT ARE FORCES? AN OVERVIEW

Learning Objective: Define a force and recognize the proper notation for force quantities.

Engineers must consider how forces affect the structures and machines they design. For example, the engineer designing a dam would think about the water pushing against the dam and ask, "Will the steel anchors connected to the bedrock be strong enough?" An engineer analyzing the landing gear for an airplane would think about the forces applied during landing and ask, "Will the size of the gear forging be sufficient to prevent failure after repeated landings?"

In everyday life, you must consider forces whenever you prop a ladder against a house to wash a window or against a tree to rescue a kitten. If the angle the ladder makes relative to the ground is too large, the top of the ladder will tend to tip away from the house or tree; if the angle is not large enough, the ladder's feet will tend to slide away from the house or tree. The question that you might ask is, "Will I be able to get just the right position for the ladder so that I can safely accomplish my task?" In asking this question, you are implicitly considering the forces acting on the ladder and their relationship to one another.

A **force** is any interaction between an object and the rest of the world that tends to affect the state of motion of the object. The strength of a force is related to the extent of its effect on the object. You cannot see forces, but if you've ever seen a car crash or felt a rush of air escaping from a balloon, you have experienced the effects of forces.

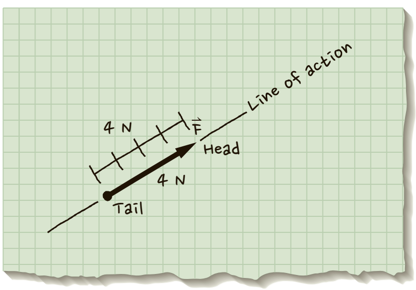


Figure 2.1.1 We draw an arrow to represent the magnitude and direction of a force.

Forces range from very small (e.g., 0.0000005 N for the gravitational pull exerted by Mars on an Earth-bound engineering student) to very large (e.g., 10 tons = 88,960 N for the weight of a large farm tractor). Forces are vector quantities, which means they have both **magnitude** (size) and **direction** associated with them. Graphically, we represent a force by an arrow (**Figure 2.1.1**). The direction from tail to head represents the direction of the force, and we commonly draw the length of the arrow proportional to the magnitude of the force. If we know the magnitude of the force, we write it (including units)

next to the arrow. The line along which the force $\boldsymbol{F}$ acts is called the **line of action** of the force.

In typed material, we show the symbol for a vector in boldface italic—$\boldsymbol{F}$—and with the Euclidean norm—$||\boldsymbol{F}||$—when only the magnitude is of concern. Conventions for representing these concepts in handwritten work are summarized in **Table 2.1**.

Table 2.1 Representation of Force Concepts

Force Concept	Symbols in Typed Material	Symbols in Handwritten Work
Force Vector	$\boldsymbol{F}$	underscore $\underline{F}$ or half arrow over the letter $\vec{F}$
Magnitude of Force	$\|\|\boldsymbol{F}\|\|$	$\|\|\underline{\boldsymbol{F}}\|\|$ or $\|\|\vec{\boldsymbol{F}}\|\|$

Physicists have traditionally identified four fundamental forces: gravitational, electromagnetic, weak, and strong. The relative strengths of these forces are strong 1, electromagnetic 10^{-2}, weak 10^{-7}, and gravitational 10^{-38}. Generally, engineers considering the forces acting on a system are concerned with **gravitational forces** that result from the pull of Earth on objects. They are also concerned with the electromagnetic forces that result from the interaction of electrical and magnetic fields at the atomic and subatomic levels—we refer to these as **contact forces**. The strong force (which keeps every atomic nucleus intact) and the weak force (a factor in radioactive decay) are significant only at the subatomic level, so we will not consider them further in this book.

In the remainder of this chapter, we look at the two fundamental forces that engineers must deal with every day: gravitational forces and contact forces.

2.2 GRAVITATIONAL FORCES

Learning Objective: Calculate gravitational forces.

Every object in the universe exerts a force on every other object in the universe, and we call this force a **gravitational force**. Because every object exerts such a force on every other object, gravitational forces always come in pairs, as **Figure 2.2.1** shows. There you see two objects, 1 and 2, and the gravitational force exerted by each on the other. The gravitational force is always an attractive force. As a result, the direction of the force exerted by object 2 on object 1 is from 1 to 2 and the direction of the force exerted by object 1 on object 2 is from 2 to 1.

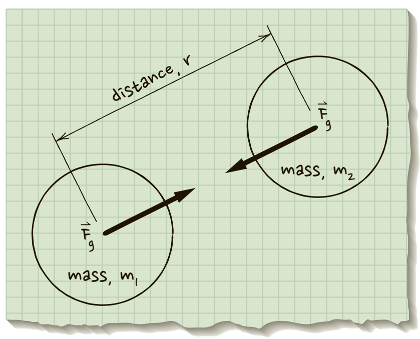


Figure 2.2.1 Gravitational forces exerted by two objects on each other.

The magnitude of the gravitational force exerted by one object on the other is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

The **universal gravitational constant** G changes the proportionality to an equality:

$$\|\boldsymbol{F}_g\| = \frac{Gm_1m_2}{r^2} \tag{2.1}$$

where

- $\|\boldsymbol{F}_g\|$ is the magnitude of the gravitational force (in newtons)
- G is the universal gravitational constant (values are found in Table 2.2)
- m_1 and m_2 are the masses of the two objects (in kilograms)
- r is the distance between their centers of mass (in meters), as shown in **Figure 2.2.1**

Table 2.2 **Values of Universal Gravitational Constant**

Measurement System	G	Units
SI	6.67×10^{-11}	$m^3/kg \cdot s^2$
U.S. Customary	3.44×10^{-8}	$ft^3/slug \cdot s^2$

***While the gravitational constant can be displayed to more significant digits, in this text we will just use three.*

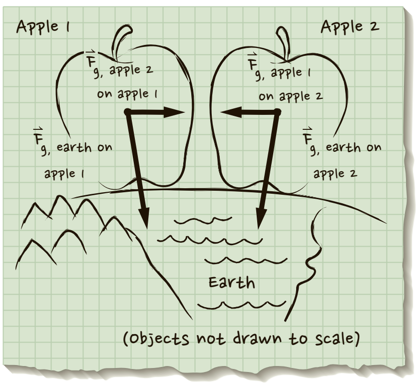


Figure 2.2.2 Gravitational forces exerted by one apple on another compared with gravitational force exerted by Earth on either apple.

Because the mass of Earth is at least 20 orders of magnitude greater than the mass of most objects on the planet, the gravitational attraction between any two objects at or near Earth's surface is negligible relative to the gravitational attraction between either object and Earth. For example, the magnitude of the gravitational force exerted by one average-size apple[3] on another is a mere 0.000 000 000 066 7 N, compared with the 0.98 N gravitational force exerted by Earth on either apple[4] (Figure 2.2.2). Therefore, the gravitational force exerted by Earth is an important force acting on all objects at or near Earth's surface.

When analyzing a gravitational force where one of the two interacting objects is Earth, we can simplify (2.1) by realizing that (a) the mass of Earth is constant and (b) for any object either on or not far from Earth's surface, the distance from Earth's center of mass to the object's center of mass can be taken to be Earth's average radius. In other words, once we arbitrarily say that m_1 in (2.1) is the mass of Earth, we can group our three constants—G, m_1, and r^2—into a new constant g. Therefore, (2.1) can be rewritten as

$$\|\boldsymbol{F}_g\| = \frac{Gm_1m_2}{r^2} = m_2\underbrace{\left(\frac{Gm_1}{r^2}\right)}_{g} = m_2 g \tag{2.2}$$

[3]This force was calculated using (2.1) with $m_1 = m_2 = 0.1$ kg, r = radius of apple 1 + radius of apple 2 = 0.050 m + 0.050 m = 0.100 m.

[4]This force was calculated using (2.1) with m_1 = mass of Earth = 5.98×10^{24} kg, m_2 = mass of apple = 0.1 kg, r = radius of Earth + radius of apple = 6.37×10^6 m + 0.05 m = 6.37×10^6 m.

where

$$g = \frac{Gm_1}{r^2}$$

m (= m_2) is the mass in kilograms of the object feeling the gravitational force exerted by Earth.

By substituting the values of G, m_1 for Earth (= 5.973×10^{24} kg), and the mean radius of Earth (= 6.37×10^6 m)[5] into (2.2), we find that at or near Earth's surface the magnitude of the gravitational force (in newtons) exerted by Earth on the object of mass m (in kilograms) is

$$\|\boldsymbol{F}_g\| = m(9.81 \text{ m/s}^2) \tag{2.3A}$$

This gravitational force exerted by Earth on an object is given a special name—the object's **weight on Earth** ($\boldsymbol{W_{Earth}}$). The magnitude of this weight force is the product of the mass of the object and the acceleration due to gravity g (= 9.81 m/s^2 = 32.2 ft/s^2):

$$\|\boldsymbol{W_{Earth}}\| = \|\boldsymbol{F}_g\| = m(9.81 \text{ m/s}^2) \tag{2.3B}$$

Occasionally you may see mass and force units seemingly used to mean the same thing. An example is the label on the candy bar shown in **Figure 2.2.3(*a*)**, where the net weight is given as 104.9 g and 3.70 oz. Grams and ounces are both mass units, but ounce is also used as a force unit, so it can be confusing. It gets even more confusing when we talk about *fig newton cookies*, because the weight of a *fig newton* is only 0.015 N; in other words seven *fig newtons* weigh about 1 newton on Earth (**Figure 2.2.3(*b*)**).

Thalia Anagnos

(*a*) (*b*)

Figure 2.2.3 (*a*) Product label contains weight and mass information; (*b*) On Earth, seven fig newton cookies weigh approximately 1 N.

[5]Earth is not a perfect sphere. Rather, it is an ellipsoid, flattened at the poles and bulging at the equator. Its equatorial radius is greater than its polar radius by 21 km (see Halliday, Resnick, and Krane, *Physics*, 5th Edition, John Wiley & Sons). What this means is that the gravitational force exerted by Earth is slightly greater at the poles (g = 9.835 m/s^2) than at the equator (9.78 m/s^2). For most engineering work, this difference is insignificant, and we use a value of g based on the mean radius of Earth.

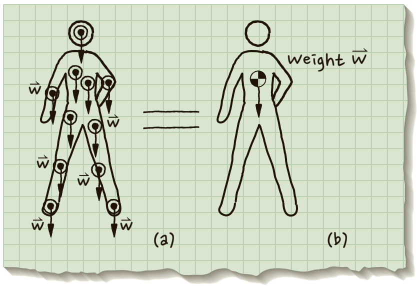


Figure 2.2.4 (*a*) The individual weights associated with Adam's atoms; (*b*) the total weight of Adam lumped at his center of mass.

Weight is a **body force**. This means that the gravitational force exerted by Earth on an object exists between every atom in the object and every atom in Earth (**Figure 2.2.4*a***). There will be times when the distributed nature of the gravitational force needs to be considered in engineering practice, and Chapter 6 deals with formal procedures for doing this. Far more frequently, however, it is sufficient to lump all the distributed gravitational forces acting on an object into a single force. This single force, which represents the weight of the object, is directed from the center of mass of the object to the center of mass of Earth (**Figure 2.2.4*b***).

Check out the following examples of applications of this material.

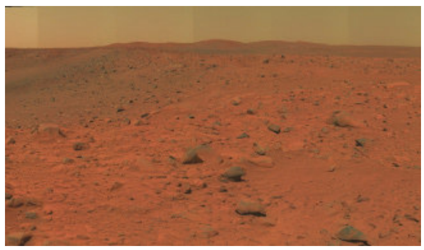
- **Example 2.2.1 Gravity, Weight, and Mass**
- **Example 2.2.2 Is Assuming Gravity is a Constant Reasonable?**
- **Example 2.2.3 Gravitational Pull from Two Planets**

EXAMPLE 2.2.1

NASA's Mars rover Spirit (**Figure 1**) was launched toward Mars on June 10, 2003 in search of answers about the history of water on the Red Planet. It landed on January 3, 2004, and in its wanderings on the planet sent back pictures such as the one shown in **Figure 2**. Spirit is about the size of a golf cart and has a mass of 174 kg. The mass of Mars is 6.39×10^{23} kg, and its mean radius is 3.39×10^{6} m (see Appendix B).

NASA Jet Propulsion Laboratory

Figure 1 Spirit on surface of Mars.



NASA/JPL/Cornell

Figure 2 Photograph of surface of Mars taken by Spirit.

Determine (a) the weight of Spirit on Earth in newtons and in pounds, and name two objects that weigh approximately the same amount on Earth as Spirit, (b) the weight of Spirit on Mars in newtons, (c) the mass of Spirit in slugs.

Goal Find the weight of rover Spirit on Earth and on Mars, and specify the mass of Spirit in slugs.

Given Mass of Spirit in kilograms.

Assume We can ignore the distance from the planet surface to the center of mass of Spirit when calculating distances in (2.1). We can also ignore the gravitational pull of other planets on Spirit.

Draw No drawings are required to address this problem.

Formulate Equations and Solve

(a) *The weight of Spirit on Earth can be found using (2.3B)*:

$$\|\boldsymbol{W}_{Earth}\| = \|\boldsymbol{F}_g\| = m(9.81\ \text{m/s}^2) \tag{2.3B}$$

$$= (174\ \text{kg})(9.81\ \text{m/s}^2) = 1707\ \text{N} \Rightarrow \boxed{\|\boldsymbol{W}_{Earth}\| = 1707\ \text{N}}$$

To convert from newtons to pounds we multiply by 0.2248 lb/N (Table 1.3).

$$\|\boldsymbol{W}_{Earth}\| = 1707\ \text{N}\ (0.2248\ \text{lb/N}) = 384\ \text{lb} \Rightarrow \boxed{\|\boldsymbol{W}_{Earth}\| = 384\ \text{lb}}$$

Answers to (a) The weight on Earth of the rover Spirit is 1707 N (384 lb), which is approximately the weight of a motorcycle or baby grand piano.

(b) *The weight of Spirit on Mars can be found using (2.1)*:

$$\|\boldsymbol{F}_g\| = \frac{Gm_1m_2}{r^2} \tag{2.1}$$

$$\|\boldsymbol{F}_{g,\,Mars}\| = \frac{(6.67\times10^{-11}\ \text{m}^2/\text{kg}\cdot\text{s}^2)(174\ \text{kg})(6.39\times10^{23}\ \text{kg})}{(3.39\times10^{6}\ \text{m})^2}$$

$$\boxed{\|\boldsymbol{F}_{g,\,Mars}\| = 646\ \text{N}}$$

This is the force exerted by Mars on Spirit as the rover sits on the Martian surface. In completing this calculation, we have assumed we can ignore the distance from the Martian surface to the center of mass of Spirit because the radius of Mars is so much larger than this distance. We have also ignored the pull of the other planets on Spirit as it sits on Mars. Is this a reasonable assumption? How could you verify this assumption?

(c) *The mass of Spirit in slugs.*

Using the units conversion in Table 1.3, we write

$$m = (174\ \text{kg})(1\ \text{slug}/14.593\ \text{kg}) \Rightarrow \boxed{m = 11.92\ \text{slugs}}$$

Check Our results say that an object that weighs 1707 N on Earth weighs 646 N on Mars. These weights are in the ratio of 1.00:0.378. Now we arrange the data on the mass and radius of Earth and Mars in a table to confirm this ratio. The last column of the table gives us these same ratios and provides a check to our calculations.

	Mean radius (r in km)	**Mass** (m in kg)	**Value of** m/r^2 (in kg/m^2 as seen in (2.1))	**Ratio** $\frac{m_{Mars}/r^2_{Mars}}{m_{Earth}/r^2_{Earth}}$
Mars	3390	6.39×10^{23}	5.56×10^{10}	0.378
Earth	6370	5.97×10^{24}	1.47×10^{10}	1.00

EXAMPLE 2.2.2

Jason Maehl/Getty Images

Figure 1 Mountain climbers at the summit of Mt. Everest.

We typically assume the same value of gravity, no matter where we are on Earth (9.81 m/s^2 or 32.2 ft/s^2). Let's analyze this assumption with respect to a mountain climber as shown in **Figure 1** of mass 75 kg who reaches the summit of Mount Everest (elevation 8,848 m).

Calculate the gravitational constant at sea level and at the summit, and compare the weight of the mountain climber at the two locations. Based on what you find, is the assumption of gravitational acceleration as a constant reasonable?

Goal Explore the assumption of a constant gravitational acceleration by comparing the weight of a mountain climber at sea level and at the top of Mt. Everest.

Given Mass of climber. Height of Mt. Everest.

Assume Earth is a perfect sphere, use mean radius of Earth, sea level is at 0 meters.

Draw No drawings are required to address this problem.

Formulate Equations and Solve

The weight of the climber at sea level can be found using (2.1):

$$\|\boldsymbol{F}_{gSL}\| = \frac{Gm_1m_2}{r^2}$$

$$= \frac{\left(6.673 \times 10^{-11}\ \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2}\right)(75\ \text{kg})(5.973 \times 10^{24})}{(6.371 \times 10^6\ \text{m})^2} \Rightarrow \boxed{\|\boldsymbol{F}_{gSL}\| = 736.5\ \text{N}}$$

The weight of the climber at the mountain summit

$$\|\boldsymbol{F}_{g\text{Summit}}\| = \frac{Gm_2m_2}{r^2}$$

$$= \frac{\left(6.673 \times 10^{-11}\ \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2}\right)(75\ \text{kg})(5.976 \times 10^{24})}{(6.371 \times 10^6\ \text{m} + 8848\ \text{m})^2} \Rightarrow \boxed{\|\boldsymbol{F}_{g\text{Summit}}\| = 734.4\ \text{N}}$$

Answers The difference in weight between sea level and the highest place on Earth is 2.1 N or 0.3%. We can conclude that assuming a constant value of gravity is a reasonable assumption.

Check The only check for this problem is to recheck the values of the variables in (2.1) and to recalculate the numbers to determine that we haven't made an error.

Explore This Further

If you were orbiting 200 km above Earth, would constant gravity be a good assumption? How much less would you weigh?

EXAMPLE 2.2.3

NASA's Mars rover Spirit is about the size of a golf cart and has a mass of 174 kg (see Figure 1). As Spirit travels from Earth to Mars, determine at what distance, measured from Earth, the gravitational force exerted by Earth on Spirit is equal to the gravitational force exerted by Mars on Spirit. Planetary data can be found in Appendix B.

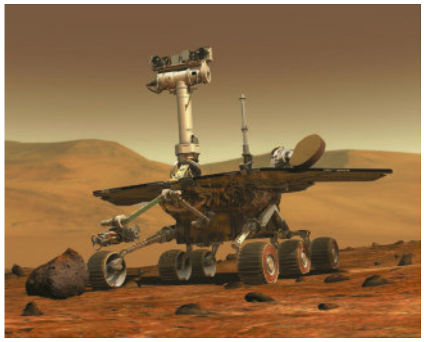


Figure 1 Spirit on surface of Mars.

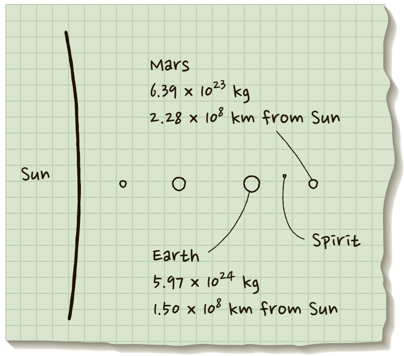


Figure 2 Spirit traveling between Earth and Mars.

Goal Determine the distance from Earth at which the Earth's gravitational force on Spirit is equal to Mars' gravitational force on it.

Given The mass of Spirit and where to find planetary data.

Assume Use the mean distances of Mars and Earth from the sun (planetary orbits are not circular). Strictly speaking, our calculation is correct only when the sun, Earth, and Mars are aligned (in that order).

Draw The various bodies and their masses and distances are shown in Figure 2.

Formulate Equations and Solve
We first write (2.1) or the gravitational force exerted by Earth on Spirit:

$$\|\boldsymbol{F}_{g,\text{ Earth on Spirit}}\| = \frac{Gm_1m_2}{r^2} \tag{2.1}$$

$$= \frac{G(5.973\times10^{24}\text{ kg})(174\text{ kg})}{r^2} \tag{1}$$

where r = distance from Earth's center of mass to Spirit's center of mass.

Now we write (2.1) for the gravitational force exerted by Mars on Spirit. We take the distance from Mars' center of mass to Spirit's center of mass to be the distance between Mars and Earth (78.3×10^6 km) minus the distance between Earth and Spirit. The distance between Mars and Earth can be calculated from data in Appendix B:

$$\|\boldsymbol{F}_{g,\text{ Mars on Spirit}}\| = \frac{G(6.391\times10^{23}\text{ kg})(174\text{ kg})}{(78.3\times10^6\text{ km}-r)^2} \tag{2}$$

Since we are interested in finding where the gravitational force exerted by Earth on Spirit is equal to the gravitational force exerted by Mars on Spirit, we equate (1) and (2):

$$\frac{G(5.973\times10^{24}\text{ kg})(174\text{ kg})}{r^2} = \frac{G(6.391\times10^{23}\text{ kg})(174\text{ kg})}{(78.3\times10^6\text{ km}-r)^2} \Rightarrow \boxed{r = 59.0\times10^6\text{ km}}$$

Check One good check is to plug the calculated value of r into (1) and (2) and confirm that each equation yields the same value of gravitational force on Spirit. If you don't get the same value, check to make sure you have used all the correct planetary data and repeat the calculations to confirm that all of the numbers were properly entered into a calculator.

Answers When Spirit is at a distance of $r = 59.0 \times 10^6$ km from the center of Earth, the gravitational pull on Spirit from Earth is equal to the pull from Mars. Note that the distance we calculated is independent of the mass of the rover.

EXERCISES 2.2

2.2.1. [*] The planet Venus has a diameter of 7700 miles and a mass of $3.34(10^{23})$ slugs.

a. Determine the gravitational acceleration at the surface of the planet. Express your answer in units of (ft/s^2).

b. Your answer in **a** is what fraction of the gravitational acceleration at Earth's surface?

2.2.2. [*] Determine the gravitational force exerted by the moon on Earth, using the following data. Make sure that you show your work.

Mass of moon: m_{moon}	$7.35(10^{22})$ kg
Mass of Earth: m_{Earth}	$5.97(10^{24})$ kg
Radius of moon: r_{moon}	$1.74(10^6)$ m
Radius of Earth: r_{Earth}	$6.37(10^6)$ m
Distance between moon and Earth	$3.84(10^8)$ m

2.2.3. [*] Determine the gravitational force exerted by the sun on Earth, using the following data. Be sure to show your work.

Mass of Earth: m_{Earth}	$5.97(10^{24})$ kg
Mass of sun: m_{sun}	$1.99(10^{30})$ kg
Radius of Earth: r_{Earth}	$6.37(10^6)$ m
Radius of sun: r_{sun}	$6.96(10^8)$ m
Distance between sun and Earth	$1.50(10^{11})$ m

2.2.4. [*] Determine the force of gravity acting on a satellite when it is in orbit 20.2×10^6 m above the surface of Earth. Its weight when on the surface of Earth is 8450 N. Use the following data as needed.

Mass of Earth: m_{Earth}	$5.97(10^{24})$ kg
Radius of Earth: r_{Earth}	$6.37(10^6)$ m

2.2.5. [**] At what distance, in kilometers, from the surface of Earth on a line from center to center would the gravitational force of Earth on a body be exactly balanced by the gravitational force of the moon on the body? Use the following data as needed.

Mass of moon: m_{moon}	$7.35(10^{22})$ kg
Mass of Earth: m_{Earth}	$5.97(10^{24})$ kg
Radius of moon: r_{moon}	$1.74(10^6)$ m
Radius of Earth: r_{Earth}	$6.37(10^6)$ m
Distance between moon and Earth	$3.84(10^8)$ m

2.3 CONTACT FORCES

Learning Objective: Identify the different types of contact forces.

As noted in Section 2.1, contact forces result from the electrical and magnetic interactions that are responsible for the bonding of atoms. Under this general heading are:

- Normal contact force
- Friction force

- Fluid contact force
- Tension force
- Shear force

We will consider each of these in turn, in the subsections that follow.

Normal Contact Force

Whenever two solid objects are in contact with each other, each exerts on the other a force that is perpendicular to the two contacting surfaces and is called a **normal contact force**. For example, when a pianist hits a piano key, his fingertip exerts a normal contact force on the key and the key exerts an equal and opposite normal contact force on his fingertip (**Figure 2.3.1**). Similarly, a book lying on your desk exerts a normal contact force on your desk, and the desk exerts an equal and opposite normal contact force on the book. Normal contact forces are directed so as to bring the two solids together. What this means in practical terms is that a clean fingertip contacting the top surface of a piano key can push on the key but can't pull it, as illustrated in **Figure 2.3.2**.

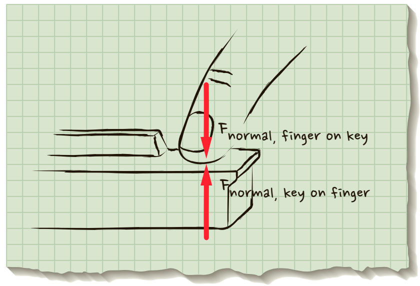


Figure 2.3.1 A fingertip pushing on a piano key causes a normal force to push back on the fingertip.

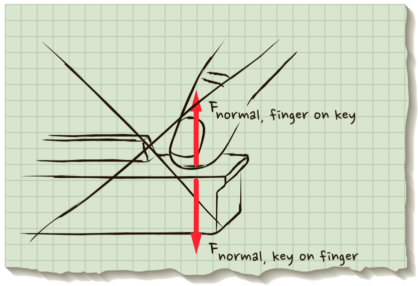


Figure 2.3.2 A fingertip cannot pull on a piano key.

Friction Force

If you attempt to slide one solid object over another, the motion is resisted by interactions between the surfaces of the two objects. This resistance is a **friction force** and is oriented parallel to the two contacting surfaces in a direction opposite the direction of (pending) motion. For example, if you push on an edge of a book as it rests on a table, as in **Figure 2.3.3*a***, the friction force exerted by the table on the book is in the direction opposite the sliding direction. An equal and opposite friction force is exerted by the book on the table (per Newton's third law).

Friction forces are related to and limited by the normal contact forces and the characteristics (e.g., smoothness) of the objects in contact. A normal contact force must be present for a friction force to be present (but not vice versa; compare **Figure 2.3.3*a*** with **Figure 2.3.3*b***).

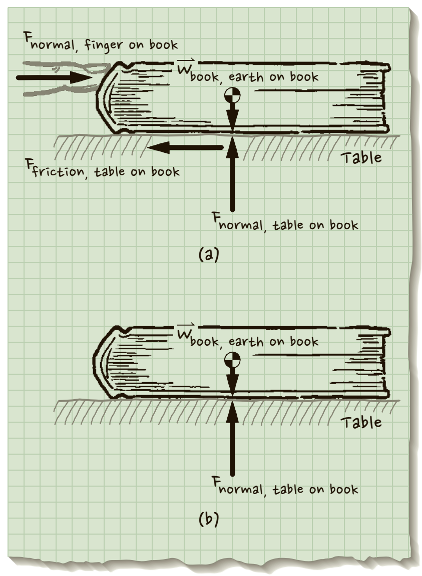


Figure 2.3.3 (*a*) Forces acting on a book that is pushed across a table; (*b*) forces acting on a book that is not pushed across a table.

Fluid Contact Force

As a fluid presses on or moves past a solid object, the fluid exerts a force on the surface of the object; we call this force a **fluid contact force**. (**Fluid** is the general term for gases and liquids—substances that change shape to fill a volume.) You have experienced a fluid contact force if you have ever put your hand out the window of a moving car—there is a definite force pushing on your hand (**Figure 2.3.4**). When we refer to the interaction between a fluid and a solid, we will typically be talking in terms of the **fluid contact pressure**, which describes the fluid contact force acting over a surface area. The dimensions of pressure are force/area, and so pressure units are N/m^2 in the SI system and $lb/in.^2$ (sometimes written as psi) in the U.S. customary system.

The fluid contact pressures engineers work with may be very small (1000 N/m^2 for the water pressure exerted on the bottom of a full tea

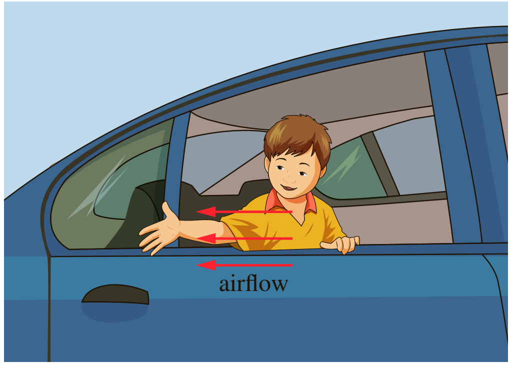


Figure 2.3.4 Fluid contact pressure acting on a hand stuck out the window of a moving vehicle.

kettle), medium size (500,000 N/m^2 for the air pressure in a high-performance bicycle tire), or very large (4,000,000 N/m^2 for the air pressure in a scuba tank). It may also be important to think about how fluid pressure varies with depth, as would be the case of considering water pressure acting on a dam, or how pressure acting on the hull of a submarine increases with dive depth.

Tension Force

A **tension force** (or simply **tension**) is contact force caused by an object's atoms pulling on one another. A cable attached to a solid object and pulled taut is said to be under tension. For example, consider a cable holding up a crate, as in **Figure 2.3.5*a***. The tension force in the cable is transmitted along the cable (**Figure 2.3.5*b***). Microscopically, each atom of the cable "pulls" on the atom next to it and is in turn pulled by that atom, according to Newton's third law (**Figure 2.3.5*c***). As a result of this atom-to-atom contact within the solid, the force pulling on one end of the cable is transmitted to the object on the other end. If we were to cut

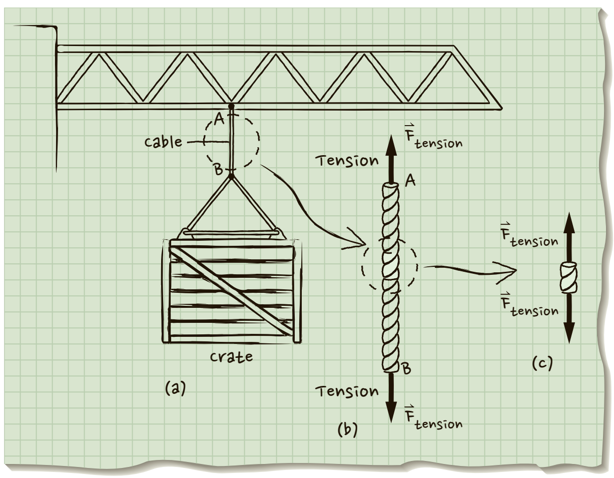


Figure 2.3.5 (*a*) A crane holds a crate with a cable; (*b*) looking more closely at the cable; (*c*) tension is transmitted along the length of the cable.

the cable at any point and insert a spring scale at the cut ends, the spring scale would read the tension force $\boldsymbol{F}_{\text{tension}}$ directly. Tension forces may be very small (0.001 N for a spider swinging on its web) or very large (on the order of 300,000,000 N tension in the main cables of the Golden Gate Bridge). Tension forces are also present in ropes, chains, bicycle spokes, rubber bands, and bungee cords.

Compressive Force

When the atoms in a solid object are pushed closer together, they experience a contact force called a **compressive force** (or simply **compression**). For example, consider a vertical column holding up a wooden deck, as in **Figure 2.3.6*a***. Compression is transmitted along the column as the deck pushes down from the top and the support pillar pushes up from the bottom. As with a cable in tension, adjacent atoms in the column exert forces on each other. In the case of the column in compression, the atoms "push" on each other with compressive force $\boldsymbol{F}_{\text{compression}}$ (**Figure 2.3.6*b***).

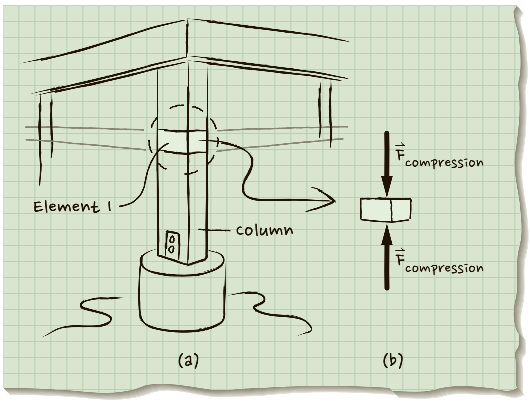


Figure 2.3.6 (*a*) A column holds up a deck; (*b*) compression is transmitted along the length of the column.

Compression forces in objects may be very small (0.5 N compression applied by household tweezers) to very large (1,000,000 N compression applied during sheet metal stamping).

Do not confuse compressive forces with normal contact forces. A compressive force is *within* an object and is due to the atoms that make up the object pushing on one another. A normal contact force acts on an object's *surface* and comes about when that object is pushed on by another object.

Shear Force

When the atoms that make up a solid object are shifted relative to one another, they experience a contact force called a **shear force** (or simply **shear**). For example, consider a rock climber standing on a small rock toehold (**Figure 2.3.7*a***). More specifically, consider the interface between the toehold and the larger rock mass that supports it (**Figure 2.3.7*b***). At the interface, a shear force is transmitted. The atoms on the

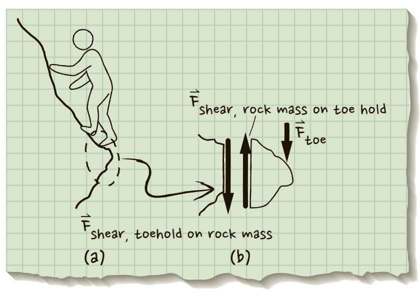


Figure 2.3.7 (*a*) A rock outcropping holds up a climber; (*b*) shear forces are transmitted across the interface.

right of the interface are pushed downward (ever so slightly) relative to the atoms on the left. This shift results in an upward shear force acting on the toehold and an equal and opposite downward shear force acting on the larger rock mass. Notice that the shear force is parallel to the interface.

Let's clarify the distinction between shear forces and friction forces. A shear force is *within* an object and is due to the atoms that make up the object tending to shift relative to one another. A friction force acts on the object's *surface* and comes about when that object is positioned to slide relative to another object.

Check out the following example of an application of this material.
- **Example 2.3.1 Identifying Types of Forces**

EXAMPLE 2.3.1

Dave and Les Jacobs/Getty Images

Figure 1 Many forces act on a moving bicycle and cyclist.

Identify some of the gravitational and contact forces associated with the moving bicycle and its rider shown in **Figure 1**.

Goal Identify forces associated with a bicycle and rider and categorize them as gravitational or contact forces.

Given A picture of a bicycle and cyclist.

Assume No assumptions needed.

Draw No drawing needed.

Formulate Equations and Solve To complete this exercise, we do not need to set up any equations. We do need to consider various components of a bicycle–cyclist system and the forces present. We use **Figure 1** to prompt our thinking (or even better, we could closely examine a real bicycle). Although we are not explicitly told how many forces we should identify, we choose to find at least two examples of each type of force.

Gravitational Forces:

Weight of cyclist; weight of bicycle. The weight of the bicycle could be broken out into the weights of its various components.

Contact Forces:

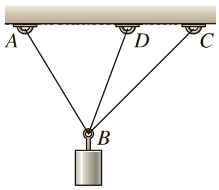
- **Normal Contact Force**—between road and tire, between cyclist and seat, between cyclist's feet and pedals, between cyclist's hands and handlebars
- **Friction Force**—between cyclist and seat (prevents sliding on the seat), between cyclist's hands and handlebars (prevents sliding on the

handlebars), between rear wheel and ground tangent to wheel circumference

- **Fluid Contact Force**—air pushing on inside of inflated tires, air moving past bicycle
- **Tension Force**—in chain, in brake cables, in shifter cables, in muscles in cyclist's legs and arms, in wheel spokes
- **Compressive Force**—in front fork, in seat tube, in tire rubber where it meets ground
- **Shear Force**—in axle adjacent to bottom bracket, in handlebars adjacent to stem

EXERCISES 2.3

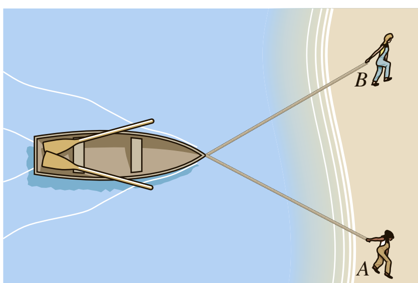
2.3.1. [*] Consider the system shown. Match each part of the system listed on the left with the type of force that is operating listed on the right.



EX 2.3.1

Part of System	Type of Force
Cable *AB*-eyelet interface at *B*	Normal Contact Force
Within cable *BD*	Tension Force
Weight of object	Gravity Force
	Compression Force
	Friction Force

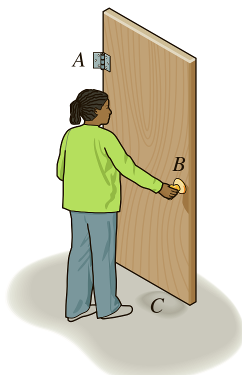
2.3.2. [*] Consider the boat being pulled ashore by two people as shown. Match the type of force that is operating listed on the left with a part of the system listed on the right. You may use a part of the system more than once.



EX 2.3.2

Type of Force	Part of System
Normal Contact Force	Boat water interface
Tension Force	Within rope *B*
Gravity Force	Person A-ground interface
Friction Force	Weight of boat
Fluid Contact Force	

2.3.3. [*] Consider the woman turning a door handle as shown. Match the type of force that is operating listed on the left with a part of the system listed on the right. You may use a part of the system more than once.



EX 2.3.3

Type of Force	Part of System
Normal Contact Force	Hand-door knob interface
Tension Force	Within woman's arm muscle
Gravity Force	Within woman's leg bone
Friction Force	Door weight
Compression Force	

2.4 IDENTIFYING FORCES FOR ANALYSIS

Learning Objective: Isolate a system from the rest of the world and identify the types of external forces acting on it.

The ability to identify one small part of the world that is relevant to some particular engineering problem and then to identify the forces acting on that part are key skills when addressing questions concerning structural integrity and performance. We refer to the part of the world being studied as the **system** and to the forces acting ON the system as **external forces**. External forces may be any of the types we have discussed previously—gravitational, normal contact, friction, fluid contact, tension, compressive, or shear. **Internal forces** are those that exist INSIDE the system in equal and opposite pairs (Newton's third law).

Whether a particular force we are looking at in a problem is external or internal depends on how the system is defined. For example, if we define our system as being two stacked books resting on a table (**Figure 2.4.1*a***), the external forces acting on the system are the weights of the two books and the normal contact force exerted by the table on the lower book. The normal contact forces between the two books (the push exerted by the lower book on the upper book and the equal and opposite push exerted by the upper book on the lower book) are internal forces (**Figure 2.4.1*b***). Because they are equal in magnitude and opposite in direction, the two members of any pair of internal forces sum to zero and therefore cancel each other.

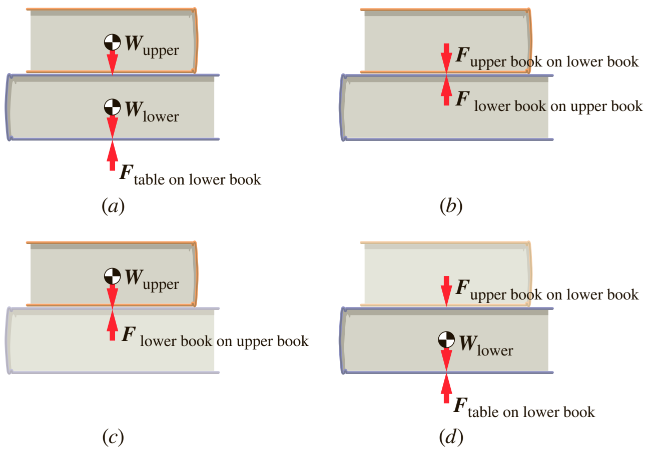


Figure 2.4.1 (*a*) External forces acting on a system of two books; (*b*) internal forces between the two books; (*c*) external forces acting on a system defined as the upper book; (*d*) external forces acting on the lower book system.

If we define our system to be only the upper book, the external forces acting on the system are the weight of the upper book and the normal contact force exerted by the lower book on the upper book (**Figure 2.4.1*c***).

Finally, if we define our system to be only the lower book, the external forces acting on this system are the weight of the lower book and the normal contact forces exerted by the upper book and by the table on the lower book (**Figure 2.4.1*d***). Notice that if we overlay **Figures 2.4.1*c*** and **2.4.1*d*** we get the two-book system in **Figure 2.4.1*a*** (remember that the normal forces between the two books sum to zero).

The distinction between external and internal forces in a given situation is further illustrated by the example of a person standing on a ladder leaning against a building. The ladder consists of two long stringers connected by eight rungs (**Figure 2.4.2**). Normal contact forces and friction forces exist between ladder and wall and between ladder and ground. Normal contact forces and friction forces are also present between the person's hands and one of the rungs and between the person's feet and another rung. Which of these forces we need to consider depends on what we want to know about the situation. For example:

Case 1: If we want to know whether the base of the ladder will begin sliding away from the building (not a desirable state of affairs!), we could define our system to be person + ladder. The external forces acting on this system are the weights of the person and ladder and the normal contact forces and friction forces that the wall and ground exert on the ladder. The normal contact forces and friction forces between the person and the ladder are internal to our system. **Figure 2.4.3** depicts the external forces in Case 1.

Case 2: Alternatively, we could take the ladder alone as our system in determining whether the ladder will slide. The external forces acting on this system are the weight of the ladder, the normal contact forces and friction forces exerted by hands and feet on the rungs, and the normal contact forces and friction forces exerted by the wall and ground on the ladder. **Figure 2.4.4** depicts the external forces in Case 2.

Case 3: To determine whether the connections between a rung and the stringers are strong enough, we would take the system to be the rung

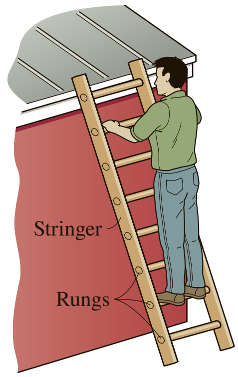


Figure 2.4.2 A person climbing a ladder.

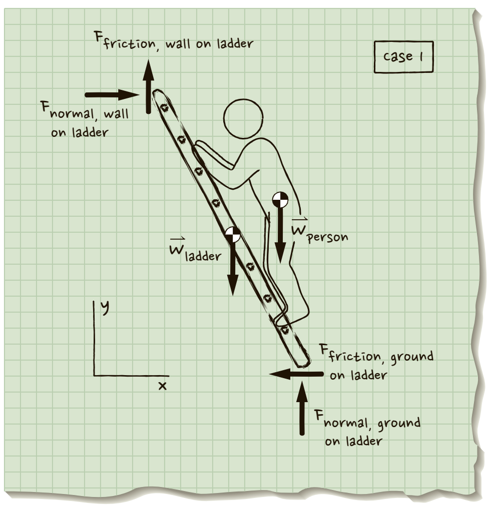


Figure 2.4.3 Case 1: External forces acting on the person-ladder system.

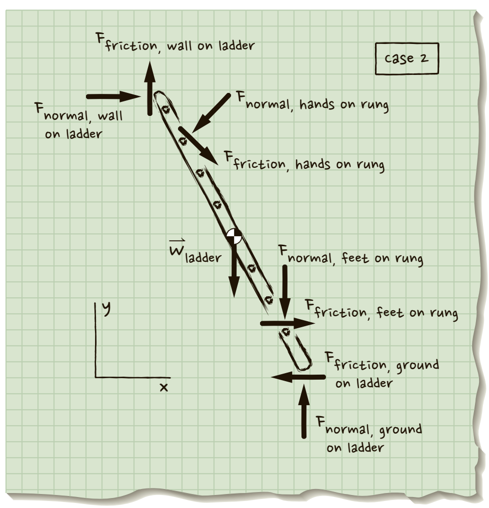


Figure 2.4.4 Case 2: External forces acting on the ladder system.

on which the person is standing. The external forces exerted on this rung are the normal contact forces and friction forces exerted by the feet plus the tension, compression, and shear forces that the stringers apply to the rung. **Figure 2.4.5** depicts the external forces in Case 3.

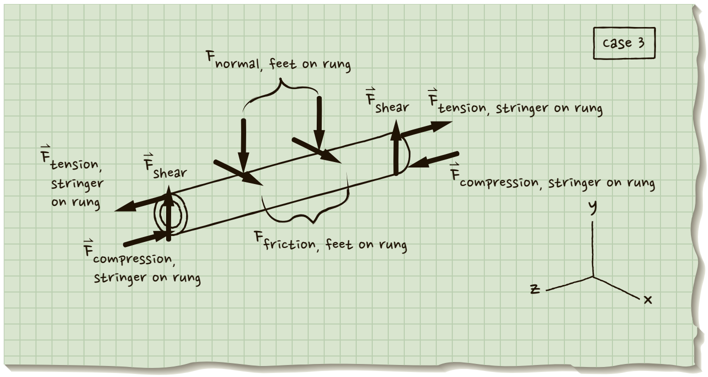


Figure 2.4.5 Case 3: External forces acting on the rung system.

Case 4: If we want to know about the forces exerted on the person's lower back, we could begin by defining the system as the person (Case 4A). The external forces exerted on this system are the person's weight and the normal contact forces and friction forces exerted by the ladder on the hands and feet (**Figure 2.4.6*a***). After analyzing this system, we would analyze another system—the upper body of the person (Case 4B). The external forces exerted on this system (shown in **Figure 2.4.6*b***) are the weight of the upper body, the normal contact forces and friction forces exerted by the ladder on the hands, and the compression and tension forces exerted by the lower body on the upper body. It is these latter external forces that are carried by the lower back. This case illustrates that sometimes we may need to take an **iterative approach** to analysis, first looking at one system and then looking at a second system that is some portion of the first system.

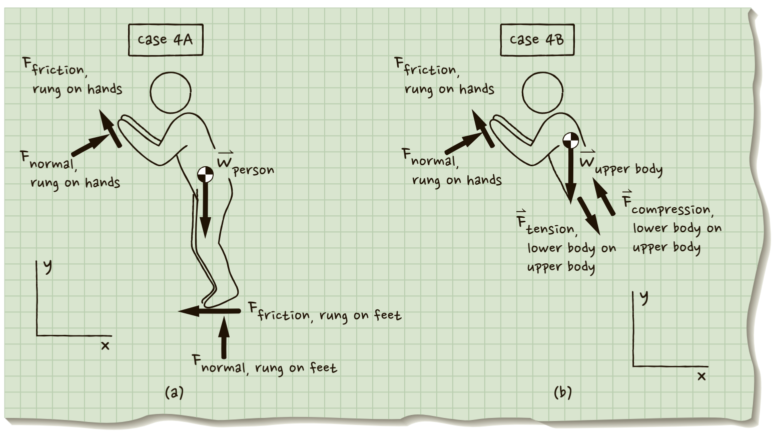


Figure 2.4.6 (*a*) Case 4A: External forces acting on the person system; (*b*) Case 4B: External forces acting on the upper body of the person.

In each of these four cases, we zoomed in and isolated a part of the world that was relevant to the question we were asking. We called this part the system. We then identified the forces acting on that system and called these the external forces. It is usually most convenient to define your system so that the forces you are trying to find are external ones. As noted earlier, internal forces, because they come in pairs, sum to zero and therefore cancel each other, so we do not consider them when analyzing a system.

Notice that as we went from Case 1 to Case 4, forces that were internal to some systems became external to others. Once again, whether a force is external or internal depends on the system of interest.

The process of defining a system and then identifying external forces acting on that system is critical in evaluating how the system performs. We presented this process more formally in the Goal, Given, Assume . . . steps laid out in the Engineering Analysis Procedure overview.

Zooming in and identifying external forces leads to the Draw step of the procedure. The drawing we commonly create is called a **free-body diagram**—"free" because an imaginary boundary around our system cuts off the system from the world around it, "body" because we have defined a specific system (a body or, sometimes, bodies) to focus on, and "diagram" to emphasize the importance of a visual representation of the system and the external forces acting on it. A free-body diagram uses vector arrows to represent the external forces acting on the system. Each force is shown on the diagram at its **point of application**; this is the point on the system where the force acts. **Figures 2.4.3** through **2.4.6** are examples of free-body diagrams.

In Chapter 4 we will have a lot more to say about creating free-body diagrams.

Check out the following example of an application of this material.
- **Example 2.4.1 Defining a System for Analysis**

EXAMPLE 2.4.1

A pallet of tiles weighing 200 lb sits on a roof with a 3 : 4 pitch (rise over run) and is held in place with a cable attached to the upper roof structure as shown in Figure 1. The pallet is sitting on loose tar paper, so it is reasonable to assume there is no friction force between the pallet and the roof. Given that the sum of the forces acting on the pallet-tile unit is zero, you wish to find the magnitude of the force exerted by the cable on the pallet.

Review the analysis steps presented in the **Engineering Analysis Procedure Overview (Box 1.1)**. Then define a system needed to analyze this problem and draw the loads acting on that system.

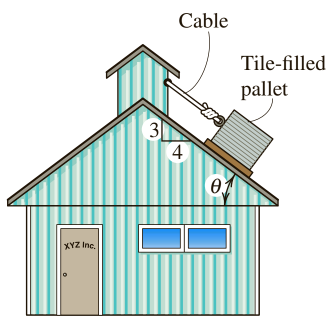


Figure 1 Pallet of tiles sitting on an inclined roof.

Goal (a) Define a system to study the forces acting on the pallet of tiles and then (b) draw the loads acting on the system. Once we complete our task, we'll have a drawing (a free-body diagram) that can be used to develop relevant equations to analyze the situation.

Given The slope of the roof and the combined weight of the pallet and tiles. The pallet sits on a frictionless surface, and the sum of the forces acting on the pallet-tile unit is zero.

Assume Any other forces acting on the pallet-tile unit (such as wind) are negligible. The cable is parallel to the roof and the forces acting on the pallet-tile unit lie in a single plane. All forces act through the geometric center of the pallet-tile unit. (We would need more information to make any other assumption.)

Draw To analyze the forces acting on the pallet-tile unit, we define it as our system by drawing a boundary to isolate it, as shown in Figure 2*a*. Notice that the boundary cuts through the cable and runs between the pallet and the roof.

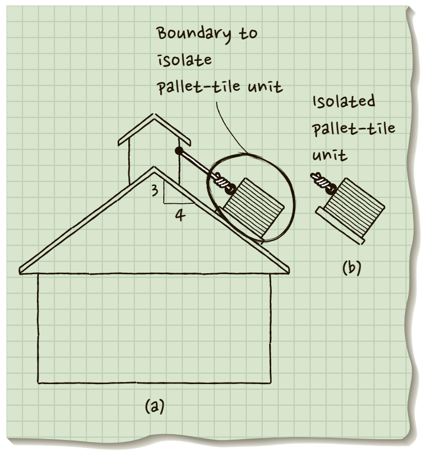


Figure 2 Isolate the pallet-tile unit from its surroundings to define the system for analysis.

Answer Part (a) Figure 2*b* is the isolated unit.

The external loads acting on our system are the cable tension ($\boldsymbol{T}_{\text{cable}}$), the gravitational force acting on the pallet and tiles ($\boldsymbol{W}$), and the normal contact force exerted by the roof on the pallet ($\boldsymbol{F}_{\text{normal}}$).

In creating the drawing of the loads acting on the isolated system in **Figure 2*b*** we assume that the normal contact force acts at the center of the bottom of the pallet. We draw each force in the direction we think it acts on the unit. Finally, we place a set of coordinate axes with the origin at the center of the unit. We could orient these axes horizontally and vertically, but orienting them along the roof pitch will make force addition easier when we do our analysis in Example 2.6.6.

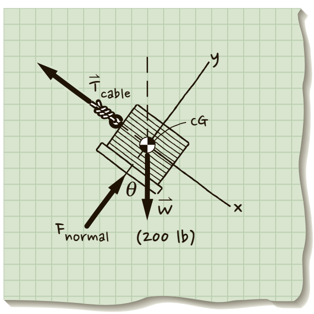
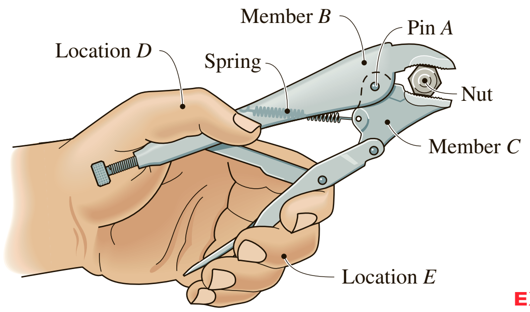


Figure 3 External forces acting on the isolated pallet.

Answer Part (b) Figure 3 is a free-body diagram of the isolated pallet-tile unit.

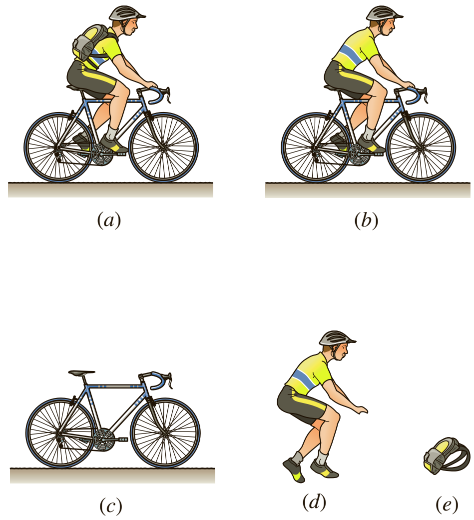
EXERCISES 2.4

2.4.1. [**] A person grips a pair of locking jaw pliers (more commonly known as vise-grips), as shown, in order to tighten a nut onto a bolt. Three cases (systems) are described in the top row of **Table EX 2.4.1**. For each system, identify the external (E) and internal (I) forces by completing **Table EX 2.4.1**. Leave blank those squares associated with a force not applied to the system under consideration.



EX 2.4.1

2.4.2. [**] A cyclist is bicycling down the road. Five cases (systems) are described in the top row of **Table EX 2.4.2** and in the figure. For each system, identify the external (E) and internal (I) forces by completing **Table EX 2.4.2**. Leave blank the squares associated with a force not applied to the system under consideration.



EX 2.4.2

Table Ex. 2.4.1

Forces	**Case 1** Vise-grips and nut defined as system	**Case 2** Vise-grips defined as system	**Case 3** Nut defined as system
Tension in spring			
Normal contact force between pin *A* and member *B*			
Nomal contact force between nut and member *B*			
Normal contact force between nut and member *C*			
Normal contact and friction forces between thumb and vise-grips at *D*			
Normal and friction forces between fingers and vise-grips at *E*			
Weight of vise-grips			
Weight of nut			
Normal contact and friction forces between nut and bolt			

Table of EX 2.4.1

Table Ex. 2.4.2

Forces	**Case 1** (*a*) Cyclist (including backpack) and bicycle defined as system	**Case 2** (*b*) Cyclist (not including backpack) and bicycle defined as system	**Case 3** (*c*) Bicycle defined as system	**Case 4** (*d*) Cyclist defined as system	**Case 5** (*e*) Backpack defined as system
Chain tension					
Weight of bicycle					
Weight of cyclist					
Weight of backpack					
Normal contact and friction forces between rider's hands and handlebars					
Normal contact and friction forces between rider's bottom and bicycle seat					
Normal contact force where rider's foot presses on pedal					
Normal contact force between front fork and front wheel hub					
Normal contact force between cyclist's back and backpack					
Tension in backpack shoulder strap					
Normal contact forces between tires and road					
Wind force acting on bicycle					
Wind force acting on cyclist					
Tension in cyclist's back muscles					

Table of EX 2.4.2

2.5 REPRESENTING FORCE VECTORS

Learning Objective: Represent a force mathematically and be able to convert between different representations.

Working with free-body diagrams involves representing and manipulating forces. Therefore we now consider how to formally work with the vector quantity of force. Although this section and the next are framed in terms of force, our comments apply to any vector quantity.

Suppose you are asked to remove a tent stake from the ground, as shown in Figure 2.5.1*a*. How would you pull on the rope? You likely would pull, using both arms, with the rope aligned with the long axis of the stake and the magnitude of your pull force about 50% of your weight.[6] We can represent this pulling force graphically (Figure 2.5.1*b*), but we can also represent it mathematically. We now present three math-based approaches to representing the magnitude and direction of a force. Table 2.3 provides an overall summary of these three approaches.

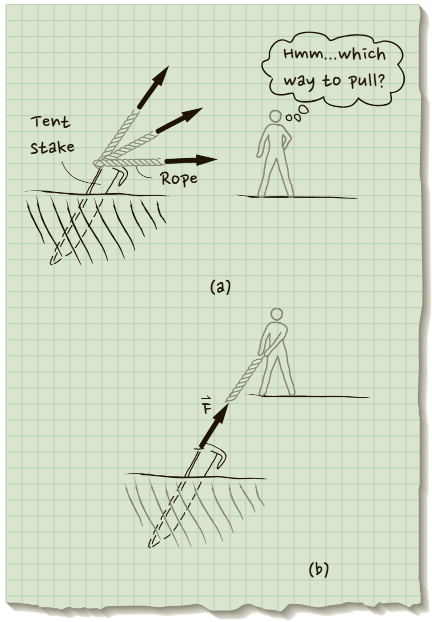


Figure 2.5.1 (*a*) Person considering how to pull on a stake; (*b*) person pulls along the stake's long axis.

2.5.1 Rectangular Component Representation

Consider a force $\boldsymbol{F}$ of known orientation and magnitude $\|\boldsymbol{F}\|$. This force can be specified by its three **rectangular component vectors** F_x, F_y, and F_z relative to a right-handed coordinate system. This is the same as specifying the "hike" you would take in the x, y, and z directions to get from

Table 2.3 Summary of Force Vector Representations

Rectangular Component Representation		
Force in terms of rectangular component vectors	$\boldsymbol{F} = F_x + F_y + F_z$ $\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k}$	(2.4A) (2.4B)
Magnitude of the force in terms of rectangular component vectors	$\|\boldsymbol{F}\| = \sqrt{F_x^2 + F_y^2 + F_z^2}$	(2.5)
Force in terms of unit vector along its line of action	$\boldsymbol{F} = \|\boldsymbol{F}\|\boldsymbol{u} = \|\boldsymbol{F}\|(u_x\boldsymbol{i} + u_y\boldsymbol{j} + u_z\boldsymbol{k})$ $u_x = \left(\dfrac{x_B - x_A}{L}\right)$ $u_y = \left(\dfrac{y_B - y_A}{L}\right)$ $u_z = \left(\dfrac{z_B - z_A}{L}\right)$ where $L = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$ is the distance from A to B.	(2.6) (2.7)
Force scalar components in terms of force magnitude and unit vector components	$F_x = \|\boldsymbol{F}\|u_x$ $F_y = \|\boldsymbol{F}\|u_y$ $F_z = \|\boldsymbol{F}\|u_z$ $u_x = \dfrac{F_x}{\|\boldsymbol{F}\|}$ $u_y = \dfrac{F_y}{\|\boldsymbol{F}\|}$ $u_z = \dfrac{F_z}{\|\boldsymbol{F}\|}$	(2.8) (2.9)

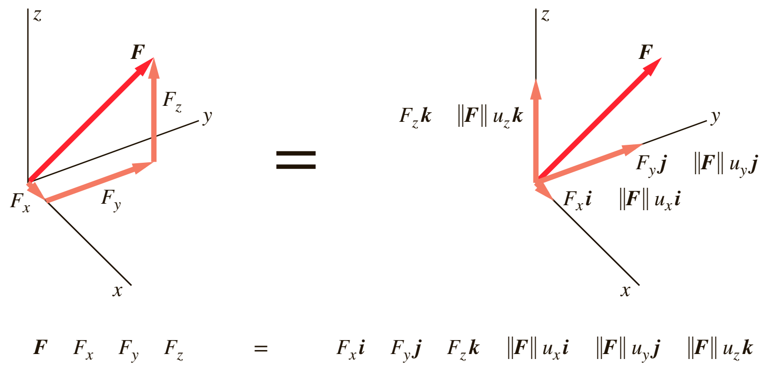
[6]Based on data gathered on nine college sophomores in October 2002.

Table 2.3 Summary of Force Vector Representations (*Continued*)

Planar force in *xy* plane	$F = F_x i + F_y j$ (2.10A) $F = \|\|F\|\|(u_x i + u_y j)$ (2.10B)
Space Angle Representation	
Space angles defined in terms of points on a line of action	$\theta_x = \cos^{-1}\left(\frac{x_B - x_A}{L}\right)$ $\theta_y = \cos^{-1}\left(\frac{y_B - y_A}{L}\right)$ (2.11) $\theta_z = \cos^{-1}\left(\frac{z_B - z_A}{L}\right)$ where $L = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$ is the distance from A to B.
Direction cosines of space angles	$\sqrt{(\cos\theta_x)^2 + (\cos\theta_y)^2 + (\cos\theta_z)^2} = 1$ (2.12A) $\cos\theta_x = \left(\frac{x_B - x_A}{L}\right)$ $\cos\theta_y = \left(\frac{y_B - y_A}{L}\right)$ (2.12B) $\cos\theta_z = \left(\frac{z_B - z_A}{L}\right)$ $\cos\theta_x = u_x$ $\cos\theta_y = u_y$ (2.13) $\cos\theta_z = u_z$
Unit vector along line of action, defined in terms of direction cosines	$u = (\cos\theta_x i + \cos\theta_y j + \cos\theta_z k)$ (2.14)
Force in terms of direction cosines	$F = \|\|F\|\|u = F(\cos\theta_x i + \cos\theta_y j + \cos\theta_z k)$ (2.15)
Force scalar components in terms of force magnitude and direct cosines	$F_x = \|\|F\|\|\cos\theta_x$ $F_y = \|\|F\|\|\cos\theta_y$ (2.16) $F_z = \|\|F\|\|\cos\theta_z$ $u_x = \cos\theta_x = \frac{F_x}{\|\|F\|\|}$ $u_y = \cos\theta_y = \frac{F_y}{\|\|F\|\|}$ (2.17) $u_z = \cos\theta_z = \frac{F_z}{\|\|F\|\|}$
Planar force in *xy* plane	$\sqrt{(\cos\theta_x)^2 + (\cos\theta_y)^2} = 1$ (2.18) $F = \|\|F\|\|u = \|\|F\|\|(\cos\theta_x i + \cos\theta_y j)$ (2.19)
Spherical Angle Representation	
Relationships between direction cosines and space angles	$\cos\theta_x = (\sin\phi)(\cos\theta)$ $\cos\theta_y = (\sin\phi)(\sin\theta)$ (2.20A) $\cos\theta_z = (\cos\theta)$ $\cos\phi = (\cos\theta_z)$ $\tan\theta = \frac{\cos\theta_y}{\cos\theta_x}$ (2.20B)

the tail of the force vector to its head (**Figure 2.5.1.1*a***). The force is the vector sum of its component vectors:

$$\boldsymbol{F} = \boldsymbol{F}_x + \boldsymbol{F}_y + \boldsymbol{F}_z \tag{2.4A}$$



(*a*) (*b*)

Figure 2.5.1.1 (*a*) $\boldsymbol{F}$ defined in terms of rectangular component vectors $\boldsymbol{F}_x$, $\boldsymbol{F}_y$, and $\boldsymbol{F}_z$; (*b*) the rectangular component vectors can be defined in terms of the force magnitude and unit vector $\boldsymbol{u}$ along the line of action of $\boldsymbol{F}$.

This equation can be rewritten in terms of the Cartesian unit vectors $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ (aligned with the x, y, and z axes) as

$$\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k} \tag{2.4B}$$

where F_x, F_y, and F_z are the **scalar components** of $\boldsymbol{F}$. Each scalar component tells us, for a particular direction, the length of a leg of the "hike" described in **Figure 2.5.1.1*a***. In fact, the scalar components F_x, F_y, and F_z are the projections of $\boldsymbol{F}$ onto the x, y, and z axes, respectively. Unlike magnitude, a scalar component has a sign associated with it. In engineering analysis, you are just as likely to be required to combine component vectors $\boldsymbol{F}_x$, $\boldsymbol{F}_y$, and $\boldsymbol{F}_z$ into the force $\boldsymbol{F}$ as to decompose $\boldsymbol{F}$ into its component vectors $\boldsymbol{F}_x$, $\boldsymbol{F}_y$, and $\boldsymbol{F}_z$. The scalar components can be combined to determine the magnitude $\|\boldsymbol{F}\|$ of the force $\boldsymbol{F}$:

$$\|\boldsymbol{F}\| = \sqrt{F_x^2 + F_y^2 + F_z^2} \tag{2.5}$$

as illustrated in **Figure 2.5.1.2**. Equation (2.5) says that the magnitude of $\boldsymbol{F}$ is equal to the positive square root of the sum of the squares of its scalar components.

We can also break $\boldsymbol{F}$ into just two vector components, one parallel to the xy plane ($\boldsymbol{F}_1$) and the other perpendicular to the xy plane ($\boldsymbol{F}_z$), as shown in **Figure 2.5.1.2*b***. Then the magnitude of $\boldsymbol{F}_1$ is found by combining F_x and F_y so that $\|\boldsymbol{F}_1\| = \sqrt{F_x^2 + F_y^2}$. Substituting into (2.5) we see that

$$\|\boldsymbol{F}\| = \sqrt{F_1^2 + F_z^2}$$

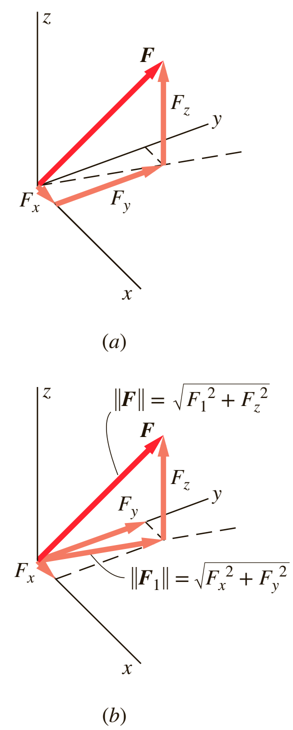


Figure 2.5.1.2 (*a*) Scalar components; (*b*) how the scalar components combine to form the magnitude of the force.

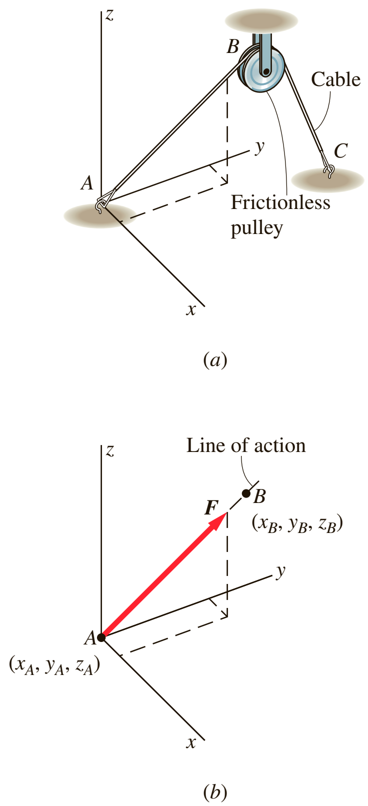


Figure 2.5.1.3 (*a*) A cable passes around a frictionless pulley; (*b*) the cable force acting on the hook at A.

We can also write $\boldsymbol{F}$ in terms of its magnitude and a unit vector, $\boldsymbol{u}$ (as defined in Section 1.4) along the line of action of $\boldsymbol{F}$:

$$\boldsymbol{F} = ||\boldsymbol{F}||\boldsymbol{u} = ||\boldsymbol{F}||(u_x\boldsymbol{i} + u_y\boldsymbol{j} + u_z\boldsymbol{k}) \tag{2.6}$$

where u_x, u_y, and u_z are the components of the unit vector along its line of action, as illustrated in **Figure 2.5.1.1*b***. The unit vector along the line of action of a force can be determined from system geometry. For example, in **Figure 2.5.1.3*a*** a cable runs from a hook at A, around a pulley at B, and to another hook at C. If we are given the coordinates of A (x_A, y_A, z_A) and B (x_B, y_B, z_B), we can find the components u_x, u_y, and u_z of the unit vector along the line of action of the cable force acting on the hook at A (**Figure 2.5.1.3*b***) from these coordinates:

$$\begin{aligned} u_x &= \left(\frac{x_B - x_A}{L}\right) \\ u_y &= \left(\frac{y_B - y_A}{L}\right) \\ u_z &= \left(\frac{z_B - z_A}{L}\right) \end{aligned} \tag{2.7}$$

where $L = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$ is the distance from A to B.

The scalar force components can therefore be written as

$$\begin{aligned} F_x &= ||\boldsymbol{F}||u_x \\ F_y &= ||\boldsymbol{F}||u_y \\ F_z &= ||\boldsymbol{F}||u_z \end{aligned} \tag{2.8}$$

If we are given $\boldsymbol{F}$ in terms of its scalar components, we see from rearranging (2.8) that we can determine the scalar components of $\boldsymbol{u}$ from the magnitude and components of $\boldsymbol{F}$:

$$\begin{aligned} u_x &= \frac{F_x}{||\boldsymbol{F}||} \\ u_y &= \frac{F_y}{||\boldsymbol{F}||} \\ u_z &= \frac{F_z}{||\boldsymbol{F}||} \end{aligned} \tag{2.9}$$

Sometimes we are given information about force components and we use (2.4B) and (2.5) to find the total force. Other times we may need to find the force components based on the force magnitude and a unit vector aligned with the force; then (2.7) and (2.8) are particularly useful.

A Note on Planar and Nonplanar Forces. We call a force in the plane of two coordinate axes a **planar force** or sometimes a **two-dimensional force**; otherwise it is called a **nonplanar force** or a **three-dimensional force**. The discussion of rectangular representation of forces up to this point has been in terms of nonplanar forces. If a force is planar, then all of the nonplanar equations we have just derived can be simplified. If you are working with a force that lies, for example, in the xy plane, $F_z = 0$ and $u_z = 0$. Then, as shown in **Figure 2.5.1.4**, (2.4B) simplifies to

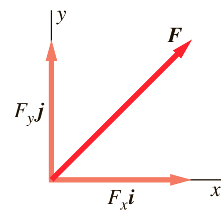


Figure 2.5.1.4 Illustration of rectangular components for a planar force.

$$\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j} \tag{2.10A}$$

and (2.6) becomes

$$\boldsymbol{F} = \|F\|(u_x\boldsymbol{i} + u_y\boldsymbol{j}) \tag{2.10B}$$

Check out the following examples of applications of this material.

- **Example 2.5.1 Rectangular Components of a Nonplanar Force Given Its Line of Action**
- **Example 2.5.2 Representing Nonplanar Forces with Rectangular Coordinates**
- **Example 2.5.3 Representing a Planar Force in Skewed Coordinate System**

EXAMPLE 2.5.1

The cable in **Figure 1** holds up a hinged door. The cable force $\boldsymbol{T}$ has a magnitude of 500 lb and acts along line AB. Consider the cable force acting on the hook at B. Show the component vectors of $\boldsymbol{T}$ graphically and write the force in Cartesian vector notation.

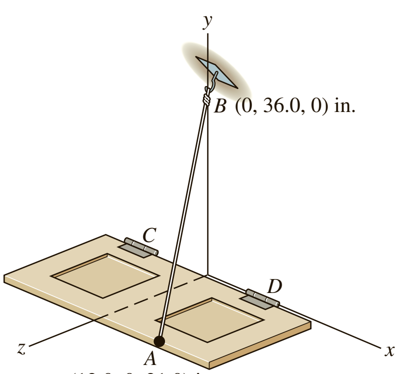


Figure 1 Cable holding up a hinged door.

Goal Show the component vectors of $\boldsymbol{T}$ graphically and write $\boldsymbol{T}$ in terms of $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$.

Given Coordinates of two points along the line of action of the force, the magnitude of the cable force, and a right-handed coordinate system, which is established in **Figure 1**.

Assume No assumptions are needed.

Draw We draw $\boldsymbol{T}$ as applied to the hook at B (**Figure 2**). Inspection of this figure shows us that T_x and T_z are positive, and T_y is negative.

Formulate Equations and Solve

(a) Using Unit Vector: Before we can create a drawing that shows component vectors, we need to calculate the scalar components, which we find using (2.8). In order to apply (2.8), however, we must determine

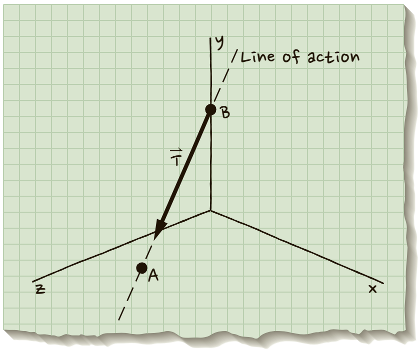


Figure 2 Line of action of cable force $\boldsymbol{T}$ acting on the hook at B.

the unit vector $\boldsymbol{u}$ pointing from B to A along the line of action of $\boldsymbol{T}$. To find $\boldsymbol{u}$, we first find L_{AB} (the distance between points A and B) and then apply (2.7):

$$L_{AB} = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2}$$

$$L_{AB} = \sqrt{(12.0 - 0)^2 + (0 - 36.0)^2 + (21.0 - 0)^2} = 43.4 \text{ in.}$$

$$u_x = \frac{(x_A - x_B)}{L_{AB}} = \frac{12.0 \text{ in.}}{43.4 \text{ in.}} = 0.277$$

$$u_y = \frac{(y_A - y_B)}{L_{AB}} = \frac{-36.0 \text{ in.}}{43.4 \text{ in.}} = -0.830$$

$$u_z = \frac{(z_A - z_B)}{L_{AB}} = \frac{21.0 \text{ in.}}{43.4 \text{ in.}} = 0.485$$

Therefore, application of (2.8) with $\|\boldsymbol{T}\| = 500$ lb results in these scalar components:

$$T_x = \|\boldsymbol{T}\| u_x = (500 \text{ lb})(0.277) = +139 \text{ lb}$$

$$T_y = \|\boldsymbol{T}\| u_y = (500 \text{ lb})(-0.830) = -415 \text{ lb}$$

$$T_z = \|\boldsymbol{T}\| u_z = (500 \text{ lb})(0.485) = +243 \text{ lb}$$

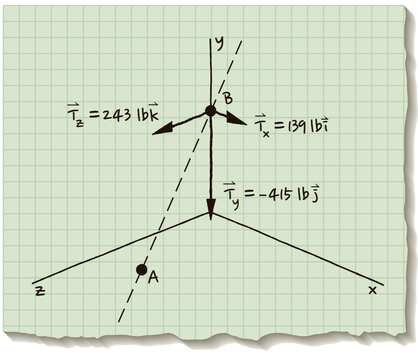


Figure 3 x, y, and z components of the force $\boldsymbol{T}$ acting on the hook at B.

Now that we know the scalar components of $\boldsymbol{T}$, we show them in terms of component vectors in the $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ directions graphically in Figure 3.

Check A check is that the square root of the sum of the squares of the components is equal to the magnitude of the force:

$$\sqrt{(139 \text{ lb})^2 + (-415 \text{ lb})^2 + (243 \text{ lb})^2} = 500 \text{ lb}$$

EXAMPLE 2.5.2

A force is specified as $\boldsymbol{F} = -10 \text{ N}\boldsymbol{i} + 15 \text{ N}\boldsymbol{j} + 5 \text{ N}\boldsymbol{k}$. Draw a graphical representation of the force, and define a unit vector $\boldsymbol{u}$ that lies along the line of action of $\boldsymbol{F}$.

Goal Draw the specified force with respect to a coordinate system and find a unit vector along its line of action.

Given A force in Cartesian vector notation.

Assume No assumptions are needed.

Draw A graphical representation of $\boldsymbol{F}$ can be drawn either as a single vector arrow representing $\boldsymbol{F}$ or as three vector arrows representing the component vectors F_x, F_y, and F_z. We will show both.

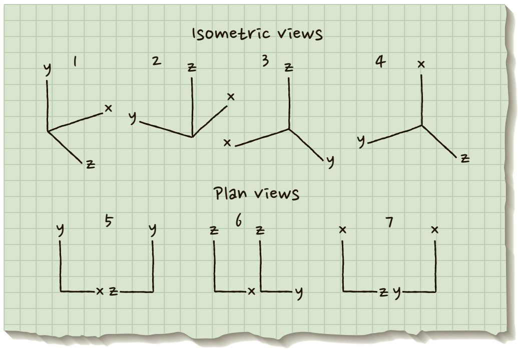


Figure 1 Right-handed coordinate systems.

First we need to decide which view to show. In other words, how should the x, y, and z axes be oriented on the paper to best illustrate the force? A number of possible orientations of right-handed coordinate systems are shown in **Figure 1**. Orientations 1–4 are called isometric views, and orientations 5–7 are plan views.

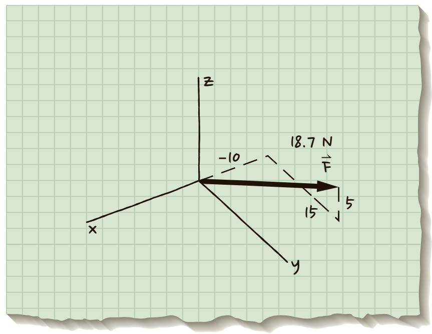


Figure 2 Force $\boldsymbol{F}$ shown in three dimensions.

Approach 1 (Showing Force *F* in Isometric View): **(a)** Several axis orientations would work for clearly showing $\boldsymbol{F}$, and we have decided on orientation 3 (**Figure 2**). A few dashed lines parallel to the axes aid in correctly orienting $\boldsymbol{F}$. The magnitude of $\boldsymbol{F}$ is given by (2.5) as

$$\|\boldsymbol{F}\| = \sqrt{(-10\,\text{N})^2 + (15\,\text{N})^2 + (5\,\text{N})^2} = 18.7\,\text{N}$$

Approach 2 (Showing Components in Isometric View): **(a)** Several of the axis orientations shown in **Figure 1** would work for showing components; we have decided on orientation 2 (**Figure 3**).

Approach 3 (Showing Components in Plan View): **(a)** Because this is a nonplanar force, two plan views are needed, with the y component vector ($\boldsymbol{F}_y = 15\text{ N}\boldsymbol{j}$) shown in both. We choose orientation 5 (**Figure 4**).

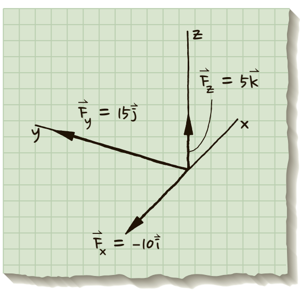


Figure 3 The x, y, and z component vectors of $\boldsymbol{F}$.

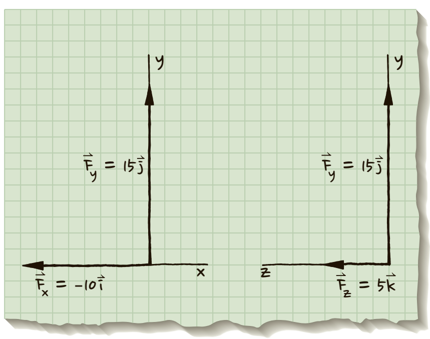


Figure 4 Components of $\boldsymbol{F}$ shown in plan view.

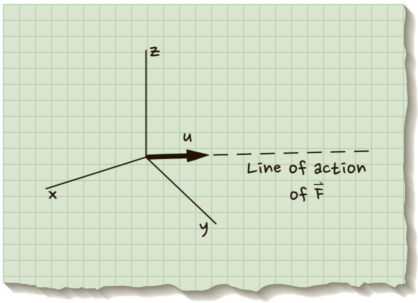


Figure 5 The unit vector $\boldsymbol{u}$ coincides with the line of action of $\boldsymbol{F}$.

All three answers convey the same information and are all equally valid vector representations of the force.

Formulate Equations and Solve (b) Define a unit vector $\boldsymbol{u}$ that lies along the line of action of $\boldsymbol{F}$ as shown in **Figure 5**. Based on (2.9) we write:

$$u_x = \frac{F_x}{\|\boldsymbol{F}\|} = \frac{-10\,\text{N}}{18.7\,\text{N}} = -0.535$$

$$u_y = \frac{F_y}{\|\boldsymbol{F}\|} = \frac{15\,\text{N}}{18.7\,\text{N}} = 0.802$$

$$u_z = \frac{F_z}{\|\boldsymbol{F}\|} = \frac{5\,\text{N}}{18.7\,\text{N}} = 0.267$$

Therefore, the unit vector is $\boldsymbol{u} = -0.535\boldsymbol{i} + 0.802\boldsymbol{j} + 0.267\boldsymbol{k}$

Check Since $\boldsymbol{u}$ is a unit vector, let's check that its magnitude is indeed one: $\sqrt{(-0.535)^2 + (0.802)^2 + (0.267)^2} = 1$! We could also check that the values of the space angles seem reasonable given the orientation of $\boldsymbol{F}$ shown in **Figure 2**; they do.

EXAMPLE 2.5.3

A force $\boldsymbol{F}$, applied via a cable wrapped around a frictionless pulley, acts on a bracket as shown in **Figure 1**. When the cable force that pulls on the bracket at O is resolved into its scalar components in the x and y directions, we write the force as $\boldsymbol{F} = 82.0\ \text{N}\boldsymbol{i} - 493\ \text{N}\boldsymbol{j}$. Resolve the same force into components in the x^* and y^* directions and write your answer as a Cartesian vector in terms of the unit vectors $\boldsymbol{i}^*$ and $\boldsymbol{j}^*$ aligned with the x^* and y^* axes.

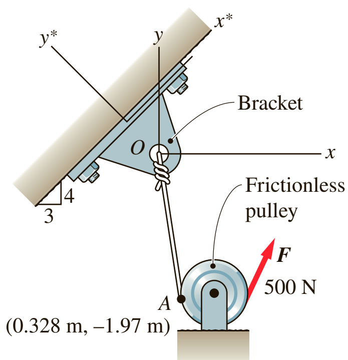


Figure 1 500 N force applied to a bracket through a pulley.

Goal Resolve the force $\boldsymbol{F}$ into components in the x^* and y^* directions, and use a Cartesian vector to represent $\boldsymbol{F}$.

Given Coordinates at two points along the line of action of the force where it acts on the bracket (the origin and point A), 4:3 orientation of the bracket, magnitude of the applied force, and pulley is frictionless. Since the problem statement indicates that $\boldsymbol{F}$ can be represented with unit vectors in the x and y direction, $\boldsymbol{F}$ lies in the xy plane and has no z component.

Assume Since the pulley is frictionless, there is a 500-N force along the entire length of the cable (we will prove this in the chapter

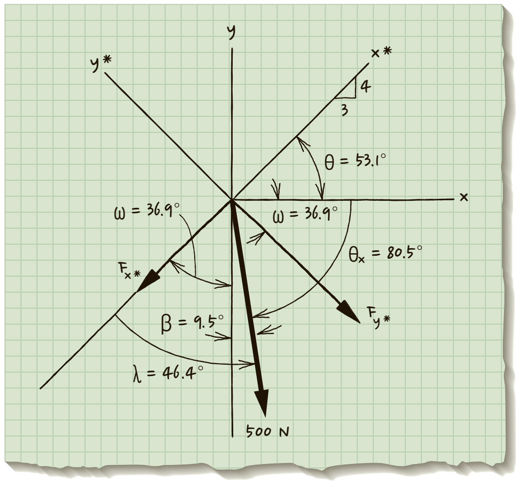


Figure 2 Force acting on bracket is broken into components perpendicular and parallel to the supporting wall.

on equilibirum). Therefore the cable applies a 500-N force to the bracket.

Draw We draw the force applied to the bracket in the x^*y^* coordinate system as shown in **Figure 2**.

Formulate Equations and Solve First we establish some angles. Based on **Figure 2**, θ_x, the angle between $\boldsymbol{F}$ and the x axis, is 80.5°. The angle between the horizontal and the x^* axis (call this angle θ) is found based on the 4:3 orientation of the bracket:

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.1°$$

Knowing θ_x and θ, we can find the angles ω (= 36.9° = 180° – 53.1° – 90°) and λ (= 46.4° = 36.9° + 9.5°) (**Figure 2**). We now have to make a choice in how we solve this problem.

Approach 1: From the geometry of **Figure 2**, we can write the starred components in terms of $||\boldsymbol{F}||$:

$$F_{x*} = -||\boldsymbol{F}||\cos\lambda = -||\boldsymbol{F}||\cos 46.4° = (-500\,\text{N})(0.690) = -345\,\text{N}$$
$$F_{y*} = -||\boldsymbol{F}||\sin\lambda = -||\boldsymbol{F}||\sin 46.4° = (-500\,\text{N})(0.724) = -362\,\text{N}$$

resulting in the vector $\boldsymbol{F} = -345\,\text{N}\,\boldsymbol{i}^* - 362\,\text{N}\,\boldsymbol{j}^*$

Approach 2: We can use the scalar components called out in the problem statement ($F_x = 82.0$ N and $F_y = -493$ N) to find F_{x*} and F_{y*}:

$$F_{x*} = +F_x \cos\theta + F_y \cos\omega$$
$$= (82.0\,\text{N})\cos 53.1° + (-493\,\text{N})\cos 36.9° = -345\,\text{N}$$
$$F_{y*} = -F_x \sin\theta + F_y \sin\omega$$
$$= (-82.0\,\text{N})\sin 53.1° + (-493\,\text{N})\sin 36.9° = -362\,\text{N}$$

Again using unit vectors $\boldsymbol{i}^*$ and $\boldsymbol{j}^*$, we write $\boldsymbol{F}$ in Cartesian vector notation as $\boldsymbol{F} = -345\,\text{N}\,\boldsymbol{i}^* - 362\,\text{N}\,\boldsymbol{j}^*$.

Check We could use Approach 2 as a check for the answer from Approach 1. We can also qualitatively check the relative magnitudes of F_{x*} and F_{y*}. Because we are resolving $\boldsymbol{F}$ into two components that are roughly 45° to $\boldsymbol{F}$, we could expect the two components to be of similar size, which they are!

It is a good idea to calculate the magnitude of $\boldsymbol{F}$ from its components: $\sqrt{(-345\,\text{N})^2 + (-362\,\text{N})^2 + (0\,\text{N})^2} = 500\,\text{N}$, which is the magnitude given in the problem statement.

2.5.2 Space-Angle Representation

Another way of representing $\boldsymbol{F}$ is by specifying its angular orientation relative to a set of right-handed coordinate axes. We specify its angular orientation with the angles θ_x, θ_y, and θ_z from the x, y, and z axes, respectively, as illustrated in Figure 2.5.2.1, and refer to them as the **space angles**. They can be specified either in degrees or in radians, and their numeric values can generally be found from the geometry of the situation. For example, let's return to the cable and pulley system we studied earlier (Figure 2.5.2.2*a*). Using the coordinates of A (x_A, y_A, z_A) and B (x_B, y_B, z_B), we can find the space angles of the cable force acting on

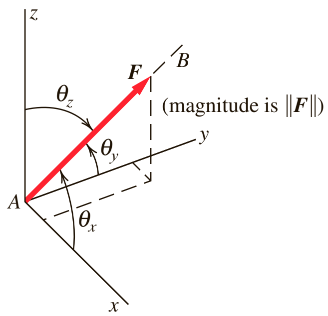


Figure 2.5.2.1 Space angles defined.

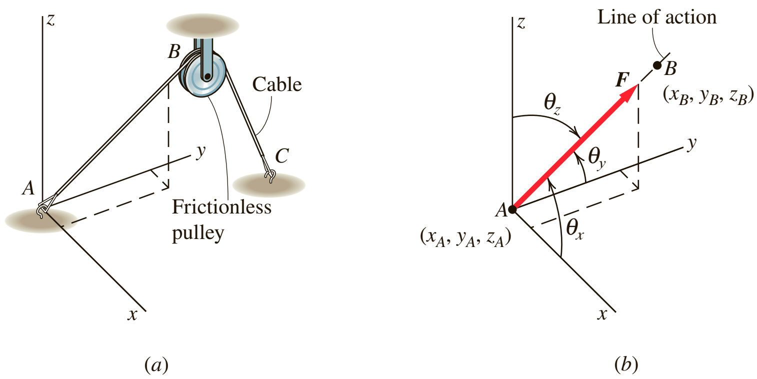


Figure 2.5.2.2 (*a*) A cable passes around a frictionless pulley; (*b*) the cable force acting on the hook at A can be represented in terms of θ_x, θ_y, and θ_z.

the hook at A (**Figure 2.5.2.2*b***) from these relationships:

$$\theta_x = \cos^{-1}\left(\frac{x_B - x_A}{L}\right)$$
$$\theta_y = \cos^{-1}\left(\frac{y_B - y_A}{L}\right) \tag{2.11}$$
$$\theta_z = \cos^{-1}\left(\frac{z_B - z_A}{L}\right)$$

where $L = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$ is the distance from A to B.

When working with space angles, consider that

1. The angles θ_x, θ_y, and θ_z are not independent of one another. They are related by the expression

$$\sqrt{(\cos\theta_x)^2 + (\cos\theta_y)^2 + (\cos\theta_z)^2} = 1 \tag{2.12A}$$

where $\cos\theta_x$, $\cos\theta_x$, and $\cos\theta_z$ are the **direction cosines** and are defined (based on (2.11)) as

$$\cos\theta_x = \left(\frac{x_B - x_A}{L}\right)$$
$$\cos\theta_y = \left(\frac{y_B - y_A}{L}\right) \tag{2.12B}$$
$$\cos\theta_z = \left(\frac{z_B - z_A}{L}\right)$$

Therefore, if we know two of the space angles, the cosine of the third angle is automatically defined by (2.12A), within ±180 degrees.

2. The space angles are always defined as positive angles between zero and 180°.
3. Comparing (2.12B) with (2.7) indicates that direction cosines are equal to the rectangular components of the unit vector ($\boldsymbol{u}$) along the line of action of $\boldsymbol{F}$

$$\cos\theta_x = u_x$$
$$\cos\theta_y = u_y \tag{2.13}$$
$$\cos\theta_z = u_z$$

so that we can write $\boldsymbol{u}$ in terms of the direction cosines as

$$\boldsymbol{u} = (\cos\theta_x\boldsymbol{i} + \cos\theta_y\boldsymbol{j} + \cos\theta_z\boldsymbol{k}) \tag{2.14}$$

Figure 2.5.2.3 provides a graphical representation of $\boldsymbol{u}$ in terms of direction cosines.

Substituting from (2.14) into (2.6) gives:

$$\boldsymbol{F} = \|\boldsymbol{F}\|\boldsymbol{u} = \|\boldsymbol{F}\|(\cos\theta_x\boldsymbol{i} + \cos\theta_y\boldsymbol{j} + \cos\theta_z\boldsymbol{k}) \tag{2.15}$$

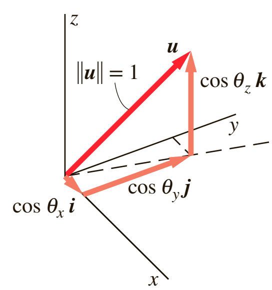


Figure 2.5.2.3 Rectangular component vectors of the unit vector $\boldsymbol{u}$ aligned with $\boldsymbol{F}$.

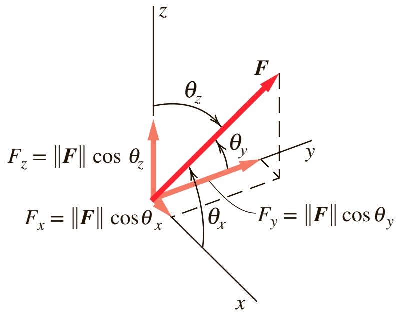


Figure 2.5.2.4 Representation of $\boldsymbol{F}$ in terms of space angles and directions cosines (see (2.15)).

This representation of $\boldsymbol{F}$ in terms of space angles and direction cosines is illustrated in **Figure 2.5.2.4**. The scalar components F_x, F_y, and F_z are projections of $\boldsymbol{F}$ onto the x, y, and z axes, respectively. We can write these projections as

$$\begin{aligned} F_x &= \|\boldsymbol{F}\|\cos\theta_x \\ F_y &= \|\boldsymbol{F}\|\cos\theta_y \\ F_z &= \|\boldsymbol{F}\|\cos\theta_z \end{aligned} \tag{2.16}$$

It is common in engineering analysis to use the scalar components F_x, F_y, F_z of a force to find the direction cosines of the force:

$$\begin{aligned} u_x &= \cos\theta_x = \frac{F_x}{\|\boldsymbol{F}\|} \\ u_y &= \cos\theta_y = \frac{F_y}{\|\boldsymbol{F}\|} \\ u_z &= \cos\theta_z = \frac{F_z}{\|\boldsymbol{F}\|} \end{aligned} \tag{2.17}$$

where $\|\boldsymbol{F}\|$ is given in (2.5). Note the similarity between (2.17) and (2.9). You will use (2.4) to (2.17) in various ways as you work with forces and free-body diagrams. Sometimes the scalar components of a force will be known, and you will be interested in finding the magnitude of that force (where (2.5) will come in handy). At other times, the force direction and magnitude will be known, and you will need to find the scalar components (using (2.8) or (2.16)). In any case, you need to feel comfortable manipulating forces and their components. If you understand the principles behind the various ways of representing a force in terms of its components, finding magnitudes and directions will become straightforward with a little practice.

Planar and Nonplanar Forces If a force is in the plane defined by just two of the coordinate axes, one of the space angles is 90°. For example, if a force is in the xy plane, the angle θ_z between the z axis and the force is 90° (**Figure 2.5.2.5**). For a planar force in the xy plane, (2.12A) simplifies to

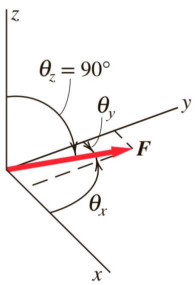


Figure 2.5.2.5 Space angles for a force in the xy plane.

$$\sqrt{(\cos\theta_x)^2 + (\cos\theta_y)^2} = 1 \tag{2.18}$$

since $\theta_z = 90°$ and therefore $\cos 90° = 0$. Equation (2.15) reduces to

$$\boldsymbol{F} = \|\boldsymbol{F}\|\boldsymbol{u} = \|\boldsymbol{F}\|(\cos\theta_x\boldsymbol{i} + \cos\theta_y\boldsymbol{j}) \tag{2.19}$$

Furthermore, for a planar force in the xy plane, $\cos\theta_y = \pm\sin\theta_x$ (the plus or minus sign depending on which quadrant the force lies in). Examples of planar and nonplanar forces specified with space angles are shown in **Figure 2.5.2.6**.

We explore the space angle representation of forces in Examples 2.5.4 and 2.5.5.

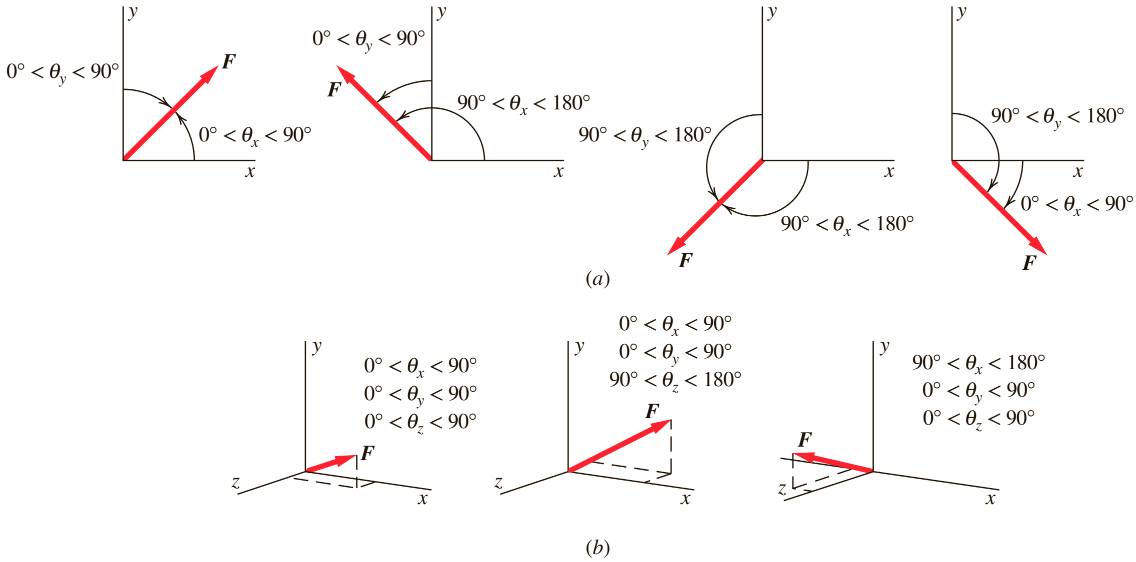


Figure 2.5.2.6 Examples of space angles used to specify (*a*) planar (two-dimensional) forces with $\theta_z = 90°$; (*b*) nonplanar (three-dimensional) forces.

Check out the following examples of applications of this material.

- **Example 2.5.4 Representing Direction of a Planar Force**
- **Example 2.5.5 Scalar Components of a Planar Force**

EXAMPLE 2.5.4

A cable force $\boldsymbol{F}$ acts on a hook as shown in **Figure 1**. Describe the direction of the force relative to horizontal and vertical axes in terms of (a) direction cosines and (b) a unit vector along its line of action.

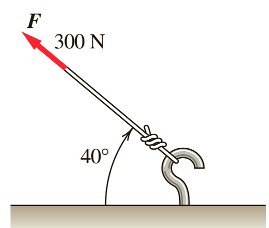


Figure 1 Cable pulling on a hook.

Goal Determine the direction cosines (a) and unit vector (b) associated with a force $\boldsymbol{F}$.

Given The magnitude and angle of orientation of $\boldsymbol{F}$ relative to the horizontal axis.

Assume Assume $\boldsymbol{F}$ is a planar force (in the plane of the page) because we are not given any information about the out-of-plane direction.

Draw We define a coordinate system with x along the horizontal axis and y along the vertical axis (**Figure 2**), and arbitrarily place the origin at the point where the cable attaches to the hook. We draw $\boldsymbol{F}$, label its magnitude, and show the 40° angle, as well as θ_x and θ_y. Since the force is assumed to be planar and in the xy plane, $\theta_z = 90°$.

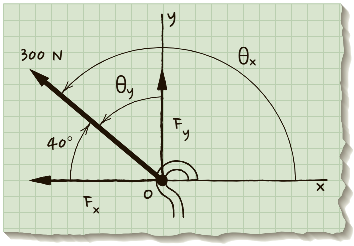


Figure 2 Force acting on hook is broken into its horizontal and vertical components.

Formulate Equations and Solve

(a) We can read the angles θ_x and θ_y from **Figure 2** and take the cosines:

θ_x, the angle between the positive x axis and $\boldsymbol{F}$, is 140°;

$$\cos\theta_x = \cos 140^\circ \quad\Rightarrow\quad \boxed{\cos\theta_x = -0.766}$$

θ_y, the angle between the positive y axis and $\boldsymbol{F}$, is $140^\circ - 90^\circ = 50^\circ$;

$$\cos\theta_y = \cos 50^\circ \quad\Rightarrow\quad \boxed{\cos\theta_y = 0.643}$$

θ_z, the angle between the positive z axis and $\boldsymbol{F}$, is 90° (because the force is in the xy plane);

$$\cos\theta_z = \cos 90^\circ \quad\Rightarrow\quad \boxed{\cos\theta_z = 0}$$

(b) By (2.14), a unit vector $\boldsymbol{u}$ along the line of action of a force is $\boldsymbol{u} = \cos\theta_x\boldsymbol{i} + \cos\theta_y\boldsymbol{j} + \cos\theta_z\boldsymbol{k}$. We then use the answer for (a) to write $\boldsymbol{u} = -0.766\boldsymbol{i} + 0.643\boldsymbol{j}$.

Check We can check that the direction cosines obey (2.12A):

$$\sqrt{(-0.766)^2 + (0.643)^2 + (0)^2} = 1$$

A qualitative check involves observing that the x-component of $\boldsymbol{u}$ is in the negative x direction and the y-component is in the positive y direction, and that the magnitude of the x-component is somewhat larger than that of the y-component. Inspection of **Figure 1** or **2** shows that these observations are consistent with the physical problem.

Comment: $\boldsymbol{F}$ in this example lies in a coordinate plane (in this case, the xy plane) and so is a planar force. Therefore, we could have used the vector notation equations using only x and y terms ((2.18) and (2.19)). We chose not to do this and instead worked with the equations for nonplanar force representation, noting where some terms are zero because the force is planar. The advantage of this approach is that with one general set of equations we can write any force in vector notation, regardless of whether it is nonplanar or planar in any of the three coordinate planes.

EXAMPLE 2.5.5

A cable force $\boldsymbol{F}$ acts on a hook as shown in **Figure 1**. Describe the direction of the force in terms of its scalar components and unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$.

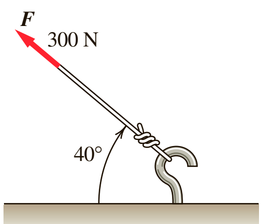


Figure 1 Cable pulling on a hook.

Goal Determine scalar components associated with a force $\boldsymbol{F}$, and represent the force as a vector.

Given The magnitude and angle of orientation of $\boldsymbol{F}$ relative to the horizontal axis.

Assume Assume $\boldsymbol{F}$ is a planar force (in the plane of the page) because we are not given any information about the out-of-plane direction.

Draw We define a coordinate system with x along the horizontal axis and y along the vertical axis (**Figure 2**), and arbitrarily place the origin at the point where the cable attaches to the hook. We draw $\boldsymbol{F}$, label its magnitude, and show the 40° angle, as well as θ_x and θ_y. Since the force is assumed to be planar and in the xy plane, $\theta_z = 90°$.

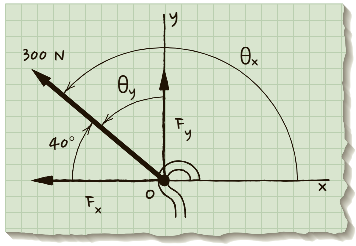


Figure 2 Force acting on hook is broken into its horizontal and vertical components.

Formulate Equations and Solve It is not uncommon for there to be more than one approach to setting up and solving equations. The selection of an approach is often based on personal preference. As you will see, some approaches involve less work than others. Regardless of which approach you use, you should get the same answer.

Approach 1: We can write $\boldsymbol{F} = F_x + F_y + F_z$ and find each of the component vectors. Because $\boldsymbol{F}$ lies in the xy plane, its z component vector is zero ($F_z = 0$).

We have added the two nonzero component vectors F_x and F_y to our drawing **(Figure 2)**.

Based on the development of the direction cosines in Example 2.5.4, we can write that $\theta_x = 140°$, $\theta_y = 50°$, $\theta_z = 90°$, and therefore by (2.16)

$$F_x = ||\boldsymbol{F}||\cos 140° = (300\,\text{N})(-0.766) = -230\,\text{N}$$
$$F_y = ||\boldsymbol{F}||\cos 50° = (300\,\text{N})(0.643) = 193\,\text{N}$$
$$F_z = ||\boldsymbol{F}||\cos 90° = (300\,\text{N})(0) = 0\,\text{N}$$

and, $\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j} = -230\text{ N }\boldsymbol{i} + 193\text{ N }\boldsymbol{j}$

Approach 2: Based on (2.15), the product of the magnitude of $\boldsymbol{F}$ and the unit vector $\boldsymbol{u}$ (found in Example 2.5.4) expresses the force in terms of its rectangular components.

$$\boldsymbol{F} = ||\boldsymbol{F}||\boldsymbol{u} = (300\,\text{N})(-0.766\,\boldsymbol{i} + 0.643\,\boldsymbol{j})$$
$$\Rightarrow \boldsymbol{F} = -230\,\text{N}\,\boldsymbol{i} + 193\,\text{N}\boldsymbol{j}$$

Approach 3: Drawing $\boldsymbol{F}$ to scale and in its proper orientation, we can then read the values of F_x and F_y (**Figure 3**) as −230 N and 193 N, respectively. The accuracy of this graphical approach depends on the precision of the drawing.

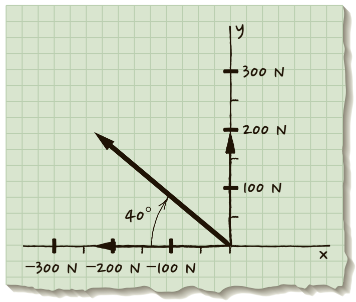


Figure 3 Determining force components using a graphical approach.

Check This answer can be checked by calculating the magnitude of $\boldsymbol{F}$ from its components. Using (2.5), we find $\sqrt{(-230\text{ N})^2 + (139\text{ N})^2 + (0)^2} =$ 300 N. A less quantitative check would be to look at the relative sizes of $||\boldsymbol{F}_x||$ and $||\boldsymbol{F}_y||$ both are smaller than $||\boldsymbol{F}||$ (which is to be expected), and $||\boldsymbol{F}_x|| > ||\boldsymbol{F}_y||$ (which is also to be expected because we are dealing with an angle that, relative to the horizontal, is less than 45°).

2.5.3 Spherical-Angle Representation

Another way to describe the direction of a force $\boldsymbol{F}$ is to use the two angles θ and ϕ associated with spherical coordinates, as illustrated in **Figure 2.5.3.1*a***. The spherical angle θ defines the sweep from the x axis to the projection of $\boldsymbol{F}$ onto the xy plane. (You can think of the projection as the "shadow" that $\boldsymbol{F}$ would cast on the xy plane if a light source were sitting far out on the z axis. This shadow is shown as a gray line in the figure.) The spherical angle ϕ defines the sweep from the z axis to $\boldsymbol{F}$, and you may recognize that it is the space angle θ_z. The angles θ and ϕ can be specified either in degrees or in radians.

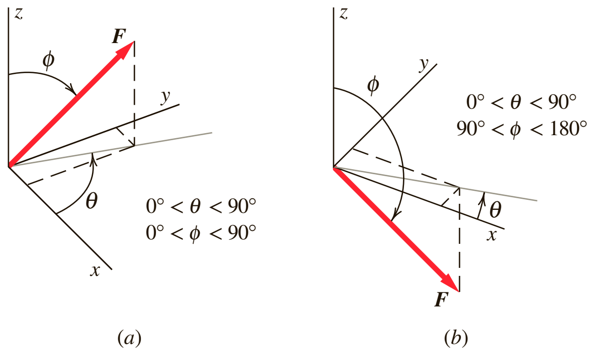


Figure 2.5.3.1 (*a*) Spherical angles (θ, ϕ) defined; (*b*) example of spherical angles defining the direction of a force with $\phi > 90°$.

When working with angles θ and ϕ, consider that:

1. The angle θ is a positive angle between zero and 360° ($0 \le \theta \le 2\pi$), measured counterclockwise from the x axis. The angle ϕ is between zero and 180° ($0 \le \phi \le \pi$), and is measured from the z axis. A force directed with θ between zero and 90° and ϕ between 90° and 180° is shown in **Figure 2.5.3.1*b***.
2. The spherical angles ϕ and θ and the space angles θ_x, θ_y, and θ_z are related to one another by the expressions

$$\begin{aligned} \cos\theta_x &= (\sin\phi)(\cos\theta) \\ \cos\theta_y &= (\sin\phi)(\sin\theta) \\ \cos\theta_z &= (\cos\theta) \end{aligned} \tag{2.20A}$$

$$\cos\phi = (\cos\theta_z) \qquad \tan\theta = \frac{\cos\theta_y}{\cos\theta_x} \tag{2.20B}$$

If a force is in the xy plane (and therefore is a planar force), ϕ is 90°. Examples of planar forces defined by spherical angles are illustrated in Figure 2.5.3.2.

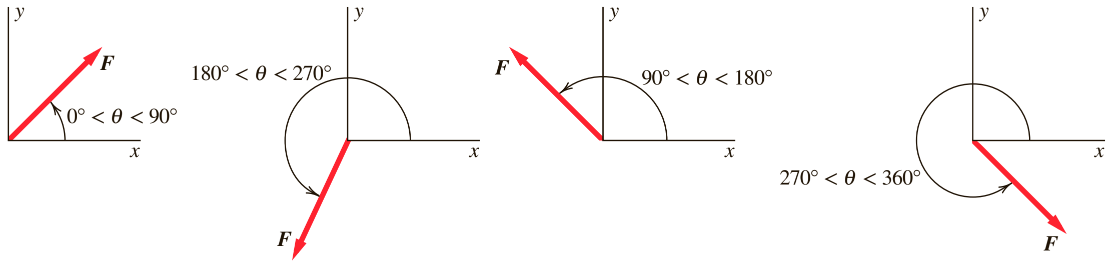


Figure 2.5.3.2 Examples of planar forces defined in terms of spherical angles. All of these forces are in the xy plane; therefore $\phi = 90°$.

Check out the following examples of applications of this material.

- **Example 2.5.6 Representing a Planar Force with Spherical Coordinates**
- **Example 2.5.7 Representing Nonplanar Forces with Spherical Angles**

EXAMPLE 2.5.6

A cable force $\boldsymbol{F}$ acts on a hook as shown in Figure 1. Represent the force in terms of the spherical coordinate angles ϕ and θ.

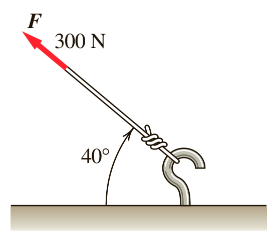


Figure 1 Cable pulling on a hook.

Goal Represent the force $\boldsymbol{F}$ in spherical coordinates.

Given The magnitude and angle of orientation of $\boldsymbol{F}$ relative to the horizontal axis.

Assume Assume $\boldsymbol{F}$ is a planar force (in the plane of the page) because we are not given any information about the out-of-plane direction.

Draw We define a coordinate system with x along the horizontal axis and y along the vertical axis (Figure 2), and arbitrarily place the origin at the point where the cable attaches to the hook. We draw $\boldsymbol{F}$, label its magnitude, and show the 40° angle, as well as θ_x and θ_y. Since the force is assumed to be planar and in the xy plane, $\theta_z = 90°$.

Formulate Equations and Solve We do not need to formulate and solve any equations to find the spherical coordinate angles ϕ and θ. A drawing that shows these answers is useful (Figure 3).

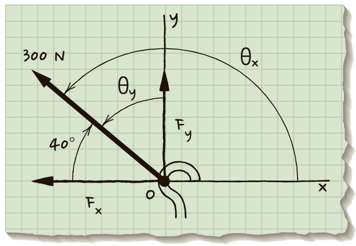


Figure 2 Force acting on hook is broken into its horizontal and vertical components.

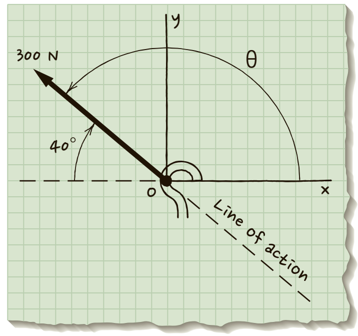


Figure 3 Describing the 300 N force using spherical coordinates.

Because $\boldsymbol{F}$ is a planar force, $\phi = 90°$, the angle θ is measured counterclockwise from the positive x axis. From the geometry shown in the figure, we see that $\theta = 180° - 40° = 140°$.

The magnitude of the force is given in the figure: $\|\boldsymbol{F}\| = 300$ N

Check As a check, (2.20B) could be used to convert from the space angles θ_x, θ_y, and θ_z found in Example 2.5.4 to the angles ϕ and θ.

EXAMPLE 2.5.7

Determine the spherical angles that describe the direction of the force $\boldsymbol{F} = -10\ \text{N}\boldsymbol{i} + 15\ \text{N}\boldsymbol{j} + 5\ \text{N}\boldsymbol{k}$.

Goal Determine angles ϕ and θ that define the direction of $\boldsymbol{F}$.

Given A force in Cartesian vector notation.

Assume No assumptions are needed.

Draw To get a sense of the orientation of $\boldsymbol{F}$ let's start by drawing $\boldsymbol{F}$ relative to the xyz coordinate system as shown in **Figure 1**. The angles θ and ϕ are as defined in **Figure 2.5.3.1*a*** and shown for this example in **Figure 2**. Referring to **Figure 1** leads us to expect $90° < \theta < 180°$ and $0° < \phi < 90°$.

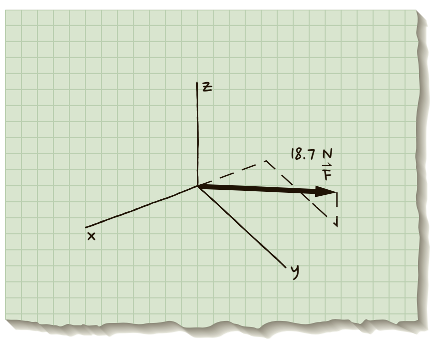


Figure 1 Sketch of $\boldsymbol{F}$.

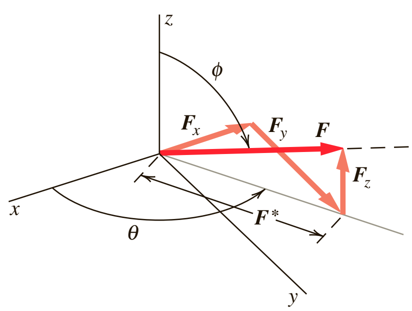


Figure 2 The spherical coordinates that describe $\boldsymbol{F}$.

Formulate Equations and Solve The first approach shown below requires the magnitude of $\boldsymbol{F}$, which we found in Example 2.5.2 to be 18.7 N.

Approach 1 (Find θ and ϕ Directly): Based on the geometry shown in **Figure 2**, we can write

$$\theta = \cos^{-1}\left(\frac{F_x}{F^*}\right) = \cos^{-1}\left(\frac{F_x}{\sqrt{F_x^2 + F_y^2}}\right)$$

$$= \cos^{-1}\left(\frac{-10\,\text{N}}{\sqrt{(-10\,\text{N})^2 + (15\,\text{N})^2}}\right) \Rightarrow \theta = 124°$$

$$\phi = \cos^{-1}\left(\frac{F_z}{\|\boldsymbol{F}\|}\right) = \cos^{-1}\left(\frac{5\,\text{N}}{18.7\,\text{N}}\right) \Rightarrow \phi = 74.5°$$

Approach 2 (Find θ and ϕ Based on the Direction Cosines): We use (2.17) to calculate the direction cosines, and from the direction cosines we find the space angles θ_x, θ_y, and θ_z.

$$\cos\theta_x = \frac{F_x}{\|\boldsymbol{F}\|} = \frac{-10\,\text{N}}{18.7\,\text{N}} = -0.535 \Rightarrow \theta_x = 122.3°$$

$$\cos\theta_y = \frac{F_y}{\|\boldsymbol{F}\|} = \frac{15\,\text{N}}{18.7\,\text{N}} = 0.802 \Rightarrow \theta_y = 36.7°$$

$$\cos\theta_z = \frac{F_z}{\|\boldsymbol{F}\|} = \frac{5\,\text{N}}{18.7\,\text{N}} = 0.267 \Rightarrow \theta_z = 74.5°$$

Now we use (2.20B) to convert θ_x, θ_y, θ_z to the spherical angles θ and ϕ:

$$\theta = \tan^{-1}\left(\frac{\cos 36.7°}{\cos 122°}\right) = \tan^{-1}\left(\frac{0.802}{-0.534}\right) \Rightarrow \theta = 124°$$

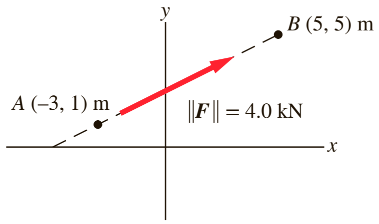
(Remember that $0 \le \theta \le 2\pi$.)

$$\phi = \theta_z \Rightarrow \phi = 74.5°$$

Check The values of θ and ϕ are consistent with our expectations that $90° < \theta < 180°$ and $0° < \phi < 90°$. The two approaches can be used as cross-checks for each other.

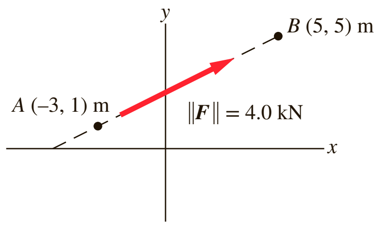
EXERCISES 2.5

2.5.1. [*] The line of action of the 4.0 kN force $\boldsymbol{F}$ runs through the points A and B as shown. Find the unit vector $\boldsymbol{u}$ in the direction of $\boldsymbol{F}$ in terms of unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$.



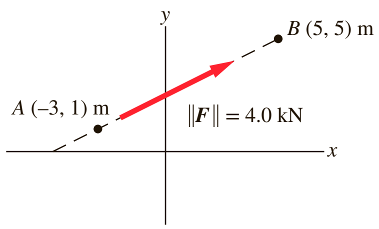
EX 2.5.1

2.5.2. [*] The line of action of the 4.0 kN force $\boldsymbol{F}$ runs through the points A and B as shown. Express $\boldsymbol{F}$ in Cartesian vector notation.



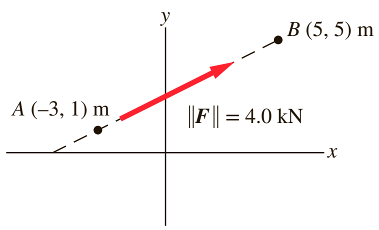
EX 2.5.2

2.5.3. [*] The line of action of the 4.0 kN force $\boldsymbol{F}$ runs through the points A and B as shown. Find the space angles that describe the direction of the force vector.



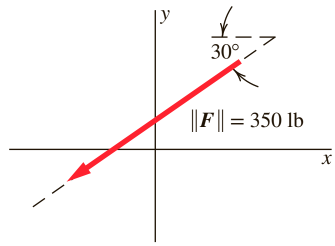
EX 2.5.3

2.5.4. [*] The line of action of the 4.0 kN force $\boldsymbol{F}$ runs through the points A and B as shown. Express $\boldsymbol{F}$ in terms of its magnitude and the angles θ and ϕ using spherical coordinates.



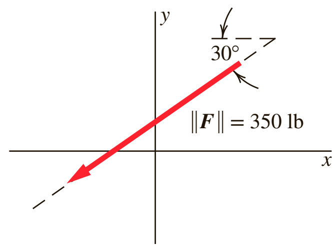
EX 2.5.4

2.5.5. [*] The line of action of the 350 lb force $\boldsymbol{F}$ is oriented at 30° to the horizontal as shown. Find the unit vector $\boldsymbol{u}$ in the direction of $\boldsymbol{F}$ in terms of unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$.



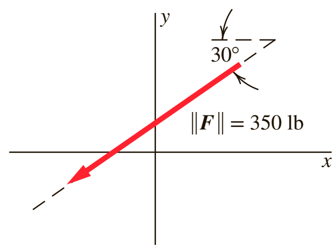
EX 2.5.5

2.5.6. [*] The line of action of the 350 lb force $\boldsymbol{F}$ is oriented at 30° to the horizontal as shown. Express $\boldsymbol{F}$ in terms of the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$.



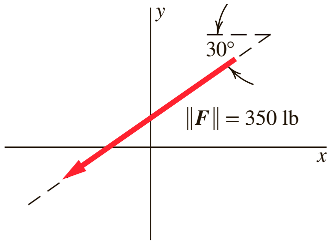
EX 2.5.6

2.5.7. [*] The line of action of the 350 lb force $\boldsymbol{F}$ is oriented at 30° to the horizontal as shown. Find the space angles that describe the direction of the force vector.



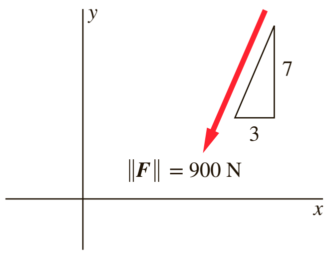
EX 2.5.7

2.5.8. [*] The line of action of the 350 lb force $\boldsymbol{F}$ is oriented at 30° to the horizontal as shown. Express $\boldsymbol{F}$ in terms of its magnitude and the angles θ and ϕ using spherical coordinates.



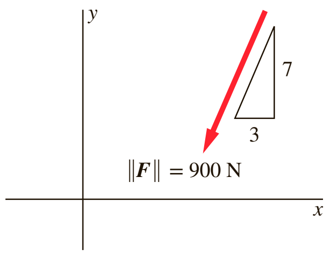
EX 2.5.8

2.5.9. [*] The slope of the 900 N force is specified as shown. Find the unit vector $\boldsymbol{u}$ in the direction of $\boldsymbol{F}$ in terms of unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$.



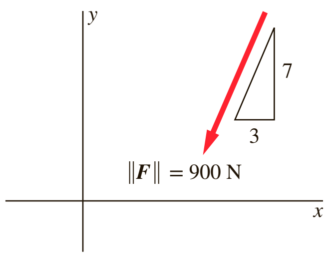
EX 2.5.9

2.5.10. [*] The slope of the 900 N force is specified as shown. Express $\boldsymbol{F}$ in terms of the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$.



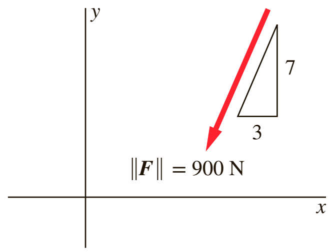
EX 2.5.10

2.5.11. [*] The slope of the 900 N force is specified as shown. Find the space angles that describe the direction of the force vector.



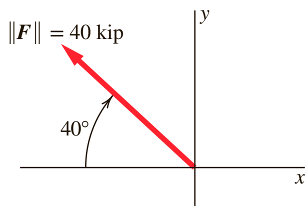
EX 2.5.11

2.5.12. [*] The slope of the 900 N force is specified as shown. Express $\boldsymbol{F}$ in terms of its magnitude and the angles θ and ϕ using spherical coordinates.



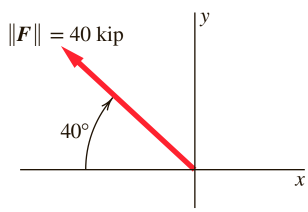
EX 2.5.12

2.5.13. [*] The 40 kip force is oriented at 40° relative to the horizontal, as shown (1 kip = 1000 lb). Find the unit vector $\boldsymbol{u}$ in the direction of $\boldsymbol{F}$ in terms of unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$.



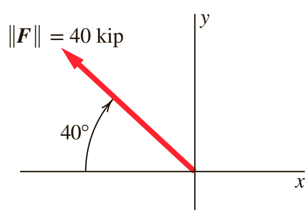
EX 2.5.13

2.5.14. [*] The 40 kip force is oriented at 40° relative to the horizontal, as shown (1 kip = 1000 lb). Express $\boldsymbol{F}$ in terms of the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$.



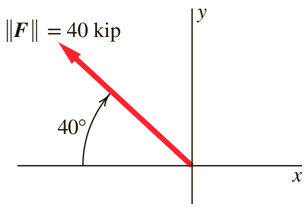
EX 2.5.14

2.5.15. [*] The 40 kip force is oriented at 40° relative to the horizontal, as shown (1 kip = 1000 lb). Find the space angles that describe the direction of the force vector.



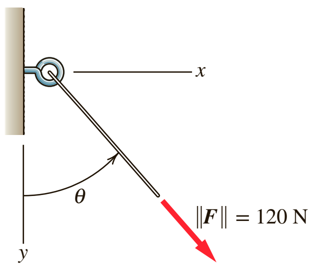
EX 2.5.15

2.5.16. [*] The 40 kip force is oriented at 40° relative to the horizontal, as shown (1 kip = 1000 lb). Express $\boldsymbol{F}$ in terms of its magnitude and the angles θ and ϕ using spherical coordinates.



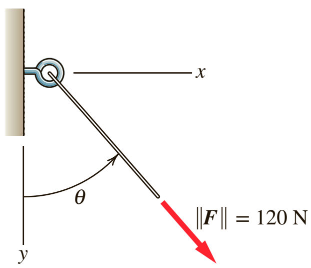
EX 2.5.16

2.5.17. [*] The cable pulls on the eyebolt with a tensile force of 120 N, and $\theta = 30°$. Determine the scalar components of the force with respect to the *xyz* coordinate system. Redraw the figure, showing the tension force in terms of its rectangular component vectors.



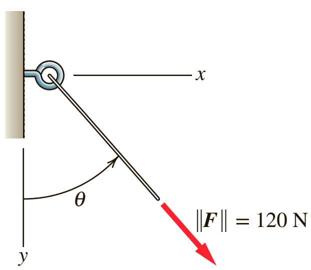
EX 2.5.17

2.5.18. [*] The cable pulls on the eyebolt with a tensile force of 120 N, and $\theta = 30°$. Find the unit vector $\boldsymbol{u}$ in the direction of $\boldsymbol{F}$ in terms of unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$.



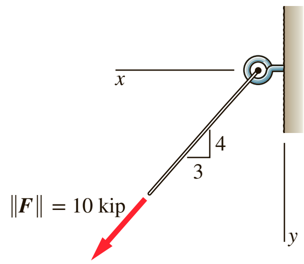
EX 2.5.18

2.5.19. [*] The cable pulls on the eyebolt with a tensile force of 120 N, and $\theta = 30°$. Determine the space angles that describe the direction of the force vector.



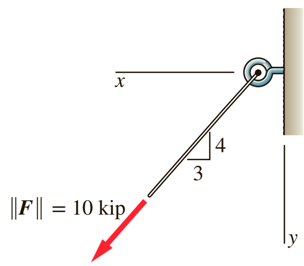
EX 2.5.19

2.5.20. [*] The cable pulls on the eyebolt with a tensile force of 10 kip as shown. Determine the unit vector along the line of action of the force.



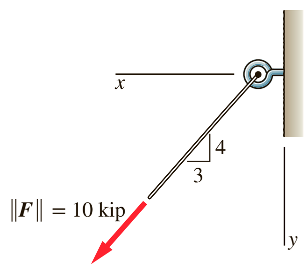
EX 2.5.20

2.5.21. [*] The cable pulls on the eyebolt with a tensile force of 10 kip as shown. Determine the scalar components of the force with respect to the *xyz* coordinate system. Redraw the figure, showing the tension force in terms of its rectangular component vectors.



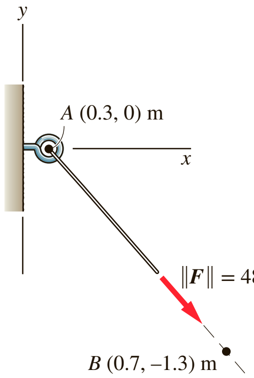
EX 2.5.21

2.5.22. [*] The cable pulls on the eyebolt with a tensile force of 10 kip as shown. Find the space angles that describe the direction of the force vector.



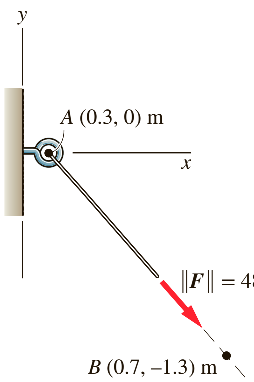
EX 2.5.22

2.5.23. [*] The cable pulls on the eyebolt with a tensile force of 480 N. Coordinates along the line of action of the force are as given in the figure. Determine the unit vector along the line of action of the force.



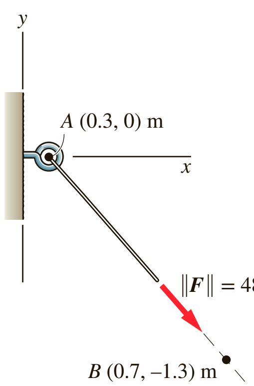
EX 2.5.23

2.5.24. [*] The cable pulls on the eyebolt with a tensile force of 480 N. Coordinates along the line of action of the force are as given in the figure. Determine the scalar components of the force with respect to the *xyz* coordinate system.



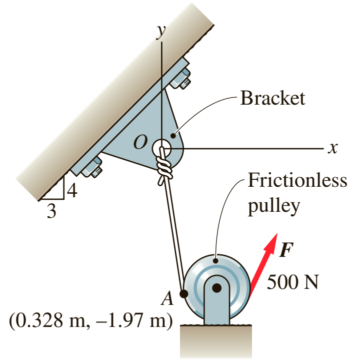
EX 2.5.24

2.5.25. [*] The cable pulls on the eyebolt with a tensile force of 480 N. Coordinates along the line of action of the force are as given in the figure. Find the space angles that describe the direction of the force vector.



EX 2.5.25

2.5.26. [*] A cable wrapped around a frictionless pulley pulls on the bracket at O with a force of 500 N. Express $\boldsymbol{F}_{OA}$ in terms of the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$.



EX 2.5.26

2.5.27. [*] Researchers tested the seismic resistance of a scaled model of a bridge bent by pulling the specimen to the side with force $\boldsymbol{F}$, which was applied via a cable as shown. If the line of action of $\boldsymbol{F}$ is $\boldsymbol{u} = 0.993\boldsymbol{i} - 0.120\boldsymbol{j}$, and the horizontal component of $\boldsymbol{F}$ required to pull the bent over is 37.7 kip, determine $\|\boldsymbol{F}\|$.

Sharon Wood

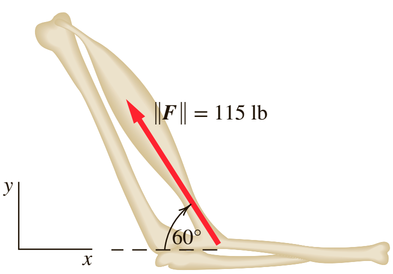
EX 2.5.27

2.5.28. [*] A student, training for a triathlon, lifts a 10 lb weight and generates a 115 lb force in the biceps brachii as shown.

a. Determine the unit vector along the line of action of, and with the same sense as, the muscle force.

b. Find the space angles that describe the direction of the force vector.

c. Write the force in Cartesian vector notation.



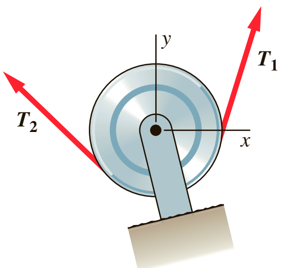
EX 2.5.28

2.5.29. **[*]** After a fire destroyed a building's roof, shoring consisting of steel pipes was used to stabilize the walls as shown in **Figure 1**. The shoring is attached to the wall at 20 ft above the ground and at 30° from vertical and lies in a plane perpendicular to the wall. The force applied by the shoring acts along the along the axis of the pipe and applies a 2250 lb horizontal force to the wall at B. Find the Cartesian vector $\boldsymbol{F}$ that describes the force applied by the shoring to the wall.

Thalia Anagnos

Figure 1 Shoring stabilizing building.

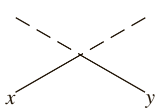
2.5.30. **[*]** The frictionless pulley shown is such that the magnitude of $\boldsymbol{T}_1$ is equal to the magnitude of $\boldsymbol{T}_2$. For $\boldsymbol{T}_1 = 25.9\ \text{kN}\boldsymbol{i} + 96.6\ \text{kN}\boldsymbol{j}$ and $\boldsymbol{T}_2 = -90.6\ \text{kN}\boldsymbol{i} + 42.3\ \text{kN}\boldsymbol{j}$, prove that the magnitudes of these two forces are equal.



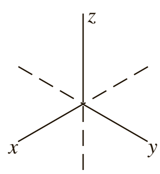
EX 2.5.30

2.5.31. **[*]** For each of the partial coordinate systems shown, add the missing axis so as to form a right-handed orthogonal coordinate system. Select which of the complete coordinate systems is a right-handed orthogonal coordinate system.

Example



Answer



EX 2.5.31

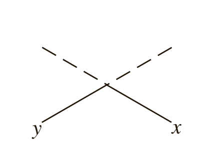
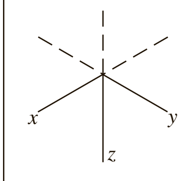
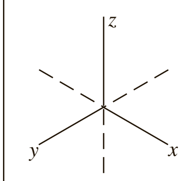
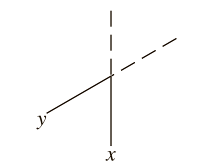
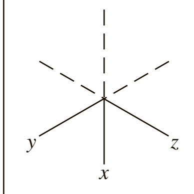
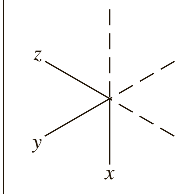
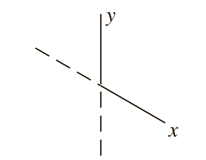
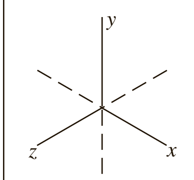
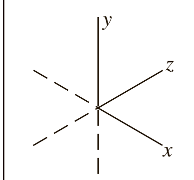
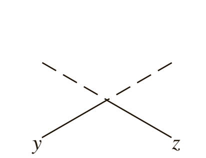
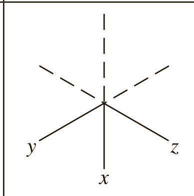
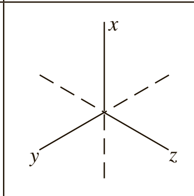
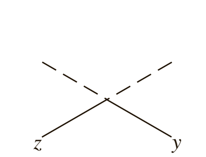
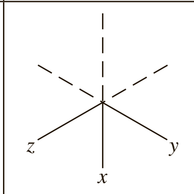
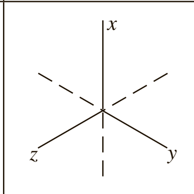
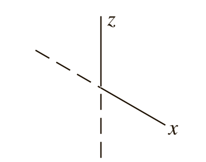
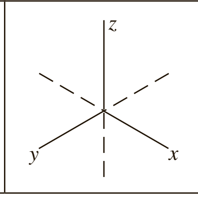
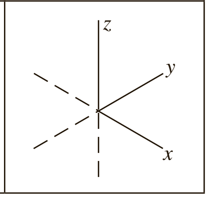
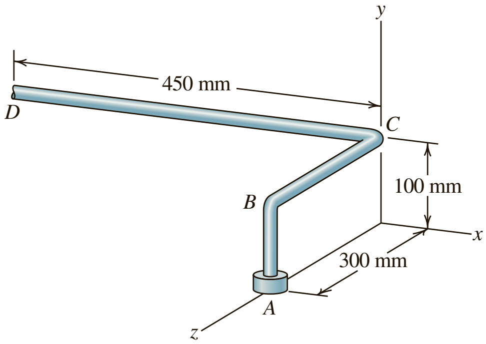
Partial coordinate system	Choice 1	Choice 2
 (a)		
 (b)		
 (c)		
 (d)		
 (e)		
 (f)		

Table of EX 2.5.31

2.5.32. **[*]** For the pipe assembly shown determine the unit vector that lies along a line between

a. points A and B
b. points A and C
c. points A and D



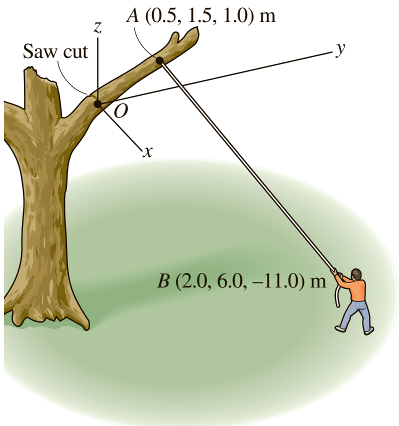
EX 2.5.32

2.5.33. [*] The tree trimmer is pulling on the branch with a force of 150 N.

a. find the unit vector along the line of action of this force

b. determine the *x, y,* and *z* components of the force pulling on the branch and write the force in Cartesian vector notation

c. how would your answer change if you wrote the force vector acting on the man instead of on the branch?



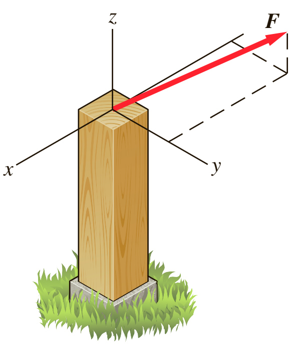
EX 2.5.33

2.5.34. [*] A student applies a force $\boldsymbol{F} = (-16 \text{ lb } \boldsymbol{i} + 4 \text{ lb } \boldsymbol{j} + 6 \text{ lb } \boldsymbol{k})$ to the top of a fence post.

a. The post will fail by tipping if the magnitude of the applied force is greater than 50 lb. Determine if the student should be worried about the post failing.

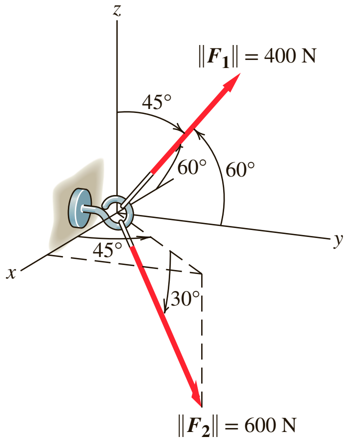
b. Find the unit vector along the line of action of the force.

c. Find the space angles that describe the direction of the force vector.



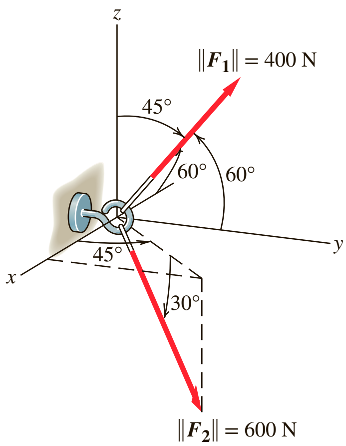
EX 2.5.34

2.5.35. [*] Two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are applied to the hook, as shown. Express $\boldsymbol{F}_1$ in Cartesian vector notation. Note the orientation of the axes.



EX 2.5.35

2.5.36. [*] Two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are applied to the hook, as shown. Express $\boldsymbol{F}_2$ in Cartesian vector notation. Note the orientation of the axes.



EX 2.5.36

2.5.37. [**] The *x* component of a 120-lb force $\boldsymbol{F}$ is twice as large as the *y* component and the *z* component is zero.

a. Find the unit vector along the line of action of the force.

b. Express $\boldsymbol{F}$ in terms of the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$.

c. Find the space angles that describe the direction of the force vector.

d. Express $\boldsymbol{F}$ in terms of its magnitude and the angles θ and ϕ.

2.5.38. [**] If the magnitude of the *x* component of a 240-N force is three times as large as the magnitude of the *y* component and the force lies in the *xy* plane, determine all possible values of space angle θ_x.

2.5.39. [**] If the magnitude of the y component of a 25-lb force is five times as large as the magnitude of the x component and the force lies in the xy plane, determine all possible values of space angle θ_x.

2.5.40. [**] A 500-N force $\boldsymbol{F}$ is applied to the post shown.

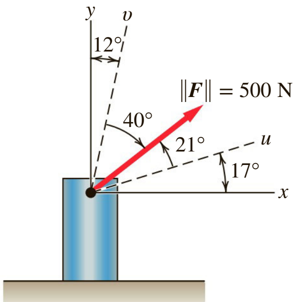
a. Determine the magnitude of the x and y components of $\boldsymbol{F}$.

b. Determine the magnitude of the u and v components of $\boldsymbol{F}$.

c. Comment on whether the following are true or false mathematical statements:

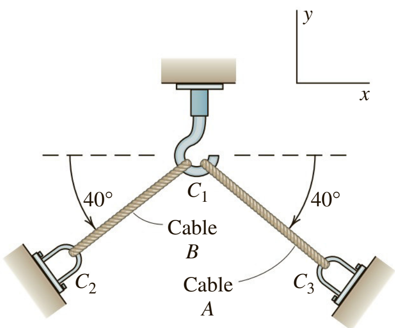
$$\|\boldsymbol{F}\| = \sqrt{(F_x)^2 + (F_y)^2}$$

$$\|\boldsymbol{F}\| = \sqrt{(F_u)^2 + (F_v)^2}$$



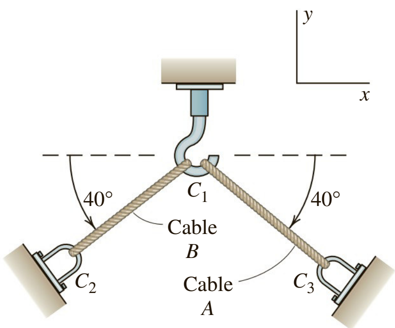
EX 2.5.40

2.5.41. [**] Cables A and B act on the hook at C_1, as shown. The tension in Cable A is 500 lb and the tension in Cable B is 400 lb. Express in Cartesian vector notation the force that Cable A applies on the hook at C_1.



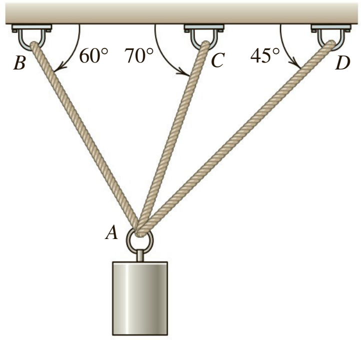
EX 2.5.41

2.5.42. [**] Cables A and B act on the hook at C_1 as shown. The tension in Cable A is 500 lb and the tension in Cable B is 400 lb. Express in Cartesian vector notation the force that Cable B applies on the bracket at C_2.



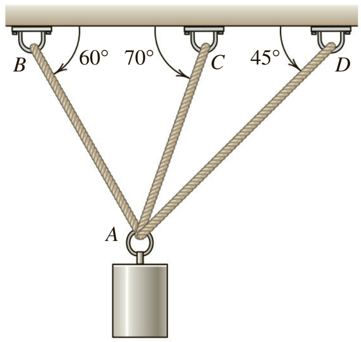
EX 2.5.42

2.5.43. [**] The tension in the supporting cable AB is 20 kip. Write the cable tension force in vector notation.



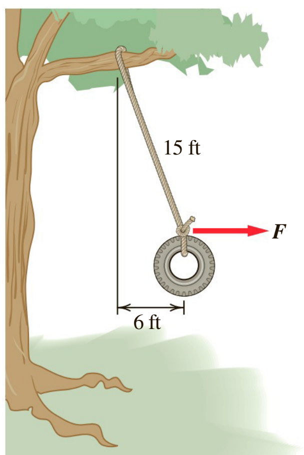
EX 2.5.43

2.5.44. [**] The tension in the supporting cable AD is 10 kN. Write the cable tension force acting on the ring at A in Cartesian vector notation.



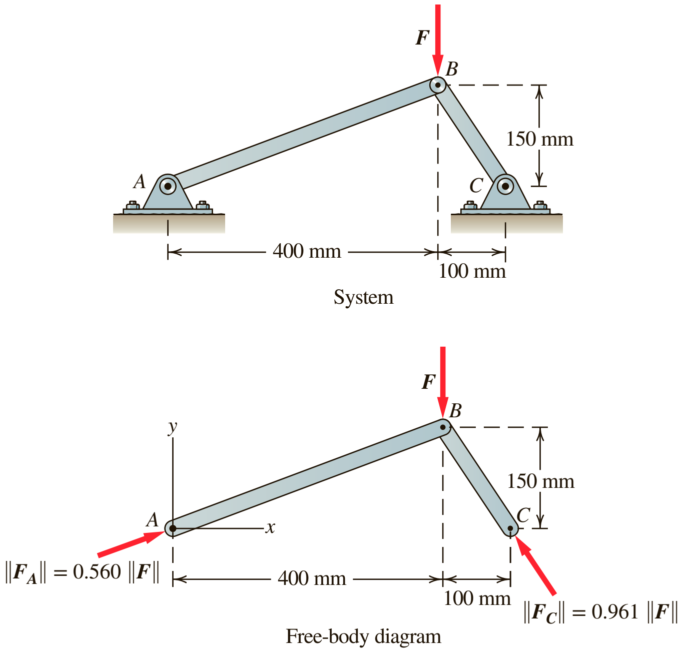
EX 2.5.44

2.5.45. [**] A tire swing hanging on a 15 ft rope is pulled 6 ft to the side by a 22-lb horizontal force $\boldsymbol{F}$ as shown. The horizontal component of the force in the rope holding up the swing is equal to $\|\boldsymbol{F}\|$. Determine the tension force holding up the swing, and write it in Cartesian vector notation.



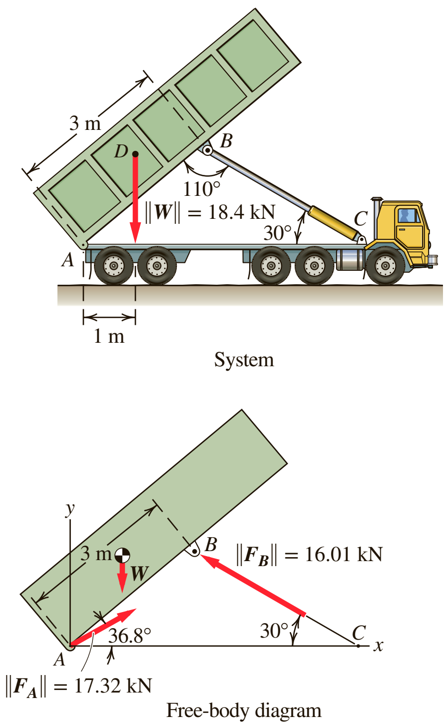
EX 2.5.45

2.5.46. [**] Structure ABC is loaded by a force $\boldsymbol{F}$. The free-body diagram shows the forces acting on links AB and BC. Express each force in terms of its rectangular component vectors.



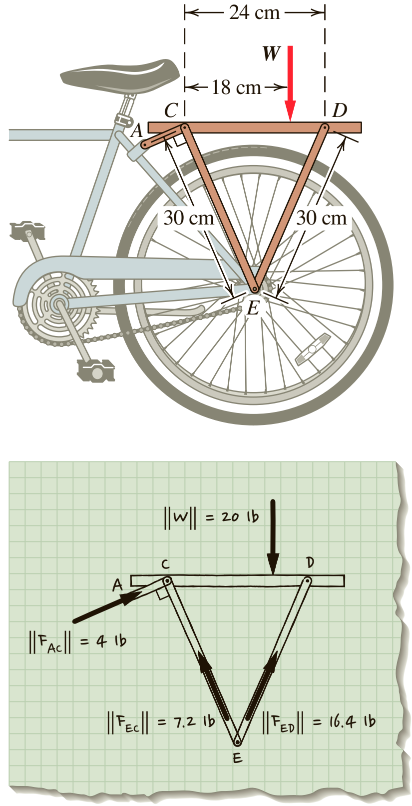
EX 2.5.46

2.5.47. [**] The free-body diagram shows the forces acting on the bed of a dump truck. Express each force in terms of its rectangular component vectors.



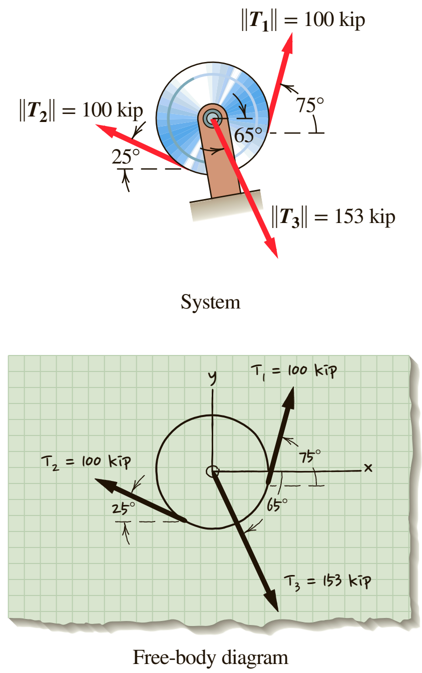
EX 2.5.47

2.5.48. [**] The free-body diagram shows the forces acting on a bike rack. Express each force in terms of its rectangular component vectors.



EX 2.5.48

2.5.49. [**] The free-body diagram shows the forces acting on a frictionless pulley. Express each force in terms of its rectangular component vectors.



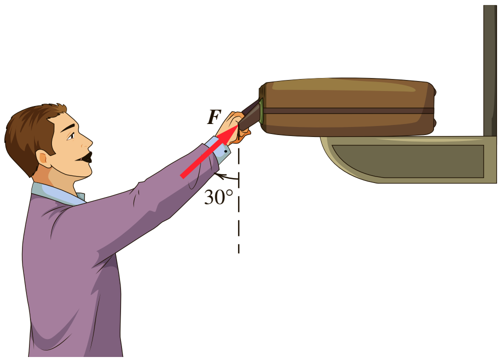
EX 2.5.49

2.5.50. [**] The airline passenger is pushing a suitcase into an overhead bin. The vertical component of the pushing force is 20 lb.

a. What is the total force $\boldsymbol{F}$ the passenger is pushing with?

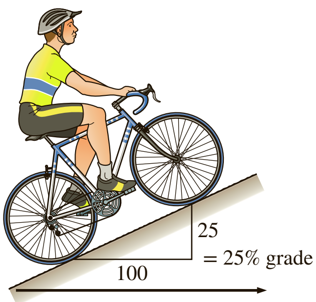
b. Write the force $\boldsymbol{F}$ in Cartesian vector notation.

c. Assuming the vertical component remains the same, how does $\boldsymbol{F}$ change, if the passenger increases his arm angle to 45°?



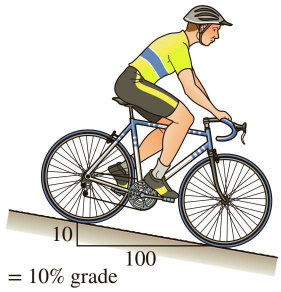
EX 2.5.50

2.5.51. [**] The bicycle and cyclist have a combined weight of 200 lb. As the bicycle moves up the 25% grade, how much of the 200-lb gravity force is directed down the incline? How much is directed perpendicular to the incline? "Percent grade" defines the rise (upward movement) relative to the run (horizontal movement). For example, a 25% grade means that for every 100 ft of horizontal movement (run) there is 25 ft of upward movement (rise).



EX 2.5.51

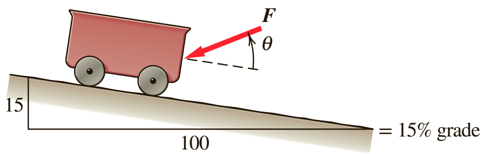
2.5.52. [**] The bicyle and cyclist have a combined weight of 200 lb. As the bicycle moves down the 10% grade how much of the 200-lb gravity force is directed down the incline? How much is directed perpendicular to the incline?



EX 2.5.52

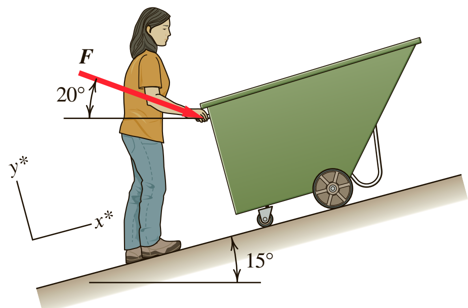
2.5.53. [**] A cart weighing 400 N is pushed up a 15% grade at a constant speed by the force $\boldsymbol{F}$ shown. Find the scalar components of the 400 N cart weight that are parallel and perpendicular to the slope.

"Percent grade" defines the rise (upward movement) relative to the run (horizontal movement). For example, a 25% grade means that for every 100 m of horizontal movement (run) there is 25 m of upward movement (rise).



EX 2.5.53

2.5.54. [**] In order to move a refuse container up a ramp at a constant velocity, a worker applies a 250-N force $\boldsymbol{F}$, as shown. Determine the components of $\boldsymbol{F}$ that are parallel to the ramp (x^* direction) and normal to the ramp (y^* direction).



EX 2.5.54

2.5.55. [**] Review the list of forces in the table shown.

a. Which of the forces have the same magnitude?

b. Which items in the table represent the same force?

Item	Force
(1)	$\boldsymbol{F} = 28\text{ kN}\boldsymbol{i} + 18\text{ kN}\boldsymbol{j} - 38\text{ kN}\boldsymbol{k}$
(2)	$\lVert\boldsymbol{F}\rVert = 50.5\text{ kN}, \cos\theta_x = -0.554, \cos\theta_y = 0.356, \cos\theta_z = 0.752$
(3)	$\lVert\boldsymbol{F}\rVert = 50.5\text{ kN}, \theta_x = 56.4°, \theta_y = 69.1°, \theta_z = 138.8°$
(4)	$\boldsymbol{F} = -28\text{ kN}\boldsymbol{i} + 18\text{ kN}\boldsymbol{j} + 38\text{ kN}\boldsymbol{k}$
(5)	$\lVert\boldsymbol{F}\rVert = 50.5\text{ kN}, \cos\theta_x = 0.554, \cos\theta_y = 0.356, \cos\theta_z = -0.752$
(6)	$\lVert\boldsymbol{F}\rVert = 50.5\text{ kN}, \theta_x = 124°, \theta_y = 69.1°, \theta_z = 41.2°$

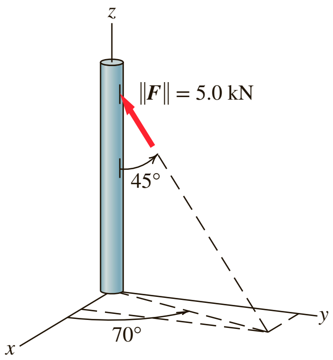
Table 2.5.55

2.5.56. [**] A 5.0-kN force $\boldsymbol{F}$ acts on the vertical pole shown.

a. Find the unit vector $\boldsymbol{u}$ in the direction of $\boldsymbol{F}$ in terms of right-handed orthogonal unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$.

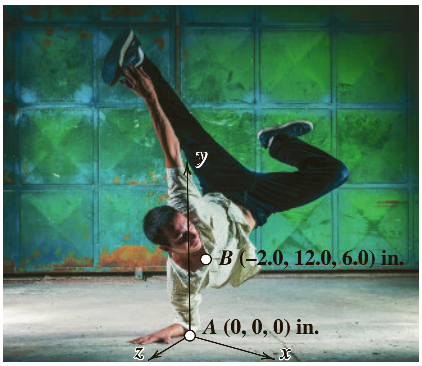
b. Express $\boldsymbol{F}$ in Cartesian vector notation (use your answer from **a** as the starting point).

c. Find the space angles that describe the direction of $\boldsymbol{F}$.



EX 2.5.56

2.5.57. [**] A break dancer is performing a trick in which he balances his entire body on one arm as shown. A compressive force $\boldsymbol{F}$ of magnitude 190 lb acts in his upper arm along a line defined by A at his elbow and B at his shoulder. Use Cartesian vector notation to express the compressive force acting on his shoulder at B.



Notorious91/Getty Images

EX 2.5.57

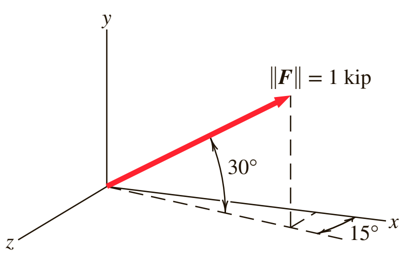
2.5.58. [**] Consider the 1-kip force $\boldsymbol{F}$ shown. Write this force

a. in terms of a unit vector along the line of action of $\boldsymbol{F}$, and a magnitude

b. in Cartesian vector notation

c. as magnitude and space angles

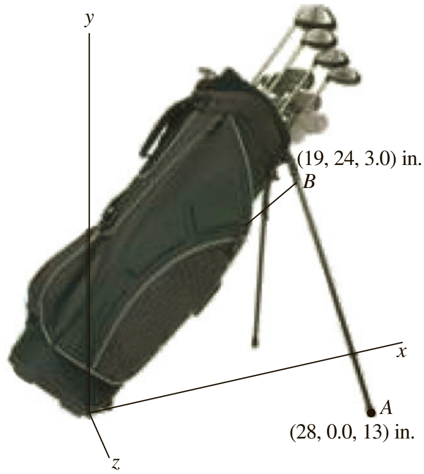
d. in terms of its magnitude and the angles θ and ϕ.



EX 2.5.58

2.5.59. [**] The x, y, and z scalar components of a 140-lb force are in the proportion of 3 : –2 : 6. Determine these components and the space angles between the force and the reference axes.

2.5.60. [**] A golf bag filled with clubs leans on two legs as shown. Assume the bag is symmetrical about the xy plane and the force pushing on the bag at B acts along the line of action of leg AB. If the maximum total force each leg can support without failing is 100 lb, determine the maximum vertical force that the two legs can exert on the bag.



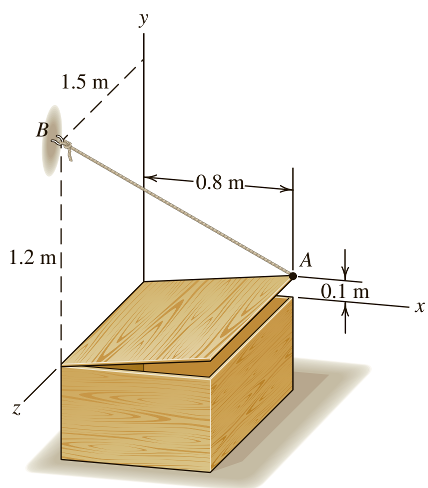
EX 2.5.60

2.5.61. [**] The lid of the wooden crate shown is held open by a rope attached at corner A and hooked to the wall at B. The tension in the rope is 30 N.

a. Write an expression for the rope force pulling on the lid at A. Call this force $\boldsymbol{F}_1$.

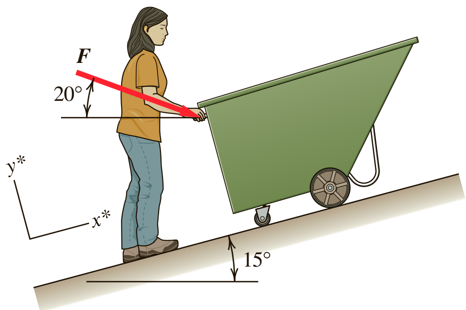
b. Write an expression for the rope force pulling on the wall at B. Call this force $\boldsymbol{F}_2$.

c. Show that $\boldsymbol{F}_1 = -\boldsymbol{F}_2$.



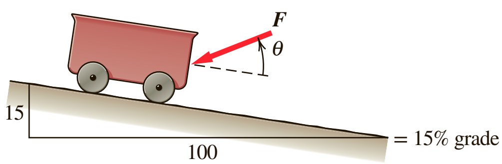
EX 2.5.61

2.5.62. [***] A worker applies a 250 N force ***F*** to push a refuse container up a ramp. For the worker and refuse container to move at constant velocity up the ramp, the magnitude of the component of ***F*** parallel to the ramp must be equal to the magnitude of the component of the weight of the container parallel to the ramp. What is the weight of the container?



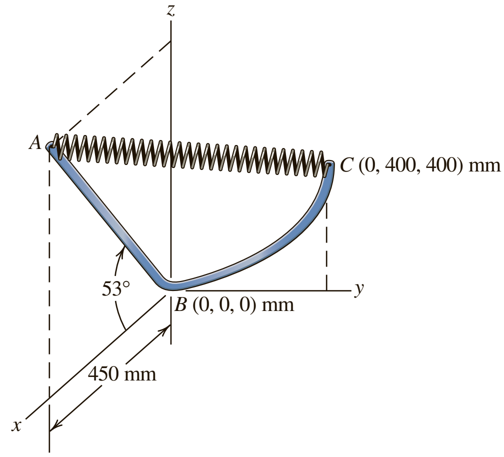
EX 2.5.62

2.5.63. [***] A cart weighing 400 N is pushed up a 15% grade at a constant speed by the force ***F*** shown. If ***F*** has magnitude of 200 N, determine the angle θ such that the scalar component of ***F*** parallel to the slope is equal in magnitude but opposite in direction to the component of the cart weight that is parallel to the slope.



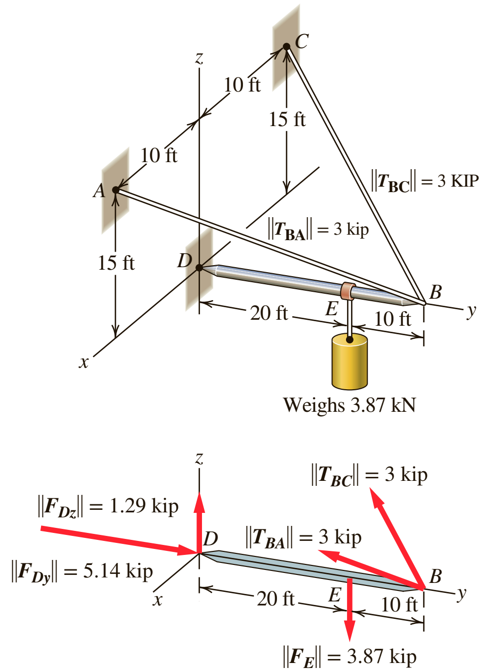
EX 2.5.63

2.5.64. [***] A spring whose stiffness is 250 N/m and whose unstretched length is 400 mm (0.400 m) is stretched between ends A and C of a bent rod ABC, as shown. If the magnitude of the spring force is the product of the spring stiffness and the stretch, determine the force exerted by the spring on end C of the rod. Write your answer in Cartesian vector notation.



EX 2.5.64

2.5.65. [***] The forces acting on member BD are shown in the free-body diagram. Express each force acting on BD in terms of rectangular component vectors.



EX 2.5.65

2.6 RESULTANT FORCE—VECTOR ADDITION

Learning Objective: Determine the resultant of forces using vector addition.

Consider a situation in which you and a friend are pulling on a tent stake, as shown in Figure 2.6.1*a*. The total force exerted on the stake is the vector sum of these two forces; we call this total force the **resultant force** $\boldsymbol{F}_R$, as indicated in Figure 2.6.1*b*. What are the magnitude and direction of $\boldsymbol{F}_R$? In this section we present the vector techniques for answering this question.

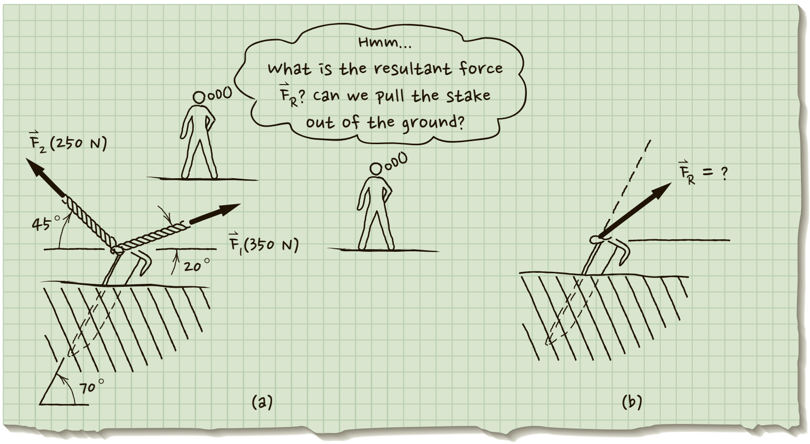


Figure 2.6.1. (*a*) Two people considering how their pulls will affect the stake; (*b*) the resultant force from the two pulls.

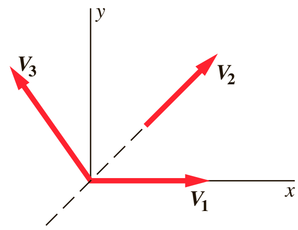


Figure 2.6.2. Forces are concurrent if their lines of action meet at a point.

Two forces that have the same point of application are called **concurrent forces**—$\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ in **Figure 2.6.1*a*** are examples of concurrent forces. More generally, **concurrent forces are any number of forces such that their lines of action intersect at a single point**, as illustrated in **Figure 2.6.2**. It is important to note that the forces do not have to be applied at the same point to be concurrent. The only requirement for concurrent forces is that their lines of action meet at a point. Any number of concurrent forces can be added together and expressed as a single resultant force, and we now present several methods for doing this. **Table 2.4** summarizes the advantages and disadvantages of each method.

Table 2.4 Advantages and Disadvantages of Different Vector Addition Methods

Method	Advantages	Disadvantages
Component Addition	• Straightforward for adding any number of planar and nonplanar forces. • Easy to generalize into an algorithmic/procedural form appropriate for computers.	• May not give you a feel for the physical situation described in the problem and for how the individual forces contribute to the resultant force.
Graphical Vector Addition	• Gives you a feel for the physical situation described in the problem and for how the individual forces contribute to the resultant force. • With proper drawing/drafting tools and skills, is straightforward when adding planar forces.	• Resultant only as accurate as the drawing. • Process becomes complex when more than three or four forces must be added or when the forces are nonplanar. • Difficult to generalize into an algorithmic/procedural form appropriate for computers.
Geometric/Trigonometric Addition	• Is straightforward when we are adding two planar forces, particularly if the directions of the two forces are defined in terms of angular orientation.	• Becomes complex with more than two forces and with nonplanar forces. • Difficult to generalize into an algorithmic/procedural form appropriate for computers.

2.6.1 Vector Component Addition

Suppose we want to find the resultant vector $\boldsymbol{V}_R$ of adding the vectors $\boldsymbol{V}_1$, $\boldsymbol{V}_2$, and $\boldsymbol{V}_3$ in **Figure 2.6.2**. Component addition is the method we

will likely use most often. It is based on the idea that if two vectors, such as $\boldsymbol{V_R}$ and $\boldsymbol{V_1} + \boldsymbol{V_2} + \boldsymbol{V_3}$, are equal, they must have the same magnitude and must point in the same direction. This can happen only if their components are equal. We now illustrate the idea of component equality. A resultant general vector $\boldsymbol{V_R}$ is composed of component vectors:

$$\boldsymbol{V_R} = V_{Rx}\boldsymbol{i} + V_{Ry}\boldsymbol{j} + V_{Rz}\boldsymbol{k} \tag{2.21}$$

and $\boldsymbol{V_1}$, $\boldsymbol{V_2}$, and $\boldsymbol{V_3}$ are also composed of component vectors:

$$\boldsymbol{V_1} = V_{1x}\,\boldsymbol{i} + V_{1y}\boldsymbol{j} + V_{1z}\,\boldsymbol{k} \tag{2.22A}$$
$$\boldsymbol{V_2} = V_{2x}\,\boldsymbol{i} + V_{2y}\boldsymbol{j} + V_{2z}\,\boldsymbol{k} \tag{2.22B}$$
$$\boldsymbol{V_3} = V_{3x}\,\boldsymbol{i} + V_{3y}\boldsymbol{j} + V_{3z}\,\boldsymbol{k} \tag{2.22C}$$

To add $\boldsymbol{V_1}$, $\boldsymbol{V_2}$, and $\boldsymbol{V_3}$, we write

$$\boldsymbol{V_1} + \boldsymbol{V_2} + \boldsymbol{V_3} = \underbrace{V_{1x}\boldsymbol{i} + V_{1y}\boldsymbol{j} + V_{1z}\boldsymbol{k}}_{\boldsymbol{V_1}} + \underbrace{V_{2x}\boldsymbol{i} + V_{2y}\boldsymbol{j} + V_{2z}\boldsymbol{k}}_{\boldsymbol{V_2}} + \underbrace{V_{3x}\boldsymbol{i} + V_{3y}\boldsymbol{j} + V_{3z}\boldsymbol{k}}_{\boldsymbol{V_3}} \tag{2.23A}$$

which can be rearranged to

$$\boldsymbol{V_1} + \boldsymbol{V_2} + \boldsymbol{V_3} = \underbrace{(V_{1x} + V_{2x} + V_{3x})\boldsymbol{i}}_{\boldsymbol{V_{Rx}}} + \underbrace{(V_{1y} + V_{2y} + V_{3y})\boldsymbol{j}}_{\boldsymbol{V_{Ry}}} + \underbrace{(V_{1z} + V_{2z} + V_{3z})\boldsymbol{k}}_{\boldsymbol{V_{Rz}}} \tag{2.23B}$$

In order for $\boldsymbol{V_R}$ to be equal to $\boldsymbol{V_1} + \boldsymbol{V_2} + \boldsymbol{V_3}$ the component vectors of $\boldsymbol{V_R}$ (from (2.21)) must be equal to the component vectors of $\boldsymbol{V_1} + \boldsymbol{V_2} + \boldsymbol{V_3}$ (from (2.23B)):

$$V_{Rx}\boldsymbol{i} + V_{Ry}\boldsymbol{j} + V_{Rz}\boldsymbol{k} = \underbrace{(V_{1x} + V_{2x} + V_{3x})\boldsymbol{i}}_{\boldsymbol{V_{Rx}}} + \underbrace{(V_{1y} + V_{2y} + V_{3y})\boldsymbol{j}}_{\boldsymbol{V_{Ry}}} + \underbrace{(V_{1z} + V_{2z} + V_{3z})\boldsymbol{k}}_{\boldsymbol{V_{Rz}}}$$

Therefore, equating the scalar components, we get

$$\begin{aligned} V_{Rx} &= (V_{1x} + V_{2x} + V_{3x}) \\ V_{Ry} &= (V_{1y} + V_{2y} + V_{3y}) \\ V_{Rz} &= (V_{1z} + V_{2z} + V_{3z}) \end{aligned} \tag{2.24}$$

This result implies that if we know the scalar components of the vectors we want to add, we can add these components to find the scalar components of the resultant. This method works for any number of vectors being added. **Figure 2.6.1.1** shows the addition of the components of two forces you and your friend apply to the stake of **Figure 2.6.2**.

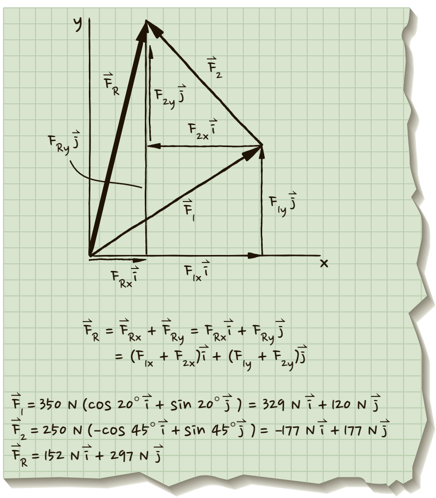


Figure 2.6.1.1 Adding the x and y components of two force vectors to obtain a resultant force $\boldsymbol{F}_R$.

Check out the following examples of applications of this material.
- **Example 2.6.1 Component Addition: Planar**
- **Example 2.6.2 Component Addition: Nonplanar**

EXAMPLE 2.6.1

Two forces act on a gusset plate as shown in Figure 1. The forces lie in the plane of the page. Find the resultant $\boldsymbol{F}_R$ of these two forces, using component addition.

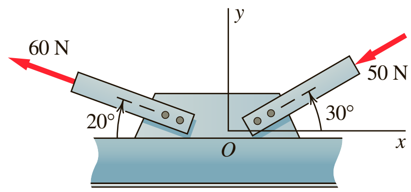


Figure 1 Truss forces, concurrent at O, acting on a gusset plate.

Goal Find the sum of two forces using component addition.

Given Magnitude and direction of two planar concurrent forces. Axes have been specified.

Assume Forces lie in xy plane.

Draw We make some sketches of the forces and the x- and y-components to help us understand the equations (Figure 2).

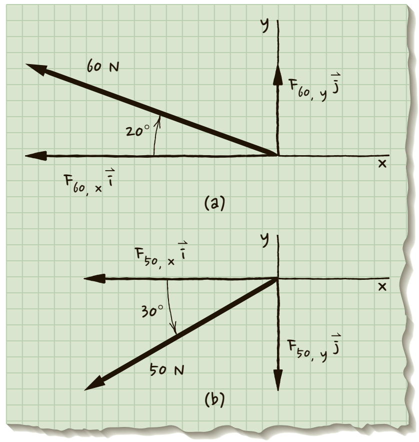


Figure 2 Sketch supporting component addition.

Formulate Equations and Solve We first need to find the scalar components of each force. For the 60-N force, we find its x-component $F_{60,x}$ and its y-component $F_{60,y}$ (**Figure 2*a***):

$$F_{60,x} = (-60 \text{ N})\cos 20° = -56.4 \text{ N}$$

$$F_{60,y} = (60 \text{ N})\sin 20° = 20.5 \text{ N}$$

For the 50-N force, the components are (**Figure 2*b***):

$$F_{50,x} = (-50 \text{ N})\cos 30° = -43.3 \text{ N}$$

$$F_{50,y} = (-50 \text{ N})\sin 30° = -25.0 \text{ N}$$

Next we combine these components to find the components of the resultant force $\boldsymbol{F}_R$:

$$F_{R,x} = F_{60,x} + F_{50,x} = -56.4 \text{ N} - 43.3 \text{ N} = -99.7 \text{ N}$$

$$F_{R,y} = F_{60,y} + F_{50,y} = 20.5 \text{ N} - 25.0 \text{ N} = -4.5 \text{ N}$$

The magnitude of $\boldsymbol{F}_R$ is found using (2.5):

$$\|\boldsymbol{F}_R\| = \sqrt{(-99.7 \text{ N})^2 + (-4.5 \text{ N})^2} \Rightarrow \boxed{\|\boldsymbol{F}_R\| = 99.8 \text{ N}}$$

From Figure 3, we find $\omega = \tan^{-1}(F_{R,y}/F_{R,x}) = 2.58°$. Therefore $\theta = 180° + 2.58° \Rightarrow \boxed{\theta = 182.6°}$.

Check We could check the answer by using a different method of addition, for example, a graphical method. A more qualitative check involves noting that, given the orientations and magnitudes of the 50-N and 60-N forces, the x- and y-components of the resultant should both be negative (they are -99.7 N and -4.5 N), and the magnitude of the x-component should be significantly bigger than the magnitude of the y-component (it is 99.7 N vs. 4.5 N).

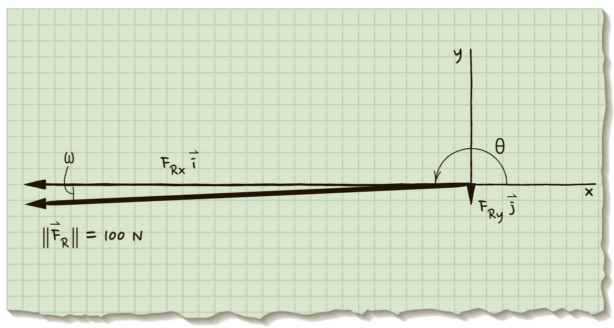


Figure 3 Resultant force $\boldsymbol{F}_R$.

Comment: Component addition gives a more accurate solution than graphical addition because the accuracy of the graphical solution is limited by the accuracy of the drawings. However, drawings are useful and recommended no matter which approach you use.

EXAMPLE 2.6.2

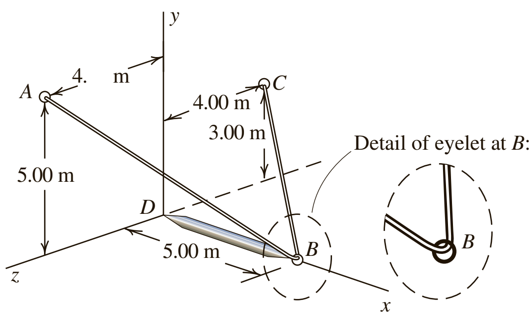


Figure 1 Boom held up by a cable.

The boom *BD* in **Figure 1** is held up by a continuous cable that runs from *A* through an eyelet at *B* to *C*. The cable is attached to the wall via hooks at *A* and *C*. The tension in the cable is 5 kN. Find the resultant force the cable exerts on the boom, reporting your answer in Cartesian vector notation.

Goal Find the Cartesian vector that represents the resultant force exerted by the cable attached to the boom at *B*.

Given The locations of points *A*, *B*, and *C*, as well as the tension in the continuous cable.

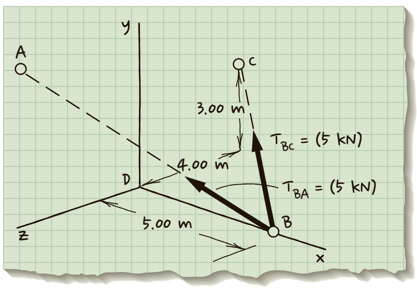


Figure 2 Cable forces acting on the eyelet at *B*.

Assume We are told the tension in the cable is 5 kN. For the tension in the portion of the cable that runs from *B* to *A* to be the same as the tension in the cable between *B* and *C*, we must assume that there is no friction between the cable and the ring at *B*.

Draw The cable exerts two 5-kN forces at *B*—one with a line of action that runs from *B* to *A*, and the other with a line of action that runs from *B* to *C* (**Figure 2**). Notice that since $\boldsymbol{T_{BA}}$ and $\boldsymbol{T_{BC}}$ are tension forces, they must pull on the eyelet at *B*.

Formulate Equations and Solve We will find the scalar components of the two forces acting at *B* ($\boldsymbol{T_{BA}}$ and $\boldsymbol{T_{BC}}$), then add the components to find the resultant force at *B*.

First consider $\boldsymbol{T_{BC}}$. Its line of action is along *BC*. First we find the direction cosines for this line of action, then use them to find the scalar components of $\boldsymbol{T_{BC}}$. The length L_{BC} from *B* to *C* is (**Figure 3*a***):

$$L_{BC} = \sqrt{(x_C - x_B)^2 + (y_C - y_B)^2 + (z_C - z_B)^2}$$

$$L_{BC} = \sqrt{(0 - 5.00)^2 + (3.00 - 0)^2 + (-4.00 - 0)^2}\ \text{m} = 7.07\ \text{m}$$

From (2.6) we can write $\boldsymbol{T_{BC}}$ as

$$\boldsymbol{T_{BC}} = \|\boldsymbol{T_{BC}}\|\boldsymbol{u_{BC}}$$

where $\boldsymbol{u_{BC}}$ is a unit vector in the direction from *B* to *C*. The scalar components of $\boldsymbol{u_{BC}}$ are, from (2.7),

$$u_{BCx} = \frac{x_C - x_B}{L_{BC}} = \frac{-5.00\,\text{m}}{7.07\,\text{m}} = -0.707$$

$$u_{BCy} = \frac{y_C - y_B}{L_{BC}} = \frac{3.00\,\text{m}}{7.07\,\text{m}} = 0.424$$

$$u_{BCz} = \frac{z_C - z_B}{L_{BC}} = \frac{-4.00\,\text{m}}{7.07\,\text{m}} = -0.566$$

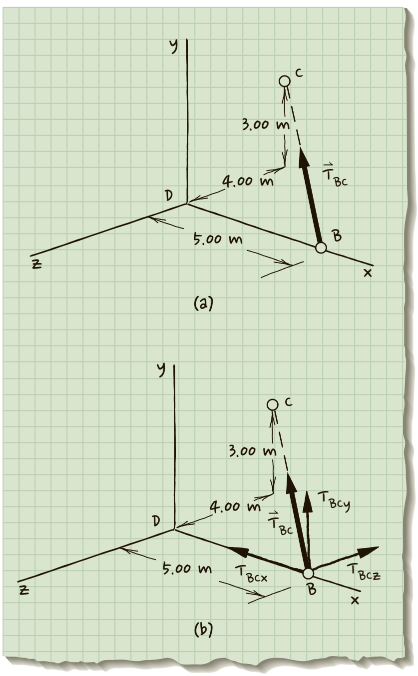


Figure 3 (*a*) Line of action of cable force $\boldsymbol{T_{BC}}$; (*b*) Scalar components of $\boldsymbol{T_{BC}}$.

Therefore we can write $\boldsymbol{T_{BC}}$ as

$$\begin{aligned}\boldsymbol{T_{BC}} &= 5\text{ kN}\left(-0.707\boldsymbol{i} + 0.424\boldsymbol{j} - 0.566\boldsymbol{k}\right)\\ &= -3.54\text{ kN}\boldsymbol{i} + 2.12\text{ kN}\boldsymbol{j} - 2.83\text{ kN}\boldsymbol{k}\end{aligned}$$

See **Figure 3*b***, and check **Figure 2** to confirm that the size and direction of these components make physical sense.

The line of action of $\boldsymbol{T_{BA}}$ is along BA. Using the direction cosines for this line of action, we find the scalar components of $\boldsymbol{T_{BA}}$. The length L_{BA} from B to A is (**Figure 4**):

$$L_{BA} = \sqrt{\left(x_A - x_B\right)^2 + \left(y_A - y_B\right)^2 + \left(z_A - z_B\right)^2}$$

$$L_{BA} = \sqrt{(0 - 5.00)^2 + (5.00 - 0)^2 + (4.84 - 0)^2\text{ m}} = 8.57\text{ m}$$

We find the unit vector $\boldsymbol{u_{BA}}$ in the direction from B to A. The scalar components of $\boldsymbol{u_{BA}}$ are

$$u_{BAx} = \frac{x_A - x_B}{L_{BA}} = \frac{-5.00\text{ m}}{8.57\text{ m}} = -0.583$$

$$u_{BAy} = \frac{y_A - y_B}{L_{BA}} = \frac{5.00\text{ m}}{8.57\text{ m}} = 0.583$$

$$u_{BAz} = \frac{z_A - z_B}{L_{BA}} = \frac{-4.84\text{ m}}{8.57\text{ m}} = 0.565$$

$\boldsymbol{T_{BA}}$ in Cartesian vector notation is (**Figure 4*b***)

$$\begin{aligned}\boldsymbol{T_{BA}} = \|T_{BA}\|\boldsymbol{u_{BA}} &= 5\text{ kN}\left(-0.583\boldsymbol{i} + 0.583\boldsymbol{j} + 0.565\boldsymbol{k}\right)\\ &= -2.92\text{ kN}\boldsymbol{i} + 2.92\text{ kN}\boldsymbol{j} + 2.83\text{ kN}\boldsymbol{k}\end{aligned}$$

Now we add the scalar components of $\boldsymbol{T_{BC}}$ and $\boldsymbol{T_{BA}}$ to find the resultant force $\boldsymbol{T_R}$ at B:

$$T_{Rx} = T_{BCx} + T_{BAx} = -3.54\text{ kN} - 2.92\text{ kN} = -6.46\text{ kN}$$

$$T_{Ry} = T_{BCy} + T_{BAy} = 2.12\text{ kN} + 2.92\text{ kN} = 5.04\text{ kN}$$

$$T_{Rz} = T_{BCz} + T_{BAz} = -2.83\text{ kN} + 2.83\text{ kN} = 0\text{ kN}$$

The resultant vector (**Figure 5**) is $\boldsymbol{T_R} = -6.46\text{ kN}\boldsymbol{i} + 5.04\text{ kN}\boldsymbol{j}$

Check Since the ring at B is frictionless it makes sense that the z component of the resultant is zero. Also revisit **Figure 3** to confirm that the sizes of the space angles make physical sense.

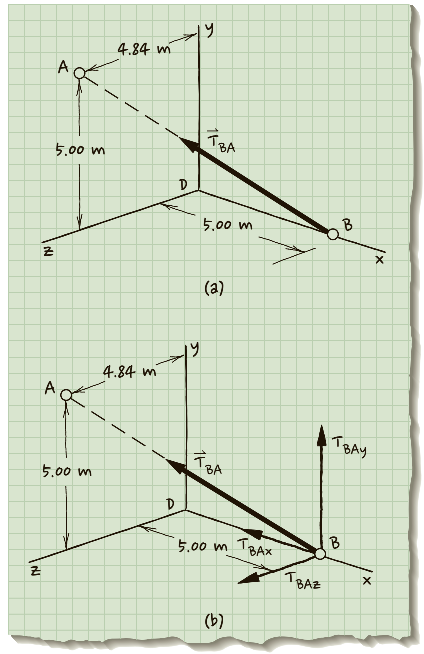


Figure 4 (*a*) Line of action of cable force $\boldsymbol{T_{BA}}$; (*b*) Scalar components of $\boldsymbol{T_{BA}}$.

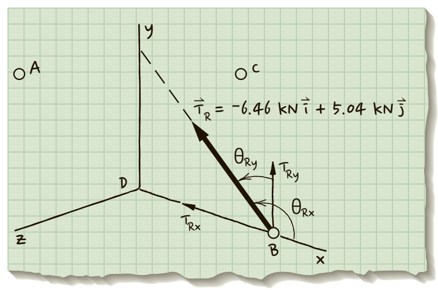


Figure 5 Resultant force acting on eyelet at B.

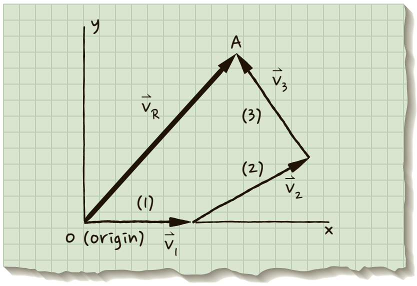


Figure 2.6.2.1 The head-to-tail approach to vector addition forms a force polygon.

2.6.2 Graphical Vector Addition

Adding vectors graphically involves laying out the vectors to be added to scale on graph paper. One way to add force vectors graphically is to measure their combined length after laying them out head to tail. We call this the **head-to-tail approach**. Using our previous example of adding the vectors in Figure 2.6.2, start by laying out $\boldsymbol{V_1}$ in its correct orientation and to scale on graph paper with its tail at the origin, as in **Figure 2.6.2.1**. Then slide $\boldsymbol{V_2}$ parallel to itself until its tail touches the head of $\boldsymbol{V_1}$, and slide $\boldsymbol{V_3}$ parallel to itself until its tail touches the head of $\boldsymbol{V_2}$. If you are working with more than three vectors, continue this process until all the vectors you are adding have been drawn. Assign the label A to the head of the final vector you've laid out. The resultant vector $\boldsymbol{V_R}$ is drawn from the origin O to A, forming a **force polygon**.

When adding only two forces the force polygon reduces to a **force triangle**. The vector length is the magnitude of the resultant force $\|\boldsymbol{V_R}\|$, and its angular orientation (measured with a protractor from a set of reference axes) defines the resultant force direction.

An alternative to the head-to-tail approach is the **parallelogram law of addition**. First select any two of the vectors to be added. Let's begin with $\boldsymbol{V_1}$ and $\boldsymbol{V_2}$ in **Figure 2.6.2.2*a***. Keeping one of the vectors in its original position, slide the other vector perpendicular to its line of action until its tail touches the tail of the first vector, assigning the label A to the point where the two tails touch (**Figure 2.6.2.2*b***). Then, beginning at the head of $\boldsymbol{V_2}$, draw a dashed line that extends in the direction of $\boldsymbol{V_1}$ and is parallel to $\boldsymbol{V_1}$. Draw a second dashed line that begins at the head of $\boldsymbol{V_1}$, extends in the direction of $\boldsymbol{V_2}$, and is parallel to $\boldsymbol{V_2}$. Call the point where these two lines intersect A^*. You have now formed a parallelogram created by sides $\boldsymbol{V_1}$ and $\boldsymbol{V_2}$ plus the two lines parallel to these vectors. A vector extending from A to A^* is the resultant of $\boldsymbol{V_1} + \boldsymbol{V_2}$, and we now use this intermediate resultant, which we call $\boldsymbol{V_{inter}}$, to find the resultant of $\boldsymbol{V_1} + \boldsymbol{V_2} + \boldsymbol{V_3}$. To find $\boldsymbol{V_{inter}} + \boldsymbol{V_3}$, leave $\boldsymbol{V_{inter}}$ where it is, slide $\boldsymbol{V_3}$ perpendicular to its line of action until its tail is touching the tail of $\boldsymbol{V_{inter}}$, and construct another parallelogram just as you did when adding $\boldsymbol{V_1}$ and $\boldsymbol{V_2}$ (**Figure 2.6.2.2*c***). Call the intersection of the two dashed lines point A^{**}.

The final resultant of adding $\boldsymbol{V_1}$, $\boldsymbol{V_2}$, and $\boldsymbol{V_3}$ (actually the resultant of $\boldsymbol{V_{inter}}$ and $\boldsymbol{V_3}$, which amounts to the same thing) is a vector from A to A^{**}, as shown in **Figure 2.6.2.2*c***.

Now back to the tent stake example of **Figure 2.6.1**. The forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ exerted by you and your friend are shown in **Figure 2.6.2.3**.

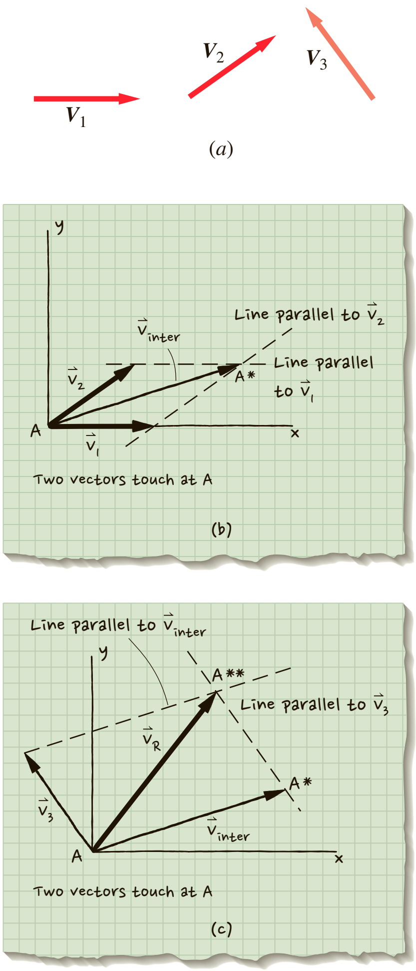


Figure 2.6.2.2 The parallelogram law of addition.

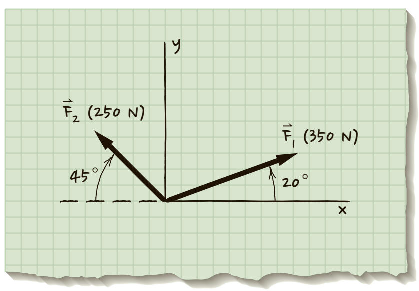


Figure 2.6.2.3 Forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ applied to the tent stake of **Figure 2.6.1**.

Graphical addition of these forces is illustrated using the head-to-tail approach in **Figure 2.6.2.4*a*** and using the parallelogram law in **Figure 2.6.2.4*b***. The length of $\boldsymbol{F}_R$ is the magnitude of the resultant force (333 N), and the angles θ_x and θ_y define its direction ($\theta_x = 62.8°$, $\theta_y = 27.2°$).

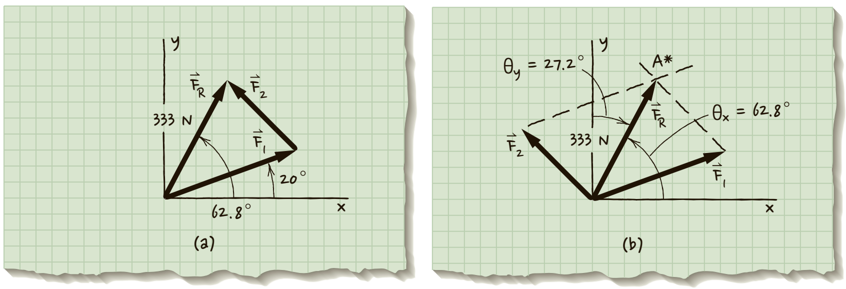


Figure 2.6.2.4 Adding forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ using (*a*) the head-to-tail approach and (*b*) the parallelogram law.

Check out the following examples of applications of this material.

- **Example 2.6.3 Graphical Addition Using Force Triangle**
- **Example 2.6.4 Graphical Addition Using Parallelogram Law**

EXAMPLE 2.6.3

Two forces act on a gusset plate as shown in **Figure 1**. The forces lie in the plane of the page. Find the resultant $\boldsymbol{F}_R$ of these two forces, using graphical addition.

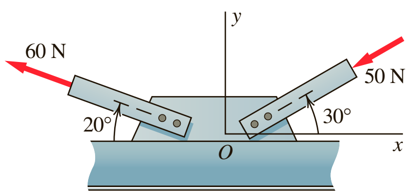


Figure 1 Truss forces, concurrent at *O*, acting on a gusset plate.

Goal Find the sum of two forces using graphical addition.

Given Magnitude and direction of two planar concurrent forces. Axes have been specified.

Assume Forces lie in *xy* plane.

Draw In order to add the forces graphically, we lay them out in a scaled drawing (**Figure 2**).

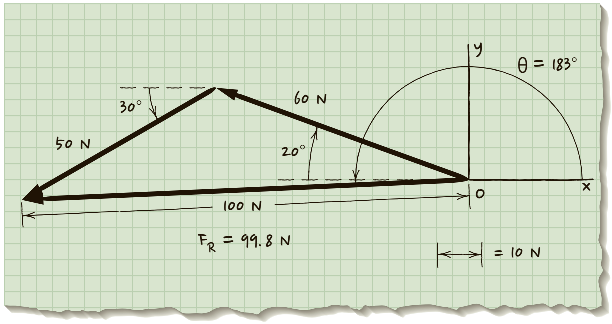


Figure 2 Head-to-tail approach for adding the two forces of **Figure 1**.

Graphical Addition (Head-to-Tail Approach) First, lay out a coordinate system on a sheet of graph paper and establish an origin O, as in **Figure 2**. Also establish a scale (e.g., 1 cm = 10 N). Then draw a vector arrow representing either the 60 N or 50 N force. It doesn't matter which we begin with, so let's arbitrarily consider the 60 N force first. Place the tail of the arrow at the origin and use a protractor to measure 20° up from the $-x$ axis. Now draw the 60 N force from the origin at the 20° orientation. Make sure that this 60 N force is drawn to scale. Next draw the 50 N force at 30° below the horizontal so that its tail coincides with the head of the 60 N force.

The resultant force is represented by an arrow drawn from the tail of the first vector to the head of the second. The length of this arrow is the magnitude of the resultant force acting on the gusset plate (100 N), and the angle θ (as measured with a protractor) is the direction of that force (183° counterclockwise from the $+x$ axis). If we were asked to find the x and y scalar components of $\boldsymbol{F}_R$, we could measure them from the figure and would find that $F_{Rx} = -99.7$ N and $F_{Ry} = -4.5$ N.

Check We could check the answer by using a different methods of addition, for example, a component addition.

Comment: Component addition gives a more accurate solution than graphical addition because the accuracy of the graphical solution is limited by the accuracy of the drawings. However, drawings are useful and recommended no matter which approach you use.

EXAMPLE 2.6.4

Two forces act on a gusset plate as shown in **Figure 1**. The forces lie in the plane of the page. Find the resultant $\boldsymbol{F}_R$ of these two forces, using graphical addition and the parallelogram law.

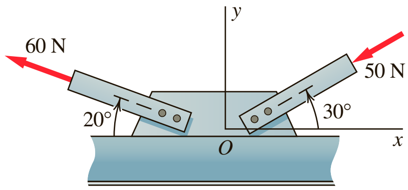


Figure 1 Truss forces, concurrent at O, acting on a gusset plate.

Goal Find the sum of two forces using graphical addition and the parallelogram law.

Given Magnitude and direction of two planar concurrent forces. Axes have been specified.

Assume Forces lie in xy plane.

Draw Lay out arrows representing the forces as shown in **Figure 2**—tail to tail and each making the proper angle with the x axis. Now create a parallelogram having these two arrows as two of its sides.

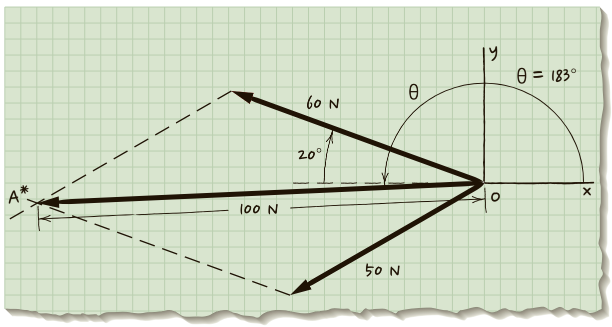


Figure 2 Parallelogram law approach.

Formulate Equations and Solve The resultant force is the parallelogram diagonal $OA'*$. The magnitude $\|\boldsymbol{F}_R\| = 100$ N and the force is oriented at $\theta = 183°$ (as measured with a protractor). Alternately, we could measure the x and y scalar components of $\boldsymbol{F}_R$ from the figure and would find that $F_{Rx} = -99.7$ N and $F_{Ry} = -4.5$ N.

Check We could check the answer by using a different methods of addition, for example, a component addition.

Comment: It would take very careful layout work to get this level of accuracy in the answer using this graphical approach.

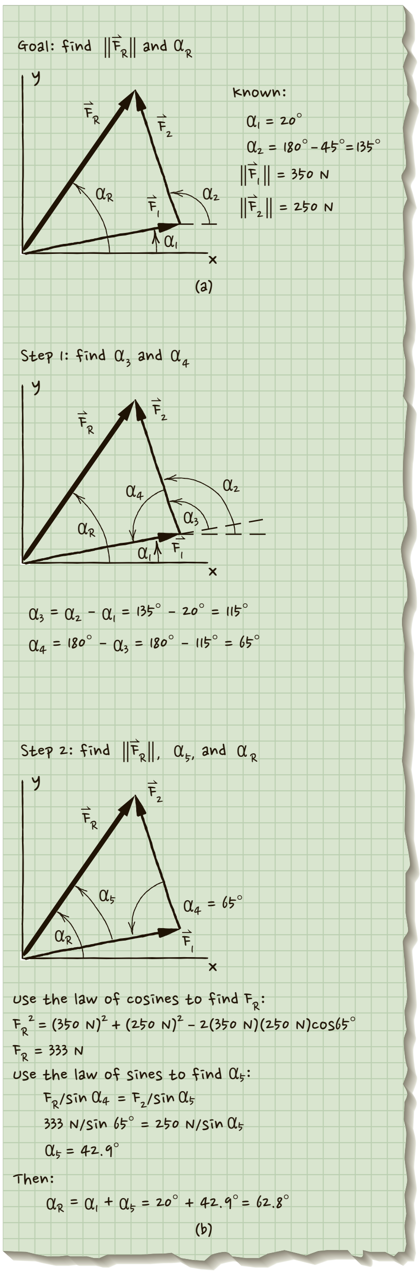


Figure 2.6.3.1 (*a*) $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, and $\boldsymbol{F}_R$ form a triangle; (*b*) using trigonometry to find $\boldsymbol{F}_R$.

2.6.3 Geometric/Trigonometric Addition

We can also add vectors by taking advantage of trigonometric identities. With this approach, it is useful to begin by sketching the vectors head to tail, as with the graphical method. Now, however, it is far less critical that the drawing be precise because the sketch serves only to show the spatial relationships between the vectors. Figure 2.6.3.1*a* shows that $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, and $\boldsymbol{F}_R$ from the tent stake example form a triangle. Using the law of sines and the law of cosines, we can "operate" on the triangle to determine the magnitude and direction of $\boldsymbol{F}_R$, as shown in Figure 2.6.3.1*b*. The laws of sines and cosines, along with other useful trigonometric identities, are presented in Appendix A.

Check out the following examples of applications of this material.

- **Example 2.6.5 Resultant of Two Forces Using a Trigonometric Approach**
- **Example 2.6.6 Analyzing a System: Trigonometric Addition**
- **Example 2.6.7 Analyzing a System: Trigonometric Approach**

EXAMPLE 2.6.5

Two forces act on a frame as shown in Figure 1. Using a trigonometric approach, find the resultant force due to these two forces.

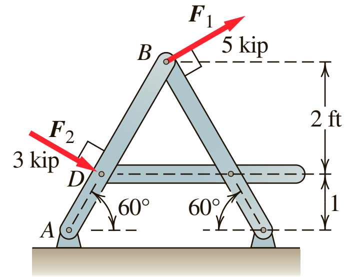


Figure 1 Forces acting on a frame. (1 kip = 1000 lb, i.e. a "*ki*lo*pound*.")

Goal Find the resultant force $\boldsymbol{F}_R$ due to the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$. In other words, find $\boldsymbol{F}_R = \boldsymbol{F}_1 + \boldsymbol{F}_2$.

Given Magnitude, orientation, and point of application of two forces acting on a frame of specified dimensions.

Assume The frame and forces all lie in the plane of the page. $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are concurrent.

Draw By extending the line of action of $\boldsymbol{F}_1$ toward $\boldsymbol{F}_2$, we confirm that these two forces are concurrent. We draw the diagram of Figure 2, establishing a right-handed coordinate system with its origin arbitrarily placed at A.

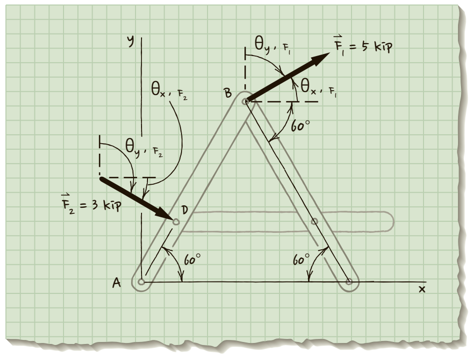


Figure 2 Establish a coordinate system and determine the space angles for applied forces.

Formulate Equations and Solve We add $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ using geometry and trigonometry to find their resultant. First we define the forces relative to the coordinate system. Based on the geometry in **Figure 2** and the assumption that $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ lie in the xy plane

$$\theta_{x,F_1} = 30°,\ \theta_{y,F_1} = 60°,\ \theta_{z,F_1} = 90°$$
$$\theta_{x,F_2} = 30°,\ \theta_{y,F_2} = 120°,\ \theta_{z,F_2} = 90°$$

We lay out the forces head to tail in a sketch (Figure 3). Because the addition of vectors is commutative, either **Figure 3*a*** or ***b*** can be used; we arbitrarily choose **Figure 3*a***. Based on the geometry in **Figure 3*a***, the included angle β_1 between $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ is 120° (**Figure 3*c***). Now we use the law of cosines to find the magnitude of $\boldsymbol{F}_R$:

$$\|\boldsymbol{F}_R\|^2 = \|\boldsymbol{F}_1\|^2 + \|\boldsymbol{F}_2\|^2 - 2\|\boldsymbol{F}_1\|\|\boldsymbol{F}_2\|\cos\beta_1$$
$$\|\boldsymbol{F}_R\|^2 = (5\text{ kip})^2 + (3\text{ kip})^2 - 2(5\text{ kip})(3\text{ kip})\cos 120°$$
$$\|\boldsymbol{F}_R\| = 7\text{ kip}$$

Using the law of sines, we find the direction (in terms of the space angles) of $\boldsymbol{F}_R$ (**Figure 3*a***):

$$\frac{\sin\beta_1}{\|\boldsymbol{F}_R\|} = \frac{\sin\beta_2}{\|\boldsymbol{F}_2\|}$$
$$\frac{\sin 120°}{7\text{ kip}} = \frac{\sin\beta_2}{3\text{ kip}}$$
$$\beta_2 = \sin^{-1}\left(\frac{3\text{ kip}}{7\text{ kip}} \times \sin 120°\right) = 21.8°$$

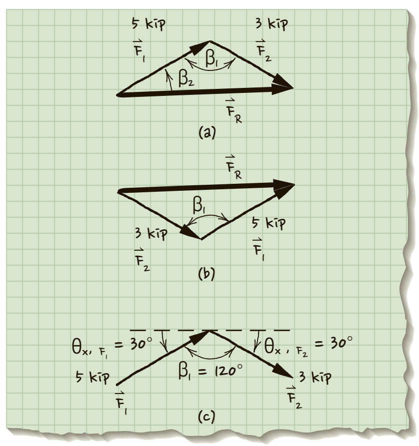


Figure 3 (*a*) and (*b*) The head-to-tail approach creates a force triangle; (*c*) Determining the angle β_1 between $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$.

Therefore, the space angles (Figure 4) associated with $\boldsymbol{F}_R$ are

$$\theta_{x,F_R} = 30° - 21.8° = 8.2°$$
$$\theta_{y,F_R} = 90° - 8.2° = 81.8°$$

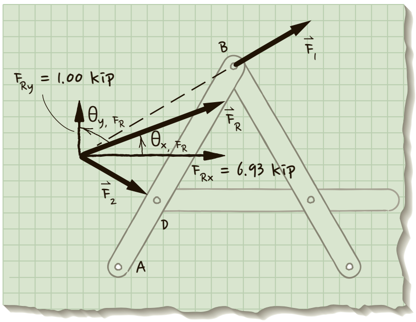


Figure 4 Resultant force $\boldsymbol{F}_R$ shown with its components.

We can specify $\boldsymbol{F}_R$ in terms of magnitude and space angles;

$$\|\boldsymbol{F_R}\| = 7\,\text{kip}, \quad \theta_{x,F_R} = 8.2°, \theta_{y,F_R} = 81.8°, \theta_{z,F_R} = 90°$$

or in Cartesian vector notation;

$$\boldsymbol{F_R} = 7 \text{ kip} \left(\cos 8.2° \boldsymbol{i} + \cos 81.8° \boldsymbol{j}\right)$$

$$\boldsymbol{F_R} = 6.93 \text{ kip}\,\boldsymbol{i} + 1.00 \text{ kip}\,\boldsymbol{j}$$

Check We check our result using component addition. Referring to **Figure 2** and the text accompanying that illustration:

$$F_{1x} = (5 \text{ kip})\cos 30° = +4.33 \text{ kip}$$
$$F_{1y} = (5 \text{ kip})\sin 30° = +2.50 \text{ kip}$$
$$F_{2x} = (3 \text{ kip})\cos 30° = +2.60 \text{ kip}$$
$$F_{2y} = (-3 \text{ kip})\sin 30° = -1.50 \text{ kip}$$

Combining these components to find the resultant force $\boldsymbol{F}_R = F_{Rx}\boldsymbol{i} + F_{Ry}\boldsymbol{j}$, we get

$$F_{Rx} = F_{1x} + F_{2x} = 4.33 \text{ kip} + 2.60 \text{ kip} = 6.93 \text{ kip}$$
$$F_{Ry} = F_{1y} + F_{2y} = +2.50 \text{ kip} - 1.50 \text{ kip} = 1.00 \text{ kip}$$
$$\boldsymbol{F_R} = 6.93 \text{ kip}\,\boldsymbol{i} + 1.00 \text{ kip}\,\boldsymbol{j}$$

as shown in **Figure 4**. We now use (2.5) to get the magnitude of $\boldsymbol{F}_R$:

$$\|\boldsymbol{F_R}\| = \sqrt{6.93^2 + 1.00^2} \text{ kip} = 7.00 \text{ kip}$$

From **Figure 4**, we find that $\theta_{x,F_R} = \tan^{-1}(F_{Ry}/F_{Rx}) = 8.2°$ and $\theta_{y,F_R} = 90° - 8.2° = 81.8°$.

If you are wondering why the tail of $\boldsymbol{F}_R$ is located at what seems like an arbitrary point in **Figure 4**, recall that forces must be concurrent if we are to add them. As **Figure 4** shows, the tail of $\boldsymbol{F}_R$ lies at the point where the lines of action of concurrent forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ intersect.

Finally, we can check the answer for reasonableness. Given the orientations of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ in **Figure 1**, we expect their resultant to have positive x and y components; it does. Also based on the sketch, we expect θ_{x,F_R} to be a small angle; it is.

EXAMPLE 2.6.6

In Example 2.4.1 we drew a free-body diagram of a pallet of tiles sitting on a roof. The pallet is sitting on loose tar paper and held in place with a cable as shown in Figure 1. If the sum of the forces acting on the pallet is zero and the pallet and tiles have a combined weight of 200 lb, what is the magnitude of the force exerted by the cable on the pallet?

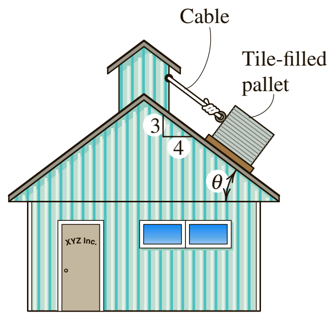


Figure 1 Pallet of tiles sitting on an inclined roof.

Goal Find the force exerted by the cable on the pallet.

Given The slope of the roof and the combined weight of the pallet and tiles. The pallet sits on a frictionless surface, and the sum of the forces acting on the pallet–tile unit is zero.

Assume Any other forces acting on the pallet-tile unit (such as wind) are negligible so the forces acting on the pallet–tile unit lie in a single plane. All forces act through the geometric center of the pallet-tile unit. (We would need more information to make any other assumption.)

Draw In Example 2.4.1 we isolated the pallet-tile unit from its surroundings (Figure 2) and drew a free-body diagram of the pallet-tile unit. This is repeated in Figure 3.

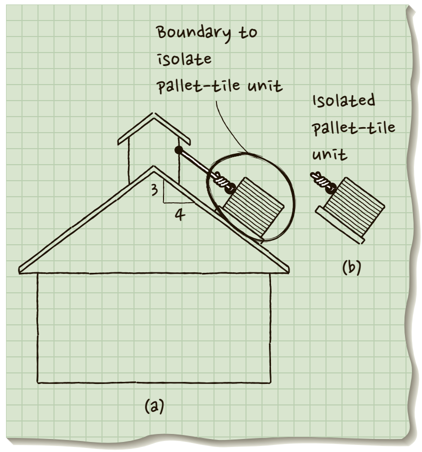


Figure 2 (*a*) Isolating the pallet-tile unit from its surroundings; (*b*) The isolated pallet–tile unit.

Formulate Equations and Solve Applying the requirement that the sum of the forces is zero, means that the resultant force acting on the pallet-tile unit $\boldsymbol{F}_R$ is zero. We use the head-to-tail approach for adding the forces acting on the pallet-tile unit, arbitrarily starting with $\boldsymbol{W}$. We then place the tail of $\boldsymbol{T}_{\text{cable}}$ at the head of $\boldsymbol{W}$, and the tail of $\boldsymbol{F}_{\text{normal}}$ at head of $\boldsymbol{T}_{\text{cable}}$. Because the length of $\boldsymbol{F}_{\text{R}}$ is zero, the head of $\boldsymbol{F}_{\text{normal}}$ touches the tail $\boldsymbol{W}$, creating a force triangle as shown in Figure 4.

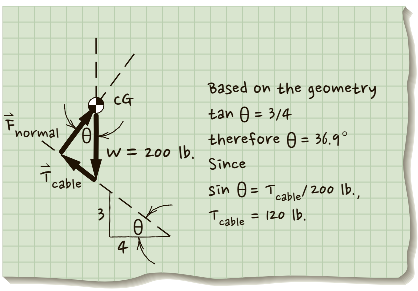


Figure 4 Force triangle formed by normal force, pallet-tile weight, and cable force.

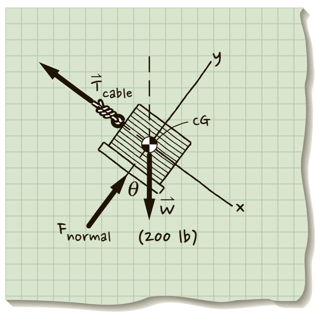


Figure 3 Free-body diagram of isolated pallet.

To find the magnitude $\|\boldsymbol{T}_{\text{cable}}\|$, we use the fact that the force triangle is a right triangle with its hypotenuse defined by $\boldsymbol{W}$:

$$\|\boldsymbol{T}_{\text{cable}}\| = \|\boldsymbol{W}\|\sin\theta = (200\,\text{lb})\sin 36.9^\circ \quad \Rightarrow \quad \boxed{\|\boldsymbol{T}_{\text{cable}}\| = 120\ \text{lb}}$$

Check One way to check your answer is to use two different approaches for adding forces, then compare the answers. Answers should also be checked for reasonableness; does it seem reasonable that the cable tension would be 120 lb for a pallet weighing 200 lb?

EXAMPLE 2.6.7

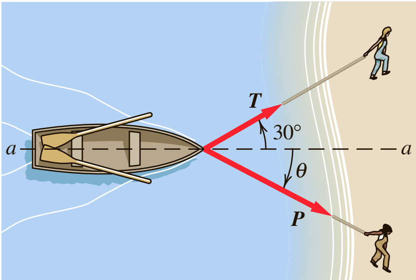


Figure 1 Pulling on a boat with two ropes.

Two ropes are used to pull a boat to shore. The force $\boldsymbol{T}$ exerted by one of the ropes acts at 30° to the keel line *a-a*, as shown in **Figure 1**. Call the force exerted by the other rope $\boldsymbol{P}$. The resultant of forces $\boldsymbol{T}$ and $\boldsymbol{P}$ has a magnitude of 300 lb and is directed along the keel line. Find the magnitudes of $\boldsymbol{T}$ and $\boldsymbol{P}$ and the angle θ at which $||\boldsymbol{P}||$ is a minimum.

Goal Find the magnitudes of the tensions in the two ropes such that the resultant force has a magnitude of 300 lb and is directed along *a-a*. Also, find θ (the angle between $\boldsymbol{P}$ and the keel line) such that $||\boldsymbol{P}||$ is minimized.

Given The direction of $\boldsymbol{T}$ relative to *a-a* and the magnitude and direction of the resultant of $\boldsymbol{T}$ and $\boldsymbol{P}$.

Assume Because we are given no information about the height of the people or the keel, we assume that they are all the same height so that $\boldsymbol{P}$, $\boldsymbol{T}$, and the keel line all lie in the same plane. Without this assumption we do not have enough information to solve the problem.

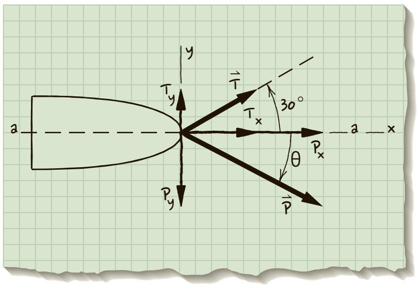


Figure 2 x and y components of forces acting on the bow of the boat.

Draw Define a set of reference axes, as shown in **Figure 2**. Aligning the x axis with the resultant force and placing the origin at the point where the tension forces are concurrent make setting up the equations straightforward.

Formulate Equations and Solve The three conditions of the problem are that the resultant force has a magnitude of 300 lb, it is directed along the x axis, and $||P||$ is as small as possible. By enforcing these three conditions, we will be able to find the three unknown quantities, $||\boldsymbol{T}||$, $||\boldsymbol{P}||$, and θ

We want the sum of the x components $\boldsymbol{P}$ and $\boldsymbol{T}$ to be 300 lb (first condition) and the sum of the y components to be zero (second condition). These two conditions give us two equations:

$$T_x + P_x = ||\boldsymbol{T}||\cos 30° + ||P||\cos\theta = 300\,\text{lb} \qquad (1)$$

$$T_y + P_y = ||\boldsymbol{T}||\sin 30° - ||P||\sin\theta = 0\,\text{lb} \qquad (2)$$

We can meet the third condition by solving these two equations for $||\boldsymbol{P}||$ as a function of θ.

$$\text{Solve (2) for } ||\boldsymbol{T}||: \qquad ||\boldsymbol{T}|| = \frac{||\boldsymbol{P}||\sin\theta}{\sin 30°} \qquad (2')$$

Substitute (2′) into (1) and rearrange:

$$||\boldsymbol{P}|| = \frac{300\,\text{lb}}{\left(\dfrac{\sin\theta}{\tan 30°} + \cos\theta\right)} \qquad (3)$$

We now use a graphical approach to find the combination of θ and $||\boldsymbol{P}||$ that minimizes $||\boldsymbol{P}||$. We substitute various values of θ into equation

Table 1

θ (degrees)	$\|\|\boldsymbol{P}\|\|$ from (3) (lb)
10	233
20	196
30	173
40	160
50	152
60	150
70	152
80	160
90	173

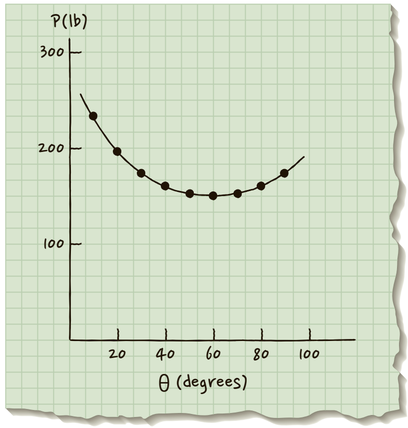


Figure 3 The minimum value of $||\boldsymbol{P}||$ occurs when the rope is at an angle of 60° to line *a-a*.

(3) and solve for $||\boldsymbol{P}||$ creating **Table 1**. We expect $0° < \theta \leq 90°$. (Why don't we look at $\theta = 0$?)

We plot these combinations of θ and $||\boldsymbol{P}||$ to see where $||\boldsymbol{P}||$ is a minimum. From **Figure 3** we see that $||\boldsymbol{P}||$ is smallest for:

$$\boxed{\theta = 60°; \quad ||\boldsymbol{P}|| = 150 \text{ lb}}$$

We find $||\boldsymbol{T}||$ by substituting the minimum value of $||\boldsymbol{P}|| = 150$ lb and $\theta = 60°$ into (2′):

$$||\boldsymbol{T}|| = \frac{(150 \text{ lb}) \sin 60°}{\sin 30°} \quad \Rightarrow \quad \boxed{||\boldsymbol{T}|| = 260 \text{ lb}}$$

We also could find the minimum of the function described by (3) by taking its derivative and setting it to zero.

Check The answers can be checked by substituting the values of θ, $||\boldsymbol{P}||$, and $||\boldsymbol{T}||$ into (1) and (2). **Figure 4** shows a force triangle formed by $\boldsymbol{P}$, $\boldsymbol{T}$, and their resultant $\boldsymbol{R}$. To meet the problem conditions, the resultant must lie along line *a-a*. Notice that the angle between $\boldsymbol{T}$ and $\boldsymbol{P}$ is 90°; does this make physical sense? If $\theta < 60°$, $||\boldsymbol{P}||$ will need to be larger than 150 lb so that its *y* component cancels the *y* component of $\boldsymbol{T}$ (because of the second condition). At $\theta < 60°$, $||\boldsymbol{P}||$ will need to be larger than 150 lb so that its *x* component when added to the *x* component of $\boldsymbol{T}$ sums to 300 lb (because of the first condition). Always check your answers to make sure they make physical sense.

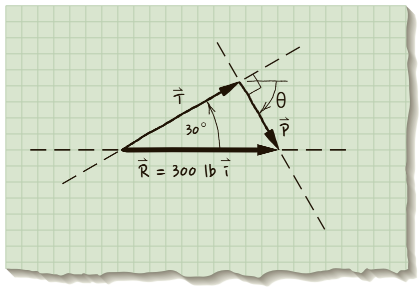
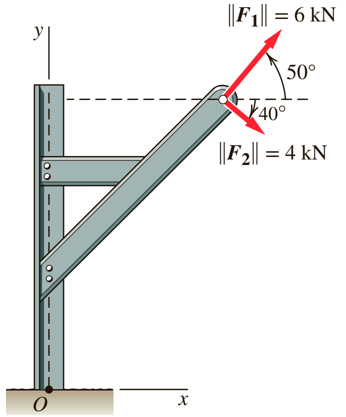


Figure 4 Force triangle when $||\boldsymbol{P}||$ is at a minimum.

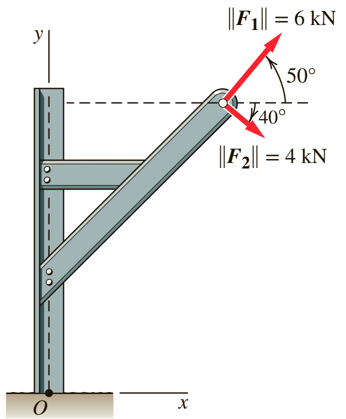
EXERCISES 2.6

2.6.1. [*] Two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are applied to the frame as shown. Write the resultant $\boldsymbol{F}_R$ (where $\boldsymbol{F}_R = \boldsymbol{F}_1 + \boldsymbol{F}_2$) in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$. Use graphical force addition.



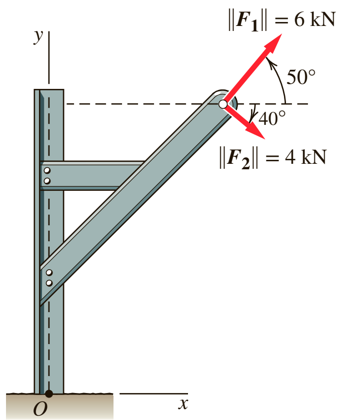
EX 2.6.1

2.6.2. [*] Two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are applied to the frame as shown. Write the resultant $\boldsymbol{F}_R$ (where $\boldsymbol{F}_R = \boldsymbol{F}_1 + \boldsymbol{F}_2$) in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$. Use geometrical/trigonometric force addition.



EX 2.6.2

2.6.3. [*] Two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are applied to the frame as shown. Write the resultant $\boldsymbol{F}_R$ (where $\boldsymbol{F}_R = \boldsymbol{F}_1 + \boldsymbol{F}_2$) in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$. Use component force addition.



EX 2.6.3

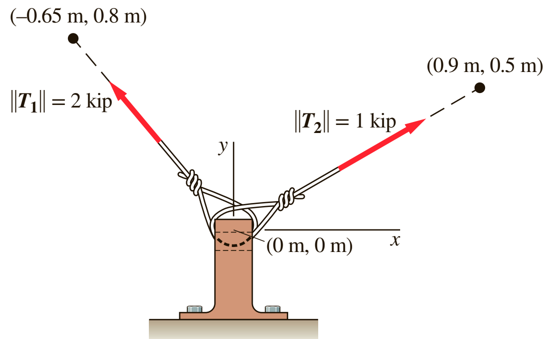
2.6.4. [*] If $\boldsymbol{F}_A = 1.00$ kN $\boldsymbol{i} - 4.5$ kN $\boldsymbol{j}$ and $\boldsymbol{F}_B = -2$ kN $\boldsymbol{i} -$ 2 kN $\boldsymbol{j}$, what is the magnitude of $\boldsymbol{F} = 6\boldsymbol{F}_A + 4\boldsymbol{F}_B$?

2.6.5. [*] Find the magnitude of $\boldsymbol{F} = 3\boldsymbol{F}_A - 2\boldsymbol{F}_B$ for the two forces $\boldsymbol{F}_A = -11.0$ kip $\boldsymbol{i} - 15.0$ kip $\boldsymbol{j}$ and $\boldsymbol{F}_B = 8.0$ kip $\boldsymbol{i} -$ 8.0 kip $\boldsymbol{j}$.

2.6.6. [*] Two support cables are tethered to the ground. The tensions in the cables are shown as well as points on the lines of action of $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$.

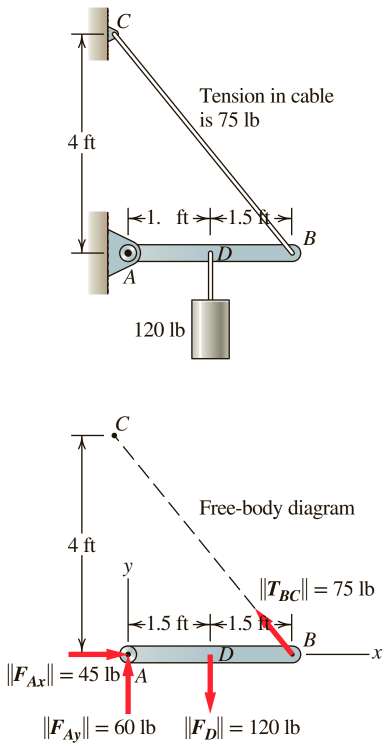
a. Write the resultant $\boldsymbol{T}_R$ (where $\boldsymbol{T}_R = \boldsymbol{T}_1 + \boldsymbol{T}_2$) in vector notation.

b. Determine the magnitude of $\boldsymbol{T}_R$ and its space angles.



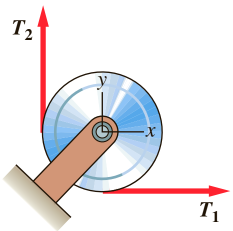
EX 2.6.6

2.6.7. [*] Redraw the free-body diagram of member AB to show $\boldsymbol{F}_A$ as a single force acting at A. Indicate the magnitude and space angle that define $\boldsymbol{F}_A$.



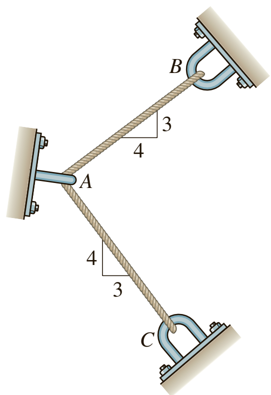
EX 2.6.7

2.6.8. [*] Consider the frictionless pulley shown. Because the pulley is frictionless, $\|\boldsymbol{T}_1\| = \|\boldsymbol{T}_2\|$. Express the resultant $\boldsymbol{R}$ of $\boldsymbol{T}_1 + \boldsymbol{T}_2$ in terms of rectangular components. Also show that the line of action of $\boldsymbol{R}$ bisects the angle between $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$.



EX 2.6.8

2.6.9. [*] The tension in each of the two supporting cables AB and AC is 10 lb. Determine the resultant force acting at A due to the two cables.

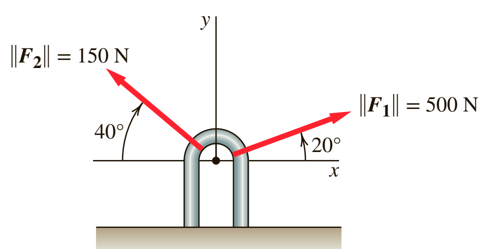


EX 2.6.9

2.6.10. [*] Two forces act on a u-bolt as shown.

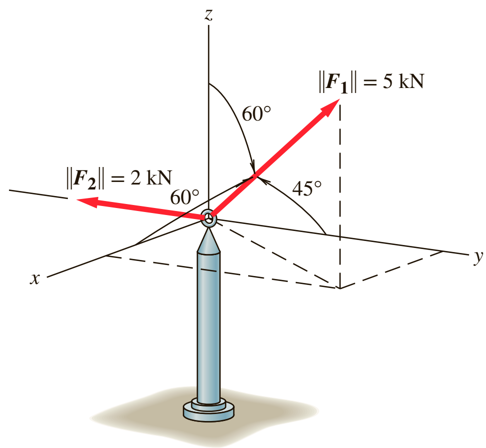
a. Find the resultant of the two forces and express it in vector notation.

b. Find the space angles associated with the resultant.



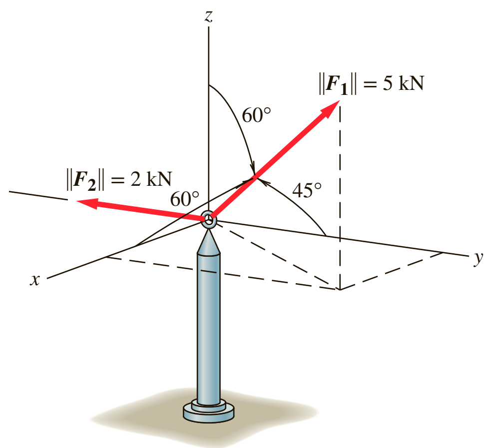
EX 2.6.10

2.6.11. [*] Two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are applied to a post as shown. Write the resultant $\boldsymbol{F}_R$ (= $\boldsymbol{F}_1 + \boldsymbol{F}_2$) in terms of vector notation.



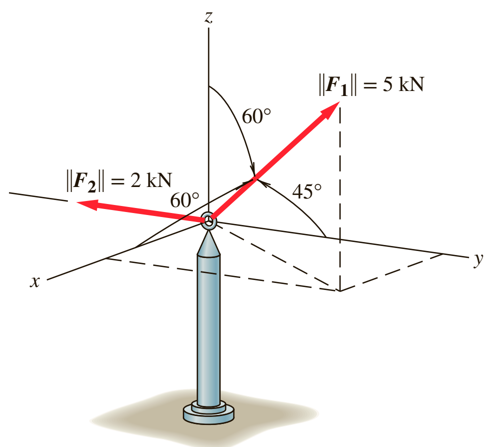
EX 2.6.11

2.6.12. [*] Two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are applied to a post as shown. Write the resultant $\boldsymbol{F}_R$ (= $\boldsymbol{F}_1 + \boldsymbol{F}_2$) in terms of magnitude and space angles.



EX 2.6.12

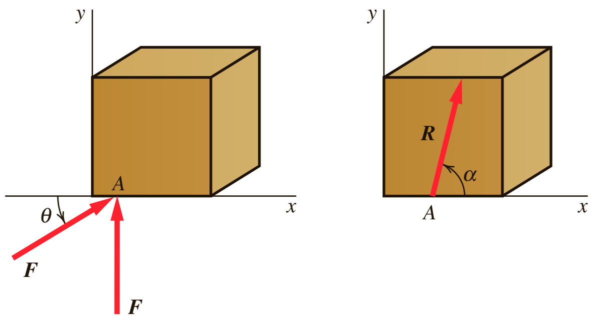
2.6.13. [*] Two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are applied to a post as shown. Write the vector expression for the unit vector along the line of action of $\boldsymbol{F}_R$ (= $\boldsymbol{F}_1 + \boldsymbol{F}_2$).



EX 2.6.13

2.6.14. [**] Two forces of equal magnitude are applied to a block in the xy plane at point A as shown in (a). If the magnitude of $\boldsymbol{F}$ remains constant and the angle θ is reduced, what will happen to the magnitude of the resultant force $\boldsymbol{R}$ as shown in (b)?

a. $\boldsymbol{R}$ will get smaller
b. $\boldsymbol{R}$ will remain the same
c. $\boldsymbol{R}$ will get larger
Justify your answer.

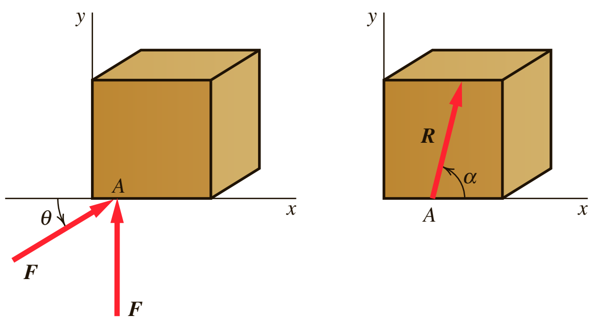


(a) (b)

EX 2.6.14

2.6.15. [**] Two forces of equal magnitude are applied to a block in the xy plane at point A as shown in (a). If the magnitude of $\boldsymbol{F}$ remains constant and the angle θ is reduced, what will happen to the angle α that the resultant force $\boldsymbol{R}$ makes with the x-axis?

a. α will get smaller
b. α will remain the same
c. α will get larger
Justify your answer.



(a) (b)

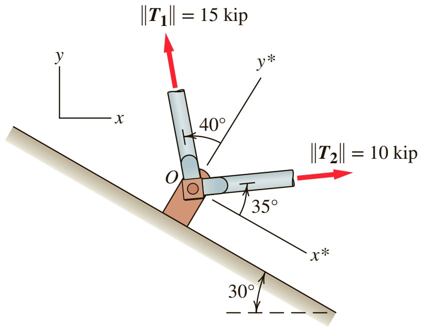
EX 2.6.15

2.6.16. [**] Two structural members are pinned to the support at O. Tension forces $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ act on the members as shown.

a. Write $\boldsymbol{T}_R = \boldsymbol{T}_1 + \boldsymbol{T}_2$ in vector notation based on the x and y axes shown.

b. Write $\boldsymbol{T}_R^* = \boldsymbol{T}_1 + \boldsymbol{T}_2$ in vector notation based on the x^* and y^* axes shown. (Call the unit vector aligned with x^*, $\boldsymbol{i}^*$, and the unit vector aligned with y^*, $\boldsymbol{j}^*$.)

c. Determine the magnitude of $\boldsymbol{T}_R$ and the magnitude of $\boldsymbol{T}_R^*$. Are they equal?

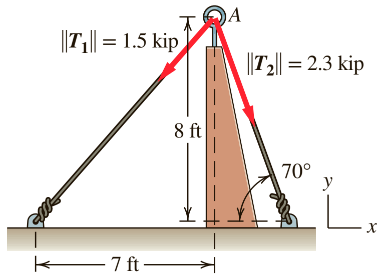


EX 2.6.16

2.6.17. [**] The tensions of two cables ($\boldsymbol{T}_1$ and $\boldsymbol{T}_2$) act on the top of a wall at A, as shown. What is their resultant $\boldsymbol{R}$?

a. Express $\boldsymbol{R}$ in terms of its components along the x and y axes shown. Include a sketch of the components.

b. Express $\boldsymbol{R}$ in terms of its magnitude and space angles. Include a sketch that shows the space angles.

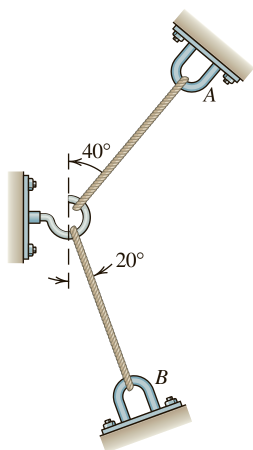


EX 2.6.17

2.6.18. [**] The cables A and B exert forces $\boldsymbol{F}_A$ and $\boldsymbol{F}_B$ on the hook. The magnitude of $\boldsymbol{F}_A$ is 100 N. The tension in cable B has been adjusted so that the resultant force $(\boldsymbol{F}_A + \boldsymbol{F}_B)$ is perpendicular to the wall to which the hook is attached.

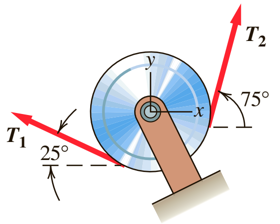
a. Find the magnitude of $\boldsymbol{F}_B$.

b. Specify the resultant force $\boldsymbol{F}_R$ in vector notation.



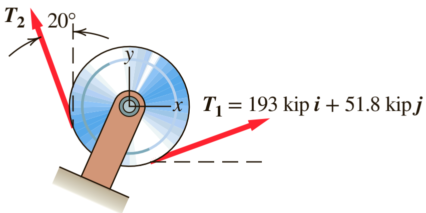
EX 2.6.18

2.6.19. [**] Consider the frictionless pulley shown. Because the pulley is frictionless, $\|\boldsymbol{T}_1\| = \|\boldsymbol{T}_2\|$. Express the resultant $\boldsymbol{R}$ of $\boldsymbol{T}_1 + \boldsymbol{T}_2$ in terms of rectangular components. Also show that the line of action of $\boldsymbol{R}$ bisects the angle between $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$.



EX 2.6.19

2.6.20. [**] The frictionless pulley shown is such that the magnitude of $\boldsymbol{T}_1$ is equal to the magnitude of $\boldsymbol{T}_2$. Based on the information given in the figure, what are the scalar components of $\boldsymbol{T}_2$?

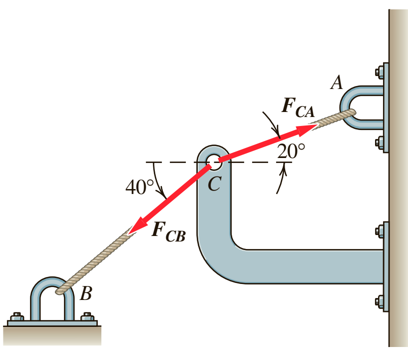


EX 2.6.20

2.6.21. [**] An angle bracket is bolted to a wall as shown. The cables AC and BC exert forces $\boldsymbol{F}_{CA}$ and $\boldsymbol{F}_{CB}$ on the bracket. The magnitude of $\boldsymbol{F}_{CA}$ is 200 lb. The tension in cable BC has been adjusted so that the resultant force $(\boldsymbol{F}_{CA} + \boldsymbol{F}_{CB})$ is vertical.

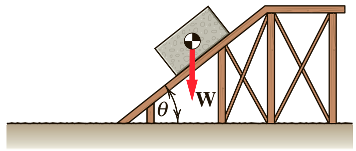
a. Find the magnitude of $\boldsymbol{F}_{CB}$.

b. Specify the resultant force $\boldsymbol{F}_R$ in vector notation.



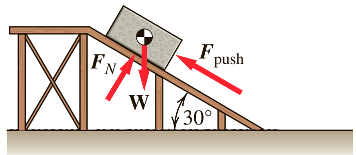
EX 2.6.21

2.6.22. [**] A block of weight W sits on an inclined surface. The block will slide down the surface if the magnitude of $\boldsymbol{F}_{\text{friction}}$ (the friction force between the block and the surface) exceeds 1/3 of the magnitude of the normal force. What is the maximum value of the incline angle θ at which the block will not slide down the surface?



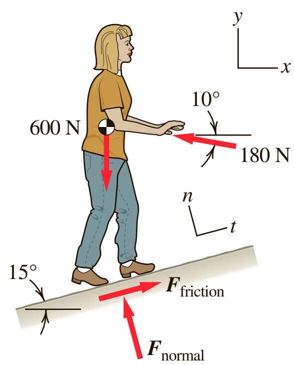
EX 2.6.22

2.6.23. [**] The forces acting on a block are gravity ($\boldsymbol{W}$, magnitude 10 lb), a normal contact force ($\boldsymbol{F}_N$), and a push force ($\boldsymbol{F}_{\text{push}}$), as shown. The surface on which the block rests is smooth, so there is no friction force. If the resultant of the forces acting on the block is zero, use component force addition to determine the magnitudes of $\boldsymbol{F}_N$ and $\boldsymbol{F}_{\text{push}}$.



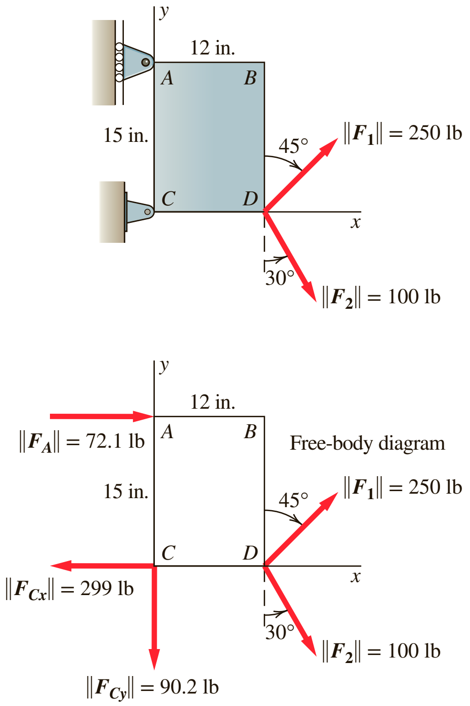
EX 2.6.23

2.6.24. [**] A person who weighs 600 N is pushing a recycling container up a 15° incline. The forces acting on the person are shown. If the resultant force in direction n is zero, what is the magnitude of $\boldsymbol{F}_{\text{normal}}$?



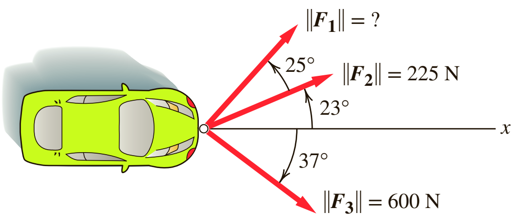
EX 2.6.24

2.6.25. [**] Redraw the free-body diagram shown to represent $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ as a single force $\boldsymbol{F}_R$ acting at D and $\boldsymbol{F}_{Cx}$ and $\boldsymbol{F}_{Cy}$ as a single force $\boldsymbol{F}_C$ acting at C. Write $\boldsymbol{F}_R$ and $\boldsymbol{F}_C$ in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$.



EX 2.6.25

2.6.26. [**] Three ropes attached to a stalled car apply the forces shown. Determine the magnitude of $\boldsymbol{F}_1$ and the magnitude of the resultant $\boldsymbol{F}_R$ if the line of action of the resultant is along the x axis.

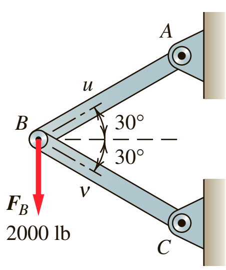


EX 2.6.26

2.6.27. [**] A 2000-lb force is resisted by two pipe struts as shown. Three forces act on the pin at B (the applied load $\boldsymbol{F}_B$, $\boldsymbol{F}_u$, and $\boldsymbol{F}_v$). Given the resultant force acting on the pin is zero, determine

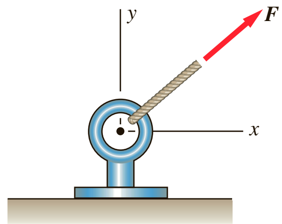
a. the magnitude of $\boldsymbol{F}_u$, the force along the axis of strut AB

b. the magnitude of $\boldsymbol{F}_v$, the force along the axis of strut BC



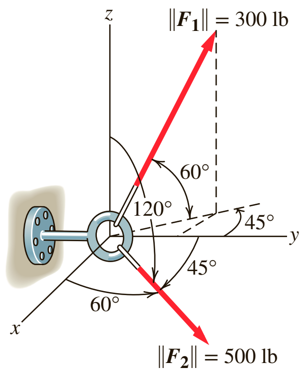
EX 2.6.27

2.6.28. [**] You have designed a bracket to support a force $\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j}$. The design is such that the magnitude of $\boldsymbol{F}$ should not exceed 1000 N. If F_x ranges from -750 N $\leq F_x \leq 500$ N, what is the range of F_y that can be safely supported by the bracket? (Assume that F_x and F_y are independent of each other.)



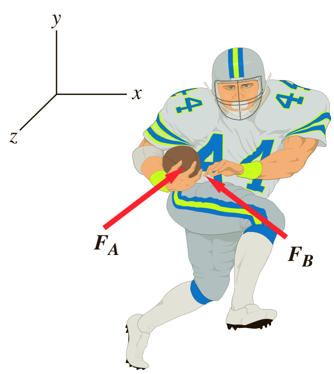
EX 2.6.28

2.6.29. [**] Two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are applied to an eyelet as shown. Determine the resultant $\boldsymbol{T}_R$ $(= \boldsymbol{F}_1 + \boldsymbol{F}_2)$, and write it in vector notation.



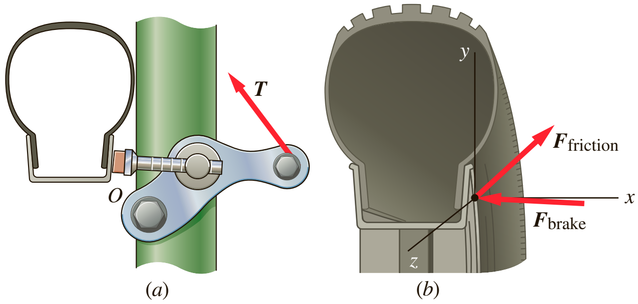
EX 2.6.29

2.6.30. [**] A football player is tackled simultaneously by two players as shown. Tackler A exerts a force $\boldsymbol{F}_A$ on the player of 1000 N along a line of action described by $\boldsymbol{u}_A = 0.216\boldsymbol{i} + 0.108\boldsymbol{j} - 0.970\boldsymbol{k}$. If the resultant force on the player is 1980 N acting along $\boldsymbol{u}_R = -0.139\boldsymbol{i} + 0.096\boldsymbol{j} - 0.986\boldsymbol{k}$, find the magnitude of $\boldsymbol{F}_B$ and write the force in vector notation.



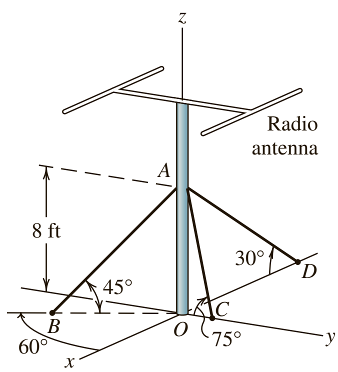
EX 2.6.30

2.6.31. [**] When the brake is applied to the bicycle wheel in (*a*) it generates a force $\boldsymbol{F}_{\text{brake}}$ normal to the rim and a force $\boldsymbol{F}_{\text{friction}}$ parallel to the rim (*b*). If $\|\boldsymbol{F}_{\text{brake}}\|$ is 15 lb and is oriented 3° with respect to the *x*-axis, and $\|\boldsymbol{F}_{\text{friction}}\|$ is 8.25 lb, determine the resultant $\boldsymbol{R}$ force acting on the rim and write it in vector notation.



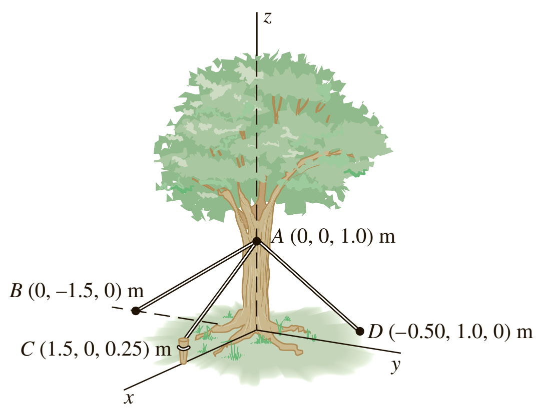
EX 2.6.31

2.6.32. [**] A radio antenna is supported by three guy wires. The tensile force in wires *AB*, *AC*, and *AD* are 20 kN, 25 kN, and 30 kN, respectively. Determine the resultant $\boldsymbol{T_R}$ $(= \boldsymbol{T_{AB}} + \boldsymbol{T_{AC}} + \boldsymbol{T_{AD}})$ acting at *A*, and write it in vector notation.



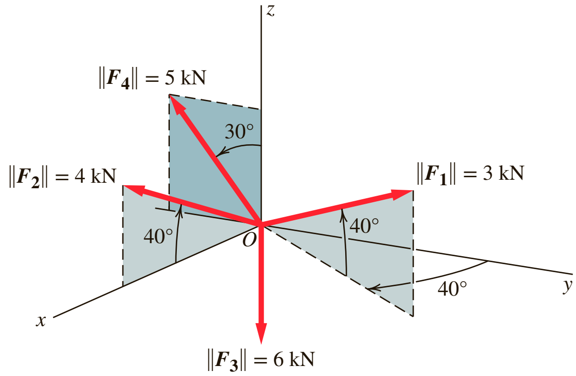
EX 2.6.32

2.6.33. [**] Three cables brace a newly planted tree as shown. Each cable has a tension of 250 N. Determine the resultant force $\boldsymbol{F}_R$ acting on the tree as a result of the three cables. Express $\boldsymbol{F}_R$ in terms of the right-handed orthogonal unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$.



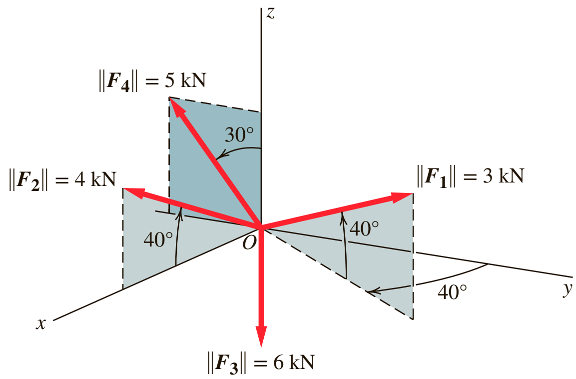
EX 2.6.33

2.6.34. [**] Four forces applied to an object are concurrent at *O* as shown. Determine the magnitude of the resultant $\boldsymbol{F}_R$ of the four forces.



EX 2.6.34

2.6.35. [**] Four forces applied to an object are concurrent at *O* as shown. Determine the space angles θ_x, θ_y, and θ_z between the line of action of the resultant and the positive *x*, *y*, and *z* coordinate axes.



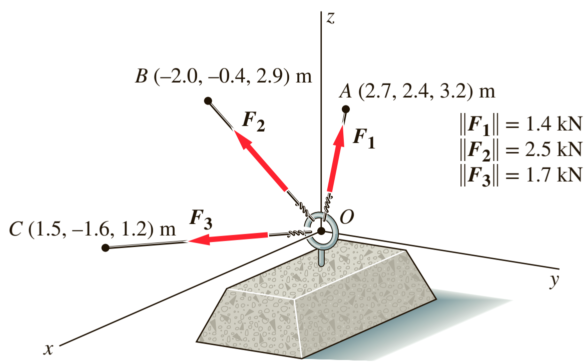
EX 2.6.35

2.6.36. [**] Three forces are applied with cables to the concrete block. Determine

a. the resultant $\boldsymbol{F}_R$ in terms of its rectangular components

b. the magnitude of the force that tends to pull the block upward (in the *z* direction)

c. the magnitude of the force that tends to slide the block along the ground (in the *xy* plane)



EX 2.6.36

2.6.37. [**] Two ropes are used to pull a boat to shore, as shown in Figure 1. The resultant of forces $\boldsymbol{T}$ and $\boldsymbol{P}$ has a magnitude of 300 lb and is directed along the keel line *a-a*. Plot magnitude $\|\boldsymbol{T}\|$ as a function of the angle θ. Where is $\|\boldsymbol{T}\|$ a minimum? Compare your results with Figure 2.

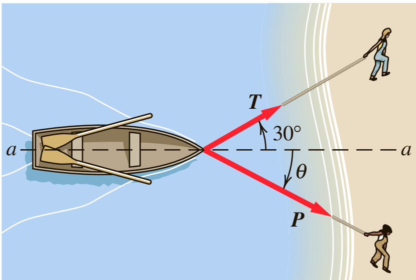


Figure 1

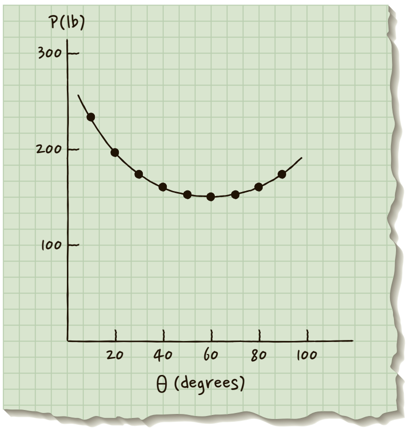
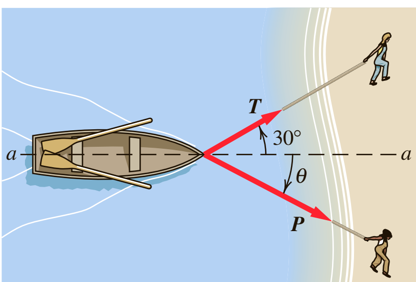


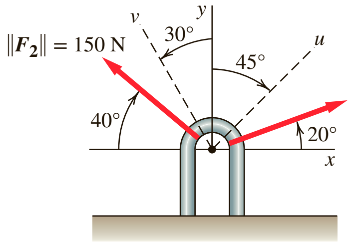
Figure 2

2.6.38. [***] For the boat shown the resultant of forces $\boldsymbol{T}$ and $\boldsymbol{P}$ has a magnitude of 300 lb and is directed along the keel line *a-a*. Write $\boldsymbol{P}$ as a function of θ and use calculus to find θ so that $\boldsymbol{P}$ is minimized. Compare your results with Example 2.6.7.



EX 2.6.38

2.6.39. [***] The two forces acting on a u-bolt have a resultant $\boldsymbol{F}_R = \boldsymbol{F}_1 + \boldsymbol{F}_2$. Find the magnitudes of two other forces $\boldsymbol{F}_u$ and $\boldsymbol{F}_v$ acting in the direction of the u and v axes respectively, which would have the same resultant $\boldsymbol{F}_R$.

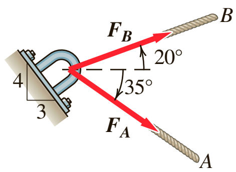


EX 2.6.39

2.6.40. [***] Cables A and B act on a bracket as shown. The tension in cable A is 500 N, oriented as shown. The orientation of $\boldsymbol{F}_B$ is at $\theta = 20°$, as shown.

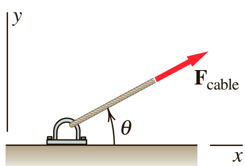
a. Write an expression (in vector notation) for the resultant force $\boldsymbol{F}_R$ acting on the bracket. Use a coordinate system perpendicular and parallel to the inclined surface.

b. If the bracket will pull away from the wall when the component of $\boldsymbol{F}_R$ parallel to the wall is greater than or equal to 550 N and the component of $\boldsymbol{F}_R$ perpendicular to the wall is greater than or equal to 600 N, how large can the magnitude of $\boldsymbol{F}_B$ be?



EX 2.6.40

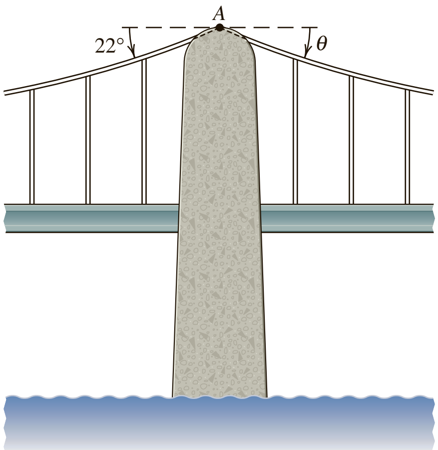
2.6.41. [***] A bracket is glued to the floor, and a cable is attached as shown. If $\boldsymbol{F}_{\text{shear}}$ (the shear force between the bracket and the floor) is not to exceed 50 N and $\boldsymbol{F}_{\text{tension}}$ (the tension force between the bracket and the floor) is not to exceed 100 N, what is the maximum allowable tension in the cable? (*Hint:* Graphical addition of $\boldsymbol{F}_{\text{shear}}$, $\boldsymbol{F}_{\text{tension}}$, and $\boldsymbol{F}_{\text{cable}}$ should result in a triangle.)



EX 2.6.41

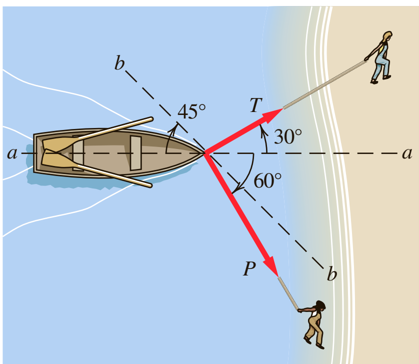
2.6.42. [***] A net downward force of 150 MN acts on a suspension bridge tower as a result of the main cable

passing over the top of the tower at A. If the tension in the cable just to the left of the tower is 220 MN, determine the tension in the cable just to the right of the tower as well as the angle of the cable relative to horizontal.



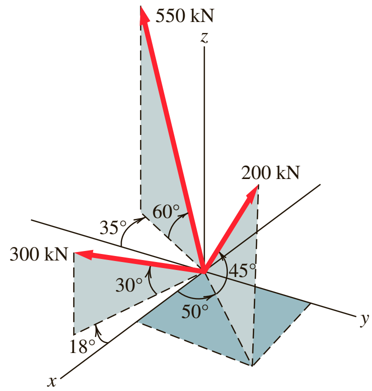
EX 2.6.42

2.6.43. [***] The boat tenders want to turn the boat towards its starboard side by pulling the boat with a resultant force of 300 lb directed along a line b-b that is 45° clockwise from a-a. Find the required forces, $\|\boldsymbol{T}\|$ and $\|\boldsymbol{P}\|$, to produce this resultant.



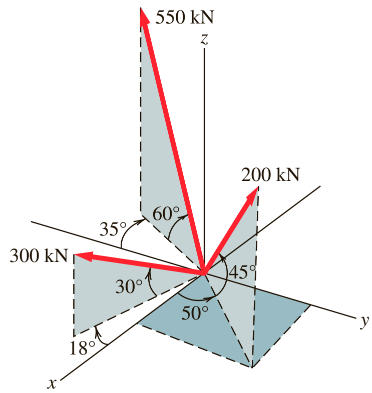
EX 2.6.43

2.6.44. [***] Three forces act as shown. Use rectangular component addition to determine the magnitude of the resultant $\boldsymbol{F}_R$ of the three forces.



EX 2.6.44

2.6.45. [***] Three forces act as shown. Determine the space angles θ_x, θ_y, and θ_z between the line of action of the resultant and the positive x, y, and z coordinate axes.



EX 2.6.45

2.7 ANGLE BETWEEN TWO FORCES—THE DOT PRODUCT

Learning Objective: Use the dot product to find the components of a force vector or the angle between two forces.

Let's return to the situation of pulling on a tent stake (**Figure 2.6.1**). We have learned how to find the resultant force $\boldsymbol{F}_R$ by adding $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ using graphical addition, geometric/trigonometric addition, and component addition. In this section we ask the question, "Will the force you and your friend exert be greater than the gripping force holding the stake in the ground?"

We can answer this question if we realize that the scalar component of the resultant force $\boldsymbol{F}_R$ along the axis of the stake pulls the stake out of the ground, whereas the component perpendicular to the stake pushes the stake against the side of the hole. As noted in **Figure 2.7.1**, we call the scalar component along the axis of the stake F_p, where the subscript "p" stands for projection. If we define a unit vector $\boldsymbol{u}$ along the stake axis, the scalar quantity F_p is the **projection of $\boldsymbol{F}_R$ onto the line defined by $\boldsymbol{u}$**. The geometry in **Figure 2.7.1** shows that the magnitude of this projection is

$$F_p = \|\boldsymbol{F}_R\|\cos\theta \tag{2.25}$$

where θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{F}_R$.

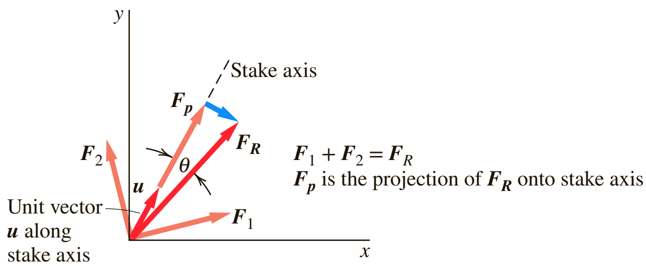


Figure 2.7.1 Finding the projection of $\boldsymbol{F}_R$ onto $\boldsymbol{u}$.

In terms of formal mathematical constructs, we have just taken the **dot product**, or **scalar product**, of vectors $\boldsymbol{F}_R$ and $\boldsymbol{u}$. The dot product of two vectors $\boldsymbol{V_1}$ and $\boldsymbol{V_2}$ is a scalar quantity and is formally written

$$\text{dot product} = \boldsymbol{V_1} \bullet \boldsymbol{V_2} = \boldsymbol{V_2} \bullet \boldsymbol{V_1} = \|\boldsymbol{V_1}\| \underbrace{\|\boldsymbol{V_2}\|\cos\theta}_{\substack{\text{component of } \boldsymbol{V_2} \\ \text{in direction of } \boldsymbol{V_1}}} \tag{2.26}$$

where θ is the angle between the two vectors and $0° \le \theta \le 180°$. Thus the dot product can be interpreted as a means of finding the component of $\boldsymbol{V_1}$ in the direction of $\boldsymbol{V_2}$, multiplied by the magnitude of $\boldsymbol{V_2}$ (or the component of $\boldsymbol{V_2}$ in the direction of $\boldsymbol{V_1}$ multiplied by the magnitude of $\boldsymbol{V_1}$).

In the case of you and your friend pulling on the tent stake, we are interested in the component of $\boldsymbol{V}_R$ in the direction of $\boldsymbol{u}$. The dot product gives:

$$F_p = \boldsymbol{u} \bullet \boldsymbol{F}_R = \|\boldsymbol{u}\|\,\|\boldsymbol{F}_R\|\cos\theta$$

This expression is the same as (2.25) because $\|\boldsymbol{u}\| = 1$. The dot product of two vectors $\boldsymbol{V_1}$ and $\boldsymbol{V_2}$ can also be found by considering the vectors in terms of their scalar components. For example, if

$$\boldsymbol{V_1} = V_{1x}\boldsymbol{i} + V_{1y}\boldsymbol{j} + V_{1z}\boldsymbol{k} \quad \text{and} \quad \boldsymbol{V_2} = V_{2x}\boldsymbol{i} + V_{2y}\boldsymbol{j} + V_{2z}\boldsymbol{k} \tag{2.27}$$

then by the distributive law

$$\begin{aligned}\boldsymbol{V_1} \bullet \boldsymbol{V_2} &= (V_{1x}\boldsymbol{i} + V_{1y}\boldsymbol{j} + V_{1z}\boldsymbol{k}) \bullet (V_{2x}\boldsymbol{i} + V_{2y}\boldsymbol{j} + V_{2z}\boldsymbol{k}) \\ &= V_{1x}V_{2x}(\boldsymbol{i}\bullet\boldsymbol{i}) + V_{1x}V_{2y}(\boldsymbol{i}\bullet\boldsymbol{j}) + V_{1x}V_{2z}(\boldsymbol{i}\bullet\boldsymbol{k}) \\ &\quad + V_{1y}V_{2x}(\boldsymbol{j}\bullet\boldsymbol{i}) + V_{1y}V_{2y}(\boldsymbol{j}\bullet\boldsymbol{j}) + V_{1y}V_{2z}(\boldsymbol{j}\bullet\boldsymbol{k}) \\ &\quad + V_{1z}V_{2x}(\boldsymbol{k}\bullet\boldsymbol{i}) + V_{1z}V_{2y}(\boldsymbol{k}\bullet\boldsymbol{j}) + V_{1z}V_{2z}(\boldsymbol{k}\bullet\boldsymbol{k})\end{aligned} \tag{2.28}$$

Once we carry out the dot-product operations, (2.28) becomes

$$\boldsymbol{V_1} \bullet \boldsymbol{V_2} = V_{1x}V_{2x} + V_{1y}V_{2y} + V_{1z}V_{2z} \tag{2.29}$$

Equations (2.26) and (2.29) are equivalent expressions for the dot product of two vectors that we can set equal to one another:

$$\boldsymbol{V_1} \bullet \boldsymbol{V_2} = \|\boldsymbol{V_1}\|\|\boldsymbol{V_2}\|\cos\theta = V_{1x}V_{2x} + V_{1y}V_{2y} + V_{1z}V_{2z} \tag{2.30}$$

Rearranging this expression and solving for the angle θ between the tails of the two vectors, we have

$$\cos\theta = \frac{V_{1x}V_{2x} + V_{1y}V_{2y} + V_{1z}V_{2z}}{\|\boldsymbol{V_1}\|\|\boldsymbol{V_2}\|}; \qquad 0° < \theta < 180° \tag{2.31}$$

If we know $\boldsymbol{V_1}$ and $\boldsymbol{V_2}$ in terms of their scalar components, this expression can be used to find the angle θ between the two vectors.

Now back to the tent stake and our desire to find F_p, the scalar component that is the projection of $\boldsymbol{F_R}$ onto the tent stake axis. If we know θ, the angle between $\boldsymbol{F_R}$ and the stake axis, it makes most sense to use (2.26) to find F_p.

If, instead, we know the scalar components of $\boldsymbol{F_R}$ and the components of $\boldsymbol{u}$, it makes more sense to use (2.29) to find F_p. Figure 2.7.2 illustrates how to use the dot product to find F_p, based on our previously found value $\|\boldsymbol{F_R}\| = 333$ N ($\|\boldsymbol{F_R}\|$ is determined in **Figure 2.6.2.4**).

Once we have found F_p, we can compare its absolute value $|F_p|$ with the absolute value $|F_{grip}|$ of the grip force F_{grip} exerted on the stake by the ground (**Figure 2.7.2**). If $|F_p| \geq |F_{grip}|$, you and your friend should be able to remove the stake from the ground simply by pulling. If $|F_p| < |F_{grip}|$ and you still want to remove the stake from the ground, you could increase $|F_p|$ and/or reduce $|F_{grip}|$. What are several ways you might do this?

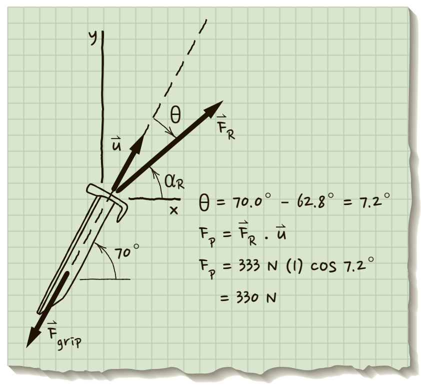


Figure 2.7.2 Components of force parallel F_p and normal F_N to the axis of the tent stake.

Check out the following examples of applications of this material.

- **Example 2.7.1 Projection of a Vector in Two Dimensions**
- **Example 2.7.2 Projection of a Vector in Three Dimensions**
- **Example 2.7.3 Angle between Two Vectors**

EXAMPLE 2.7.1

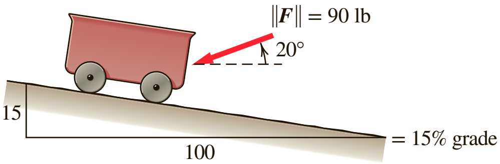


Figure 1 A cart being pushed up a 15% grade.

A cart is pushed up a hill with a 15% grade at a constant speed by a 90-lb force $\boldsymbol{F}$, as shown in **Figure 1**. The force is applied to the cart at an angle of 20° relative to the horizontal. Determine the magnitude and direction of the component vector of $\boldsymbol{F}$ that is parallel to the grade.

Goal Find the magnitude and direction of the component vector of $\boldsymbol{F}$ parallel to the grade.

Given The magnitude of $\boldsymbol{F}$ and its orientation relative to the horizontal, as well as the grade of the hill.

Assume $\boldsymbol{F}$ lies in xy plane so that the problem is planar.

Draw Because we were given the orientation of $\boldsymbol{F}$ with respect to the horizontal, establishing a coordinate system as shown in **Figure 2** will allow us to easily calculate the scalar components of $\boldsymbol{F}$.

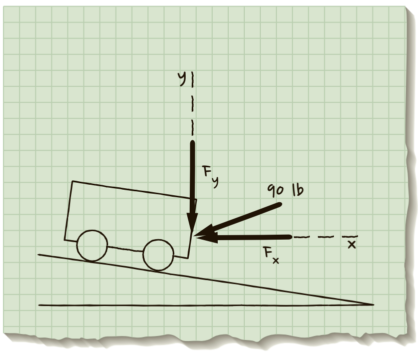


Figure 2 The components of $\boldsymbol{F}$ relative to the defined coordinate system.

Formulate Equations and Solve We calculate the scalar components of $\boldsymbol{F}$:

$$F_x = \|\boldsymbol{F}\|\cos\theta = (90.0\,\text{lb})(\cos 20°) = 84.6\,\text{lb}$$
$$F_y = \|\boldsymbol{F}\|\sin\theta = (90.0\,\text{lb})(\sin 20°) = 30.8\,\text{lb}$$

This allows us to write $\boldsymbol{F}$ as $\boldsymbol{F} = -84.6\text{ lb }\boldsymbol{i} - 30.8\text{ lb }\boldsymbol{j}$.

The vector component we must find is the projection of $\boldsymbol{F}$ onto an axis parallel to the grade. We determine the angle α associated with 15% grade from the geometry shown in **Figure 1**:

$$\alpha = \arctan\frac{15}{100} = 8.53°$$

The direction parallel to the surface is defined by a unit vector $\boldsymbol{u}$ (**Figure 3**), which we arbitrarily choose to point to the right (down the slope). The rectangular components of $\boldsymbol{u} = u_x\boldsymbol{i} + u_y\boldsymbol{j}$ are

$$u_x = \cos\alpha = \cos 8.53° = 0.989$$
$$u_y = -\sin\alpha = -\sin 8.53° = -0.148$$

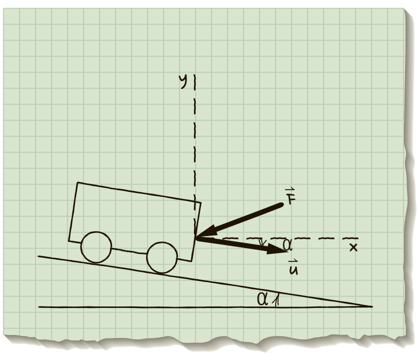


Figure 3 Unit vector acting parallel to sloped surface.

The projection of $\boldsymbol{F}$ onto $\boldsymbol{u}$ is the dot product $\boldsymbol{u}\cdot\boldsymbol{F}$. Based on (2.29) we write

$$\left\|\boldsymbol{F}_{\text{parallel}}\right\| = \boldsymbol{u}\cdot\boldsymbol{F} = u_xF_x + u_yF_y = 0.989(-84.6\,\text{lb}) + (-0.148)(-30.8\,\text{lb})$$
$$\Rightarrow \boxed{\left\|\boldsymbol{F}_{\text{parallel}}\right\| = -79.1\,\text{lb}}$$

This dot product has a magnitude of 79.1 lb and is negative (meaning it is along the -$\boldsymbol{u}$ direction, or up the slope).

Check The magnitude and direction we found for $\left\|\boldsymbol{F}_{\text{parallel}}\right\|$ are consistent with the problem statement. Since $\boldsymbol{F}$ is at an angle of only 28.5° above the slope, the component parallel to the slope is relatively large, which makes sense.

EXAMPLE 2.7.2

It is common practice in describing the resultant force acting on a surface to refer to the component vector normal to the surface as a pull force ($\boldsymbol{F}_{\text{pull}}$) if it points away from the surface and as a push force ($\boldsymbol{F}_{\text{push}}$) if it points toward the surface. The component vector of the resultant force parallel to the surface is commonly called the tangential force ($\boldsymbol{F}_{\text{tang}}$).

Consider the force $\boldsymbol{F}$ acting on the sloped rectangular surface shown in Figure 1. $\boldsymbol{F}$ is oriented at an angle α relative to the xz plane and β relative to the x axis, and its magnitude is $\|\boldsymbol{F}\| = 6.00\,\text{kN}$. Determine the magnitudes of the pull (or push) and tangential forces acting on the surface.

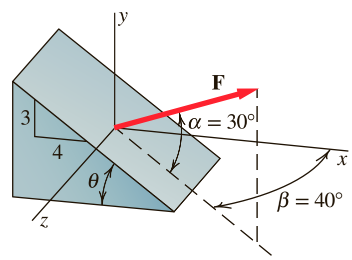


Figure 1 A 6.00-kN force acting on a sloped surface.

Goal Find the components of $\boldsymbol{F}$ perpendicular to and tangential to the plane of the sloped surface.

Given $\|\boldsymbol{F}\| = 6.00\,\text{kN}$; the orientation of $\boldsymbol{F}$ and surface relative to a right-handed coordinate system.

Assume No assumptions are required.

Draw We draw $\boldsymbol{F}$ in terms of its scalar components F_x, F_y, F_z (Figure 2).

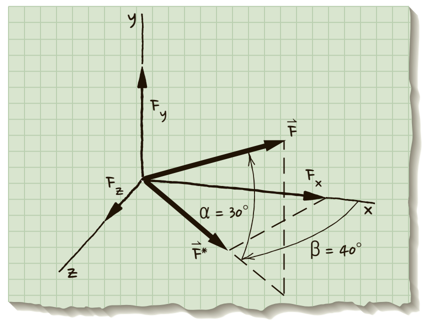


Figure 2 The scalar components of $\boldsymbol{F}$ relative to the coordinate system defined in **Figure 1**.

Formulate Equations and Solve Because we are given orientation information of $\boldsymbol{F}$ based on a horizontal x axis and a vertical y axis, we start by finding the components of $\boldsymbol{F}$ relative to the x, y, and z axes.

We first find the scalar components of $\boldsymbol{F}$ parallel to the y axis (F_y) and in the xz plane (F^*). We then divide F^* into components in the x and z directions. Based on the geometry in **Figure 2**, the scalar projection of $\boldsymbol{F}$ onto the xz plane is:

$$F^* = \|\boldsymbol{F}\|\cos\alpha = (6.00\,\text{kN})(\cos 30°) = 5.20\,\text{kN}$$

(What we have just done is take the dot product of $\boldsymbol{F}$ and a unit vector along $\boldsymbol{F}^*$.) Next we find the scalar components of $\boldsymbol{F}$ in the x, z, and y directions:

$$F_x = F^*\cos\beta = (5.20\,\text{kN})(\cos 40°) = 3.98\,\text{kN}$$
$$F_z = F^*\sin\beta = (5.20\,\text{kN})(\sin 40°) = 3.34\,\text{kN}$$
$$F_y = \|\boldsymbol{F}\|\,\sin\alpha = (6.00\,\text{kN})(\sin 30°) = 3.00\,\text{kN}$$

This allows us to write $\boldsymbol{F}$ as

$$\boldsymbol{F} = 3.98\,\text{kN}\boldsymbol{i} + 3.00\,\text{kN}\boldsymbol{j} + 3.34\,\text{kN}\boldsymbol{k}$$

Now we find the projection of $\boldsymbol{F}$ onto an axis perpendicular to the sloped surface. As shown in Figure 3, the surface is at an angle θ of 36.9° to the horizontal. The direction perpendicular to the surface is defined by the unit vector $\boldsymbol{u}$. The scalar components of $\boldsymbol{u}$ are

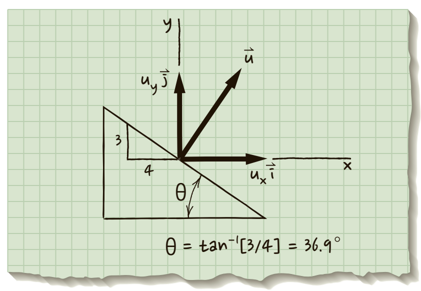


Figure 3 The direction perpendicular to the sloped surface is defined by the unit vector $\boldsymbol{u}$.

$$u_x = \sin\theta = \sin 36.9° = 0.600$$
$$u_y = \cos\theta = \cos 36.9° = 0.800$$
$$u_z = 0$$

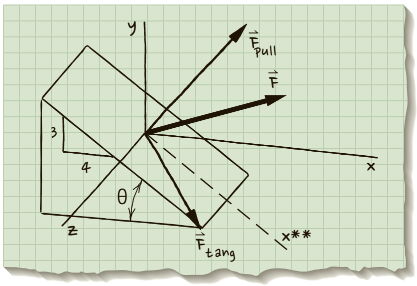


Figure 4 Pull force acts normal to surface and tangential force acts parallel to surface.

or in Cartesian vector notation:

$$u = u_x\boldsymbol{i} + u_y\boldsymbol{j} + u_z\boldsymbol{k} = 0.600\,\boldsymbol{i} + 0.800\,\boldsymbol{j}$$

The projection of $\boldsymbol{F}$ onto $\boldsymbol{u}$ is the dot product $\boldsymbol{u}\cdot\boldsymbol{F}$. Based on (2.30) the dot product is given as

$$\begin{aligned}\boldsymbol{u}\cdot\boldsymbol{F} &= u_xF_x + u_yF_y + u_zF_z\\ &= 0.600(3.98\,\text{kN}) + 0.800(3.00\,\text{kN}) + 0(3.34\,\text{kN})\\ &= 4.79\,\text{kN}\end{aligned}$$

This dot product has a magnitude of 4.79 kN and is positive (meaning it is along the $+\boldsymbol{u}$ direction and, as such, indicates a pull force). Therefore we denote the force perpendicular to the surface as $\boldsymbol{F}_{\text{pull}}$, with

$$\boxed{\|\boldsymbol{F}_{\text{pull}}\| = 4.79\,\text{kN}.}$$

The tangential force, $\boldsymbol{F}_{\text{tang}}$, is the component vector of $\boldsymbol{F}$ that is parallel to the sloped surface and therefore perpendicular to $\boldsymbol{F}_{\text{pull}}$. Using the Pythagorean theorem, we can write

$$\|\boldsymbol{F}_{\text{tang}}\| = \sqrt{\|\boldsymbol{F}\|^2 - \|\boldsymbol{F}_{\text{pull}}\|^2} = \sqrt{(6.00\,\text{kN})^2 - (4.79\,\text{kN})^2}$$

$$\Rightarrow \boxed{\|\boldsymbol{F}_{\text{tang}}\| = 3.61\,\text{kN}}$$

Alternately, we could have found $\|\boldsymbol{F}_{\text{tang}}\|$ by finding the projections of $\boldsymbol{F}$ onto the z and x^{**} axes (as defined in Figure 4) using the dot product two times, then summing these projections using the Pythagorean theorem—but this would have involved a lot more work!

Check To check our results we could use an alternate method to calculate the components.

EXAMPLE 2.7.3

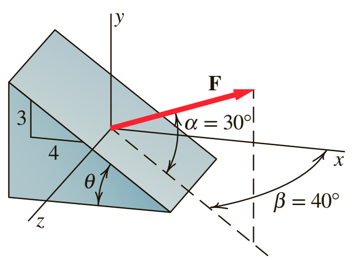


Figure 1 A 6.00-kN force acting on a sloped surface.

Consider the 6.00-kN force $\boldsymbol{F}$ acting on the sloped surface in Figure 1. Determine the angle between the line of action of $\boldsymbol{F}$ and the x axis.

Goal Find the angle between the line of action of $\boldsymbol{F}$ and the x axis.

Given $\|\boldsymbol{F}\| = 6.00\,\text{kN}$; the orientation of $\boldsymbol{F}$ and surface relative to a right-handed coordinate system.

Assume No assumptions are required.

Draw We draw $\boldsymbol{F}$ in terms of its scalar components F_x, F_y, F_z (Figure 2).

Formulate Equations and Solve Another way of stating our goal is that we are to find the angle between $\boldsymbol{F}$ and the vector $\boldsymbol{i}$ (a unit vector along the x axis). Equation (2.31) enables us to find the angle between two vectors based on their components. Therefore, the angle θ

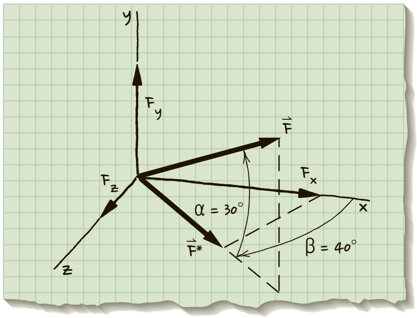


Figure 2 The scalar components of ***F*** relative to the coordinate system defined in **Figure 1**.

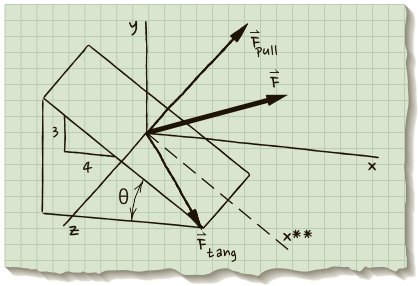


Figure 3 Pull force acts normal to surface and tangential force acts parallel to surface.

between $\boldsymbol{F}(= 3.98\text{ kN}\boldsymbol{i} + 3.00\text{ kN}\boldsymbol{j} + 3.34\text{ kN}\boldsymbol{k})$ (from Example 2.7.2) and $\boldsymbol{i} = 1\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}$ is

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{F}\cdot\boldsymbol{i}}{\|\boldsymbol{F}\|\|\boldsymbol{i}\|}\right) = \cos^{-1}\left(\frac{F_x(1) + F_y(0) + F_z(0)}{\|\boldsymbol{F}\|(1)}\right)$$

$$= \cos^{-1}\left(\frac{3.98\,\text{kN}(1)}{6.00\,\text{kN}(1)}\right) \Rightarrow \boxed{\theta = 48.4^\circ}$$

Check Inspection of **Figure 3** shows that θ is the space angle θ_x, and so we can calculate this space angle using the scalar component F_x to confirm that it is 48.4°:

$$\cos\theta_x = \frac{F_x}{\|\boldsymbol{F}\|} = \frac{3.98\,\text{N}}{6.00\,\text{N}} \Rightarrow \theta_x = 48.4^\circ$$

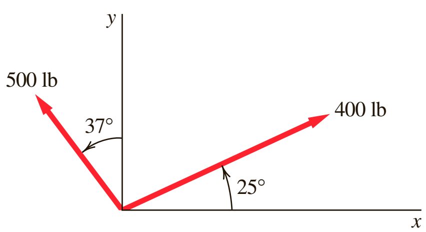
This is the same answer we found before.

EXERCISES 2.7

2.7.1. [*] Determine the dot product of the two forces shown by

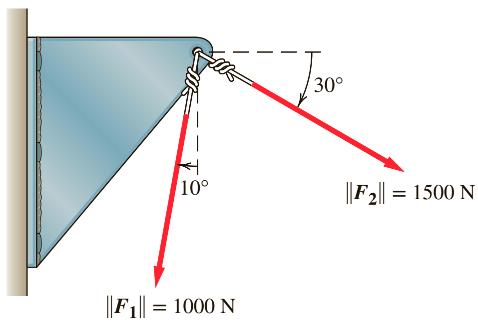
a. using the definition given in (2.26)

b. using the definition given in (2.29)



EX 2.7.1

2.7.2. [*] Determine the dot product of the two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ applied to the bracket. Use this dot product to show that the angle between the two forces is 70°.

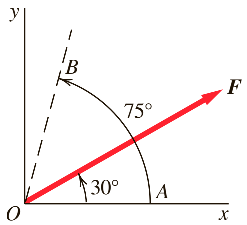


EX 2.7.2

2.7.3. [*] Consider the 800-lb force $\boldsymbol{F}$.

a. Write an expression for the projection of $\boldsymbol{F}$ in the direction OA. Use this expression to determine the value of the projection.

b. Write an expression for the projection of $\boldsymbol{F}$ in the direction OB. Use this expression to determine the value of the projection.



EX 2.7.3

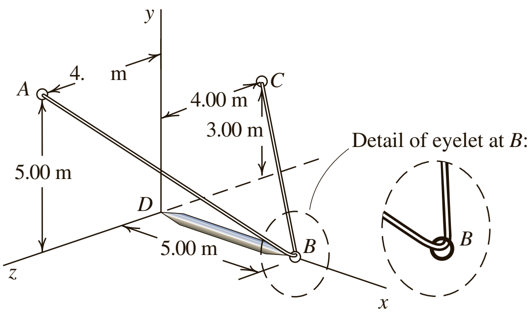
2.7.4. [*] Let $\boldsymbol{A} = t\boldsymbol{i} - 3\boldsymbol{j}$ and $\boldsymbol{B} = 5\boldsymbol{i} + 7\boldsymbol{j}$, where t is a scalar. Find t so that $\boldsymbol{A}$ and $\boldsymbol{B}$ are perpendicular to each other.

2.7.5. [*] Let $\boldsymbol{A} = 3\boldsymbol{i} + 6\boldsymbol{j} - 2\boldsymbol{k}$, $\boldsymbol{B} = -3\boldsymbol{j} + 4\boldsymbol{k}$, and $\boldsymbol{C} = 1\boldsymbol{i} - 7\boldsymbol{j} + 4\boldsymbol{k}$. Determine

a. $(\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}$

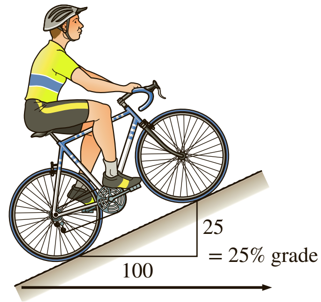
b. $\boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{C})$ $(-\boldsymbol{A}) \cdot (3\boldsymbol{B} - 5\boldsymbol{C})$

2.7.6. [*] Boom BD is held up by a continuous cable that runs from A through an eyelet at B to C. The cable is attached to the wall via hooks at A and C. Denote as $\boldsymbol{T}_{BC}$ the force applied by cable BC on member BD. Determine the magnitude of the projection of $\boldsymbol{T}_{BC}$ along line AB. The tension in the cable is 5 kN.



EX 2.7.6

2.7.7. [**] The bicycle and cyclist moving up the incline have a combined weight of 180 lb. Use the dot product to determine the magnitude of the component of the 180-lb gravity force directed down the incline. "Percent grade" defines the rise (upward movement) relative to the run (horizontal movement). For example, a 25% grade means that for every 100 ft of horizontal movement (run) there is 25 ft of upward movement (rise).



EX 2.7.7

2.7.8. [**] Use the properties of the dot product to prove the identity

$$(\boldsymbol{A} + \boldsymbol{B}) \cdot (\boldsymbol{A} - \boldsymbol{B}) = \|\boldsymbol{A}\|^2 - \|\boldsymbol{B}\|^2$$

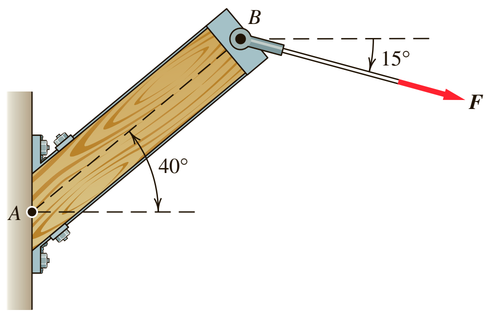
2.7.9. [**] Use the dot product to show that the four points $P = (1, 2)$, $Q = (2, 3)$, $R = (1, 4)$, and $S = (0, 3)$, are the vertices of a square.

2.7.10. [**] A force $\boldsymbol{F}$ acts on the end of the beam in the direction shown.

a. Find the component of $\boldsymbol{F}$ in the direction AB. Call this value F_{AB}.

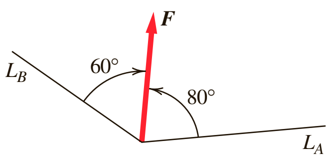
b. Find the component of $\boldsymbol{F}$ in the direction perpendicular to AB. Call this value F_{perp}.

c. For the beam design to be acceptable, neither F_{AB} nor F_{perp} should exceed 2.0 kN. What is the largest acceptable magnitude of $\boldsymbol{F}$?



EX 2.7.10

2.7.11. [**] A force $\boldsymbol{F}$ lies in the plane defined by the lines L_A and L_B. Its magnitude is 400 N. Resolve $\boldsymbol{F}$ into projections parallel to L_A and L_B.



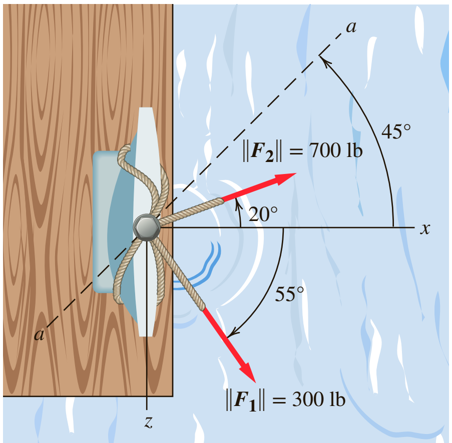
EX 2.7.11

2.7.12. [**] Two boats are attached to a cleat on a dock by their bow lines as shown.

a. Determine the magnitude of the projection of $\boldsymbol{F}_1$ along a-a.

b. Determine the magnitude of the projection of $\boldsymbol{F}_2$ along *a-a*.

c. Use the results from **a** and **b** to find the total force at O in the *a-a* direction due to $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$.



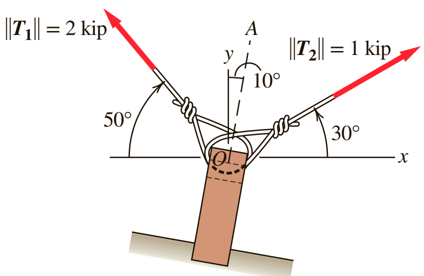
EX 2.7.12

2.7.13. [**] Two support cables are tethered to a support at O. The support is embedded in concrete as shown.

a. Determine the magnitude of the projection of $\boldsymbol{T}_1$ onto O-A.

b. Determine the magnitude of the projection of $\boldsymbol{T}_2$ onto O-A.

c. Engineers are concerned that the support will pull out of the concrete if the force pulling along the axis of the support is greater than 2500 lb. Use the results from ***a*** and ***b*** to analyze the situation and determine if failure will occur.



EX 2.7.13

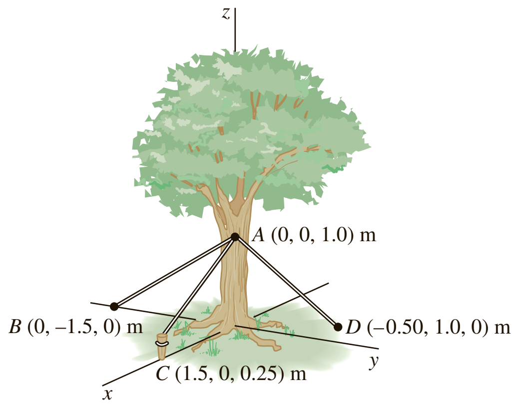
2.7.14. [**] Let $\boldsymbol{A} = \boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k}$ and $\boldsymbol{B} = 4\boldsymbol{i} - \boldsymbol{j} + 7\boldsymbol{k}$. Determine the scalar component of $\boldsymbol{A}$ in the direction of $\boldsymbol{B}$.

2.7.15. [**] Three cables are attached to a tree as shown. Determine the angles between

a. cables AB and AC

b. cables AB and AD

c. cables AC and AD



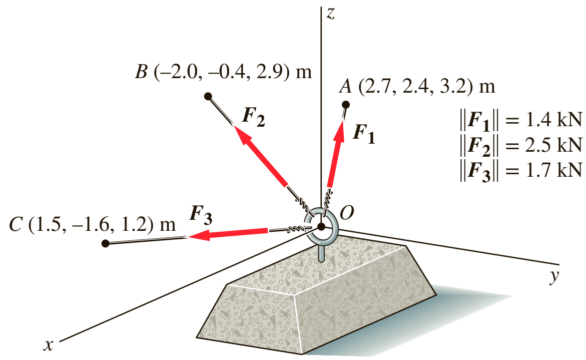
EX 2.7.15

2.7.16. [***] Thee cables are attached to a concrete block as shown. Determine the angles between

a. cables OA and OB

b. cables OA and OC

c. cables OB and OC

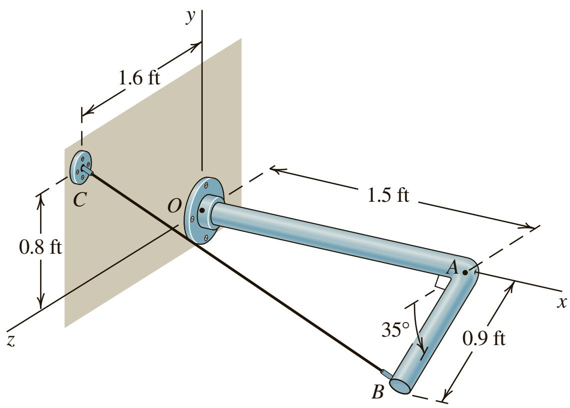


EX 2.7.16

2.7.17. [***] A cable is attached at B to support the frame OAB. The tension in the cable is 500 lb.

a. Determine the angle between the cable and member BA.

b. Determine the components of $\boldsymbol{F}_{BC}$ (the cable force acting at B) parallel and perpendicular to member BA.



EX 2.7.17

2.8 JUST THE FACTS

What Are Forces? An Overview

A **force** is any interaction between an object and the rest of the world that tends to affect the state of motion of the object. It is a vector quantity and is therefore specified in terms of **magnitude** and **direction**. In this text we denote a force in boldface italic (e.g., $\boldsymbol{F}$) and its magnitude as $\|\boldsymbol{F}\|$. In drawings we use an arrow to represent a force. The line along which the force acts is called its **line of action**. The point on which it acts on an object is the force's **point of application**.

Gravitational Forces

Gravitational force is the attractive force between any two objects. Its magnitude is related to the masses of the two objects and the distance between their centers of mass by

$$\|\boldsymbol{F}_g\| = \frac{Gm_1 m_2}{2} \tag{2.1}$$

and its line of action is along a line connecting the centers of mass. This expression for gravitational force is used to determine an expression for the gravitational force acting on an object on or near Earth. We refer to this gravitational force as an object's **weight** on Earth, and it is given in newtons as

$$\|\boldsymbol{W}_{Earth}\| = \|\boldsymbol{F}_g\| = m(9.81 \text{ m/s}^2) \tag{2.3B}$$

Contact Forces

Contact forces include **normal contact forces**, **friction forces**, **fluid contact forces, tension forces** (forces from atoms within an object pulling on one another), **compression forces**, and **shear forces**. The normal contact force, friction force, and fluid contact force act on the *surface* of an object and result from contact between the object and the rest of the world. The tension force, compression force, and shear force occur *within* an object and result from the interactions of the atoms that make up the object.

Identifying Forces for Analysis

In this chapter we also looked at zooming in on a part of the world (we call that small part our **system**), drawing an imaginary boundary around the system to isolate it from the rest of the world, and identifying the **external forces** acting on it, which may be gravitational forces, contact forces, or both.

Representing Force Vectors

Since a force is a vector quantity, it is specified by a magnitude and direction. There are three commonly used ways of specifying this information;

rectangular components, **space angles**, and **spherical-angles**. Key equations related to these representations are summarized in Table 2.3.

Representing Force Vectors: Rectangular Component Representation

A force $\boldsymbol{F}$ may be represented by specifying its three **rectangular component vectors** F_x, F_y, and F_z relative to a right-handed coordinate system. These component vectors consist of **scalar components** and unit vectors in the $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ directions:

$$\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k} \tag{2.4B}$$

The scalar components F_x, F_y, and F_z are the projections of $\boldsymbol{F}$ onto the x, y, and z axes, respectively, and the sum of their squares is related to the magnitude of the force:

$$\|\boldsymbol{F}\| = \sqrt{F_x^2 + F_y^2 + F_z^2} \tag{2.5}$$

A force may also be specified in terms of its magnitude and a unit vector $\boldsymbol{u}$ aligned with the line of action of $\boldsymbol{F}$.

$$\boldsymbol{F} = \|\boldsymbol{F}\|\boldsymbol{u} = \|\boldsymbol{F}\|(u_x\boldsymbol{i} + u_y\boldsymbol{j} + u_z\boldsymbol{k}) \tag{2.6}$$

The components of $\boldsymbol{u}$ can be determined from system geometry

$$\begin{aligned} u_x &= \left(\frac{x_B - x_A}{L}\right) \\ u_y &= \left(\frac{y_B - y_A}{L}\right) \\ u_z &= \left(\frac{z_B - z_A}{L}\right) \end{aligned} \tag{2.7}$$

where $L = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$ is the distance from A to B.

We call a force that lies in the plane of two of the reference axes either a **planar force** or a **two-dimensional force**; otherwise it is called either a **nonplanar force** or a **three-dimensional force**. A planar force has at least one zero force component. For example, a force that lies in the xy plane would have a zero z-component:

$$\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j} \tag{2.10A}$$

$$\boldsymbol{F} = \|\boldsymbol{F}\|(u_x\boldsymbol{i} + u_y\boldsymbol{j}) \tag{2.10B}$$

Key equations related to rectangular component representation are summarized in Table 2.3.

Representing Force Vectors: Space Angle Representation

The direction of a force also can be specified in terms of space angles relative to a right-handed coordinate system. **Space angles** are given in terms of two points along the line of action of the force as

$$\theta_x = \cos^{-1}\left(\frac{x_B - x_A}{L}\right)$$
$$\theta_y = \cos^{-1}\left(\frac{y_B - y_A}{L}\right) \qquad (2.11)$$
$$\theta_z = \cos^{-1}\left(\frac{z_B - z_A}{L}\right)$$

where $L = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$ is the distance from A to B (two points along the line of action of the force).

The angles θ_x, θ_y, and θ_z are not independent of one another. They are related by the expression

$$\sqrt{(\cos\theta_x)^2 + (\cos\theta_y)^2 + (\cos\theta_z)^2} = 1 \qquad (2.12\text{A})$$

where $\cos\theta_x$, $\cos\theta_y$, and $\cos\theta_z$ are the **direction cosines** of θ_x, θ_y, and θ_z and are defined as

$$\cos\theta_x = \left(\frac{x_B - x_A}{L}\right)$$
$$\cos\theta_y = \left(\frac{y_B - y_A}{L}\right) \qquad (2.12\text{B})$$
$$\cos\theta_z = \left(\frac{z_B - z_A}{L}\right)$$

The direction cosines are another way to define the unit vector $\boldsymbol{u}$ in the direction of $\boldsymbol{F}$:

$$\boldsymbol{u} = (\cos\theta_x\boldsymbol{i} + \cos\theta_y\boldsymbol{j} + \cos\theta_z\boldsymbol{k}) \qquad (2.14)$$

Therefore, we can write force $\boldsymbol{F}$ in terms of the space angles and direction cosines as:

$$\boldsymbol{F} = \|\boldsymbol{F}\|\boldsymbol{u} = F(\cos\theta_x\boldsymbol{i} + \cos\theta_y\boldsymbol{j} + \cos\theta_z\boldsymbol{k}) \qquad (2.15)$$

For a planar force, one of the space angles is 90°. Because the cosine of this space angle is zero, and the scalar component associated with the space angle is zero.

Key equations related to space angle representation are summarized in Table 2.3.

Representing Force Vectors: Spherical Coordinates

Another way to describe the direction of a force $\boldsymbol{F}$ is to use the two angles θ and ϕ associated with spherical coordinates. The spherical angles ϕ and θ and the space angles θ_x, θ_y, and θ_z are related to one another by the expressions

$$\begin{aligned} \cos\theta_x &= (\sin\phi)(\cos\theta) \\ \cos\theta_y &= (\sin\phi)(\sin\theta) \\ \cos\theta_z &= (\cos\theta) \end{aligned} \tag{2.20A}$$

$$\begin{aligned} \cos\phi &= (\cos\theta_z) \\ \tan\theta &= \frac{\cos\theta_y}{\cos\theta_x} \end{aligned} \tag{2.20B}$$

Key equations related to spherical coordinate representation are summarized in Table 2.3.

Resultant Force—Vector Addition

Engineering analysis commonly involves adding forces (or other vectors) to find their resultant. We presented three approaches to vector addition: **component addition**, **graphical vector addition**, and **geometric/trigonometric addition**. All three methods can be used to find either the magnitude and direction of a resultant force or the scalar components of a resultant force. They can also be used when two forces are being added but all you know is the magnitude and direction of one force, the direction the other force, and the direction of the resultant. With any of these four pieces of information you can find the unknown magnitudes. In fact, all of these methods work any time there are at most two unknowns when adding planar forces and at most three unknowns when adding nonplanar forces.

Angle between Two Forces—The Dot Product

The **dot product** is a convenient mathematical construct for finding the projection of a vector in a particular direction and for finding the angle between two vectors. The dot product of vectors $\boldsymbol{V_1}$ and $\boldsymbol{V_2}$ can be calculated by

$$\boldsymbol{V_1} \bullet \boldsymbol{V_2} = \|\boldsymbol{V_1}\|\,\|\boldsymbol{V_2}\|\cos\theta = V_{1x}V_{2x} + V_{1y}V_{2y} + V_{1z}V_{2z} \tag{2.30}$$

Rearranging this expression and solving for the angle θ between the tails of the two vectors, we have

$$\cos\theta = \frac{V_{1x}V_{2x} + V_{1y}V_{2y} + V_{1z}V_{2z}}{\|\boldsymbol{V_1}\|\|\boldsymbol{V_2}\|}; \qquad 0° < \theta < 180° \tag{2.31}$$

SYSTEM ANALYSIS (SA) EXERCISES

SA2.1 Calibrating Your Capacity

It is important for engineers to have a sense of how large forces are. One way of developing this sense is to develop reference frames for force comparison; one such reference frame is your physical capacity. To this end, find a weight room on campus and report the following information. If you are personally not able to complete the exercises, observe someone else engaged in them and record that person's data. Before attempting any of these exercises, make sure that you have received instruction on how properly to operate the equipment.

(a) Maximum weight you are able to bench press without hurting yourself

(b) Weight you are able to bench press repetitively 20 times without hurting yourself

(c) Maximum weight you are able to leg press without hurting yourself

(d) Weight you are able to leg press repetitively 20 times without hurting yourself

(e) Maximum weight you are able to arm curl without hurting yourself

(f) Weight you are able to arm curl repetitively 20 times without hurting yourself

Complete the table below with your measurements from (a)–(f):

Exercise	Average College Sophomore Male (N)	Average College Sophomore Female (N)	Your measurements (lb)	Your measurements (N)	Your data expressed as a percentage of average college sophomore (select male or female, as appropriate)
Bench press (max.)	753*	328*			
Bench press (repetitive)	609*	243*			
Leg press (max.)	1459*	633*			
Leg press (repetitive)	1168*	504*			
Arm curl (max.)	104*	57**			
Arm curl (repetitive)	—	—			

*Based on data gathered from 177 college students, fall 2001 and 2002. The complete data set is shown in Figure SA2.1.1.
**Based on data gathered from 12 college students, fall of 2002.

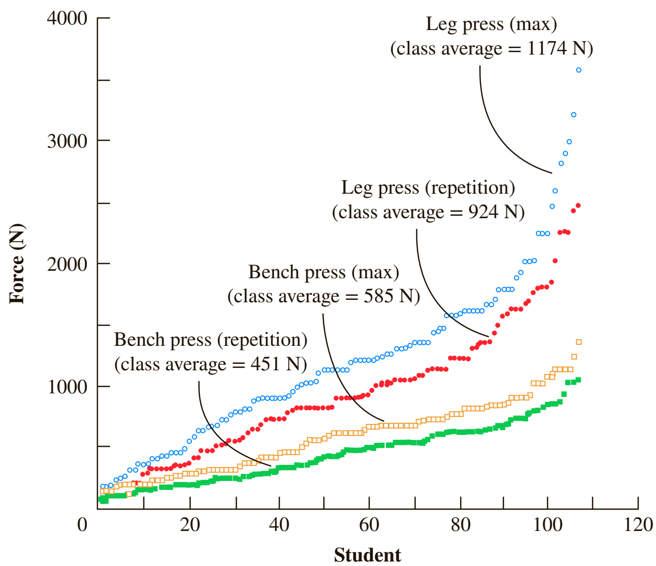


Figure SA2.1.1 Force capacity for college students for various weight-lifting exercises)

SA2.2 Estimating Force Values

As noted in SA2.1, it is important for engineers to be able to estimate the forces involved in various situations. In this exercise you are asked to estimate forces. If you are not familiar with a particular situation, you may want to consult with experts, visit the library, and/or research the Web for information. Make sure to record your sources of information.

(a) Order the following list of forces from smallest to largest magnitude.

(b) If possible, provide an order of magnitude estimate of each force magnitude.

System		Situation
A.	Adult 1	Gravitational attractive force of adult 2 on adult 1 (standing close to each other)
B.	Bicycle pedal	Force exerted by trained athlete on pedal
C.	Bus	Drag force on a bus when traveling at 50 mph
D.	Fully loaded commercial aircraft	Weight of fully loaded aircraft
E.	People held by Golden Gate Bridge	Weight of people on bridge for 50th anniversary celebration
F.	Leg press	Force exerted on press by trained athlete
G.	Leg press	Force exerted on press by average engineering student
H.	Leg press	Total force applied by all students in your statics class acting simultaneously on press
I.	Ninety-fifth percentile adult female in the United States	Weight of one woman in that percentile
J.	Ninety-fifth percentile adult male in the United States	Weight of one man in that percentile
K.	Sports car	Drag force when car is traveling at 20 mph
L.	Sports car	Drag force on car when traveling at 60 mph
M.	Touring bicycle	Drag force on a bicycle when traveling at 20 mph

SA2.3 Forces Holding a Scoreboard in Place

The basketball facility inside the Reynolds Coliseum at North Carolina State University contains a scoreboard suspended from the ceiling with two cables (Figure SA2.3.1). The 4500-N scoreboard can be lowered using two winches attached to the bottom flange of a main roof beam girder. Figure SA2.3.2 presents a view of the key suspension elements as viewed from directly underneath the board.

Leonhard Bernold

Figure SA2.3.1 View of the 49-meter-wide field with suspended scoreboard

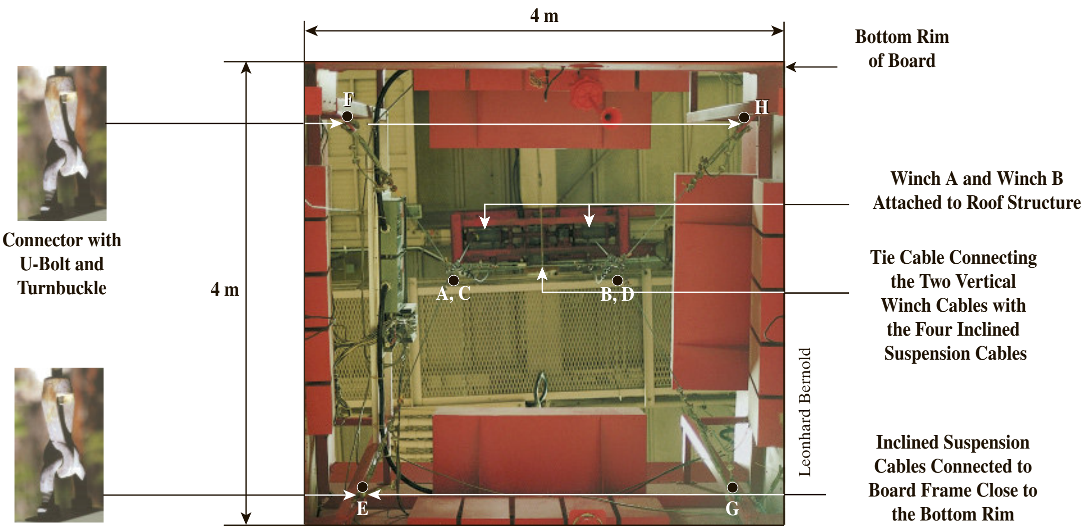


Figure SA2.3.2 View into the scoreboard from the floor

Assume you are attending a basketball game with your buddy George, who has not studied statics. Before the game, you are explaining how the heavy scoreboard is secured in the air, which stimulates him to ask whether the maximum tension in the cables that run from E, F, G, and H in Figure SA2.3.3 to tie cable locations C and D will be 1/4 of the weight of the scoreboard. He also wonders what the purpose is of tie cable CD. Having just studied this chapter, you should have no problem answering his questions. Consider George's curiosity as a wonderful opportunity to understand the material, since we all know that the best way to learn something well is by teaching it.

Here is how you work with George to answer his questions:

1. Figure SA2.3.4 shows the forces acting at the point labeled D in **Figures SA2.3.2** and **SA2.3.3**.

   (a) Find magnitude and direction of forces $\boldsymbol{F}_{DC}$, $\boldsymbol{F}_{DB}$, $\boldsymbol{F}_{DG}$, and $\boldsymbol{F}_{DH}$. Write your answers in Cartesian vector notation.

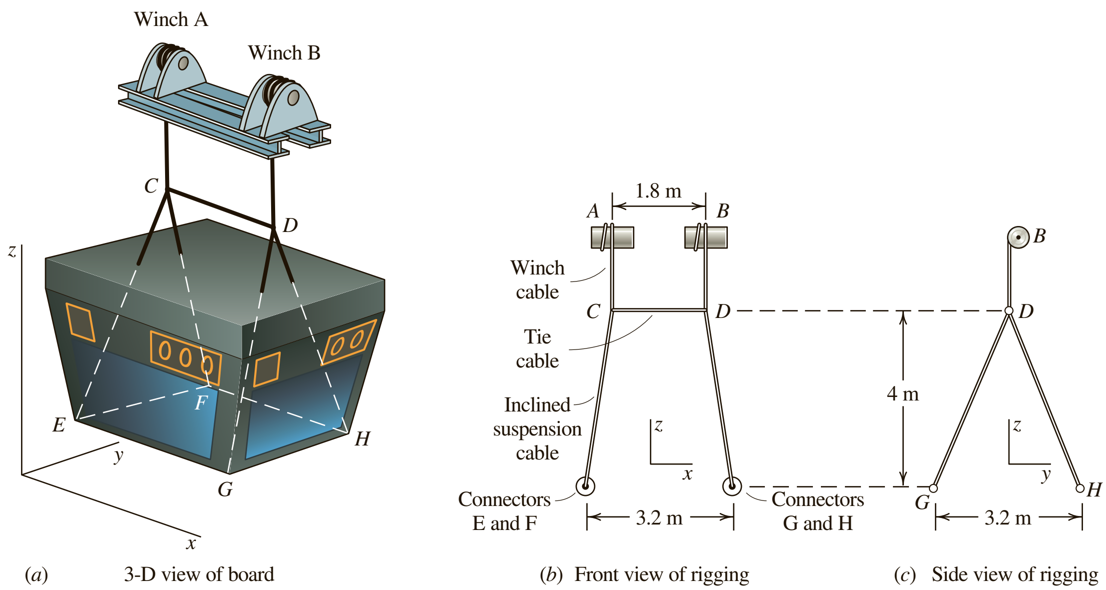


Figure SA2.3.3 Models of the rigging system suspending the scoreboard from the roof

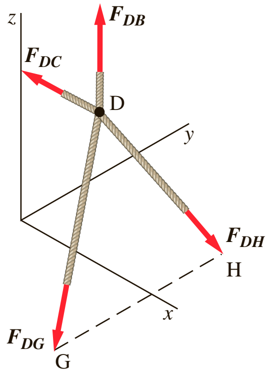


Figure SA2.3.4 Free-body diagram for Point D

(b) If $\|F_{DB}\| = (4500/2)$ N (is this reasonable?) and the sum of the forces acting at D is zero, what are the magnitudes of F_{DC}, F_{DG}, and F_{DH}?

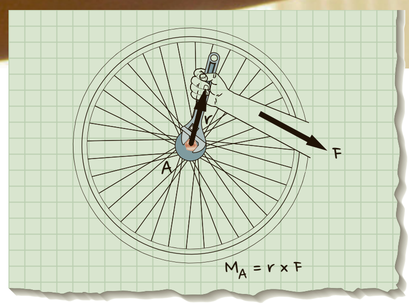
(c) Based on your findings in (b), help George answer his question about whether the maximum tension in the cables that run from E, F, G, and H to the winch connections at C and D are each (4500/4) N.

2. What do you think the function of the tie cable CD is? If the tie cable were not present, would the maximum tension in the cables from E, F, G and H to C and D increase, decrease, or remain the same? Explain your answer.

C H A P T E R 3

MOMENTS

Frank Gaglione/Getty Images



In Chapter 2 we considered the forces that push and pull on a system. Now we consider how forces not only push and pull but also tend to twist, tip, turn, and rock the systems on which they act. These types of loads are called moments. Some designs incorporate the principles of moments into their function—see-saws, balance scales, can openers, and torsion-bar suspensions are a few examples. Other designs, such as skyscrapers, diving boards, and airplane wings, must be designed to resist moments.

We begin this chapter by presenting the properties and characteristics of moments. Then we outline two formal mathematical methods for calculating moments and address situations involving multiple forces that cause moments.

LEARNING OBJECTIVES

On completion of this chapter, you will be able to:

- **Calculate a moment using perpendicular distance from moment center to load.** (3.1)
- **Calculate a moment using cross products.** (3.2)
- **Calculate moment components in a particular direction.** (3.3)
- **Calculate a couple moment.** (3.4)
- **Find the equivalent moment and equivalent force due to multiple loads acting on a system.** (3.5)

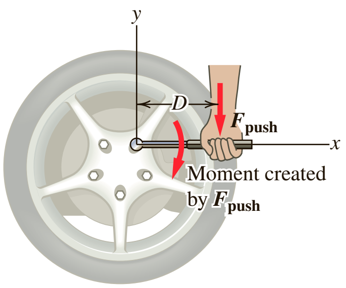


Figure 3.1.1 Pushing down on the wrench creates a moment.

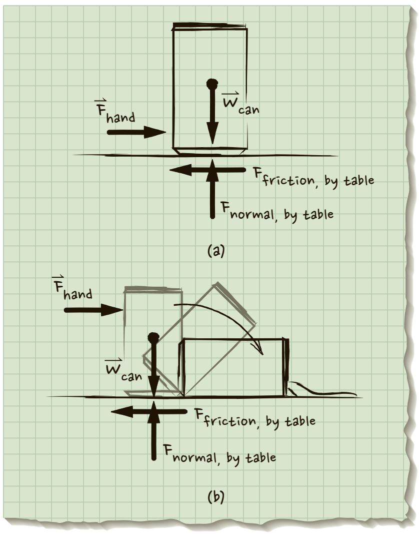


Figure 3.1.2 (*a*)When push force is applied near the bottom of the can, can slides. (*b*) When push force is applied near the top, can turns over.

3.1 WHAT ARE MOMENTS?

Learning Objective: Calculate a moment using perpendicular distance from moment center to load.

Imagine you have just changed a flat tire on your car and are replacing the lug nuts that bolt the wheel to the hub. You use a wrench to tighten each nut onto a bolt, as in Figure 3.1.1. By pushing downward on the wrench, you are applying a **moment*** to the nut. This moment is created by the force exerted by your hand on the wrench and is about the axis of the bolt (which in the figure is perpendicular and "into" the page). The magnitude of the moment is the product of the magnitude of the applied force and the distance from the center of the bolt to your hand.

The effects of an applied moment depend in part on where a force is applied to an object or system. Consider for example the can illustrated in Figure 3.1.2. If you place your hand near the bottom of the can and push, the can does not tip over. If you place your hand near the top of the can and push, the can turns over due to the moment you have applied.

A **moment** is a load created by a force that is offset relative to a point in space. It is a vector quantity and therefore has both magnitude and direction. In working with moments, we are concerned with the force's position relative to this point in space, the magnitude, direction, and sense of the moment, and its graphical depiction. Let's look at these elements one at a time.

Moment Center

The point from which the force is offset is the **moment center** (or MC for short). The moment center can be specified by x, y, and z coordinates in a Cartesian coordinate system.

Position Vector

A **position vector** is any vector that runs from the moment center to a point on the line of action of the force (Figure 3.1.3). Position vectors are commonly specified in vector notation (e.g., position vector $\boldsymbol{r}$ is specified as $r_x\boldsymbol{i} + r_y\boldsymbol{j} + r_z\boldsymbol{k}$, where r_x, r_y, and r_z are scalar components in the $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ directions, respectively, from the MC to the point on the line of action).

*Some physics textbooks use the word **torque** for what we are calling moment—namely, the tendency of a force to cause tipping or turning. As we will see in Chapter 9, engineers generally use the word torque to describe a moment created in conjunction with a machine.

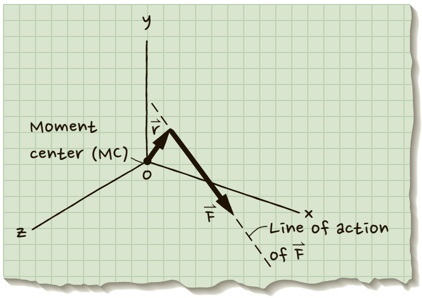


Figure 3.1.3 Position vector runs from the MC to line of action of the force.

A position vector for the wrench force $\boldsymbol{F}_{\text{push}}$ relative to a moment center at the center of the bolt is illustrated in **Figure 3.1.4*a*** as a vector that runs from the moment center (MC) to the point of application of $\boldsymbol{F}_{\text{push}}$. With the coordinate system shown in **Figure 3.1.4**, this vector is defined as $\boldsymbol{r}_1 = 250 \text{ mm } \boldsymbol{i}$. It is just one of a family of position vectors because *any* vector from the MC to *any* point on the line of action of the force is a position vector. For example, another position vector for $\boldsymbol{F}_{\text{push}}$ is $\boldsymbol{r}_2 = 250 \text{ mm } \boldsymbol{i} + 100 \text{ mm } \boldsymbol{j}$ (**Figure 3.1.4*b***).

We now consider defining a position vector for the situation depicted in **Figure 3.1.5*a***, where an individual pulls on a rope tied to a branch. (He is attempting to pull down the nearly-sawn-through branch. This seems sort of dangerous; maybe he should wear a helmet!). We can represent this force as $\boldsymbol{F} = \|\boldsymbol{F}\|(0.116\,\boldsymbol{i} + 0.349\,\boldsymbol{j} - 0.930\,\boldsymbol{k})$, as shown in **Figure 3.1.5*b***. If we are interested in the moment created by this force at the saw cut, we establish the saw cut as the MC. Possible position vectors from the MC to the line of action of the force include $\boldsymbol{r}_1 = 2 \text{ m}\,\boldsymbol{i} + 6 \text{ m}\,\boldsymbol{j} - 11 \text{ m}\,\boldsymbol{k}$ (which runs from MC to B) and $\boldsymbol{r}_2 = 0.5 \text{ m}\,\boldsymbol{i} + 1.5 \text{ m}\,\boldsymbol{j} + 1.0 \text{ m}\,\boldsymbol{k}$ (which runs from MC to A), as shown in **Figure 3.1.5*c***.

IMPORTANT NOTE! A position vector is ANY vector that runs from the moment center to the line of action of the force. When choosing $\boldsymbol{r}$, keep in mind that it need not lie along a physical connection between the MC and the line of action of $\boldsymbol{F}$. Generally, you should choose the $\boldsymbol{r}$ that is easiest to define with the information you know about the physical system.

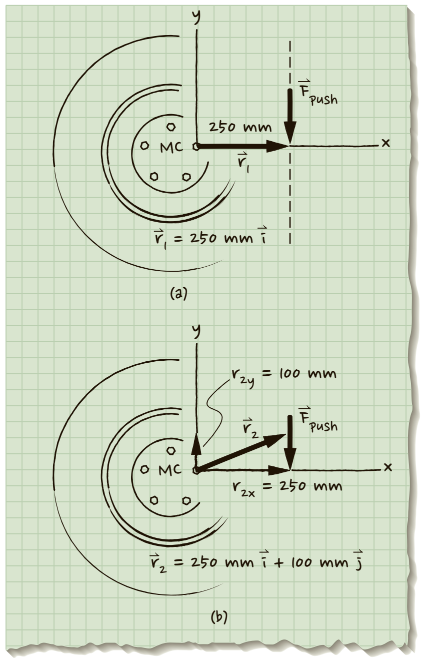


Figure 3.1.4 Two possible position vectors for the moment on the lug nut.

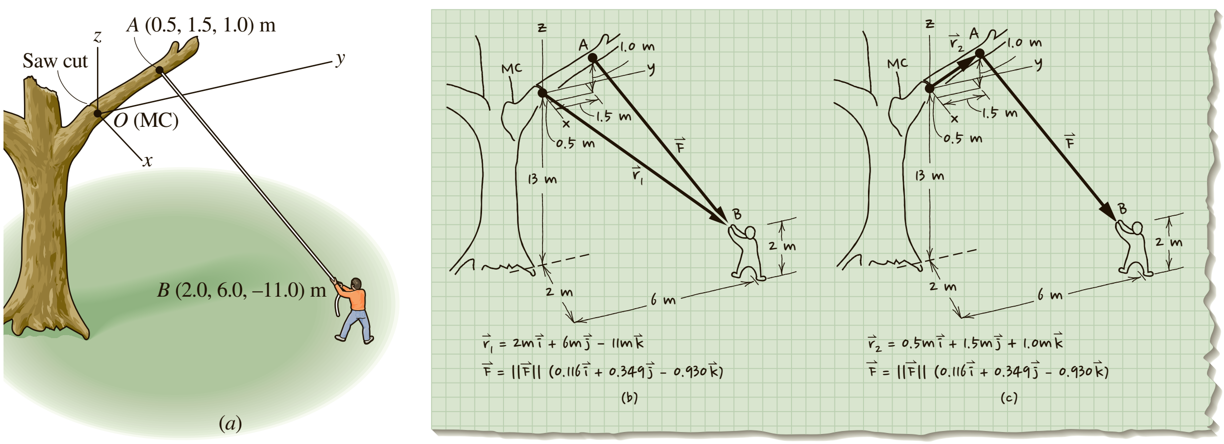


Figure 3.1.5 A person pulling on a rope attached to a tree branch and two possible position vectors.

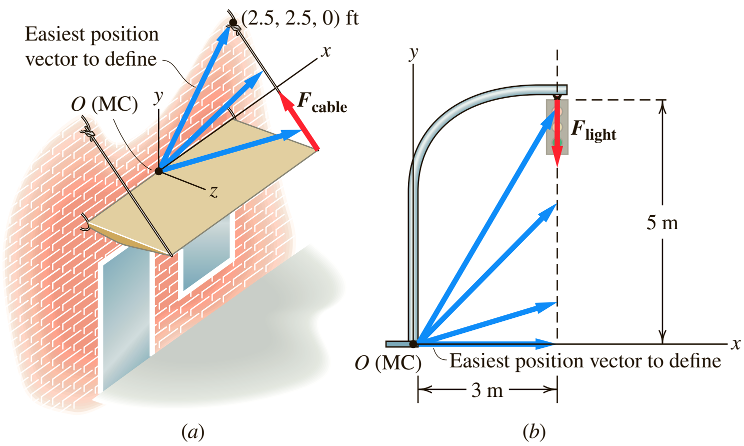


Figure 3.1.6 There are an infinite number of position vectors. Use the one that is easiest to define.

Figure 3.1.6 shows position vectors in two systems; in each case, the position vector that is easiest to define with available dimensions is labeled.

Synonyms commonly used in engineering practice for the position vector that is perpendicular to the line of action of the force are moment arm vector, moment arm, lever arm, and offset.

Example 3.1.1 explores position vectors for planar systems and Example 3.1.2 explores position vectors for nonplanar systems.

Sense and Direction of Moment

Now back to our tire iron. To tighten the lug nut onto the bolt, a force is applied at the end of the wrench so that the resulting moment twists the nut clockwise about a *z* axis, as viewed from in front of the wheel. To loosen the nut, we would apply the force so that the resulting moment twists the nut counterclockwise about a *z* axis. The terms *clockwise* and *counterclockwise* refer to the rotational **sense** of the moment about an axis. If we establish a local coordinate system at the moment center, we define the sense of the moment by standing on the positive axis (in this case, the positive *z* axis) and looking back at the moment center, as illustrated in **Figure 3.1.7**. The axis that we are standing on defines the **direction** of the moment, as detailed below.

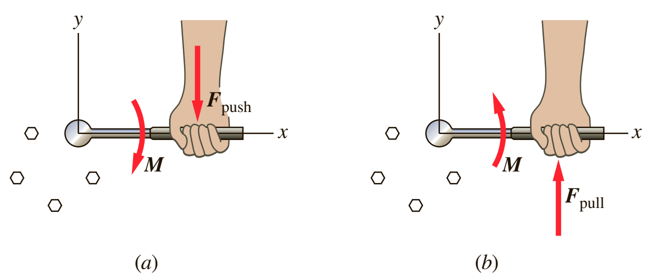
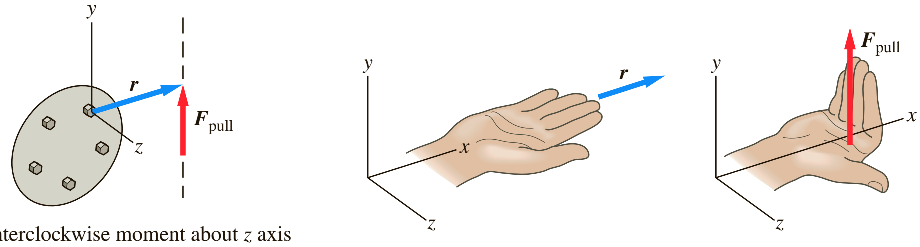


Figure 3.1.7 (*a*) Clockwise sense, tightening, direction is $-\boldsymbol{k}$; (*b*) counterclockwise sense, loosening, direction is $+\boldsymbol{k}$.



Counterclockwise moment about z axis

(a) (b) (c)

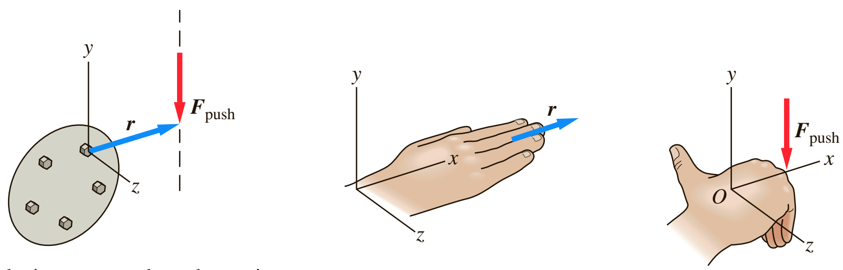
Figure 3.1.8 Orienting right-hand fingers and thumb to show moment sense is counterclockwise.

A right-hand rule enables us to formalize a procedure for determining the direction of a moment. Consider a force applied to the wrench that results in a counterclockwise (*loosening*) rotation (**Figure 3.1.8*a***). Align the fingers of your right hand with the position vector (**Figure 3.1.8*b***), with your palm in the position that allows you to curl your four fingers so that they point in the direction of the force (**Figure 3.1.8*c***). Now move your thumb to make a 90° angle with your palm. The direction in which your thumb points is the direction of the vector $\boldsymbol{M}$, which in this example is the $+\boldsymbol{k}$ direction. Consistent with this, we say that the sense of the moment is counterclockwise about a z axis.

When you follow the right-hand rule for a force applied to the wrench that results in a clockwise (*tightening*) rotation, you find you have to rotate your hand such that your thumb aligns with the negative z axis (**Figure 3.1.9**). Therefore, the moment vector is in the $-\boldsymbol{k}$ direction. Consistent with this, we say that the sense of the moment is clockwise about a z axis.

The right-hand rule also works when determining the direction of moments about the x and y axes. For example, a moment direction of $+\boldsymbol{i}$ or $-\boldsymbol{i}$ refers to a counterclockwise or clockwise sense, respectively, about the x axis, as illustrated in **Figure 3.1.10*a***. Similarly, $+\boldsymbol{j}$ or $-\boldsymbol{j}$ refers to a counterclockwise or clockwise sense, respectively, about the y axis, as illustrated in **Figure 3.1.10*b***.

The position vector and force vector define a plane, and the moment is in the direction perpendicular to this plane (the direction of your thumb). Therefore, the direction of the moment is perpendicular to both the direction of the position vector and the direction of the force vector.



Clockwise moment about the z axis

(a) (b) (c)

Figure 3.1.9 Orienting right-hand fingers and thumb to show moment sense is clockwise.

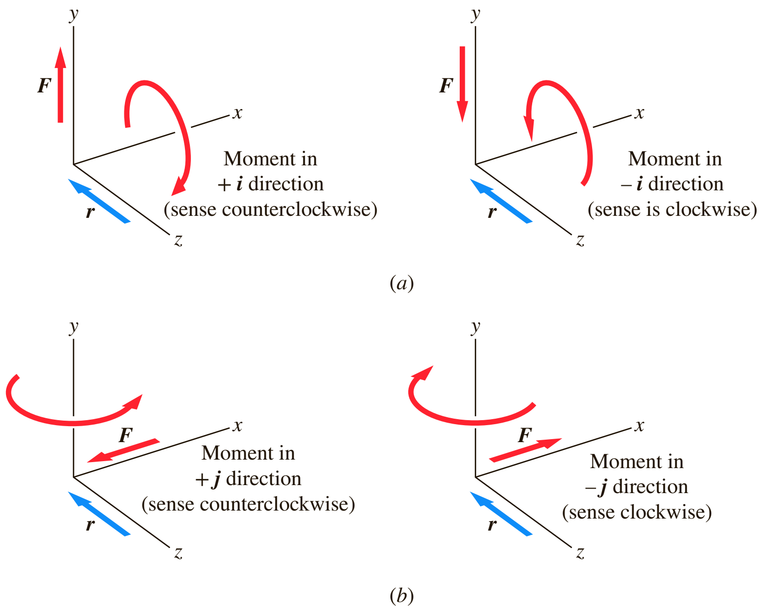


Figure 3.1.10 (*a*) Moments about the *x* axis; (*b*) moments about the *y* axis.

Graphical Representation of a Moment

We have been representing moments in drawings (e.g., **Figure 3.1.10**) with an arrow-headed arc. The arrow shows the sense of the moment. The magnitude of the moment (if known) is written next to the arc, as illustrated in **Figure 3.1.11*a***.

A moment can also be represented with a double-headed arrow (**Figure 3.1.11*b***). The direction of the arrow represents the direction of the moment; the moment is positive if the arrow points along the positive rotational axis

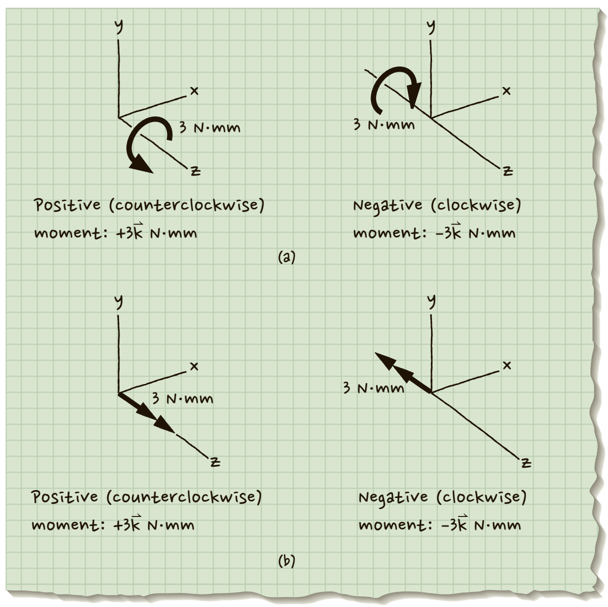


Figure 3.1.11 Representing moments graphically as (*a*) an arrow-headed arc or (*b*) a double-headed arrow.

and negative if the arrow points along the negative rotational axis. If the magnitude of the moment is known, it is written next to the arrow and/or the length of the arrow is drawn in proportion to the magnitude.

These various representations of moments in **Figure 3.1.11** enable moments to be readily distinguished from forces (which are depicted as single-headed, straight arrows).

Magnitude and Direction of the Moment Vector

The **magnitude of a moment** $\|\boldsymbol{M}\|$ is the product of the magnitude of the force* $\|\boldsymbol{F}\|$ and the *perpendicular distance to the line of action of the force*. This perpendicular distance is $\|\boldsymbol{r}\| \sin\theta$, where $\|\boldsymbol{r}\|$ is the magnitude of the position vector,† and θ is the angle between $\boldsymbol{r}$ and $\boldsymbol{F}$ when they are placed tail to tail such that $0° < \theta < 180°$, as illustrated in **Figure 3.1.12**. Therefore, we write the magnitude of the moment as

$$\|\boldsymbol{M}\| = \|\boldsymbol{F}\| \, (\|\boldsymbol{r}\| \sin\theta) \tag{3.1}$$

We can also represent $\|\boldsymbol{M}\|$ as the product of the magnitude of the position vector $\|\boldsymbol{r}\|$ and the *force component perpendicular to the position vector*. As shown in **Figure 3.1.12**, the perpendicular force component is $\|\boldsymbol{F}\| \sin\theta$. The resulting equation is the same as (3.1).

$$\|\boldsymbol{M}\| = \|\boldsymbol{r}\| \, (\|\boldsymbol{F}\| \sin\theta)$$

Use whichever interpretation makes the most sense in any given situation.

Figure 3.1.13 illustrates how to use (3.1) to determine the magnitude of the moment for our lug nut example from **Figure 3.1.4**, with two position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. With $\boldsymbol{r}_1$, the angle θ is 90°, which means the perpendicular distance to $\boldsymbol{F}_{push}$ is simply $\|\boldsymbol{r}_1\|$. With $\boldsymbol{r}_2$, we can see from the geometry that the perpendicular distance is $\|\boldsymbol{r}\| \sin 111.8° = \|\boldsymbol{r}\| \sin 68.2° = 250$ mm. Alternatively we can calculate the component of $\boldsymbol{F}_{push}$ perpendicular to $\boldsymbol{r}_2$ as $\|\boldsymbol{F}\| \sin 111.8°$. *Remember that only the component of* $\boldsymbol{F}$ *perpendicular to* $\boldsymbol{r}$ *creates the moment. The component of* $\boldsymbol{F}$ *along* $\boldsymbol{r}$ *does* not *create a moment.*

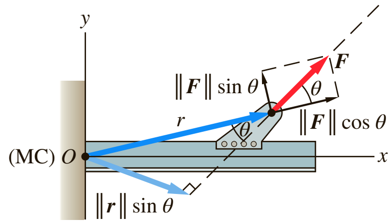


Figure 3.1.12 The moment can be developed in two ways: (a) the magnitude of $\boldsymbol{F}$ times the perpendicular distance from the MC to the line of action of the force, and (b) the component of the force perpendicular to the position vector times the magnitude of $\boldsymbol{r}$.

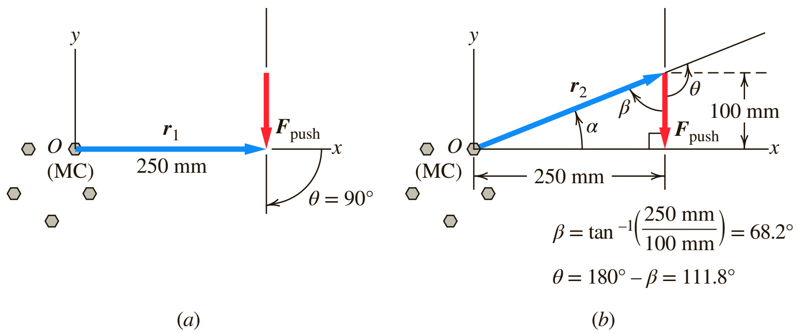


Figure 3.1.13 Calculating the magnitude of the moment on the lug nut based on (*a*) $\boldsymbol{r}_1$ and (*b*) $\boldsymbol{r}_2$.

*For a force vector expressed in scalar components, $\|\boldsymbol{F}\| = \sqrt{F_x^2 + F_y^2 + F_z^2}$.

†For a position vector expressed in scalar components, $\|\boldsymbol{r}\| = \sqrt{r_x^2 + r_y^2 + r_z^2}$.

We end up with the same value $||\boldsymbol{M}||$ no matter which position vector we use. This leads us to conclude that *the magnitude of the moment created by a force is independent of the choice of position vector used to calculate the magnitude.*

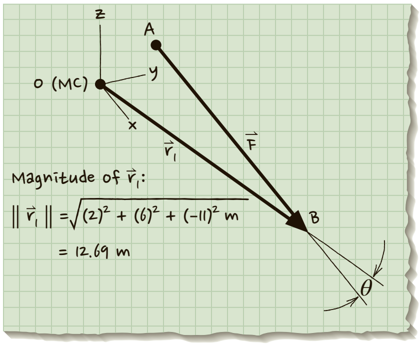


Figure 3.1.14 The angle θ between position vector and force.

Figure 3.1.14 shows the angle θ in (3.1) for the moment created by the rope tension pulling on the branch in **Figure 3.1.5**. With $\boldsymbol{F}$ and $\boldsymbol{r}_1$ known in terms of their components, the dot product would be a straightforward way to find θ. Recall that the dot product, introduced in Section 2.7, allows us to find the projection of one vector onto another.

We now express moment in terms of its magnitude (given by Equation (3.1)) and its direction as:

$$\boldsymbol{M} = ||\boldsymbol{M}||\boldsymbol{u} = ||\boldsymbol{F}||\,||\boldsymbol{r}||\sin\theta\boldsymbol{u}$$

where $\boldsymbol{u}$ is a unit vector aligned with the axis about which the moment acts. This unit vector can be written in terms of its direction cosines as:

$$\boldsymbol{u} = \cos\theta_x\boldsymbol{i} + \cos\theta_y\boldsymbol{j} + \cos\theta_z\boldsymbol{k}$$

Therefore, we can write the moment vector as:

$$\boldsymbol{M} = ||\boldsymbol{M}||\boldsymbol{u} = ||\boldsymbol{F}||\,||\boldsymbol{r}||\sin\theta(\cos\theta_x\boldsymbol{i} + \cos\theta_y\boldsymbol{j} + \cos\theta_z\boldsymbol{k}) \tag{3.2}$$

Calculating the moment created by a force $\boldsymbol{F}$ relative to a moment center with Equation (3.2) is generally straightforward when the force and position vectors are contained in a single coordinate plane (for example, the xy plane). In this planar situation one can identify by inspection that the axis of rotation will be perpendicular to the plane (e.g., the z-axis, if the force and position vector are in the xy plane), with sense/direction determined as described above. The magnitude of the moment can be calculated using (3.1), and then the sense/direction and magnitude are combined into an expression for the moment $\boldsymbol{M}$.

When the force and position vectors are not contained in a single coordinate plane, applying Equation (3.2) to write the moment created by the force is likely to be more complicated. Finding the angle θ (perhaps involving the dot product) and the direction cosine angles θ_x, θ_y, and θ_z probably involves a series of calculations. In the next section we present an alternative expression that is often more useful in calculating moment in these non-planar situations.

Check out the following examples of applications of this material.

- **Example 3.1.1 Specifying the Position Vector - Planar**
- **Example 3.1.2 Specifying the Position Vector - Nonplanar**
- **Example 3.1.3 The Magnitude of a Moment - Planar**
- **Example 3.1.4 The Magnitude of a Moment - Nonplanar**
- **Example 3.1.5 Moment Center on the Line of Action of Force**

EXAMPLE 3.1.1

Consider two gears that contact at point P (**Figure 1*a***). Gear B pushes on gear A with a force of 100 N at P (**Figure 1*b***). The radius of gear A is 100 mm. Specify in vector notation (a) the position vector $\boldsymbol{r}_1$ and (b) the position vector $\boldsymbol{r}_2$, as shown in **Figure 1**:

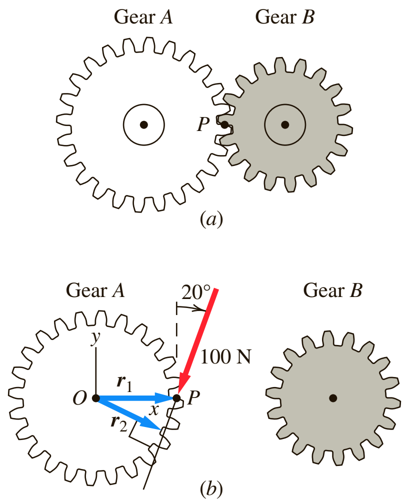


Figure 1 a) Two gears in contact at point P; b) Applied force on gear A and position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.

Goal Define two position vectors ($\boldsymbol{r}_1$, $\boldsymbol{r}_2$) in vector notation for a gear force relative to a moment center at O.

Given The magnitude and angle of force $\boldsymbol{F}$ applied at P that represents gear B as it pushes on gear A. The radius of gear A, and the x and y axes have been specified. Position vectors are defined in **Figure 1*b***.

Assume The gear force lies in the xy plane.

Draw, Formulate Equations, and Solve **(a)** Position vector $\boldsymbol{r}_1$ is aligned with the x axis and goes from O to point P (a distance of 100 mm).

Therefore, by inspection of **Figure 1**, we write $\Rightarrow$ $\boldsymbol{r}_1 = 100\,\text{mm}\,\boldsymbol{i}$

(b) Position vector $\boldsymbol{r}_2$ goes from the moment center at O to the line of action of the force and is perpendicular to the line of action. Based on the drawing of $\boldsymbol{r}_2$ in **Figure 2*a***, we can write

$$\|\boldsymbol{r}_2\| = 100\,\text{mm}\cos 20° = 94.0\,\text{mm}$$

Based on **Figure 2*b***, the scalar components can be written as

$$r_{2x} = \|\boldsymbol{r}_2\|\cos 20° = 94.0\,\text{mm}\cos 20° = 88.3\,\text{mm}$$

$$r_{2y} = -\|\boldsymbol{r}_2\|\sin 20° = -94.0\,\text{mm}\sin 20° = -32.1\,\text{mm}$$

We then write $\boldsymbol{r}_2$ in vector notation as $\Rightarrow$ $\boldsymbol{r}_2 = 88.3\,\text{mm}\,\boldsymbol{i} - 32.1\,\text{mm}\,\boldsymbol{j}$

Check The answer for (a) can be checked by inspection; when we look at **Figure 2*b*** we see that $\boldsymbol{r}_1$ is aligned with the positive x axis. Furthermore, it is along a radius of gear A. Therefore, the answer of $\boldsymbol{r}_1 = 100\,\text{mm}\,\boldsymbol{i}$ makes physical sense.

We check the answer for (b) by calculating the magnitude of $\boldsymbol{r}_2$ from its components:

$$\sqrt{(88.3\,\text{mm})^2 + (-32.1\,\text{mm})^2} = 94.0\,\text{mm}$$

Comment: The position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are both valid for locating $\boldsymbol{F}$ relative to a moment center at the center of gear A since each of them goes from the moment center to the line of action of $\boldsymbol{F}$. They are only two examples of the infinite number of position vectors for locating $\boldsymbol{F}$ relative to O.

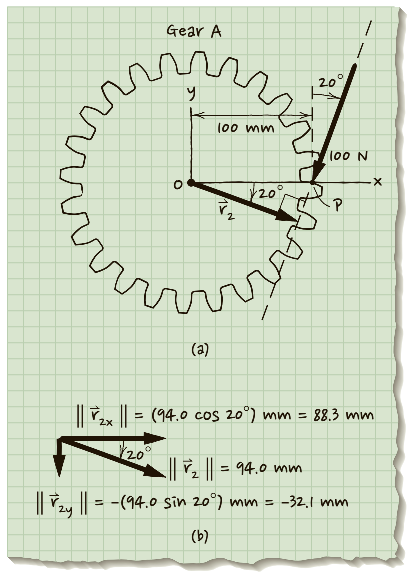


Figure 2 a) Orientation of $\boldsymbol{r}_2$; b) x and y components of $\boldsymbol{r}_2$.

EXAMPLE 3.1.2

A cable exerts a force $\boldsymbol{F}$ of 500 lb on the right-angle pipe shown in **Figure 1**. Specify position vectors (a) $\boldsymbol{r}_1$ and (b) $\boldsymbol{r}_2$ in vector notation for the force relative to the moment center at O.

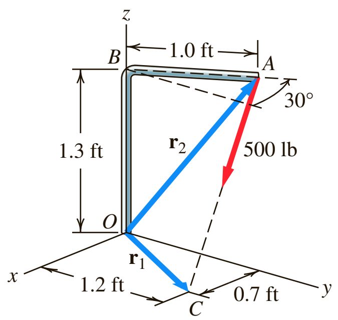


Figure 1 Cable AC pulls on a pipe at point A.

Goal Define two position vectors ($\boldsymbol{r}_1$, $\boldsymbol{r}_2$) in vector notation for cable force $\boldsymbol{F}$.

Given Magnitude and orientation of $\boldsymbol{F}$, dimensions of the pipe on which $\boldsymbol{F}$ acts.

Assume No assumptions are required.

Draw The sketch of pipe element BA in **Figure 2** will facilitate our finding the coordinates of point A.

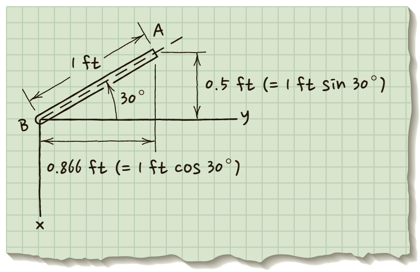


Figure 2 Plan view of pipe element BA.

Formulate Equations and Solve **(a)** Position vector $\boldsymbol{r}_1$ in **Figure 1** goes from the moment center O to point C on the line of action of the force. By inspection of **Figure 1**, we determine that

- the coordinates of O are $(x, y, z) = (0, 0, 0)$ ft, and
- the coordinates of point C are $(x, y, z) = (0.70, 1.20, 0)$ ft.

This allows us to write $\boldsymbol{r}_1 = (0.70 \text{ ft} - 0 \text{ ft})\,\boldsymbol{i} + (1.20 \text{ ft} - 0 \text{ ft})\,\boldsymbol{j}$,

which simplifies to $\Rightarrow$ $\boldsymbol{r}_1 = 0.70 \text{ ft}\,\boldsymbol{i} + 1.20 \text{ ft}\,\boldsymbol{j}$

(b) Position vector $\boldsymbol{r}_2$ goes from the moment center O to the point of application of $\boldsymbol{F}$ at A. By inspection of **Figures 1** and **2**

- the coordinates of point A are $(x, y, z) = (-1.0 \sin 30°, 1.0 \cos 30°, 1.30)$ ft $= (-0.50, 0.866, 1.30)$ ft.

  We can then write

$$\boldsymbol{r}_2 = (-0.50 \text{ ft} - 0 \text{ ft})\,\boldsymbol{i} + (0.866 \text{ ft} - 0 \text{ ft})\,\boldsymbol{j} + (1.30 \text{ ft} - 0 \text{ ft})\,\boldsymbol{k}$$

Which simplifies to $\Rightarrow$ $\boldsymbol{r}_2 = -0.50 \text{ ft}\,\boldsymbol{i} + 0.87 \text{ ft}\,\boldsymbol{j} + 1.30 \text{ ft}\,\boldsymbol{k}$

Check We check our answers by inspection. Looking at **Figure 1**, we see that to go from O to C we "walk" 0.70 ft in the $\boldsymbol{i}$ direction and 1.20 ft in the $\boldsymbol{j}$ direction; therefore, our answer of $\boldsymbol{r}_1 = 0.70 \text{ ft}\,\boldsymbol{i} + 1.20 \text{ ft}\,\boldsymbol{j}$ makes sense.

Using **Figures 1** and **2**, we see that to go from O to A we "walk" in the negative $\boldsymbol{i}$ direction a distance of 1.0 ft sin 30°, in the positive $\boldsymbol{j}$ direction a distance of 1.0 ft cos 30°, and in the positive $\boldsymbol{k}$ direction a distance of 1.3 ft; therefore, our answer of $\boldsymbol{r}_2 = -0.50 \text{ ft}\,\boldsymbol{i} + 0.87 \text{ ft}\,\boldsymbol{j} + 1.30 \text{ ft}\,\boldsymbol{k}$ makes sense.

EXAMPLE 3.1.3

Gear B pushes on gear A with a force of 100 N at point P (Figure 1). The radius of gear A is 100 mm. Find the magnitude of the moment that the force acting on gear A creates about an axis perpendicular to gear A that passes through its center.

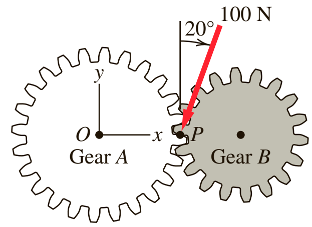


Figure 1 Gear B pushes on gear A at point P.

Goal Calculate the $||M||$ about a moment center at O by the force of gear B acting on gear A.

Given The magnitude, direction, and point of application of the force gear B applies to gear A. The radius of gear A and the x and y axes have been specified.

Assume Assume the gear force lies in the xy plane.

Draw We sketch gear A, force $\boldsymbol{F}$ (100 N), and its line of action. Inspection of Figure 2 shows that a logical position vector to consider is from the moment center O to point P; call this $\boldsymbol{r}$. We could have used any number of position vectors. The decision to work with the one from O to P was a pragmatic one; we wanted to make the moment calculation as simple as possible!

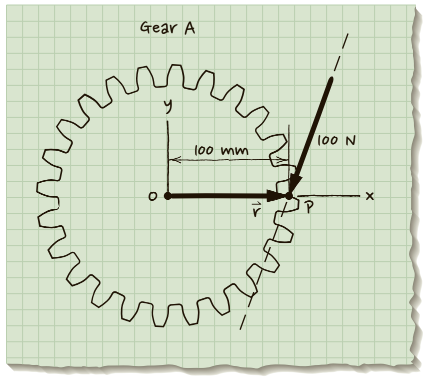


Figure 2 The line of action of the 100 N force.

Formulate Equations and Solve Equation (3.1) states

$$||\boldsymbol{M}|| = ||\boldsymbol{r}|| \, (||\boldsymbol{F}|| \sin \theta)$$

We first find $||\boldsymbol{r}||$ (the magnitude of the position vector), which is simply the radius of gear A. Therefore,

$$||\boldsymbol{r}|| = 100 \text{ mm}$$

The angle θ is the angle between the position vector and the force when they are placed end to end. Based on the geometry depicted in Figure 3,

$$\theta = 110°$$

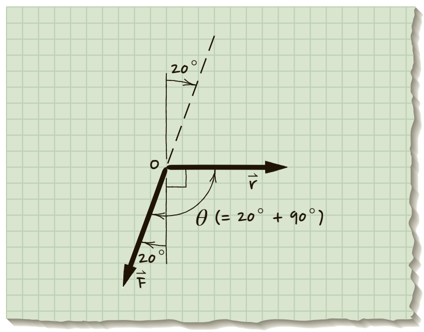


Figure 3 The angle *theta* between the position vector and the force.

The magnitude of the moment (3.1) yields

$$||\boldsymbol{M}|| = ||\boldsymbol{r}|| \, (||\boldsymbol{F}|| \sin \theta) = (100 \text{ mm})(100 \text{ N}) \sin 110° \Rightarrow \boxed{||\boldsymbol{M}|| = 9400 \text{ N} \cdot \text{mm}}$$

Check We present an accuracy check to confirm that the calculations were set up and carried out correctly. We do this by finding $||\boldsymbol{M}||$ using a different position vector to confirm the answer. For example, if $\boldsymbol{r}_2$ found in Example 3.1.1 is used,

$$||\boldsymbol{M}|| = ||\boldsymbol{r}_2|| \, (||\boldsymbol{F}|| \sin 90°) = (94.0 \text{ mm})(100 \text{ N}) \sin 90° = 9400 \text{ N} \cdot \text{mm}$$

EXAMPLE 3.1.4

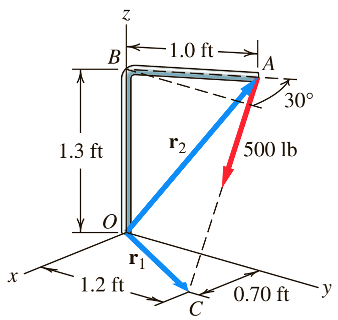


Figure 1 A 500-lb force applied at point A.

A cable exerts a force $\boldsymbol{F}$ of 500 lb on the right-angle pipe shown in **Figure 1**. Find the moment the force creates about a moment center at O based on two position vectors (a) $\boldsymbol{r}_1$, and (b) $\boldsymbol{r}_2$.

Goal Calculate the magnitude of the moment about a moment center at O created by the 500-lb force acting at A using two different position vectors.

Given The dimensions of the right-angle pipe, an xyz coordinate system, and the magnitude and direction of a force acting at A.

Assume No assumptions are required.

Formulate Equations and Solve Before we actually start any calculations, we note that we expect the answers for parts (a) and (b) to be the same because the moment a force creates about a particular moment center is independent of the position vector. We will use this as a check of our calculations.

(a) To apply (3.1) we must find $\|\boldsymbol{r}_1\|$ and the angle θ between $\boldsymbol{r}_1$ and $\boldsymbol{F}$.

The position vector $\boldsymbol{r}_1$ was defined in Example 3.1.2a as $\boldsymbol{r}_1 = 0.70 \text{ ft } \boldsymbol{i} + 1.20 \text{ ft } \boldsymbol{j}$. Its magnitude is therefore

$$\|\boldsymbol{r}_1\| = \sqrt{(0.70\,\text{ft})^2 + (1.20\,\text{ft})^2} = 1.39\,\text{ft}$$

The angle θ is the angle between $\boldsymbol{r}_1$ and $\boldsymbol{F}$ when they are placed tail to tail. Because the problem is nonplanar, we cannot simply read θ from the figure. We can, however, use (2.26), which says that the angle between the lines of action of two vectors $\boldsymbol{V}_1 = V_{1x}\boldsymbol{i} + V_{1y}\boldsymbol{j} + V_{1z}\boldsymbol{k}$ and $\boldsymbol{V}_2 = V_{2x}\boldsymbol{i} + V_{2y}\boldsymbol{j} + V_{2z}\boldsymbol{k}$ is

$$\cos\theta = \frac{V_{1x}V_{2x} + V_{1y}V_{2y} + V_{1z}V_{2z}}{\|\boldsymbol{V}_1\|\ \|\boldsymbol{V}_2\|}$$

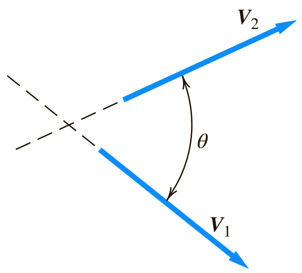


Figure 2 The angle θ between two vectors.

where $0° \le \theta \le 180°$, as illustrated in **Figure 2**.

In the current problem we use (2.26) to find the angle between the vectors $\boldsymbol{r}_1$ and $\boldsymbol{F}$. To write $\boldsymbol{F}$ in vector notation, we begin by finding the vector that goes from A to C. Based on the geometry in **Figure 1** and reviewing the coordinates found in Example 3.1.2, the vector that runs from A to C is

$$\begin{aligned}\boldsymbol{V}_{AC} &= (x_C - x_A)\boldsymbol{i} + (y_C - y_A)\boldsymbol{j} + (z_C - z_A)\boldsymbol{k} \\ &= 1.20\,\text{ft}\,\boldsymbol{i} + 0.334\,\text{ft}\,\boldsymbol{j} - 1.30\,\text{ft}\,\boldsymbol{k}\end{aligned}$$

with a magnitude of

$$\|\boldsymbol{V}_{AC}\| = \sqrt{(1.20\,\text{ft})^2 + (0.334\,\text{ft})^2 + (-1.30\,\text{ft})^2} = 1.80\,\text{ft}$$

The vector $\boldsymbol{V}_{AC}$ can be converted into a unit vector $\boldsymbol{u}_{AC}$ by dividing each scalar component by the magnitude of $\boldsymbol{V}_{AC}$:

$$\begin{aligned}\boldsymbol{u}_{AC} &= \frac{\boldsymbol{V}_{AC}}{\|\boldsymbol{V}_{AC}\|} = \frac{1.20\text{ ft}}{1.80\text{ ft}}\boldsymbol{i} + \frac{0.334\text{ ft}}{1.80\text{ ft}}\boldsymbol{j} - \frac{1.30\text{ ft}}{1.80\text{ ft}}\boldsymbol{k} \\ &= 0.667\,\boldsymbol{i} + 0.186\,\boldsymbol{j} - 0.722\,\boldsymbol{k}\end{aligned}$$

The force $\boldsymbol{F}$ can now be written as the product of the magnitude of the force (given as 500 lb) and $\boldsymbol{u}_{AC}$:

$$\boldsymbol{F} = (500\,\text{lb})\boldsymbol{u}_{AC} = 334\,\text{lb}\,\boldsymbol{i} + 93.0\,\text{lb}\,\boldsymbol{j} - 361\,\text{lb}\,\boldsymbol{k}$$

We are now (finally!) in a position to apply (2.26) to find the angle θ between $\boldsymbol{r}_1 = 0.70\text{ ft }\boldsymbol{i} + 1.20\text{ ft }\boldsymbol{j}$ and $\boldsymbol{F} = 334\text{ lb }\boldsymbol{i} + 93.0\text{ lb }\boldsymbol{j} - 361\text{ lb }\boldsymbol{k}$:

$$\theta = \cos^{-1}\left[\frac{(0.70\,\text{ft})(334\,\text{lb}) + (1.20\,\text{ft})(93.0\,\text{lb}) + (0\,\text{ft})(-361\,\text{lb})}{(1.39\,\text{ft})(500\,\text{lb})}\right]$$

$$\theta = 60.2°$$

This angle is illustrated in **Figure 3**.

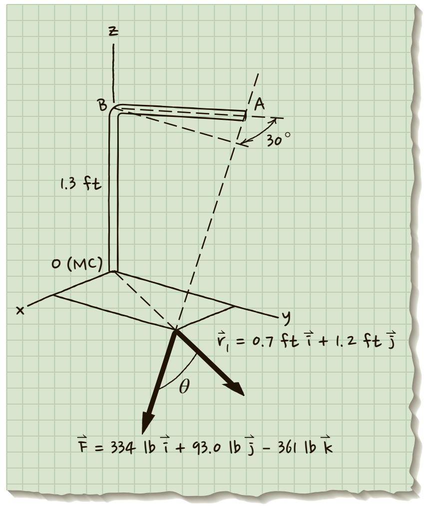


Figure 3 The angle θ between $\boldsymbol{r}_1$ and $\boldsymbol{F}$.

Intermediate check*: Based on the geometry depicted in **Figure 3**, we would expect $0° < \theta < 90°$. This is indeed what we found with $\theta = 60.2°$.

The magnitude of the moment: Substituting into (3.1) yields

$$\|\boldsymbol{M}\| = \|\boldsymbol{r}_1\|\,(\|\boldsymbol{F}\|\sin\theta) = (1.39\text{ ft})\,(500\text{ lb})\sin 60.2° \Rightarrow \boxed{\|\boldsymbol{M}\| = 603\text{ ft}\cdot\text{lb}}$$

(b) We use the same approach with the position vector $\boldsymbol{r}_2$.

The position vector $\boldsymbol{r}_2$ was defined in Example 3.1.2. Its magnitude is

$$\|\boldsymbol{r}_2\| = \sqrt{(-0.50\,\text{ft})^2 + (0.87\,\text{ft})^2 + (1.30\,\text{ft})^2} = 1.64\,\text{ft}$$

The angle θ: We use (2.26) to find the angle between position vector $\boldsymbol{r}_2 = -0.50\text{ ft }\boldsymbol{i} + 0.87\text{ ft }\boldsymbol{j} + 1.30\text{ ft }\boldsymbol{k}$ and $\boldsymbol{F} = 334\text{ lb }\boldsymbol{i} + 93.0\text{ lb }\boldsymbol{j} - 361\text{ lb }\boldsymbol{k}$:

$$\theta = \cos^{-1}\left[\frac{(-0.50\,\text{ft})(334\,\text{lb}) + (0.87\,\text{ft})(93.0\,\text{lb}) + (1.30\,\text{ft})(-361\,\text{lb})}{(1.64\,\text{ft})(500\,\text{lb})}\right]$$

$$\theta = 132.6°$$

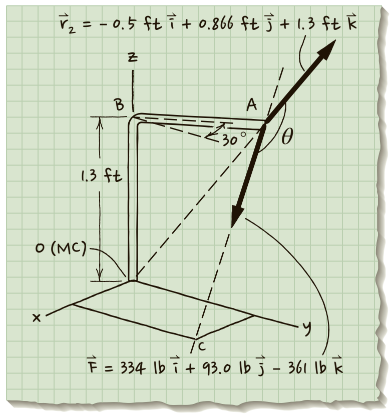


Figure 4 The angle θ between $\boldsymbol{r}_2$ and $\boldsymbol{F}$.

Intermediate check: Based on the geometry depicted in **Figure 4**, we would expect $90° < \theta < 180°$. This is indeed what we found with $\theta = 132.6°$.

The magnitude of the moment: Substituting into (3.1) yields

$$\|\boldsymbol{M}\| = \|\boldsymbol{r}_2\|\,(\|\boldsymbol{F}\|\sin\theta) = (1.64\text{ ft})(500\text{ lb})\sin 132.6° \Rightarrow \boxed{\|\boldsymbol{M}\| = 604\text{ ft}\cdot\text{lb}}$$

Check Parts (a) and (b) can be used as checks for one another. The slight differences in the answers from (a) and (b) are due to round-off in intermediate calculation steps.

Comment: This example illustrates that for a particular moment center, the magnitude of the moment is independent of the position vector.

*As you go along, check your intermediate calculations whenever possible. This will both increase the likelihood that you will get the right answer and make it easier to find where you went wrong if you get an incorrect answer.

EXAMPLE 3.1.5

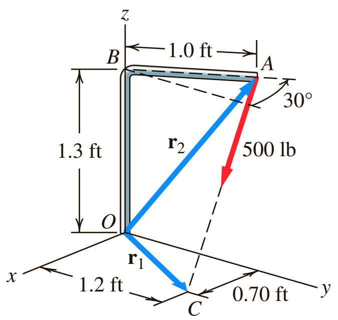


Figure 1 Cable AC pulls on a pipe at point A.

Let's again analyze the right-angle pipe with a 500-lb cable force applied at point A as shown in **Figure 1**. By calculating the moment the force creates about a moment center (a) at A and (b) at C, show that locating the moment center on the line of action of the force results in $\|\boldsymbol{M}\| = 0$.

Goal Calculate the magnitude of the moment at (a) moment center at A and (b) moment center at C due to the 500-lb force acting at A.

Given The dimensions of the right-angle pipe, an xyz coordinate system, and the magnitude and orientation of a force acting at the end of the pipe.

Assume No assumptions are required.

Formulate Equations and Solve **(a)** First we locate the moment center at A. If we attempt to draw a position vector from the moment center (at point A) to the point of application of the force (also point A), we find that the position vector has zero length—in other words, $\|\boldsymbol{r}_{AA}\| = 0$. Substituting this into (3.1), we find that the magnitude of the moment created about a moment center at A by the force is zero.

$$\|\boldsymbol{M}_A\| = \|\boldsymbol{r}_{AA}\|\,(\|\boldsymbol{F}\|\sin\theta) = (0\text{ ft})(500\text{ lb})\sin\theta \Rightarrow \boxed{\|\boldsymbol{M}_A\| = 0\text{ ft}\cdot\text{lb}}$$

(b) To find the magnitude of the moment the force creates about a moment center at C, we draw a position vector from point C to the point of application of the force (point A) (**Figure 2**). This position vector is

$$\begin{aligned}\boldsymbol{r}_{CA} &= (x_A - x_C)\boldsymbol{i} + (y_A - y_C)\boldsymbol{j} + (z_A - z_C)\boldsymbol{k}\\ &= -1.20\text{ ft}\,\boldsymbol{i} - 0.334\text{ ft}\,\boldsymbol{j} + 1.30\text{ ft}\,\boldsymbol{k}\end{aligned}$$

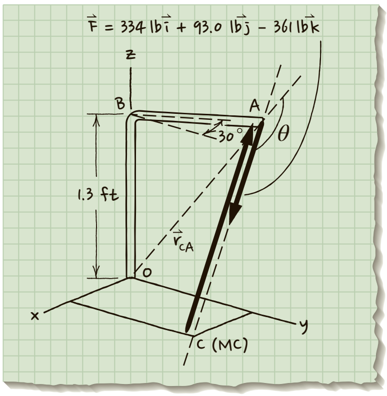


Figure 2 Position vector for a moment center at C.

Now, using (2.26) with expressions for the position vector ($\boldsymbol{r}_{CA}$) and force ($\boldsymbol{F} = 334$ lb $\boldsymbol{i}$ + 93.0 lb $\boldsymbol{j}$ – 361 lb $\boldsymbol{k}$) or by inspection of **Figure 2**, we find that $\theta = 180°$. Since $\sin 180° = 0$, substituting into (3.1) leads us to the result that the moment created about a moment center at C by the force is zero.

$$\|\boldsymbol{M}_C\| = \|\boldsymbol{r}_{CA}\|(\|\boldsymbol{F}\|\sin\theta) = (1.80\text{ ft})(500\text{ lb})\sin 180° \Rightarrow \boxed{\|\boldsymbol{M}_C\| = 0\text{ ft}\cdot\text{lb}}$$

Alternately, we could avoid all calculations by noting that a position vector drawn from point C (the moment center) to the point C (a point on the line of action of the force) has zero length. Then, using the same reasoning as in (a), we could conclude that the moment created about a moment center at C by the force is zero.

Comment: This example illustrates that a force creates no moment at a moment center located anywhere on the line of action of the force. This concept is illustrated in **Figure 3**.

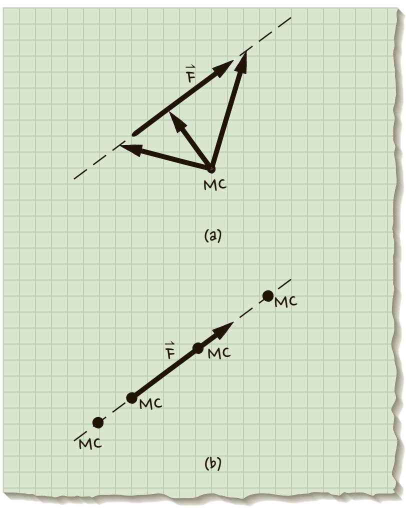
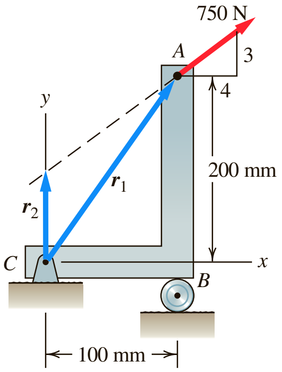


Figure 3 (*a*) Any position vector from MC to the line of action $\boldsymbol{F}$ is valid; (*b*) When MC is on the line of action of $\boldsymbol{F}$ no moment is created.

EXERCISES 3.1

3.1.1. [*] For the load applied at A, write the two position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in terms of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.



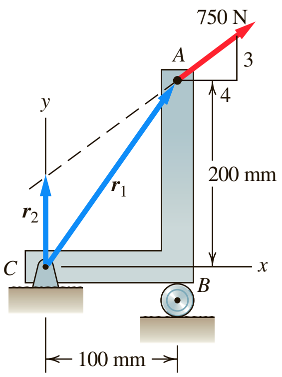
EX 3.1.1

3.1.2. [*] For the load applied at A, find

a. the magnitude of $\boldsymbol{r}_1$

b. θ (the angle between $\boldsymbol{r}_1$ and the 750-N force)

c. $\|\boldsymbol{M}\|$ for a moment center at C



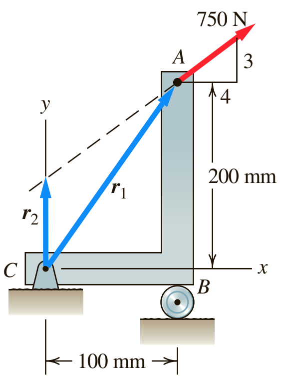
EX 3.1.2

3.1.3. [*] For the load applied at A, find

a. the magnitude of $\boldsymbol{r}_2$

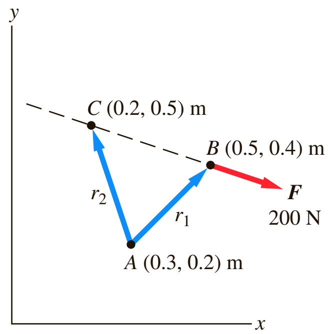
b. θ (the angle between $\boldsymbol{r}_2$ and the 750-N force)

c. $\|\boldsymbol{M}\|$ for a moment center at C



EX 3.1.3

3.1.4. [*] For the situation shown, write the two position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in vector notation.



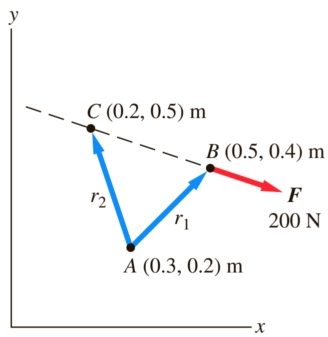
EX 3.1.4

3.1.5. [*] For the situation shown, find

a. the magnitude of $\boldsymbol{r}_1$

b. θ (the angle between $\boldsymbol{r}_1$ and the 200-N force)

c. $\|\boldsymbol{M}\|$ for a moment center at A



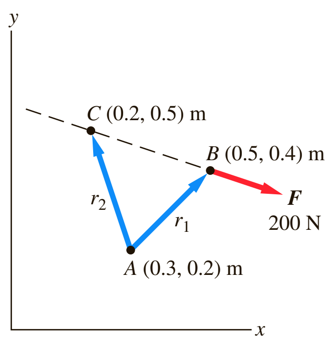
EX 3.1.5

3.1.6. [*] For the situation shown, find

a. the magnitude of $\boldsymbol{r}_2$

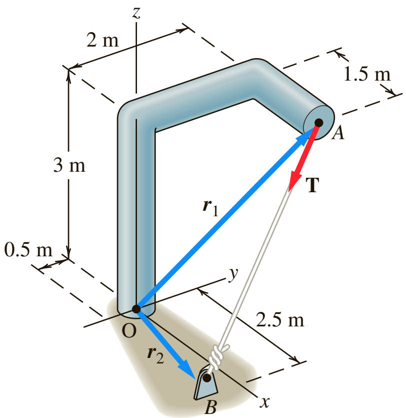
b. θ (the angle between $\boldsymbol{r}_2$ and the 200-N force)

c. $\|\boldsymbol{M}\|$ for a moment center at A



EX 3.1.6

3.1.7. [*] To determine the moment of $\boldsymbol{T}$ about moment center O, write the two position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in vector notation.



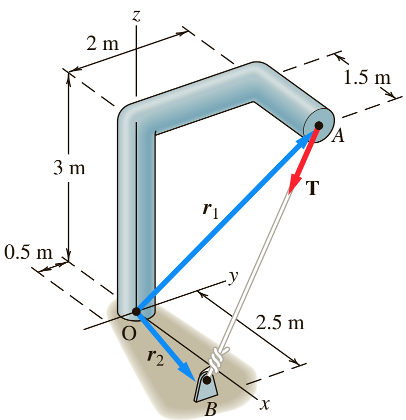
EX 3.1.7

3.1.8. [*] For the tension force $\boldsymbol{T}$ applied to assembly AO, find

a. the magnitude of $\boldsymbol{r}_1$

b. θ (the angle between $\boldsymbol{r}_1$ and the 1.2-kN tension force)

c. $\|\boldsymbol{M}\|$ for a moment center at O



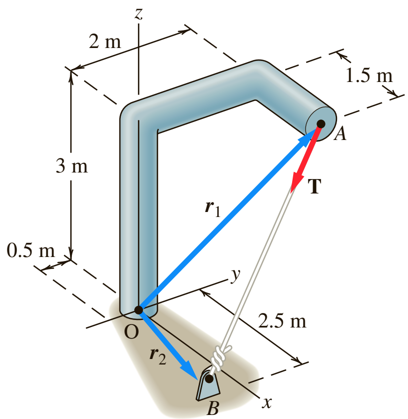
EX 3.1.8

3.1.9. [*] For the tension force $\boldsymbol{T}$ applied to assembly AO, find

a. the magnitude of $\boldsymbol{r}_2$

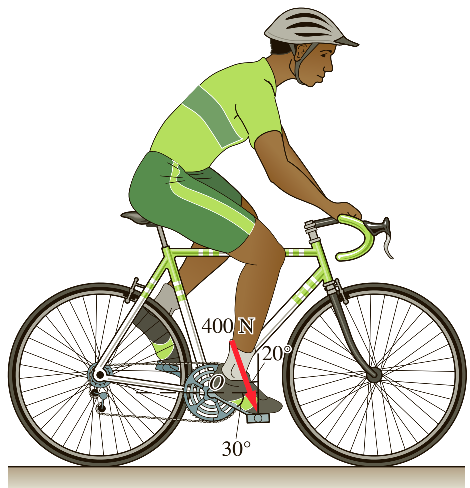
b. θ (the angle between $\boldsymbol{r}_2$ and the 1.2-kN tension force)

c. the magnitude of the moment $\boldsymbol{M}$ for a moment center at O



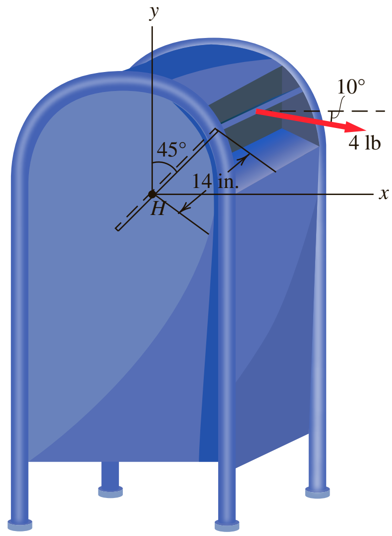
EX 3.1.9

3.1.10. [*] When Merrill is riding his bicycle and the crank is at an angle of 30° relative to the horizontal, he is applying a force of 400 N to the pedal at 20° relative to the vertical as shown. The length of the crank is 175 mm. Determine the magnitude and direction of the moment the 400-N force creates at O, the center of the chain ring.



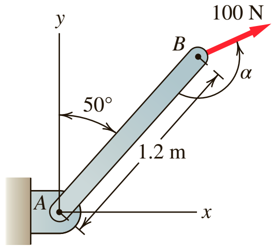
EX 3.1.10

3.1.11. [*] A person pulls on the door of a mailbox with a force of 4 lb parallel to the xy plane. When the door is at 45° to the vertical, the force acts at 10° to the horizontal. Calculate the magnitude of the moment the force creates at the moment center at H, the door hinge.



EX 3.1.11

3.1.12. [*] Control bar AB is inclined at 50° from the vertical. A 100-N force is applied to the bar at end B.



EX 3.1.12 and EX 3.1.13

Determine the magnitude and direction of the moment this force creates at a moment center at A if

a. $\alpha = 140°$
b. $\alpha = 210°$
c. $\alpha = 180°$

3.1.13. [**] For control bar AB, determine the value of the angle α for which the 100-N force exerts

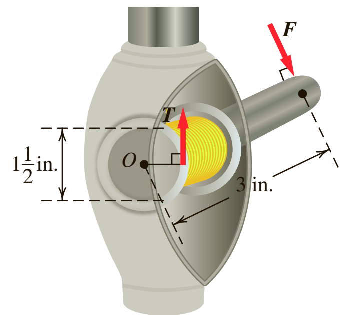
a. the largest counterclockwise moment at a moment center at A (Also determine the corresponding magnitude of the moment.)

b. the largest clockwise moment at a moment center at A (Also determine the corresponding magnitude of the moment.)

c. zero moment at a moment center at A

3.1.14. [**] When a force $\boldsymbol{F}$ is applied to the 3-inch long umbrella crank in **Figure *a***, it turns a 1 1/2-inch diameter spool inside the casing that winds or unwinds a cord (see **Figure *b***). The cord then raises or lowers the umbrella. If the magnitude of the moment at O due to the tension in the cord $\boldsymbol{T}$ is equal to that created by $\boldsymbol{F}$, determine the magnitude of $\boldsymbol{F}$ in terms of $\boldsymbol{T}$.

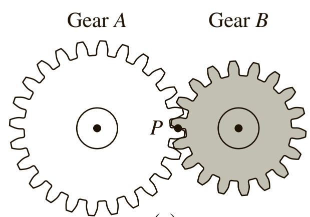
(*a*)



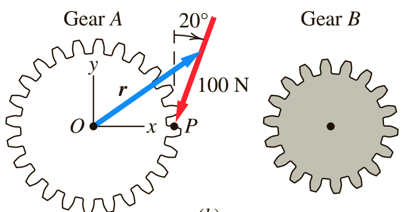
(*b*)

EX 3.1.14

3.1.15. [**] Gear B pushes on gear A at point P with a force of 100 N at P as shown in **Figure *a***. The radius of gear A is 100 mm. Specify in vector notation the position vector $\boldsymbol{r}$ shown in **Figure *b***, which has a magnitude of 150 mm.



(*a*)



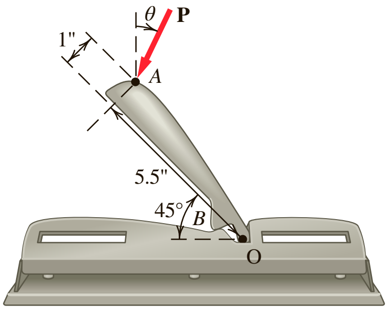
(*b*)

EX 3.1.15

3.1.16. [**] A lever has been added on a 3-hole punch to make it easier to punch holes, as shown. A hand applies a force $\boldsymbol{P}$ to the lever.

a. If $\theta = 20°$, find the moment created by $\boldsymbol{P}$ at a moment center at O. Express it as a function of $\boldsymbol{P}$.

b. For each of the magnitudes indicate whether the force is a reasonable (realistic) value of $\boldsymbol{P}$ for an average adult to apply to the hole punch. **a.** 0.01 lb **b.** 2.0 lb **c.** 0.5 lb



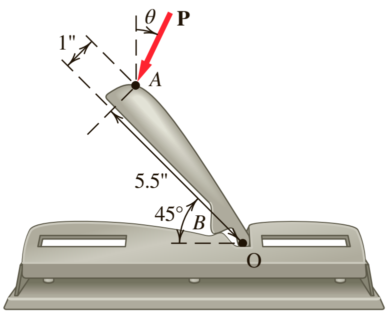
EX 3.1.16

3.1.17. [**] For the 3-hole punch shown, θ can vary between 0° and 90°. Determine the angle θ for which the magnitude of the moment created by force $\boldsymbol{P}$ at O is

a. a minimum

b. a maximum

c. Explain why it is or is not possible for the force $\boldsymbol{P}$ to create zero moment at the O.

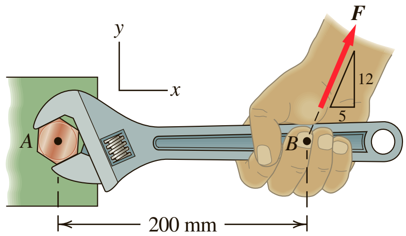


EX 3.1.17

3.1.18. [**] A person applies a 200-N force to a wrench, as shown.

a. What is the magnitude of the moment that this force creates at a moment center at A?

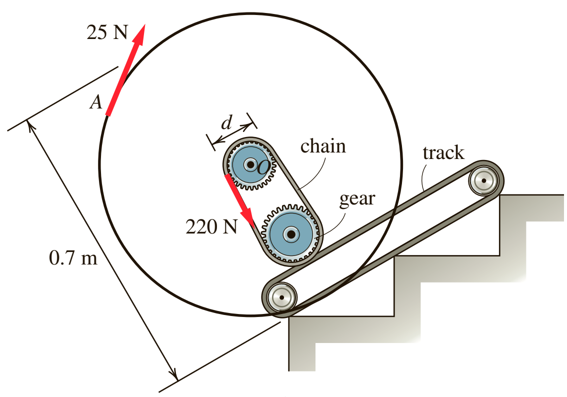
b. Two redesigns to the wrench handle are proposed to allow the 200-N force to create a larger moment. For each design change, determine how much the moment will be increased, expressed as a percentage of your answer in **a.** (1) Increase the handle length so that distance from A to B is 250 mm. (2) Angle the wrench handle so that the force is naturally applied perpendicular to the handle



EX 3.1.18

3.1.19. [**] Engineering students at University of California, Irvine designed the stair climbing wheel chair shown in **Figure *a***. Turning the chair wheel drives a bicycle chain that is attached to a gear. The gear drives a continuous track that grabs the stairs as it moves. A simplified schematic of the wheel and track mechanism is shown in **Figure *b***. When the student applies a 25 N force tangent to the wheel at A, determine the diameter of the chain ring, d, so that moment at O due to the 220 N chain force is equal and opposite to the moment due to the applied wheel force.

(*a*)



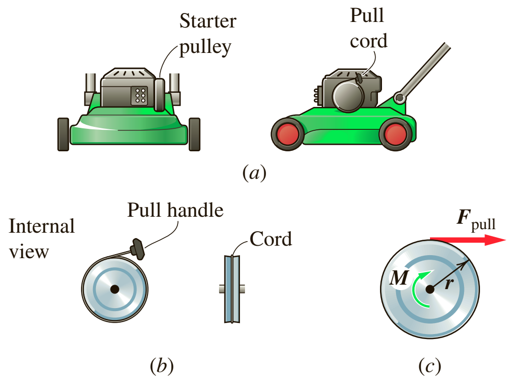
(*b*)

EX 3.1.19

3.1.20. [**] Starting a lawn mower engine requires you to pull on a cord wrapped around a pulley, as shown. The diameter of the pulley is 25 cm, and the moment required to start the mower is 5 N · m.

a. What force $\boldsymbol{F}_{\text{pull}}$ must be applied on the cord in order to start the mower?

b. Suppose as you start the motor, you pull the cord slightly to the side instead of in the plane of the pulley. Is more or less force required than in (**a**)? Why?

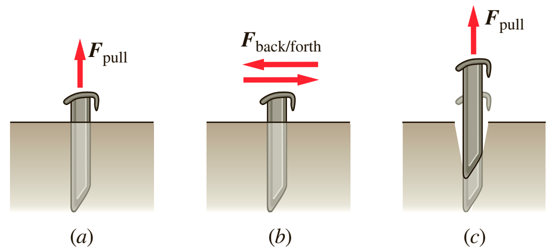


EX 3.1.20

3.1.21. [***] A friend tells you that she had trouble pulling a tent stake out of the ground. The ground consists of firm mud. She tried pulling straight up on the stake (**Figure *a***), and she could not pull it out of the ground. Then she applied a back-and-forth force (**Figure *b***). After this, she was able to remove the stake by pulling straight up (**Figure *c***).

a. What did she accomplish by applying the back-and-forth force?

b. Speculate how effective this procedure would be for removing the stake if it had been in loose sand instead of firm mud.

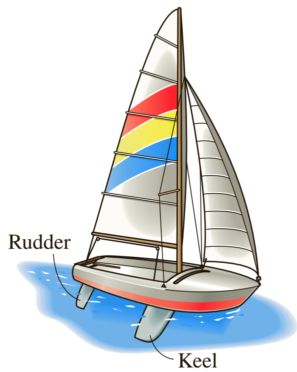


EX 3.1.21

3.1.22. [***] Racing yachts have both large sails and deep keels. Knowing that the rudder, rather than the keel, is used to steer the boat:

a. What is the purpose of the keel? (Think about the moments applied to the yacht.)

b. What do you think the relationship is between the depth of the keel, the size of the sail, and the speed of the boat?



EX 3.1.22

3.1.23. [***] You are cycling down the road, and there's a big hill up ahead. Knowing that you have a long way to go to get home, you would like to have an easy ride up the hill.

a. Downshifting on your bike's rear derailleur makes pedaling easier. Using two or three sentences explain in terms of moments why this is so. Use sketches as necessary.

b. What is the side effect of downshifting? Explain in two or three sentences why this is so. Use sketches as necessary.

3.2 MATHEMATICAL REPRESENTATION OF A MOMENT

Learning Objective: Calculate a moment using cross products.

It is frequently convenient to represent a moment in terms of its scalar components:

$$\boldsymbol{M} = M_x \boldsymbol{i} + M_y \boldsymbol{j} + M_z \boldsymbol{k} \tag{3.3}$$

The scalar components define the magnitude of the moment, $\|\boldsymbol{M}\|$, as shown in **Figure 3.2.1*a***:

$$\|\boldsymbol{M}\| = \sqrt{M_x^2 + M_y^2 + M_z^2} \tag{3.4}$$

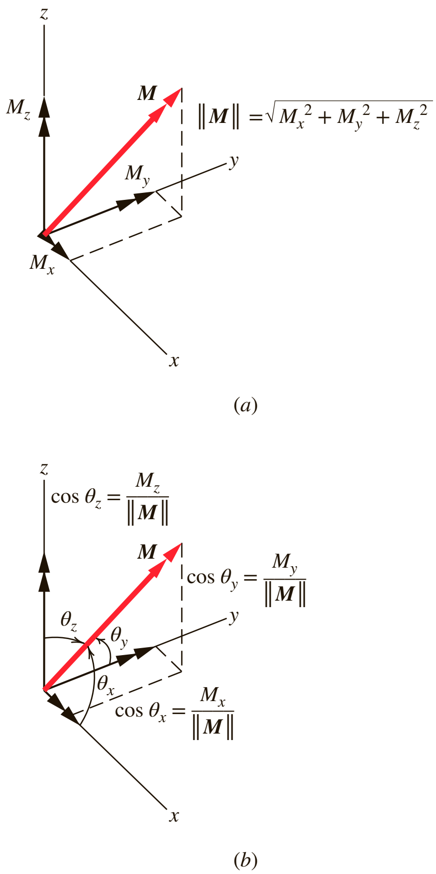


Figure 3.2.1 (*a*) Finding the magnitude of the moment; (*b*) moment direction defined by space angles.

The direction of the moment can be defined in terms of the direction cosines (**Figure 3.2.1*b***):

$$\cos\theta_x = \frac{M_x}{\|\boldsymbol{M}\|} \qquad \cos\theta_y = \frac{M_y}{\|\boldsymbol{M}\|} \qquad \cos\theta_z = \frac{M_z}{\|\boldsymbol{M}\|} \tag{3.5}$$

The moment's direction can also be described by a unit vector $\boldsymbol{u}$:

$$\boldsymbol{u} = \cos\theta_x\,\boldsymbol{i} + \cos\theta_y\,\boldsymbol{j} + \cos\theta_z\,\boldsymbol{k} \tag{3.6}$$

The direction of a vector (in this case, $\boldsymbol{M}$) can be represented in terms of its direction cosines or in terms of a unit vector directed along its line of action, as we saw in Section 2.5. The unit vector in (3.6) can also be interpreted as the rotational axis of the moment.

Our goal now is to rewrite the expressions for moment in (3.3), (3.4), and (3.5) in terms of the force vector and position vector. In other words, we will develop a procedure for finding the scalar components M_x, M_y, and M_z of a moment created by a force $\boldsymbol{F}$ offset from a moment center by position vector $\boldsymbol{r}$. We start by making the following observations about position vectors, forces, and moments:

- All are vector quantities.
- A force offset from a moment center creates a moment about an axis that is perpendicular to the plane defined by the position vector and the force vector (**Figure 3.2.2*a***) and has a direction (+ or −) given by a right-hand rule.
- Only the force component $\|\boldsymbol{F}\|\sin\theta$ perpendicular to the position vector creates a moment (**Figure 3.2.2*b***), and the magnitude of the moment is $\|\boldsymbol{M}\| = \|\boldsymbol{r}\|(\|\boldsymbol{F}\|\sin\theta)$. Another way of looking at this is that only the position vector component $\|\boldsymbol{r}\|\sin\theta$ perpendicular to the force creates a moment (**Figure 3.2.2*c***).

A mathematical construct that captures these observations is called the **vector product** or the **cross product**.

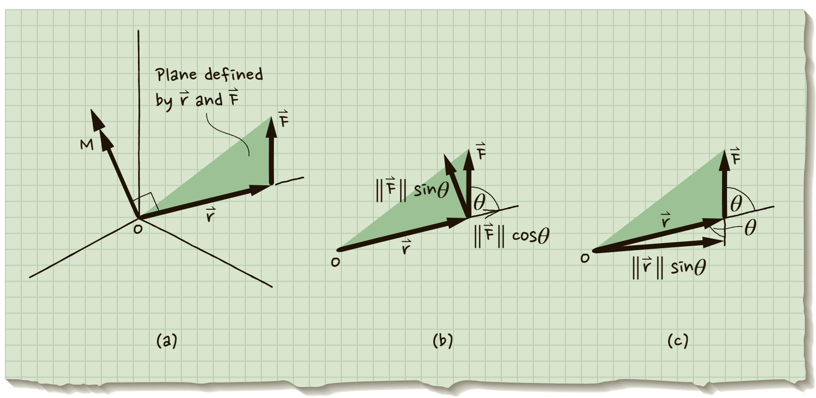


Figure 3.2.2 (*a*) Vectors $\boldsymbol{r}$ and $\boldsymbol{F}$ form a plane. Moment is perpendicular to this plane. (*b*) Component of force perpendicular to position vector creates moment. (*c*) Component of position vector perpendicular to force creates moment.

The cross product of two vectors (in the case of a moment, $\boldsymbol{r}$ and $\boldsymbol{F}$, in that order) is, by definition, a third vector that is perpendicular to the plane defined by the two vectors. The third vector that we are dealing with is the moment $\boldsymbol{M}$. The magnitude of this vector is $(\|\boldsymbol{r}\|\,\|\boldsymbol{F}\|\sin\theta)$, where θ is the angle between $\boldsymbol{r}$ and the line of action of $\boldsymbol{F}$ when $\boldsymbol{r}$ and $\boldsymbol{F}$ are placed tail to tail:

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = (\|\boldsymbol{r}\|\ \|\boldsymbol{F}\| \sin\theta)\,\boldsymbol{u} \tag{3.7A}$$

where $\|\boldsymbol{r}\|$ and $\|\boldsymbol{F}\|$ are the magnitudes of the position vector and force vector, respectively, and $\boldsymbol{u}$ is a unit vector perpendicular to the plane defined by $\boldsymbol{r}$ and $\boldsymbol{F}$. This expression, "$\boldsymbol{r}$ crossed with $\boldsymbol{F}$," is illustrated in **Figure 3.2.3*a***. The cross product is not commutative, meaning that $\boldsymbol{r} \times \boldsymbol{F}$ is not equal to $\boldsymbol{F} \times \boldsymbol{r}$. (Compare **Figure 3.2.3*a*** with **Figure 3.2.3*b***.)

It is generally more convenient to work with the position vector and force vectors in terms of their components. For the position vector and force vectors in **Figure 3.2.4*a***, (3.7A) can be rewritten as

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = (r_x\boldsymbol{i} + r_y\boldsymbol{j} + r_z\boldsymbol{k}) \times (F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k}) \tag{3.7B}$$

By applying the distributive and associative laws of vector multiplication to (3.7B) and noting that $\boldsymbol{i} \times \boldsymbol{i} = 0$, $\boldsymbol{j} \times \boldsymbol{j} = 0$, $\boldsymbol{k} \times \boldsymbol{k} = 0$, $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$, $\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$, and so on, we can write

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = \underbrace{(+r_yF_z - r_zF_y)}_{M_x}\boldsymbol{i} + \underbrace{(+r_zF_x - r_xF_z)}_{M_y}\boldsymbol{j} + \underbrace{(+r_xF_y - r_yF_x)}_{M_z}\boldsymbol{k} \tag{3.8}$$

Equation (3.8) is an expression of **Varignon's Theorem**, which states that the moment of a force at a particular moment center is equal to the sum of the moments of the components of the force at the same moment center (**Figure 3.2.4*b***).

If we substitute the moment scalar components M_x, M_y, and M_z defined in (3.8) into (3.4), the magnitude of the moment can be written as

$$\|\boldsymbol{M}\| = \sqrt{(+r_yF_z - r_zF_y)^2 + (+r_zF_x - r_xF_z)^2 + (+r_xF_y - r_yF_x)^2} \tag{3.9}$$

We can also use the moment scalar components in (3.8) to rewrite the direction cosines in (3.5) that define the direction of the moment:

$$\begin{aligned}
\cos\theta_x &= \frac{M_x}{\|\boldsymbol{M}\|} = \frac{+r_yF_z - r_zF_y}{\|\boldsymbol{M}\|} \\
\cos\theta_y &= \frac{M_y}{\|\boldsymbol{M}\|} = \frac{+r_zF_x - r_xF_z}{\|\boldsymbol{M}\|} \\
\cos\theta_z &= \frac{M_z}{\|\boldsymbol{M}\|} = \frac{+r_xF_y - r_yF_x}{\|\boldsymbol{M}\|}
\end{aligned} \tag{3.10}$$

The direction of the moment is defined by the unit vector $\boldsymbol{u}$ (see (3.6)) based on the direction cosines in (3.10) as

$$\boldsymbol{u} = \frac{+r_yF_z - r_zF_y}{\|\boldsymbol{M}\|}\boldsymbol{i} + \frac{+r_zF_x - r_xF_z}{\|\boldsymbol{M}\|}\boldsymbol{j} + \frac{+r_xF_y - r_yF_x}{\|\boldsymbol{M}\|}\boldsymbol{k} \tag{3.11}$$

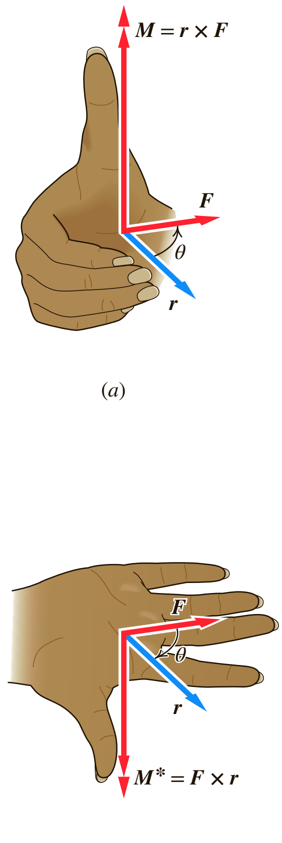


Figure 3.2.3 (a) Right-hand rule used to find direction of moment, (b) $\boldsymbol{r} \times \boldsymbol{F}$ is not equal to $\boldsymbol{F} \times \boldsymbol{r}$.

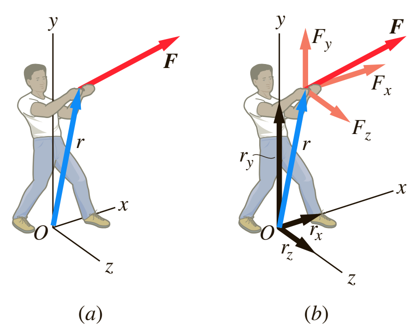


Figure 3.2.4 (*a*) Position vector and force; (*b*) components of position and force vectors.

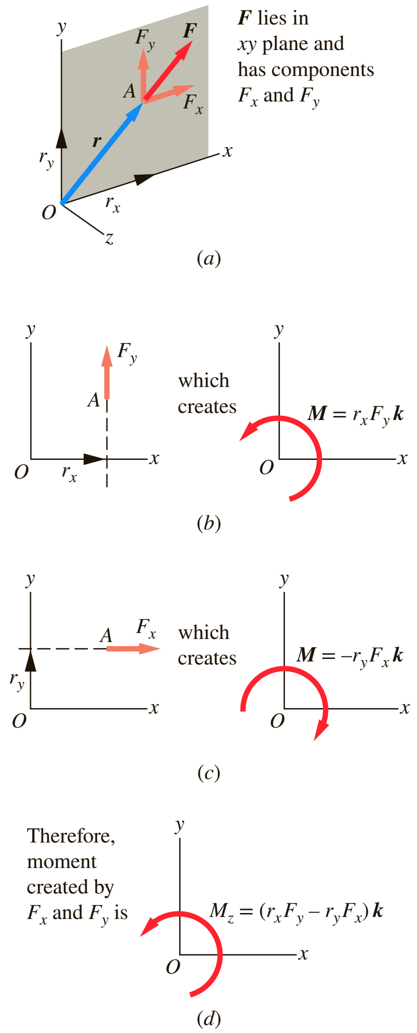


Figure 3.2.5 (*a*) Finding the moment created by a force with x and y components; (*b*) consider component F_y; (*c*) consider component F_x; (*d*) moment created by F_y and F_x.

Now we return to the cross-product form of moment calculation, as presented in (3.7B). More generally, the cross product of the position vector and the force can be written in matrix form as

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \tag{3.12}$$

The form in (3.12) specifies that the **determinant** of the matrix be taken. (If you are unfamiliar with this form see Appendix A.4.)

The expression for moment $\boldsymbol{M}$ in (3.12) is equivalent to the expression in (3.8). The beauty of (3.12) is that the various combinations of position vector and force components (r_x, r_y, r_z, F_x, F_y, F_z) needed to correctly calculate the moment come out naturally in taking the determinant, so there is no need to memorize (3.8). Simply by arranging the position vector and force vector in the matrix form in (3.12) and taking the determinant, we calculate the moment. Using the cross-product is the second approach to calculating the moment. While the first approach (presented in the prior section) is often more straightforward in situations where the position and force vectors are in a single coordinate plane, the cross-product approach is useful when they are not.

Moments about the *x*, *y*, or *z* Axis

Notice that if the force and position vectors lie in the xy, yz, or zx plane, the moment found by applying (3.8) will be about the z, x, or y axis, respectively. More specifically:

For a force vector and position vector lying in the same xy plane, we can write

$$\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j}, \qquad \boldsymbol{r} = r_x\boldsymbol{i} + r_y\boldsymbol{j}$$

The moment at a moment center at O is about the z axis and is then (based on **Figure 3.2.5**):

$$\boldsymbol{M}_z = M_z\boldsymbol{k} = (+r_xF_y - r_yF_x)\boldsymbol{k} \tag{3.13A}$$

This is the third term on the right-hand side of (3.8). Because no moment is created about the x or y axis, $\boldsymbol{M}_x = \boldsymbol{M}_y = 0$. The magnitude of the moment (based on (3.4)) is $|+r_xF_y - r_yF_x|$. Also, because $\theta_z = 0°$, $\cos\theta_z = 1$.

For a force vector and position vector lying in the same yz plane (**Figure 3.2.6**), we find that the moment at a moment center at O is about the x axis:

$$\boldsymbol{M}_x = M_x\boldsymbol{i} = (+r_yF_z - r_zF_y)\boldsymbol{i} \tag{3.13B}$$

This is the first term on the right-hand side of (3.8). No moment is created about the y or z axis, so $\boldsymbol{M}_y = \boldsymbol{M}_z = 0$. The magnitude of the moment (based on (3.4)) is $|+r_yF_z - r_zF_y|$, resulting in $\theta_x = 0°$ and $\cos\theta_x = 1$.

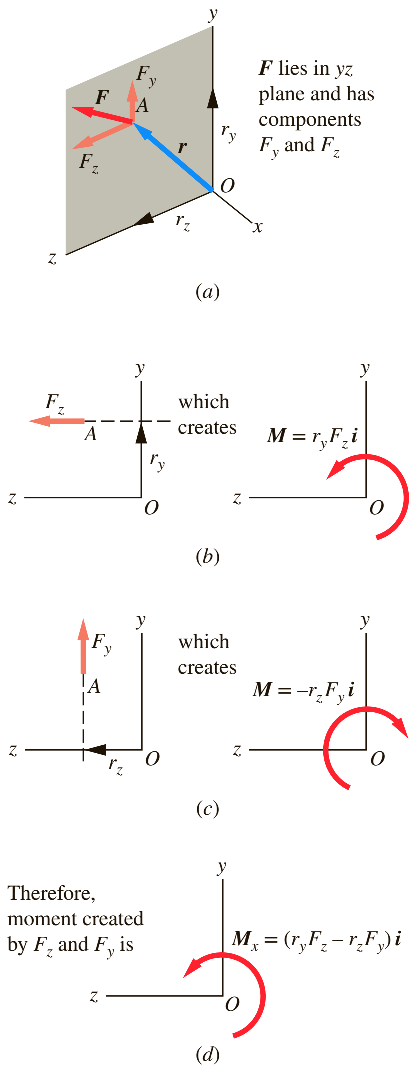


Figure 3.2.6 (*a*) Finding the moment created by a force with y and z components; (*b*) consider component F_z; (*c*) consider component F_y; (*d*) moment created by F_z and F_y.

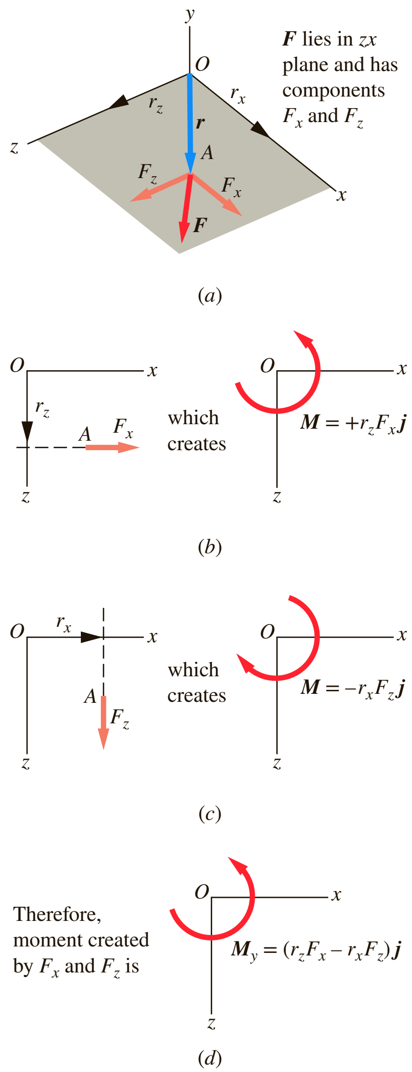


Figure 3.2.7 (*a*) Finding the moment created by a force with z and x components; (*b*) consider component F_x; (*c*) consider component F_z; (*d*) moment created by F_x and F_z.

Similarly, *for a force vector and position vector lying in the same zx plane* (Figure 3.2.7), the moment at a moment center O is about the y axis:

$$\boldsymbol{M}_y = M_y\boldsymbol{j} = (+r_zF_x - r_xF_z)\boldsymbol{j} \qquad (3.13\text{C})$$

This is the second term on the right-hand side of (3.8). The magnitude of the moment (based on (3.4)) is $|+r_zF_x - r_xF_z|$, and $\theta_y = 0°$, so $\cos\theta_y = 1$.

The equations can also be written in matrix form. For example, for a force and position vector in the same xy plane, (3.13A) can be written as:

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ r_x & r_y & 0 \\ F_x & F_y & 0 \end{vmatrix} \tag{3.14}$$

However, for situations where the position vector and force are in the same plane, you might find application of one of the expressions in (3.13) more straightforward.

Check out the following examples of applications of this material.

- **Example 3.2.1 Calculating the Moment About the *z* Axis with a Vector-Based Approach**
- **Example 3.2.2 Calculating the Moment About the *z* Axis with the Component of the Force Perpendicular to the Position Vector**
- **Example 3.2.3 Calculating the Moment - Nonplanar**
- **Example 3.2.4 Calculating the Magnitude and Direction of a Moment - Nonplanar**
- **Example 3.2.5 Finding the Force to Create A Moment - Nonplanar**

EXAMPLE 3.2.1

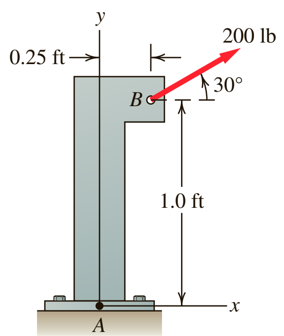


Figure 1 A force applied at B creates a moment about the z axis at A.

A cable pulls on the L-bracket shown in **Figure 1** at B with a force $\|\boldsymbol{F}\| = 200$ lb. Find the moment the cable force creates about a moment center at A.

Goal Find $\boldsymbol{M}$ about a moment center at A due to a 200-lb force applied at B.

Given The dimensions of the L-bracket, the locations of points A and B, the orientation and magnitude of the force at B, and the xy axes are specified.

Assume The 200-lb force and the L-bracket lie in the xy plane.

Formulate Equations and Solve While multiple approaches are possible to solve this problem, we demonstrate a vector-based approach $(\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F})$.

The position vector $\boldsymbol{r}$: We arbitrarily define the position vector as a vector from A (the moment center) to B, the point of force application. This choice for the position vector makes it easy to write $\boldsymbol{r}$ in terms of its components based on the dimensions given in **Figure 2*b***:

$$\boldsymbol{r} = r_x \boldsymbol{i} + r_y \boldsymbol{j} = 0.25\,\text{ft}\,\boldsymbol{i} + 1.00\,\text{ft}\,\boldsymbol{j}$$

The force $\boldsymbol{F}$ (**Figure 2*a***):

$$\begin{aligned} \boldsymbol{F} &= F_x\boldsymbol{i} + F_y\boldsymbol{j} = (200\text{ lb})(\cos 30°)\boldsymbol{i} + (200\text{ lb})(\sin 30°)\boldsymbol{j} \\ &= 173.2\text{ lb}\,\boldsymbol{i} + 100.0\text{ lb}\,\boldsymbol{j} \end{aligned}$$

Noting that the position vector $\boldsymbol{r}$ and $\boldsymbol{F}$ lie in the xy plane, we know from (3.13A) that the resulting moment will be about the z axis.

$$\boldsymbol{M}_z = M_z\boldsymbol{k} = (+r_xF_y - r_yF_x)\boldsymbol{k}$$

Substituting the values for F_x, F_y, r_x, and r_y in (3.13A), we determine that

$$\boldsymbol{M}_z = [(0.25\text{ ft})(100.0\text{ lb}) - (1.00\text{ ft})(173.2\text{ lb})]\boldsymbol{k} \quad\Rightarrow\quad \boxed{\boldsymbol{M}_z = -148\text{ ft}\cdot\text{lb}\,\boldsymbol{k}}$$

The moment is negative because the sense of the rotation caused by $\boldsymbol{M}_z$ is clockwise about the z axis. The negative sign also indicates that the moment vector is in the $-\boldsymbol{k}$ direction.

Check The negative sign for the moment can be confirmed by looking at the relative sizes of F_x, F_y and r_x, r_y. Looking at **Figure 2*a***, we see that F_x causes a negative (clockwise) moment about the z axis while F_y causes a positive (counterclockwise) moment. Because $F_x > F_y$ and $r_y > r_x$ (meaning that the position vector associated with F_x is larger than the one associated with F_y), the moment resulting from F_x and F_y will be negative.

This check does not involve any additional calculations, but involves reasoning about relative sizes of values. This type of check complements those that require additional calculations. Remember that checks are done to convince yourself that you made reasonable assumptions, set the problem up properly, and carried out the calculations correctly.

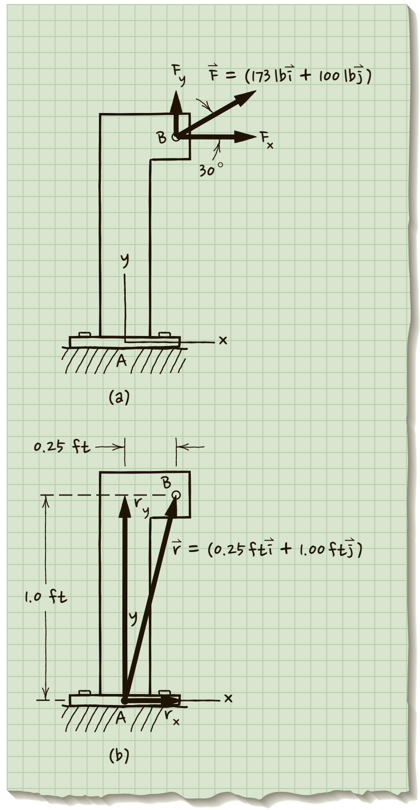


Figure 2 (a) $\boldsymbol{F}$ is divided into its x and y components, (b) We choose to draw the position vector $\boldsymbol{r}$ from A to B.

EXAMPLE 3.2.2

Check the results for the moment about A found in Example 3.2.1 by using the component of the force perpendicular to the position vector.

Goal Find the moment at a moment center at A created by a 200-lb force applied at B using (3.1), an approach that is different from Example 3.2.1.

Given and Assume These are the same as in Example 3.2.1.

Draw We arbitrarily choose the position vector to be from A (moment center) to B. We draw the position vector as well as the angle θ between $\boldsymbol{r}$ and $\boldsymbol{F}$ (**Figure 2**).

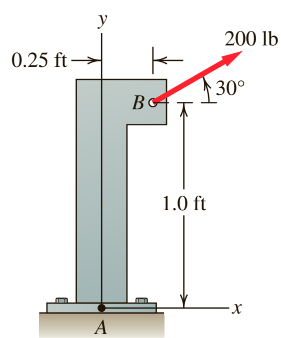


Figure 1 A force applied at B creates a moment about the z axis at A.

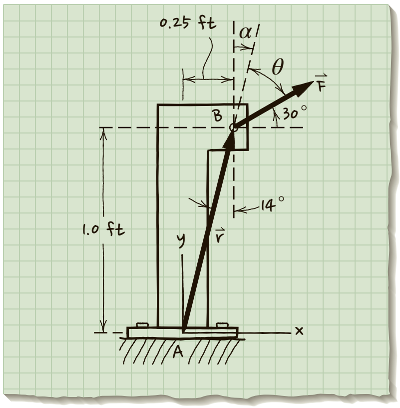


Figure 2 The position vector is drawn from A to B.

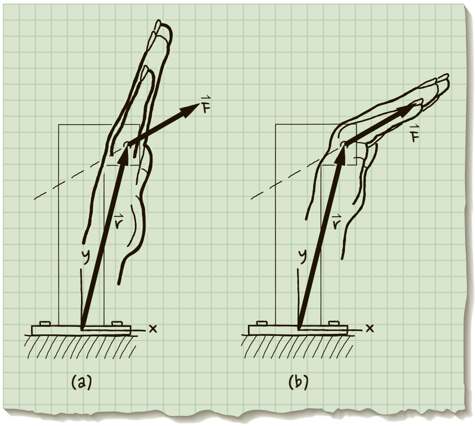


Figure 3 Based on the right-hand rule, the resulting moment is clockwise.

Formulate Equations and Solve The cable tension creates a moment about the z axis. Equation (3.1) says that the magnitude of this moment is

$$||\boldsymbol{M}|| = ||\boldsymbol{r}||\,(||\boldsymbol{F}||\,\sin\theta)$$

Magnitude of $\boldsymbol{r}$: From the dimensions in **Figure 2** we calculate the length of $\boldsymbol{r}$:

$$||\boldsymbol{r}|| = \sqrt{(0.25\text{ ft})^2 + (1.00\text{ ft})^2} = 1.031\text{ ft}$$

The angle θ between $\boldsymbol{r}$ *and* $\boldsymbol{F}$: Based on the geometry shown in **Figure 2**, we find that $\theta = 90° - \alpha - 30°$. Because $\alpha = \tan^{-1}(0.25\text{ ft}/1.00\text{ ft}) = 14°$, we have $\theta = 46°$.

Substituting these values of $||\boldsymbol{r}||$, $||\boldsymbol{F}||$, and θ into (3.1), we find that

$$||\boldsymbol{M}|| = (1.031\text{ ft})(200\text{ lb})\sin 46° \quad\Rightarrow\quad ||\boldsymbol{M}|| = 148\text{ ft}\cdot\text{lb}$$

Use the right-hand rule to determine direction of the moment about the z axis (**Figure 3**). Align your right-hand palm with $\boldsymbol{r}$ in a way that will allow you to curl your fingers to align with $\boldsymbol{F}$, and then move your thumb to make a 90° angle with your palm. The "curl" sense is clockwise when viewed from far out on the z axis and defines the direction of the moment as being along an axis aligned with your thumb (which is in the negative z direction). Therefore, the direction of the moment vector is $-\boldsymbol{k}$ and the sense of rotation is clockwise.

This agrees with our results for 3.2.1. In Examples 3.2.1 and 3.2.2 we show two approaches for finding the moment vector. Both result in the same answer!

The selection of an approach is often up to personal preference. As you will see, some approaches involve less work. The key is that whatever approach you use, you should get the same answer.

EXAMPLE 3.2.3

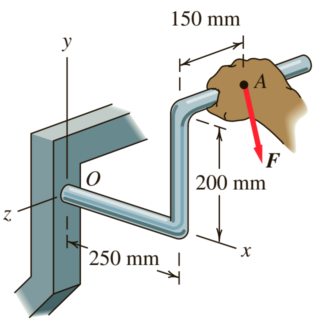


Figure 1 A hand pulls on a handlebar at point A with force $\boldsymbol{F}$.

The hand in **Figure 1** pulls on the handlebar at point A with force $\boldsymbol{F}$. The magnitude of the force is 110 N, and its direction is described by the unit vector $\boldsymbol{u} = (2/3)\boldsymbol{i} - (2/3)\boldsymbol{j} - (1/3)\boldsymbol{k}$. Determine the moment $\boldsymbol{M}_O$ created by $\boldsymbol{F}$ about a moment center at O. Present the answer in vector notation and graphically.

Goal Find $\boldsymbol{M}_O$ created by a specified force and present the answer both in vector notation and graphically.

Given Dimensions of the handlebar, $||\boldsymbol{F}||$, and the unit vector that defines the line of action of $\boldsymbol{F}$. A coordinate system also has been established.

Assume No assumptions are necessary.

Formulate Equations and Solve We will use the cross-product (3.12) to calculate $\boldsymbol{M}_O$. To apply (3.12), we determine the components of $\boldsymbol{F}$ and those of the position vector $\boldsymbol{r}$.

The force in vector notation:

$$\boldsymbol{F} = 110\,\text{N}\,\boldsymbol{u} = 110\,\text{N}\left(\frac{2}{3}\boldsymbol{i} - \frac{2}{3}\boldsymbol{j} - \frac{1}{3}\boldsymbol{k}\right) = 73.3\,\text{N}\,\boldsymbol{i} - 73.3\,\text{N}\,\boldsymbol{j} - 36.7\,\text{N}\,\boldsymbol{k}$$

These components are illustrated in **Figure 2**.

Position vector: The most straightforward position vector to define is one from O to A, where A is the point of application of the force. The scalar components of $\boldsymbol{r}$ are illustrated in **Figure 3**. Reading off the dimensions given in the figure, we find

$$\boldsymbol{r} = 250\,\text{mm}\,\boldsymbol{i} + 200\,\text{mm}\,\boldsymbol{j} - 150\,\text{mm}\,\boldsymbol{k}$$

Now we write (3.12) for $\boldsymbol{M}_O$ in terms of $\boldsymbol{r}$ and $\boldsymbol{F}$:

$$\boldsymbol{M}_O = \boldsymbol{r} \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 250\,\text{mm} & 200\,\text{mm} & -150\,\text{mm} \\ 73.3\,\text{N} & -73.3\,\text{N} & -36.7\,\text{N} \end{vmatrix}$$

Taking the determinant of this (which is what the cross-product is) yields

$$\begin{aligned} \boldsymbol{M}_O = \{&[(200)(-36.7) - (-150)(-73.3)]\boldsymbol{i} \\ &+ [(-150)(73.3) - (250)(-36.7)]\boldsymbol{j} \\ &+ [(250)(-73.3) - (200)(73.3)]\,\boldsymbol{k}\}\ \text{N}\cdot\text{mm} \end{aligned}$$

$$\boldsymbol{M}_O = -18.33\,\text{N}\cdot\text{m}\,\boldsymbol{i} - 1.820\ \text{N}\cdot\text{m}\ \boldsymbol{j} - 33.0\ \text{N}\cdot\text{m}\ \boldsymbol{k}$$

This result is shown graphically in **Figure 4**.

Check We can check our answer by reviewing the signs on each of the position vector and force components relative to **Figure 1** to make sure that we got all of the signs right in our calculations. We also review the signs and magnitudes of the scalar components of the moment. Does it make sense that the magnitude of M_z is almost twice that of M_x? Does it make sense that the magnitude of M_y is so much smaller than that of M_z and M_x?

Comment: Earlier in the chapter, we noted that the cross product is not commutative. This means that $\boldsymbol{F} \times \boldsymbol{r} \neq \boldsymbol{r} \times \boldsymbol{F}$. This can be proved by finding $\boldsymbol{F} \times \boldsymbol{r}$ and comparing the answer to $\boldsymbol{r} \times \boldsymbol{F}$. Find $\boldsymbol{F} \times \boldsymbol{r}$:

$$\boldsymbol{F} \times \boldsymbol{r} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ F_x & F_y & F_z \\ r_x & r_y & r_z \end{vmatrix} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 73.3\,\text{N} & -73.3\,\text{N} & -36.7\,\text{N} \\ 250\,\text{mm} & 200\,\text{mm} & -150\,\text{mm} \end{vmatrix}$$

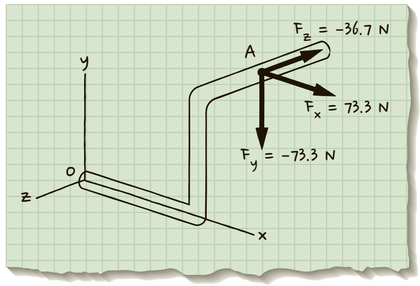


Figure 2 The x, y, and z components of $\boldsymbol{F}$.

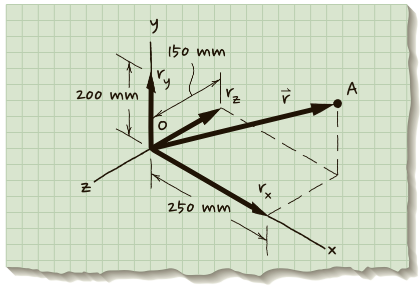


Figure 3 The x, y, and z components of $\boldsymbol{r}$.

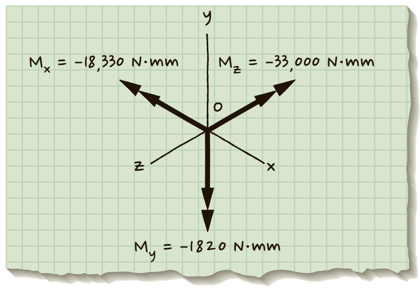


Figure 4 Components of $\boldsymbol{M}_O$ in the x, y, and z directions.

Again using the rules of determinants, we can expand this to yield

$$\begin{aligned}\boldsymbol{F}\times\boldsymbol{r} = &\left[(-73.3\,\text{N})(-150\,\text{mm}) - (-36.7\,\text{N})(200\,\text{mm})\right]\boldsymbol{i}\\ &+\left[(-36.7\,\text{N})(250\,\text{mm}) - (73.3\,\text{N})(-150\,\text{mm})\right]\boldsymbol{j}\\ &+\left[(73.3\,\text{N})(200\,\text{mm}) - (-73.3\,\text{N})(250\,\text{mm})\right]\boldsymbol{k}\end{aligned}$$

Carrying out the arithmetic gives us

$$\boldsymbol{F}\times\boldsymbol{r} = 18.33\,\text{N}\cdot\text{m}\,\boldsymbol{i} + 1.820\,\text{N}\cdot\text{m}\,\boldsymbol{j} + 33.0\,\text{N}\cdot\text{m}\,\boldsymbol{k}$$

This is not the same as $\boldsymbol{r}\times\boldsymbol{F}$, but it does show that

$$-(\boldsymbol{F}\times\boldsymbol{r}) = \boldsymbol{r}\times\boldsymbol{F}$$

EXAMPLE 3.2.4

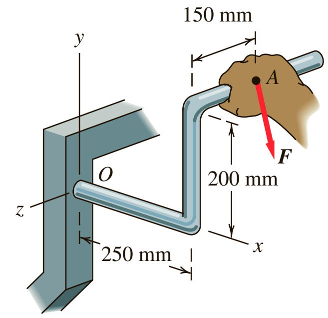


Figure 1 A hand pulls on a handlebar at point A with force $\boldsymbol{F}$.

Revisit the the handlebar with 110-N force $\boldsymbol{F}$ applied at point A that we analyzed in Example 3.2.3 (**Figure 1**). For $\boldsymbol{M}_O$ calculated in that example, (a) determine the magnitude of the moment, and (b) find the direction cosines associated with the moment, and the unit vector that describes the direction of the moment.

Goal Find (a) the magnitude of $\boldsymbol{M}_O$, the moment about moment center (O) created by $\boldsymbol{F}$. In addition, determine (b) its direction cosines as well as the unit vector aligned with it.

Given Dimensions of the handlebar, $||\boldsymbol{F}||$, and the unit vector that defines the line of action of $\boldsymbol{F}$. A coordinate system also has been established.

Assume No assumptions are necessary.

Draw No additional drawings are needed.

Formulate Equations and Solve **(a)** We know the moment components from Example 3.2.3 (also see **Figure 2**).

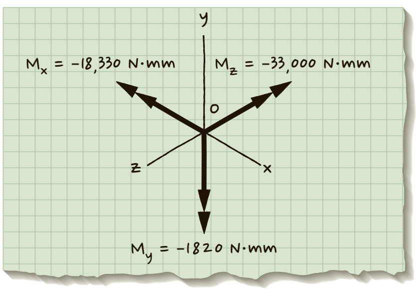


Figure 2 Components of $\boldsymbol{M}_O$ in the x, y, and z directions.

$$\boldsymbol{M}_O = -18.33\,\text{N}\cdot\text{m}\,\boldsymbol{i} - 1.820\,\text{N}\cdot\text{m}\,\boldsymbol{j} - 33.0\,\text{N}\cdot\text{m}\,\boldsymbol{k}$$

The most straightforward approach to finding the magnitude of the moment is to apply (3.4):

$$||\boldsymbol{M}_O|| = \sqrt{M_x^2 + M_y^2 + M_z^2} = \sqrt{(-18.33\,\text{N}\cdot\text{m})^2 + (-1.820\,\text{N}\cdot\text{m})^2 + (-33.0\,\text{N}\cdot\text{m})^2}$$

$$||\boldsymbol{M}_O|| = 37.8\,\text{N}\cdot\text{m}$$

(b) We use (3.10) to find the direction cosines of the moment, which describe the axis about which the moment acts

$$\cos\theta_x = \frac{M_x}{\|\boldsymbol{M}\|} = \frac{-18.33\ \text{N}\cdot\text{m}}{37.8\ \text{N}\cdot\text{m}} \Rightarrow \boxed{\cos\theta_x = -0.485}$$

$$\cos\theta_y = \frac{M_y}{\|\boldsymbol{M}\|} = \frac{-1.820\ \text{N}\cdot\text{m}}{37.8\ \text{N}\cdot\text{m}} \Rightarrow \boxed{\cos\theta_y = -0.0481}$$

$$\cos\theta_z = \frac{M_z}{\|\boldsymbol{M}\|} = \frac{-33.0\ \text{N}\cdot\text{m}}{37.8\ \text{N}\cdot\text{m}} \Rightarrow \boxed{\cos\theta_y = -0.873}$$

Now we use the direction cosines to describe a unit vector along the axis of $\boldsymbol{M}_O$, per (3.6):

$$\boldsymbol{u}_M = \cos\theta_x\,\boldsymbol{i} + \cos\theta_y\,\boldsymbol{j} + \cos\theta_z\,\boldsymbol{k} \qquad \Rightarrow \qquad \boxed{\boldsymbol{u}_M = -0.485\,\boldsymbol{i} - 0.0481\,\boldsymbol{j} - 0.873\,\boldsymbol{k}}$$

This unit vector is perpendicular to the plane defined by the position vector and the force.

Check There are no great checks for this problem except to repeat the calculations to check that we correctly entered the numbers into the calculator.

EXAMPLE 3.2.5

Consider the person trying to break off the tree branch in **Figure 1**. Assuming that the magnitude of the moment necessary to break the branch is 300 N · m, what force is required to break the branch?

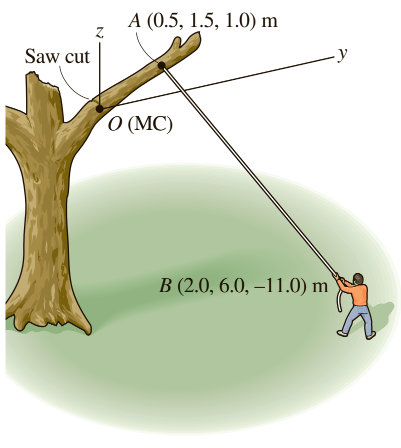


Figure 1 A person trying to apply a 300 N · m moment with a rope.

Goal Find the magnitude of force necessary to break the tree branch.

Given The magnitude of moment necessary to break the branch, the direction and point of application of the force, and relevant dimensions.

Assume No assumptions are necessary.

Draw We draw $\boldsymbol{F}$ and $\boldsymbol{r}$ in **Figure 2**.

Formulate Equations and Solve We use (3.1) and the breaking moment $\|\boldsymbol{M}_O\|$ to solve for the required force.

First we write the force in terms of its rectangular components, and find the angle between $\boldsymbol{r}$ and $\boldsymbol{F}$. Using the coordinates of points A and B to determine a unit vector from A to B, we can represent the applied force in terms of a unit vector as

$$\boldsymbol{F} = \|\boldsymbol{F}\|\,\boldsymbol{u}_F = \|\boldsymbol{F}\|(0.116\,\boldsymbol{i} + 0.349\,\boldsymbol{j} - 0.930\,\boldsymbol{k})$$

We define a position vector that runs from O to A. Based on the information in **Figure 2**, we write

$$\boldsymbol{r} = 0.5\ \text{m}\,\boldsymbol{i} + 1.5\ \text{m}\,\boldsymbol{j} + 1.0\ \text{m}\,\boldsymbol{k} \quad \Rightarrow \quad \|\boldsymbol{r}\| = 1.87\ \text{m}$$

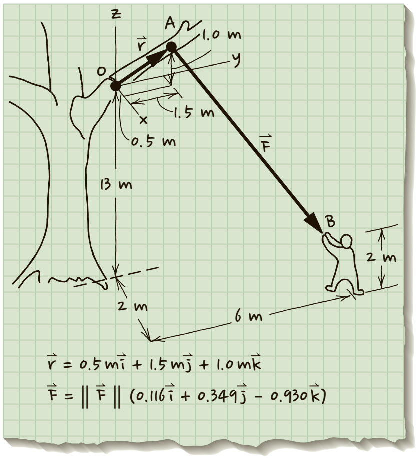


Figure 2 We draw a position vector from O to A.

Now we can use (2.26) to determine the angle between $\boldsymbol{F}$ and $\boldsymbol{r}$:

$$\begin{aligned}\theta &= \cos^{-1}\left[\frac{||\boldsymbol{F}||((0.116)(0.5\text{ m}) + (0.349)(1.5\text{ m}) + (-0.930)(+1.0\text{ m}))}{||\boldsymbol{F}||\,(1.87\text{ m})}\right]\\ &= 100.7°\end{aligned}$$

We use (3.1) to determine the magnitude of the force:

$$||\boldsymbol{M}_O|| = 300\text{ N}\cdot\text{m} = ||\boldsymbol{r}||\,||\boldsymbol{F}||\sin\theta = (1.87\text{ m})||\boldsymbol{F}||\sin 100.7°$$

Solving for $||\boldsymbol{F}||$ we have:

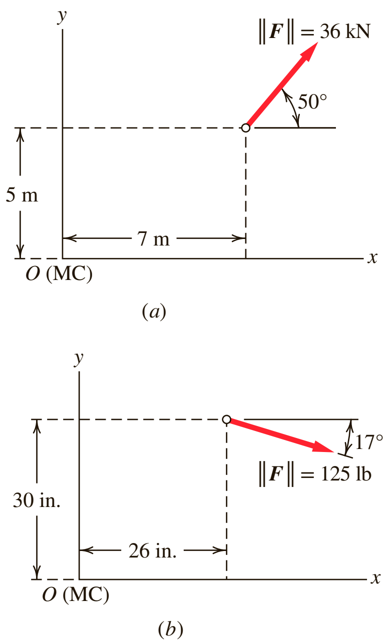
$$||\boldsymbol{F}|| = 163\text{ N}$$

Check To check the answer we could use an alternative approach. For example, we could use the cross product (3.12) to find the moment based on the force and position vectors given above. We would then equate this expression to 300 N·m to find that $||\boldsymbol{F}|| = 163$ N.

Comment: Is 163 N a small or large force? Would you be able to apply it to break off the branch? How should $\boldsymbol{F}$ be oriented so as to minimize its required magnitude?

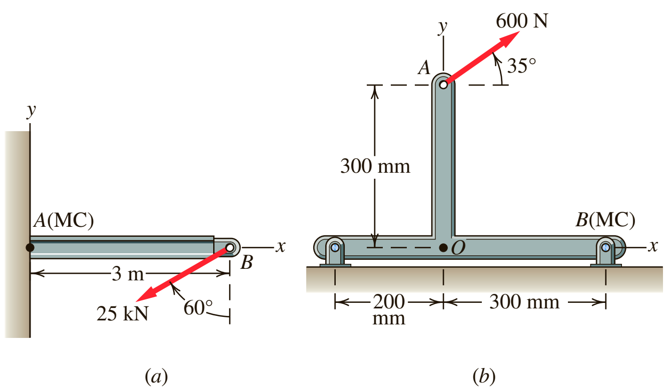
EXERCISES 3.2

3.2.1. [*] For each of two situations shown, calculate the moment the force creates at the moment center (MC) indicated in the figure.



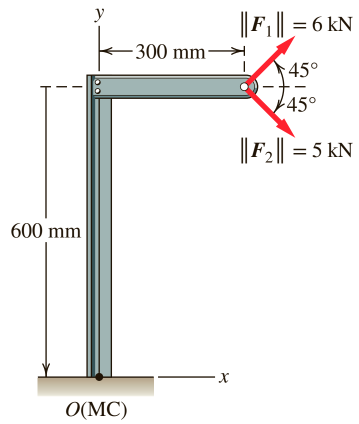
EX 3.2.1

3.2.2. [*] For each of the two situations shown, calculate the moment the force creates at the moment center (MC) indicated in the figure.



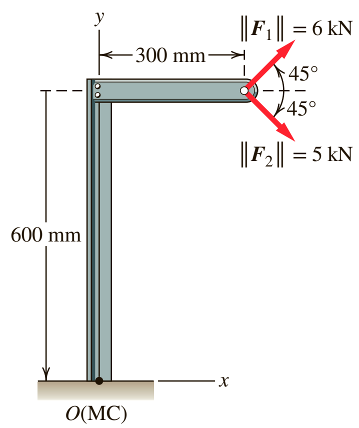
EX 3.2.2

3.2.3. [*] For the frame and loading as shown, consider $\boldsymbol{F}_1$ and moment center O and write (**a**) symbolic expressions for the force and position vector, and then (**b**) an expression for the moment created by the force at the indicated moment center.



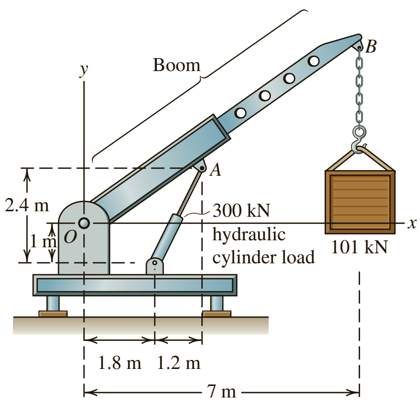
EX 3.2.3

3.2.4. [*] For the frame and loading as shown, consider $\boldsymbol{F}_2$ and moment center O and write (**a**) symbolic expressions for the force and position vector, and then (**b**) an expression for the moment created by the force at the indicated moment center.



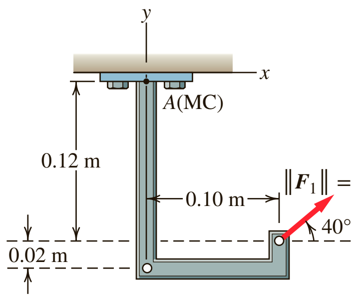
EX 3.2.4

3.2.5. [*] For boom OAB, consider $\boldsymbol{F}_H$, the load in the hydraulic cylinder pushing on the boom at A, and moment center O and write (**a**) symbolic expressions for the force and position vector, and then (**b**) an expression for the moment created by the force at the indicated moment center.



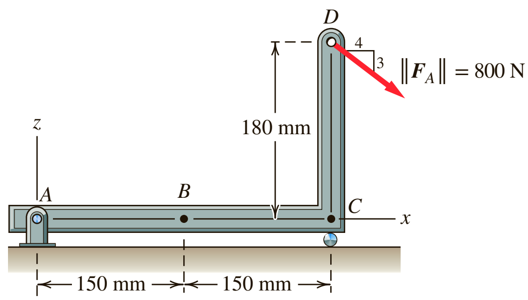
EX 3.2.5

3.2.6. [*] For the frame and loading as shown, consider $\boldsymbol{F}_1$ and moment center A and write (**a**) symbolic expressions for the force and position vector, and then (**b**) an expression for the moment at A created by the force.



EX 3.2.6

3.2.7. [*] For the frame and loading as shown, consider $\boldsymbol{F}_A$ and moment center O, and write (**a**) symbolic expressions for the force and position vector, and then (**b**) an expression for the moment created by the force at the indicated moment center.

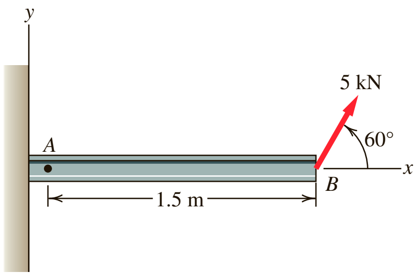


EX 3.2.7

3.2.8. [*] A 5-kN force is applied to the end of an I-beam, as shown.

a. Use the definition of $\|\boldsymbol{M}\|$ in (3.1) to determine the magnitude of the moment at A created by the force. About which axis is the moment?

b. Determine the moment $\boldsymbol{M}$ at A created by the force by using the appropriate expression from (3.13A), (3.13B), or (3.13C). Confirm that the magnitude of this moment is the same as found in **a**.



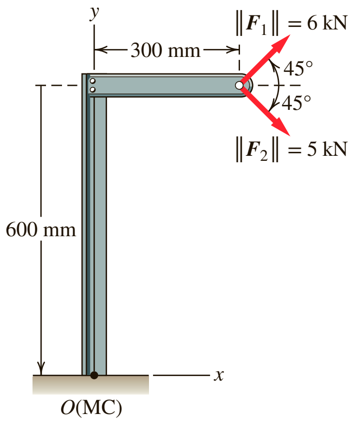
EX 3.2.8

3.2.9. [*] Consider the frame shown. Determine at moment center *O*:

a. the moment $\boldsymbol{M}_1$ created by $\boldsymbol{F}_1$

b. the moment $\boldsymbol{M}_2$ created by $\boldsymbol{F}_2$

c. the sum of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$



EX 3.2.9

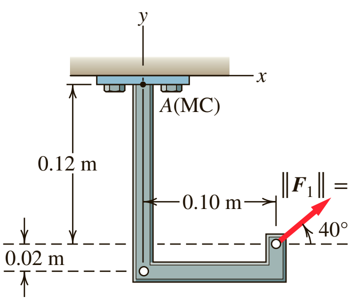
3.2.10. [*] Consider the frame shown. Determine at moment center *A*

a. the moment $\boldsymbol{M}_1$ created by $\boldsymbol{F}_1$

b. the moment $\boldsymbol{M}_2$ created by $\boldsymbol{F}_2$

c. the moment $\boldsymbol{M}_3$ created by $\boldsymbol{F}_3$

d. the sum $\boldsymbol{M}_1 + \boldsymbol{M}_2 + \boldsymbol{M}_3$



EX 3.2.10

3.2.11. [*] A 10-kN force acts on a bracket.

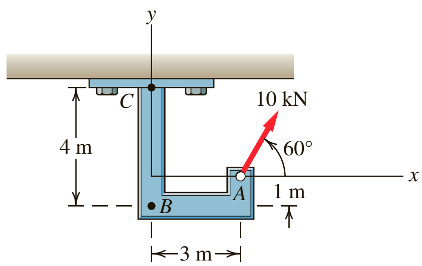
a. Speculate on whether the force creates a greater moment about a moment center at *B* or a moment center at *C*.

Calculate the moment the force creates:

b. at a moment center at *B*.

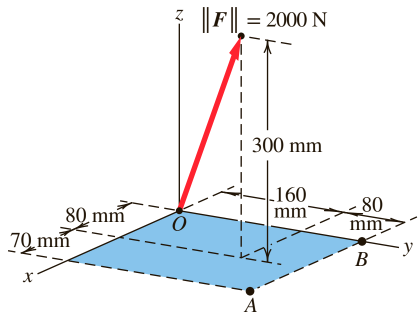
c. at a moment center at *C*.

d. Compare your answers in **b** and **c** with your speculation in **a**.



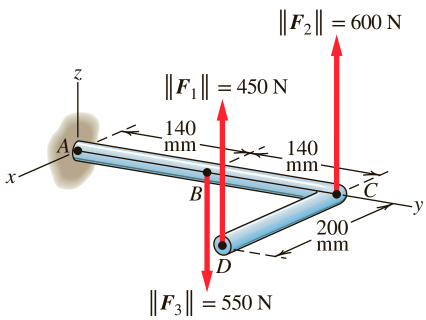
EX 3.2.11

3.2.12. [*] Consider $\boldsymbol{F}$ and a moment center at *A* and write (**a**) symbolic expressions for the force and position vector, then (**b**) an expression for the moment created by the force at the indicated moment center.



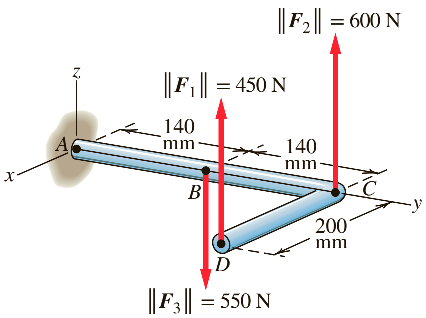
EX 3.2.12

3.2.13. [*] Consider $\boldsymbol{F}_1$ and a moment center at *A* and write (a) symbolic expressions for the force and position vector, then (b) an expression for the moment created by the force at the indicated moment center.



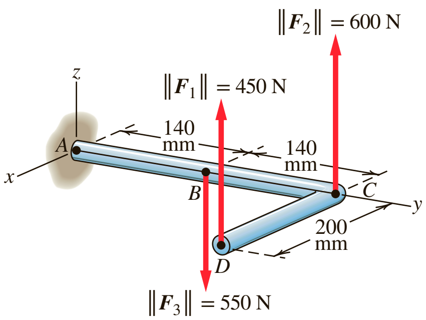
EX 3.2.13

3.2.14. [*] Consider F_2 and a moment center at A and write (a) symbolic expressions for the force and position vector, then (b) an expression for the moment created by the force at the indicated moment center.



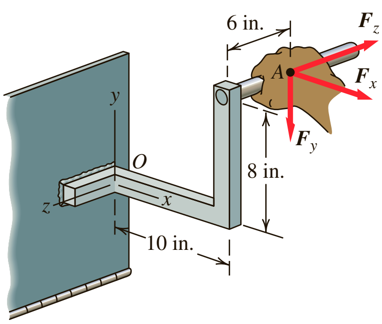
EX 3.2.14

3.2.15. [*] Consider F_3 and a moment center at A and write (a) symbolic expressions for the force and position vector, then (b) an expression for the moment created by the force at the indicated moment center.



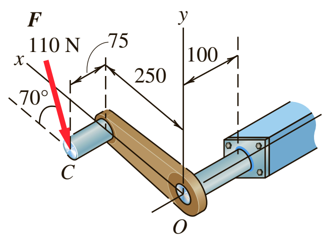
EX 3.2.15

3.2.16. [*] Consider F_x F_y, and F_z, and moment center O and write (a) symbolic expressions for the force and position vector, then (b) an expression for the moment created by the force at the indicated moment center.



EX 3.2.16

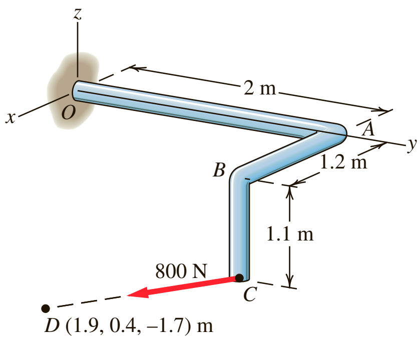
3.2.17 [*] A crank is used to adjust an ergonomic desk. The applied load $\boldsymbol{F}$ is parallel to the xy plane. For a moment center at O, write (a) symbolic expressions for the force and position vector, then (b) a numeric expression for the moment created by the force at the indicated moment center.



Dimensions in millimeters

EX 3.2.17

3.2.18. [*] Use the cross product to determine the moment at O created by the 800-N force acting at C.

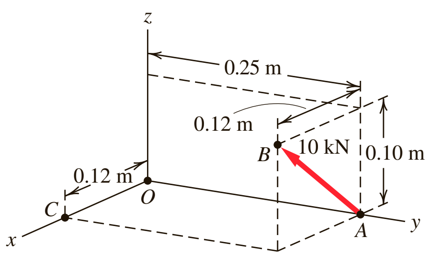


EX 3.2.18

3.2.19. [*] A 10-kN force acts at point A.

a. Use the cross product to determine the moment the 10-kN force creates at a moment center at C.

b. Find the space angles associated with the moment vector in **a**.

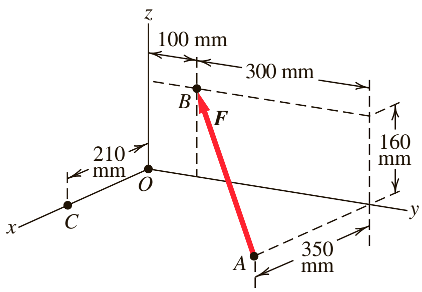


EX 3.2.19

3.2.20. [*] The magnitude of the force ***F*** is 20 N.

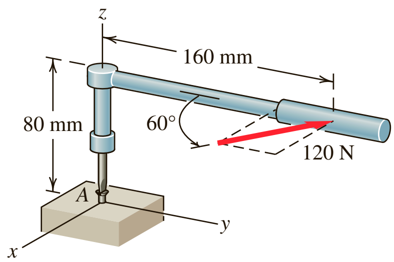
a. Use the cross product to determine the moment the 20-N force creates at a moment center at *C*.

b. What are the space angles associated with the moment?



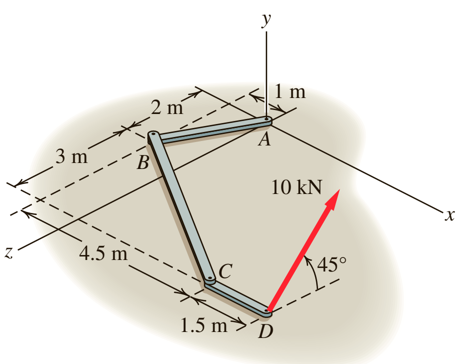
EX 3.2.20

3.2.21. [*] A socket wrench is used to loosen a screw. A 120-N horizontal force is applied to the handle as shown. Determine the moment this force creates at a moment center at *A* (the center of the head of the screw). Which component of the moment is acting to loosen the screw?



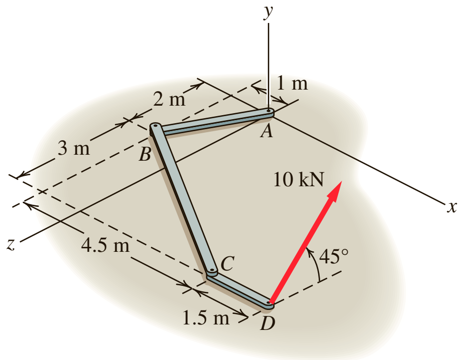
EX 3.2.21

3.2.22. [*] A 10-kN force acts on the linkage at point *D* parallel to the *yz* plane. Use the cross product to compute the moment this force creates at a moment center at *A*.



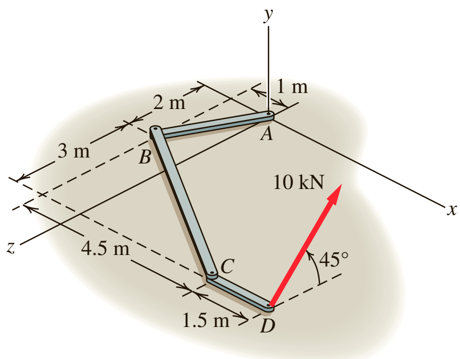
EX 3.2.22

3.2.23. [*] A 10-kN force acts on the linkage at point *D* parallel to the *yz* plane. Use the cross product to compute the moment this force creates at a moment center at *B*.



EX 3.2.23

3.2.24. [*] A 10-kN force acts on the linkage at point *D* parallel to the *yz* plane. Use the cross product to compute the moment this force creates at a moment center at *C*.

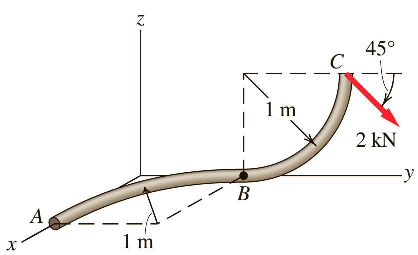


EX 3.2.24

3.2.25. [*] A 2-kN force acts on one end of the curved rod shown. Section *AB* of the rod lies in the *xy* plane, and section *BC* lies in the *zy* plane. Use the cross product to determine the moment created by the force

a. at a moment center at *A*

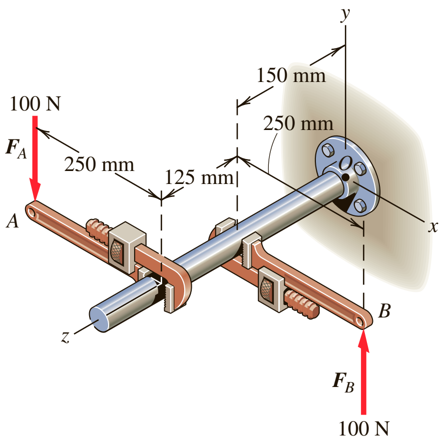
b. at a moment center at the midpoint *B* of the rod



EX 3.2.25

3.2.26. [*] Two 100-N vertical forces act on the pipe wrenches, as shown. Use the cross product to determine

a. the moment created by $\boldsymbol{F}_A$ at O
b. the moment created by $\boldsymbol{F}_B$ at O
c. the sum of the moments found in **a** and **b**



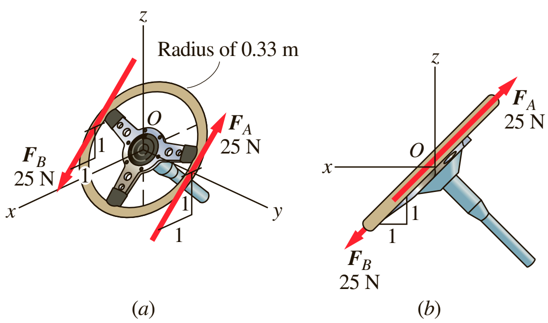
EX 3.2.26

3.2.27. [*] The 25-N forces shown in **Figure *a*** are applied to a bus's steering wheel. A side view of the wheel is shown in **Figure *b***. Find

a. the moment created by force $\boldsymbol{F}_A$ at a moment center at O

b. the moment created by force $\boldsymbol{F}_B$ at a moment center at O

c. the sum of the moments found in **a** and **b**



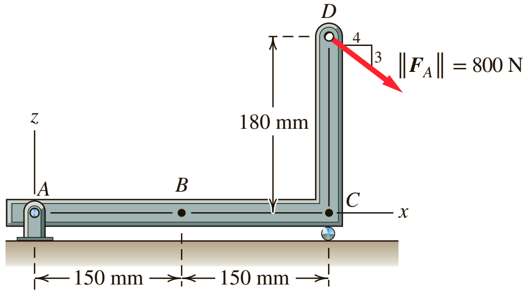
EX 3.2.27

3.2.28. [**] A force with a magnitude of 800 N acts as shown.

a. Speculate on where the force creates the largest moment (in terms of magnitude): at a moment center at A, at B, at C, or at D.

Calculate the moment the force creates:

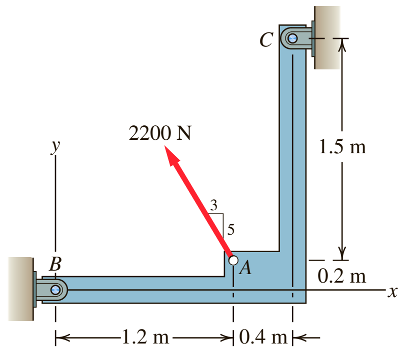
b. at a moment center at A.
c. at a moment center at B.
d. at a moment center at C.
e. at a moment center at D.
f. Compare your answers in **b**–**e** with your speculation in **a**.



EX 3.2.28

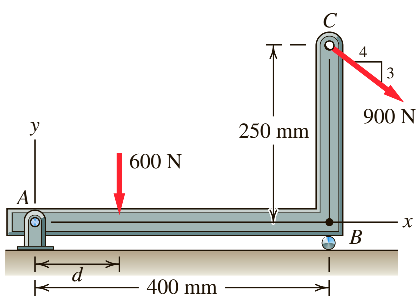
3.2.29. [**] A force with a magnitude of 2200 N acts on a bracket as shown. Determine the moment the force creates at a moment center at

a. point B
b. point C



EX 3.2.29

3.2.30. [**] A 900-N force acts at C on the frame. A 600-N force acts at a distance d from point A. Determine d so that the moment at A due to the two forces is 550 N·m.



EX 3.2.30

3.2.31 [**] A force $\boldsymbol{F}_1$ acts permanently at point C on the cantilever beam shown. A second force, $\boldsymbol{F}_2$, moves along

the beam from A to B ($x = 0$ to $x = L$). $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are of equal magnitude ($2P$).

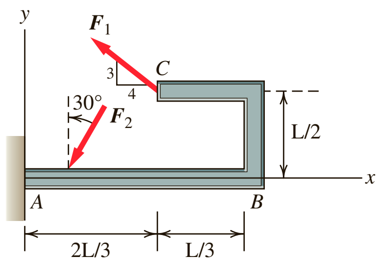
a. Write an equation in terms of P, L, and x for the moment at A due to the two forces as a function of x, the distance from A to the point of application of $\boldsymbol{F}_2$.

Draw a graph of the moment at A as a function of x to answer the following questions.

b. Which of the following represents the correct shape of the graph of M_A as a function of x? Define a counterclockwise moment about the z axis to be positive.

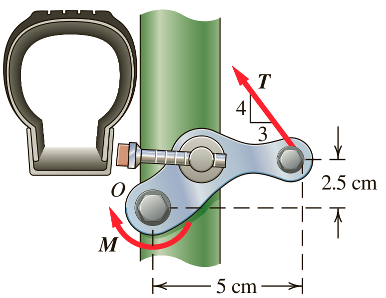
c. What is the magnitude of the maximum clockwise moment at A? State your answer in terms of PL.

d. What is the magnitude of the maximum counterclockwise moment at A? State your answer in terms of PL.



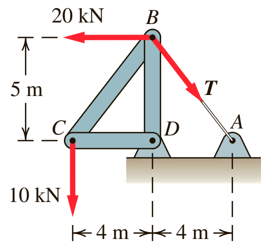
EX 3.2.31

3.2.32. [**] The arm of a cantilever bicycle brake pivots about O. A torsional spring exerts a restoring moment of 1.65 N-m on the brake when a force $\boldsymbol{T}$ is applied to the brake cable. Determine the force $\boldsymbol{T}$ so that the moment about O due to the torsional spring and the brake cable is zero.



EX 3.2.32

3.2.33. [**] For the truss shown, calculate the magnitude of the cable force $\boldsymbol{T}$ so that the moment about D due to $\boldsymbol{T}$, the 10-kN force, and the 20-kN force is zero.



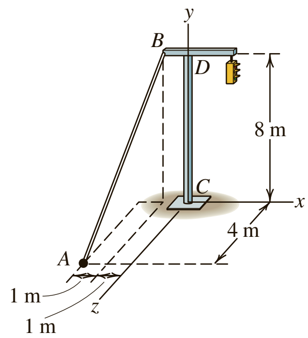
EX 3.2.33

3.2.34. [**] A traffic light is steadied by cable AB. If the tension in this cable is 4 kN, determine

a. the moment $\boldsymbol{M}$ the cable tension creates at C, expressed in vector notation

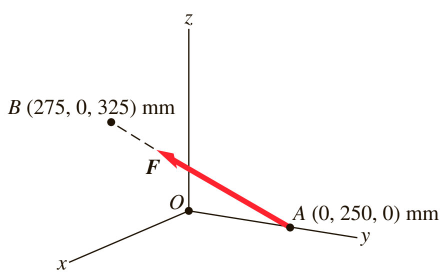
b. the moment $\boldsymbol{M}$ the cable tension creates at C, expressed in terms of a magnitude and a unit vector $\boldsymbol{u}$

c. the scalar component of the moment about the long axis of the pole, CD



EX 3.2.34

3.2.35. [**] The point of application of a force of magnitude 1,720 N is at A. Determine the moment $\boldsymbol{M}_O$ the force creates at O. Express your answer in vector notation. In addition, write a unit vector that defines the axis of $\boldsymbol{M}_O$.



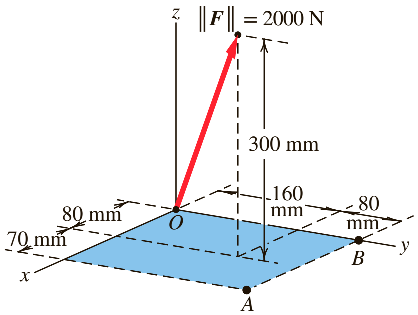
EX 3.2.35

3.2.36. [**] Consider the situation shown, and determine

a. the moment $\boldsymbol{M}_B$ the force creates at B, expressed in vector notation

b. the magnitude of $\boldsymbol{M}_B$ and a unit vector $\boldsymbol{u}$ that describes its direction

c. the space angles associated with the moment vector



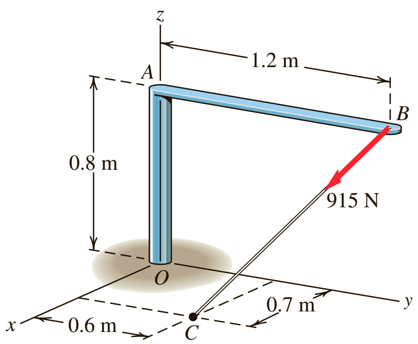
EX 3.2.36

3.2.37. [**] A 915 N force acts on a lever attached to a post as shown. Determine

a. the moment $\boldsymbol{M}_O$ the force creates at a moment center at O

b. the space angles associated with the moment vector

c. the unit vector that defines the axis of $\boldsymbol{M}_O$

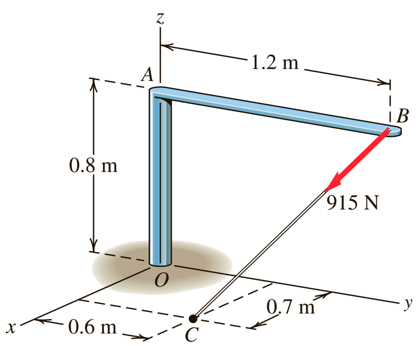


EX 3.2.37

3.2.38. [**] A 915 N force acts on a lever attached to a post as shown. Determine

a. the moment $\boldsymbol{M}_A$ the force creates at a moment center at A.

b. the moment $\boldsymbol{M}_O$ the force creates at a moment center O. Which moment ($\boldsymbol{M}_O$ or $\boldsymbol{M}_A$) has the greater magnitude?



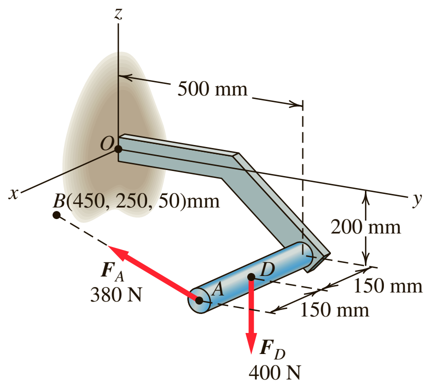
EX 3.2.38

3.2.39. [**] Consider the bracket shown. Determine at a moment center O

a. the moment $\boldsymbol{M}_D$ created by $\boldsymbol{F}_D$

b. the moment $\boldsymbol{M}_A$ created by $\boldsymbol{F}_A$

c. the sum of $\boldsymbol{M}_D$ and $\boldsymbol{M}_A$



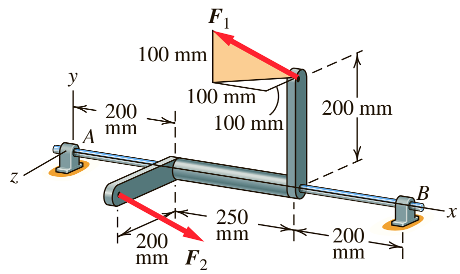
EX 3.2.39

3.2.40. [**] On the winding mechanism shown, $\boldsymbol{F}_2$ is parallel to and opposite $\boldsymbol{F}_1$. $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are each of magnitude 750 N. Determine at a moment center at A:

a. the moment $\boldsymbol{M}_1$ created by $\boldsymbol{F}_1$

b. the moment $\boldsymbol{M}_2$ created by $\boldsymbol{F}_2$

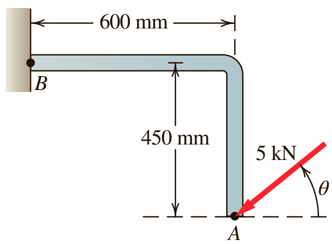
c. the sum of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$



EX 3.2.40

3.2.41. [**] A 5-kN force acts on the bent arm at point A.

a. Use the cross product to write an expression for the moment $\boldsymbol{M}$ that the force creates at a moment center at B. This expression will contain the angle θ.



E3.2.41

b. Determine the angle θ such that the force at A exerts the maximum moment at B.

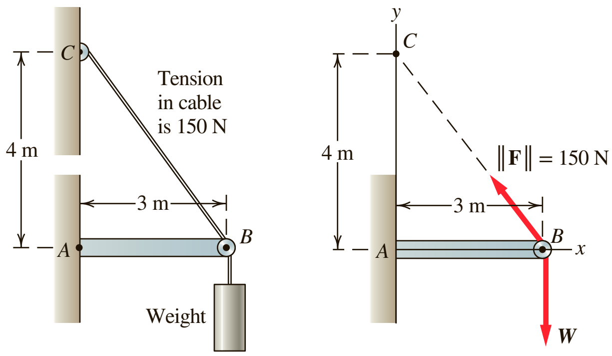
c. What is the magnitude of the corresponding moment? Also write the corresponding moment in vector notation.

3.2.42. [**] A cable and weight act on a beam as shown in **Figure *a***. They can be represented by the forces $\boldsymbol{F}$ and $\boldsymbol{W}$ in **Figure *b***.

a. Determine the moment $\boldsymbol{M}_1$ that $\boldsymbol{W}$ creates at A. Your answer will be in terms of $\|\boldsymbol{W}\|$.

b. Determine the moment $\boldsymbol{M}_2$ that $\boldsymbol{F}$ creates at A.

c. If the magnitude of $\boldsymbol{M}_1 + \boldsymbol{M}_2$ is limited to be less than or equal to 1 kN · m, what is the maximum allowable value of $\boldsymbol{W}$?



(*a*) (*b*)

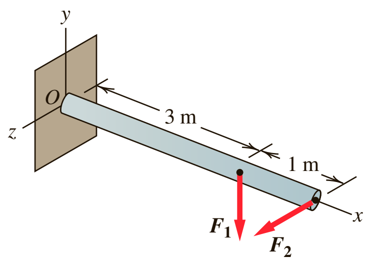
EX 3.2.42

3.2.43. [**] The forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ act on a cantilever beam.

a. Determine the moment $\boldsymbol{M}_1$ that $\boldsymbol{F}_1$ creates at O.

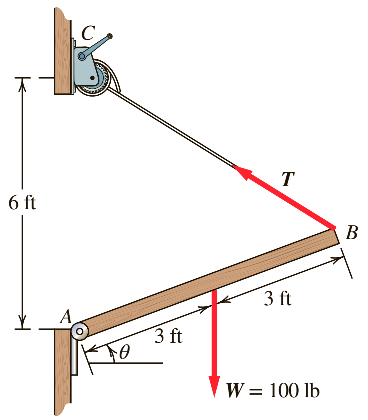
b. Determine the moment $\boldsymbol{M}_2$ that $\boldsymbol{F}_2$ creates at O.

c. If the magnitude of $\boldsymbol{M}_1 + \boldsymbol{M}_2$ is limited to be less than or equal to 100 kN·m, what is the maximum allowable magnitude of $\boldsymbol{F}_2$ given that the magnitude of $\boldsymbol{F}_1$ is fixed at 20 kN?



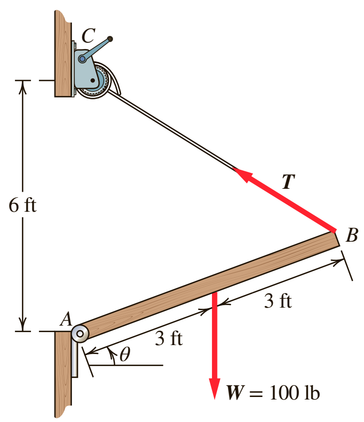
EX 3.2.43

3.2.44. [**] The door of a tree house is opened and closed using a hand operated winch to unwind and wind a cable that runs from B to C. The cable pulls on the door at B with a force $\boldsymbol{T}$. When $\theta = 20°$, find the magnitude of $\boldsymbol{T}$ so that the moment at A due to $\boldsymbol{T}$ and $\boldsymbol{W}$ is zero.



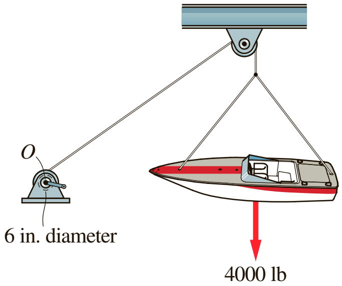
EX 3.2.44

3.2.45. [***] Consider the tree house door attached to a cable at B, that winds around a winch at C. The cable force at B is $\|\boldsymbol{T}\| = 50$ lb. Find the angle θ so that the moment at A due to $\boldsymbol{T}$ and $\boldsymbol{W}$ is zero. (Hint: you may want to use MATLAB, the Newton-Raphson method, or numerical techniques to help solve your equation.)



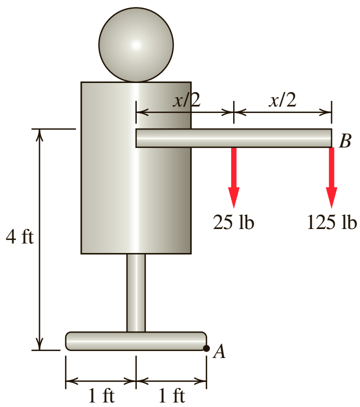
EX 3.2.45

3.2.46. [***] A system for lifting a 4000-lb sport boat out of water consisting of a pulley, winch, and 1/2 inch diameter cable is shown. The winch drum rotates about point O and has a diameter of 6 inches. With every 10 turns of the crank the cable winds onto a new layer. The winch is designed to hold a maximum of five layers of cable. Assuming the force is transferred through the center of the cable, write an expression for the moment about O as a function of the number of layers of cable on the winch.



EX 3.2.46

3.2.47. **[***]** Engineering students have entered a competition to design a robot that can lift an object equal to its own weight and extend it as far away from the robot as possible. The prototype robot has an extendable arm that weighs 25 lb and picks up a 125-lb load at B. If the robot will start to overturn when the moment at A is 100 ft · lb, how far out (x) can the students extend the arm?



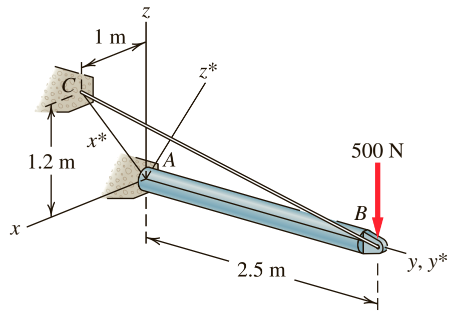
EX 3.2.47

3.2.48. **[***]** A 500-N force acts at the end of a cable-supported bar as shown.

a. Use the cross product to determine the moment the 500-N force creates at a moment center at A ($\boldsymbol{M}_A$). Write the moment in vector notation based on the xyz coordinate axes.

b. Using the $x^*y^*z^*$ coordinate axes and the cross product, again determine $\boldsymbol{M}_A$. Write $\boldsymbol{M}_A$ in vector notation.

c. Confirm that the moments found in **a** and **b** have the same magnitude.



EX 3.2.48

3.3 FINDING MOMENT COMPONENTS IN A PARTICULAR DIRECTION

Learning Objective: Calculate moment components in a particular direction.

The previous section presented relationships for calculating the moment created by a force offset from a moment center (MC). For example, we could use either (3.8) or (3.12) to find the moment $\boldsymbol{M}$ about a MC at A created by $\boldsymbol{F}_{\text{cable}}$ in **Figure 3.3.1*a***.

Now let's say that we are interested in finding the component of $\boldsymbol{M}$ in the direction of the axis AB of the post in **Figure 3.3.1*a***, which

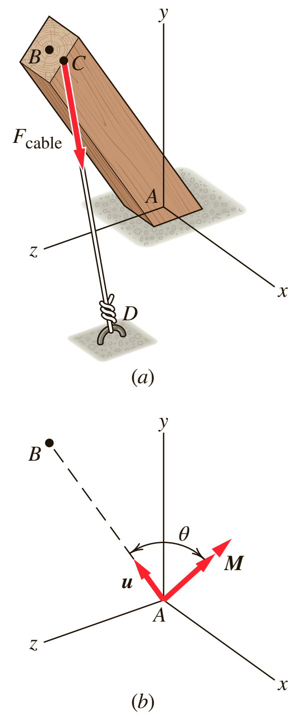


Figure 3.3.1 (*a*) Cable force creates a moment at *A*; (*b*) We use a dot product to find the component of that moment aligned with the pole.

is the component of $\boldsymbol{M}$ causing twisting around axis AB. We might be interested in this because if the component becomes too large, the post may be twisted out of its concrete mount. The component of $\boldsymbol{M}$ in the direction of AB is simply the dot product of $\boldsymbol{u}$ and $\boldsymbol{M}$, where $\boldsymbol{u}$ is a unit vector along AB (**Figure 3.3.1*b***). Calling the component of interest M_{axis}, we can write it as

$$M_{\text{axis}} = \underbrace{\boldsymbol{u} \bullet \boldsymbol{M}}_{\substack{\text{projection of} \\ \boldsymbol{M} \text{ in the direction} \\ \text{of unit vector } \boldsymbol{u}}} = \|\boldsymbol{u}\|\,\|\boldsymbol{M}\| \cos\theta \tag{3.15}$$

where θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{M}$, as indicated in **Figure 3.3.1*b***. The dot product, enables us to find the component or projection of a vector in a particular direction.

If we write $\boldsymbol{u}$ and $\boldsymbol{M}$ in terms of their scalar components, the dot product in (3.15) can be restated as

$$M_{\text{axis}} = \boldsymbol{u} \bullet \boldsymbol{M} = u_x M_x + u_y M_y + u_z M_z \tag{3.16}$$

If we revisit the definition of the moment as a cross product (3.8), the calculation of M_{axis} can be written as

$$M_{\text{axis}} = \boldsymbol{u} \bullet \boldsymbol{M} = \boldsymbol{u} \bullet (\boldsymbol{r} \times \boldsymbol{F}) \tag{3.17}$$

Substituting the determinant formulation of the moment (3.12), we can restate (3.15) as

$$M_{\text{axis}} = \boldsymbol{u} \bullet (\boldsymbol{r} \times \boldsymbol{F}) = \begin{vmatrix} u_x & u_y & u_z \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \tag{3.18}$$

The choice of using (3.15), (3.16) or (3.18) to find the component of $\boldsymbol{M}$ in the direction of $\boldsymbol{u}$ will generally be a matter of convenience and/or personal preference.

Finally, the angle θ between $\boldsymbol{u}$ and $\boldsymbol{M}$, based on the development of the dot product, is given as

$$\cos\theta = \frac{u_x M_x + u_y M_y + u_z M_z}{\|\boldsymbol{u}\|\;\|\boldsymbol{M}\|}, \qquad \text{with } 0^\circ \le \theta \le 180^\circ \tag{3.19}$$

An important concept to understand as you use the dot product to find the component of the moment in a particular direction, is that the resulting quantity is the moment about an axis. The moment about an axis physically represents the component of the moment that would tend to make a rigid body rotate about that axis. Let's look at a familiar moment, $\boldsymbol{M} = M_x\boldsymbol{i} + M_x\boldsymbol{j} + M_x\boldsymbol{k}$; in this case M_x is the magnitude

of the moment about the x axis. You can prove this to yourself by calculating

$$M_{x\text{ axis}} = \boldsymbol{i} \bullet \boldsymbol{M} = M_x$$

noting that $\boldsymbol{i} \bullet \boldsymbol{i} = 1$, $\boldsymbol{i} \bullet \boldsymbol{j} = 0$, and $\boldsymbol{i} \bullet \boldsymbol{k} = 0$. If M_{axis} is zero for a specified direction, then the moment is not causing any rotation about that axis.

Check out the following examples of applications of this material.

- **Example 3.3.1 Finding the Moment About the *z* Axis**
- **Example 3.3.2 Finding the Moment in a Particular Direction**

EXAMPLE 3.3.1

A person is opening the lid of the plywood box in Figure 1 with a force $\boldsymbol{F} = -8\text{N}\,\boldsymbol{i} + 25\text{N}\,\boldsymbol{j} + 4\text{N}\,\boldsymbol{k}$. Find the moment of $\boldsymbol{F}$ about the z axis.

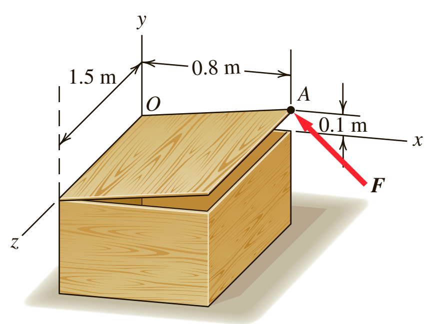


Figure 1 A force applied at A opens the lid of a plywood box.

Goal Find the moment about the z axis due to a force applied to the lid of the box at point A.

Given A coordinate system, $\boldsymbol{F}$ in vector notation, and the dimensions of the box

Assume No assumptions are necessary.

Draw As discussed below, we draw a position vector from O to A.

Formulate Equations and Solve First we need to find the moment $\boldsymbol{M}$ at a moment center anywhere along the z axis. We choose point O because $\boldsymbol{r}_{OA}$ is easy to determine from the information in Figure 2. The moment at O is then

$$\boldsymbol{M}_O = (\boldsymbol{r}_{OA} \times \boldsymbol{F}) \tag{1}$$

where

$$\boldsymbol{r}_{OA} = 0.8\text{ m}\,\boldsymbol{i} + 0.1\text{ m}\,\boldsymbol{j}$$

The unit vector in the direction of the z axis is $\boldsymbol{k}$, so the dot product $(\boldsymbol{k} \bullet \boldsymbol{M}_O)$ will give us the component of $\boldsymbol{M}_O$ in the direction of the z axis (M_z). Substituting from (1) for $\boldsymbol{M}_O$ gives $M_z = \boldsymbol{k} \bullet (\boldsymbol{r}_{OA} \times \boldsymbol{F})$. Using (3.18) we get

$$M_z = \boldsymbol{k} \bullet (\boldsymbol{r}_{OA} \times \boldsymbol{F}) = \begin{vmatrix} u_x & u_y & u_z \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 0.8\text{m} & 0.1\text{m} & 0\text{m} \\ -8\text{N} & 25\text{N} & 4\text{N} \end{vmatrix}$$

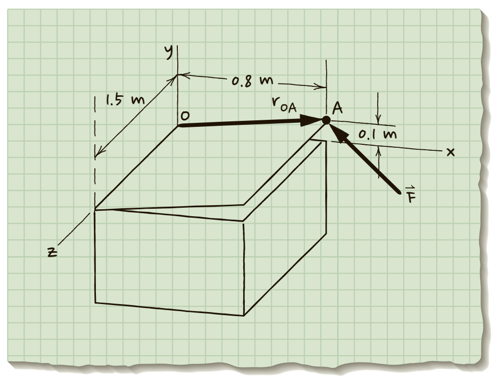


Figure 2 We can draw a position vector from any point on the z axis to point A; we select $\boldsymbol{r}$ from O to A.

This calculation results in

$$M_z = 20.8\,\text{N·m} \qquad \text{counterclockwise about } z \text{ axis}$$

Check One check would be to calculate the moment due to each scalar component of $\boldsymbol{F}$ that is perpendicular to the z axis and to sum up the resulting moments. Both F_x and F_y are perpendicular to the z axis. Using F_x and F_y, we calculate two moments about the z axis: $M_{z1} = (8\text{ N})(0.1\text{ m}) = 0.8\text{ N·m}$ and $M_{z2} = (25\text{ N})(0.8\text{ m}) = 20.0\text{ N·m}$. When we sum these up, we get 20.8 N·m, the same result as before.

Comment: The only component of $\boldsymbol{M}_O$ that is acting to open the lid is M_z, which is the moment component parallel to the z axis. M_x tends to "peel" the lid off the box from end OA, and M_y tends to rotate the lid about the y axis.

EXAMPLE 3.3.2

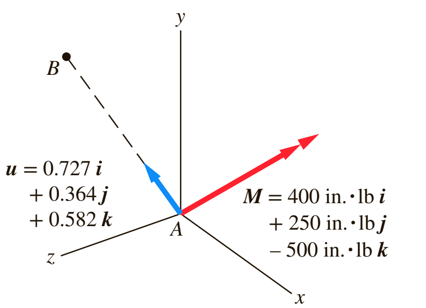


Figure 1 A moment is applied at point A.

Consider the moment $\boldsymbol{M} = 400\text{ in.·lb }\boldsymbol{i} + 250\text{ in.·lb }\boldsymbol{j} - 500\text{ in.·lb }\boldsymbol{k}$, as shown in **Figure 1**.

(a) Find the component of $\boldsymbol{M}$ in the direction defined by the unit vector $\boldsymbol{u} = 0.727\,\boldsymbol{i} + 0.364\,\boldsymbol{j} + 0.582\,\boldsymbol{k}$.

(b) Write the component vector of $\boldsymbol{M}$ in the direction of $\boldsymbol{u}$.

Goal Find the component of $\boldsymbol{M}$ in the direction specified by $\boldsymbol{u}$, which is to say, find the projection of $\boldsymbol{M}$ in the direction of $\boldsymbol{u}$.

Given $\boldsymbol{u}$ and $\boldsymbol{M}$ in vector notation

Assume No assumptions are necessary.

Draw We redraw **Figure 1** to show the angle between $\boldsymbol{u}$ and $\boldsymbol{M}$, and the components of these two vectors (**Figure 2**).

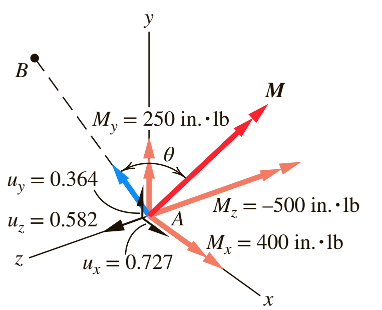


Figure 2 We indicate the components of $\boldsymbol{M}$ and $\boldsymbol{u}$ on our drawing.

Formulate Equations and Solve **(a)** According to (3.15) the dot product $(\boldsymbol{u} \bullet \boldsymbol{M})$ will give us the component of $\boldsymbol{M}$ in the direction of $\boldsymbol{u}$, multiplied by $\|\boldsymbol{u}\|$ (which is simply 1, since $\boldsymbol{u}$ is a unit vector).

Since we are given $\boldsymbol{u}$ and $\boldsymbol{M}$ in terms of their components, the most straight forward approach is to find the dot product using (2.30):

$$\boldsymbol{V}_1 \bullet \boldsymbol{V}_2 = V_{1x}V_{2x} + V_{1y}V_{2y} + V_{1z}V_{2z}$$

Substituting the components of $\boldsymbol{u}$ and the components of $\boldsymbol{M}$ into (3.14):

$$M_u = \boldsymbol{u} \bullet \boldsymbol{M} = 0.727(400\text{ in.·lb}) + 0.364(250\text{ in.·lb}) + 0.582(-500\text{ in.·lb})$$

$$M_u = 90.8\,\text{in.·lb}$$

(b) The component vector of $\boldsymbol{M}$ in the direction of $\boldsymbol{u}$ is

$$\boldsymbol{M}_u = M_u\boldsymbol{u} = 90.8 \text{ in.}\cdot\text{lb} \,(0.727\,\boldsymbol{i} + 0.364\,\boldsymbol{j} + 0.582\,\boldsymbol{k})$$

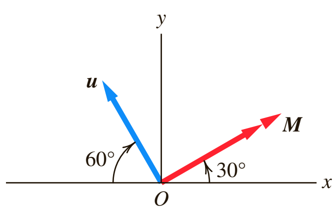
$$\boldsymbol{M}_u = 66.0\,\text{in.}\cdot\text{lb}\,\boldsymbol{i} + 33.1\,\text{in.}\cdot\text{lb}\,\boldsymbol{j} + 52.8\,\text{in.}\cdot\text{lb}\,\boldsymbol{k}$$

Check An inspection of **Figure 2** shows that one would expect the projection of $\boldsymbol{M}$ onto $\boldsymbol{u}$ to be approximately 1/6 of the size of $\boldsymbol{M}$; indeed, 90.8 is 0.132 of 687 (the magnitude of $\boldsymbol{M}$). Also, we could recalculate the moment using cos θ to find the dot product using (2.30).

Comment: Would it have made a difference in our answer if we had taken $\boldsymbol{M} \bullet \boldsymbol{u}$ instead of $\boldsymbol{u} \bullet \boldsymbol{M}$?

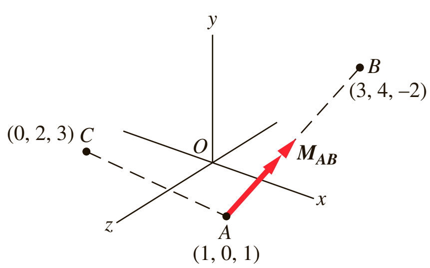
EXERCISES 3.3

3.3.1. [*] For the moment $\boldsymbol{M}$ shown, find the component in the direction of $\boldsymbol{u}$. What does your answer tell you about $\boldsymbol{M}$ and $\boldsymbol{u}$?



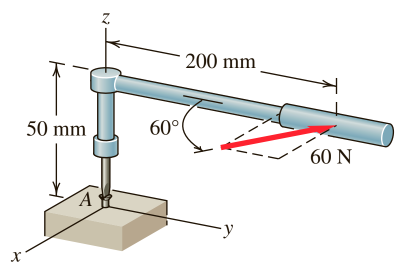
EX 3.3.1

3.3.2. [*] The moment $\boldsymbol{M}_{AB}$ has a magnitude of 50 kN·m. Find the magnitude of the moment about line AC. What does your answer tell you about $\boldsymbol{M}_{AB}$ and line AC?



EX 3.3.2

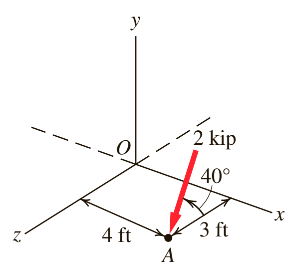
3.3.3. [*] A 60-N horizontal force is applied to the handle. Determine the moment about the z-axis, $\boldsymbol{M}_z$. What effect does $\boldsymbol{M}_z$ have on the screw?



EX 3.3.3

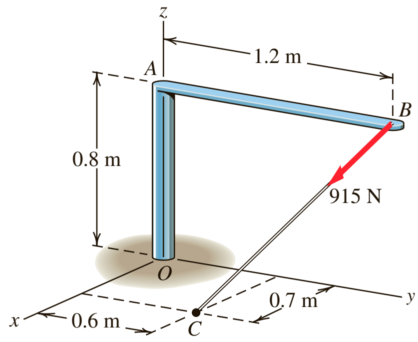
3.3.4. [*] The 2 kip force is parallel to the yz plane. It is applied to point A at 40° relative to the xz plane. Determine

a. the moment about the x axis.
b. the moment about the y axis.



EX 3.3.4

3.3.5. [*] Determine the moment about a moment center at O created by the 915-N force acting on frame OAB. Then determine the scalar component of this moment about the z axis.



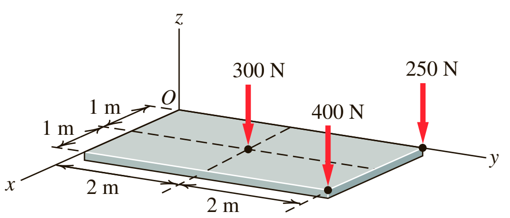
EX 3.3.5

3.3.6. [*] Three parallel forces act on the plate shown.

a. Determine the moment about the x axis.

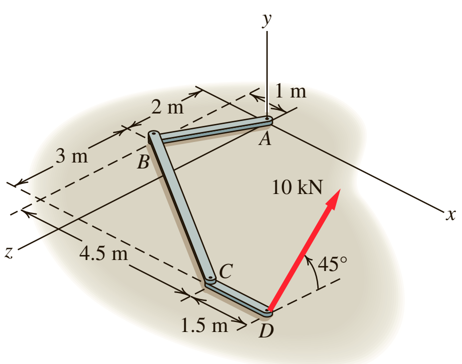
b. Determine the moment about the y axis.

c. Why is the moment about the z axis zero?



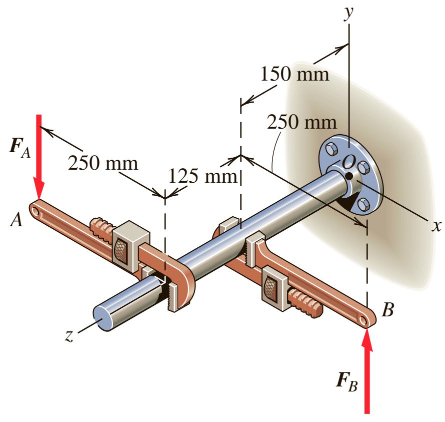
EX 3.3.6

3.3.7. [*] A 10-kN force acts at point D parallel to the yz plane on the linkage shown. Find the scalar component aligned with member BC of the moment created at moment center B.



EX 3.3.7

3.3.8. [*] The forces $\boldsymbol{F}_A$ and $\boldsymbol{F}_B$ applied to the pipe wrenches are of equal magnitude. If the moment about the axis of the pipe must be at least 60 N · m in order to loosen the pipe from the threaded hole, what is the minimum required magnitude of $\boldsymbol{F}_A$ and $\boldsymbol{F}_B$? Use the 60 N · m as a torque specification for tightening the fitting.

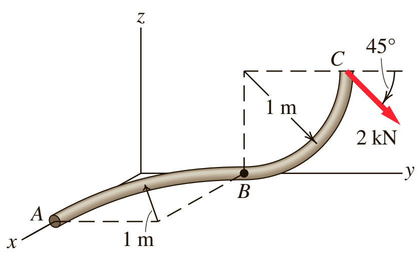


EX 3.3.8

3.3.9. [*] Segment AB of the curved rod is in the xy plane, and segment BC is in the yz plane. Determine the magnitude of the moment created by the 2-kN force

a. about an axis defined by AB. (One possible strategy is to find the moment at A and then find the dot product of the moment components in the direction defined by AB.)

b. about an axis defined by BC. Why is this moment zero?



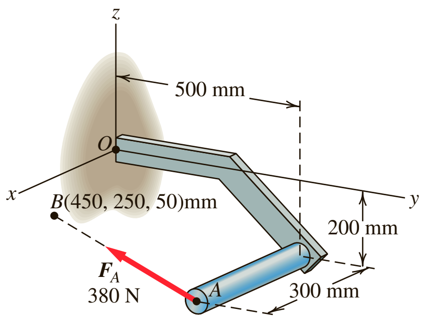
EX 3.3.9

3.3.10. [**] Consider the 380-lb force applied to the bracket shown. For the moment $\boldsymbol{M}_A$ created at O by $\boldsymbol{F}_A$, determine the scalar component of this moment about the

a. x axis

b. y axis

c. z axis



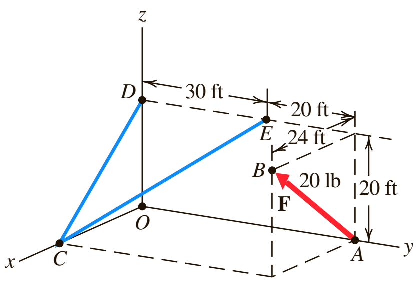
EX 3.3.10

3.3.11. [**] Consider the 20-lb force applied at A.

a. Find the moment created at a moment center at C by the 20-lb force. Call this moment $\boldsymbol{M}_C$.

b. Determine the scalar component of $\boldsymbol{M}_C$ that is in the direction of CE.

c. Determine the scalar component of $\boldsymbol{M}_C$ that is in the direction of CD.



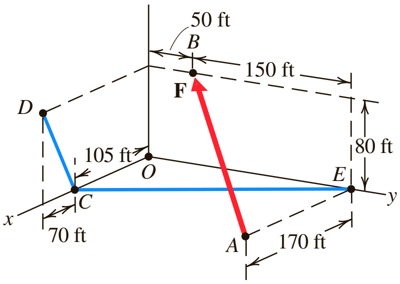
EX 3.3.11

3.3.12. [**] Consider the 2-kip force $\boldsymbol{F}$ applied at A.

a. Find the moment created at C by the force $\boldsymbol{F}$. Call this moment $\boldsymbol{M}_C$.

b. Determine the scalar component of $\boldsymbol{M}_C$ that is in the direction of CE.

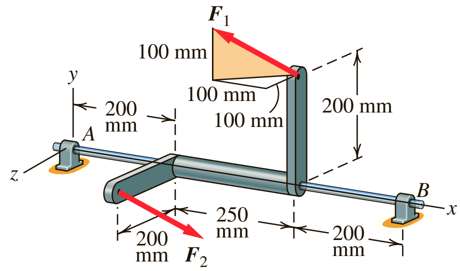
c. Determine the scalar component of $\boldsymbol{M}_C$ that is in the direction of CD.



EX 3.3.12

3.3.13. [**] Determine the sum $\boldsymbol{M}_1 + \boldsymbol{M}_2$ for the moments created at A by the two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ acting on the structure shown. Call this sum $\boldsymbol{M}_3$. $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are both of magnitude 750 N.

What is the scalar component of $\boldsymbol{M}_3$ about the axis of the shaft AB?



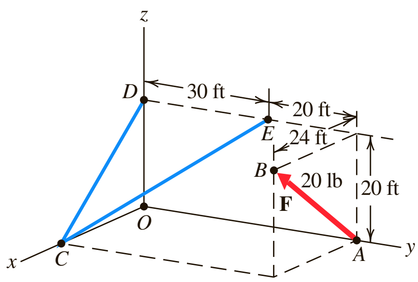
EX 3.3.13

3.3.14. [***] Consider the 20-lb force acting at point A.

a. Find the moment created at a moment center at C by the 20-lb force. Call this moment $\boldsymbol{M}_C$.

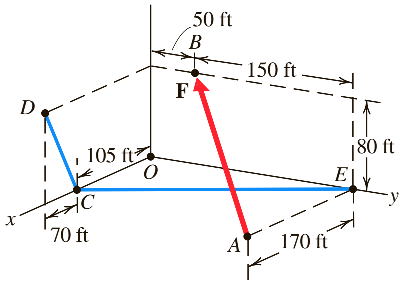
b. For the moment $\boldsymbol{M}_C$, find the scalar component that is parallel to line CD.

c. Find the component of $\boldsymbol{M}_C$ that is perpendicular to line CD.



EX 3.3.14

3.3.15. [***] The magnitude of the force $\boldsymbol{F}$ is 10 lb.



EX 3.3.15

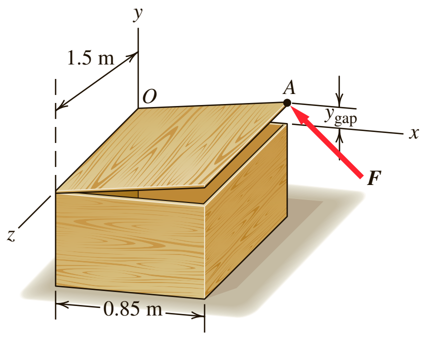
a. Find the moment created at a moment center at C by the 10-lb force. Call this moment $\boldsymbol{M}_C$.

b. For the moment $\boldsymbol{M}_C$, find the scalar component that is parallel to line CE.

c. Write the moment about the CE axis in vector form.

d. Find the component of the moment that is perpendicular to line CE.

3.3.16. [***] Consider the plywood box being opened with a force $\boldsymbol{F} = -8\text{ N}\boldsymbol{i} + 25\text{ N}\boldsymbol{j} + 4\text{ N}\boldsymbol{k}$ as shown. The magnitude of M_z varies with the height that the lid is lifted (y_{gap}). For what value of y_{gap} is M_z a maximum? *Hint*: This can be solved using calculus or by doing a computer analysis in Excel, MATLAB, or other analysis program.



EX 3.3.16

3.4 WHEN ARE TWO FORCES EQUAL TO A MOMENT? (WHEN THEY ARE A COUPLE)

Learning Objective: Calculate a couple moment.

A star wrench (**Figure 3.4.1*a***) allows the user to simultaneously exert a downward force with one hand and an upward force with the other to tighten a bolt. Assume that $\boldsymbol{F}_{\text{LH}}$ and $\boldsymbol{F}_{\text{RH}}$ are equal in magnitude and opposite in direction. In such a case, the sum of the two forces acting on the wrench is zero. Because the resultant force is zero, the wrench does not tend to cause any *translation* of the nut. On the other hand, the pair of forces does make the nut want to *rotate*. Why is that?

The answer is: the two forces on the wrench are applying a moment to the nut. This moment is found by calculating the moment created by each force relative to a moment center. Let's select the moment center at O, then

$$\boldsymbol{M}_O = \boldsymbol{r}_{OA} \times \boldsymbol{F}_{\text{LH}} + \boldsymbol{r}_{OB} \times \boldsymbol{F}_{\text{RH}}$$

From the geometry of **Figure 3.4.1*a*** we see that the position vectors are perpendicular to the applied forces, and that the forces are of equal magnitude, $\boldsymbol{M}_O$ becomes

$$\boldsymbol{M}_O = -D\boldsymbol{i} \times \|\boldsymbol{F}\|\,\boldsymbol{j} + D\boldsymbol{i} \times -\|\boldsymbol{F}\|\,\boldsymbol{j}$$

$$\boldsymbol{M}_O = -(D\|\boldsymbol{F}\|)\boldsymbol{k} - (D\|\boldsymbol{F}\|)\boldsymbol{k} = -2D\|\boldsymbol{F}\|\boldsymbol{k} \tag{3.20}$$

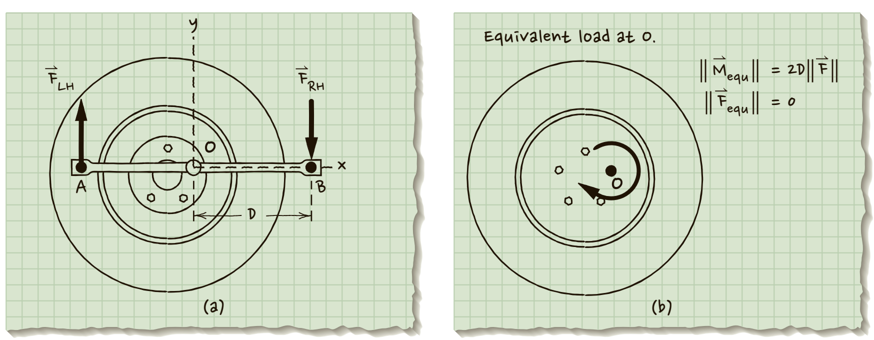


Figure 3.4.1 (*a*) A couple; (*b*) the couple moment at moment center O.

This moment, which is clockwise about the z axis, is shown in **Figure 3.4.1*b***.

Two parallel forces that are equal in magnitude and opposite in sign, such as those in **Figure 3.4.1*a***, are referred to as a **couple**. The moment they create, which we calculated in (3.20), is called a **couple moment** (or simply a moment). The direction of the couple moment is perpendicular to the plane containing the forces that make up the couple.

To explore the characteristics of a couple, let's revisit **Figure 3.4.1*a*** but this time calculate the moment at a moment center at A:

$$\boldsymbol{M}_A = \boldsymbol{r}_{AA} \times \boldsymbol{F}_{\text{LH}} + \boldsymbol{r}_{AB} \times \boldsymbol{F}_{\text{RH}}$$

Substituting the values for the position vectors ($\boldsymbol{r}_{AA} = 0\ \boldsymbol{i}$, and $\boldsymbol{r}_{AB} = 2D\ \boldsymbol{i}$) results in

$$\boldsymbol{M}_A = (0\|\boldsymbol{F}\|)\ \boldsymbol{k} + (2D)(-\|\boldsymbol{F}\|)\ \boldsymbol{k} = -2D\|\boldsymbol{F}\|\boldsymbol{k}$$

This is the same moment we calculated in (3.20). This illustrates a notable property of a couple: the associated couple moment is *independent of the moment center* used. No matter where we place the moment center, the couple moment we calculate will have the same magnitude and direction.

What are other implications of the couple being independent of the moment center? In **Figure 3.4.2** we have replaced the star wrench with a socket wrench that has a handle of length 2D. By applying equal and opposite forces to the wrench as shown, we have moved the couple in **Figure 3.4.1** to a new location. Once again let's calculate the moment, using O as the moment center:

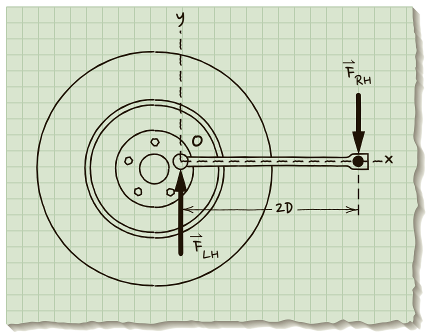


Figure 3.4.2 Move the couple to a new location by applying equal and opposite forces to a wrench with a handle of length $2D$.

$$\boldsymbol{M}_O = 0\ \boldsymbol{i} \times \|\boldsymbol{F}\|\ \boldsymbol{j} + 2D\,\boldsymbol{i} \times -\|\boldsymbol{F}\|\ \boldsymbol{j}$$

$$\boldsymbol{M}_O = (0\|\boldsymbol{F}\|)\boldsymbol{k} - (2D\|\boldsymbol{F}\|)\ \boldsymbol{k} = -2D\|\boldsymbol{F}\|\boldsymbol{k} \tag{3.21}$$

This is the same moment we calculated in (3.20). Equations (3.20) and (3.21) use the same moment center (O) to calculate the moment created by a couple applied to two different locations. As we saw, in both cases the couple moments had the same magnitude and sense. This shows another important property of a couple: we can *move a couple anywhere in the plane of the couple, or in a parallel plane, without changing the effect of the couple*.

To summarize what we have learned about couples, we examine a general case of two equal and opposite forces applied to a body located somewhere in space (**Figure 3.4.3**). The lines of action of these two forces are parallel and at a distance d from each other. The resultant force on the body due to the couple is zero. We can calculate the moment created by the couple at any moment center in space, and the result will be the same. In **Figure 3.4.3** we arbitrarily select a moment center somewhere along the line of action of one force, and draw a

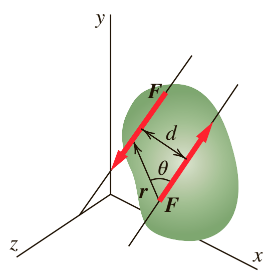


Figure 3.4.3 Two equal and opposite parallel forces make a couple.

position vector to the line of action of the other force. The resulting moment is

$$\boldsymbol{M}_{\text{couple}} = \boldsymbol{r} \times \boldsymbol{F} \tag{3.22}$$

$\boldsymbol{M}_{\text{couple}}$ is perpendicular to the plane created by the pair of forces.

Returning to (3.1), we calculate the magnitude of the couple moment

$$\left\|\boldsymbol{M}_{\text{couple}}\right\| = \|\boldsymbol{r}\|(\|\boldsymbol{F}\| \sin\theta)$$

which, using the commutative property of multiplication, can be rewritten $\left\|\boldsymbol{M}_{\text{couple}}\right\| = \|\boldsymbol{F}\|(\|\boldsymbol{r}\| \sin\theta)$. This is the same expression (3.1) developed in Section 3.1 for a moment's magnitude. From **Figure 3.4.3** we note that $\|\boldsymbol{r}\| \sin\theta = d$, which leads us to:

$$\left\|\boldsymbol{M}_{\text{couple}}\right\| = \|\boldsymbol{F}\| d \tag{3.23}$$

Equation (3.23) means that the magnitude of a couple moment is calculated by multiplying the magnitude of the applied forces by the perpendicular distance between the two forces.

Check out the following examples of applications of this material.

- **Example 3.4.1 A Couple in the *xy* Plane**
- **Example 3.4.2 Working with Couples**

EXAMPLE 3.4.1

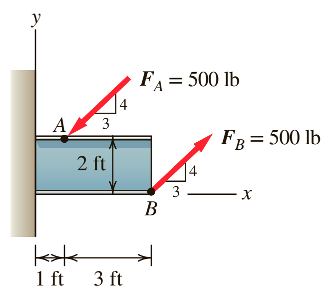


Figure 1 Two equal and opposite parallel forces are applied to a structural member.

Two forces act on the member shown in **Figure 1**. Find the moment created by the two forces. Do the two forces represent a couple?

Goal Find the moment created by two forces acting on a member.

Given The dimensions of the member, coordinate axes, and point of application, direction, and magnitude of the two forces.

Assume The two forces lie in the *xy* plane, as defined in **Figure 1**.

Formulate Equations and Solve Because the forces $\boldsymbol{F}_A$ and $\boldsymbol{F}_B$ are parallel, are of the same magnitude, and act in opposite directions, they form a couple. Recall that we calculate the same couple moment regardless of where we choose to locate the moment center.

According to (3.23) the magnitude of the couple is given by $\|\boldsymbol{M}\| = \|\boldsymbol{F}\| d$. In this case, the perpendicular distance between the forces d is somewhat difficult to find. It is easier to break each force into its x and y components, find the moment associated with each, and then sum them together.

First, resolve the force $\boldsymbol{F}_A$ acting at point A into its scalar components:

$$F_{Ax} = -\frac{3}{5}(500 \text{ lb}) = -300 \text{ lb} \quad \text{and} \quad F_{Ay} = -\frac{4}{5}(500 \text{ lb}) = -400 \text{ lb}$$

Then, resolve the force $\boldsymbol{F}_B$ acting at point B into its scalar components:

$$F_{Bx} = \frac{3}{5}(500\text{ lb}) = 300\text{ lb} \quad \text{and} \quad F_{By} = \frac{4}{5}(500\text{ lb}) = 400\text{ lb}$$

We now redraw the applied loads as shown in **Figure 2**.

Using (3.23), we write the moment associated with the couple that consists of F_{Ax} and F_{Bx}. Note that the moment is perpendicular to the xy-plane, therefore in the $\boldsymbol{k}$ direction.

$$\boldsymbol{M}_1 = \|\boldsymbol{F}\|d\,\boldsymbol{k} = +(2\text{ ft})(300\text{ lb})\boldsymbol{k} = +600\text{ ft·lb}\,\boldsymbol{k}$$

We write the magnitude of the moment associated with the couple that consists of F_{Ay} and F_{By} as

$$\boldsymbol{M}_2 = \|\boldsymbol{F}\|d\,\boldsymbol{k} = +(3\text{ ft})(400\text{ lb})\boldsymbol{k} = +1200\text{ ft·lb}\,\boldsymbol{k}$$

Finally, to find the overall moment, we simply sum $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$.

$$\boldsymbol{M}_{\text{couple}} = \boldsymbol{M}_1 + \boldsymbol{M}_2 \quad \Rightarrow \quad \boxed{\boldsymbol{M}_{\text{couple}} = +1800\text{ ft·lb}\,\boldsymbol{k}}$$

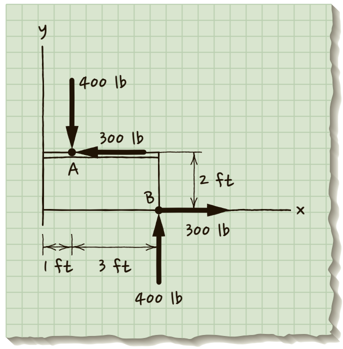


Figure 2 Break $\boldsymbol{F}_A$ and $\boldsymbol{F}_B$ into components parallel to the x and y axes.

Check Since the couple creates a moment that is independent of the moment center, we arbitrarily select a moment center at A and proceed to find the moment due to $\boldsymbol{F}_A$ and $\boldsymbol{F}_B$. Notice that $\boldsymbol{F}_A$ creates zero moment at A. The moment $\boldsymbol{M}$ created by $\boldsymbol{F}_A$ and $\boldsymbol{F}_B$ is the same as we calculated previously.

$$\boldsymbol{M}_{\text{couple}} = [(2\text{ ft})(F_{Bx}) + (3\text{ ft})(F_{By})]\boldsymbol{k} = [(2\text{ ft})(300\text{ lb}) + (3\text{ ft})(400\text{ lb})]\boldsymbol{k} = +1800\text{ ft·lb}\,\boldsymbol{k}$$

EXAMPLE 3.4.2

A T-joint is acted on by the two couples shown in **Figure 1**. Determine the magnitude and direction of the resultant couple moment. Where does the resultant moment act?

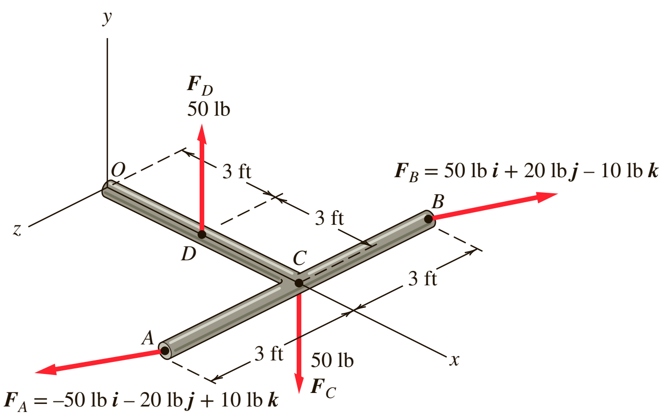


Figure 1 Two couples are applied to a T-joint: (1) $\boldsymbol{F}_A$ and $\boldsymbol{F}_B$, (2) $\boldsymbol{F}_C$ and $\boldsymbol{F}_D$.

Goal Find the couple moment created by the forces acting on the T-joint shown in **Figure 1**.

Given Dimensions of the T-joint, the points of application, direction, and magnitude of two couples, and a coordinate system.

Assume No assumptions are necessary.

Draw No additional drawings are needed.

Formulate Equations and Solve We will determine the moment due to $\boldsymbol{F}_A$ and $\boldsymbol{F}_B$, and then the moment due to $\boldsymbol{F}_C$ and $\boldsymbol{F}_D$. Summing the two couple moments together produces the resultant moment acting on the system.

First we consider the couple consisting of $\boldsymbol{F}_A$ and $\boldsymbol{F}_B$ and determine its moment using the cross product (3.22). Keeping in mind that for a couple we can select any point for the moment center*, we select point A and draw the position vector from A to B.

$$\boldsymbol{r}_{AB} = -6\text{ ft }\boldsymbol{k}$$

Then using (3.22) we calculate the couple moment at A:

$$\boldsymbol{M}_{AB} = \boldsymbol{r}_{AB} \times \boldsymbol{F}_B = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ r_{ABx} & r_{ABy} & r_{ABz} \\ F_{Bx} & F_{By} & F_{Bz} \end{vmatrix} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0\text{ ft} & 0\text{ ft} & -6\text{ ft} \\ 50\text{ lb} & 20\text{ lb} & -10\text{ lb} \end{vmatrix}$$

so that

$$\boldsymbol{M}_{AB} = -(-6\text{ ft})(20\text{ lb})\,\boldsymbol{i} - 6\text{ ft}(50\text{ lb})\,\boldsymbol{j} = 120\text{ ft·lb}\,\boldsymbol{i} - 300\text{ ft·lb}\,\boldsymbol{j}$$

Next we consider the couple consisting of $\boldsymbol{F}_C$ and $\boldsymbol{F}_D$, choosing C as the moment center. The moment at C created by $\boldsymbol{F}_C$ and $\boldsymbol{F}_D$ is

$$\boldsymbol{M}_{CD} = \boldsymbol{r}_{CD} \times \boldsymbol{F}_D = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ r_{CDx} & r_{CDy} & r_{CDz} \\ F_{Dx} & F_{Dy} & F_{Dz} \end{vmatrix} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -3\text{ ft} & 0\text{ ft} & 0\text{ ft} \\ 0\text{ lb} & 50\text{ lb} & 0\text{ lb} \end{vmatrix}$$

To determine the magnitude and direction of the resultant moment for the loading in **Figure 1**, we move $\boldsymbol{M}_{AB}$ to C and sum it with $\boldsymbol{M}_{CD}$

$$\boldsymbol{M}_{\text{couple}} = \boldsymbol{M}_{AB} + \boldsymbol{M}_{CD} = (120\text{ ft·lb}\,\boldsymbol{i} - 300\text{ ft·lb}\,\boldsymbol{j}) - 150\text{ ft·lb}\,\boldsymbol{k}$$

$$\boxed{\boldsymbol{M}_{\text{couple}} = 120\text{ ft·lb}\,\boldsymbol{i} - 300\text{ ft·lb}\,\boldsymbol{j} - 150\text{ ft·lb}\,\boldsymbol{k}}$$

In **Figure 2** we show the resultant moment acting at point C. Alternatively, in **Figures 3** and **4** we show the resultant moment acting at point O. Since we are working with couples, we can choose any point, on or off the structure, to represent our resultant moment.

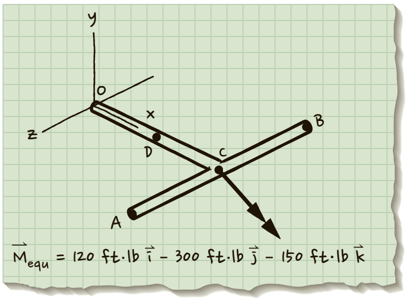


Figure 2 The resultant couple moment is placed at C.

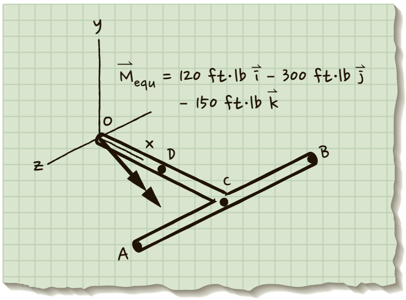


Figure 3 The resulting couple moment is placed at point O.

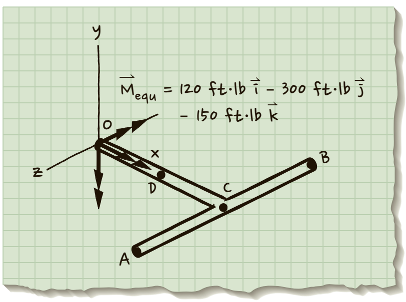


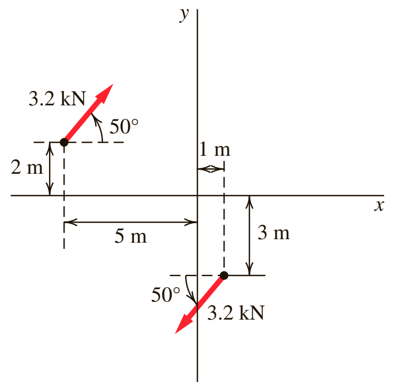
Figure 4 The resultant moment placed at O and represented in terms of its x, y, and z components.

Check We could check our result by calculating $\boldsymbol{M}_{AB}$ and $\boldsymbol{M}_{CD}$ using different moment centers. We also could have used the approach from a previous example in which we divided $\boldsymbol{F}_A$, $\boldsymbol{F}_B$, $\boldsymbol{F}_C$, and $\boldsymbol{F}_D$ into scalar components and used the perpendicular distances to find the moments. The same answer as above would have resulted.

*Since we are dealing with couples, we could have chosen any point as the moment center. The important thing to remember is that the position vector is defined from the moment center; the moment center is not required to be the origin of the coordinate axes.

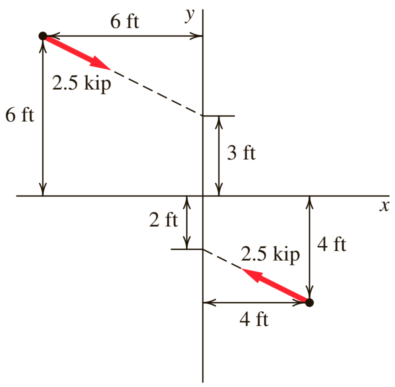
EXERCISES 3.4

3.4.1. **[*]** Determine the magnitude and sense of the moment created by the couple shown.



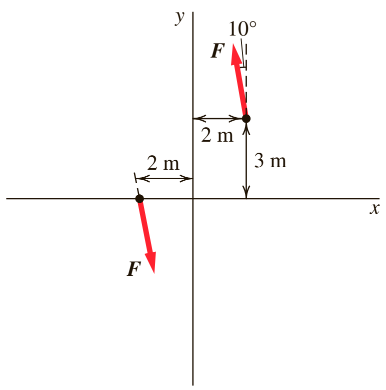
EX 3.4.1

3.4.2. **[*]** Determine the magnitude and sense of the moment created by the couple shown.



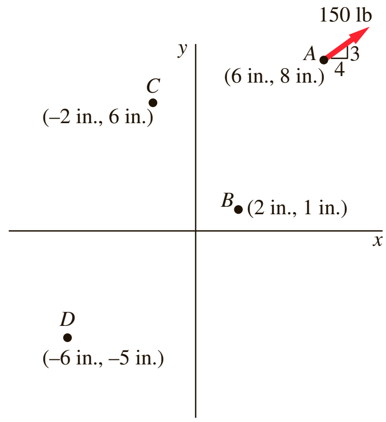
EX 3.4.2

3.4.3. **[*]** For the couple shown, determine the magnitude of $\boldsymbol{F}$ needed to create a 4000-N·m moment.



EX 3.4.3

3.4.4. **[*]** A 150-lb force is applied to point A. At which of the three points B, C, or D should an equal and opposite force be placed to create a moment of +480 in.·lb $\boldsymbol{k}$?

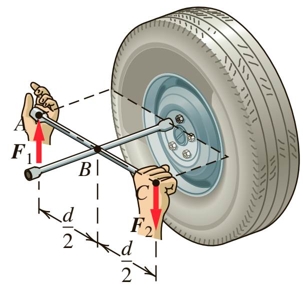


EX 3.4.4

3.4.5. **[*]** A lug wrench is being used to tighten a lug nut on an automobile wheel, as shown. Two equal-magnitude parallel forces of opposite sign are applied to the wrench.

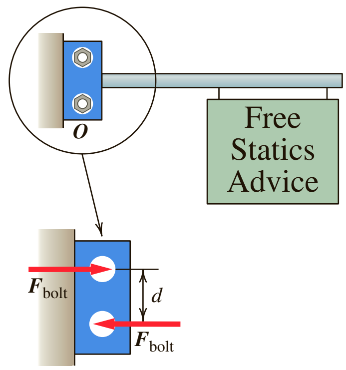
a. Replace $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ by a moment at B.

b. Now replace $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ by a moment at A. How does the magnitude of the moment at A compare with the magnitude of the moment at B?



EX 3.4.5

3.4.6. **[*]** Students have placed a new sign outside the tutoring center. The weight of the sign and the supporting aluminum beam create a moment $\boldsymbol{M}_O = 117$ in.·lb at O. $\boldsymbol{M}_O$ is transferred to the wall connection by bolt forces ($\boldsymbol{F}_{\text{bolt}}$), which form a couple. The bolts are spaced 3 inches apart.



EX 3.4.6

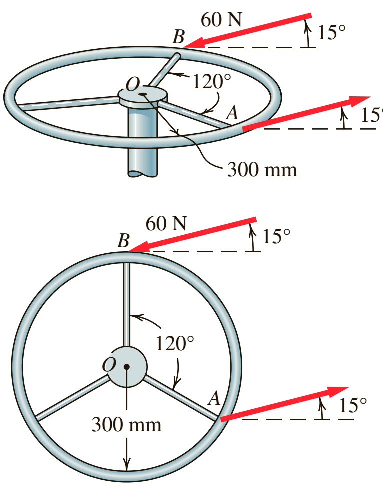
a. Determine the force F_{bolt} transferred by each bolt.

b. If the allowable load for each bolt is 150 lb, what is the minimum spacing of the bolts d_{min} to ensure the connection is safely designed?

3.4.7. [*] A bus driver applies two 60-N forces to the steering wheel shown. These two forces are parallel and are in the plane of the steering wheel.

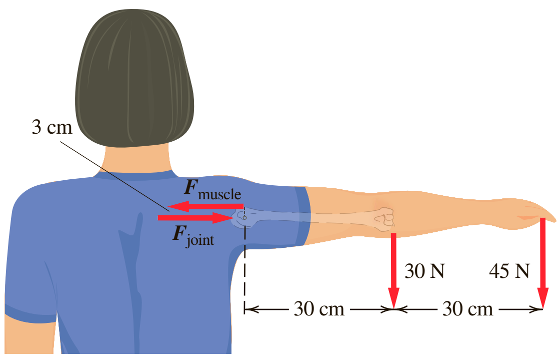
a. What is the magnitude of the applied couple moment?

b. What is the magnitude of the component of this couple moment that has the effect of turning the steering wheel?



EX 3.4.7

3.4.8. [**] The forces acting on the glenohumeral joint (in the shoulder) can be represented by the simplified model shown. The deltoid pulls on the humerus (arm) with a force F_{muscle} and the joint pushes back with a force F_{joint} forming a couple. A woman is holding a 45-N weight in her hand. Her arm weighs 30 N (represented by a single force at 30 cm from the joint). Determine the magnitude of the forces acting at the glenohumeral joint if the sum of all the moments acting at the joint is zero.



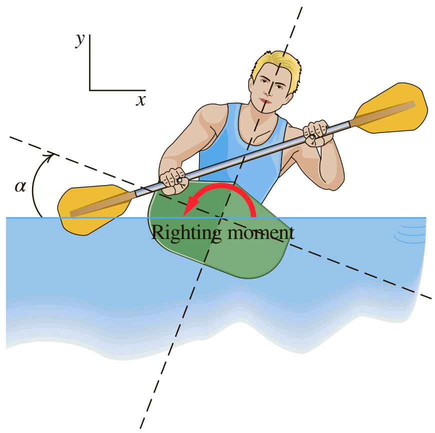
EX 3.4.8

3.4.9. [**] The canoe shown in **Figure *a*** has been designed so that when it tilts at an angle α, a *righting moment* returns it to a horizontal stable position. The righting moment is formed by a couple as shown in **Figure *b***, consisting of F_1 (the weight of the canoe and paddler) and F_2 (a buoyancy force due to the water pushing up on the canoe, which is described in more detail in a later chapter). When the canoe is horizontal ($\alpha = 0$), F_1 and F_2 line up ($d = 0$) and the righting moment is zero. Assume the combined weight of the paddler and canoe is 900 N.

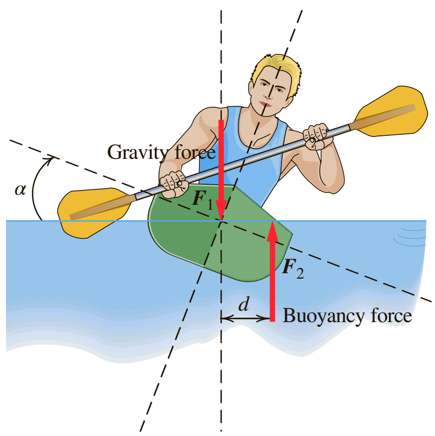
a. If d increases 5 mm per degree of tilt, develop an equation that describes the righting moment as a function of α. Define clockwise tilt as positive α.

b. Determine the righting moment when $\alpha = 20°$.

c. Make a plot of righting moment versus α. Stop your graph at 30°, because at this angle the canoe starts to take on water!



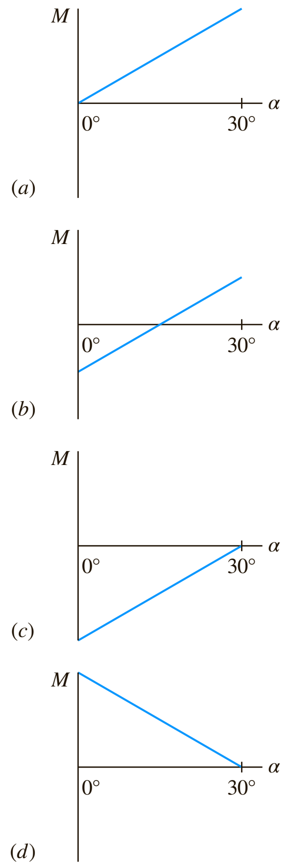
EX 3.4.9(*a*)



EX 3.4.9(*b*)

Answer the following question.

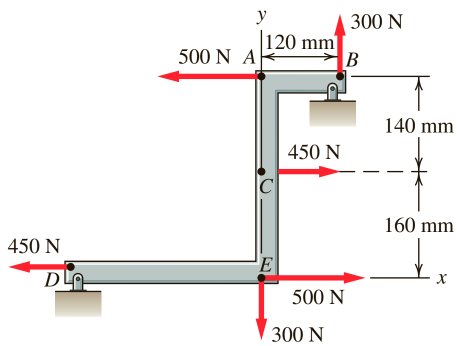
Which of the following represents the correct shape of the graph of M as a function of α? Define counterclockwise moment about the z axis to be positive.



3.4.10. [*] Six forces acting on the bracket form three couples.

a. Determine the resultant couple acting on the bracket.

b. Where should you place the resultant couple?

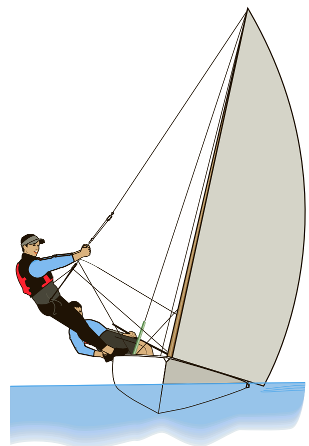


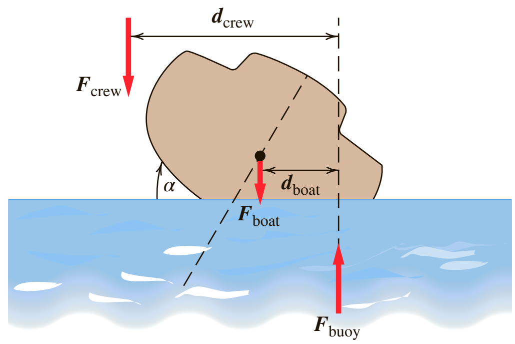
EX 3.4.10

3.4.11. [**] The sailors in **Figure *a*** are *hiking out* to help stabilize the boat. Two couples are acting on the boat in **Figure *b*** to create the resultant *righting moment* when the boat is heeling at an angle α. When the sail boat tips, the *righting moment* returns the boat to a horizontal stable position. One couple is created by the weight of the crew $\boldsymbol{F}_{\text{crew}}$ and the second by the weight of the boat $\boldsymbol{F}_{\text{boat}}$. (Note: $\boldsymbol{F}_{\text{buoy}} = \boldsymbol{F}_{\text{crew}} + \boldsymbol{F}_{\text{boat}}$.) Hiking out increases the righting moment by increasing the contribution from the couple moment due to $\boldsymbol{F}_{\text{crew}}$. Assume the boat weighs 200 lb, the crew weighs 250 lb, $\alpha = 20°$, and $d_{\text{boat}} = 0.75$ ft.

a. Determine the righting moment when the crew is sitting in the middle of the boat ($d_{\text{crew}} = 0.75$ ft).

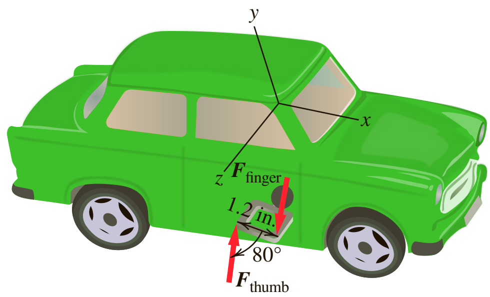
b. Determine the righting moment when the crew is hiking out ($d_{\text{crew}} = 5$ ft).





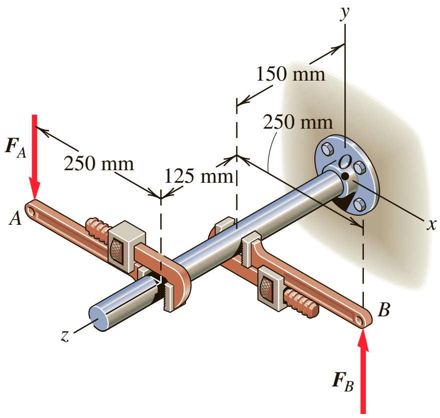
EX 3.4.11

3.4.12. [*] To wind up the car shown, you apply a couple to the key using your finger and thumb. Assume $\boldsymbol{F}_{\text{finger}}$ and $\boldsymbol{F}_{\text{thumb}}$ are parallel to the side of the car and at an angle of 80° relative to the face of the key. The key is 1.2 in. wide. Determine the magnitude and sense of the moment when you apply 0.1 lb to the key.



EX 3.4.12

3.4.13. [**] For the two pipe-wrench forces shown, determine the magnitude of $\boldsymbol{F}_A$ and $\boldsymbol{F}_B$ if the magnitude of the resulting couple is 40 N·m.

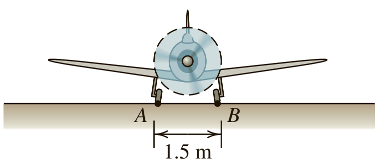


EX 3.4.13

3.4.14. [**] The drive shaft from the engine of a small airplane applies to the propeller a moment that is counterclockwise as viewed from the front of the plane. The magnitude of the moment is 1.8 kN · m.

a. This moment is transferred to the ground at the wheels by a 1.8 kN · m-couple consisting of vertical forces applied by the wheels to the ground at points A and B. Determine the magnitude of the forces.

b. Both of the vertical forces in **a** happen at the interface between the ground and the tires. The vertical force at B seems to indicate that there is a tensile normal force between the ground and the tire. According to our presentation in the chapter on forces, however, normal forces are compressive. What is really happening?

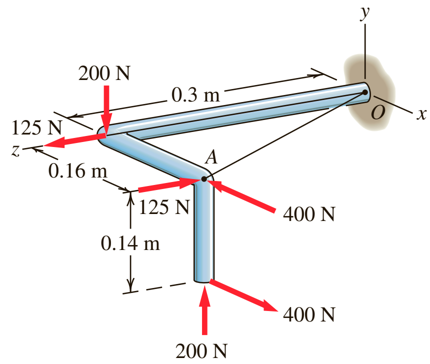


EX 3.4.14

3.4.15. [**] Three couples are applied to a bent bar as shown. Determine

a. the resultant loading (Present your answer in the form of a vector.)

b. the scalar component of the resultant moment about the line OA.

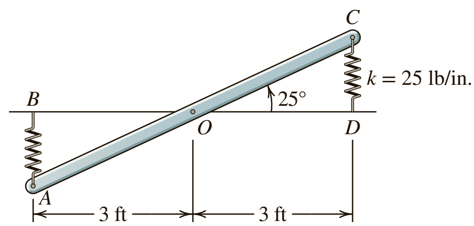


EX 3.4.15

3.4.16. [***] Bar AOC has been rotated counterclockwise to an angle of 25°, which has stretched springs AB and CD. The springs are vertical and are 12 in. long when not stretched. Determine

a. the couple that acts on bar AOC due to the stretched springs.

b. the magnitude and sense of the moment created by the couple.

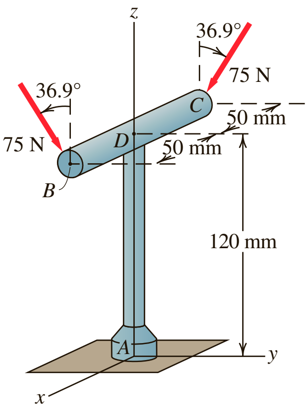


EX 3.4.16

3.4.17. [***] A T-bar tool is used to loosen a nut by applying 75-N forces at B and C, as shown.

a. Determine the moment created by the couple acting on the T-bar. What is the effect of the moment?

b. What is the resultant force acting on the T-bar? What is the effect of the resultant force?



EX 3.4.17

3.5 EQUIVALENT LOADS

Learning Objective: Find the equivalent moment and equivalent force due to multiple loads acting on a system.

We have discussed the moment created by a force offset from a moment center. For example, $\boldsymbol{F}_{\text{push}}$ applied at A in **Figure 3.5.1*a*** creates a negative moment at a moment center at B of $-250 \text{ mm} \left\|\boldsymbol{F}_{\text{push}}\right\| \boldsymbol{k}$. Now we consider that we can replace $\boldsymbol{F}_{\text{push}}$ applied at A with a loading applied at B consisting of a force $\boldsymbol{F}_{\text{push}}$ and a moment $-250 \text{ mm} \left\|\boldsymbol{F}_{\text{push}}\right\| \boldsymbol{k}$ (**Figure 3.5.1*b***). The loading in **Figure 3.5.1*a*** is equivalent to the loading in **Figure 3.5.1*b***; we call them **equivalent loads**. Equivalent loads each have the same influence with regard to pushing, pulling, twisting, tipping, turning, and/or rocking the system on which they act. The loading in **Figure 3.5.1*b*** (consisting of a force and a moment applied at B) is equivalent to the loading in **Figure 3.5.1*a*** (consisting of only a force applied at A).

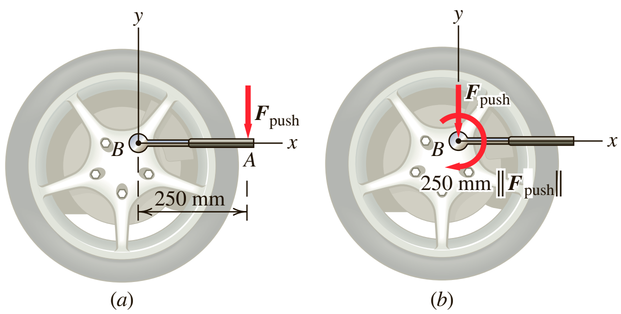


Figure 3.5.1 A force applied at the end of the wrench (*a*) is equivalent to a force and a moment applied at the center of the lug nut (*b*).

We can generalize our observations from **Figure 3.5.1** as follows:

Key Concept: For any load (consisting of forces and/or moments) applied to a system, we can replace that load with an equivalent load (consisting of a force and a moment) applied at a single point.

We call this the **equivalent load principle**.

Now let's apply the principle to a system with a number of forces acting on it. For example, the forces acting on the supporting frame for a traffic light (**Figure 3.5.2*a***) are the weight of the light, a drag force resulting from wind, and a tension force exerted by the cable attached to the sign (**Figure 3.5.2*b***). If we wish to find the loading applied at O that is equivalent to these three forces, we add the forces to come up with an **equivalent force**, and we add the moments created by the individual forces about a moment center at O to come up with an **equivalent moment**.

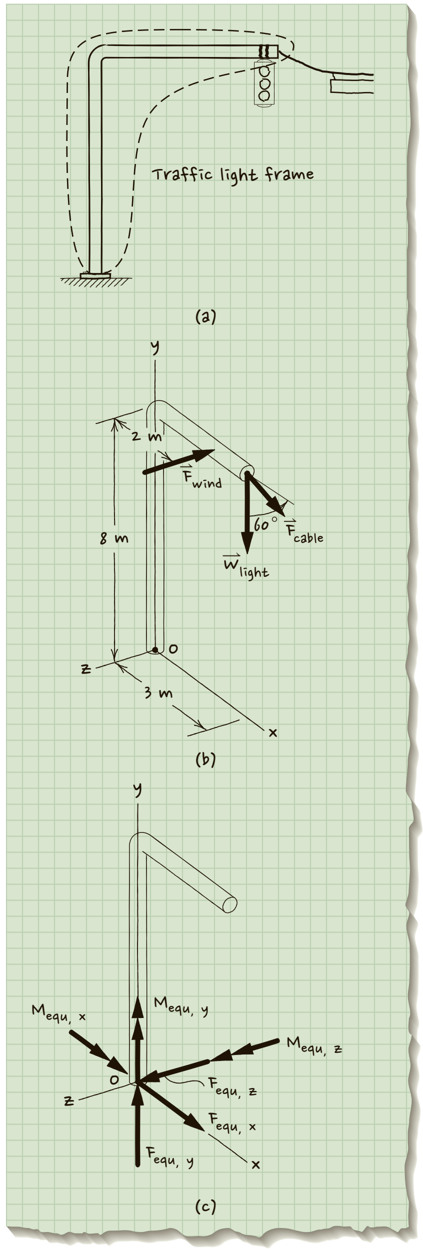


Figure 3.5.2 Equivalent loads: the load on the traffic light shown in (b) is equivalent to the load in (c).

The equivalent force is simply the vector sum of the forces acting on the frame:

$$\boldsymbol{F}_{\text{equ @ }O} = \underbrace{\|\boldsymbol{F}_{\text{cable}}\| \sin 60° \boldsymbol{i}}_{F_{equ,x}} - \underbrace{(\|\boldsymbol{F}_{\text{cable}}\| \cos 60° + \|\boldsymbol{W}_{\text{light}}\|)\boldsymbol{j}}_{F_{equ,y}} - \underbrace{\|\boldsymbol{F}_{\text{wind}}\|\boldsymbol{k}}_{F_{equ,z}}$$

The most straightforward approach to finding the equivalent moment at a moment center at O is to calculate the moment created by each force relative to this moment center, then add the moments together. We find that the equivalent moment due to the forces $\boldsymbol{F}_{\text{wind}}$, $\boldsymbol{F}_{\text{cable}}$, and $\boldsymbol{W}_{\text{light}}$ acting on the supporting frame is*

$$\boldsymbol{M}_{\text{equ @ }O} = \underbrace{-8\text{ m}\|\boldsymbol{F}_{\text{wind}}\|\boldsymbol{i}}_{M_{equ,x}} + \underbrace{2\text{ m}\|\boldsymbol{F}_{\text{wind}}\|\boldsymbol{j}}_{M_{equ,y}} - \underbrace{(3\text{ m}\|\boldsymbol{W}_{\text{light}}\| + 8.43\text{ m}\|\boldsymbol{F}_{\text{cable}}\|)\boldsymbol{k}}_{M_{equ,z}}$$

We show the equivalent moment and equivalent force on a drawing of the system in **Figure 3.5.2*c***. Notice that the equivalent moment and equivalent force are placed at the moment center O. The loads in **Figure 3.5.2*b*** and those in **Figure 3.5.2*c*** are equivalent loads.

The analysis in Section 3.4 to calculate couple moments resulting from applied couples is another example of finding equivalent loads. A couple moment and a couple are equivalent loads. Reconsider the star wrench in **Figure 3.5.3*a*** in Section 3.4. The couple moment at O shown in **Figure 3.5.3*b*** is really just an equivalent load that replaces the couple formed by $\boldsymbol{F}_{\text{LH}}$ and $\boldsymbol{F}_{\text{RH}}$. In this case, the equivalent force (which is the vector sum of the forces acting on the system) is zero since the two forces are equal in magnitude and opposite in direction. As we found in Section 3.4, the equivalent moment is $\boldsymbol{M}_{\text{equ}} = -(2D\|\boldsymbol{F}\|)\boldsymbol{k}$.

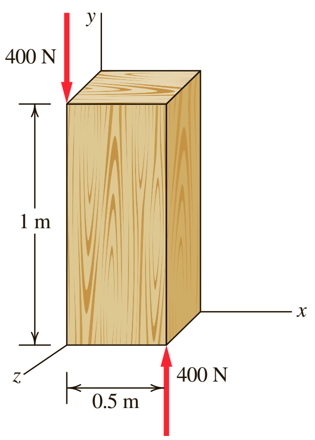


Figure 3.5.4 A couple creating a moment of 200 N·m $\boldsymbol{k}$ is applied to a block.

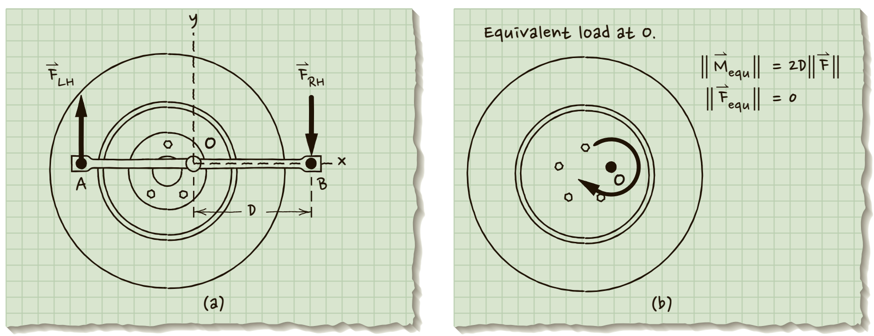


Figure 3.5.3 (*a*) A couple; (*b*) the equivalent load at O consisting of only a couple moment and no force.

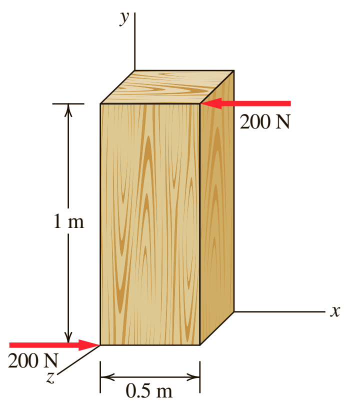


Figure 3.5.5 An equivalent couple creating a moment of 200 N·m $\boldsymbol{k}$ is applied to a block.

Two couples are said to be **equivalent couples** if they produce the same moment (magnitude and direction). The couple consisting of two equal and opposite *vertical* forces in **Figure 3.5.4** produces a moment of 200 N·m about the z axis. **Figure 3.5.5** shows an equivalent couple consisting of two

*Note that in the following equation, 8.43 m is the perpendicular distance between point O and the line of action of $\boldsymbol{F}_{\text{cable}}$.

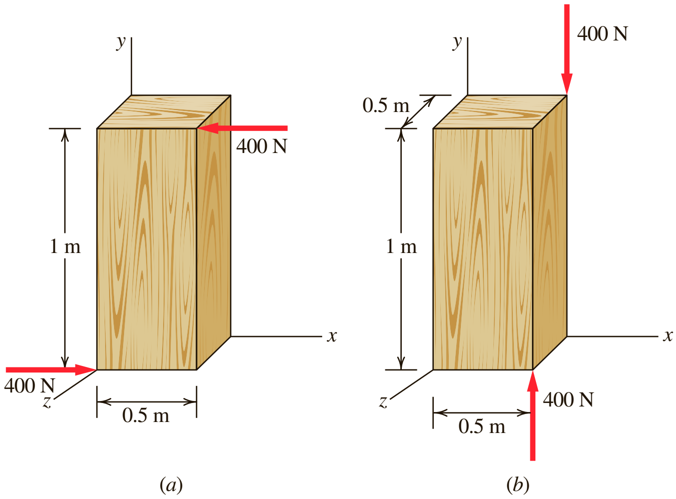


Figure 3.5.6 Neither couple, (a) nor (b), is equivalent to the couple in **Figure 3.5.4**.

equal and opposite *horizontal* forces. In these two cases, even though the forces are applied to different points on the block, and have different magnitudes and directions, each couple produces a moment of 200 N·m $\boldsymbol{k}$, which would tend to make the block rotate counterclockwise about the z axis. Therefore these two couples are equivalent.

The two couples shown in **Figure 3.5.6** are not equivalent to the couple in **Figure 3.5.4**. The couple in **Figure 3.5.6*a*** is in the same direction ($\boldsymbol{k}$) as **Figure 3.5.4** but has a different magnitude (400 N·m). The couple in **Figure 3.5.6*b*** has the same magnitude (200 N·m) but would cause the block to rotate counterclockwise about the x axis (rather than the z axis).

Check out the following examples of applications of this material.
- **Example 3.5.1 Equivalent Moment and Equivalent Force - Planar**
- **Example 3.5.2 Equivalent Moment and Equivalent Force - Nonplanar**
- **Example 5.3.3 Equivalent Load for an Applied Couple**

EXAMPLE 3.5.1

A beam is held in place by a pin at A and a rope attached at B and C. A block weighing 1000 lb hangs at end D, as shown in **Figure 1**.

(a) If the equivalent moment at A due to the rope tension and block load is zero, what is the tension in the rope?

(b) What is the equivalent loading at A acting on the beam due to the rope tension and block load?

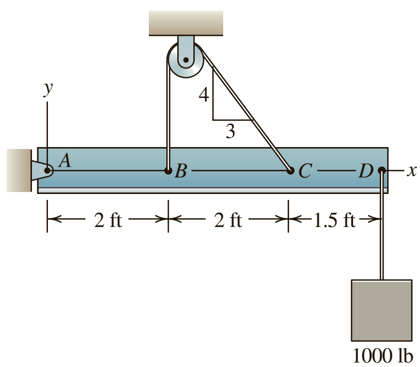


Figure 1 Beam is supported by a pin at A and a rope attached at B and C.

Goal (a) Find the tension in the rope such that the equivalent moment at A due to the rope tension and the weight of the block acting on the beam has zero magnitude. (b) Using this rope force, find the equivalent force at A.

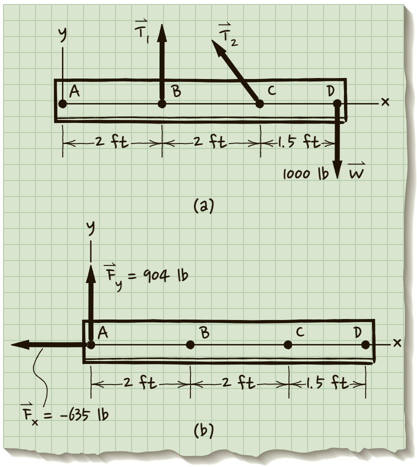


Figure 2 (a) Loads acting on the beam at B, C, and D; (b) equivalent loading on the beam at A.

Given A coordinate system, the geometry of the system and its components, the beam loads, and the equivalent moment at A due to the loadings.

Assume The beam, block, rope, and pulley all lie in the xy plane, the pulley is frictionless, and the weight of beam AD is negligible.

Draw We begin by drawing a diagram of the beam, showing external loads $\boldsymbol{T}_1$, $\boldsymbol{T}_2$, and $\boldsymbol{W}(= -1000 \text{ lb } \boldsymbol{j})$; see Figure 2*a*.

Formulate Equations and Solve **(a)** Because a single rope is attached at B and C and passes over a frictionless pulley, $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ have the same magnitude; call it $\|\boldsymbol{T}\|$.

We proceed by finding the moment at A created by each force ($\boldsymbol{T}_1$, $\boldsymbol{T}_2$, and $\boldsymbol{W}$), then summing these moments; their sum should be equal to zero. Because $\boldsymbol{T}_1$, $\boldsymbol{T}_2$, and $\boldsymbol{W}$ lie in the xy plane, the resulting moments should all be about the z-axis.

Moment $\boldsymbol{M}_1$ about A created by $\boldsymbol{T}_1$: With $\boldsymbol{T}_1 = \|\boldsymbol{T}\|\,\boldsymbol{j}$ and $\boldsymbol{r}_1 = 2 \text{ ft } \boldsymbol{i}$

$$\boldsymbol{M}_1 = \boldsymbol{r}_1 \times \boldsymbol{T}_1 = 2\|\boldsymbol{T}\| \text{ft } \boldsymbol{k}$$

Moment $\boldsymbol{M}_2$ about A created by $\boldsymbol{T}_2$: $\boldsymbol{T}_2 = -3/5\|\boldsymbol{T}\|\boldsymbol{i} + 4/5\|\boldsymbol{T}\|\boldsymbol{j}$, and $\boldsymbol{r}_2 = 4 \text{ ft } \boldsymbol{i}$

$$\boldsymbol{M}_2 = \boldsymbol{r}_2 \times \boldsymbol{T}_2 = (4 \text{ ft})\left(\frac{4}{5}\|\boldsymbol{T}\|\right)\boldsymbol{k} = \frac{16}{5}\|\boldsymbol{T}\| \text{ft } \boldsymbol{k}$$

(Notice that the x component of $\boldsymbol{T}_2 = -3/5\|\boldsymbol{T}\|\boldsymbol{i}$, does not create a moment about A because its line of action intersects A.)

Moment $\boldsymbol{M}_3$ about A created by $\boldsymbol{W}$: $\boldsymbol{W} = -1000 \text{ lb } \boldsymbol{j}$ and $\boldsymbol{r}_3 = 5.5 \text{ ft } \boldsymbol{i}$

$$\boldsymbol{M}_3 = \boldsymbol{r}_3 \times \boldsymbol{W} = -5500 \text{ lb} \cdot \text{ft } \boldsymbol{k}$$

Equivalent moment at A: The equivalent moment $\boldsymbol{M}_{\text{equ}} = \boldsymbol{M}_1 + \boldsymbol{M}_2 + \boldsymbol{M}_3$:

$$\boldsymbol{M}_{\text{equ @ } A} = \left(2\|\boldsymbol{T}\| + \frac{16}{5}\|\boldsymbol{T}\|\right) \text{ft } \boldsymbol{k} - 5500 \text{ lb} \cdot \text{ft } \boldsymbol{k}$$

$$\boldsymbol{M}_{\text{equ @ } A} = \left(\frac{26}{5}\|\boldsymbol{T}\| \text{ft} - 5500 \text{ lb} \cdot \text{ft}\right)\boldsymbol{k} \tag{1}$$

The problem statement indicated the equivalent moment is equal to zero, therefore we set (1) equal to zero and solve for $\|\boldsymbol{T}\|$:

$$\left(\frac{26}{5}\|\boldsymbol{T}\| \text{ft} - 5500 \text{ lb} \cdot \text{ft}\right)\boldsymbol{k} = 0$$

$$\|\boldsymbol{T}\| = 1058 \text{ lb}$$

(b) The equivalent load at A due to the loads at B, C, and D consists of an equivalent moment and an equivalent force. Based on our work in

(a), $\boldsymbol{M}_{equ} = 0$ for a rope tension of 1058 lb. The equivalent force is equal to $\boldsymbol{T}_1 + \boldsymbol{T}_2 + \boldsymbol{W}$:

$$\begin{aligned}\boldsymbol{F}_{equ} &= \|\boldsymbol{T}\|\,\boldsymbol{j} - \frac{3}{5}\|\boldsymbol{T}\|\,\boldsymbol{i} + \frac{4}{5}\|\boldsymbol{T}\|\,\boldsymbol{j} - 1000\text{ lb}\,\boldsymbol{j}\\ &= 1058\text{ lb}\,\boldsymbol{j} - 635\text{ lb}\,\boldsymbol{i} + 846\text{ lb}\,\boldsymbol{j} - 1000\text{ lb}\,\boldsymbol{j}\end{aligned}$$

$$\boxed{\boldsymbol{F}_{equ} = -635\text{ lb}\,\boldsymbol{i} + 904\text{ lb}\,\boldsymbol{j}}$$

We draw this equivalent load in **Figure 2*b***.

Check A good check is to substitute the value of the rope tension back into (1) to confirm that the equivalent moment due to the rope tension and block weight is indeed zero.

EXAMPLE 3.5.2

A 3507 kg cylinder hangs from a roller of radius 30 mm. In addition, a cable attached to the cylinder and wound around the roller pulls with a force of 36 kN at A as shown in **Figure 1**. Find the equivalent moment and equivalent force at O due to these two loads.

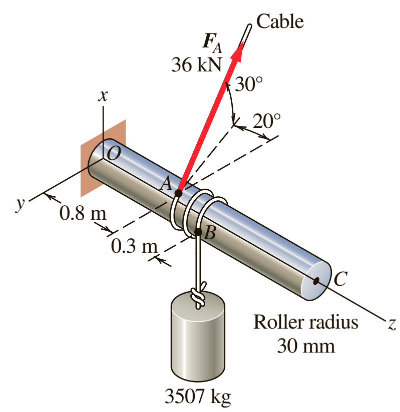


Figure 1 A cylinder hangs by a cable wound around a roller.

Goal Find the equivalent loading (moment and force) at a specified moment center (O).

Given Information on the dimensions of the roller, the force due to a cable wound around the roller, the mass of the cylinder, and a coordinate system.

Assume Earth's gravity acts in the $-x$ direction, and the weight of the roller can be ignored.

Draw In **Figure 2** we have drawn the roller and the forces acting on it due to the cable and the weight ($\boldsymbol{F}_A$ and $\boldsymbol{F}_B$, respectively). We determine the force due to the weight of the cylinder to be 34.4 kN (= 3507 kg × 9.81 m/s^2).

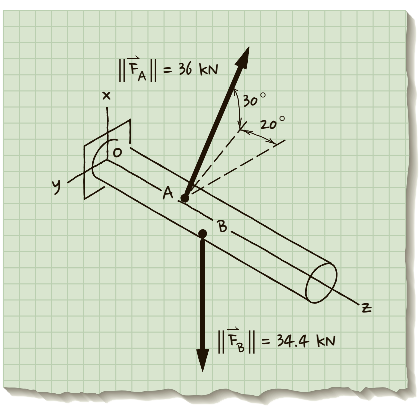


Figure 2 We draw the forces acting at A and B.

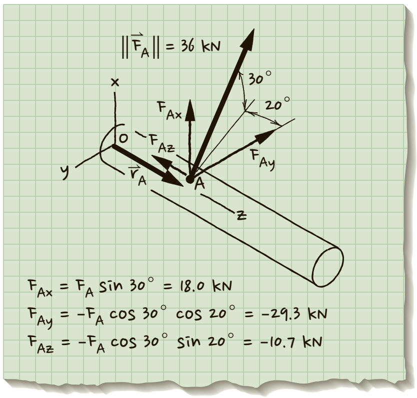


Figure 3 Calculate the scalar components of $\boldsymbol{F}_A$ and draw the position vector from O to A.

Formulate Equations and Solve **(a)** Since neither of these forces lies in a reference axes plane, we use the cross-product in (3.12) to find the moments.

Moment $\boldsymbol{M}_A$ created by $\boldsymbol{F}_A$: Using the calculations in **Figure 3**, we write $\boldsymbol{F}_A$ in vector form as

$$\boldsymbol{F}_A = 18.0\,\text{kN}\,\boldsymbol{i} - 29.3\,\text{kN}\,\boldsymbol{j} - 10.7\,\text{kN}\,\boldsymbol{k}$$

We use the fact that the rope is tangent to the cylinder at A to determine that A is located on a radius 30° counterclockwise from the x axis. Thus the position vector from O to A is $\boldsymbol{r}_A = 0.026\text{ m}\,\boldsymbol{i} + 0.015\text{ m}\boldsymbol{j} + 0.8\text{ m}\,\boldsymbol{k}$. We find the moment that $\boldsymbol{F}_A$ creates at a moment center at O to be

$$\boldsymbol{M}_A = \boldsymbol{r}_A \times \boldsymbol{F}_A = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0.026\text{ m} & 0.015\text{ m} & 0.8\text{ m} \\ 18.0\text{ kN} & -29.3\text{ kN} & -10.7\text{ kN} \end{vmatrix}$$

Expanded, this becomes

$$\boldsymbol{M}_A = (23.3\,\text{kN·m}\,\boldsymbol{i} + 14.7\,\text{kN·m}\,\boldsymbol{j} - 1.032\,\text{kN·m}\,\boldsymbol{k})$$

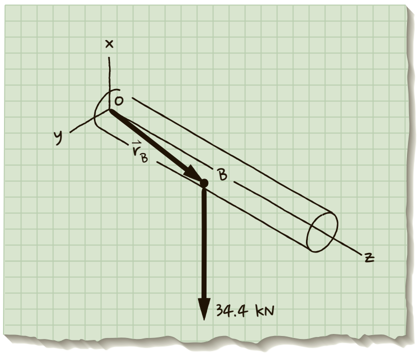


Figure 4 Define the position vector $\boldsymbol{r}_B$ from O to B.

Intermediate check: As you are proceeding through a calculation, check to make sure that the steps make sense. For example, check that the sign of each scalar component of $\boldsymbol{M}_A$ is reasonable, given the position of the force relative to the coordinate axes.

Moment $\boldsymbol{M}_B$ created by $\boldsymbol{F}_B$: Based on **Figure 4** we write the force vector position vector from O to B as:

$$\boldsymbol{r}_B = 0.03\text{ m}\,\boldsymbol{j} + 1.1\text{ m}\,\boldsymbol{k} \quad \text{and} \quad \boldsymbol{F}_B = -34.4\text{ kN}\,\boldsymbol{i}$$

We then calculate the moment that $\boldsymbol{F}_B$ creates at O to be

$$\boldsymbol{M}_B = \boldsymbol{r}_B \times \boldsymbol{F}_B = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0.0\text{ m} & 0.03\text{ m} & 1.1\text{ m} \\ -34.4\text{ kN} & 0.0\text{ kN} & 0.0\text{ kN} \end{vmatrix}$$

Expanding this becomes

$$\boldsymbol{M}_B = (-37.8\,\text{kN}\cdot m\boldsymbol{j} + 1.032\,\text{kN}\cdot m\,\boldsymbol{k})$$

The equivalent moment:

$$\boldsymbol{M}_{\text{equ@}O} = \boldsymbol{M}_A + \boldsymbol{M}_B = \overbrace{(23.3\boldsymbol{i} + 14.7\boldsymbol{j} - 1.032\,\boldsymbol{k})}^{M_A}\text{kN·m}$$

$$+\, M_B\,\overbrace{(-37.8\,\boldsymbol{j} + 1.032\,\boldsymbol{k})}\text{kN·m}$$

$$\boxed{\boldsymbol{M}_{\text{equ@}O} = (23.3\,\text{kN·m}\,\boldsymbol{i} - 23.2\,\text{kN·m}\,\boldsymbol{j})}$$

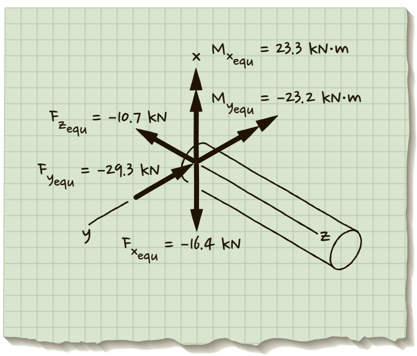


Figure 5 Equivalent loading at O.

Notice that there is no scalar moment about the z axis because the z scalar components from $\boldsymbol{M}_A$ and $\boldsymbol{M}_B$ cancel each other.

The equivalent force:

$$\boldsymbol{F}_{\text{equ}} = \boldsymbol{F}_A + \boldsymbol{F}_B = \overbrace{(18.0\boldsymbol{i} - 29.3\boldsymbol{j} - 10.7\boldsymbol{k})}^{F_A}\text{kN} + \overbrace{(-34.4\boldsymbol{i})}^{F_B}\text{kN}$$

$$\boldsymbol{F}_{\text{equ}} = (-16.4\,\text{kN}\,\boldsymbol{i} - 29.3\,\text{kN}\,\boldsymbol{j} - 10.7\,\text{kN}\,\boldsymbol{k})$$

The equivalent loading consisting of $\boldsymbol{M}_{\text{equ@}O}$ and $\boldsymbol{F}_{\text{equ}}$ is shown in **Figure 5**.

EXAMPLE 3.5.3

The rope wrapped around the frictionless pulleys at B and C in **Figure 1** applies a couple to the bracket. The tension in the rope is $\boldsymbol{T}$.

Show that you will calculate the same equivalent force and moment at A whether you treat the forces applied at B and C as individual forces, or use your knowledge of the characteristics of couples.

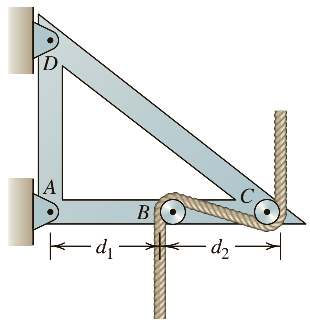


Figure 1 A rope applies a couple to a bracket.

Goal Determine the equivalent loading at a moment center at A using two approaches: (a) sum the contributions from individual forces, or (b) find the equivalent moment due to a couple

Given Key dimensions of the triangular bracket in terms of variables (d_1, d_2); the pulleys are frictionless.

Assume The forces that the rope applies to the bracket act in the vertical direction and are in the plane of the bracket. The radius of the pulleys is very small relative to d_1 and d_2.

Draw Based on the assumptions, we redraw the bracket in **Figure 2a**. Because the pulleys are frictionless, $\|\boldsymbol{T}_1\| = \|\boldsymbol{T}_2\| = \|\boldsymbol{T}\|$*. These two forces are equal in magnitude and opposite in direction, and therefore constitute a couple.

Formulate Equations and Solve (a) We will calculate the equivalent loads at A due to $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ separately. Then we will sum them to find the resultant equivalent loads at A. Force $\boldsymbol{T}_1$ creates a moment $\boldsymbol{M}_1$ at A of

$$\boldsymbol{M}_1 = -d_1\|\boldsymbol{T}\|\boldsymbol{k}$$

and $\boldsymbol{T}_2$ creates a moment $\boldsymbol{M}_2$ at A of

$$\boldsymbol{M}_2 = +(d_1 + d_2)\|\boldsymbol{T}\|\boldsymbol{k}$$

The equivalent load at A consists of $\boldsymbol{M}_{\text{equ}}$ and $\boldsymbol{F}_{\text{equ}}$ as shown in **Figure 2b**.

$$\boldsymbol{M}_{\text{equ}} = \boldsymbol{M}_1 + \boldsymbol{M}_2 \Rightarrow \boldsymbol{M}_{\text{equ}} = d_2\|\boldsymbol{T}\|\boldsymbol{k}$$

$$\boldsymbol{F}_{\text{equ}} = \boldsymbol{T}_1 + \boldsymbol{T}_2 = -\|\boldsymbol{T}\|\boldsymbol{j} + \|\boldsymbol{T}\|\boldsymbol{j} \Rightarrow \boldsymbol{F}_{\text{equ}} = 0$$

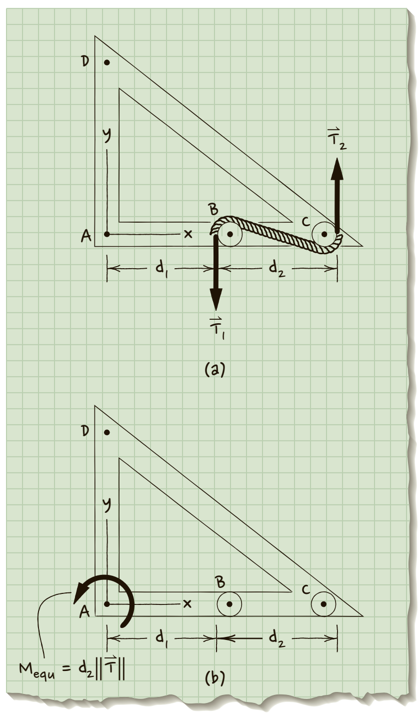


Figure 2 (a) The tension on the pulleys applies a couple to the bracket; (b) the equivalent moment at A.

*We will prove this in Chapter 5.

(b) Realizing that $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ form a couple in the xy plane, we know (because we have already learned about the characteristics of couples) that $\boldsymbol{F}_{\text{equ}} = 0$. We then use (3.23) to calculate the magnitude of the couple moment.

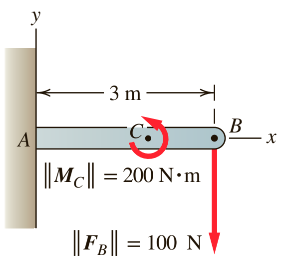
$$\left\|\boldsymbol{M}_{\text{couple}}\right\| = \|\boldsymbol{F}\|d = \|\boldsymbol{T}\|d_2$$

Using the right-hand rule, we determine that the moment is counterclockwise about the z axis (the positive $\boldsymbol{k}$ direction). Once we have calculated $\boldsymbol{M}_{\text{couple}}$ we can move it anywhere on the bracket, for example to point A.

$$\boldsymbol{M}_{\text{equ}} = d_2\|\boldsymbol{T}\|\boldsymbol{k} \quad \text{and} \quad \boldsymbol{F}_{\text{equ}} = 0$$

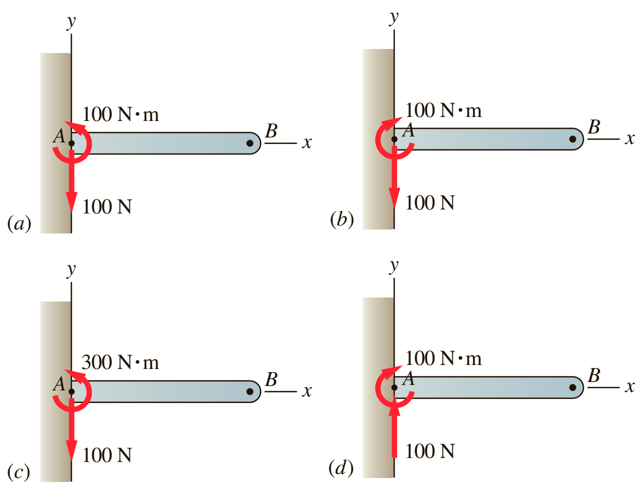
EXERCISES 3.5

3.5.1. [*] For the cantilever beam acted on by $\boldsymbol{F}_B$ and $\boldsymbol{M}_C$, replace the loads by the equivalent loading acting at A. Present your answer in vector notation and as a diagram that shows the equivalent force and moment acting at A.

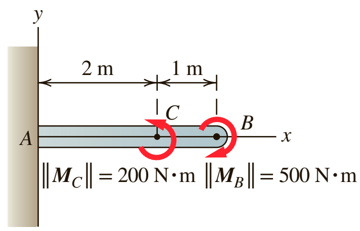


EX 3.5.1

Which of the following looks like your drawing?

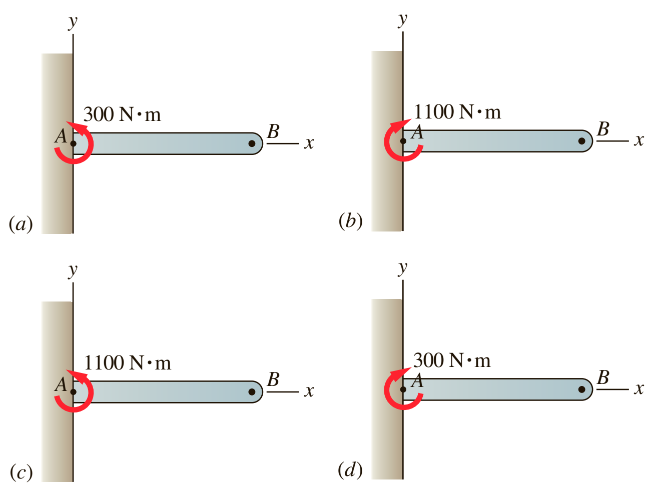


3.5.2. [*] For the cantilever beam acted on by moments $\boldsymbol{M}_B$ and $\boldsymbol{M}_C$, replace the loads by the equivalent loading acting at A. Present your answer in vector notation and as a diagram that shows the equivalent force and moment acting at A.

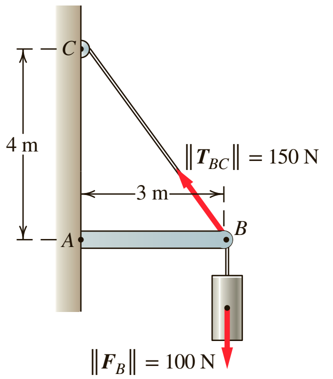


EX 3.5.2

Which of the following looks like your diagram?

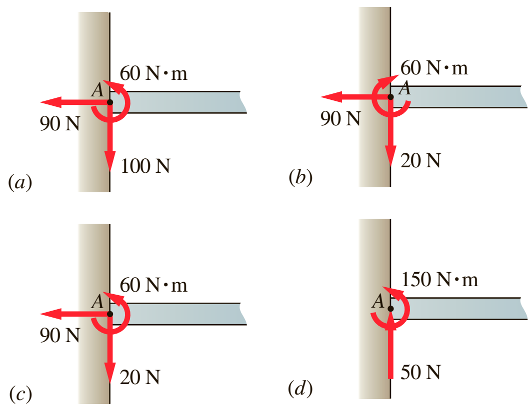


3.5.3. [*] For the cantilever beam acted on by $\boldsymbol{T}_{BC}$ and $\boldsymbol{F}_B$ replace the forces by the equivalent loading acting at A. Present your answer in vector notation and as a diagram that shows the equivalent force components and the moment acting at A.

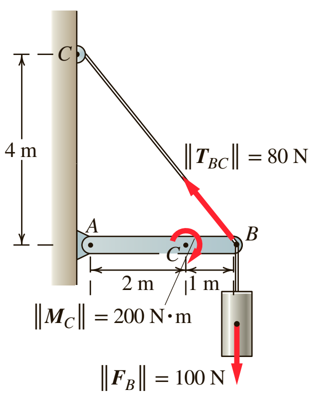


EX 3.5.3

Which of the following looks like your diagram?

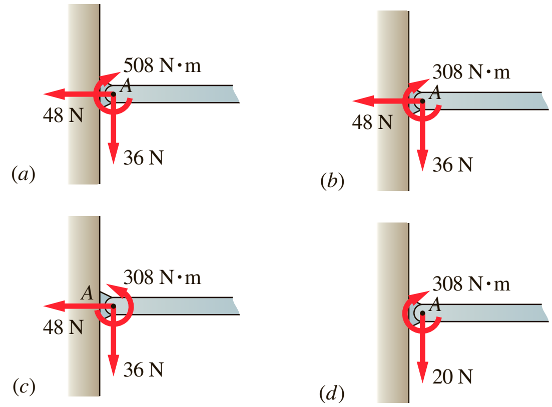


3.5.4. [*] For the beam pinned at A and acted on by $\boldsymbol{T}_{BC}$, $\boldsymbol{F}_B$, and $\boldsymbol{M}_C$, replace the loads by the equivalent load acting at A. Present your answer in vector notation and as a diagram that shows the equivalent force components and the moment acting at A.

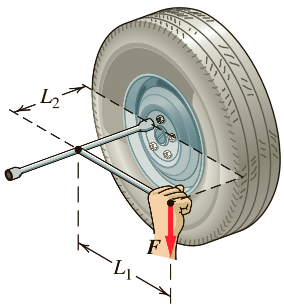


EX 3.5.4

Which of the following looks like your diagram?

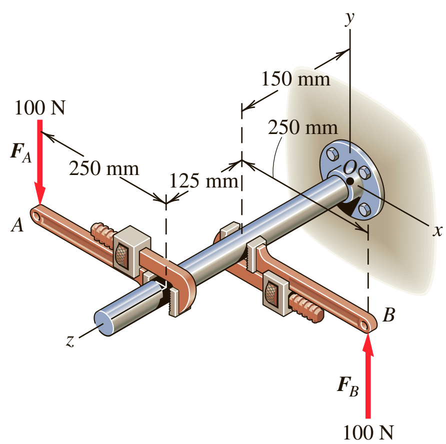


3.5.5. [*] A lug wrench is used to tighten a lug nut on an automobile wheel, as shown. Replace the force $\boldsymbol{F}$ applied by the mechanic's hand to the lug wrench by an equivalent loading at the center of the lug nut. Present your answer in vector notation and as a diagram that shows the equivalent force and moment acting at the center of the lug nut.



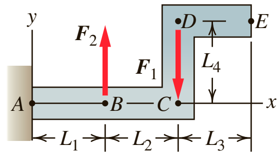
EX 3.5.5

3.5.6. [*] For the two pipe-wrench forces, determine the equivalent loading at O. Present your answer in vector notation.



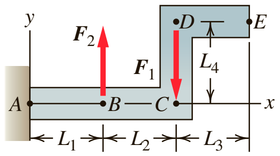
EX 3.5.6

3.5.7. [*] Consider the cantilever beam shown. The applied forces are of equal magnitude so that $\|\boldsymbol{F}_1\| = \|\boldsymbol{F}_2\| = F$. Determine the equivalent load at C. Present your answer in vector notation and as a diagram that shows the equivalent force and moment acting at C.



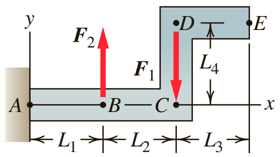
EX 3.5.7

3.5.8. [*] Consider the cantilever beam shown. The applied forces are of equal magnitude so that $\|\boldsymbol{F}_1\| = \|\boldsymbol{F}_2\| = F$. Determine the equivalent loading at B. Present your answer in vector notation and as a diagram that shows the equivalent force and moment acting at B.



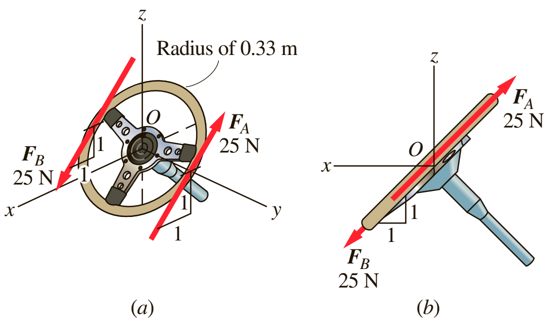
EX 3.5.8

3.5.9. [*] Consider the cantilever beam shown. The applied forces are of equal magnitude so that $\|\boldsymbol{F}_1\| = \|\boldsymbol{F}_2\| = F$. Determine the equivalent load that consists of horizontal forces applied at C and D. In addition to the magnitude, present a diagram that shows the equivalent forces acting at C and D.



EX 3.5.9

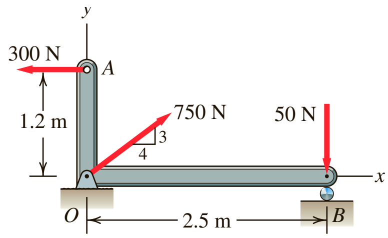
3.5.10. [*] For the steering wheel shown find the equivalent load at O for $\boldsymbol{F}_A$ and $\boldsymbol{F}_B$. Specify both $\boldsymbol{F}_{\text{equ}}$ and $\boldsymbol{M}_{\text{equ}}$ in vector notation, and show the equivalent load graphically.



EX 3.5.10

3.5.11. [**] Replace the three forces shown by the equivalent loading acting at O. Present your answer in vector notation and as a diagram that shows the equivalent force and moment acting at O.

Answer the following questions about your diagram. What is the magnitude of the equivalent force at O? What angle does the force make with the x axis? (Measure the angle positive counterclockwise from the positive x axis.)

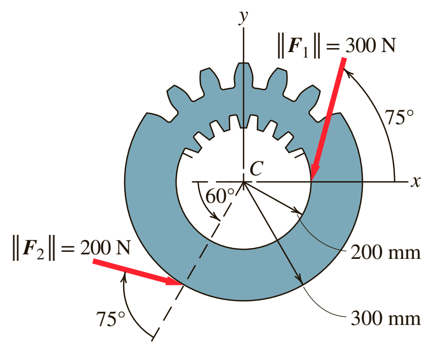


EX 3.5.11

3.5.12. [**] A stepped gear is subjected to two forces as shown. Determine the equivalent load acting at the center C of the gear. Present your answer in vector notation and as a diagram that shows the equivalent force and moment acting at C. Answer the following questions about your diagram.

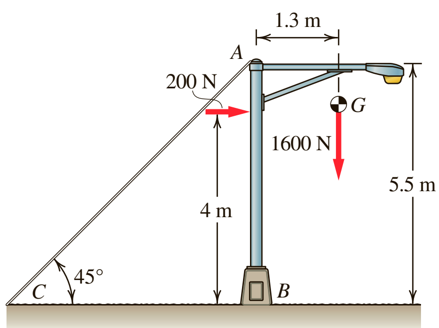
What is the magnitude of the equivalent force at C?

What angle does the force make with the x axis? (Measure the angle positive counterclockwise from the positive x axis)



EX 3.5.12

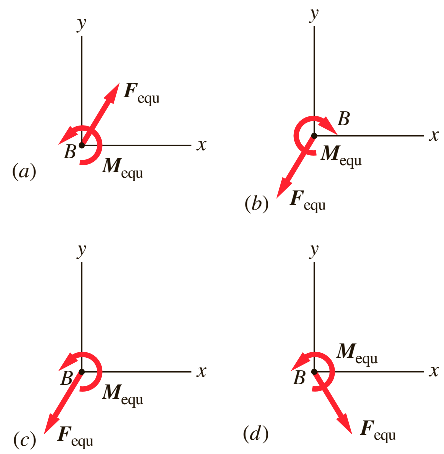
3.5.13. [**] The streetlight is supported by cable AC. The total weight of the structure is 1600 N acting at point G, and



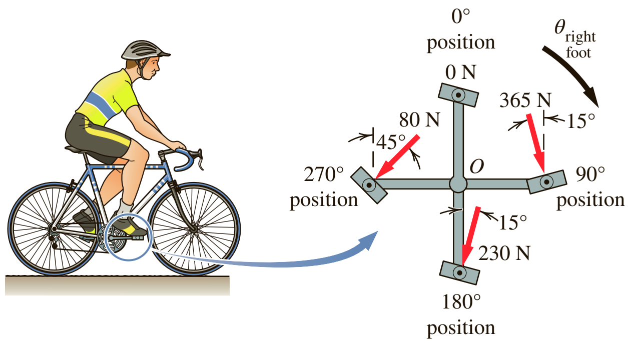
EX 3.5.13

the tension in the cable is 3600 N. The 200-N horizontal force represents the effect of the wind. Determine the equivalent loading at the base B. Present your answer in vector notation and as a diagram that shows the equivalent force and moment acting at B.

Which of the following looks like your diagram?



3.5.14. [**] For each of the right-pedal configurations shown, determine the equivalent load at the center of the bottom bracket axle. The length of the crankarm is 170 mm. Present your answers in the table.



EX 3.5.14

$\theta_{\text{right foot}}$	**Moment created by right foot at O (center of bottom bracket axle)**
90°	
180°	
270°	
0°	

Table 3.5.14

3.5.15. [**] Three parallel forces act on the plate shown.

a. Determine the equivalent load at O. Present your answer in vector notation.

b. Find the location in terms of x and y coordinates of a load equivalent to the one found in **a** that consists of an equivalent force and a *zero* equivalent moment. Present a drawing that shows your answer.

c. Intuitively, what does your answer in **b** tell you about the situation?

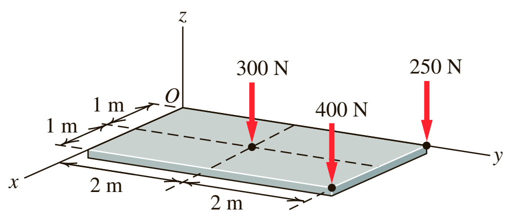
Check all that apply.

a. Nothing

b. The moment is zero, so the plate won't tip.

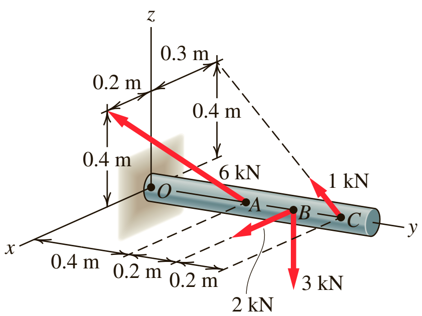
c. This is the balance point.

d. There is another unique point where the equivalent force is zero.



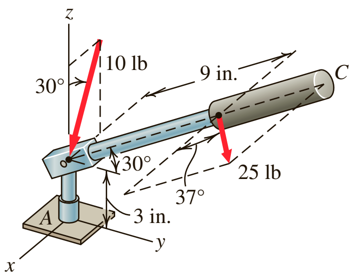
EX 3.5.15

3.5.16. [**] Forces are applied at points A, B, and C on the bar shown. Replace the forces with their equivalent load acting at O. Present your answer in vector notation.



EX 3.5.16

3.5.17. [**] A socket wrench is subjected to the forces shown. Determine the equivalent load acting at A. Present your answer in vector notation and determine the magnitude of the force and the moment.

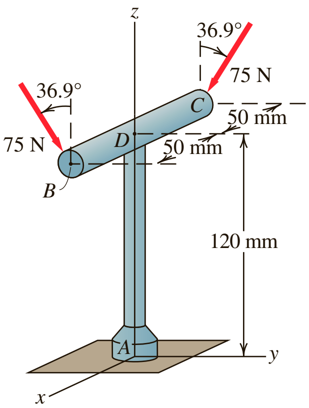


EX 3.5.17

3.5.18. [**] A T-bar tool is used to loosen a nut by applying 75-N forces at B and C, as shown.

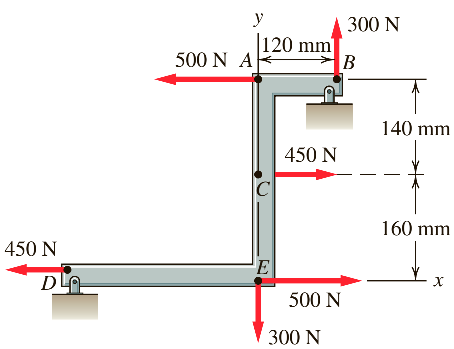
a. Determine the equivalent load acting at A.

b. If the T-bar were redesigned with the handle BDC twice as long, determine the magnitudes of the forces that you would need to apply at B and C to have the same equivalent load at A as calculated in **a**.



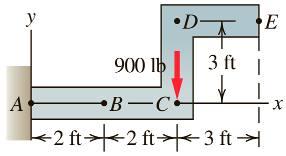
EX 3.5.18

3.5.19. [**] Three couples act on the bracket shown. Replace these three couples by a single couple consisting of forces acting at A and E.



EX 3.5.19

3.5.20. [**] A 900-lb force acts at C on the cantilever beam shown.



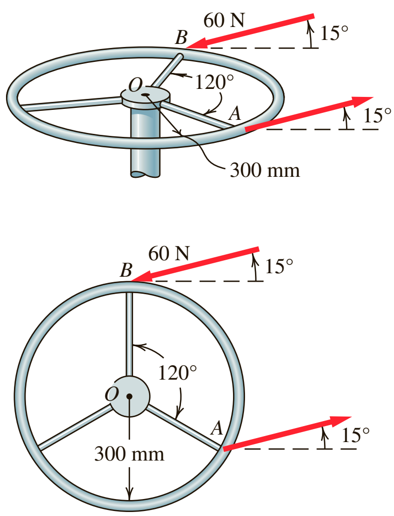
EX 3.5.20

a. Determine the equivalent load consisting of a force and a moment at A. Present your answer in vector notation and as a drawing.

b. Determine the equivalent load consisting of a force at A and a couple made up of forces applied at C and D. Present your answer as a drawing.

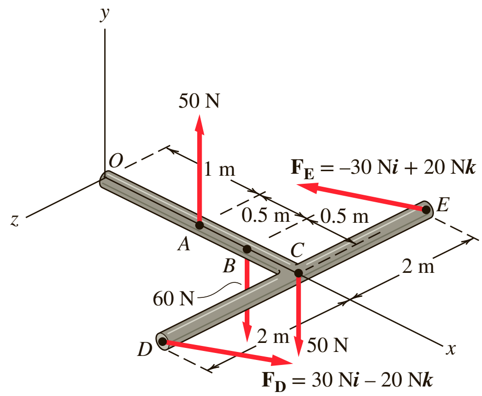
c. Determine the equivalent load consisting of a force at A and a couple consisting of forces applied at C and E. Present your answer as a drawing.

3.5.21. [**] Initially a bus driver applies two 60-N forces to the steering wheel shown. These two forces are a couple and are in the plane of the steering wheel. The bus driver moves his hands onto the steering wheel spokes OA and OB and applies forces with the same orientation as before. He now has to apply 120 N to create the same moment. Where on the spokes has the bus driver put his hands?



EX 3.5.21

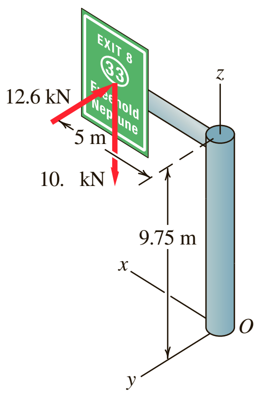
3.5.22. [**] The loadings acting on the T-joint consist of two couples and a force. Find the equivalent force and equivalent moment at B for these loads. Present your answer in vector form.



EX 3.5.22

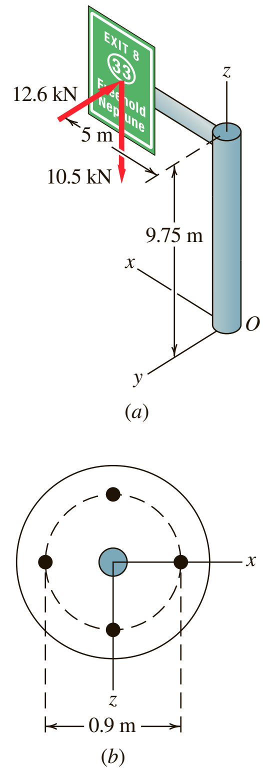
3.5.23. [**] The wind blowing on the freeway exit sign and the weight of the sign can be represented by point loadings acting at the center of the sign, as shown in the drawing. If the force of the wind is 12.6 kN and the sign weighs 10.5 kN, find the equivalent moment at O.

AA Roads.com (Alex Nitzman, Andy Field)



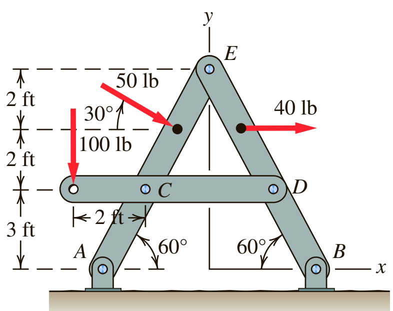
EX 3.5.23

3.5.24. [**] The base of the freeway sign in **Figure *a*** is attached to its foundation with four bolts **Figure *b***. The four bolts must be able to resist the twisting of the pole that is caused by the wind. Assume each bolt carries the same force. If the diameter of the bolt circle is 0.9 m, calculate the horizontal (shear) force on each of the bolts due to the twisting moment about the y axis.



EX 3.5.24

3.5.25. [***] Replace the three forces shown by the equivalent load acting at A. Present your answer in vector notation and as a diagram that shows the equivalent force and moment acting at A.



EX 3.5.25

3.6 JUST THE FACTS

What Is a Moment?

A **moment** is created by a force offset from a point in space called a **moment center** and is the tendency of the force to cause rotation. Moment is a vector quantity and is specified in terms of magnitude and direction. In drawings we use an arc or double-headed arrow to denote a moment.

The magnitude of a moment is given by

$$\|\boldsymbol{M}\| = \|\boldsymbol{r}\|(\|\boldsymbol{F}\|\sin\theta) \tag{3.1}$$

where $\|\boldsymbol{r}\|$ is the magnitude of the **position vector** (any vector from the moment center to the line of action of $\boldsymbol{F}$), $\|\boldsymbol{F}\|$ is the magnitude of the force vector, and θ is the angle between the position vector and force vector when these two vectors are placed tail to tail such that $0° < \theta < 180°$ (Figure 3.6.1).

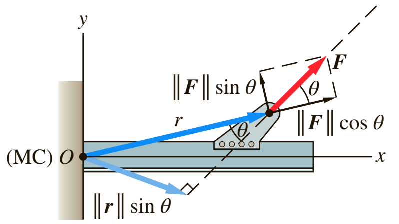


Figure 3.6.1 When the position vector and force vector are placed tail to tail, θ is the angle between $\boldsymbol{r}$ and $\boldsymbol{F}$.

- For a particular moment center, the magnitude of the moment is independent of the position vector.
- A force creates no moment about a moment center located anywhere on the line of action of the force.

The **direction** of the moment is perpendicular to the plane defined by the position vector $\boldsymbol{r}$ and the force $\boldsymbol{F}$, and is defined using the right-hand rule. This direction can also be interpreted as the axis about which the moment acts. If we establish a local coordinate system at the moment center, we define the **sense** of the moment (clockwise or counterclockwise) by standing on the positive axis and looking back at the moment center. The moment created by a force is defined by its magnitude (3.1) and a direction or sense. In vector notation a moment can be written as:

$$\boldsymbol{M} = M_x\boldsymbol{i} + M_y\boldsymbol{j} + M_z\boldsymbol{k} \tag{3.2}$$

Mathematical Representation of a Moment

A moment can be represented in terms of its scalar components:

$$\boldsymbol{M} = M_x\boldsymbol{i} + M_y\boldsymbol{j} + M_z\boldsymbol{k} \tag{3.3}$$

The magnitude of the moment $\|\boldsymbol{M}\|$ is

$$\|\boldsymbol{M}\| = \sqrt{M_x^2 + M_y^2 + M_z^2} \tag{3.4}$$

The moment's direction can be described by a unit vector $\boldsymbol{u}$:

$$\boldsymbol{u} = \cos\theta_x\,\boldsymbol{i} + \cos\theta_y\,\boldsymbol{j} + \cos\theta_z\,\boldsymbol{k} \tag{3.6}$$

with direction cosines defined as

$$\cos\theta_x = \frac{M_x}{\|\boldsymbol{M}\|} \quad \cos\theta_y = \frac{M_y}{\|\boldsymbol{M}\|} \quad \cos\theta_z = \frac{M_z}{\|\boldsymbol{M}\|} \tag{3.5}$$

We can rewrite the expressions for moment in (3.4), (3.5), and (3.6) in terms of the force vector ($\boldsymbol{F}$) and position vector ($\boldsymbol{r}$) as the vector product, or cross product, of the position vector and the force

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = (\|\boldsymbol{r}\|\ \|\boldsymbol{F}\| \sin\theta)\,\boldsymbol{u} \tag{3.7A}$$

where $\boldsymbol{u}$ is a unit vector perpendicular to the plane defined by $\boldsymbol{r}$ and $\boldsymbol{F}$. This expression is read, "$\boldsymbol{r}$ crossed with $\boldsymbol{F}$." The cross product is not commutative, meaning that $\boldsymbol{r} \times \boldsymbol{F}$ is not equal to $\boldsymbol{F} \times \boldsymbol{r}$.

It is generally more convenient to work with the position vector and force in terms of their components.

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = \underbrace{(+r_yF_z - r_zF_y)}_{M_x}\boldsymbol{i} + \underbrace{(+r_zF_x - r_xF_z)}_{M_y}\boldsymbol{j} + \underbrace{(+r_xF_y - r_yF_x)}_{M_z}\boldsymbol{k} \tag{3.8}$$

Equation (3.8) is an expression of **Varignon's Theorem**.

More generally, the cross product of the position vector and the force can be written in matrix form as

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \tag{3.12}$$

The form in (3.12) specifies that the **determinant** of the matrix be taken.

We can use the moment scalar components in (3.8) to rewrite the direction cosines in (3.5) that define the direction of the moment:

$$\begin{aligned} \cos\theta_x &= \frac{M_x}{\|\boldsymbol{M}\|} = \frac{+r_yF_z - r_zF_y}{\|\boldsymbol{M}\|} \\ \cos\theta_y &= \frac{M_y}{\|\boldsymbol{M}\|} = \frac{+r_zF_x - r_xF_z}{\|\boldsymbol{M}\|} \\ \cos\theta_z &= \frac{M_z}{\|\boldsymbol{M}\|} = \frac{+r_xF_y - r_yF_x}{\|\boldsymbol{M}\|} \end{aligned} \tag{3.10}$$

where the magnitude of the moment is

$$\|\boldsymbol{M}\| = \sqrt{\underbrace{(+r_yF_z - r_zF_y)}_{M_x}^2 + \underbrace{(+r_zF_x - r_xF_z)}_{M_y}^2 + \underbrace{(+r_xF_y - r_yF_x)}_{M_z}^2} \tag{3.9}$$

The direction of the moment is defined by the unit vector $\boldsymbol{u}$, based on the direction cosines in (3.10), as

$$\boldsymbol{u} = \frac{+r_yF_z - r_zF_y}{\|\boldsymbol{M}\|}\boldsymbol{i} + \frac{+r_zF_x - r_xF_z}{\|\boldsymbol{M}\|}\boldsymbol{j} + \frac{+r_xF_y - r_yF_x}{\|\boldsymbol{M}\|}\boldsymbol{k} \tag{3.11}$$

The expression for moment in (3.8) can be simplified if the position and force vectors lie in the same *xy*, *yz*, or *zx* plane:

For a force vector and position vector lying in the same *xy* plane, we can write

$$\boldsymbol{M}_z = M_z\boldsymbol{k} = (+r_xF_y - r_yF_x)\boldsymbol{k} \tag{3.13A}$$

For a force vector and position vector lying in the same *yz* plane, we can write

$$\boldsymbol{M}_x = M_x\boldsymbol{i} = (+r_yF_z - r_zF_y)\boldsymbol{i} \tag{3.13B}$$

For a force vector and position vector lying in the same *zx* plane, we can write

$$\boldsymbol{M}_y = M_y\boldsymbol{j} = (+r_zF_x - r_xF_z)\boldsymbol{j} \tag{3.13C}$$

Finding Moment Components in a Particular Direction

The dot product is useful in finding the component of a moment in a particular direction (or, alternately worded, about a particular axis). If $\boldsymbol{u}$ is the unit vector in the direction of a particular line or axis, then

$$M_{axis} = \underbrace{\boldsymbol{M} \bullet \boldsymbol{u}}_{\substack{\text{projection of} \\ \boldsymbol{M}\text{ in the direction} \\ \text{of unit vector }\boldsymbol{u}}} = M_xu_x + M_yu_y + M_zu_z \tag{3.14}$$

If we represent the moment as a cross product, M_{axis} can be written as

$$M_{\text{axis}} = \boldsymbol{u} \bullet (\boldsymbol{r} \times \boldsymbol{F}) = \begin{vmatrix} u_x & u_y & u_z \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \tag{3.18}$$

Couples

Two parallel forces of equal magnitude and opposite sign are called a **couple** and create a **couple moment**, which acts perpendicular to the plane created by the couple. Calculation of a couple moment is *independent* of the moment center used. This means we can calculate the moment created by the couple at any moment center in space, and the result will be the same:

$$\boldsymbol{M}_{\text{couple}} = \boldsymbol{r} \times \boldsymbol{F} \tag{3.22}$$

The magnitude of a couple can be determined by multiplying the magnitude of the applied force by the perpendicular distance *d* between the two forces:

$$\left\|\boldsymbol{M}_{\text{couple}}\right\| = \|\boldsymbol{F}\|d \tag{3.23}$$

An important property of a couple is that we can move it anywhere in the plane of the couple, or in a parallel plane, without changing the effect of the couple.

Equivalent Loads

The **equivalent load principle** states:

> *For any load (consisting of forces and/or moments) applied to a system, we can replace that load with an equivalent load (consisting of a force and a moment) applied at a single point.*

A special case of equivalent loads is **equivalent couples**. Two couples are said to be equivalent if they produce the same moments (magnitude and direction). A couple moment is the equivalent load for a couple.

SYSTEM ANALYSIS (SA) EXERCISES

SA3.1 Consideration of Left- and Right-Foot Pedaling

Multiple right pedal positions for a cyclist are shown in **Figure SA3.1.1**. The bicycle would be moving to the right.

(a) For $\theta_{\text{right foot}}$ equal to 90°, 180°, 270°, and 0°, determine the moment at O (which is the center of the bottom bracket) created by the foot force. Present your answers in column A of **Table SA3.1.1**.

(b) The action of the left pedal and foot is offset from that of the right pedal and foot by 180°. This means that, for example, when the right foot is applying 365 N at $\theta_{\text{right foot}} = 90°$, the left foot is at 270° applying 80 N. Based on this information, complete column B of **Table SA3.1.1**.

(c) Add columns A and B to determine the total moment acting at O due to simultaneously pedaling with the right foot and the left. Present your answers in column C.

(d) Based on your results in **Table SA3.1.1**, what is the maximum moment created in cycling? At what $\theta_{\text{right foot}}$ angle does it occur?

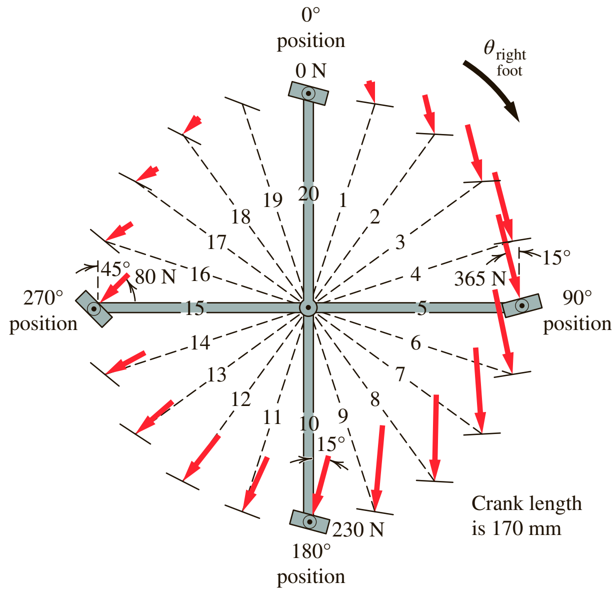


Figure SA3.1.1 Right foot force acting on bicycle pedal in various crank arm positions. Bicycle is moving to the right.

Table SA3.1.1 Summary of Moments in Various Angular positions

Angular position $\theta_{\text{right foot}}$	A Moment at O created by right foot	B Moment at O created by left foot	C Total moment at O created by pedaling with both feet
90°			
180°			
270°			
0°			

SA3.2 Vehicle Recovery: Attempt 1

In the spring of 1996, an M1 U.S. Army tank (**Figure SA3.2.1**) slid off a mountain road while on a training exercise in the Republic of Korea. The road was an unimproved dirt trail along a steep slope. The tank became precariously balanced on the edge and had to be pulled back onto the road to be recovered. Two M88 recovery vehicles (**Figure SA3.2.2**) responded and began recovery operations.

Recovery operations such as this are dangerous and require careful planning. The M1 tank weighs approximately 70 tons, and the forces in the recovery cables are significant.

Sandy Schaeffer/Mai/The LIFE Images Collection/Getty Images, Inc.

Figure SA3.2.1 M1 tank.

US Army

Figure SA3.2.2 M88 recovery vehicle.

(a) In **Figure SA3.2.3**, M88 1 is shown pulling with 10 kip of force at 70° from the horizontal; and M88 2 is pulling with 15 kip at 60° from the horizontal. Both cables are attached to the front left tow hook of the M1 tank (Point A). Calculate the resultant force vector due to the two M88 recovery vehicle forces.

(b) Assume that M88 1 cannot pull with a greater force because of its position on the road and M88 2 can pull with an increasingly greater force as needed. Graph the relationship between the magnitude of the resultant force and the force applied by M88 2. Assume that M88 2 can pull with 0 to 50 kip. Discuss the resulting graph.

(c) Assume that the applied forces by the recovery vehicles are limited to a constant magnitude. Recommend a physical change to the recovery operation that will increase the magnitude of the resultant force on the M1 tank.

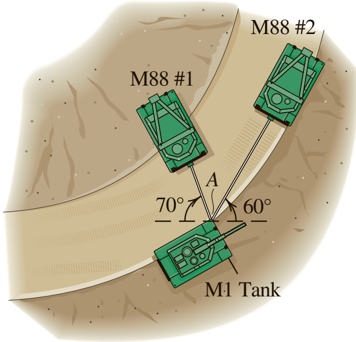


Figure SA3.2.3 Recovery operation

(d) Describe what you think will physically happen when the recovery vehicles actually apply their forces to the front left tow hook of the M1 tank. Assume that the applied forces are sufficient to cause the M1 tank to slide.

SA3.3 Vehicle Recovery: Attempt 2

Read **SA3.2** for background information.

The recovery operation from SA3.2 was unsuccessful. The M1 tank would rotate counterclockwise when the forces from the M88 recovery vehicles were applied to the front left tow hook of the tank, as shown in **Figure SA3.2.3**. This tendency for rotation caused the M1 tank to continue to slide off the road and potentially down the slope. A third M88 recovery vehicle was ordered to join the recovery operation. The third M88 was positioned behind the M1 tank as shown in **Figure SA3.3.1**.

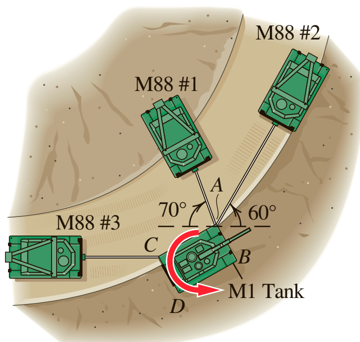


Figure SA3.3.1 Modified recovery operation

(a) Discuss attaching the third tow cable to each of the four tow hooks A through D on the corners of the M1 tank. Which tow hook position do you recommend and why? Consider the mechanical advantage and practicality of each tow hook position.

(b) Assume that M88 1 applies a 10-kip force at 70° from the horizontal, M88 2 applies a 15-kip force at 60° from the horizontal, and M88 3 applies a 20-kip force along the horizontal through the M1 tank's center of gravity, G, as shown in **Figure SA3.3.2**. Calculate the tendency of each of the recovery forces to cause rotation about the M1 tank's center of gravity. Assume that the center of gravity is located at the geometric center of the tank's area as shown in **Figure SA3.3.2**.

(c) Describe what you think will physically happen when the forces shown in **Figure SA3.3.2** are applied to the M1 tank. Assume they are sufficient to cause the tank to slide.

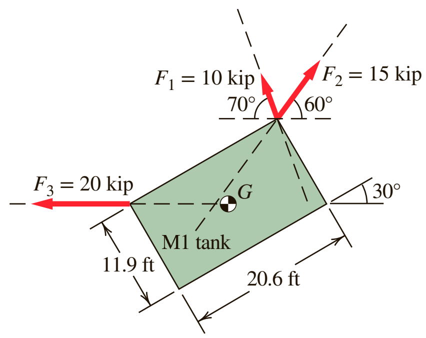


Figure SA3.3.2 Rotation about center of gravity.

SA3.4 Too Much Moment Can Topple a Crane

History of Cranes

Cranes have long been used to assist humans in building structures. For example, the Ancient Egyptians, Greeks, and Romans created cranes with which to build bridges and statues. The first mobile cranes, made of wood and iron, were invented so that military personnel could get over large walls without having to climb them, as shown in **Figure SA3.4.1*a***.

The Italian genius Leonardo Da Vinci (1452–1519) was extremely interested in cranes and helped advance the technology from simple tower cranes to large semi-mobile slewing cranes, all still built out of lumber and iron, as shown in **Figure SA3.4.1*b***. The rollers on which the slewing plate moved increased the productivity of those types of cranes dramatically—the cycle time for one lift was shortened due to the easy rotation of the jib around the main mast (= slewing).

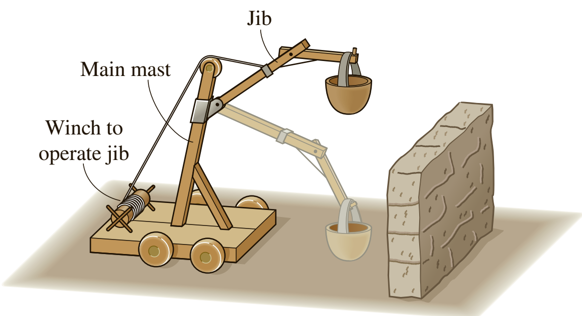
The invention of steel, wire rope, and steam engines during the middle of the 19th century did much to advance transportation systems (e.g., canals, railways, harbors) and the construction technologies used to create these systems. In particular, the invention of wire rope in 1834 (also known as cable) and steam engines enabled the design of cranes with larger load capacities and increased mobility.

At the time the Reynolds Coliseum was constructed in 1949, derrick cranes were in use. The roof of the building includes heavy precast concrete panels placed on top of the steel beams that span the large open interior space, as shown in **Figure SA3.4.2**. The panels were hoisted from the ground and then stored on the center section of the roof before being placed one by one onto the beams by one of the two derrick cranes. **Figure SA3.4.3** illustrates the basic elements and operation of a derrick crane. The placing sequence of each of the 4.0-kN precast concrete panels consisted of the following four steps: (1) hooking the panel to the spreader bar of the crane, (2) hoisting to the proper height and the start of slewing, (3) slewing, and (4) luffing of the jib (changing the boom angle) and placing the panel at the proper location.

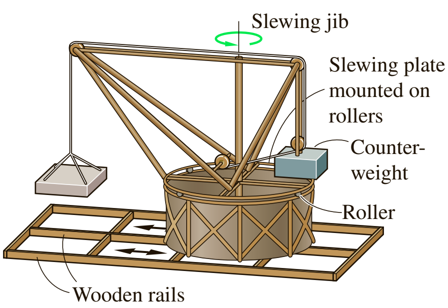
Modern Technology

Construction technology has further evolved since 1949. One example is the emergence of the tower crane presented in **Figure SA3.4.4**. The horizontal jib, also called the saddle jib, is supported by fixed cables that tie into the counterjib with ballast. The entire saddle jib, which is able to cover a large circle, rests on a slewing ring, similar to Leonardo's design. In turn, the slewing ring, located at *C* in **Figure SA3.4.4**, sits atop a fixed tower that has the capability to climb higher as the building grows. This economical design became very popular in the 1960s in Europe for the construction of apartment buildings in tight spaces.

This tower crane design has since spread all over the world.



(*a*)



(*b*)

Figure SA3.4.1 Early engineered cranes for military and construction, made of wood and iron: (*a*) first mobile cranes in 15th century; (*b*) Da Vinci's first real slewing crane with counterweight.

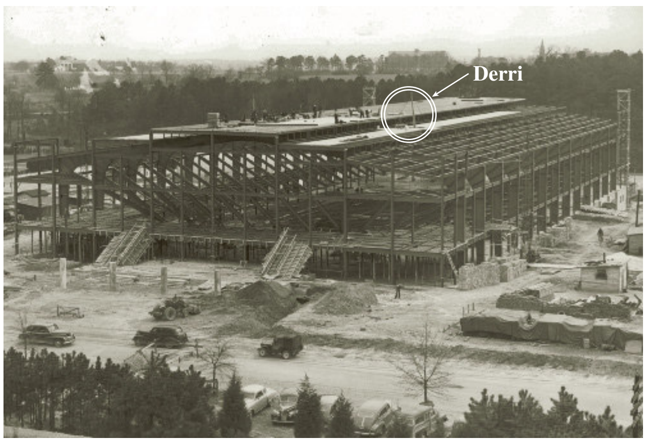


Figure SA3.4.2 Placement on concrete panels with help of two derricks on March 17, 1949.

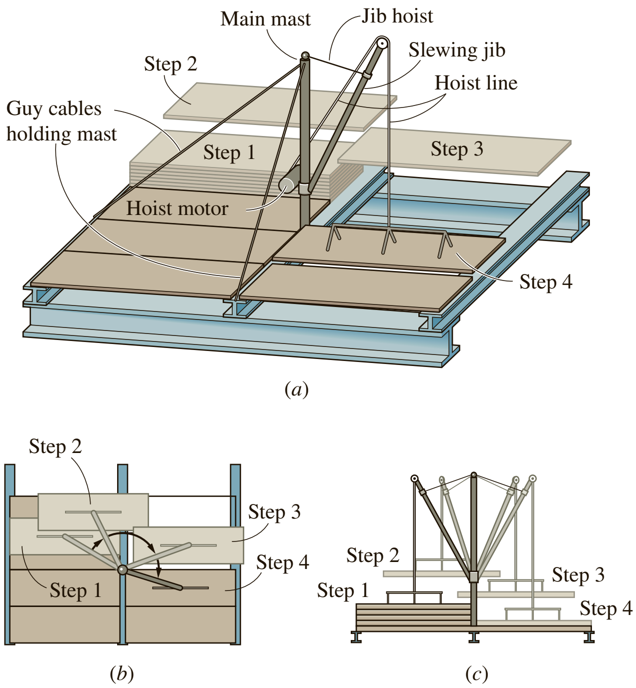


Figure SA3.4.3 Placing roof panels with a derrick crane: (*a*) layout of placing panels with derrick onto coliseum roof structure; (*b*) sequence of picking and placing one panel (top view); (*c*) sequence of picking and placing one panel (side view).

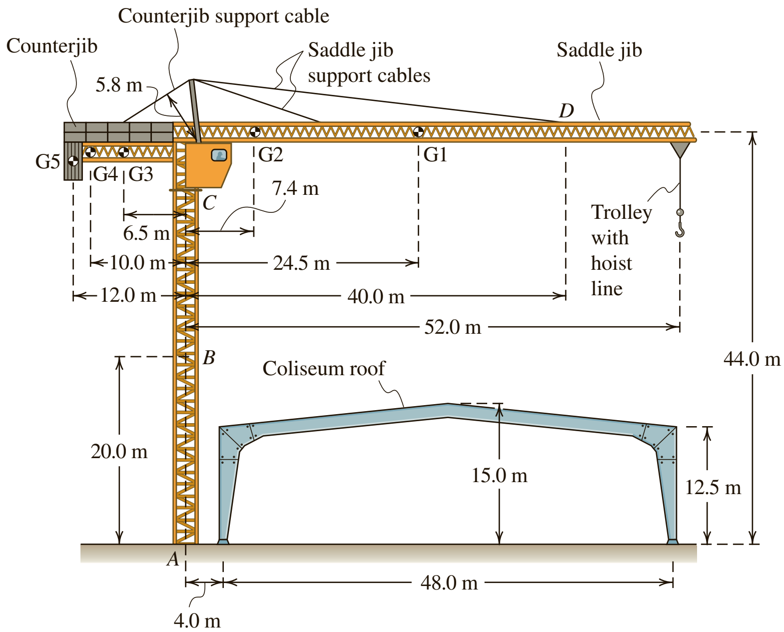


Figure SA3.4.4 Layout for placing concrete panels with fixed tower crane (Option A).

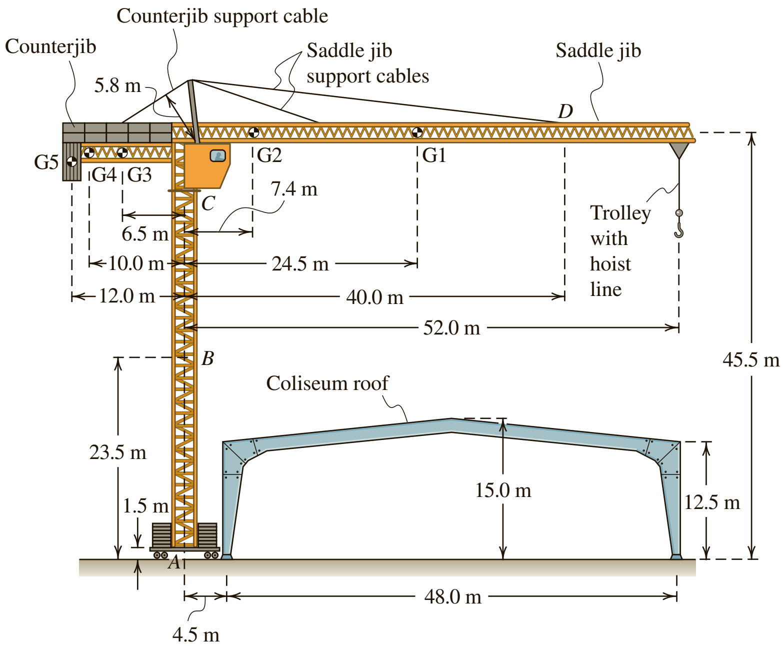


Figure SA3.4.5 Tower crane Option B on rails along the side of the building.

Your Assignment

Assume that you are hired as a summer intern by a company that won a contract to replace all the concrete panels on the Reynolds Coliseum. Your knowledge of moments is critical for your next assignment. Your boss needs your help in deciding which crane arrangement he should select. He wants you to compare two crane setups, Option A (**Figure SA3.4.4**) and Option B (**Figure SA3.4.5**).

Each panel measures 2.0 m long by 0.8 m wide and weighs 4.0 kN. **Figures SA3.4.4** and **SA3.4.5** provide the dimensions for each crane option in meters. The allowable, or rated, loading is smaller than the theoretical lifting capacity by a safety factor to allow for unforeseen conditions and imperfections in realworld components and materials. The rated loadings at 52.0 m (end of the saddle jib) are 6.0 kN for Option A and 4.0 kN for Option B. **Table SA3.4.1** provides additional data for both cranes.

Your boss would like to know which crane option he should use, considering that the building is approximately 98 m long. Since you are new to this job, he provides you with a step-by-step guideline on how to tackle this kind of problem. He expects you to do this on your own the next time around. Here is his “cookbook”:

Table SA3.4.1 Weight of Various Crane Elements

Label	Crane Element	Option A Rated 6.0 kN at 52.0 m Weight of Element (kN)	Option B Rated 4.0 kN at 52.0 m Weight of Element (kN)
G1	Saddle jib center of gravity	18.0	12.0
G2	Jib support cables center of gravity	9.4	6.3
G3	Counterjib center of gravity	17.1	11.3
G4	Counterjib winches and cables center of gravity	6.5	4.3
G5	Ballast center of gravity	45.0	30.0

I. Assessment of Option A

1. Draw the area this crane option can cover, to scale, onto the "footprint" of the building. Determine what area you will be able to cover with option A. (*Hint:* You could place the base of crane A inside the building and remove its tower after completion of the coliseum with a separate truck crane.)
2. Where would you stage/store the precast panels before hooking them to the hoist line to be placed on the roof? Show this location on your footprint drawing.
3. Redraw the crane showing all the forces at G1–G5 acting on it (**Table SA3.4.1**). The forces at G1–G5 are referred to as the "dead weights" because they are due to the weight of the crane itself.
4. Based on your drawing in 3, calculate the equivalent loading (consisting of an equivalent force and an equivalent moment) at
   (a) a moment center at A
   (b) a moment center at B
   (c) a moment center at C
   (d) a moment center at D
5. Repeat the calculation in 4 to include the rated loading listed in **Table SA3.4.1** (in addition to the dead loads). Summarize your answers from 4 and 5 in **Table SA3.4.2**.
6. Based on your answers in **Table SA3.4.2**, calculate the ratio of M_{equ}/F_{equ} and add this to **Table SA3.4.2**. Would you expect this ratio to ever be greater than or equal to 52.0 m? Why or why not?

II. Assessment of Option B

Same questions as for Option A.

III. Comparing the Two Options

1. Based on your footprint diagrams, would you recommend one option over the other? Why? What questions do you still have about the two options?
2. The crane represents an assembly of space frames held together by bolts and nuts. Locations A and B are most critical when dimensioning and maintaining the bolts. If the cross-sectional area of both towers is a square measuring 1.6 m by 1.6 m (**Figure SA3.4.6**), what is the maximum possible tension force that a bolt at each corner will experience beyond its preload (neglect the weight of the tower itself)? Which crane might possibly need two sets of bolts at each corner? Support your answer with your numerical calculations.

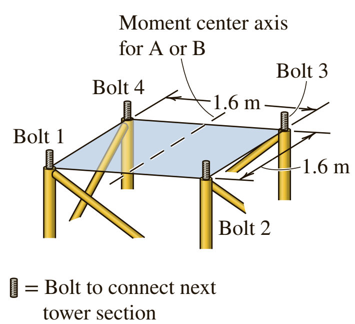
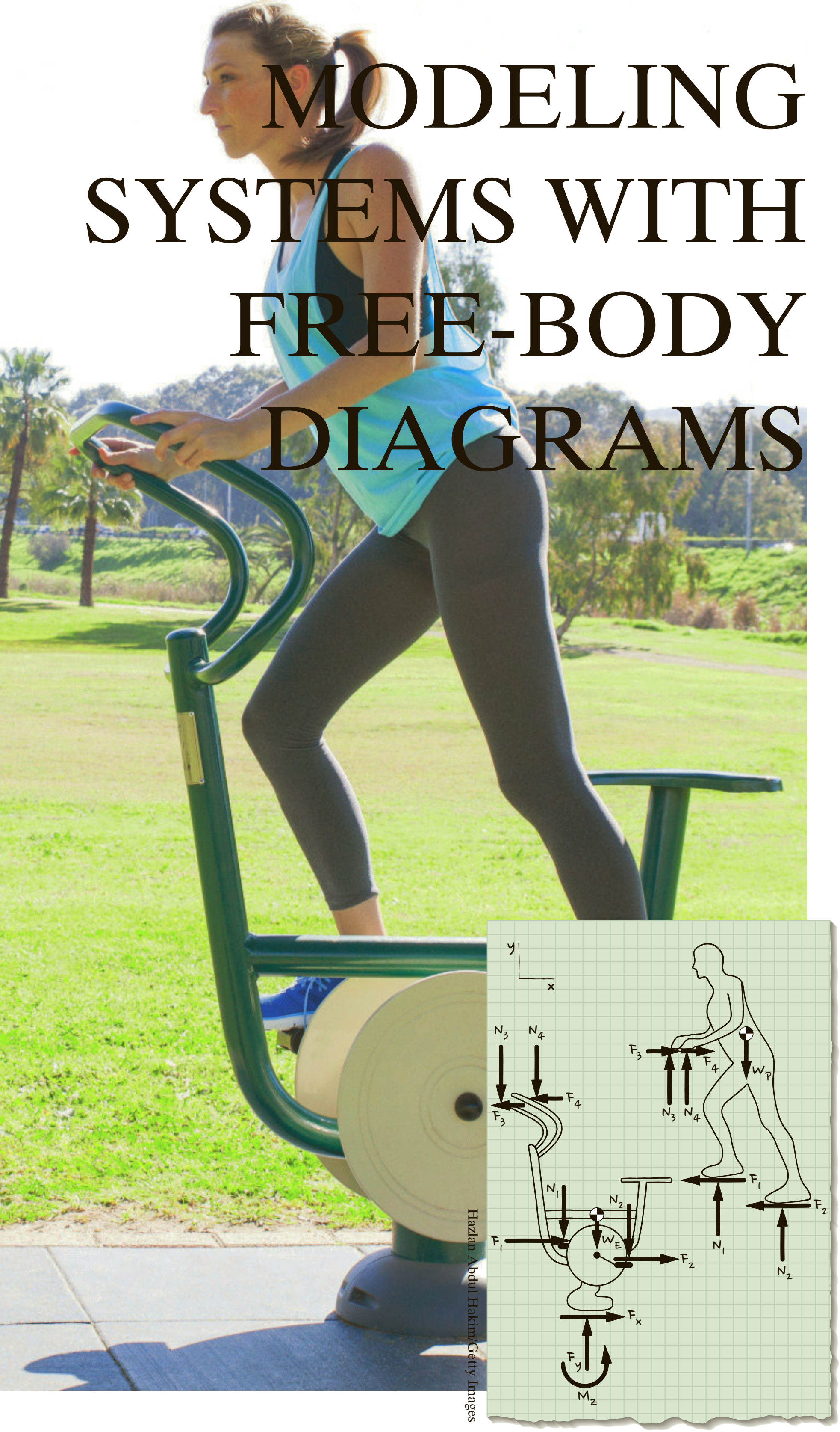


Figure SA3.4.6 Simple sketch of cross section A-A and B-B.

Table SA3.4.2 Summary of Crane Option Calculations

	Option A			Option B		
Moment Center at	**M_{equ} and F_{equ} due to Dead Loads**	**M_{equ} and F_{equ} due to Dead Loads and Rated Loads**	**M_{equ}/F_{equ} due to Dead Loads and Rated Loads**	**M_{equ} and F_{equ} due to Dead Loads**	**M_{equ} and F_{equ} due to Dead Loads and Rated Load**	**M_{equ}/F_{equ} due to Dead Loads and Rated Load**
A						
B						
C						
D						

MODELING SYSTEMS WITH FREE-BODY DIAGRAMS



Hazlan Abdul Hakim/Getty Images

The **free-body diagram** is the most important tool in this text. It is a drawing of a system and the loads acting on it. A four-step process is used to create a free-body diagram.

First, you mentally separate the system (the portion of the world you are interested in) from its surroundings (the rest of the world). Second, you draw a simplified representation of your system, noting any assumptions you have made when creating your drawing. Third, you identify all the known loads (forces and moments) acting on the system and add them to the drawing. Finally, you draw the unknown loads acting on the system at the supports or other boundaries.

This chapter is devoted exclusively to the task of creating free-body diagrams. We build on the coverage of forces (Chapter 2) and moments (Chapter 3) and on the engineering analysis procedure presented in Chapter 1. In Chapter 5 you will learn how to solve for unknown loads in your free-body diagrams.

FOUR STEPS TO CREATING A FREE-BODY DIAGRAM

1	**Study the physical situation**
2	**Draw system; state assumptions**
3	**Draw known loads**
4	**Identify and draw support loads**

LEARNING OBJECTIVES

On completion of this chapter, you will be able to:

- **Describe how external loads, including distributed loads, are depicted on free-body diagrams.** (4.1)
- **Identify the standard supports for planar systems and how they are represented on a free-body diagram.** (4.2)
- **Identify the standard supports for nonplanar systems and how they are represented on a free-body diagram.** (4.3)
- **Determine by inspection whether a system should be modeled as planar or nonplanar.** (4.4)
- **Apply a step-by-step approach to creating free-body diagrams of various systems.** (4.5)

4.1 TYPES OF EXTERNAL LOADS ACTING ON SYSTEMS

Learning Objective: Describe how external loads, including distributed loads, are depicted on free-body diagrams

Depending on the nature of an external load acting on a system, it is represented on a free-body diagram by either a vector acting at a point of application or as a distributed load acting on an area. The load is given a unique variable label, and its magnitude (if known) is written next to the vector.

In actuality, all supports consist of forces distributed over a finite surface area. For example, if you press down on a table with your hand, the force you apply to the table is distributed over a finite area (**Figure 4.1.1*a***). For many practical applications, we can "condense" this distributed force into an equivalent point force (**Figure 4.1.1*b***).

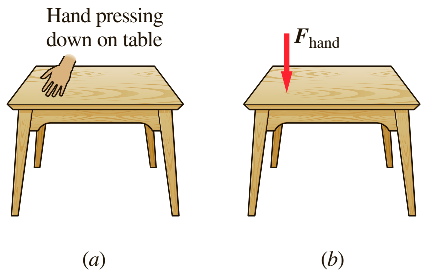


Figure 4.1.1 Hand pressing down on a table modeled as a point force.

There are, however, some situations for which we explicitly consider the loads to be distributed; **Figure 4.1.2** shows some examples. As with all loads acting on a system, distributed forces must be included in the system's free-body diagram. They can be represented in the diagram either as distributed forces (**Figure 4.1.3*b***) or as an equivalent point force (**Figure 4.1.3*c***). This equivalent point force is the total force represented by the distributed force and is located so as to create the same moment as the distributed force. For the uniformly distributed load in **Figure 4.1.3*a*** we are able to find this location by inspection. In Chapter 6 we show how to find the location for nonuniformly distributed forces. The important point to remember for your

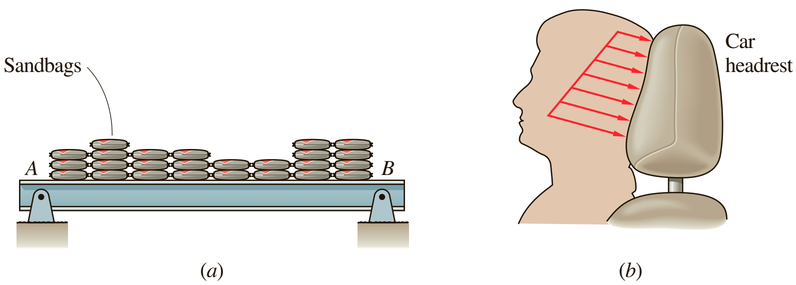


Figure 4.1.2 Distributed loads: (*a*) 60-lb bags of sand stacked on beam *AB*; (*b*) a head presses back on a head-rest.

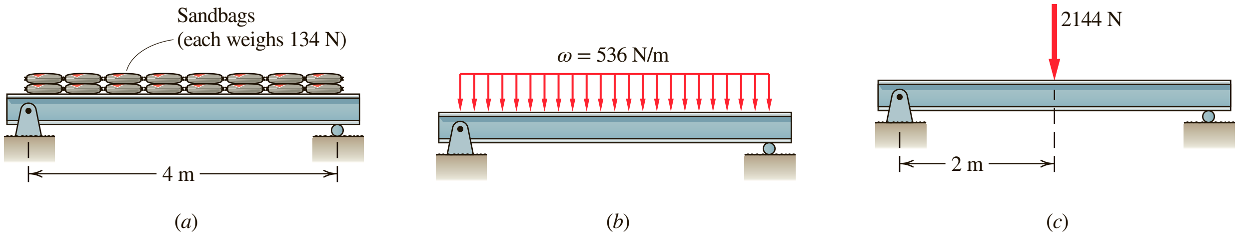


Figure 4.1.3 (*a*) Sandbags sitting on a beam; (*b*) weight of sandbags represented as disributed load; (*c*) weight of sandbags represented as an equivalent force.

current work is that these distributed forces must be included in the free-body diagram.

A free-body diagram must include all of the external loads acting on the system you are analyzing. Some of the external loads acting on a system act *across* the system boundary; the principal example of this type of load is **gravitational force** (or **gravity** for short), which manifests itself as weight. Another example is magnetic force, which results from electromagnetic field interaction. Magnetic force is what turns a motor.

Other external loads act *directly on* the boundary. Consider where:

- The boundary passes through a connection between the system and its surroundings, commonly referred to as a **boundary support** (or **support** for short). A support may be, for example, a bolt, cable, or weld, or simply where the surroundings rest against the system. We replace each support with the loads it applies to the system. These support loads consist of the contact forces discussed in Chapter 2 (normal contact, friction, tension, compression, and shear) or moments as discussed in Chapter 3. A synonym for the term *support loads* is *reactions*.
- The boundary separates the system from fluid surroundings. We refer to this as a **fluid boundary**. We replace the fluid with the load the fluid applies to the system. This load consists of the pressure (force per unit area) of the fluid pressing on and/or moving along the boundary. By their very nature, fluids acting on a system boundary are distributed. Like distributed loads associated with supports, the loads at fluid boundaries are included in a free-body diagram, either as distributed loads or as an equivalent force. In Chapter 6 we discuss in greater detail distributed loads due to fluids acting on the system.

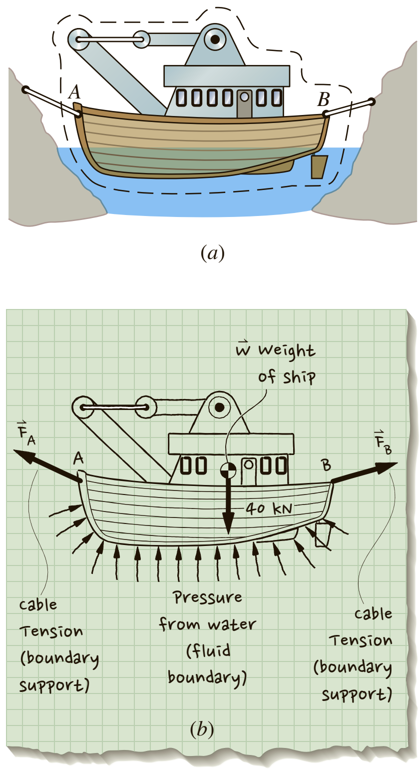


Figure 4.1.4 (*a*) Isolate the system (the ship) by drawing a boundary; (*b*) Free-body diagram of the ship.

In practice, a system may be acted on by a combination of cross-boundary loads (usually gravity), loads at supports, and loads at fluid boundaries, as illustrated in **Figure 4.1.4**. Notice that at some boundary locations no loads act. At other locations there are **known loads**—for example, in **Figure 4.1.4**, the 40-kN gravity force is a known load.

EXERCISES 4.1

4.1.1. [*] The system to be considered is a coat rack with some items hanging on it as shown. In your mind draw a boundary around the system to isolate it from its surroundings.

a. Make a sketch of the coat rack and the external loads acting on it. Show the loads as vectors and label them with variables, and where possible give word descriptions of the loads.

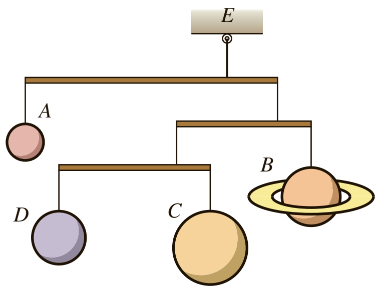
b. List any uncertainties you have about the free-body diagram you have created.

EX 4.1.1

4.1.2. [*] The system to be considered is a mobile up to the point where it is attached at the ceiling (E), as shown. In your mind draw a boundary around the system to isolate it from its surroundings.

a. Make a sketch of the mobile and the external loads acting on it. Show the loads as vectors and label them with variables, and where possible give word descriptions of the loads.

b. List any uncertainties you have about the free-body diagram you have created.



EX 4.1.2

4.1.3. [*] The system to be considered is a person and a ladder, as shown. In your mind draw a boundary around the system to isolate it from its surroundings.

a. Make a sketch of the system and the external loads acting on it. Show the loads as vectors and label them with variables, and where possible give word descriptions of the loads.

b. List any uncertainties you have about the free-body diagram you have created.

EX 4.1.3

4.1.4. [**] Visit a weight room, and take a look at one of the exercise stations—preferably one that is in use!

a. Consider where the person is standing, hanging, laying, and/or pushing on it. In your mind, draw a boundary around *the person* to define him or her as the system. Make a sketch of the system and the external loads acting on it, showing the loads as vectors with variable labels. Where possible give word descriptions of the loads. List any uncertainties you have about the free-body diagram you have created.

b. For the same situation draw a boundary around the *exercise machine* to define it as the system. Make a sketch of the system and the external loads acting on it, showing the loads as vectors with variable labels. Where possible give word descriptions of the loads. List any uncertainties you have about the free-body diagram you have created.

4.1.5. [**] Visit a local playground near campus, and take a look at a climbing structure—preferably one that is in use! Consider where the children (or adults!) are standing/hanging. In your mind, draw a boundary around the *climbing structure* to define it as your system. Make a sketch of the system and the external loads acting on it, showing the loads as vectors with variable labels. List any uncertainties you have about the free-body diagram you have created.

4.1.6. [**] Visit a local pet store or zoo and look at the fish tanks.

a. In your mind, draw a boundary around the *fish tank including the water* to define it as your system. Make a sketch of the system and the external loads acting on it,

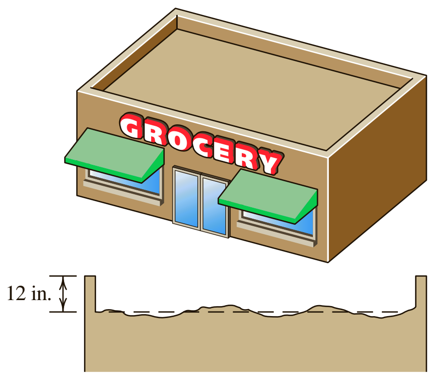
showing the loads as vectors with variable labels. List any uncertainties you have about the free-body diagram you have created.

b. In your mind, draw a boundary around the *fish tank excluding the water* to define it as your system. Make a sketch of the system and the external loads acting on it, showing the loads as vectors with variable labels. List any uncertainties you have about the free-body diagram you have created.

4.1.7. [*] Consider the coat rack. Make a sketch of the coat rack and the external loads acting on it showing the distributed load between the base of the coat rack and the floor.

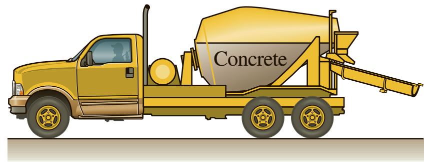
EX 4.1.7

4.1.8. [*] Consider a building with a nominally flat roof. The actual roof surface is slightly irregular and can be represented by the profile shown. After a night of heavy rain, an average rainfall total of 2 in. was recorded, causing ponding on the roof. Make a sketch of the distributed force that rain water applies to the nominally flat roof. Indicate the magnitude of the forces with the length of the vectors.



EX 4.1.8

4.1.9. [*] Consider a concrete truck with a tank that is half full. Draw the distributed load applied to the inside of the tank. Indicate the magnitude of the loads with the length of the vectors.



EX 4.1.9

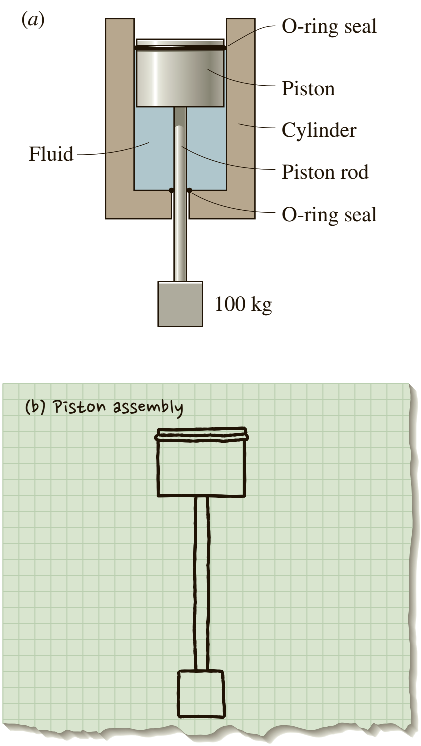
4.1.10. [*] Identify three different systems on which distributed forces act. Make a sketch of each system and show what you think the distributed forces look like; indicate the magnitude of the forces with the length of vectors.

4.1.11. [*] Consider a person wanting to cross a frozen pond. She has a choice of going on foot, wearing snow shoes, or using skis.

a. Make three sketches showing the distribution of her body weight on the frozen pond given the three types of footwear.

b. Which type of footwear would you choose? Why?

4.1.12. [*] A hydraulic cylinder works by pumping fluid in and out of a piston assembly as shown in part (a). Draw the loads acting on the piston assembly shown in part (b). Assume that the piston is oriented vertically relative to gravity.



EX 4.1.12

4.2 PLANAR SYSTEM SUPPORTS

Learning Objective: Identify the standard supports for planar systems and how they are represented in a free-body diagram

We now consider how to identify supports and represent the loads associated with them when working with **planar systems**. A system is modeled as planar if all the forces acting on it can be represented in a single plane and all moments are about an axis perpendicular to that plane. If a system is not planar it is a **nonplanar system**.

Consider the planar systems in **Figures 4.2.1–4.2.4** for which we have drawn free-body diagrams. Each has different supports that connect it to its surroundings. In each case we isolated the bar from the supports so that the system consists of a uniform bar of weight W, oriented so that gravity acts in the negative y direction. At each support we consider whether the surroundings act on the system with a force, a moment, or both.

Note that the free-body diagrams include loads due to supports, as well as the load due to gravity acting at J. Each load is represented as a vector and is given a variable label. As a general rule, *if a support prevents the translation of the system in a given direction, then a force acts on the system at the location of the support in the opposing direction. Likewise, if rotation is prevented, a moment opposing the rotation acts on the system at the location of the support.*

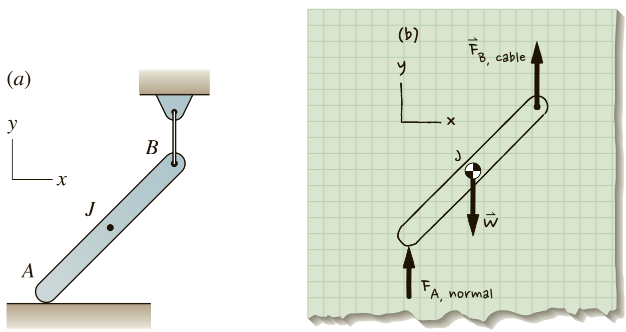


Figure 4.2.1 (*a*) The supports consist of a **normal contact without friction** at A, and a **cable** attached to the system at B; (*b*) The free-body diagram of bar AJB.

At point A in **Figure 4.2.1**, the support consists of a **normal contact without friction**. At this support the system rests against a smooth, frictionless surface. A normal force prevents the system from moving into the surface and is oriented to push on the system. Because the supporting surface at A is smooth, no friction exists between the system and its surroundings. Therefore no force component acts parallel to the surface. In the free-body diagram, the force resulting from normal contact at A is represented by $\boldsymbol{F}_{A,\text{normal}}$; we know its direction is normal to the surface so as to *push* on the system.

At Point B in **Figure 4.2.1**, the support consists of a **cable**. At this support a force acts on the system; the line of action of the force is along the cable. The force represents the cable pulling on the system because the cable can only act in tension. In the free-body diagram the force from the cable at B is represented by $\boldsymbol{F}_{B,\text{cable}}$; we know its direction is along the cable axis in the direction that allows the cable to *pull* on the system.

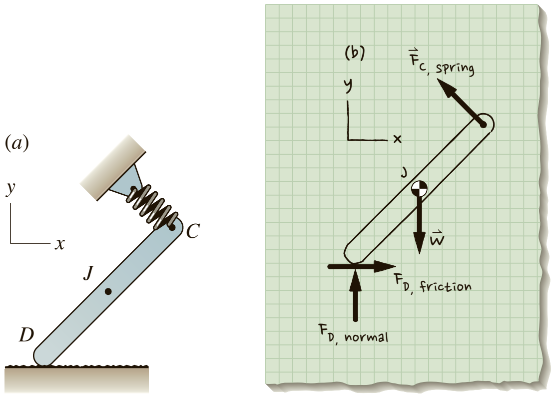


Figure 4.2.2 (*a*) The supports consist of a **spring** attached to the system at C, and a **normal contact with friction** at D; (*b*) The free-body diagram of bar CJD.

At point C in **Figure 4.2.2**, the support consists of a **spring**. At this support a force either pushes or pulls on the system; the line of action of the force is along the axis of the spring. If the spring is extended by an amount Δ, the spring is in tension and the force is oriented to pull on the system. If the spring is shortened (compressed) by an amount Δ, the force is oriented to push on the system. The magnitude of the force is proportional to the amount of spring extension or shortening, and the proportionality constant is the spring constant, k. In other words, the magnitude of the force is equal to the product of k and the spring extension or shortening:

$$F_C = k\Delta \tag{4.1}$$

where the dimensions of k are force/length (e.g., N/mm). The value of F_C in (4.1) will be positive when the spring is in tension (since Δ will be positive) and negative when the spring is compressed (since Δ will be negative).

In the free-body diagram, the spring force at C is represented by $\boldsymbol{F}_{C,\text{ spring}}$; we know its direction is along the spring axis. If the spring is in tension, the force acts to *pull* on the system. If the spring is in compression, the force acts to *push* on the system. We have drawn the direction of $\boldsymbol{F}_{C,\text{ spring}}$ to indicate that the spring is in tension. We could equally

well have chosen the direction of $\boldsymbol{F}_{C,\,\text{spring}}$ to indicate that the spring is in compression, but as we will see in Chapter 5, drawing the spring in tension will make interpreting numerical answers easier.

At point D in **Figure 4.2.2**, the support consists of a **normal contact with friction**. At this support the force acting on the system is represented by two components. One is normal to the surface (just like normal contact without friction). The second is due to friction and is parallel to the surface against which the system rests—therefore it is perpendicular to the normal force component.

The component due to friction (F_{friction}) is related to and limited by normal contact force (F_{normal}) and the characteristics of the contact. Often the relationship between F_{friction} and F_{normal} is represented in terms of the Coulomb Friction Model. This model states that if $F_{\text{friction}} < \mu_{\text{static}}$ (F_{normal}) the system will not slide relative to its surroundings, where μ_{static} is the coefficient of static friction and typically ranges from 0.01 to 0.70, depending on the characteristics of the contact. If $F_{\text{friction}} = \mu_{\text{static}}$ (F_{normal}), the condition is that of impending motion. We will have a lot more to say about friction in Chapter 7.

In **Figure 4.2.2*b*** the normal force component at D is represented by $F_{D,\text{normal}}$; we know its direction is normal to the surface so as to push on the system. The friction force component is represented by $F_{D,\text{friction}}$ and is parallel to the surface and perpendicular to the normal force. We could have drawn $F_{D,\text{friction}}$ to point to the right or to the left; we arbitrarily chose to draw it to the right.

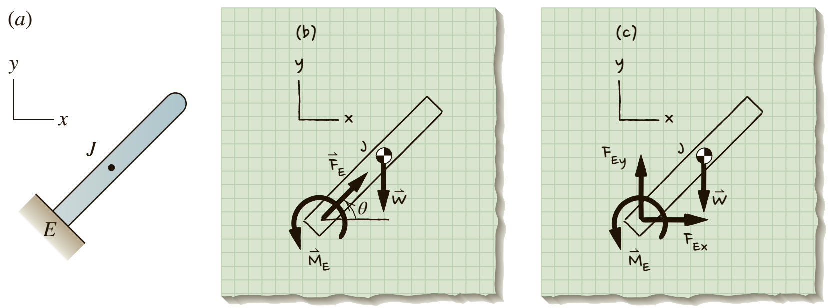


Figure 4.2.3 (*a*) The **fixed support** consists of a fixed connection to the system at *E*. (*b*) The free-body diagram of bar *EJ*, with force at *E* shown in terms of force and angle. (*c*) The free-body diagram of bar *EJ*, with force at *E* shown in terms of force components.

At point E in **Figure 4.2.3**, the support consists of a fixed connection to surroundings, referred to as a **fixed support**. At this support a force at angle θ and a moment act on the system. In the free-body diagram in **Figure 4.2.3*b*** we represent the force at E in terms of a force at unknown angle θ. Alternately, we could represent the force by its x and y components (**Figure 4.2.3*c***). In vector notation we can

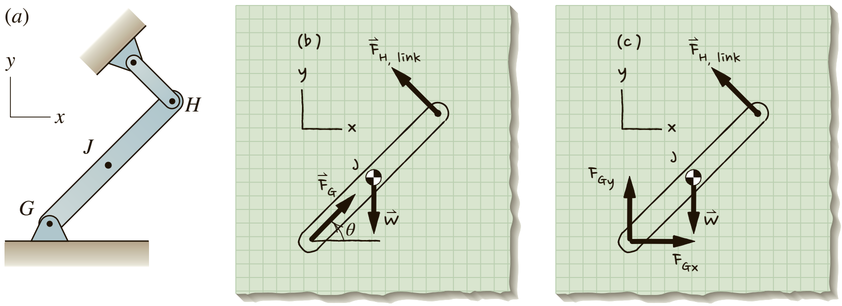


Figure 4.2.4 (*a*) The supports consist of a **pin** connected to the system at *G*, and a **link** attached to the system at *H*. (*b*) The free-body diagram of bar *GJH*. (*c*) Alternate free-body diagram, showing pin connection as two force components.

describe the loads acting at the fixed support at E as $\boldsymbol{F}_E = F_{Ex}\boldsymbol{i} + F_{Ey}\boldsymbol{j}$ and $\boldsymbol{M}_E = M_{Ez}\boldsymbol{k}$.

At point G in **Figure 4.2.4**, the support consists of a system pinned to surroundings, referred to as a **pin connection**. Support G is a pin that is loosely fitted in a hole. At this support a force acts on the system at some angle θ. We have represented the force as a force of unknown magnitude, acting at an unknown angle, as shown in **Figure 4.2.4*b***. Alternately, we could represent the force in terms of its components in the x and y directions and can describe the load acting at the pin connection at G as $\boldsymbol{F}_G = F_{Gx}\boldsymbol{i} + F_{Gy}\boldsymbol{j}$ (**Figure 4.2.4*c***). We have arbitrarily chosen to draw each component in its respective positive direction.

At point H in **Figure 4.2.4**, the support consists of a **link**. At this support a force either pushes or pulls on the system; the line of action of the force is along the axis of the link. We say much more about links in Chapter 5—for now we simply note that a link is a member with a pin connection at each end and no other loads acting on it.

In the free-body diagram the force at H is represented by $\boldsymbol{F}_{H,\text{link}}$ acting along the long axis of the link. A link may either *push or pull* on the system, and here we have chosen to assume pulling. We could equally well have chosen the direction of $\boldsymbol{F}_{H,\text{link}}$ to indicate that the link is pushing, but as we will see in Chapter 5, drawing the link as pulling will make interpreting numerical answers easier.

Table 4.1 summarizes the loads associated with the planar supports discussed already, along with some other commonly found planar supports. Don't feel that you need to memorize all the supports in this table—it is presented merely as a ready reference. On the other hand, you should be familiar with the loads associated with these fairly common planar supports.

Table 4.1 Standard Supports for Planar Systems

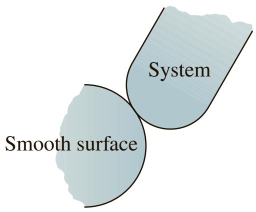
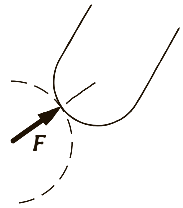
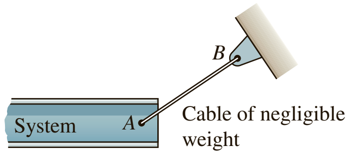
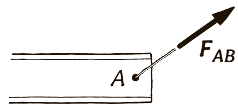
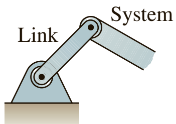
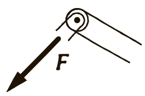
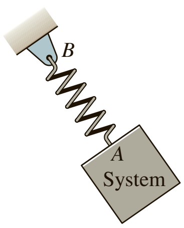
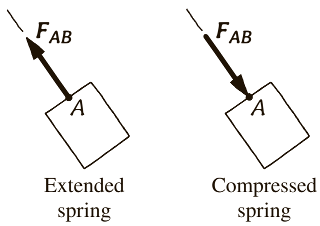
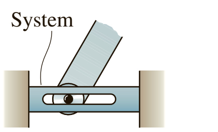
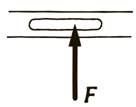
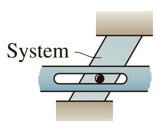
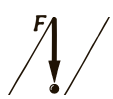
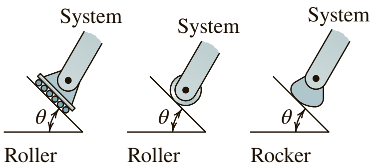
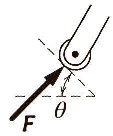
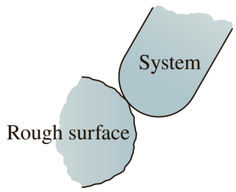
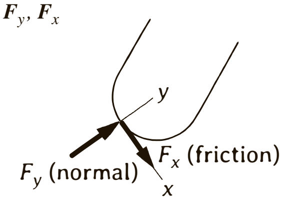
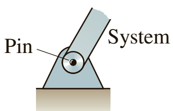
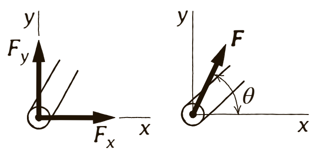
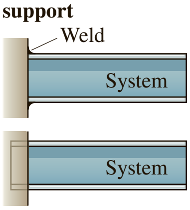
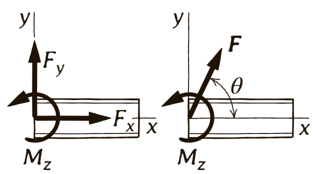
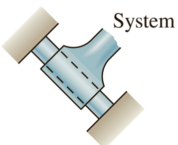
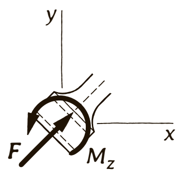
(A) Supports	Description of Loads	(B) Loads to Be Shown on Free-Body Diagram
1. Normal contact without friction 	**Force** (***F***) oriented normal to surface on which system rests. Direction is such that force pushes on system.	***F*** 
2. Cable, rope, wire 	**Force** (***F***) oriented along cable. Direction is such that cable pulls on the system.	***F*** 
3. Link 	**Force** (***F***) oriented along link length; force can push or pull on the system.	***F*** 
4. Spring 	**Force** (***F***) oriented along long axis of spring. Direction is such that spring pulls on system if spring is in tension, and pushes if spring is in compression.	***F*** 
5. Slot-on-pin (frictionless) (slotted member is part of system) 	**Force** (***F***) oriented normal to long axis of slot. Direction is such that force can pull or push on system. The slot is frictionless. Therefore no forces act parallel to the slot.	***F*** 
6. Pin-in-slot (frictionless) (pin is part of system) 	**Force** (***F***) oriented normal to long axis of slot. Direction is such that force can pull or push on system. The slot is frictionless. Therefore no forces act parallel to the slot.	***F*** 

Table 4.1 Standard Supports for Planar Systems (*Continued*)

<table>
<tr><th>(A)
Supports</th><th>Description of Loads</th><th>(B)
Loads to Be Shown on
Free-Body Diagram</th></tr>
<tr><td>7. Roller or rocker
</td><td>Force (F) oriented normal to surface on which system rests. Direction is such that force pushes on system.</td><td>F
</td></tr>
<tr><td>8. Normal contact with friction
</td><td>Two force components, one (F_y) oriented normal to surface on which the system rests so as to push on system, other force (F_x) is tangent to surface.</td><td>F_y, F_x
</td></tr>
<tr><td>9. Pin connection
(pin or hole is part of system)
</td><td>Force perpendicular to pin axis represented in terms of components F_x and F_y. Point of application is at center of pin.

Alternative representation: Force (F) oriented at angle θ with respect to coordinate system. Point of application is at center of pin.</td><td>F_x, F_y F, θ
</td></tr>
<tr><td>10. Fixed support
</td><td>Force in x-y plane represented in terms of components F_x and F_y.
Moment about z axis (M_z).

Alternative representation: Force (F) oriented at angle θ with respect to coordinate system. Moment about z axis (M_z).</td><td>F_x, F_y, M_z F, θ, M_z
</td></tr>
<tr><td>11. Smooth collar on smooth shaft
</td><td>Force (F) oriented perpendicular to long axis of shaft. Direction is such that force can pull or push on system.
Moment (M_z) about z axis.</td><td>F, M_z
</td></tr>
</table>

Check out the following examples of applications of this material.

- **Example 4.2.1 Free-Body Diagram of a Planar System**
- **Example 4.2.2 Free-Body Diagram of a Planar System with Moment**
- **Example 4.2.3 Using Questions to Determine Loads at Supports**

EXAMPLE 4.2.1

In **Figure 1**, a 100-N block is supported at several points. Gravity acts downward in the $-y$ direction at the indicated center of gravity (CG), and the surface at B is rough. Define the system as the block. (a) Explain why Figure 1 is not a free-body diagram and (b) create a free-body diagram of the system.

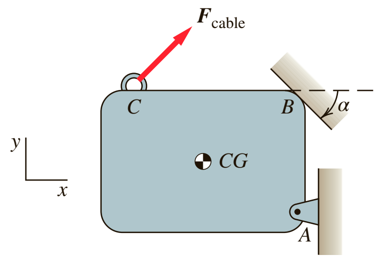


Figure 1 A supported block.

Goals Explain why the given figure is not a free-body diagram and create a complete and correct free-body diagram of the block.

Givens A system with specified loads, supports, and coordinate system.

Assumptions Model the system as planar because the known loads and the loads applied by supports all lie in a single plane.

(a) Why is Figure 1 not a free-body diagram?

Figure 1 is not a free-body diagram because we have defined the system as block ABC, but the block still is shown connected to its surroundings at A and B. The first step in creating a free-body diagram is to isolate the system being studied from its surroundings.

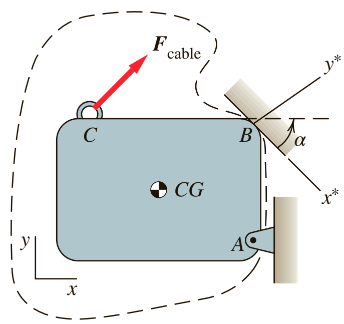


Figure 2 Boundary to isolate system.

(b) Creating the free-body diagram

To create the free-body diagram, first we isolate the system (block ABC) by drawing a boundary around it, as shown in **Figure 2**.

Next, we must replace each support by its associated loads. We use the xy coordinate system for the entire system and establish the x^*y^* coordinate system to simplify the representation of the loads applied by the support at B. Then, using **Table 4.1** as a reference, we note the following to create the free-body diagram in **Figure 3**:

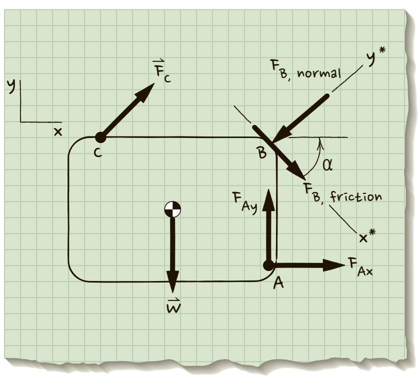


Figure 3 Free-body diagram of system.

- *At CG* (*the center of gravity*) a force $\mathbf{W}$ acts in the $-y$ direction. The magnitude of this force is 100 N, which we add to the drawing.
- *At A* the pin connection attaches the system to its surroundings and applies a force to the system. As we do not know the direction or magnitude of this force, we represent it as two components, F_{Ax} and F_{Ay}, which we arbitrarily draw in the positive x and y directions (**Figure 3**).
- *At B* the system rests against a surface inclined at angle α. Since we know that the surface is rough, we must consider the friction between the surface and the system. A normal force $F_{B,\text{normal}}$ acts on the system, oriented perpendicular to the surface so as to push on the system, as shown in **Figure 3**. We do not know its magnitude. The friction force $F_{B,\text{friction}}$ acts perpendicular to $F_{B,\text{normal}}$. We do not know the magnitude of $F_{B,\text{friction}}$ or whether it acts in the $+x^*$ or $-x^*$ direction, so we arbitrarily draw it in the $+x^*$ direction.

- *At C* a cable pulls on the system, which we represent as a force of unknown magnitude but known direction ($\boldsymbol{F}_C$).

The block presented in **Figure 3**, with forces F_{Ax}, F_{Ay}, $F_{B,\text{normal}}$, $F_{B,\text{friction}}$, $\boldsymbol{F}_C$, and $\boldsymbol{W}$ each drawn at its point of application, is a free-body diagram of the system in **Figure 1**.

EXAMPLE 4.2.2

In **Figure 1**, a block is supported at several points and subjected to a 10-lb force and 40 in.·lb moment at C. Gravity acts downward in the $-y$ direction at the indicated center of gravity (CG). Define the system as the block. Create a free-body diagram of the system.

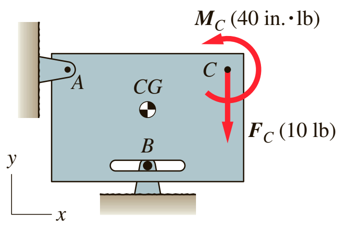


Figure 1 A supported block.

Goals Create a complete and correct free-body diagram of the block.

Givens A system with specified loads, supports, and coordinate system.

Assumptions The system can be modeled as planar because the known forces and moments, and the loads applied by supports, all lie in a single plane. Because no information is provided, assume the slot-pin connection at B in **Figure 1** is smooth (frictionless).

Free-body diagram Isolate the system (block ABC) by drawing a boundary around it, as shown in **Figure 2**. Then replace each support by its associated loads.

We establish an xy coordinate system as shown in **Figure 3**, and then we note the following:

- *At CG (the center of gravity)* a force $\boldsymbol{W}$ acts in the $-y$ direction.
- *At A* a pin connection attaches the system to its surroundings. According to **Table 4.1**, a pin connection applies a force to the system. As we do not know the direction or magnitude of this force, we represent it as two components, F_{Ax} and F_{Ay}, which we arbitrarily draw in the positive x and y directions (**Figure 3**).
- *At B* a frictionless slot in the block holds a slider that allows the block to move in the x direction but prevents movement in the y direction as shown in **Table 4.1**. We represent this by a force $\boldsymbol{F}_{By}$, which we have arbitrarily drawn in the positive y direction.
- *At C* a known force ($\boldsymbol{F}_C$) and a moment ($\boldsymbol{M}_C$) are applied. Known values are written next to the vectors.

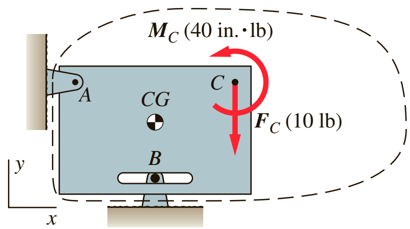


Figure 2 Boundary to isolate system.

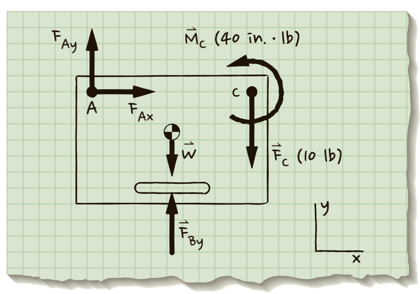


Figure 3 Free-body diagram of system.

The block presented in **Figure 3**, with forces F_{Ax}, F_{Ay}, $\boldsymbol{F}_{By}$, $\boldsymbol{F}_C$, $\boldsymbol{M}_C$, and $\boldsymbol{W}$ each drawn at its point of application, constitutes a free-body diagram of the system in **Figure 1**. Note that this diagram includes the known magnitudes of $\boldsymbol{F}_C$ and $\boldsymbol{M}_C$.

EXAMPLE 4.2.3

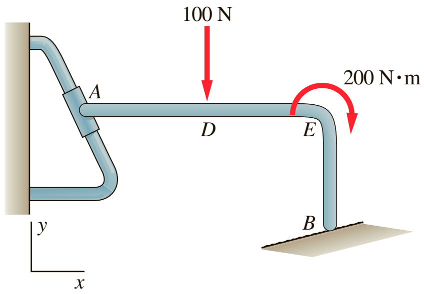


Figure 1 A bar with loads in the xy plane.

A bar is supported at A by a frictionless collar guide and rests against a rough surface at B. Known loads act at D and E, as shown in **Figure 1**.

(a) What loads act at A and B? Use the general rule about the surroundings preventing translation and/or rotation at each support to answer this question.

(b) Draw a free-body diagram of the bar.

Goals Use information about translation and rotation at supports to create a complete and correct free-body diagram of the bar.

Givens A system with specified loads, supports, and coordinate system.

Assumptions The weight of the bar can be ignored and the system (the bar) can be modeled as planar.

(a) Determine the loads at *A* and *B*

Since the bar can be modeled as planar, we need consider only translations in the x–y plane and rotations about the z axis. Isolate the bar $ADEB$ and draw coordinate systems $x'y'$ at A and x^*y^* at B that are normal and tangent to the support surfaces, as shown in **Figure 2**. These coordinate systems will help clarify our discussions of translations at these two supports.

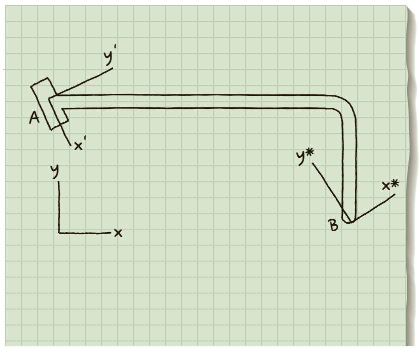


Figure 2 Axes normal and tangent to the surfaces at A and B.

At *A*: What translations and rotations are prevented by the collar at A?

Possible motion at A	Answer	Implication
Is x' translation possible?	Yes	No force acts on the bar in the x' direction, since it is frictionless.
Is y' translation possible?	No	A force $\boldsymbol{F}_{Ay'}$ acts in the $\pm y'$ direction.
Is z rotation possible?	No	A moment $\boldsymbol{M}_{Az}$ acts about the z axis in the clockwise or counterclockwise direction.

At *B*: What translations and rotations are prevented by the surface at B?

Possible motion at B	Answer	Implication
Is x^* translation possible?	No, unless the force applied in the x^* direction exceeds the maximum friction force that can be applied by the rough surface.	A force component F_{Bx^*} due to friction acts in the $\pm x^*$ *direction*.

Possible motion at *B*	Answer	Implication
Is y^* translation possible?	It is not possible in the negative y^* direction, but is possible in the positive y^* direction.	Force component F_{By^*} acts in the positive y^* direction to prevent motion in the negative y^* direction.
Is z rotation possible?	Yes.	There is no moment about the z axis.

Answers **(a)** *At A:* $\boldsymbol{F}_{Ay'}$, $\boldsymbol{M}_{Az}$

At B: F_{Bx^*}, F_{By^*} (in the positive y^* direction).

(b) Using the information from part (a), the free-body diagram of the bar is as shown in **Figure 3**.

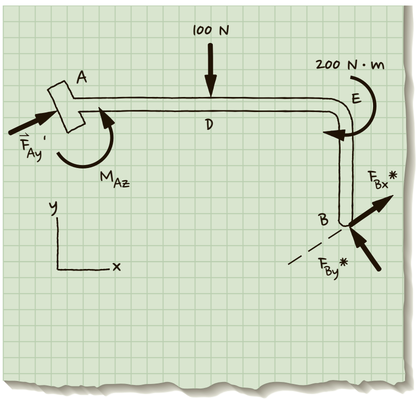
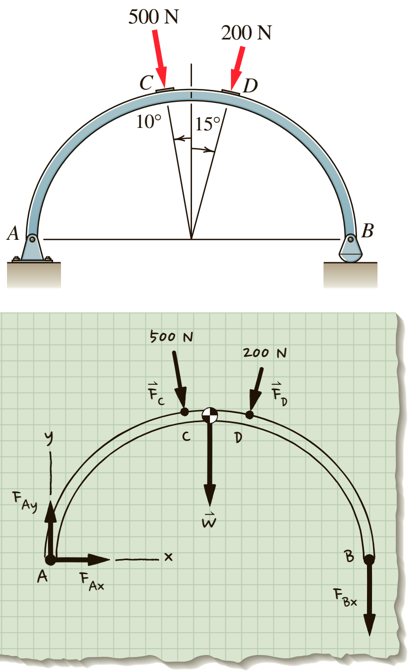


Figure 3 Free-body diagram of system.

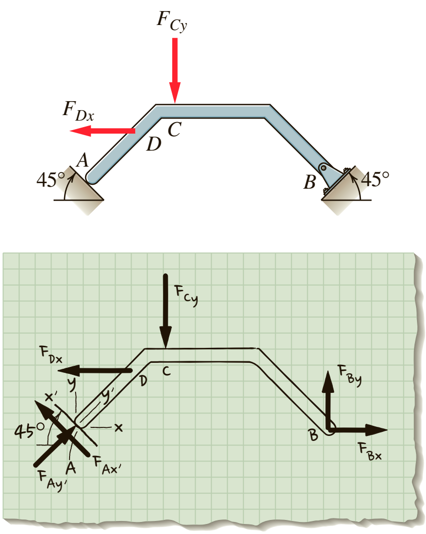
EXERCISES 4.2

4.2.1. [**] A semicircular uniform beam of weight W is supported at A by a pin connection and at B by a rocker, as shown. Is the proposed free-body diagram correct? If not, indicate what is wrong.



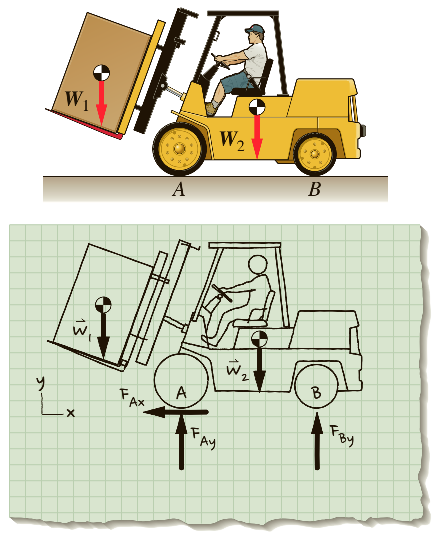
EX 4.2.1

4.2.2. [**] A frame is pinned at B and rests against a smooth incline at A as shown. The total weight of the frame is W. Is the proposed free-body diagram correct? If not, indicate what is wrong.



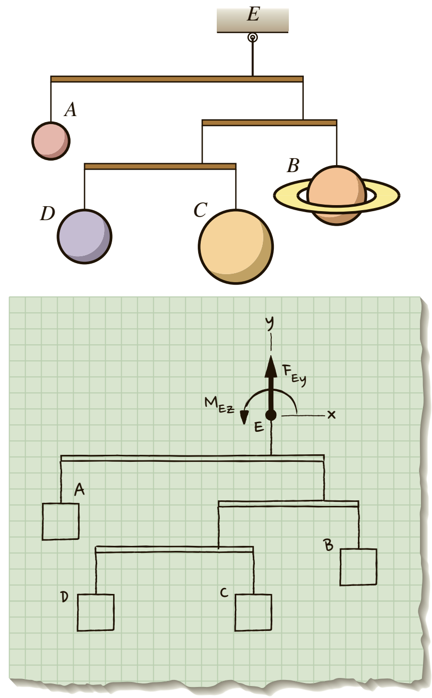
EX 4.2.2

4.2.3. [**] A forklift lifts a crate of weight W_1 as shown. The weight of the forklift is W_2. The front wheels are free to turn and the rear wheels are locked. The actual numerical values of W_1 and W_2 are 1000 lb and 5000 lb, respectively. Is the proposed free-body diagram correct? If not, indicate what is wrong.



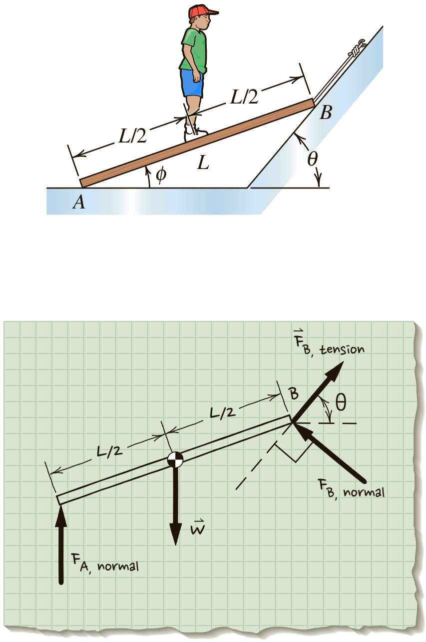
EX 4.2.3

4.2.4. [**] A mobile dangles from the ceiling from the cord as shown. Is the proposed free-body diagram correct? If not, indicate what is wrong.



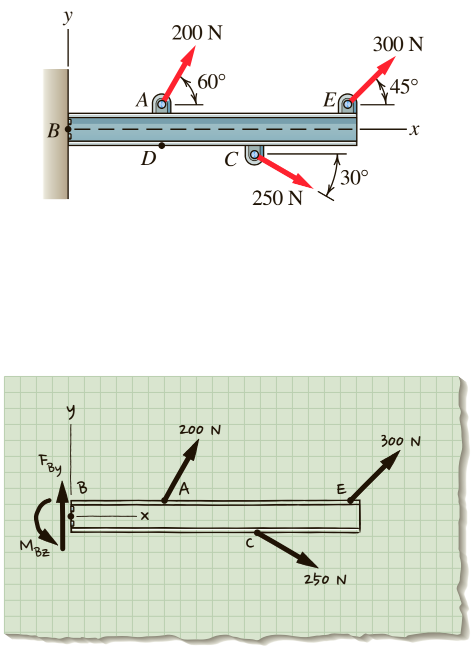
EX 4.2.4

4.2.5. [**] A child of weight W balances on the beam as shown. Planes A and B are smooth. The weight of the beam is negligible. Is the proposed free-body diagram correct? If not, indicate what is wrong.



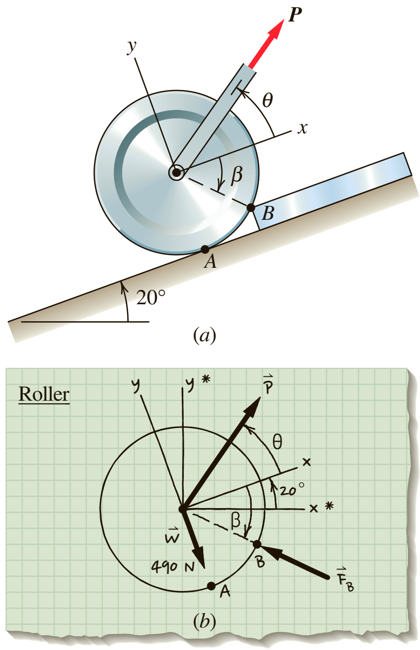
EX 4.2.5

4.2.6. [**] A beam is rigidly attached to a wall at B as shown. The weight of the beam is negligible. Is the proposed free-body diagram correct? If not, indicate what is wrong.



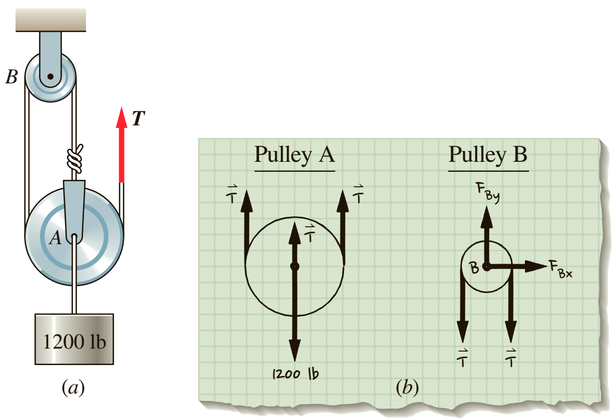
EX 4.2.6

4.2.7. [**] A 50-kg roller is pulled up a 20° incline with a force P. As it is pulled over a smooth step, all of its weight rests against the step, as shown. Is the proposed free-body diagram correct? If not, indicate what is wrong.



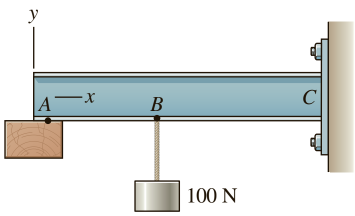
EX 4.2.7

4.2.8. [**] A 1200-lb object is held up by a force T on a rope threaded through a system of frictionless pulleys, as shown. Are the proposed free-body diagrams of Pulley A and Pulley B correct? If not, indicate what is wrong.



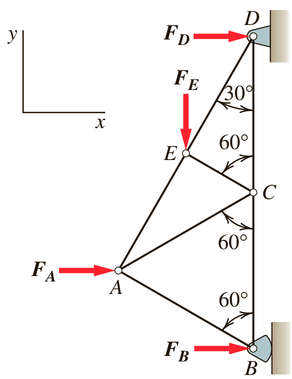
EX 4.2.8

4.2.9. [***] The beam of uniform weight is fixed to the wall at C and rests against a smooth block at A. In addition, a 100-N weight hangs from point B. Based on information in **Table 4.1**, what loads do you expect to act on the beam at C due to the fixed condition? What loads do you expect to act on the beam at A where it rests on the smooth block? Present your answer in terms of a sketch of the beam that shows the loads acting on it at A, B and C. Also comment on whether the sketch you created of the beam is or is not a free-body diagram. If not, what is missing?



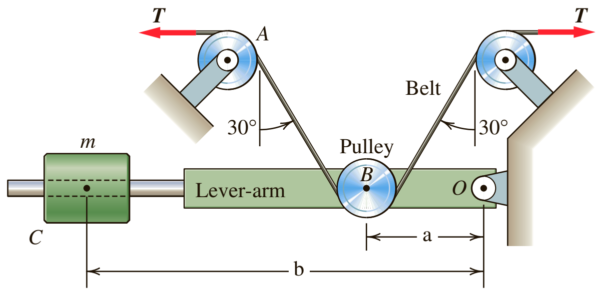
EX 4.2.9

4.2.10. [***] The truss is attached to the wall at B with a rocker and at D with a pin connection. Based on information in **Table 4.1**, what loads do you expect to act on the truss at B due to the rocker connection? What loads do you expect to act on the truss at D due to the pin connection? Present your answer in terms of a sketch of the truss that shows the loads acting on it at B and D. Also comment on whether the sketch you created of the truss is or is not a free-body diagram. If not, what is missing?



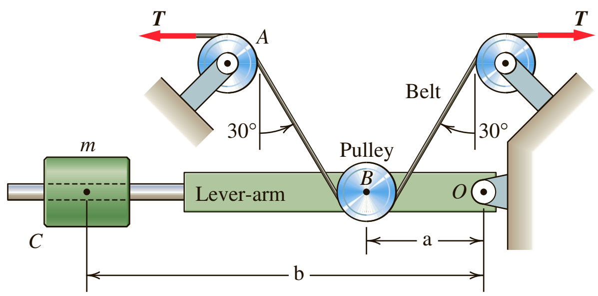
EX 4.2.10

4.2.11. [***] The belt-tensioning device is as shown. Based on information in **Table 4.1**, what loads do you expect to act on the lever-arm at pin connection O? If pulley B is pinned to the lever-arm, what loads do you expect to act on the lever-arm due to the pulley? Present your answers in terms of a sketch of the lever-arm. Also comment on whether the sketch you created of the lever-arm is or is not a free-body diagram. If not, what is missing?



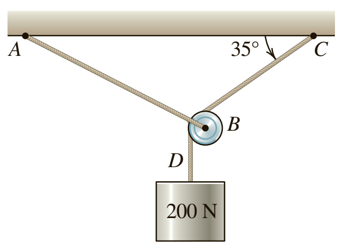
EX 4.2.11

4.2.12. [***] Consider the belt-tensioning device shown. Based on information in **Table 4.1**, what loads do you expect to act on the pulley B at its pin connection with the lever-arm? What loads do you expect to act on pulley B due to the tension in the belt? Present your answers in terms of a sketch of pulley B. Also comment on whether the sketch you created of pulley B is or is not a free-body diagram. If not, what is missing?



EX 4.2.12

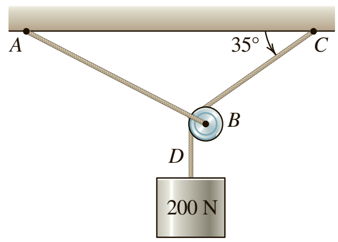
4.2.13. [***] Cable CD passes over the small frictionless pulley B without a change in tension and holds up the metal cylinder. Based on information in **Table 4.1**, what loads do you expect to act on the cylinder? Present your answer in terms of a sketch of the cylinder that shows all the loads



EX 4.2.13

acting on it. Also comment on whether the sketch you created of the cylinder is or is not a free-body diagram. If not, what is missing?

4.2.14. [***] Consider the situation shown. What loads act on pulley B due to cable CD? What loads act on pulley B due to the cable that runs from A to pulley B? Present your answer in terms of a sketch of the pulley that shows all the loads acting on it. Also comment on whether the sketch you created of the pulley is or is not a free-body diagram.



EX 4.2.14

4.3 NONPLANAR SYSTEM SUPPORTS

Learning Objective: Identify the standard supports for nonplanar systems and how they are represented in a free-body diagram

A system is modeled as **nonplanar** if all the forces acting on it cannot be represented in a single plane or all moments acting on it are not about an axis perpendicular to that plane. You will see similarities to our discussion in the prior section on planar systems and some important differences when it comes to representing them as free-body diagrams.

Consider the nonplanar systems in Figures 4.3.1 and 4.3.2 for which we have drawn free-body diagrams. Each has different supports that connect the system (in this case, a plate) to its surroundings. In each case we isolated the square plate from the supports so that the system consists of a uniform plate of weight W, oriented so that gravity acts in the negative y direction. At each support we consider whether the surroundings act on the system with a force and/or a moment.

Note that the particular supports in **Figures 4.3.1** and **4.3.2**—normal contact without friction, a cable, a spring, normal contact with friction, and a fixed support—should sound familiar from our study of planar systems in Section 4.2, and they operate in the same manner as they did with a planar system. The free-body diagrams include loads due to supports, as well as the load due to gravity acting at the plate's center of gravity (CG). Each support load is represented as a vector and is given a variable label.

As a general rule, *if a support prevents the translation of the system in a given direction, then a force acts on the system at the location of the support in the opposing direction. Likewise, if rotation is prevented, a moment opposing the rotation acts on the system at the location of the support.*

At point A in **Figure 4.3.1**, the support consists of a **normal contact without friction**. At this support the system rests against a smooth, frictionless surface. A normal force prevents the system from moving into the surface and is oriented to push on the system. Because the supporting surface at A is smooth, no friction exists between the system and its surroundings. Therefore no force component acts parallel to the surface. In the free-body diagram, the force resulting from normal contact at A is represented by $\boldsymbol{F}_{\text{normal}}$; we know its direction is normal to the surface so as to *push* on the system.

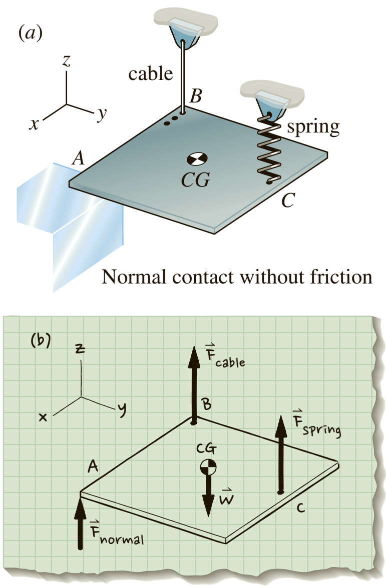


Figure 4.3.1 (a) The supports consist of a **normal contact without friction** at A, a **cable** attached to the system at B, and a **spring** attached at C; (b) The free-body diagram of plate.

At point B in **Figure 4.3.1**, the support consists of a **cable**. At this support a force acts on the system; the line of action of the force is along the cable. The force represents the cable pulling on the system because the cable can only act in tension. In the free-body diagram the force from the cable at B is represented by $\boldsymbol{F}_{\text{cable}}$; we know its direction is along the cable axis in the direction that allows the cable to *pull* on the system.

At point C in **Figure 4.3.1**, the support consists of a **spring**. At this support a force either pushes or pulls on the system; the line of action of the force is along the axis of the spring. In the free-body diagram, the spring force at C is represented by $\boldsymbol{F}_{\text{spring}}$; we know its direction is along the spring axis. If the spring is in tension, the force acts to *pull* on the system. If the spring is in compression, the force acts to *push* on the system. We have drawn the direction of $\boldsymbol{F}_{\text{spring}}$ to indicate that the spring is in tension.

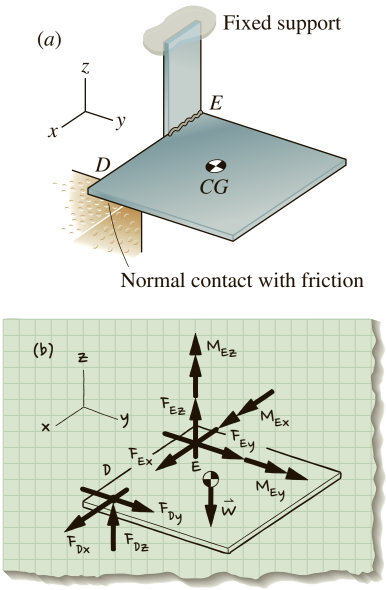


Figure 4.3.2 (*a*) The supports consist of a **normal contact with friction** at *D*, and a **fixed support** at *E*; (*b*) The free-body diagram of plate.

At point D in **Figure 4.3.2**, the support consists of a **normal contact with friction**. At this support two forces act on the system. One is a normal force (just like normal contact without friction); in this case it is directed in the z-direction (F_{Dz}). The second is due to friction ($\boldsymbol{F}_{\text{friction}}$) and is in the plane that is perpendicular to the normal force. We represent this friction force in terms of its components in the x–y plane: F_{Dx} and F_{Dy}. The force due to friction ($\boldsymbol{F}_{\text{friction}} = F_{Dx}\boldsymbol{i} + F_{Dy}\boldsymbol{j}$) is related to and limited by the normal contact force ($\boldsymbol{F}_{\text{normal}}$) and the characteristics of the contact. We will have a lot more to say about friction in Chapter 7.

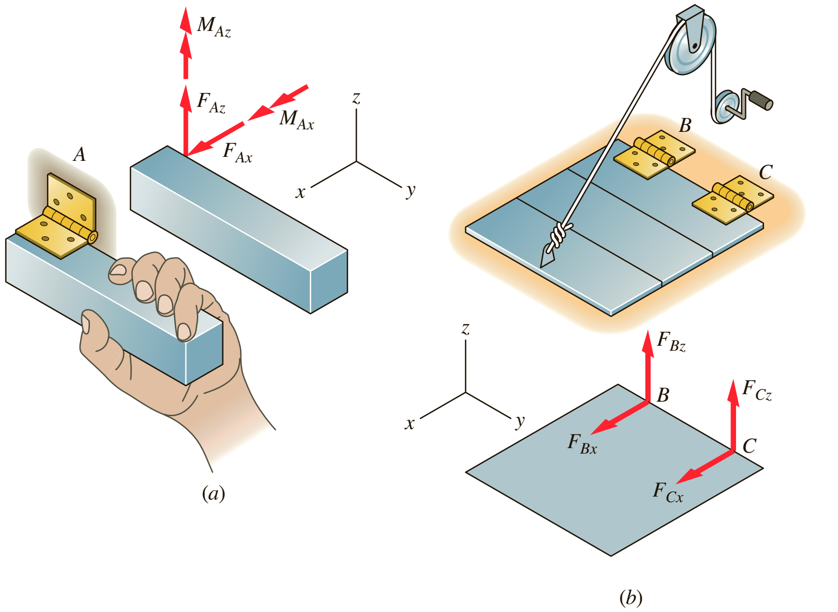
At point E in **Figure 4.3.2**, the support is a fixed connection to surroundings, referred to as a **fixed support**. At this support a force and a moment act on the system that prevent the system from translating along and rotating about any axis—therefore it involves a force with three components ($\boldsymbol{F_E} = F_{Ex}\boldsymbol{i} + F_{Ey}\boldsymbol{j} + F_{Ez}\boldsymbol{k}$) and a moment with three components ($\boldsymbol{M_E} = M_{Ex}\boldsymbol{i} + M_{Ey}\boldsymbol{j} + M_{Ez}\boldsymbol{k}$).

Other commonly found nonplanar supports are illustrated in **Figures 4.3.3 and 4.3.4** for which we have drawn the loads that these supports apply to the system.

At point A in **Figure 4.3.3*a***, the support is a **single hinge**. It does not restrict rotation of the system about the hinge pin. A single hinge applies a force (with two components) and a moment (with two components) perpendicular to the axis of the hinge. Here we represent the loads acting at the single hinge support as a force with two components ($\boldsymbol{F_A} = F_{Ax}\boldsymbol{i} + F_{Az}\boldsymbol{k}$) and a moment with two components ($\boldsymbol{M_E} = M_{Ax}\boldsymbol{i} + M_{Az}\boldsymbol{k}$).

At Points B and C in **Figure 4.3.3*b***, the support is a **multiple hinge**. If a hinge is one of several properly aligned hinges attached to a system, each hinge applies a force perpendicular to the hinge axis and no moment. Depending on the design of a hinge (and regardless of whether it is a single hinge or one of several), it may also apply a force along the axis of the pin (Fy in **Figure 4.3.3*c***). The experiments outlined in Example 4.3.1 illustrate the difference in the loads involved with single versus multiple hinges.

The **ball-and-socket support** shown in **Figure 4.3.4** restricts all translations of the system by applying a force $\boldsymbol{F} = (F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k})$ to the system, but it does not restrict rotation of the system about any axis.



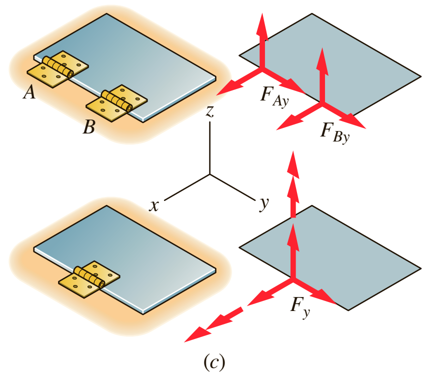


Figure 4.3.3 (*a*) A single hinge supporting the system, and its associated loads. (*b*) Two hinges supporting a system, and their associated loads. (*c*) Hinges designed to prevent motion along the hinge axis. The force component F_y is commonly referred to as the thrust force.

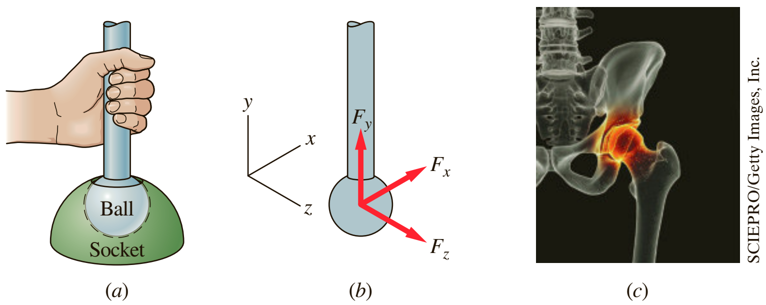


Figure 4.3.4 (*a*) A schematic of a ball-and-socket joint. (*b*) The forces the socket applies to the ball. (*c*) An example of a ball-and-socket support familiar to everyone is the human hip joint.

Table 4.2 summarizes the loads associated with the nonplanar supports discussed already, along with some other commonly found nonplanar supports. Don't feel that you need to memorize all the supports

Table 4.2 Standard Supports for Nonplanar Systems

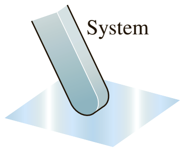
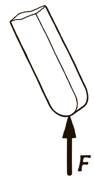
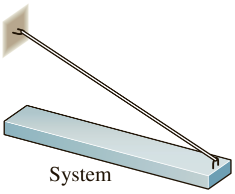
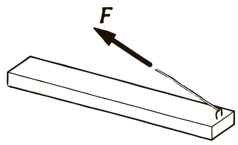
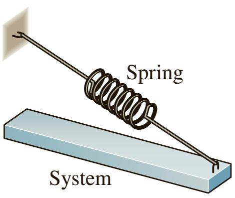
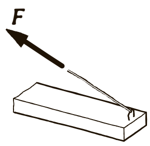
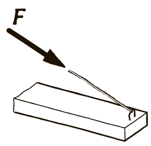
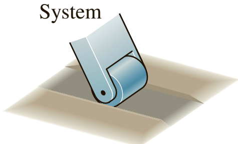
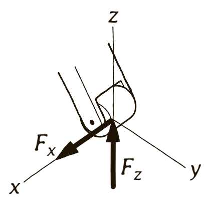
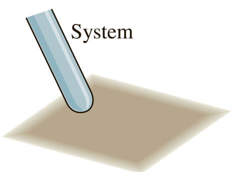
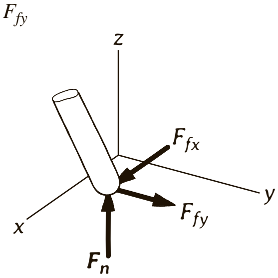
(A) Supports	Description of Boundary Loads	(B) Loads to Be Shown in Free-Body Diagram
1. Normal contact without friction 	**Force** (***F***) oriented normal to surface on which system rests. Direction is such that force pushes on system.	***F*** 
2. Cable, rope, wire 	**Force** (***F***) oriented along cable. Direction is such that force pulls on system.	***F*** 
3. Spring 	**Force** (***F***) oriented along long axis of spring. Direction is such that force pulls on system if spring is in tension and pushes if spring is in compression.	***F***  Spring in tension $+\Delta$  Spring in compression $-\Delta$
4. Smooth roller in guide 	**Force** represented as two components. One component (F_z) normal to surface on which system rests; the other is perpendicular to rolling direction (F_x).	$F_x + F_z$ 
5. Normal contact with friction 	**Two forces**, one ($\boldsymbol{F_n}$) oriented normal to surface so as to push on system, other force is tangent to surface on which the system rests and is represented in terms of its components ($F_{fx} + F_{fy}$).	$\boldsymbol{F_n}$ $F_{fx} + F_{fy}$ 

Table 4.2 Standard Supports for Nonplanar Systems (*Continued*)

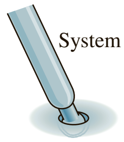
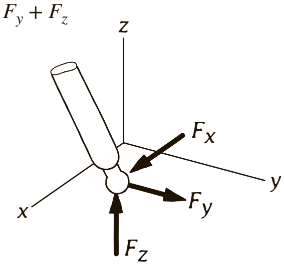
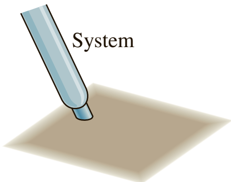
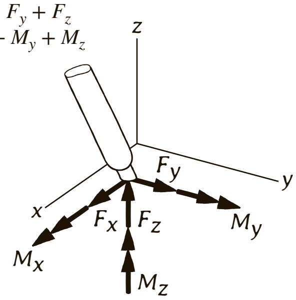
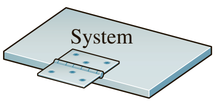
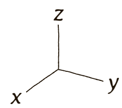
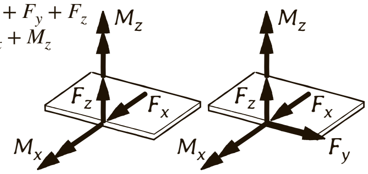
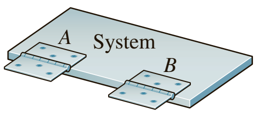
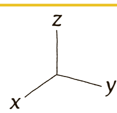
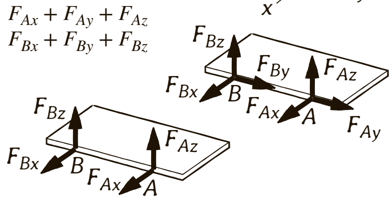
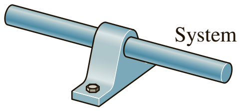
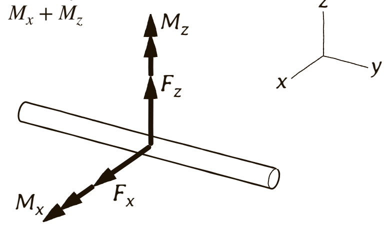
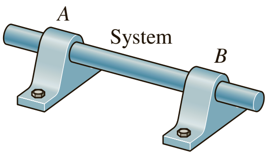
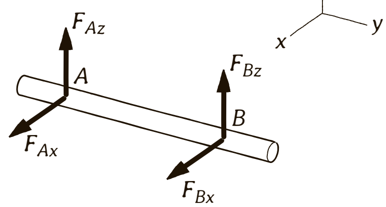
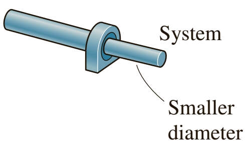
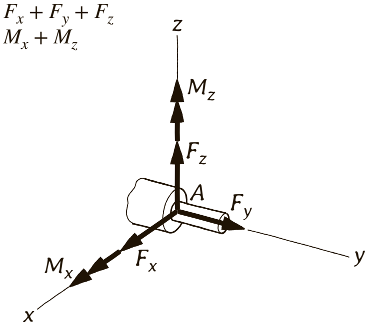
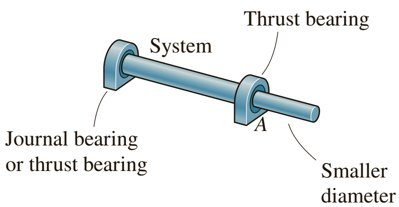
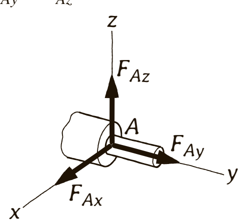
(A) Supports	Description of Boundary Loads	(B) Loads to Be Shown in Free-Body Diagram
6. Ball and socket support (ball or socket as part of system)  	**Force** represented as three components.	$F_x + F_y + F_z$ 
7. Fixed support  	**Force** represented in terms of components ($F_x + F_y + F_z$). **Moment** represented in terms of components ($M_x + M_y + M_z$).	$F_x + F_y + F_z$ $M_x + M_y + M_z$ 
8A. Single hinge (shaft and articulated collar)  	**Force** in plane perpendicular to shaft axis; represented as x and z components ($F_x + F_z$). **Moment** with components about axes perpendicular to shaft axis ($M_x + M_z$). Depending on the hinge design, may also have a **force** component along axis of shaft, (F_y).	$F_x + F_z$ $M_x + M_z$  OR $F_x + F_y + F_z$ $M_x + M_z$ 
8B. Multiple hinges (one of two or more properly aligned hinges)  	**Force** in plane normal to shaft axis represented in terms of components ($F_x + F_z$). Point of application at center of shaft. Depending on design, may also apply **force** component along axis of shaft (F_y).	At hinge A: $F_{Ax} + F_{Az}$ At hinge B: $F_{Bx} + F_{Bz}$  OR $F_{Ax} + F_{Ay} + F_{Az}$ $F_{Bx} + F_{By} + F_{Bz}$ 

Table 4.2 Standard Supports for Nonplanar Systems (*Continued*)

(A) Supports	Description of Boundary Loads	(B) Loads to Be Shown in Free-Body Diagram
9A. Single journal bearing (frictionless collar that holds a shaft)  	**Force** in plane perpendicular to shaft axis; represented as x and z components ($F_x + F_z$). **Moment** with components about axes perpendicular to shaft axis ($M_x + M_z$).	$F_x + F_z$ $M_x + M_z$ 
9B. Multiple journal bearings (two or more properly aligned journal bearings holding a shaft)  	**Force** in plane perpendicular to shaft axis represented in terms of components ($F_{Ax} + F_{Az}$). Point of application at center of shaft.	At journal bearing A: $F_{Ax} + F_{Az}$ At journal bearing B: $F_{Bx} + F_{Bz}$ 
10A. Single thrust bearing (journal bearing that also restricts motion along axis of shaft)  	**Force** represented in terms of three components ($F_x + F_y + F_z$). Component in direction of shaft axis (F_y) is sometimes referred to as the "thrust force." Point of application is at center of shaft. **Moment** with components perpendicular to shaft axis ($M_x + M_z$).	$F_x + F_y + F_z$ $M_x + M_z$ 
10B. Multiple thrust bearings (one of two or more properly aligned thrust bearings)  	**Force** represented in terms of three components ($F_x + F_y + F_z$). Component in direction of shaft axis (F_y) is sometimes referred to as the "thrust force." Point of application is at center of shaft.	At thrust bearing A: $F_{Ax} + F_{Ay} + F_{Az}$ 

in this table—it is presented merely as a ready reference. Take a few minutes to study Table 4.2 and notice the similarities and differences between hinges, journal bearings, and thrust bearings.

Check out the following example of an application of this material.

- **Example 4.3.1 Exploring Single and Double Bearings and Hinges**

EXAMPLE 4.3.1

In this example your hands serve as models of bearings and hinges so that you can gain a physical feel for the loads supplied by these types of supports. To create the situations yourself, you will need a yardstick, a rubber band, and a candy bar or other object to serve as a weight. For each of the three situations draw a free-body diagram and describe the loads involved. When considering the system, ignore the weight of the yardstick.

Situation 1: Hold the yardstick level as shown in Figure 1. The left hand is at $x = 0$ in. and the right hand is at $x = 18$ in. The candy bar is hanging at the far right.

Description Each hand acts like a bearing or hinge as described by the "multiple hinge" entry in Table 4.2.

The left-hand fingers push down on the top of the yardstick. Notice that you can move your left thumb away from the stick because there is no load on it. The right thumb pushes up on the bottom of the yardstick. Notice that you can move your right-hand fingers away from the stick because there is no load on them.

Free-Body Diagram Consider how the hands apply loads to the yardstick. Defining the yardstick as the system, draw these loads on the yardstick to create the free-body diagram (Figure 2).

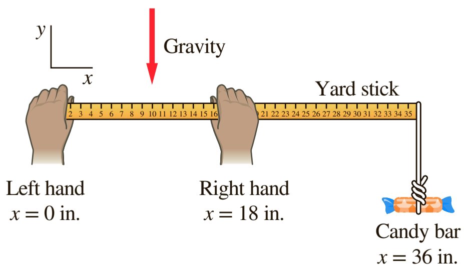


Figure 1 Situation 1.

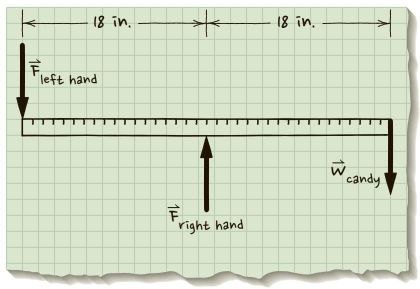


Figure 2 Free-body diagram of Situation 1.

Situation 2: Hold the yardstick level as shown in Figure 3. The left hand is at $x = 9$ in. and the right hand is at $x = 18$ in. The candy bar is hanging at the far right.

Description The description for situation still holds. That is, each hand acts like a bearing or hinge as described by the "multiple hinge" entry in Table 4.2.

You should notice a difference; in order to keep the yardstick level, the forces involved in pushing down with the left fingers and up with the right thumb are larger in magnitude than in Situation 1.

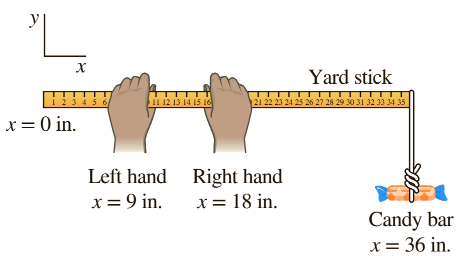


Figure 3 Situation 2.

Free-Body Diagram Defining the yardstick as the system, draw the loads that the hands apply to the yardstick to create the free-body diagram (**Figure 4**).

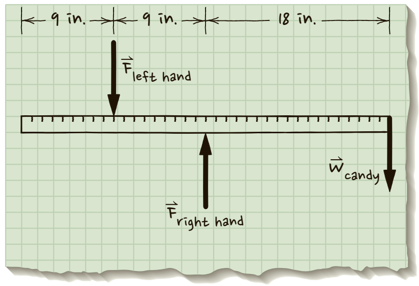


Figure 4 Free-body diagram of Situation 2.

Situation 3: Hold the yardstick level as shown in **Figure 5** with just your right hand. The right hand is at $x = 18$ in., and the candy bar is hanging at the far right.

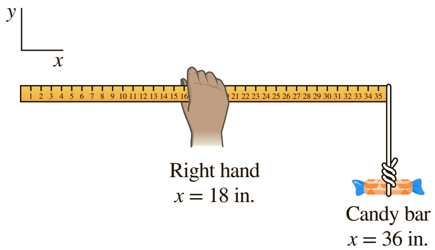


Figure 5 Situation 3.

Description The right hand acts like a single bearing or hinge as depicted in **Table 4.2**.

The weight of the candy bar will create a tendency for the yardstick to rotate clockwise. To keep it level, the thumb pushes up on the bottom of the yardstick and works in conjunction with the right-hand fingers to prevent the stick from rotating; in doing this, the right hand both applies a counterclockwise moment and pushes upward with a force.

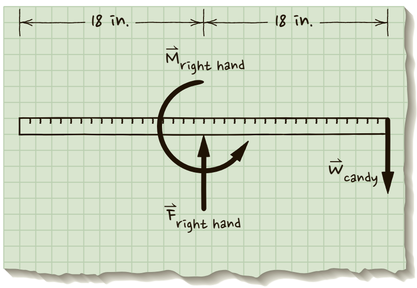


Figure 6 Free-body diagram of Situation 3.

Free-Body Diagram Consider how the hand applies loads to the yardstick. Defining the yardstick as the system, draw these loads on the yardstick to create the free-body diagram (**Figure 6**).

Summary Situations 1 and 2 are analogous to systems with two properly aligned bearings or hinges, with the hands playing the role of bearings/hinges. Although each hand applies only a force, each does create an equivalent moment at a specified moment center.

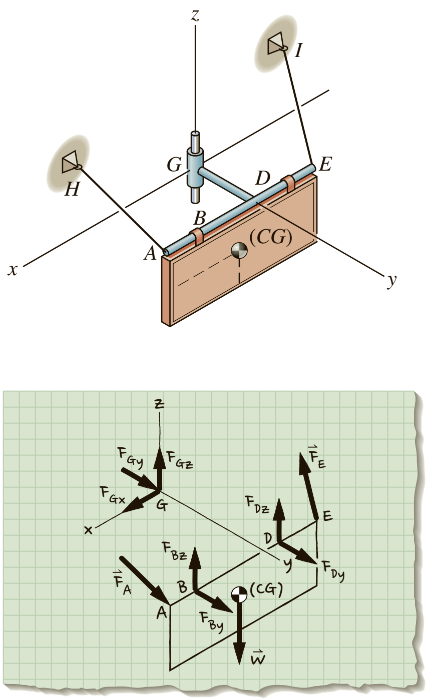
For example, if we designate $x = 18$ in. as the moment center (this is the point of application of $\boldsymbol{F}_{\text{left hand}}$) in Situation 1, $\boldsymbol{F}_{\text{left hand}}$ creates a counterclockwise equivalent moment of (18 in. $\|\boldsymbol{F}_{\text{left hand}}\|$) that opposes the clockwise equivalent moment created by the dangling candy of (18 in. $\|\boldsymbol{W}_{\text{candy}}\|$).

In Situation 2, the candy stays at the same position, creating the same clockwise equivalent moment of (18 in. $\|\boldsymbol{W}_{\text{candy}}\|$) about $x = 18$ in. Since the left hand is placed at $x = 9$ in., it must exert a larger force than in Situation 1 to maintain the same counterclockwise moment.

In Situation 3, the right hand acts like a single bearing or hinge, and must apply a force and a moment to oppose the clockwise moment created by the dangling candy that is 18 inches from the hand. Notice that in **Figure 6** the right hand applied both a force and a moment to the yard stick.

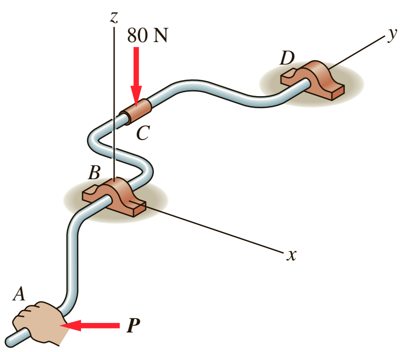
EXERCISES 4.3

4.3.1. [*] A sign of weight W (500 N) with center of gravity as shown is supported by cables and a collar joint. Is the proposed free-body diagram of the system (defined as the sign and bracket BDG suspended from cables) correct? If not, indicate what is wrong.

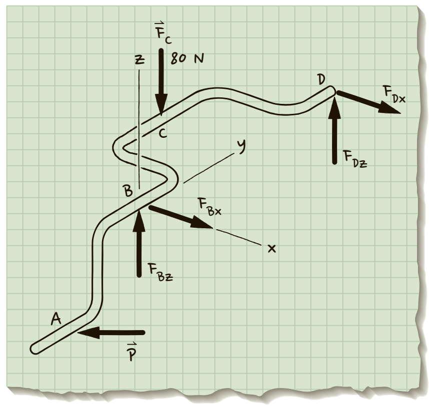


EX 4.3.1

4.3.2. [*] A crankshaft is supported by a journal bearing at B and a thrust bearing at D. Ignore the weight of the crank. Is the proposed free-body diagram of the crankshaft correct? If not, indicate what is wrong.

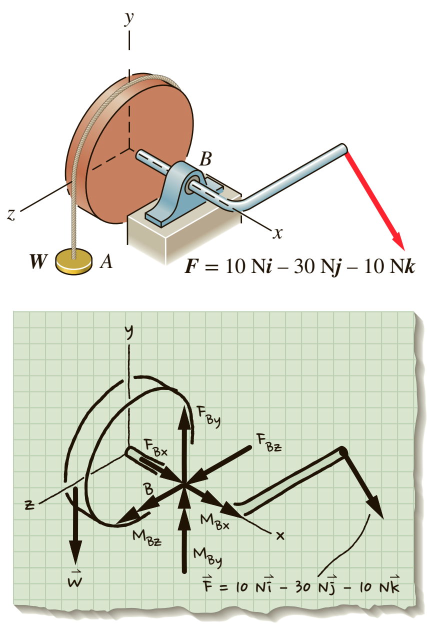


EX 4.3.2



EX 4.3.2

4.3.3. [*] A pulley is used to lift a weight W. The shaft of the pulley is supported by a journal bearing, as shown. Ignore the weights of the pulley and the shaft. Is the proposed free-body diagram of the shaft-pulley system correct? If not, indicate what is wrong.



EX 4.3.3

4.3.4. [*] Perform the experiment described below, then follow the steps to create a free-body diagram of the situation.

Materials needed: Two wire clothes hangers, a friend.

Experiment: Hook the two clothes hangers together, as shown. Have your friend push and pull on her hanger, as shown. At the same time, use your hands (in the positions shown) to keep the hanger level. Notice how your hands and those of your friend push and/or pull on the wire.

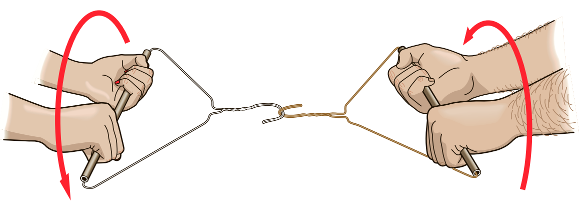
a. Consider the two hangers to be your system. **Draw** the structure. Add a coordinate system such that the y axis is aligned with gravity.

b. **List** (in words) the forces acting on the system.

c. **Draw** each of the forces listed in (b) as a vector on the drawing created in (a). Clearly mark the points of application of each force and add variable labels. If the magnitudes of any of the forces are known, include this information on the drawing. You have now created a free-body diagram of the system. Make sure to **list any assumptions** you made and any uncertainties you have.

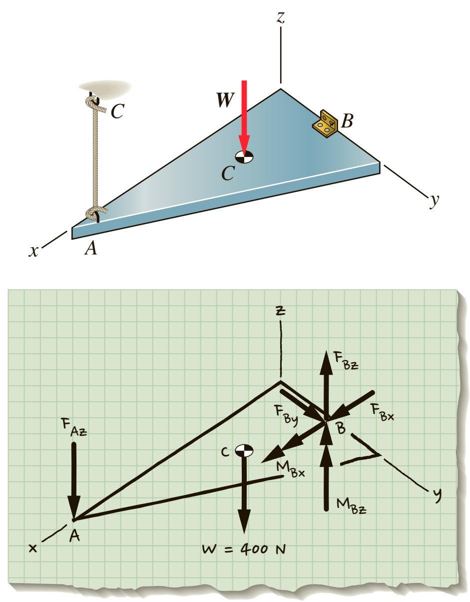
d. Study your free-body diagram and **identify any couples** (describe in words).

e. Repeat (a)–(d) if the system is defined as one of the hangers.



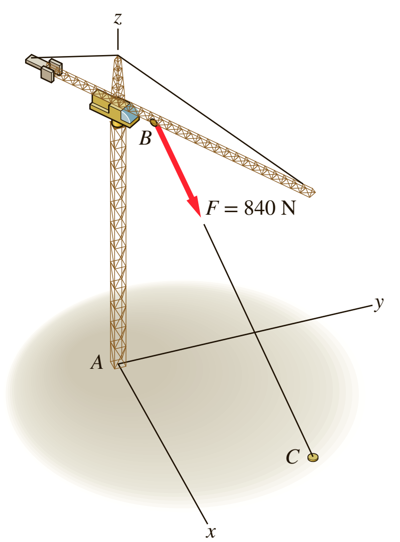
EX 4.3.4

4.3.5. **[*]** A triangular plate is supported by a rope at A and a hinge at B. Its weight of 400 N acts at the plate's center of gravity at C, as shown. Is the proposed free-body diagram of the triangular plate correct? If not, indicate what is wrong.



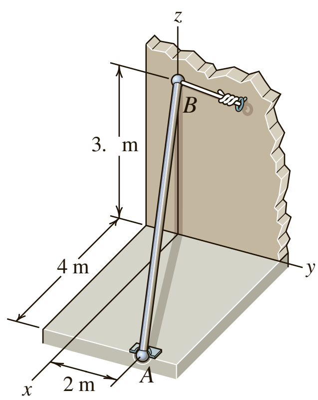
EX 4.3.5

4.3.6. **[*]** A tower crane is fixed to the ground at A as shown. Based on information in **Table 4.2**, what loads do you expect to act on the tower at A? Present your answer in terms of a sketch of the tower that shows the loads acting on it at A. Also comment on whether the sketch you created is or is not a free-body diagram.



EX 4.3.6

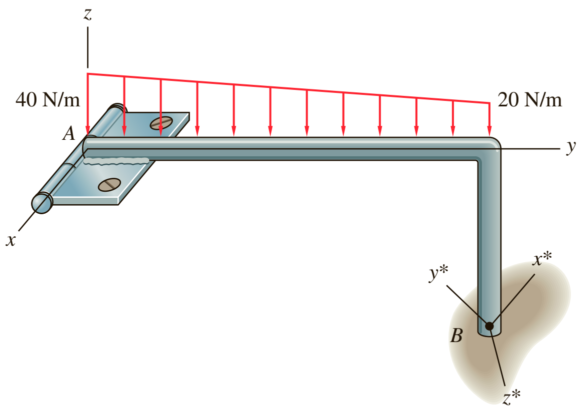
4.3.7. **[*]** The uniform aluminum shaft is supported by a ball-and-socket at A. At B it rests against a smooth wall and is tethered by a cable as shown. Based on the information in **Table 4.2**, what loads do you expect to act on the shaft at A? What loads do you expect to act on the shaft



EX 4.3.7

at B? Present your answer in terms of a sketch of the shaft that shows the loads acting on it at A and B. Also comment on whether the sketch you created is or is not a free-body diagram.

4.3.8. [*] A bar is supported at A by a hinge, and at B it rests against a rough surface. The surface is defined as the x^*z^* plane. Based on information in **Table 4.2**, what loads do you expect to act on the bar at A? What loads do you expect to act on the bar at B? Present your answer in terms of a sketch that shows the loads acting on the bar at A and B. Also comment on whether the sketch you created is or is not a free-body diagram.



EX 4.3.8

4.4 MODELING SYSTEMS AS PLANAR OR NONPLANAR

Learning Objective: Determine by inspection whether a system should be modeled as planar or nonplanar

Defining the loads at a system's boundary is simplified if we can classify the system as a planar system, which is one in which all the forces acting on the system lie in the same plane and all moments are about an axis perpendicular to that plane. In this case, cross-boundary loads (e.g., gravity), known loads, fluid boundary loads, and supports are all in a single plane. **Figure 4.4.1** shows an example of a system that can be modeled as a planar system—planar because the gravity force and supports A and B are all in a single plane. Planar systems are referred to as **two-dimensional systems**. As we saw in Section 4.2, the free-body diagram associated with a planar system typically requires only a single view of the system.

No system is really planar, because we live in a three-dimensional world. Even something as thin as a sheet of paper has a third dimension; BUT under certain conditions we can model it as planar for the purpose of static analysis.

A system in which the loads do not all lie in a single plane can be treated as a planar system for the purpose of static analysis if the system has a plane of symmetry *with regard to its geometry and the loads acting on it*. A **plane of symmetry** is one that divides the system into two sections that are mirror images of each other. None of the forces acting on the system has a component perpendicular to the plane of symmetry, and all moments acting on the system are about an axis perpendicular to the plane. **Figure 4.4.2** illustrates a system that has a plane of symmetry and therefore can be treated as a planar system.

Another example in which we have taken advantage of a plane of symmetry to classify a system is the ladder–person example in Chapter 2

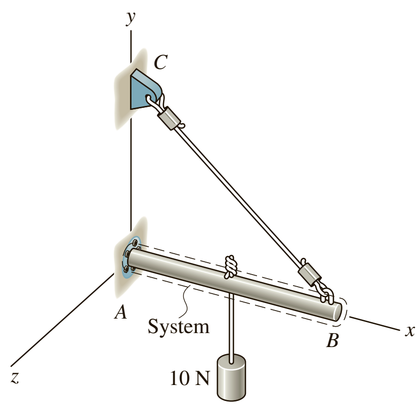


Figure 4.4.1 Points A, B, and C are in the same plane, as well as the 10-N load. This can be modeled as a planar system.

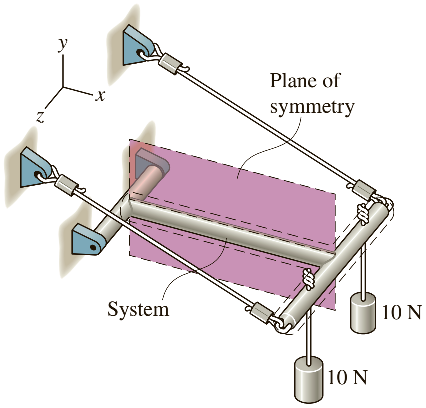


Figure 4.4.2 Since all loads are parallel to the plane of symmetry this can be modeled as a planar system.

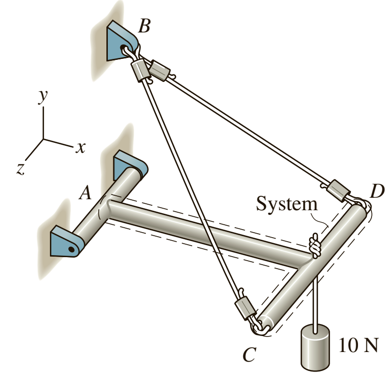


Figure 4.4.3 The loads in cables *BC* and *BD* are not parallel to the plane of symmetry so this must be modeled as a nonplanar system.

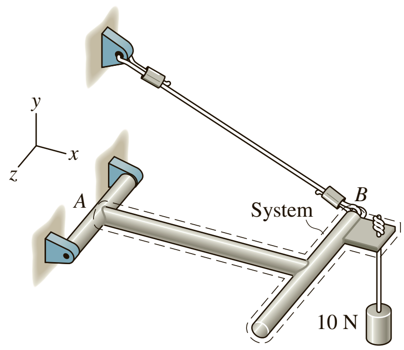


Figure 4.4.4 No plane of symmetry exists and the loads do not lie in a single plane, so this must be modeled as a nonplanar system.

(**Figures 2.4.3, 2.4.4**, and **2.4.6**, but not **Figure 2.4.5**). **Figure 4.4.3** illustrates a system with geometric symmetry, but because the cable forces acting on it have a component perpendicular to this plane, we are not able to classify the system as planar.

If it is not possible to define a single plane in which all forces and moments lie, or there is no plane of symmetry, the system is modeled as nonplanar. In the system of **Figure 4.4.4**, for instance, it is not possible to define a single plane that contains the gravity force and supports *A* and *B*, and no plane of symmetry exists. Nonplanar systems are referred to as **three-dimensional systems**. The free-body diagram associated with a nonplanar system typically requires an isometric drawing or multiple views.

In Section 4.2 we dealt exclusively with planar systems and in Section 4.3 with nonplanar systems. Drawing the free-body diagram for a planar system is generally more straightforward because only external forces in the plane and external moments about an axis perpendicular to the plane must be considered. In performing analysis in engineering practice you will not be told whether a physical situation can be modeled as a planar system or must be modeled as a nonplanar system—making this modeling judgment will be up to you. The discussion in this section gives you some guidelines for making such a decision.

Check out the following examples of applications of this material.

- **Example 4.4.1 Identifying Planar and Nonplanar Systems**
- **Example 4.4.2 Identifying Planar and Nonplanar Systems with a Plane of Symmetry**

EXAMPLE 4.4.1

The space truss in **Figure 1** is of negligible weight and is supported by a roller at B and ball-and-socket supports at C and D. A vertical 800-N force is applied at A. The space truss in **Figure 2** is identical to Figure 1 except all of the supports are on rollers. In each case, determine if the system, which is defined as the space truss, can be modeled as planar or nonplanar.

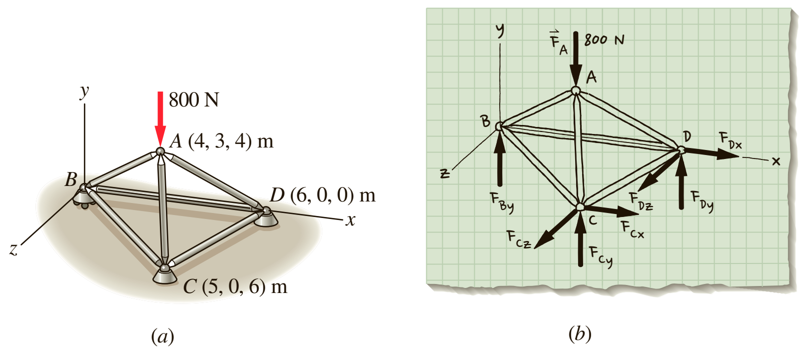


Figure 1 Case 1: (*a*) A system and (*b*) its free-body diagram.

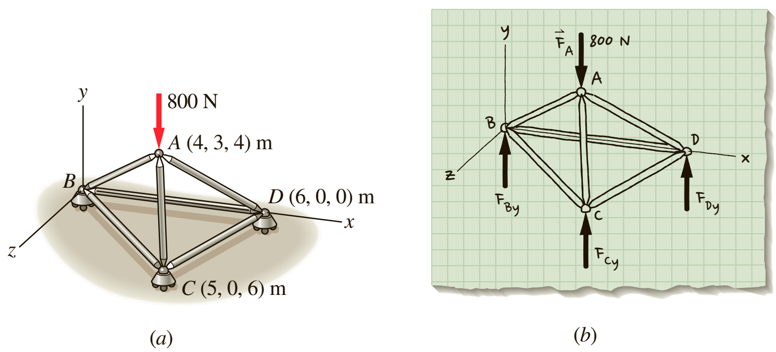


Figure 2 Case 2: (*a*) A system and (*b*) its free-body diagram.

Case 1 *Nonplanar.* The free-body diagram of the space truss is shown in **Figure 1*b***. At supports C and D, forces act in the x, y, and z directions. The forces acting on the space truss do not all act in one plane. Therefore, this system must be modeled as a nonplanar system.

Case 2 *Nonplanar.* The free-body diagram of the space truss is shown in **Figure 2*b***. Although all of the forces acting on the space truss are parallel to the y axis, it is not possible to define a single plane that contains the points of application of the supports (B, C, D) and the 800-N force. Therefore, this system must be modeled as a nonplanar system. Our answer would be unchanged if we had included gravity forces acting on the space truss.

EXAMPLE 4.4.2

The semicircular plate in **Figure 1** weighs 300 N, which is modeled by a point force at the center of mass labeled *A*, the plate center of gravity. Vertical cables support the plate at *B* and *C*, and a ball-and-socket joint supports the plate at *D*. The same plate as in **Figure 1** is supported by diagonal cables at *B* and *C*, and a ball-and-socket support at *D*, as shown in **Figure 2**. In each case, determine if the system, which is defined as the plate, can be modeled as planar or nonplanar.

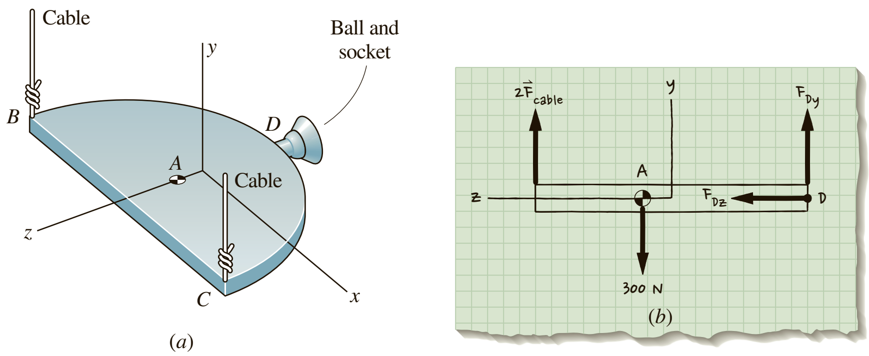


Figure 1 Case 1: (*a*) A plate supported by vertical cables and (*b*) its free-body diagram.

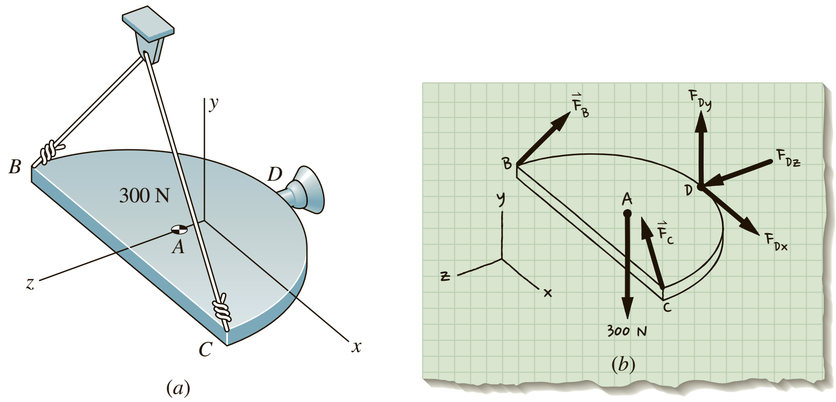


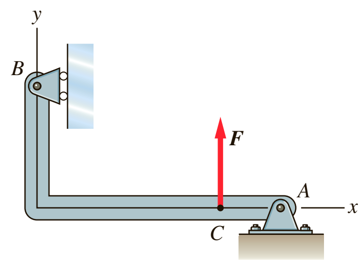
Figure 2 Case 2: (*a*) A plate supported by diagonal cables and a ball-and-socket support at *D*, and (*b*) its free-body diagram.

Case 1 *Planar.* The *y*–*z* plane is a plane of symmetry for the system (defined as the plate) and no loads act perpendicular to the *y*–*z* plane. Consequently, it is possible to represent all of the forces as projections onto the *y*–*z* plane. The free-body diagram of the plate modeled as a two-dimensional (or planar) system is shown in **Figure 1*b***.

Case 2 *Nonplanar.* As in **Case 1**, the *y*–*z* plane is a plane of symmetry for this system. However, since the cable forces have components perpendicular to the plane of symmetry, we cannot model this as a planar system. The free-body diagram of the plate is shown in **Figure 2*b***.

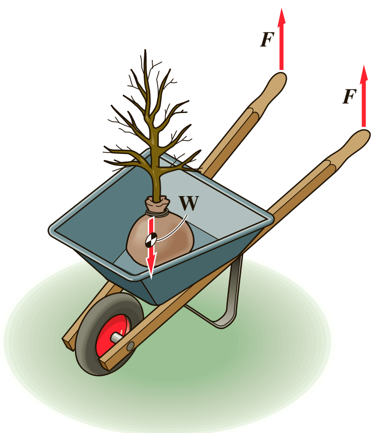
EXERCISES 4.4

4.4.1. [*] The uniform L-bar is pinned to its surroundings at A, and slides along a wall at B. A vertical force $\boldsymbol{F}$ acts at C. Gravity acts in the negative y direction. The system is taken as the L-bar. Should this system be modeled as planar or nonplanar? Present your reasoning.



EX 4.4.1

4.4.2. [*] The wheelbarrow is loaded, with its load's center of gravity as shown. The system is taken as the wheelbarrow. Should this system be modeled as planar or nonplanar? Present your reasoning.

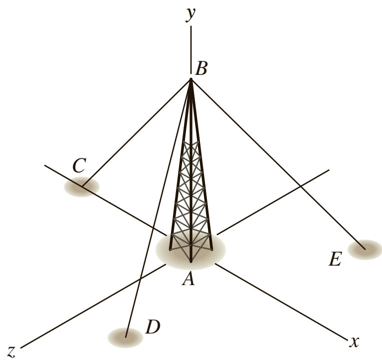


EX 4.4.2

4.4.3. [*] The wheelbarrow is loaded with the wooden stakes resting to one side. The system is taken as the wheelbarrow. Should this system be modeled as planar or nonplanar? Present your reasoning.

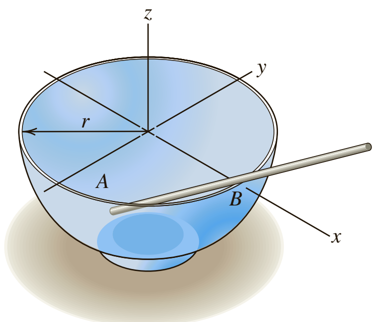
EX 4.4.3

4.4.4. [*] A tower is tethered by three cables, as shown. The system is taken as the tower. Should this system be modeled as planar or nonplanar? Present your reasoning.



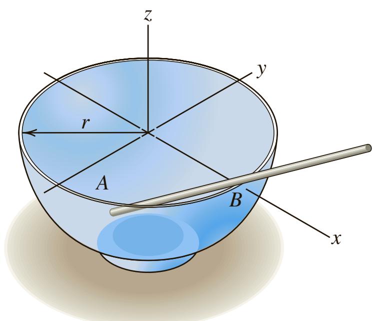
EX 4.4.4

4.4.5. [*] A uniform glass rod having a length L rests in a smooth hemispherical bowl having a radius r. The system is taken as the rod. Should this system be modeled as planar or nonplanar? Present your reasoning.



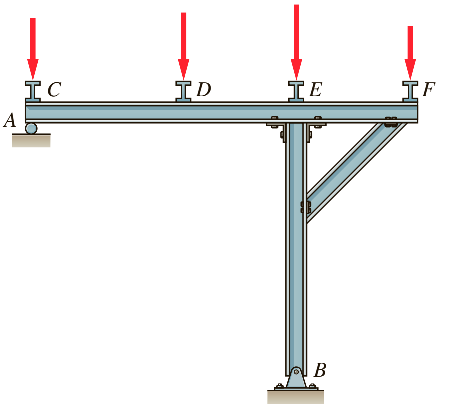
EX 4.4.5

4.4.6. [*] A uniform glass rod rests in a smooth hemispherical bowl having a radius r as shown. Define the bowl as the system. Should this system be modeled as planar or nonplanar? Present your reasoning.



EX 4.4.6

4.4.7. [*] The frame is supported at A and B. Loads act at C, D, E and F, as shown. Define the system as the frame and ignore the effect of gravity. Should this system be modeled as planar or nonplanar? Present your reasoning.

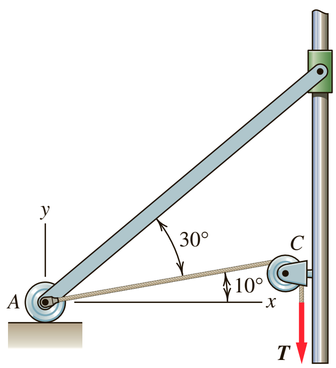


EX 4.4.7

4.4.8. [*] The 14 kN airplane sits on the tarmac. It has one front wheel and two rear wheels. Its center of gravity is as shown. If the system is the airplane, should this system be modeled as planar or nonplanar? Present your reasoning.

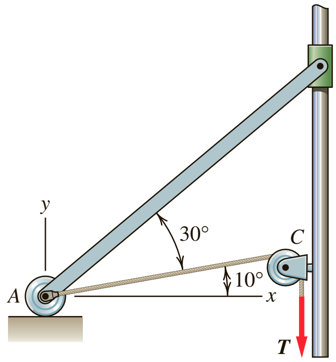
EX 4.4.8

4.4.9. [*] Uniform bar AB weighs 60 N and is pulled on by a rope at A. End B is able to slide up and down along a vertical guide. Assume gravity acts in the negative y-direction. If the system is the bar, should this system be modeled as planar or nonplanar? Present your reasoning.



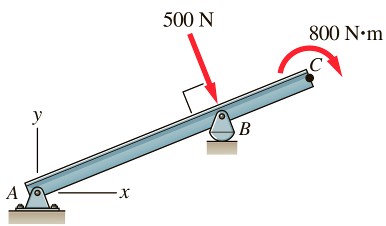
EX 4.4.9

4.4.10. [*] The uniform bar AB weighs 60 N and is pulled on by a rope at A. If the system is the assembly consisting of bar AB and the wheel at A, and gravity acts in the z-direction, would we reach the same conclusion about modeling the system as planar or nonplanar as when gravity acts in the negative y-direction? Present your reasoning.



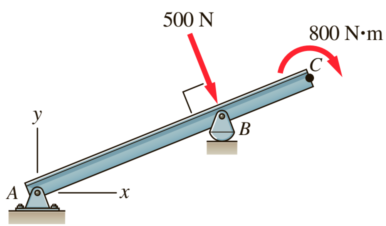
EX 4.4.10

4.4.11. [*] The beam AC is pinned to its surroundings at A and rests against a rocker at B. Ignore gravity. If the system is the beam, should the system be modeled as planar or nonplanar? Present your reasoning.



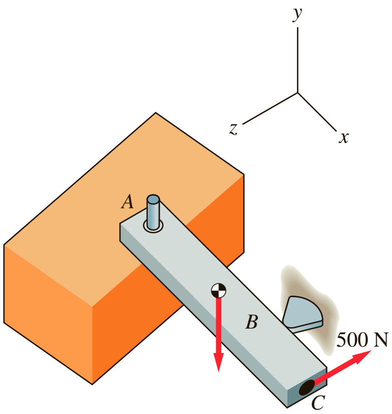
EX 4.4.11

4.4.12. [*] The beam AC is pinned to its surroundings at A and rests against a rocker at B. Define the beam as the system. If gravity is considered and acts in the negative y-direction, should the system be modeled as planar or nonplanar? Present your reasoning.



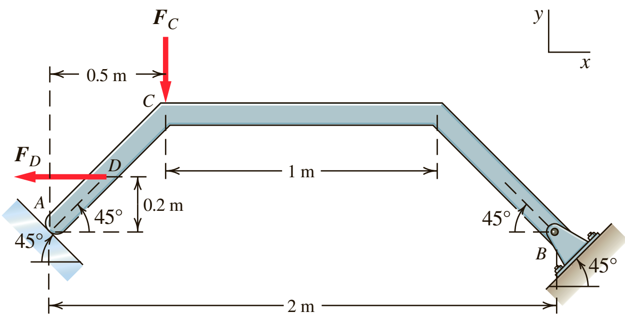
EX 4.4.12

4.4.13. [*] The beam AC rests on a block at A. In addition, there is a pin connection at A and a rocker at B. The system is the beam. Should this system be modeled as planar or nonplanar? Present your reasoning.



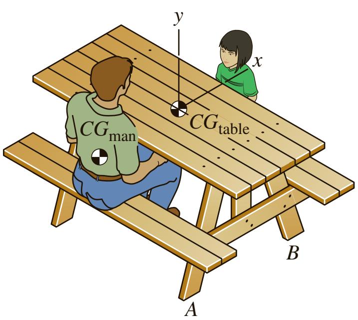
EX 4.4.13

4.4.14. [*] If gravity is considered and acts in the negative y-direction for the beam shown, should the system be modeled as planar or nonplanar? Present your reasoning.



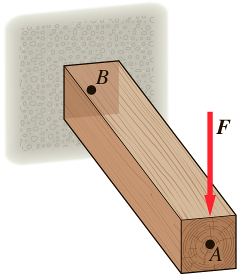
EX 4.4.14

4.4.15. [*] A child sits directly across from the man shown sitting at this picnic table. If the system is taken as the table, should it be modeled as planar or nonplanar? Present your reasoning.



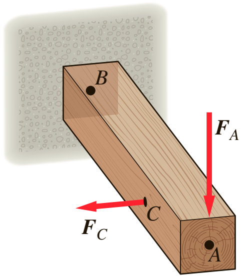
EX 4.4.15

4.4.16. [*] The bar AB is fixed to a wall at B. At end A, a force acts, as shown. Ignore gravity. Determine whether the bar can be classified as planar or nonplanar, presenting your reasoning.



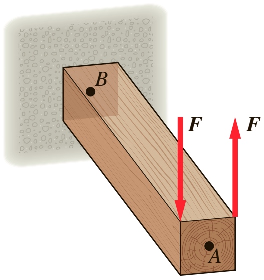
EX 4.4.16

4.4.17. [*] The bar AB is fixed to a wall at B. At end A and at C, forces act, as shown. Ignore gravity. Determine whether the bar can be classified as planar or nonplanar, presenting your reasoning.



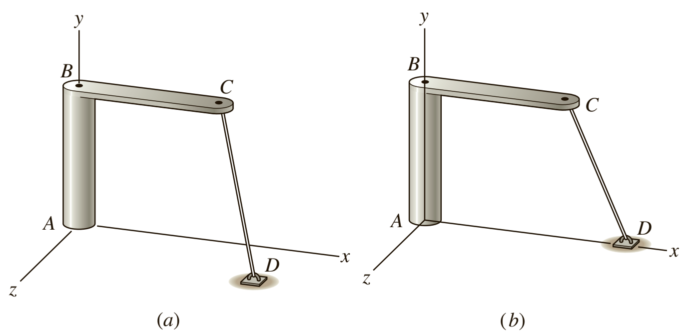
EX 4.4.17

4.4.18. [*] The bar AB is fixed to a wall at B. At end A, a force acts, as shown. Ignore gravity. Determine whether the bar can be classified as planar or nonplanar, presenting your reasoning.



EX 4.4.18

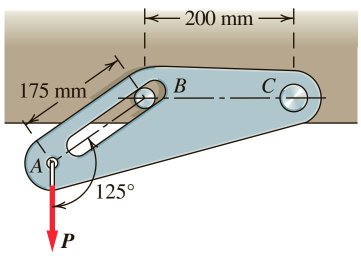
4.4.19. [*] A bracket ABC is tethered in two different configurations, as shown. Bracket ABC is the system. Determine whether each configuration should be modeled as planar or nonplanar. Describe your reasoning. Ignore the effect of gravity.



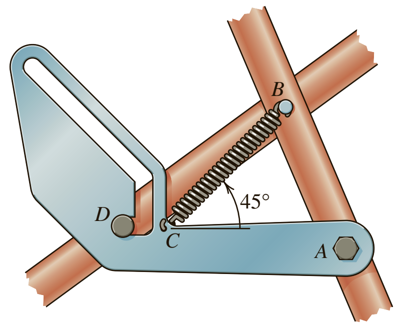
EX 4.4.19

4.4.20. [**] A cable pulls on the bracket with a force P of 3.0 kN. At C the bracket is attached to the wall with a pin connection, and B is a pin-in-slot connection. If the system is defined as the bracket and is considered to be planar, what loads act on the bracket at C and B? Use the general rule about "prevention of motion" to answer this question, then confirm your answer is consistent with information on loads given in **Table 4.1**. Present your answer as an annotated drawing of the bracket.

4.4.21. [**] The locking mechanism for a collapsible bicycle carrier locks into place when bracket AC is in the position shown, and the pin on member BD has moved down the slot to position D. The bracket is attached to member AB with a pin at A and a spring at B. If the system is defined as the bracket and is considered to be planar, what loads act on the bracket at A,C, and D? Use the general rule about "prevention of motion" to answer this question, then confirm your answer is consistent with information on loads given in **Table 4.1**. Present your answer as an annotated drawing of the bracket.



EX 4.4.20



EX 4.4.21

4.5 A STEP-BY-STEP APPROACH TO FREE-BODY DIAGRAMS

Learning Objective: Apply a step-by-step approach to creating free-body diagrams of various systems

We now outline the process for drawing a free-body diagram of a system in more detail; this is the DRAW step in our engineering analysis procedure.

Figure 4.5.1 gives a more detailed version of the four-step process introduced at the beginning of this chapter. Each step is described next.

1	**Study the physical situation**. Classify the system as a planar or nonplanar system.
2	Define system boundary and **draw the sytem** that is within the boundary. Include dimensions. **Establish coordinate system. State assumptions**.
3	Identify and draw **known loads** at their points of application (e.g., gravity forces should be at appropriate centers of gravity)
4	Identify **boundary conditions** including **supports**, drawing and labeling the loads they exert on the system. Supports may be classified as per **Table 4.1 and Table 4.2**.

Figure 4.5.1 Four Steps to Creating a Free-Body Diagram.

Step 1. Before diving into drawing, take time to **study the physical situation**. Consider what loads may be present at boundaries and ask yourself whether you have ever seen a similar support. Study actual hardware (if available); pick it up or walk around it to really get a sense of how the loads act on the system. This inspection helps in making modeling assumptions. **Classify the system as planar or nonplanar**. If the system can be modeled as planar, drawing the free-body diagram and writing and solving the conditions of equilibrium (covered in Chapter 5) all become easier. If you are unsure, consider the system to be nonplanar. Also, consider asking for advice and opinions from others.

Step 2. Define (either by imagining or actually drawing) a boundary that isolates the system from the rest of the world, and then **draw the system** that is within the boundary including important dimensions. The drawing should contain enough detail so that distances and locations of loads acting on the system can be shown accurately. Sometimes multiple views of the system will be needed, especially if the system is nonplanar. **Establish a coordinate system**. **State any assumptions** you make.

Step 3. Identify all **known loads**. Known loads may include **cross-boundary forces** acting on the system at appropriate centers of gravity as well as loads (including fluid forces) acting at the boundaries. Draw these known loads at their points (or surface areas) of application; identify each load on the drawing with a variable label and magnitude. Continue to state any assumptions you make.

Step 4. Identify the **unknown loads** associated with each **support** or **fluid boundary**, including those loads that act at discrete points and those that consist of distributed forces. If possible, classify each support as one of the standard supports (**Table 4.1** for planar systems and **Table 4.2** for nonplanar systems) to help in identifying the loads. If the support is not one of the standard supports defined in the tables, consider how that support restricts motion (either translation and/or rotation) in order to identify the loads acting on the system that prevent this motion. Add all of these loads to the drawing and identify each load on the drawing with a variable label. Continue to state any assumptions you make.

You now have a free-body diagram of a system, as well as a list of the assumptions made in creating it. The diagram consists of a depiction of the system and the external loads acting on the system. The loads are represented in the diagram as vectors or distributed forces and include variable labels and magnitudes (if known).

Engineering Judgment in Creating Your Free-Body Diagram Model

A free-body diagram is an idealized model of a real system. By making assumptions about the behavior of supports, dimensions, and the material, you are able to simplify the complexity of the real system into a model that you can analyze. You might want the model to describe the real situation exactly, but this is generally not an achievable goal, due

to limitations such as information, time, and money. What you do want, however, is a model that you can trust and that gives results that closely approximate the real situation.

In creating a model, an engineer must decide which loads acting on the system are significant. For example, a hinge on a door is often modeled as having no friction about its axis. Yet for most hinges, grease, dust, and dirt have built up, and there is actually some friction—some resistance to rotation. If friction is large enough, the engineer should include it in the model. However, if the friction is small enough that the door can still swing freely, the engineer may conclude that it is not significant for the problem at hand, and model the hinge loads as shown in Figure 4.5.2.

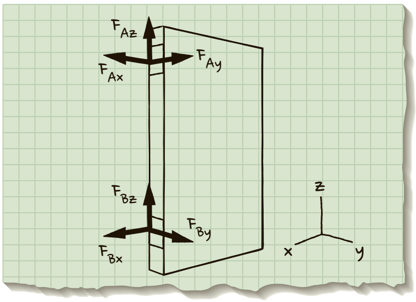


Figure 4.5.2 Forces acting at hinges A and B when hinges are frictionless.

Often the significance of loads is judged by their relative magnitude or location. For example, the weight of a sack of groceries is insignificant relative to the weight of an automobile carrying them but very significant if the vehicle is a bicycle. Whenever you are in doubt about the significance of a load, consider it significant.

Many of the examples in this book set the stage by making some of the assumptions regarding significance. In others, though, you need to judge the significance of a load based either on your own experience and/or on the advice of other engineers. Making these judgments gets easier with experience, though it is not uncommon for even expert engineers to ask for fellow engineers to affirm decisions. Any loads considered insignificant are not included in the free-body diagram and should be noted in the assumption list.

Check out the following examples of applications of this material.

- **Example 4.5.1 Creating a Free-Body Diagram of an Airplane Wing**
- **Example 4.5.2 Creating a Free-Body Diagram of a Ladder**
- **Example 4.5.3 Creating a Free-Body Diagram of a Nonplanar System**
- **Example 4.5.4 Creating a Free-Body Diagram of a Leaning Person**

EXAMPLE 4.5.1

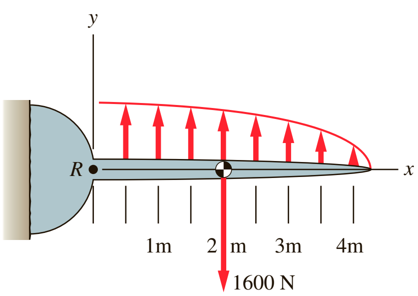


Figure 1 An airplane wing subjected to self-weight and lift.

The lift force on an airplane wing (which is actually a distributed load) can be modeled by eight forces as shown in Figure 1. The magnitude of each force is given in terms of its position x on the wing by

$$F_i = 300\sqrt{1-\left(\frac{x_i}{17}\right)^2}\ \text{N}, \quad \text{with } i = 1, 2, \ldots, 8 \tag{1}$$

where position $x_1 = 0.5$ m and $x_8 = 4.0$ m. The location $x = 0$ is at the root of the wing (location R); this where the wing connects with the fuselage. The weight of the wing $W = 1600$ N can be located at the midpoint of the wing's length. Create a free-body diagram of the wing.

Goal Draw a free-body diagram of the wing.

Given A planar view of the wing, a coordinate system, and some dimensions. We model the lift as eight forces acting upward at locations $x_i = 0.5$ m, 1.0 m, 1.5 m, . . . , 4.0 m. The weight of the wing is 1600 N, with its point of application at $x = 2.0$ m.

Create the Free-Body Diagram

Step 1—Study the physical situation Based on the information provided in **Figure 1**, all of the loads are applied in the x–y plane, so we model the wing as a planar system.

Step 2—Draw system; State assumptions Defining the wing as our system, we isolate the wing from the fuselage at its root. Ignoring any slight differences in the leading and trailing edges of the wing, we assumed the wing has a plane of symmetry (the x–y plane in **Figure 1**) allowing us to model the wing as a planar system. This means we are ignoring any twist on the wings that would occur from asymmetry. We also are assuming the wing is rigidly attached to the fuselage at R.

Step 3—Draw the known loads We determine the values of the lift force at each of the eight locations using function (1), obtaining the values shown in **Table 1**. We draw these loads along with the weight of the wing onto the isolated system as shown in **Figure 2**.

Table 1 Point Loads Representing Lift on the Wing

x_i	x(m)	F_i	F(N)
x_1	0.5	F_1	300
x_2	1	F_2	299
x_3	1.5	F_3	299
x_4	2	F_4	298
x_5	2.5	F_5	297
x_6	3	F_6	295
x_7	3.5	F_7	294
x_8	4	F_8	292

Step 4—Identify and draw support loads The boundary condition at R can be modeled as a fixed boundary condition; therefore, from **Table 4.1** we draw a force (represented as x and y components) and a moment about the z axis.

The free-body diagram is given in **Figure 3**.

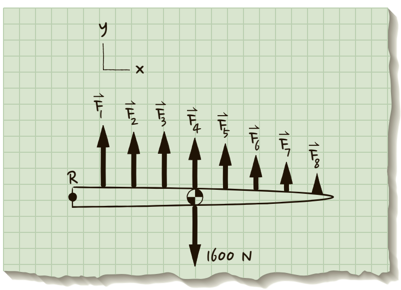


Figure 2 We draw the known loads on the isolated wing.

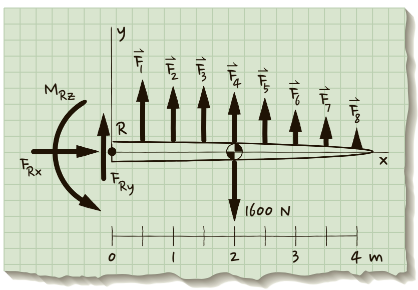


Figure 3 Free-body diagram of airplane wing.

EXAMPLE 4.5.2

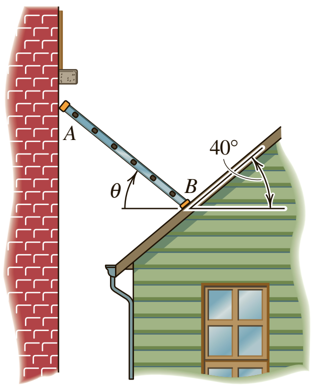


Figure 1 A ladder propped between two buildings.

The ladder in **Figure 1** rests against the wall of a building at A and on the roof of an adjacent building at B. If the ladder has a weight of 100 N and length 3 m, and the surfaces at A and B are assumed smooth, create a free-body diagram of the ladder.

Goal Draw a free-body diagram of the ladder.

Given A planar view of the ladder and some spatial information (length of ladder and angle of orientation with respect to the roof and wall). The roof angle is 40° with respect to the horizontal, and the surfaces at A and B are smooth.

Create the Free-Body Diagram

Step 1—Study the physical situation Typically a ladder has identical rungs (cross-pieces) and side rails (sides of ladder to which the rungs are attached), creating a plane of symmetry at its center. This plane of symmetry allows us to model the ladder as a planar system.

Step 2—Draw system; State assumptions We isolate the ladder from the wall (at A) and from the roof (at B). First, we assume that the weight of the ladder is significant and that gravity works downward in the vertical direction. Also, the ladder is uniform. By this we mean that the rungs are identical to one another, as are the two side rails. Because it is uniform, the ladder's center of mass can be located at its midpoint.

Step 3—Draw the known loads The only known load is the weight of the ladder, which we draw at the center of the ladder.

Step 4—Identify and draw support loads Table 4.1 for the case of normal contact without friction indicates that, at A and B, forces normal to the surface push on the ladder. No frictional force is applied at A or B because both surfaces are smooth. The resulting free-body diagram is shown in **Figure 2**. Notice the factor of two associated with both normal forces; this factor reflects that there are two side rails.

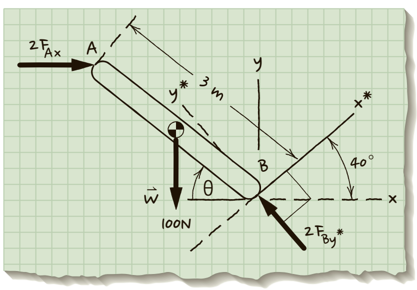


Figure 2 Free-body diagram of ladder.

EXAMPLE 4.5.3

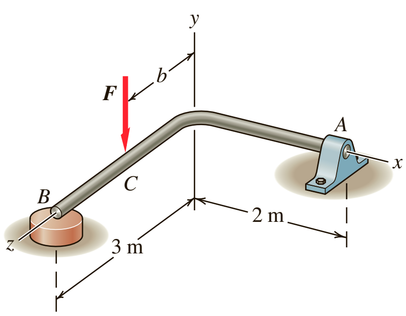


Figure 1 L-shaped bar.

The L-shaped bar in **Figure 1** is supported by a bearing at A and rests on a smooth horizontal surface at B, with $\|\mathbf{F}\| = 800$ N and $b = 1.5$ m. Ignore gravity. Create a free-body diagram of the bar.

Goal Draw a free-body diagram of the bar.

Given The dimensions of the bar, and the surface B is smooth. Support A is a bearing. Ignore the weight of the bar.

Create Free-Body Diagram

Step 1—Study the physical situation If we could closely inspect and touch the system we could determine if the bearing at A is a journal or thrust bearing. Without additional information we arbitrarily

assume it is a journal bearing. Since the loads applied at A, B, and C do not lie in a single plane, we must model the bar as a nonplanar system.

Step 2—Draw system; State assumptions We isolate the bar (the system) from the load $\boldsymbol{F}$, the surface at B, and the bearing at A. We have assumed a journal bearing at A.

Step 3—Draw the known loads The only known load is the applied 800-N load at C.

Step 4—Identify and draw support loads—The bar is supported by a single journal bearing at A; according to Table 4.2 this means two mutually perpendicular forces are applied in the y and z directions. We arbitrarily choose to draw them in the direction of the positive axes. In addition, the bearing resists rotations about the y- and z axes. Thus we draw the moments M_{Ay} and M_{Az}, also arbitrarily in the positive direction. At the smooth surface at B a normal force acts to push up on the bar. The resulting free-body diagram is shown in Figure 2.

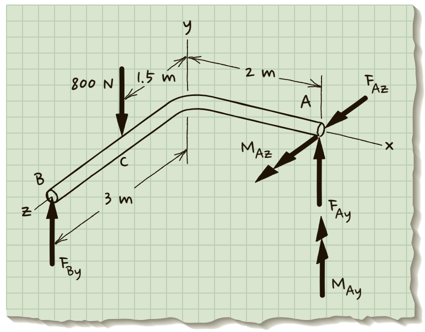


Figure 2 Free-body diagram of L-bar.

EXAMPLE 4.5.4

The 600-N person in Figure 1 is using a rope to lean back. The rope is oriented at an angle $\theta = 20°$ relative to the horizontal. Create a free-body diagram of the leaning person.

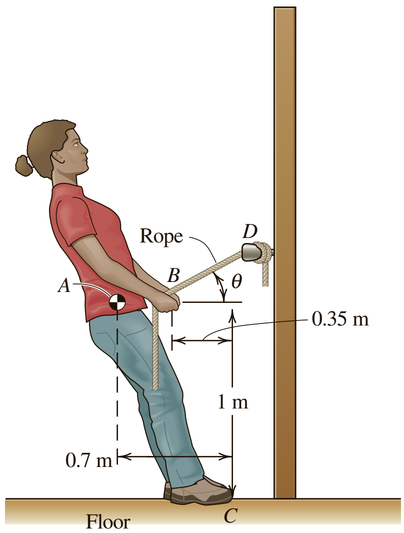


Figure 1 Person leaning back.

Goal Draw a free-body diagram of the leaning person.

Given The weight of the person and her center of mass, relevant dimensions of the situation, and the angle of the rope.

Create the Free-Body Diagram

Step 1—Study the physical situation Our system of interest is the leaning person. She is in a state of equilibrium with her feet on the floor and her hands pulling on a rope. Because all loads act in the x–y plane, we can model this as a planar system.

Step 2—Draw system; State assumptions We isolate the leaning person from the rope and the floor. We have assumed:

- Person is in equilibrium.
- Sufficient friction between the person's feet and the floor for her to not slide. (If she started to slide she would no longer be in equilibrium.)
- Rope is taut and of negligible weight. (This allows us to draw the rope tension along the line of the rope.)
- Gravity acts downward.
- System can be modeled as planar.

Step 3—Draw the known loads The only known load is the 600-N weight of the person, which we draw at A, the given center of mass.

Step 4—Identify and draw support loads We have unknown loads at C and B. The person's shoes at C are in contact with a rough surface. According to Table 4.1, the loads at C consist of a normal

force, F_{Cy}, and a friction force, F_{Cx}. Going with our intuition, we chose to draw F_{Cy} in the positive y direction and F_{Cx} in the negative x direction. When we perform an equilibrium analysis we will determine if the forces are actually in the correct directions.

At B the rope pulls on the person's hands. According to **Table 4.1** the force acts along the length of the rope, which is at an angle of 20° with respect to the horizontal.

The resulting free-body diagram is shown in **Figure 2**.

Alternatively, we could represent the rope force in terms of its x and y components as shown in **Figure 3**.

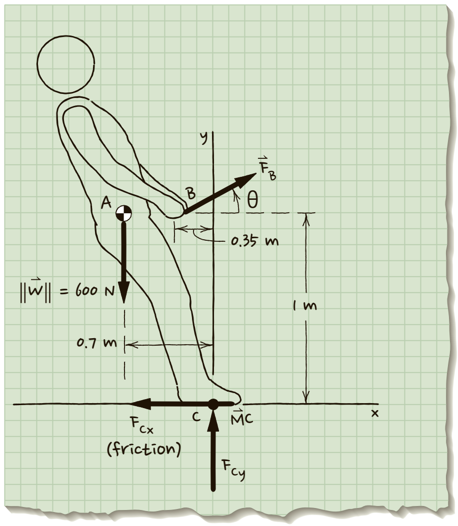


Figure 2 Free-body diagram of leaning person.

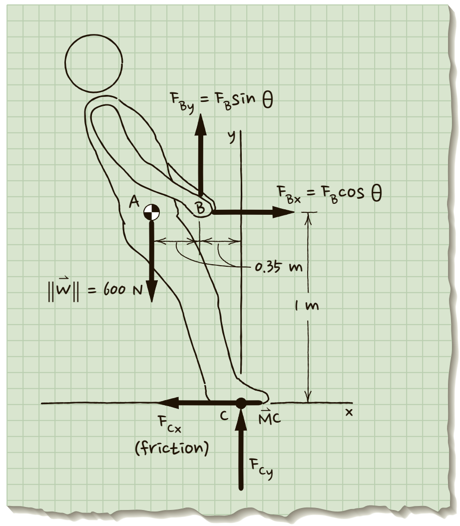
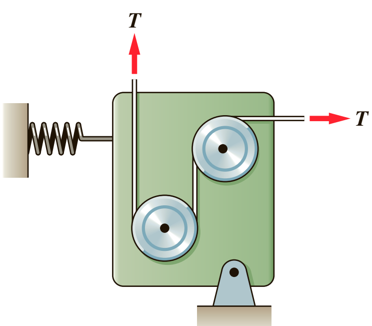


Figure 3 Free-body diagram of leaning person with rope force shown as x and y components.

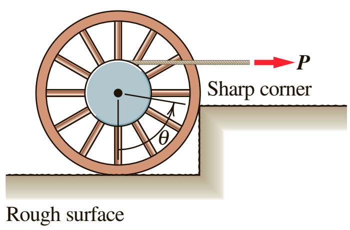
EXERCISES 4.5

4.5.1. [*] A tape guide assembly supported by a pin and spring is subjected to the loading as shown. Define the tape guide assembly as the system and draw its free-body diagram, using the four-step process presented in the chapter.



EX 4.5.1

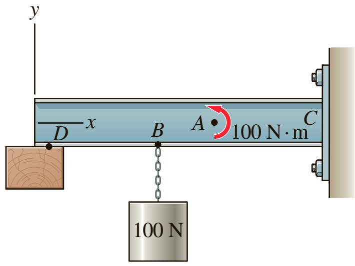
4.5.2. [*] Draw the free-body diagram of the wheel and pulley assembly shown, using the four-step process presented in the chapter.



EX 4.5.2

4.5.3. [*] A beam is fixed to the wall at C and rests against a block at D. Additional loads act on the beam, as shown. Define the beam as the system. Draw its

free-body diagram, using the four-step process presented in the chapter.

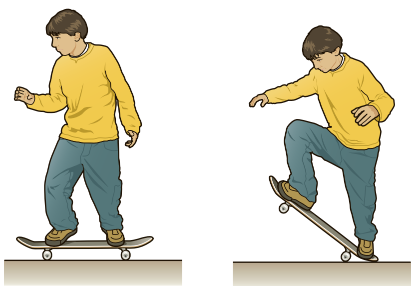


EX 4.5.3

4.5.4. [*] A skateboarder is shown in Position 1 standing on his board with his feet between the front and back wheels. In Position 2 he has moved his left foot backward and flipped the skateboard up on end. Define the skateboard as the system and draw its free-body diagram for the following cases, being sure to list the assumptions you made:

a. when it is in Position 1.

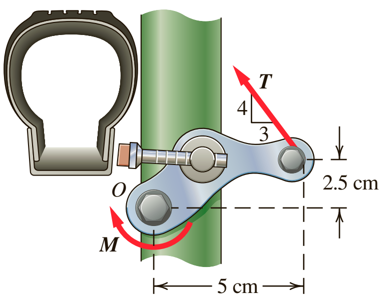
b. when it is in Position 2.



(a) Position 1 (b) Position 2

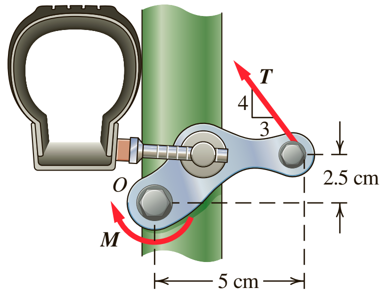
EX 4.5.4

4.5.5. [*] The arm of a cantilever bicycle brake pivots freely about O. A torsional spring at O exerts a return moment of magnitude $\|M\| = 1.65\ \text{N}\cdot\text{m}$ on the brake arm when in the position shown. Define the system as the brake arm with the brake pad. Draw its free-body diagram, using the four-step process presented in the chapter.



EX 4.5.5

4.5.6. [*] Consider the brake-arm shown when the brake pad is pressed against the wheel rim to slow the turning wheel. Define the system as the brake arm with the brake pad. Draw its free-body diagram using the four-step process presented in the chapter.



EX 4.5.6

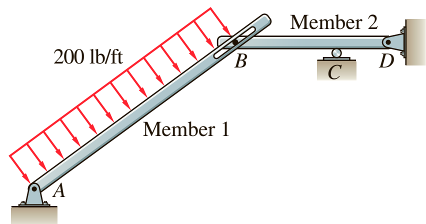
4.5.7. [*] Consider the frame shown. Draw the free-body diagram for the frame and its parts, as listed below. Make sure to

- be consistent in drawing the loads,
- follow the four-step process presented in the chapter, and
- count and list unknown loads in each diagram. How many of these loads are unique when you consider all three free-body diagrams?

  a. Frame $ABCD$ as the system

  b. Member 1 as the system

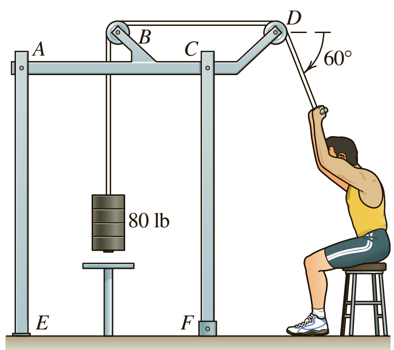
  c. Member 2 as the system



EX 4.5.7

4.5.8. [*] Consider the exercise frame that is fixed at E and pinned at F as shown. Define the entire frame as the system and draw its free-body diagram, using the four-step process presented in the chapter.

4.5.9. [*] Consider the exercise frame that is fixed at E and pinned at F as shown. Draw the free-body diagram for each part of the frame, as listed below. Make sure to

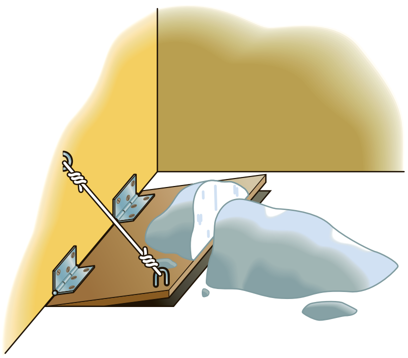


EX 4.5.8 and EX 4.5.9

- be consistent in drawing the loads,
- follow the four-step process presented in the chapter, and
- count and list unknown loads in each diagram. How many of these loads are unique when you consider all three free-body diagrams?
  a. pulley B
  b. pulley D
  c. upper frame member $ABCD$

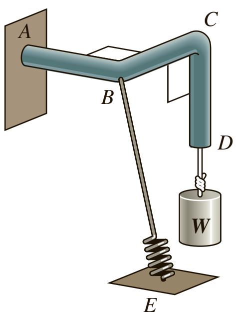
4.5.10. [*] A roof access cover is buried under a pile of snow as shown.

a. Draw the free-body diagram of the cover.
b. Describe in words how you showed the snow load and why you chose to show it in this manner.



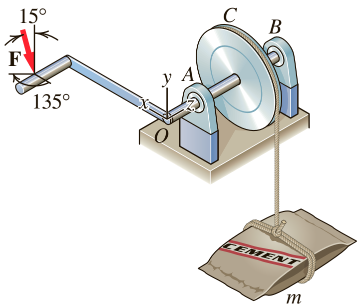
EX 4.5.10

4.5.11. [*] The bent bar is loaded and attached as shown. Draw the free-body diagram of the bar, assuming its weight can be ignored. Follow the four-step process presented in the chapter.



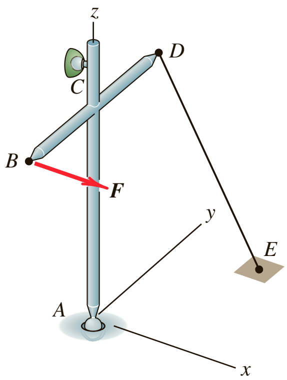
EX 4.5.11

4.5.12. [*] A 110-N force is applied to the handle of a construction site hoist, as shown. There is a journal bearing at A and a thrust bearing at B. Draw a free-body diagram of the handle—shaft—pulley assembly. Make sure to record any assumptions.



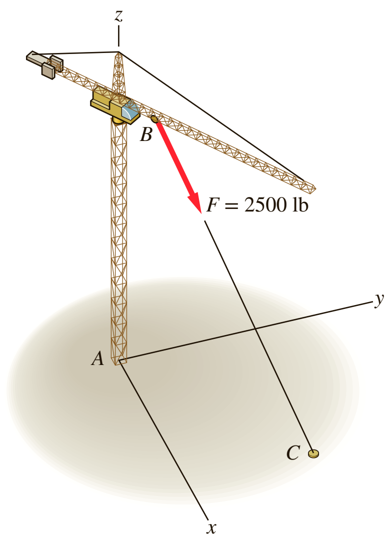
EX 4.5.12

4.5.13. [*] Consider the welded tubular frame shown with ball-and-socket supports at A and C. Draw the free-body diagram of the frame, following the four-step process outlined in the chapter.



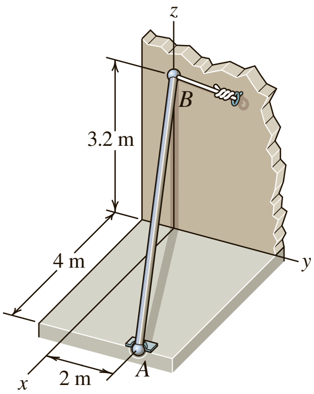
EX 4.5.13

4.5.14. [*] Consider the tower crane that fixed to the ground at A. Draw the free-body diagram of the crane, following the four-step process outlined in the chapter.



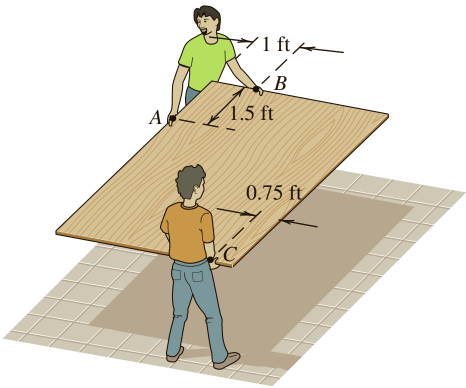
EX 4.5.14

4.5.15. [*] Consider the steel shaft shown. End A of the shaft is attached at the floor via a ball-and-socket joint, and at B it rests against a wall and is also tethered with a horizontal cable (as shown). Draw the free-body diagram of the shaft, following the four-step process presented in this chapter.



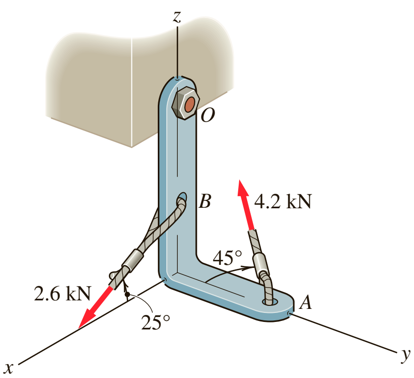
EX 4.5.15

4.5.16. [*] Two workers are carrying a 4-ft by 8-ft siding panel by grabbing the panel at the points shown. The panel weighs 80 lb. Define the panel as the system and draw a free-body diagram of the panel.



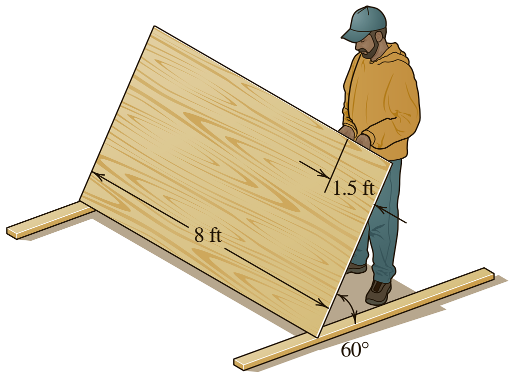
EX 4.5.16

4.5.17. [*] A bracket is bolted to the beam at O. The 2.6-kN cable force is contained in the x-z plane, and the 4.2-kN cable force in the x-y plane. Define the bracket as the system and draw a free-body diagram, following the four-step procedure presented in the chapter.



EX 4.5.17

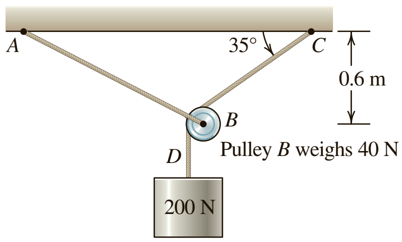
4.5.18. [*] A construction worker is tilting up a 4-ft by 8-ft sheet of plywood that has been stored on 2-in. by 4-in. skids. The sheet of plywood weighs 100 lb. In the current position shown it is tilted at 60° from horizontal. Define the sheet of plywood as the system and draw its free-body diagram.



EX 4.5.18

4.5.19. [**] Consider the cable—frictionless pulley—cylinder assembly shown. Draw the free-body diagram for parts, as listed below. Make sure to be consistent in drawing loads.

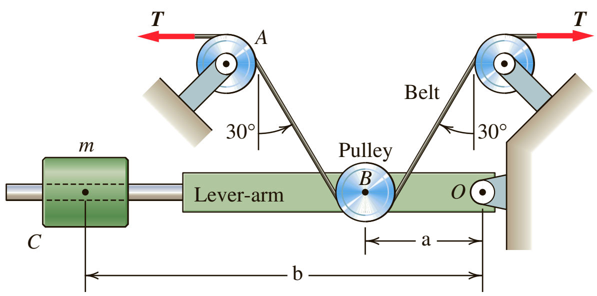
a. the cylinder as the system
b. the pulley as the system



EX 4.5.19

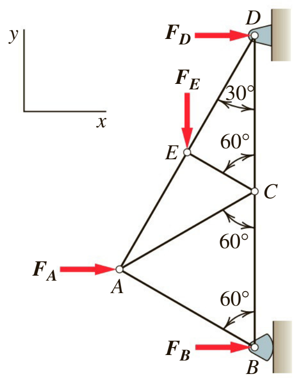
4.5.20. [**] For the belt-tensioning device shown, draw the free-body diagram for three systems, making sure to be consistent in drawing loads:

a. the pulley B as the system
b. the mass as the system
c. the pulley, lever-arm, and mass as the system



EX 4.5.20

4.5.21. [**] Consider the massless truss shown. Define the truss as the system and draw its free-body diagram.

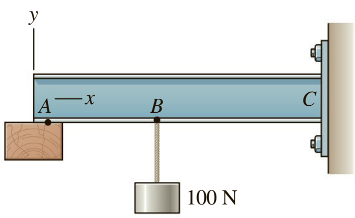


EX 4.5.21

4.5.22. [**] The beam of uniform weight is fixed at C and rests against a smooth block at A as shown. A 100-N weight hangs from point B. Draw the free-body diagram for the parts listed below, making sure to be consistent in drawing loads:

a. the beam as the system

b. the beam and the 100-N cylinder as the system



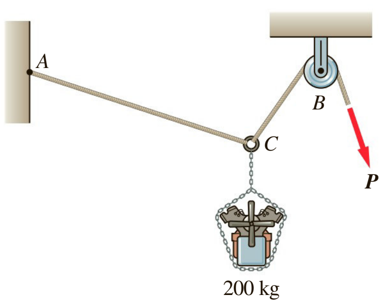
EX 4.5.22

4.5.23. [**] An engine is lifted with the pulley system shown. Draw the free-body diagram for the parts listed below, making sure to be consistent in drawing loads:

a. the pulley at B as the system

b. the engine and chain as the system

c. the ring at C as the system



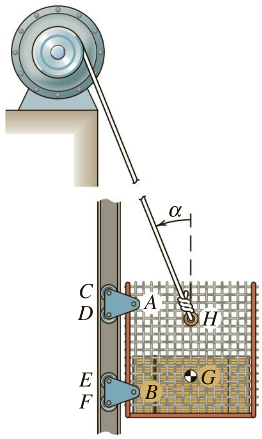
EX 4.5.23

4.5.24. [**] A construction site elevator bed and the materials it transports have a combined mass of 500 kg, with mass center at G. The elevator is raised by the cable. It is guided vertically by two sets of guide rollers at A (one on each side of the vertical guide post), and two sets at B. Draw the free-body diagram for each of the parts listed below, making sure to be consistent in drawing loads:

a. the elevator bed (including materials) minus the triangular guides at A and B as the system

b. the triangular guide at A (including the two wheels) as the system

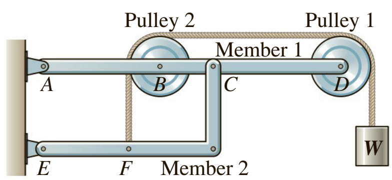
c. the elevator bed (including materials) and the triangular guides (including the wheels) at A and B as the system



EX 4.5.24

4.5.25. [**] Consider the frame shown. Draw the free-body diagrams for the entire frame, Member 1, Member 2, Pulley 1, and Pulley 2. Make sure to

- be consistent in drawing the loads,
- follow the four-step process presented in the chapter, and
- count and list unknown loads in each diagram. How many of these loads are unique when you consider all three free-body diagrams?

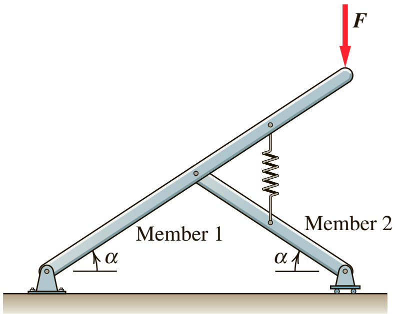


EX 4.5.25

4.5.26. [**] Consider the frame shown. Draw the free-body diagrams for the entire frame, Member 1, and Member 2. Make sure to

- be consistent in drawing the loads,
- follow the four-step process presented in the chapter, and

- count and list unknown loads in each diagram. How many of these loads are unique when you consider all three free-body diagrams?



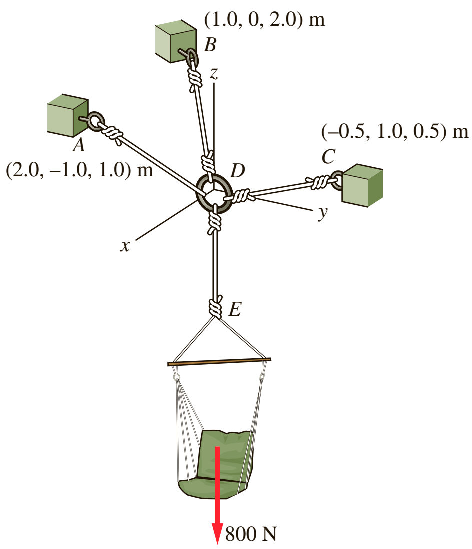
EX 4.5.26

4.5.27. [**] A hanging chair is suspended, as shown. A person weighing 800 N is sitting in the chair (but is not shown). Define

a. the ring at D as the system and draw its free-body diagram

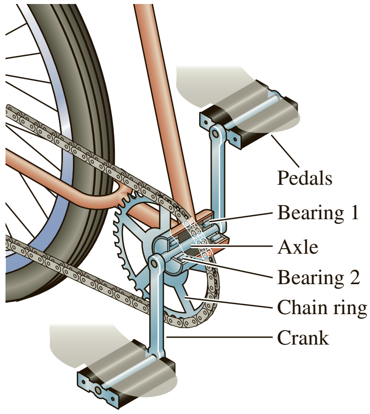
b. the chair (including the knot at E) as the system and draw its free-body diagram

c. the eyelet fastener at A as the system and draw its free-body diagram



EX 4.5.27

4.5.28. [***] Consider the bottom bracket assembly of a bicycle that is being ridden, as shown. The assembly consists of an axle, chain ring, left and right cranks, and left and right pedals. The axle is held in the frame by two sets of ball bearings. For the position shown, draw a free-body diagram of the bottom bracket assembly.

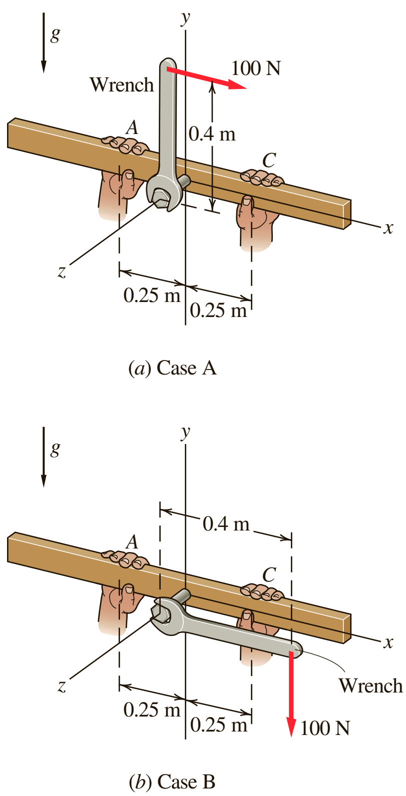


EX 4.5.28

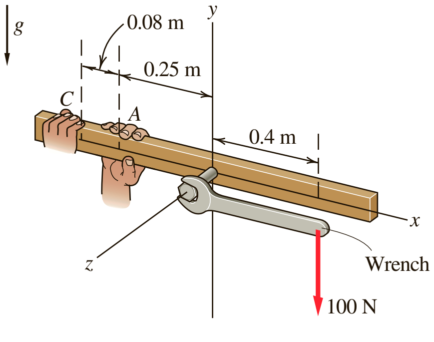
4.5.29. [***] The following three cases involve a 2×4 wooden board, a bolt and a wrench. In each case a force is applied, and hands are used to react to this force, as shown. Imagine what forces the hands would need to apply to the 2×4, then draw the free-body diagram for the following systems:

a. *Case A*. Define the system as the 2×4, bolt, and wrench.

b. *Case B*. Define the system as the 2×4, bolt, and wrench.



EX 4.5.29



(*c*) Case C

EX 4.5.29

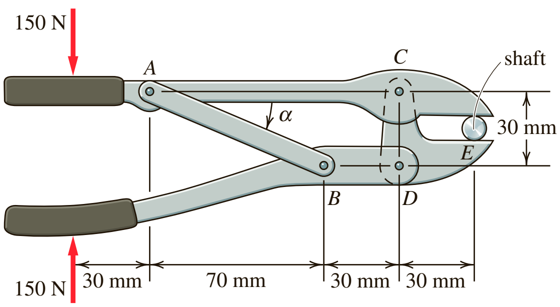
c. *Case C*. Define the system as the 2×4, bolt, and wrench.

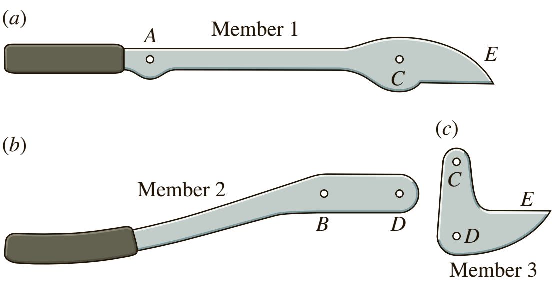
d. Repeat **a**, **b**, and **c** if the system is defined as just the 2×4 and the bolt.

4.5.30. [***] Consider the pair of pliers shown. Draw the free-body diagrams for the systems listed below, making sure to

- be consistent in drawing the loads,
- follow the four-step process presented in the chapter, and
- count and list unknown loads in each diagram. How many of these loads are unique when you consider all three free-body diagrams?

a. Member 1 (*a*) as the system
b. Member 2 (*b*) as the system
c. Member 3 (*c*) as the system
d. the entire pair of pliers as the system (minus the shaft it is clamping)

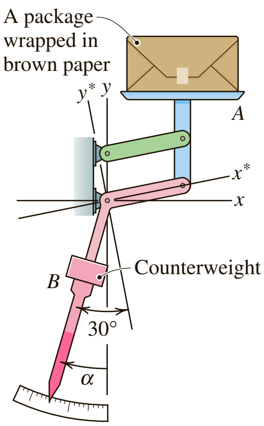




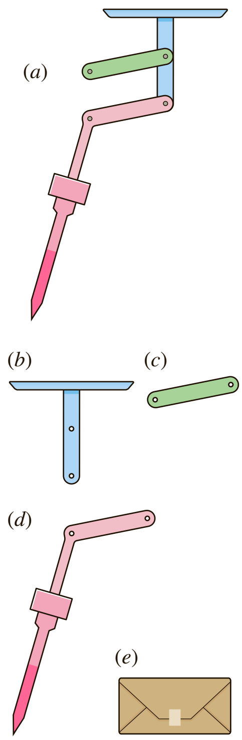
EX 4.5.30

4.5.31. [***] Consider the mechanism used to weigh mail shown. A package placed on the upper tray causes the weight pointer to rotate through an angle α. Neglect the weights of the members except for the counterweight at B, which has a mass of 4 kg. For a particular package, $\alpha = 20°$. Draw the free-body diagram for each of the defined systems, making sure to be consistent in drawing loads:

a. the system as shown in *a*
b. the system as shown in *b*
c. the system as shown in *c*
d. the system as shown in *d*
e. the system as shown in *e*



EX 4.5.31



EX 4.5.31

4.6 JUST THE FACTS

This chapter is devoted exclusively to the task of creating free-body diagrams. We build on the **engineering analysis procedure** presented in Chapter 1.

Types of External Loads Acting on Systems

A free-body diagram must include all of the external loads acting on the system you are analyzing. Broad categories of these external loads are **gravitational force** (or **gravity** for short), which manifests itself as weight, **boundary support loads** (or **supports** for short), and **fluid boundary** loads. Loads may be modelled as distributed loads or equivalent point loads.

Planar System Supports and Nonplanar System Supports

A number of standard boundary supports are commonly used in engineered structures. These supports are represented with standard idealized support loads, also called reactions. For example, if a pin connects a system to the rest of the world, the pin applies a force to the system. Standard boundary supports for planar systems are summarized in **Table 4.1** and for nonplanar systems in **Table 4.2**.

Modeling Systems as Planar or Nonplanar

Being able to distinguish between **planar** and **nonplanar** systems is important in developing the system model. As a general principle, if it is possible to define a single plane in which all forces and moments lie, then a system can be modeled as a planar system. In addition, if the system is geometrically symmetrical and the external forces acting on it are parallel to the system's symmetry plane and the external moments are all about an axis perpendicular to that plane, it can be modeled as a planar system.

A Step-by-Step Approach to Free-Body Diagrams

A four-step process for creating a **free-body diagram** was introduced in this chapter. This four-step process consists of:

- Studying the physical situation. This helps in determining whether the system can be modeled as planar or nonplanar.
- Drawing the system and stating assumptions.
- Drawing known loads acting on the system. Some of these may be point loads and others distributed loads.
- Identifying and drawing support loads. For this, **Tables 4.1** and **4.2** really come in handy.

SYSTEM ANALYSIS (SA) EXERCISES

SA4.1 Checking on the Design of a Chair

To increase the seating capacity during basketball games, collapsible and portable floors on rollers have been installed, as shown in **Figure SA4.1.1**. The chairs that go with this portable flooring system are placed onto the floors but are not connected to the floor. **Figure SA4.1.2** shows the dimensions of one of these chairs.

Situation: The basketball game of Wolfpack against UNC Chapel Hill is underway. As Sierra, an engineering student taking Statics, is cheering her team, she notices a woman sitting on the front edge of one of the chairs described above. Because of the small size of the hinge that holds the chair's seat, Sierra becomes concerned for the safety of the woman. Imagine that you are in Sierra's place and re-create what goes through her mind as she has a sudden flashback to moments and the concept of free-body diagrams covered in her Statics class.

(a) Assuming that the mass of the woman is 61 kg, what is the maximum moment that the slotted hinge has to bear? **Figure SA4.1.3** may be helpful.

(b) As indicated in **Figure SA4.1.3**, the entire moment created by the woman has to be held by the connector pins and stop-pins. Consider the chair with the dimensions in **Figures SA4.1.2**. Draw free-body diagrams of the: (1) seat with hinge bracket *A* and (2) hinge bracket *B* with 90° slot. The dimensions in **Figure SA4.1.4** will be useful. Note that the 90° slot is milled

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Figure SA4.1.1 Mobile floor and chairs in the Reynolds Coliseum

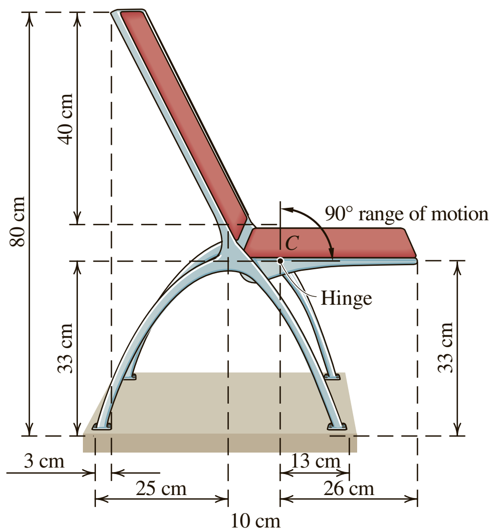


Figure SA4.1.2 Dimensions of the chair

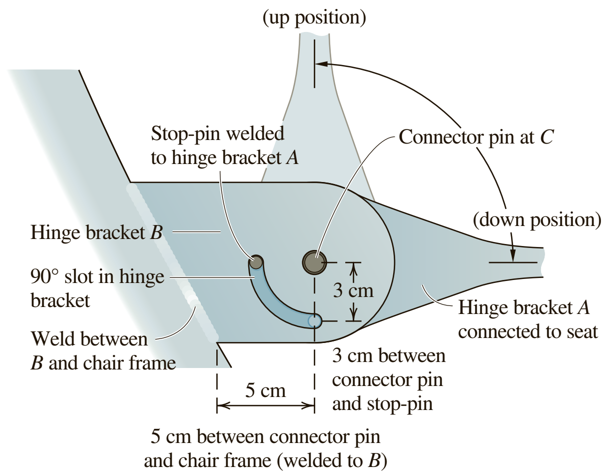


Figure SA4.1.3 Basic design of hinge for seat

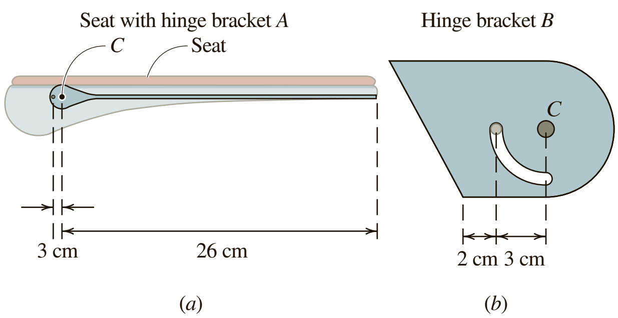


Figure SA4.1.4 Dimensions associated with (*a*) seat with hinge bracket *A*; (*b*) hinge bracket *B* with 90° slot

into bracket *B* while the stop-pin is welded to bracket *A* at a distance of 3 cm from *C*, with the center of rotation at the connector pin.

(c) Based on the free-body diagram of the seat with hinge bracket *A* created in (b), write expressions for the equivalent load (consisting of an expression for the equivalent force and an expression for the equivalent moment) acting at a moment center at *C*. If the magnitude of the expression for equivalent moment is zero, what force must the stop-pin apply to the seat plate?

(d) **Figure SA4.1.5** provides a plot that shows the maximum allowable force that the stop-pin can safely hold. What size pin is needed to ensure that the woman sitting on the chair is safe? (*Remember:* Each chair has two hinges and stop-pins.)

(e) Assume that the maximum mass of a person for which each chair should be designed is 100 kg. What pin size would you recommend?

(f) Finally, it is very likely that in the process of sitting down, a person might actually "fall" into the chair. Since you did not study the dynamic effect yet, let's assume you should triple the static force to account for the person's deceleration. What would your final recommendation be to the seat manufacturer regarding the pin size? (*Hint:* You may have to extend the plot by identifying the function that underlies the curve, which depends on the cross-sectional area of the pin and a fixed force per cm^2.)

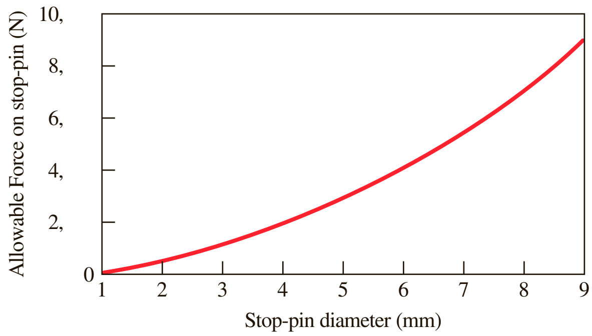


Figure SA4.1.5 Shear strength versus pin diameter

SA4.2 Following the Path of the Gravitational Force

As you are leaving the coliseum after the game, you recognize still another mechanical "beauty" in a corner—a large trash cart ready for action (**Figure SA4.2.1**). We can safely assume that the cart would be able to carry a total of 150 kg.

Leonhard Bernold

Figure SA4.2.1 Trash Cart

Part I: Lifting a load

(a) *Situation 1:* Draw a free-body diagram for the situation in which the janitor is just starting to dump a full cart by moving the handle on the back of the cart upward. The center of gravity for the full load distributed is shown in **Figure SA4.2.2**. If the equivalent moment about a moment center at *C* (the contact point between the front wheels and the ground) is zero, what is the magnitude of the force that the janitor must apply to the bin when the rear wheels just lift off the ground? State any assumptions you make.

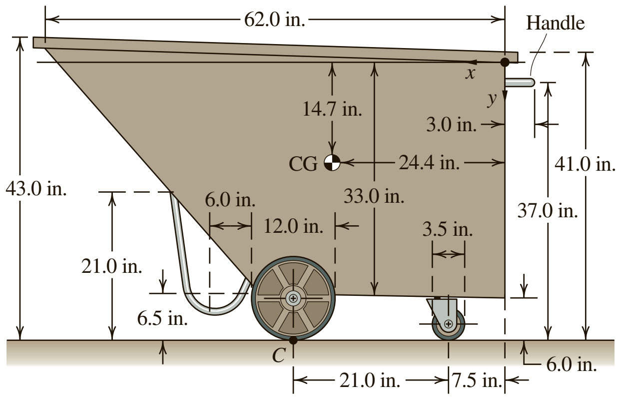


Figure SA4.2.2 Dimensions of trash cart load distributions

(b) *Situation 2:* Instead of the load shown in **Figure SA4.2.2**, the cart is loaded with several heavy concrete pieces with a total mass of 150 kg that are placed close to the handle at position *A*; see **Figure SA4.2.3*a***. Draw a free-body diagram for the situation in which the janitor is just starting to dump the cart by moving the handle on the back of the cart upward. If the equivalent moment about a moment center at *C* is

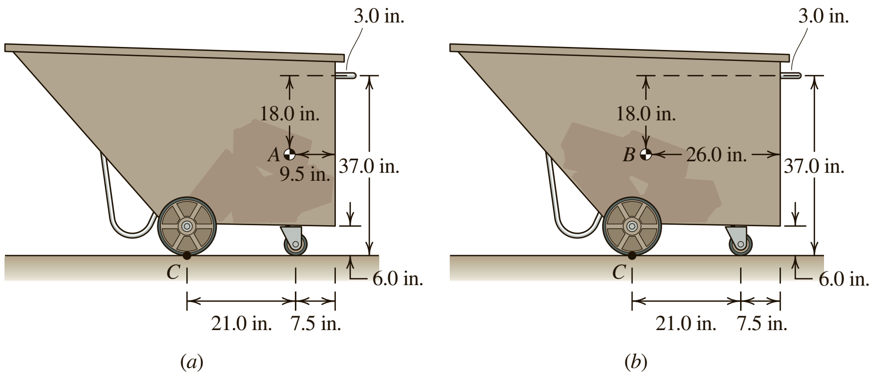


Figure SA4.2.3 The cart is carrying 150-kg mass with the center of mass at (*a*) Position *A*; (*b*) Position B

zero, what is the magnitude of the force that the janitor must apply to the bin when the rear wheels just lift off the ground? State any assumptions you make.

(c) *Situation 3:* If the heavy concrete pieces with a total mass of 150 kg are placed at *B* (**Figure SA4.2.3*b***), what force must the janitor apply to just lift the rear wheels off the ground? Should the janitor worry more about hurting his or her back in Situation 1, 2, or 3, and why?

(d) Will the magnitude of the force that the janitor is required to apply to the cart to move it from Position 1 to Position 2 (see **Figure SA4.2.4**) decrease, increase, or remain the same as the cart goes from Position 1 to Position 2? Include the rationale for your answer (no calculations are required).

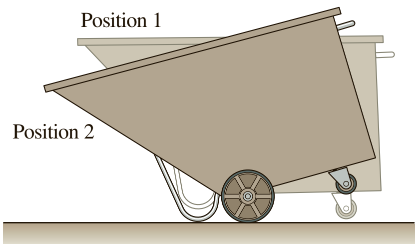


Figure SA4.2.4 Moving cart from Position 1 to Position 2

Part II: How the load is transferred to the ground

You notice that the cart has large front wheels and a sturdy main axle that connects the wheels to one another. The main axle is attached to a trash bin as shown in **Figure SA4.2.5**. Because the axle is attached to each of the front wheels with a bearing, the wheels rotate and the axle does not. Let's consider how the weight of the trash contained in the bin is transferred to the ground. To do this we will create a series of free-body diagrams. Consider the following cross-sectional view of the trash bin in **Figure SA4.2.6** with various components labeled.

Figure SA4.2.5 Cart suspension system

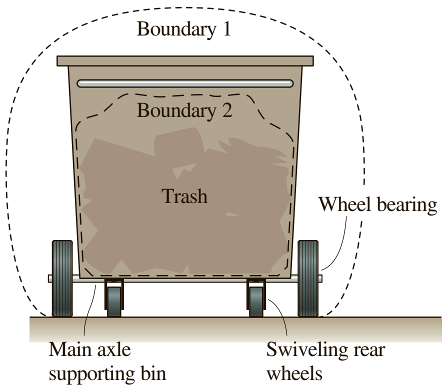


Figure SA4.2.6 Cross section through cart

(a) Draw a free-body diagram of the system defined by Boundary 1 (the cart and the trash it is holding).

(b) Draw a free-body diagram of the system defined by Boundary 2 (the trash).

(c) Draw a free-body diagram of the main axle and wheel assembly.

(d) Now we are ready to separate the main axle from the wheels by pushing the shaft out of the bearing wheel

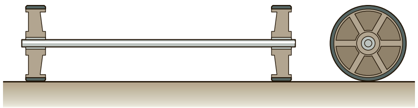


Figure SA4.2.7 Axle and wheels

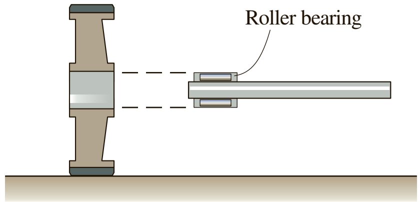


Figure SA4.2.8 Details of axle–wheel connection

hub. **Figure SA4.2.7** shows details of the axle–wheel connection and the bearings that connect the axle and the wheels. Draw free-body diagrams for the wheels and the axle after they are separated from one another. **Figure SA4.2.8** may be useful in visualizing this.

(e) Based on the free-body diagrams you created in (a)–(d), which of the schematics shown in **Figure SA4.2.9** most accurately depicts how the weight of the trash is transferred to the ground?

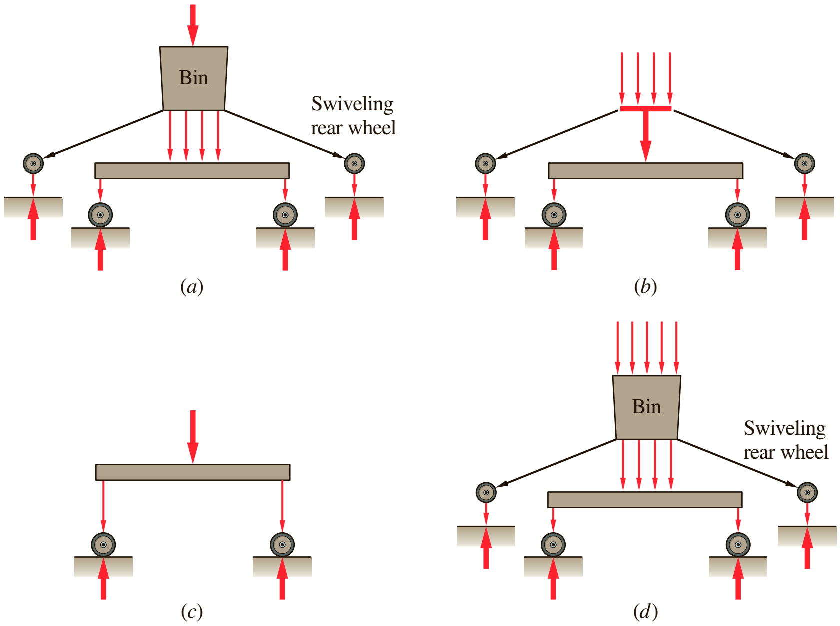


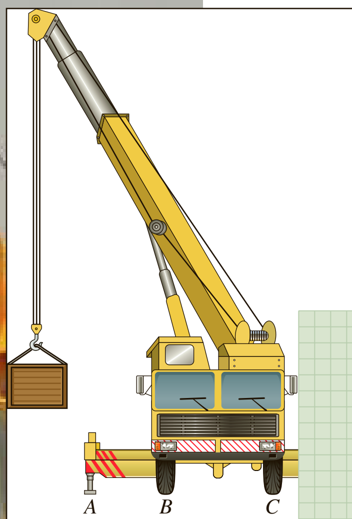
Figure SA4.2.9 Candidate load path schematics

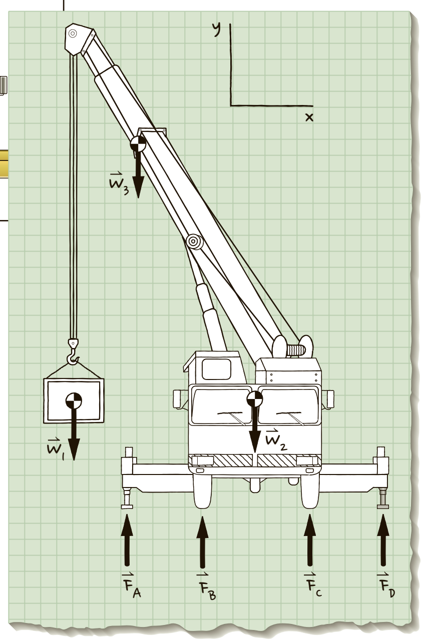
CHAPTER 5

MECHANICAL EQUILIBRIUM

Hazlan Abdul Hakim/Getty Images

In Chapter 4 we outlined the process for creating a free-body diagram of a system. This diagram describes the key geometric features of the system and the loads that act on it. In this chapter we use free-body diagrams to determine whether a system is balanced. This will allow us, for example, to figure out the weights needed to create a balanced and visually pleasing mobile. On a larger scale, it will allow us to calculate the counterweight required to balance a boom so that it doesn't tip while lifting its load.





LEARNING OBJECTIVES

On completion of this chapter, you will be able to:

- **Describe the conditions of equilibrium (5.1).**
- **Write the equations of equilibrium (5.2).**
- **Identify conditions of equilibrium for planar and nonplanar systems (5.2).**
- **Use the conditions of equilibrium to carry out structured static analysis of**
  - **Planar systems (5.3)**
  - **Nonplanar systems (5.5)**
- **Recognize when a system can be modeled as**
  - **A particle (5.4.1)**
  - **A two-force member (5.4.2)**
  - **A three-force member (5.4.3)**
- **Apply a structured analytical procedure to calculate the equilibrium of**
  - **A particle (5.4.1)**
  - **A two-force member (5.4.2)**
  - **A three-force member (5.4.3)**
  - **A frictionless pulley (5.4.4)**
- **Apply the equations of equilibrium to subsystems within larger systems (5.6).**
- **Define and identify statically determinate, statically indeterminate, and underconstrained systems. (5.7)**

5.1 CONDITIONS OF MECHANICAL EQUILIBRIUM

Learning Objective: Describe the conditions of equilibrium

A system that is in **mechanical equilibrium** (where equilibrium is a state of balance) is one that experiences zero linear acceleration and zero angular acceleration about any axis fixed in an inertial reference frame. With no acceleration, the system either does not move at all or, if it does move, has constant translational and angular velocities. A system in equilibrium has restrictions, or conditions, on the forces and moments that act on it.

Remember that the picture we have developed of a system and the loads acting on it is the **free-body diagram**. In this chapter we will see how a free-body diagram describes the **conditions of equilibrium** for a system.

When a system is in mechanical equilibrium, the loads—in the form of forces and moments—acting on the system have particular relationships to one another. These relationships can be described mathematically by Equations (5.1) and (5.2).

$$\boldsymbol{F}_{\text{net}} = \sum^{\substack{\text{All forces}\\\text{acting on}\\\text{system}}} \boldsymbol{F} = 0 \qquad \textbf{force equilibrium condition} \tag{5.1}$$

This condition represents the vector sum of all the forces acting on the system and says that this vector sum is zero. It reflects Newton's first and second laws: A system that is not accelerating has *no net force* acting on it.

$$M_{net} = \sum^{\substack{\text{All moments}\\\text{acting on}\\\text{system}}} M = 0 \qquad \textbf{moment equilibrium condition} \tag{5.2}$$

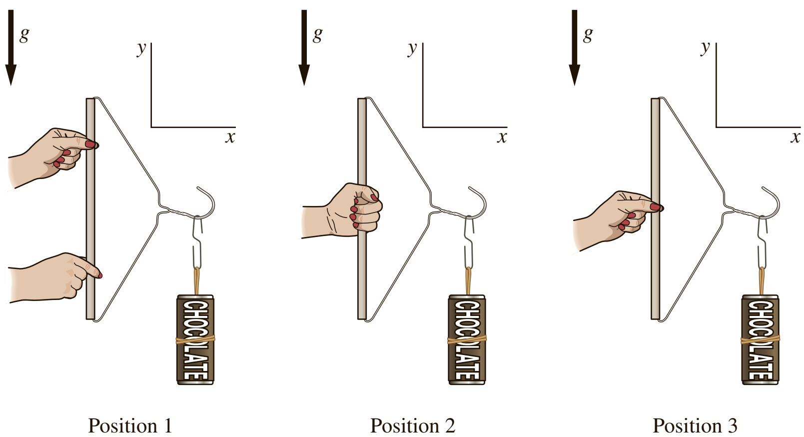
This condition represents the vector sum of the moments acting on the system and says that the vector sum of all moments is zero. That is, there is *no net moment* on the system.

Conditions (5.1) and (5.2) are necessary and sufficient conditions for a system to be in mechanical equilibrium.

EXERCISES 5.1

5.1.1. [*] For this exercise, you will need the following materials: one wire clothes hanger, a rubber band, a paper clip, a weight (a candy bar is suggested), a scale (ruler).

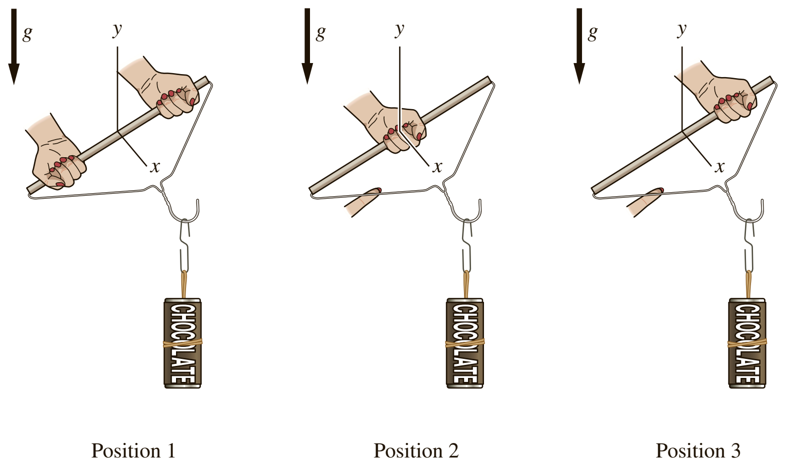
Configure the hanger, rubber band, paper clip, and weight vertically in each of the positions shown in the figure. (*Note:* Some of the positions may not be achievable if the hanger is underconstrained.) Draw a free-body diagram for the hanger in each position. Assuming that the weight of the candy bar is known, note how many unknown loads are acting on the hanger. Will the equations of equilibrium allow you to find these unknowns?



EX 5.1.1

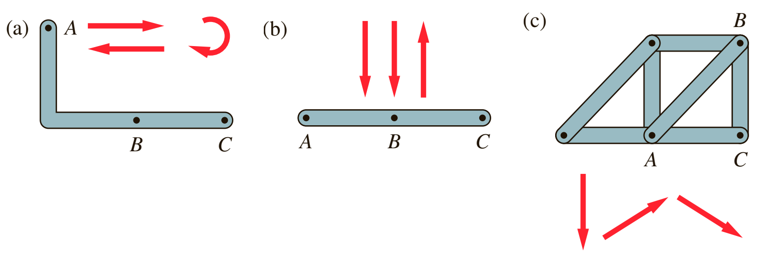
5.1.2. [*] For this exercise, you will need the following materials: one wire clothes hanger, a rubber band, a paper clip, a weight (a candy bar is suggested), a scale (ruler).

Configure the hanger, rubber band, paper clip, and weight vertically in each of the positions shown in the figure. (*Note:* Some of the positions may not be achievable if the hanger is underconstrained.) Draw a free-body diagram for the hanger in each position. Assuming that the weight of the candy bar is known, note how many unknown loads are acting on the hanger. Will the equations of equilibrium allow you to find these unknowns?



EX 5.1.2

5.1.3. [*] For each of the systems shown in (*a*)–(*c*), place the three given loads at the labeled points so that the system will be in equilibrium. (*Note:* There may be more than one answer for each system.)



EX 5.1.3

5.2 THE EQUILIBRIUM EQUATIONS

Learning Objectives

- **Write the equations of equilibrium.**
- **Identify conditions of equilibrium for planar and nonplanar systems.**

It is often more convenient to work with the equilibrium conditions in (5.1) and (5.2) in terms of vector components. We write each force in the summation in condition (5.1) in component form and then group the *x*, *y*, and *z* components as

$$\boldsymbol{F}_{\text{net}} = \sum^{\substack{\text{All forces}\\\text{acting on}\\\text{system}}} \boldsymbol{F} = (\Sigma F_x)\boldsymbol{i} + (\Sigma F_y)\boldsymbol{j} + (\Sigma F_z)\boldsymbol{k} = 0$$

where F_x, F_y, and F_z represent the force scalar components in the x, y, and z directions of an orthogonal coordinate system. For this component representation of $\boldsymbol{F}_{\text{net}}$ to be zero, each component must be zero.

$$\sum F_x = 0 \tag{5.3A}$$

$$\sum F_y = 0 \tag{5.3B}$$

$$\sum F_z = 0 \tag{5.3C}$$

These are the **force equilibrium equations**. Each equation states that if we add up all the force components acting on a system in a particular direction, the sum must be zero if there is equilibrium. Equations (5.3A)–(5.3C) comprise the requirements for **force balance**.

Similarly, we can write each moment $\boldsymbol{M}$ in the summation in condition (5.2) in terms of its vector components as

$$\boldsymbol{M}_{\text{net}} = \sum^{\substack{\text{All moments}\\\text{acting on}\\\text{system}}} \boldsymbol{M} = \left(\sum M_x\right)\boldsymbol{i} + \left(\sum M_y\right)\boldsymbol{j} + \left(\sum M_z\right)\boldsymbol{k} = 0$$

where M_x, M_y, and M_z represent the individual moment components about the x, y, and z axes of an orthogonal coordinate system, respectively. For this component representation of $\boldsymbol{M}_{\text{net}}$ to be zero, each of its components must be zero.

$$\sum M_x = 0 \tag{5.4A}$$

$$\sum M_y = 0 \tag{5.4B}$$

$$\sum M_z = 0 \tag{5.4C}$$

Equations (5.4A)–(5.4C) are the **moment equilibrium equations**. These equations for **moment balance** state that if a system is in equilibrium, the sum of the moments about each axis must be zero. To apply them, we will need to choose a **moment center**.

The six equilibrium equations in (5.3) and (5.4) are true for any system that is in mechanical equilibrium (i.e., a balanced system). We can solve the equations for the unknowns they contain. Alternatively, we can work directly with the equilibrium conditions in (5.1) and (5.2) to ensure that the vector sums of forces and moments acting on the system are zero.

Any system that is in equilibrium must satisfy all six equilibrium equations. Three of these equations, however, are automatically satisfied for a planar system. A planar system is one in which all the forces and moments involved are contained in the same plane. Planar and non-planar systems were introduced in Section 4.4.

For a planar system in the x–y plane, the three equations that must be satisfied for equilibrium are the **planar equilibrium equations**:

$$\sum F_x = 0 \tag{5.5A}$$

$$\sum F_y = 0 \tag{5.5B}$$

$$\sum M_z = 0 \tag{5.5C}$$

Check out the following example of an application of this material.

- **Example 5.2.1 Using a Free-Body Diagram to Write Equilibrium Equations**

EXAMPLE 5.2.1 USING A FREE-BODY DIAGRAM TO WRITE EQUILIBRIUM EQUATIONS

The 600-N person in **Figure 1** is using a rope to lean back. The rope is oriented at an angle $\theta = 20°$ relative to the horizontal. Is the tension in the rope less than, the same as, or greater than the tension when the rope is horizontal? If the rope fails when its tension exceeds 500 N, should the person be concerned about it failing?

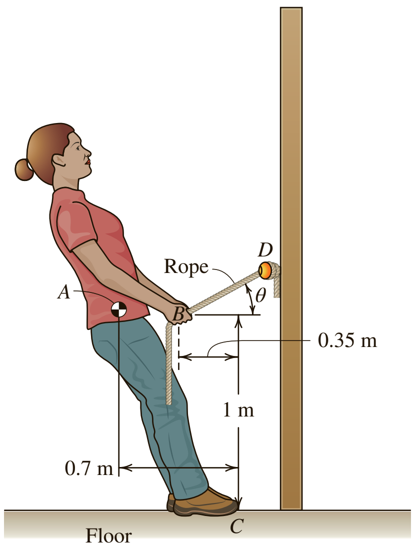


Figure 1 Person leaning back.

Strategy Overview We are being asked to find *the tension in the rope that will cause the rope to fail.* We will begin by drawing a free-body diagram of the leaning person. Then we will use the free-body diagram to write the equations of equilibrium in terms of a variable θ. Finally, we will analyze the system for $\theta = 0°$ and $\theta = 20°$ to determine whether the rope is strong enough.

Goals Compare the tension in the rope in angled and horizontal configurations. Compare this tension to the rope's 500-N design capacity, to determine whether the rope may break.

Given Dimensions associated with the situation, the weight of the person, the angle θ at 20°, and the location of her center of gravity (point A).

Assumptions

- The person is in equilibrium.
- The rope is of negligible weight, it is taut, and it is at an angle θ (=20°) relative to the horizontal.
- There is sufficient friction between the person's feet and the floor for the person not to slide.
- The system can be modeled as a planar system.

Free-Body Diagram Define the person as the system and isolate her from the surroundings. Draw a free-body diagram of the person as shown in **Figure 2**.

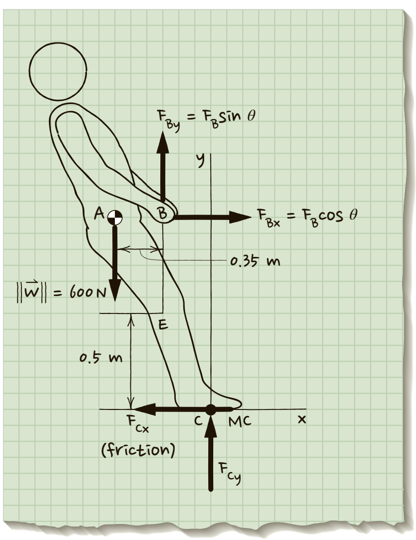


Figure 2 Free-body diagram of leaning person.

Equilibrium Equations (Option 1) Examine the free-body diagram and write the three force equilibrium equations introduced in (5.3A)–(5.3C):

$$\sum F_x = 0 \Rightarrow F_B \cos\theta - F_{Cx} = 0 \quad (1)$$

$$\sum F_y = 0 \Rightarrow -600\,\text{N} + F_B \sin\theta + F_{Cy} = 0 \quad (2)$$

$$\sum F_z = 0 \Rightarrow 0 = 0 \quad (3)$$

We write the three moment equilibrium equations (5.4A)–(5.4C), but we are strategic in selecting the location of the moment center. By choosing a moment center that eliminates several unknowns, we simplify the mathematics needed to solve for the rope tension.

We select point C for the moment center.

$$\sum M_{x@C} = 0 \Rightarrow 0 = 0 \tag{4}$$

$$\sum M_{y@C} = 0 \Rightarrow 0 = 0 \tag{5}$$

$$\sum M_{z@C} = 0 \Rightarrow +(600\,\text{N})(0.7\,\text{m}) - (F_B)(\cos\theta)(1\,\text{m}) - (F_B)(\sin\theta)(0.35\,\text{m}) = 0 \tag{6}$$

Solve With $\theta = 20°$, solve (6) and obtain $F_B = 396$ N.

Since 396 N < 500 N (the capacity of the rope), the person should not be concerned that the rope will fail under the given conditions.

Equilibrium Equations and Solve (Option 2) Alternatively, we select point E as the moment center.

$$\sum M_{x@E} = 0 \Rightarrow 0 = 0 \tag{4 alt}$$

$$\sum M_{y@E} = 0 \Rightarrow 0 = 0 \tag{5 alt}$$

$$\sum M_{z@E} = 0 \Rightarrow +(600\,\text{N})(0.35\,\text{m}) + (F_{Cy})(0.35\,\text{m}) - (F_{Cx})(0.5\,\text{m}) - (F_B)(\cos\theta)(0.5\,\text{m}) = 0 \tag{6 alt}$$

When we try to solve (6 alt), we discover that we have three unknowns (F_B, F_{Cx}, F_{Cy}). First we have to solve for two of the three unknowns using (1) and (2) and then substitute these results into (6 alt), resulting in

$$F_B = 396\text{ N},\ F_{Cx} = 373\text{ N},\ F_{Cy} = 464\text{ N},\ F_{Cz} = 0\text{ N}$$

We conclude that our first choice of point C as the moment center resulted in simpler mathematics.

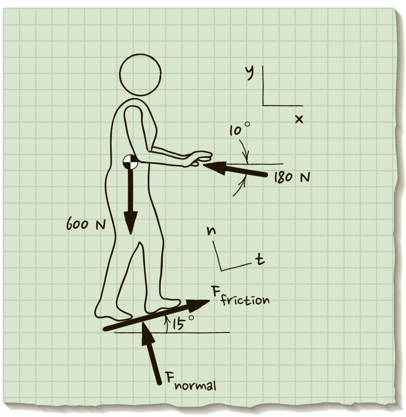
Checks We do a visual check of the calculations. Consider that in **Figure 2**, F_{Bx} and F_{Cx} are equal and opposite, resulting in no net force in the x direction. They also form a clockwise couple of 373 N·m that is countered or balanced by the counterclockwise moment created by $\boldsymbol{W}$ and F_{By}, F_{Cy}. For a system in mechanical equilibrium, there is no net moment, which is what inspection of the free-body diagram and our calculated forces show.

Answers When the rope is at 20°, the rope tension is $F_B = 396$ N. Performing the same calculations with $\theta = 0°$ results in $F_B = 420$ N. The tension in the rope in the angled position (396 N) is less than in the horizontal position (420 N). Is this what you expected?

EXERCISES 5.2

5.2.1. [*] A 600-N woman is pushing a recycling container up a hill. The forces acting on her are shown in the free-body diagram. Write the force equilibrium equations in the x and y directions for the woman.

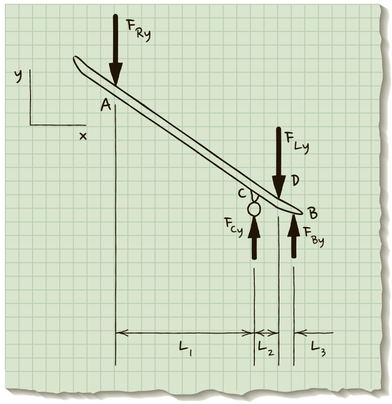
EX 5.2.1a



Free-body diagram EX 5.2.1b

5.2.2. [**] A skateboarder flips up his skateboard onto the rear wheel. The free-body diagram of the board is shown. Write applicable equilibrium equations for this free-body diagram.

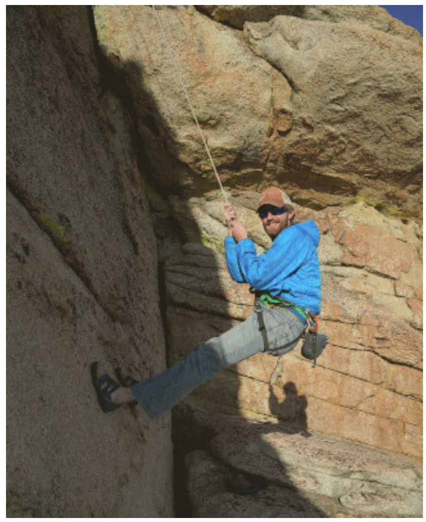
EX 5.2.2a



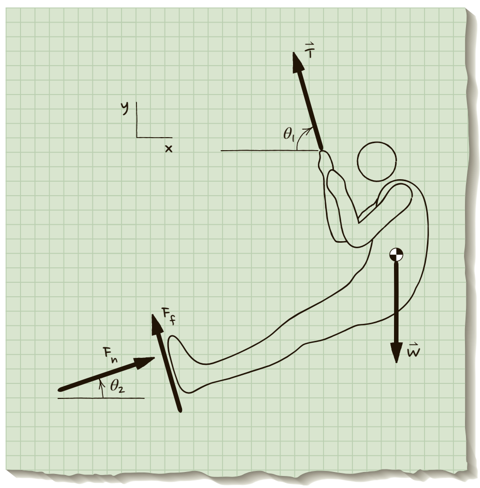
EX 5.2.2b

5.2.3. [**] A climber is being lowered down a cliff. The forces acting on him are shown on the free-body diagram. Write the force equations in the x and y directions for the free body.

Photo courtesy Daniel Merrick

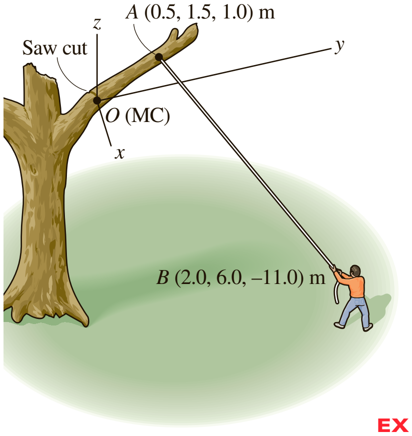


EX 5.2.3a



EX 5.2.3b

5.2.4. [**] An 800-N tree trimmer uses a rope to pull with force $\boldsymbol{F}$ on a tree branch that has been sawn nearly through. Draw a free-body diagram of the man and write the force equilibrium equations in the x, y, and z directions.



EX 5.2.4

5.2.5. [***] A vendor of violins at a festival created a display by tying a pole at points A and B on his truck and hanging five violins from it. Draw a free-body diagram of the pole by isolating it from the truck and the five violins, and write the applicable force and moment equilibrium equations. Assume that the horizontal distance between A and B is L, the distance between B and the first violin is $L/5$, and the distance between each of the subsequent violins is $L/5$.

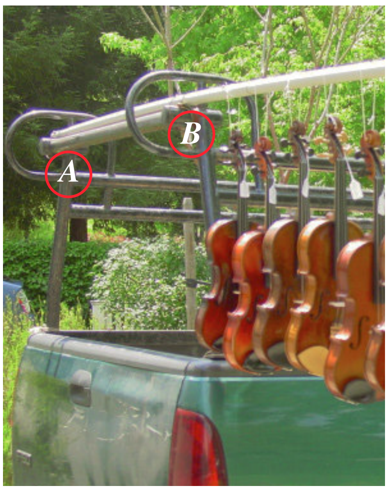


Photo by Thalia Anagnos

EX 5.2.5

5.2.6. [***] The hood of this Toyota Prius is attached to the car by hinges at A and B, and supported by a rod that is attached with a hinge at C and leans on the car at D. Isolate the hood from the car and draw its free-body diagram. Write the applicable force and moment equilibrium equations for the hood, assuming that A is the origin of the xyz coordinate system with the x axis through AB. Assume that the horizontal distance between A and B is L, and point C has coordinates (k, l, m). Use A as the moment center.

Photo by Thalia Anagnos

EX 5.2.6

5.3 APPLYING THE PLANAR EQUILIBRIUM EQUATIONS

Learning Objective: Use the conditions of equilibrium to carry out structured static analysis of planar systems.

Armed with the equilibrium equations, you are equipped to take on a wide variety of engineering analysis problems. In fact, writing the equilibrium

equations for a system is one of the critical steps in the engineering analysis procedure introduced in Chapter 1 and summarized in **Table 5.1**. The combination of free-body diagrams and equilibrium equations is perhaps the most powerful engineering tool in this course, and is at the core of investigating the integrity of load-bearing systems.

Table 5.1 **Three-Step Engineering Analysis Procedure**

Analytical Task	Solution Steps	Questions to Answer
Step 1. Understand the Problem by. . .	Defining **Goals**	Why is the analysis being undertaken? What questions do you want to answer?
	Identifying **Givens**	What information are you given?
	Listing **Assumptions**	What assumptions about the situation will you need to make in order to model the system of interest?
Step 2. Set up the model by. . .	Drawing a **Free-Body Diagram**	What is your system? What loads act on that system?
	Writing **Equilibrium Equations**	How do various loads acting on the system add together?
	Solving **Equations**	How do you solve the equations for the unknown values?
Step 3. Confirm values by. . .	Carrying out **Checks**	How do you know your values are correct?
	Summarzing **Answers**	How do the answers address the goals of the analysis?

Writing Equilibrium Equations We write the equilibrium equations by reading load information from the free-body diagram. In the case of force equilibrium equations for a planar system (5.5A and 5.5B), this is fairly straightforward (just don't forget any forces). In the case of the moment equilibrium, judicious application of Equation 5.5C will simplify the mathematics in the **solving equations** part of the analysis procedure. By choosing a point through which force lines of action pass, you can reduce the number of terms in the moment equation (because a force does not create a moment about a point that its line of action passes through). This in turn will reduce the number of simultaneous equations you must deal with.

Checking Your Results Before applying this step to some planar equilibrium problems, let's focus for a moment on Step 3 of the analysis procedure. Just solving equations and getting numbers is not enough. There needs to be a **check** of the results using technical knowledge, engineering judgment, and common sense. Use one or a combination of the following strategies to check your results.

- **Perform a visual check.** Draw a free-body diagram of the system with your final results. Look to see whether forces pointing in one direction (for example, the positive x direction) are balanced by at least one force acting in the opposite direction. Similarly, look for balancing couples.

- **Verify that the results make sense.** Look at the order of magnitude, units, and directions of loads. A negative answer for a load value means that the direction in which the load acts is opposite the direction depicted in the free-body diagram.
- **Check your algebra.** Substitute numerical answers into the equilibrium equations to verify that the equations are satisfied.
- **Select an alternate moment center.** Write the moment equilibrium equation with respect to the alternate moment center. Substitute in your previously determined numerical answers and ensure that equilibrium is satisfied.
- **Use an alternate method of solution.** You could use a graphical solution, such as a force triangle, to check vector addition.
- **Compare values with design requirements (if available).** Draw conclusions about the adequacy of the design.

Remember, checks are about confirming that you set up and solved your equations correctly, and that your approach to addressing the question or problem is reasonable. Checks are about building confidence and credibility that your work is correct.

Summarizing and Interpreting Your Results Engineering analysis is undertaken to provide new insights into how a system may behave. After drawing diagrams, solving equations, and checking answers, the engineer loops back to relate the answers to the problem's goals. For example, if there is concern that a member might break, the calculated loads acting on the member will be compared with the member's material strength. Or if the analysis is undertaken to determine whether mechanical performance meets a design requirement, the analytically predicted performance will be compared to the design requirement.

Interpreting your results for a textbook problem will often be as straightforward as clearly stating the values of the desired loads at the end of your work. In more complex problems it might involve writing a short paragraph or creating a table of results to summarize your answers.

Doing Planar Analysis The analysis procedure applies to any system, whether it is nonplanar or **planar**. If the system is planar, at most three equilibrium equations can be written. As developed in Section 5.2, for a planar system in the *x–y* plane, these equations are

$$\sum F_x = 0 \tag{5.5A}$$

$$\sum F_y = 0 \tag{5.5B}$$

$$\sum M_z = 0 \tag{5.5C}$$

For a particular system, Equations (5.5) will result in three **linearly independent equations**. *Independent* means that none of the equations can be derived by scaling and/or combining the other equations, and *linearly* means that all the unknowns (commonly, unknown loads) are to the first power. These equations can then be solved for at most three unknowns.

Check out the following examples of applications of this material.

- **Example 5.3.1 Applying the Analysis Procedure to a Planar Equilibrium Problem**
- **Example 5.3.2 Analysis of a Simple Structure**
- **Example 5.3.3 Analysis of a Planar Truss**

EXAMPLE 5.3.1 APPLYING THE ANALYSIS PROCEDURE TO A PLANAR EQUILIBRIUM PROBLEM

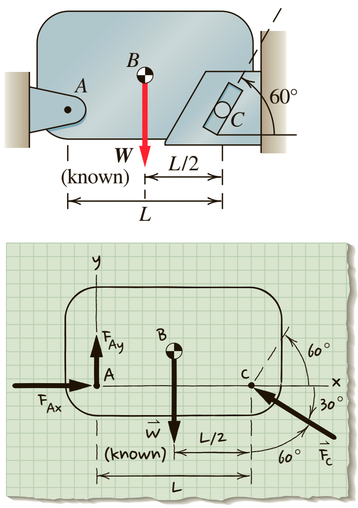


Figure 1

Consider a plate of weight W and dimension L, with a pin at A and a frictionless slider at C (**Figure 1**). Its associated planar free-body diagram is also shown in the figure.

Find the loads on the system acting at A and C.

Strategy Overview We are being asked to *find the three unknown forces F_{Ax}, F_{Ay}, and F_C*. We will use the free-body diagram to write the equilibrium equations. Then we will solve the equilibrium equations to determine the forces at A and C.

Goals Find the three unknown forces F_{Ax}, F_{Ay}, and F_C.

Given Dimensions of the plate, location of center of gravity, support A is a pin, the angle of slot C and that the slot is frictionless, the weight of the plate, the free-body diagram.

Assumptions Model the system as planar.

Free-Body Diagram The free-body diagram is given in **Figure 2**.

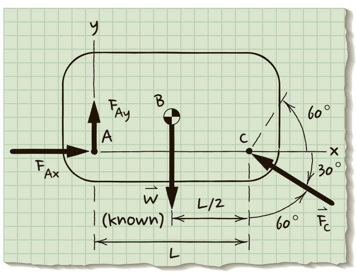


Figure 2

Equilibrium Equations Because this is a planar situation, we need to consider only the three equations introduced in (5.5A)–(5.5C):

$$\overset{+}{\rightarrow}\sum F_x = F_{Ax} - F_C \sin 60° = 0$$

$$\uparrow + \sum F_y = F_{Ay} + F_C \cos 60° = 0$$

We need to choose a moment center for the third equilibrium equation, which is related to the moment about the z axis. Any point will do, but some points will make the math easier when it comes to solving the equilibrium equations for the unknowns.

We choose A as the moment center because then our moment equation will not contain either F_{Ax} or F_{Ay}. This means that we will have only one unknown in our equation.

$$+\circlearrowleft \sum M_{z@A} = -W\left(\frac{L}{2}\right) + F_C \cos 60°(L) = 0$$

Solve Solving these three equations, we determine F_{Ax}, F_{Ay}, and F_C:

$$F_C = \frac{W}{2\cos 60^\circ}$$

$$F_{Ax} = \frac{W\tan 60^\circ}{2}$$

$$F_{Ay} = \frac{W}{2}$$

Checks We illustrate several forms of checks. Generally, we will want to do a visual check in combination with some additional calculations. For a *visual check* we note that the signs on the three calculated forces are all positive. This means that the directions of the calculated forces are all the same as we assumed on the initial free-body diagram.

This makes sense: The rightward force F_{Ax} on the free-body diagram can balance the leftward-pointing x component of $\boldsymbol{F_C}$.

Also, the upward force F_{Ay} and the y component of $\boldsymbol{F_C}$ can balance the downward weight of the plate. F_{Ay}, F_{Ax}, and $\boldsymbol{F_C}$ also can "teeter-totter" around a moment center at B, thus causing no rotation.

So our visual check indicates that things are on track.

An alternative check involves choosing a moment center other than A and seeing whether the structure is in equilibrium.

Moment center at C:

$$\sum M_{z@C} = W\left(\frac{L}{2}\right) - \underbrace{F_{Ay}}_{F_{Ay}=\frac{W}{2}}(L) \underbrace{=}_{\text{Yes, equals zero!}} 0$$

The resulting moment equals zero, confirming our calculation of F_{Ay}. Similarly, if we choose a moment center at B, we can check our calculation of F_C.

$$\sum M_{z@B} = \underbrace{F_C}_{F_C=\frac{W}{2\cos 60^\circ}} \cos 60^\circ\left(\frac{L}{2}\right) - \underbrace{F_{Ay}}_{F_{Ay}=\frac{W}{2}}\left(\frac{L}{2}\right) \underbrace{=}_{\text{Yes, equals zero!}} 0$$

The moments indeed sum to zero; our answers are confirmed.

An alternate calculation check is to add the force answers together *graphically* and see whether they indeed add to zero, as shown in Figure 3. The resulting force polygon confirms that the forces acting on the plate do add to zero.

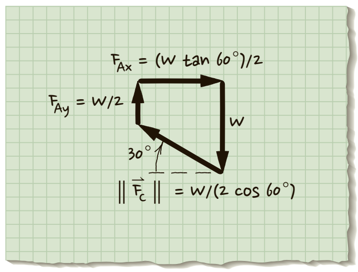


Figure 3

Answers The values are shown in Figure 4.

$$F_C = \frac{W}{2\cos 60^\circ}$$

$$F_{Ax} = \frac{W\tan 60^\circ}{2}$$

$$F_{Ay} = \frac{W}{2}$$

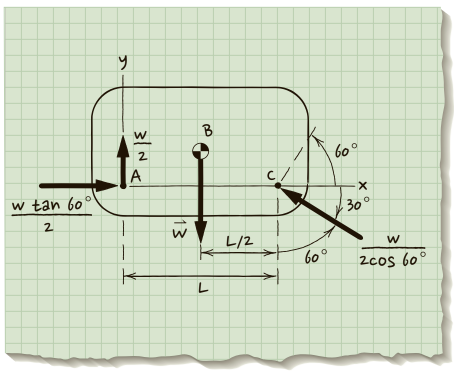


Figure 4

EXAMPLE 5.3.2 ANALYSIS OF A SIMPLE STRUCTURE

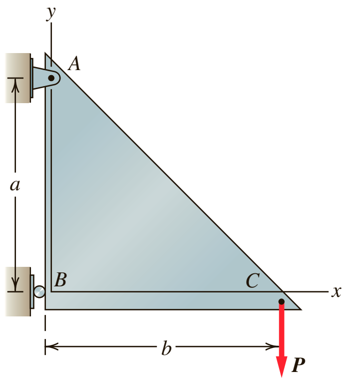


Figure 1 A triangular plate.

The triangular plate shown in **Figure 1** is pinned to its surroundings at A and rests against a roller at B. Force $\boldsymbol{P}$ acts at C. The weight of the plate can be ignored because it is much smaller than $\boldsymbol{P}$. Find the loads acting at supports A and B as a function of the geometry and loading.

Goal Find the reactions (loads) at A and B as a function of geometry (dimensions a and b) and force ($\boldsymbol{P}$).

Given Information about the structure's geometry, connections, and that the weight of the structure is negligible compared to $\boldsymbol{P}$.

Assumption The system (the triangular plate) can be modeled as planar.

Free-Body Diagram We draw a free-body diagram and arbitrarily draw the unknown loads at A and B in the directions of the positive axes (**Figure 2**). At the pin connection the loads are two forces components, and at the roller a force pushes on the system; see Table 4.1.

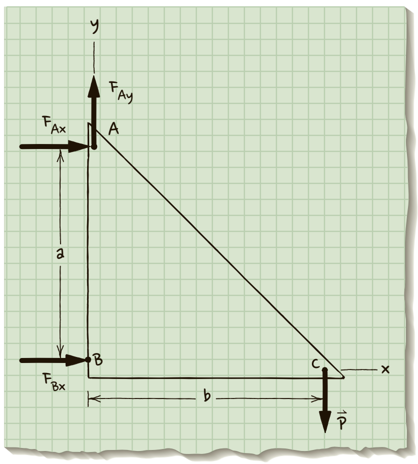


Figure 2 Free-body diagram of structure.

Formulate Equations and Solve We set up the equations for planar equilibrium (5.5) to find the unknown loads at A and B.

$$\overset{+}{\rightarrow}\circlearrowleft\Sigma F_x = 0 = F_{Ax} + F_{Bx} \Rightarrow F_{Ax} = -F_{Bx} \tag{1}$$

$$\uparrow + \circlearrowleft\Sigma F_y = 0 = F_{Ay} - P \quad \Rightarrow F_{Ay} = P \tag{2}$$

$$+\circlearrowleft\Sigma M_{z@A} = 0 = aF_{Bx} - bP \Rightarrow F_{Bx} = \left(\frac{b}{a}\right)P \tag{3}$$

Substituting (3) into (1), we find that

$$F_{Ax} = -\left(\frac{b}{a}\right)P$$

(the minus sign indicates that F_{Ax} acts in the direction opposite that shown in **Figure 2**).

Alternatively, the loads at A and B could be reported in Cartesian vector notation,

$$@A; \boldsymbol{F}_A = -\left(\frac{b}{a}\right)P\,\boldsymbol{i} + P\,\boldsymbol{j}$$

$$@B; \boldsymbol{F}_B = \left(\frac{b}{a}\right)P\,\boldsymbol{i}$$

or with a narrative description,

@A; a force acts with a component of $(b/a)P$ in the negative x direction and a component of P in the y direction

@B; a force acts with a component of $(b/a)P$ in the x direction.

Checks We redraw the free-body diagram to show the calculated values and directions of the forces (**Figure 3**). Notice that F_{Ay} balances P, and F_{Ax} balances F_{Bx}. Also, F_{Ay} and P form a clockwise couple of magnitude (bP) that balances the counterclockwise couple created by F_{Ax} and F_{Bx}. We can also check that dimension b in the numerators of F_{Ax} and F_{Bx} makes sense: As b increases, the magnitude of the couple created by F_{Ay} and P increases. If dimension a is fixed, the magnitudes of F_{Ax} and F_{Bx} must increase to balance this couple. A similar argument can be made for dimension a being in the denominator.

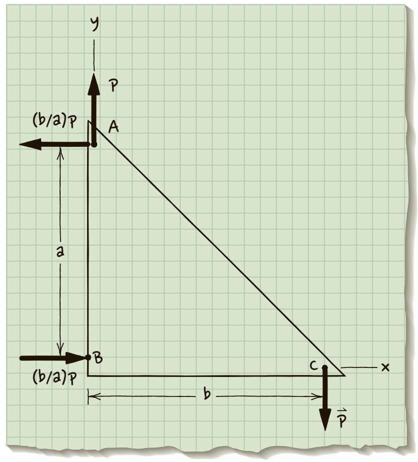


Figure 3 Free-body diagram of structure with calculated forces shown.

EXAMPLE 5.3.3

The planar truss shown in **Figure 1** is supported by a pin connection at C and a roller on a frictionless inclined plane at G. Determine the loads at these two supports.

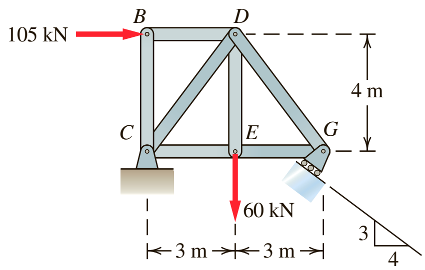


Figure 1 Planar truss supported on a pin and a roller on an inclined plane.

Goal Find the loads acting on the truss at C and G.

Given Information about the geometry of the truss, the types of supports, and the loads acting at B and E.

Assumption The member weights are negligible.

Free-Body Diagram We draw a free-body diagram of the truss (**Figure 2**) to find the loads at supports C and G.

Formulate Equations and Solve Based on the free-body diagram in **Figure 2**, we write the planar equilibrium equations (5.5):

$$\sum M_{z@C}(+\circlearrowleft) = \frac{4}{5}F_G(6\,\text{m}) - 60\,\text{kN}(3\,\text{m}) - 105\,\text{kN}(4\,\text{m}) = 0$$

$$\frac{4}{5}F_G(6\,\text{m}) = 600\,\text{kN}\cdot\text{m} \Rightarrow F_G = 125\,\text{kN}$$

$$\sum F_x(\rightarrow +) = F_{Cx} + \frac{3}{5}F_G + 105\,\text{kN} = 0$$

$$F_{Cx} + \frac{3}{5}(125\,\text{kN}) + 105\,\text{kN} = 0 \Rightarrow F_{Cx} = -180\,\text{kN}$$

$$\sum F_y(\uparrow +) = F_{Cy} + \frac{4}{5}F_G - 60\,\text{kN} = 0$$

$$F_{Cy} + \frac{4}{5}(125\,\text{kN}) - 60\,\text{kN} = 0 \Rightarrow F_{Cy} = -40.0\,\text{kN}$$

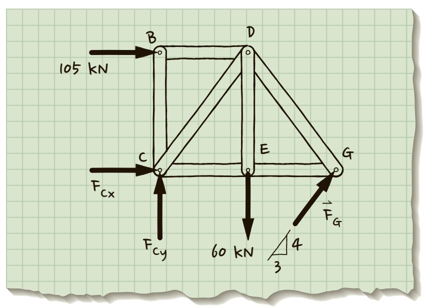


Figure 2 Free-body diagram of the truss.

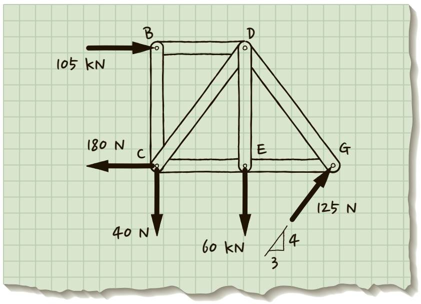


Figure 3 Free-body diagram of the truss with calculated forces shown.

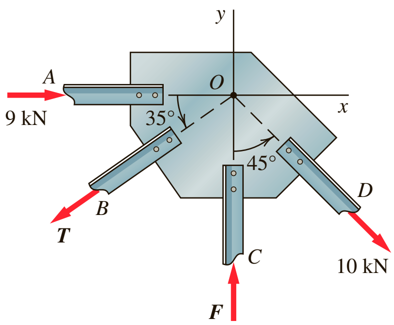
Checks As a check, we confirm that our calculated values (as indicated in **Figure 3**) maintain equilibrium with an alternate moment center:

$$\sum M_{z@G}(+\circlearrowleft) = -(4\,\text{m})(105\,\text{kN}) + (3\,\text{m})(60\,\text{kN}) + (6\,\text{m})(40\,\text{kN})$$
$$= 0 \Rightarrow 0 = 0$$

Yes, the truss is in equilibrium!

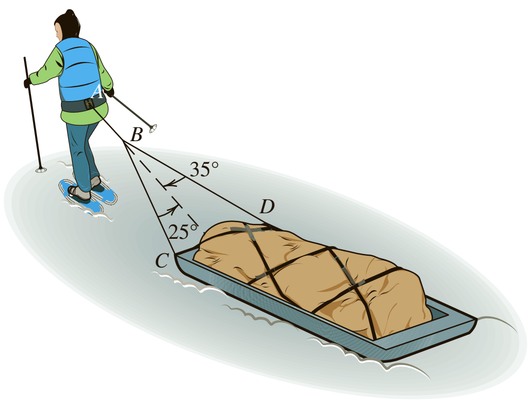
EXERCISES 5.3

5.3.1. [*] Members of a truss are connected to the gusset plate, as shown. If the forces are concurrent at point *O*, determine the magnitudes of *F* and *T* for equilibrium.



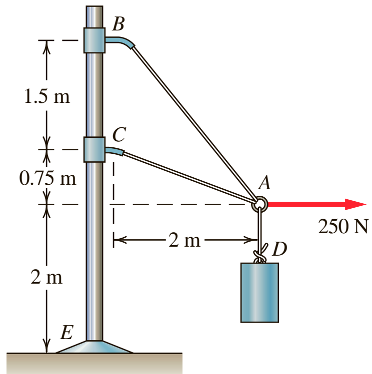
EX 5.3.1

5.3.2. [*] A snow-shoer towing a sled moves along at constant velocity. Determine the force in each of the ropes *BD* and *BC*, if the snow-shoer is pulling on rope *AB* with force of 100 N.



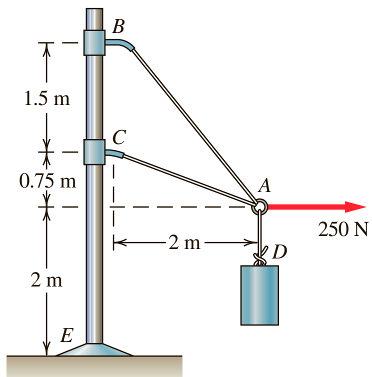
EX 5.3.2

5.3.3. [*] Determine the forces in cables *AC* and *AB* required to hold the 15-kg cylinder *D* in equilibrium.



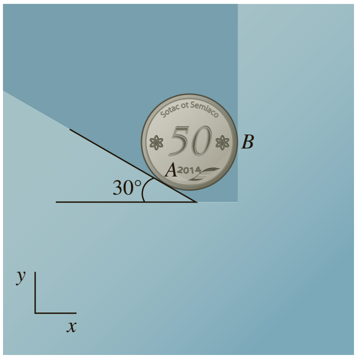
EX 5.3.3

5.3.4. [*] Consider the pole-cable system. Determine the loads acting on the pole at *E* to hold the 15-kg cylinder *D* in equilibrium.



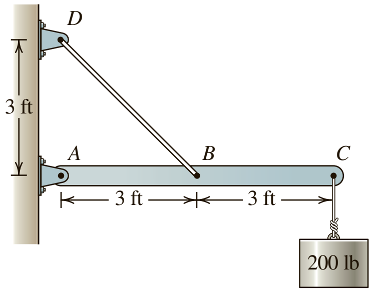
EX 5.3.4

5.3.5. [*] A coin of mass m rests in a smooth slot, as shown. The left edge of the slot is at 30° to the horizontal and the right edge of the slot is vertical. If the coin is in equilibirum, what loads act on the coin at A? What loads act on the coin at B?



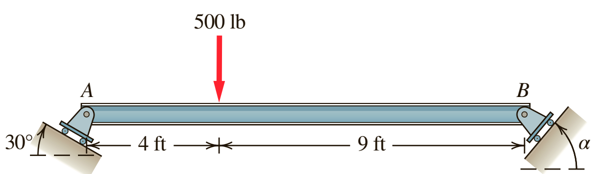
EX 5.3.5

5.3.6. [*] A weight of 200 lb hangs from a beam of negligible weight, as shown. If the beam is in equilibrium, determine the loads acting on it at A and B.



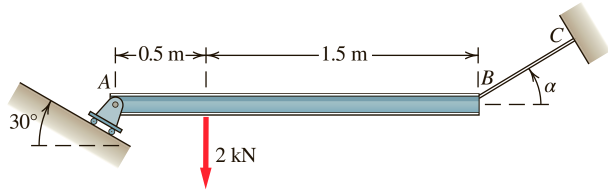
EX 5.3.6

5.3.7. [*] A force of 500 lb acts on a beam of negligible weight, as shown. Determine the angle α necessary for the beam to be in equilibrium.



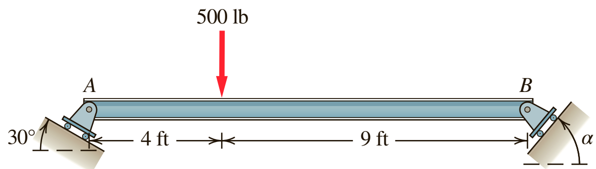
EX 5.3.7

5.3.8. [*] A force of 2 kN acts on a beam of negligible weight, as shown. Determine the angle α necessary for the beam to be in equilibrium.



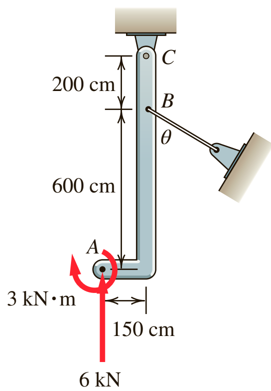
EX 5.3.8

5.3.9. [*] If the weight in the beam is not negligible determine the angle α necessary for the beam to be in equilibrium. The uniform beam weighs 4000 lb.



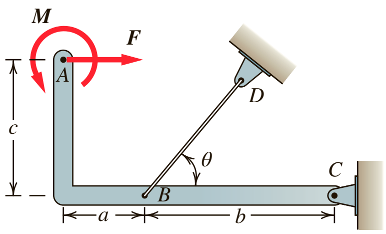
EX 5.3.9

5.3.10. [**] A force of 6 kN and a moment of 3 kN · m act on frame ABC of negligible weight, as shown. The angle between the frame and the cable at B is 60°. Determine the loads acting on the beam at B and C.



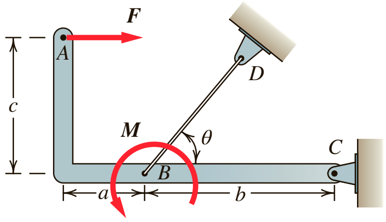
EX 5.3.10

5.3.11. [**] A force F and moment M act on frame ABC, as shown. For the case when F is zero and M is non-zero, determine the tension T in the cable BD in terms of M, a, b, c and θ.



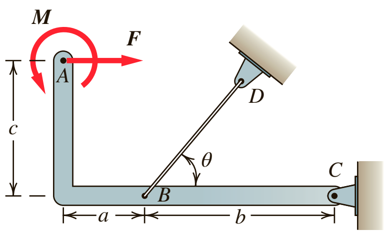
EX 5.3.11

5.3.12. [**] A force F and moment M act on frame ABC, as shown. Determine the tension T in the cable BD in terms of M, a, b, c and θ.



EX 5.3.12

5.3.13. [**] Consider the frame shown, with $\theta = 45°$ $a = b = 10$ in., and $c = 2.5$ in.



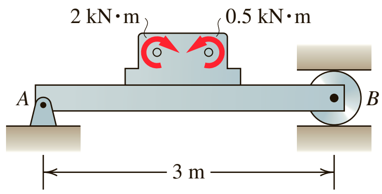
EX 5.3.13

a. If $M = 15$ in.·lb, and $F = 1$ lb, determine the tension in the cable BD.

b. If $M = 15$ in.·lb, and $F = 6.0$ lb, determine the tension in the cable BD.

c. If F is greater than 6.0 lb, is this a viable system?

5.3.14. [*] A gear box is mounted on a beam and is subjected to two moments, as shown. Determine the loads acting on the beam at A and B if the beam is in equilibrium.

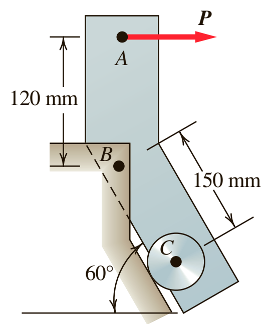


EX 5.3.14

5.3.15. [**] A cable pulls on the bracket with a horizontal force of P as shown. At C the bracket rests against a smooth surface and at B it is pinned to the wall.

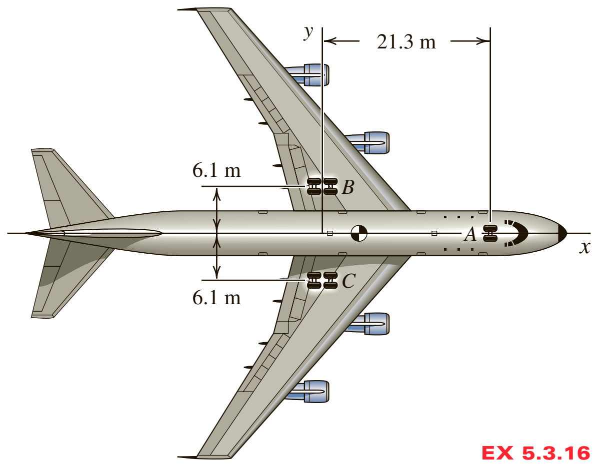
a. If the bracket is in equilibrium, what loads act on the bracket at C? What loads act on the bracket at B?

b. If the maximum allowable force at C is 500 N, what is the maximum allowable cable pull P?



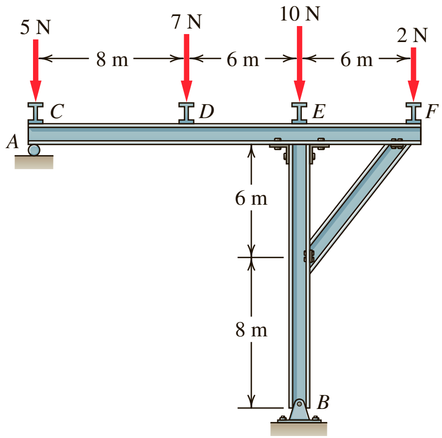
EX 5.3.15

5.3.16. [**] An airplane rests on the ground. Its landing gear wheels are at points A, B, and C. Its center of gravity is at (3.0, 0.0, 0.0) m. The airplane weighs 1.56×10^6 N. Determine the magnitudes of the forces of the ground acting on the landing gear wheels. In what direction do the forces act?



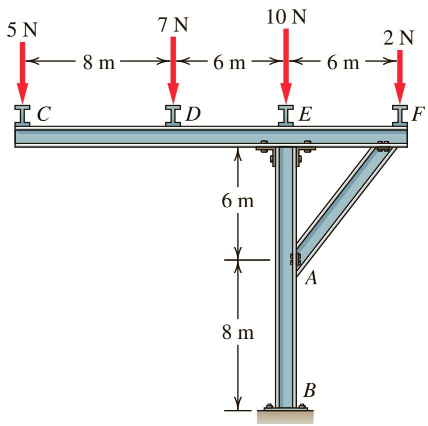
EX 5.3.16

5.3.17. [**] The frame shown is supported at A and B. Forces act at C, D, E, and F, as shown. The system is taken as the frame. If weight of the frame is negligible and the frame is in equilibrium, what loads act on the frame at A and at B?



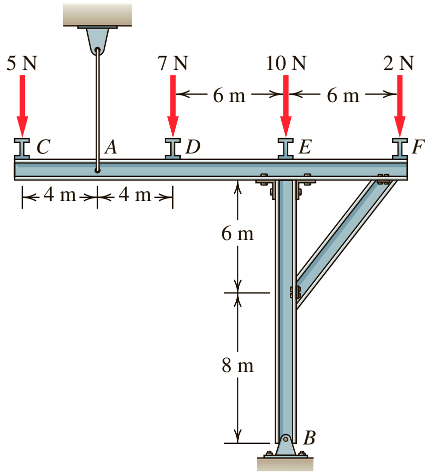
EX 5.3.17

5.3.18. [**] The frame shown is supported at B. Forces act at C, D, E, and F, as shown. The system is taken as the frame. If weight of the frame is negligible and the frame is in equilibrium, what loads act on the frame at B?



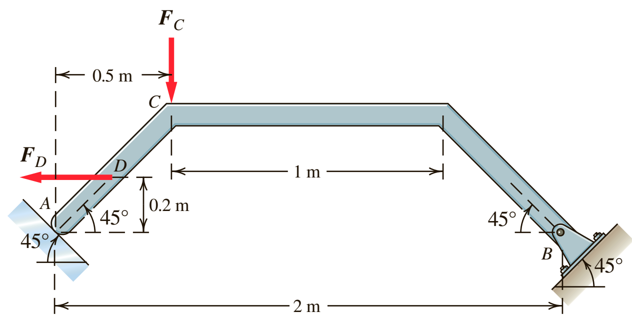
EX 5.3.18

5.3.19. [**] The frame shown is supported at A (by a vertical cable) and at B. Forces act at C, D, E, and F, as shown. The system is taken as the frame. If weight of the frame is negligible and the frame is in equilibrium, what loads act on the frame at A and at B?



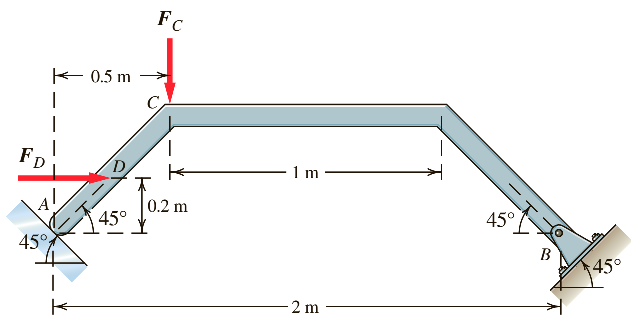
EX 5.3.19

5.3.20. [**] A uniform frame is pinned at B and rests against a smooth incline at A. Additional loads act as shown. The total weight of the frame is W. If the frame is in equilibrium, what loads act on the frame at A and at B?



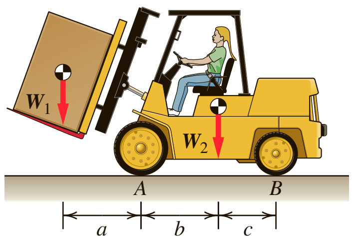
EX 5.3.20

5.3.21. [**] A frame of negligible weight is pinned at B and rests against a smooth incline at A. Two forces act on the frame, as shown. At what value of the ratio (F_D/F_C) will the frame just begin to lift-off at A?



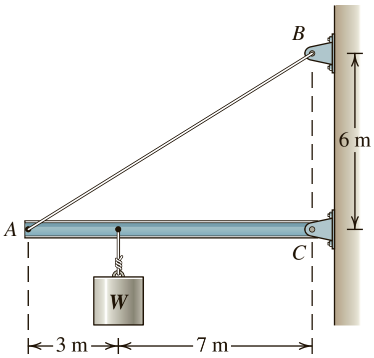
EX 5.3.21

5.3.22. [**] A forklift is lifting a crate of weight W_1 as shown. The weight of the forklift is W_2. The front wheels are free to turn, and the rear wheels are locked. If the forklift and crate are in equilibrium, what loads act on the forklift at A and at B?



EX 5.3.22

5.3.23. [**] When the cylinder of weight W is 7 m from the pin connection at C, the tension $\boldsymbol{T}$ in the cable has a magnitude of 9 kN. The beam AC is of negligible weight. If the beam is in equilibrium, determine the weight of the cylinder.

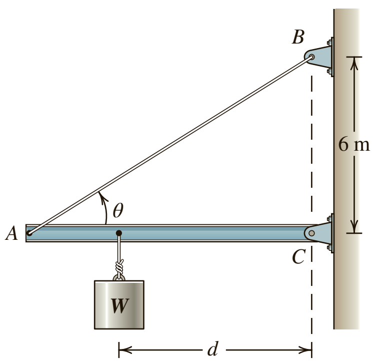


EX 5.3.23

5.3.24. [**] The cylinder of weight W is movable along along beam AC and $\theta = 35°$.

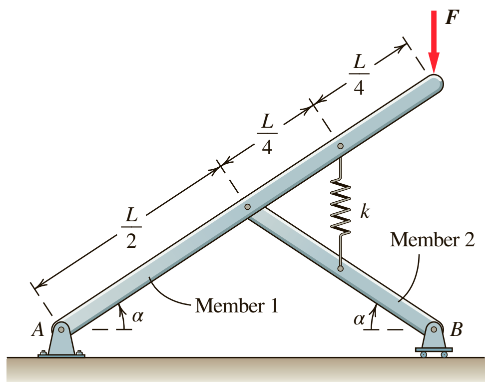
a. Write an expression for T/W as a function of d, where T is the tension in cable AB. Plot this expression.

b. From this plot determine at what position d the ratio $T/W = 1$.



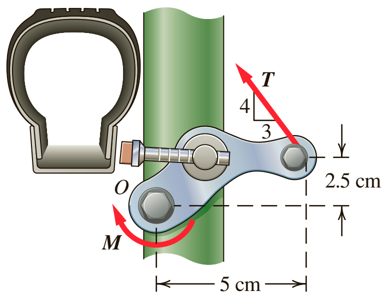
EX 5.3.24

5.3.25. [**] Consider the frame shown. If the weights of Members 1 and 2 are negligible, what loads act on the frame at A and B?



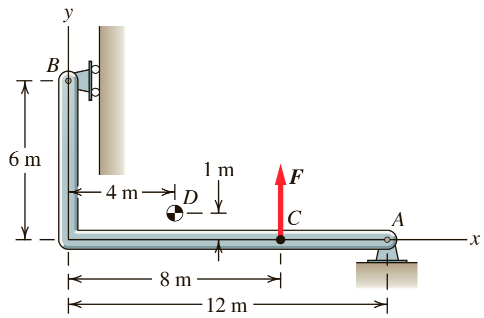
EX 5.3.25

5.3.26. [**] The arm of a cantilever bicycle brake pivots freely about O. A torsional spring at O exerts a return moment of magnitude $\|\boldsymbol{M}\| = 1.65$ N·m on the brake arm when in the position shown. If the brake arm is in equilibrium, what is the tension in the cable?



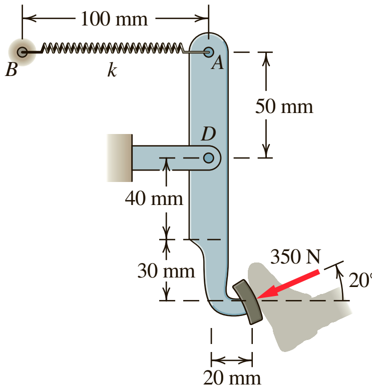
EX 5.3.26

5.3.27. [***] The uniform L-bar is pinned to its surroundings at A, and slides along a wall at B as shown. A vertical force $\boldsymbol{F}$ acts at C and the total weight of the L-bar is $\boldsymbol{W}$ with center of gravity at D. Gravity acts in the negative y direction. The system is taken as the L-bar. When the L-bar is in equilibrium, find F, A_x and B_x so that $A_y = 0$.



EX 5.3.27

5.3.28. [***] A 350-N force acts on a brake pedal, as shown. a) If the pedal is in equilibrium in the position shown, what is the tension in the spring? b) If the total deflection at A is to be limited to 5 mm when the 350-N pedal force is applied, what is the minimum required spring stiffness k?

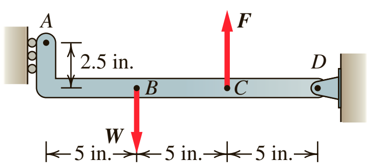


EX 5.3.28

5.3.29. [***] Consider the frame shown, which has a weight W hanging at B.

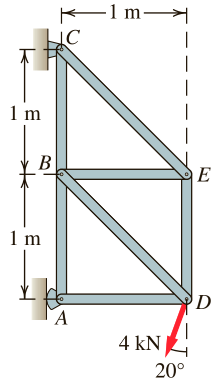
a. When $F/W = 1$, determine F_A (expressed as a function of W).

b. Determine the ratio of F/W when the load is zero at A.



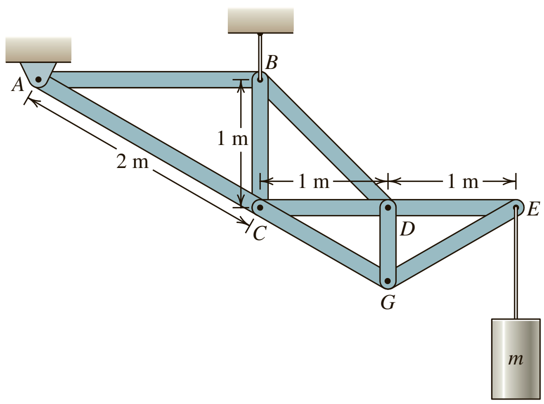
EX 5.3.29

5.3.30. [*] Determine the loads acting on the truss at A and C.



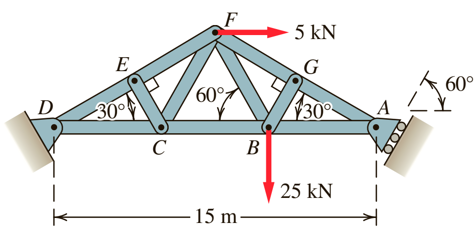
EX 5.3.30

5.3.31. [**] Determine the loads acting on the truss at A and B if the cylinder has a mass of 300 kg.



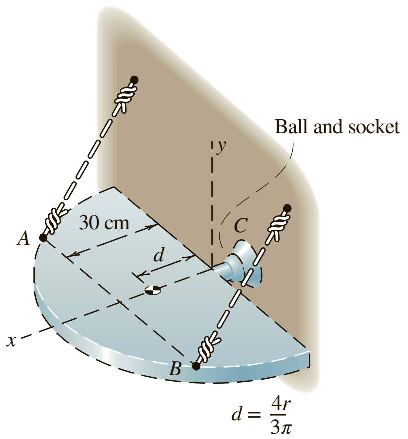
EX 5.3.31

5.3.32. [**] Determine the loads acting on the truss at A and D.



EX 5.3.32

5.3.33. [***] Consider the semi-circular plate that has a radius of 40 cm. The plate weighs 100 N with center of gravity at $(d, 0, 0)$. Determine the tension in the cables at A and B, and the loads acting on the plate at C (a ball-and-socket connection). The cables are at 60° above the horizontal.

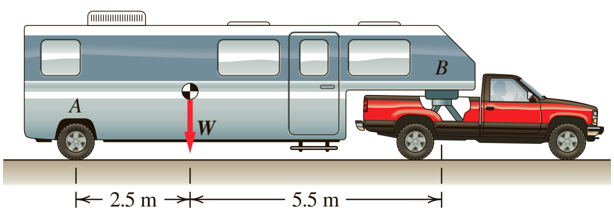


EX 5.3.33

5.3.34. [***] The truck–trailer assembly is in equilibrium on level ground. The hitch at B can be modeled as a pin connection.

a. If the weight of the trailer is 16 kN, determine the normal force exerted on the rear tires at A and the force exerted on the trailer at the pin connection B.

b. If the pin connection at B is designed to support a maximum force of 6.5 kN, what is the weight of the heaviest trailer that should be safely attached to the truck? Clearly state all assumptions in addressing this question.

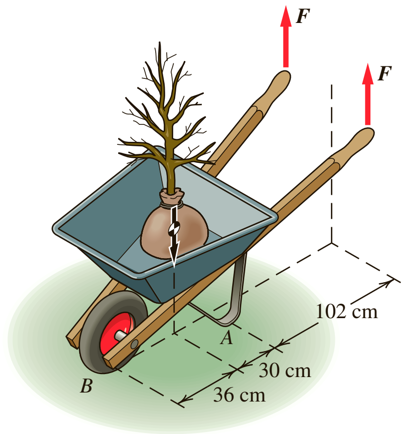


EX 5.3.34

5.3.35. [***] A wheelbarrow loaded with a tree weighs 400 N and has a center of mass as shown. The wheelbarrow sits on level ground and is in equilibrium.

a. What are the loads at A and B if $\|\boldsymbol{F}\| = 0$?

b. What is the value of $\|\boldsymbol{F}\|$ necessary to just lift the wheelbarrow off the ground at A? Do you think that you could personally supply this magnitude of force?

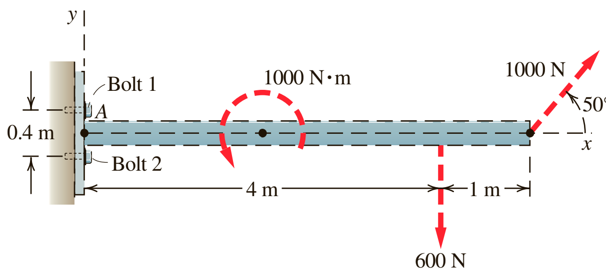


EX 5.3.35

5.3.36. [***] A beam is bolted to a wall at end A and loaded as shown.

a. Determine the loads acting on the beam at A if the beam is in equilibrium.

b. Replace the moment you found at end A with a couple, such that the forces act through the centers of the two bolts. What is the magnitude of the force? This couple results in increased tension in which of the two bolts?



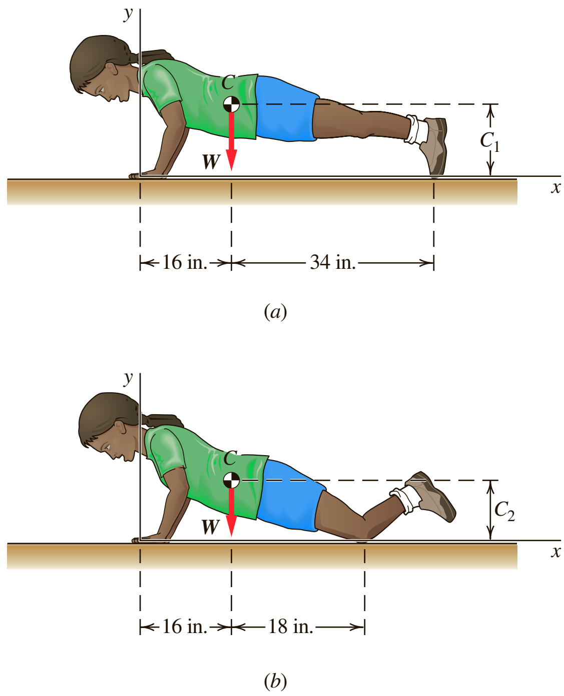
EX 5.3.36

5.3.37. [***] A child who weighs 400 N is doing push-ups.

a. When the child does standard push-ups and is in the position shown in a, determine the loads acting on each hand and on each foot.

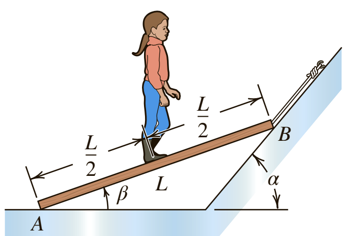
b. When the child does modified push-ups and is in the position shown in b, determine the loads acting on each hand and on each knee.

c. Compare the loads acting on each hand in a and b by finding the loads in b as a percentage of the loads in a.



EX 5.3.37

5.3.38. [***] A child of weight W is standing at the center of a plank in a playground. The child and plank are in equilibrium. Planes A and B are smooth.



EX 5.3.38

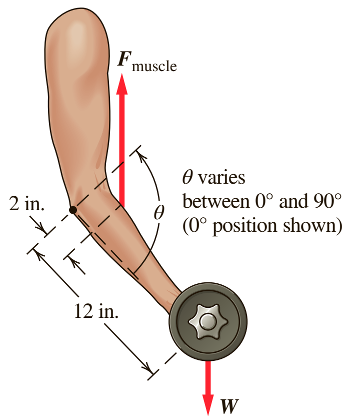
a. Determine the tension in the cable (F_{Bt}) and the ground normal force at B (F_{Bn}), in terms of L, W, α, and β. Present your answers as equations.

b. If $L = 3$ m, $W = 400$ N, and $\beta = 25°$, create plots of F_{Bt} and F_{Bn} as α varies from 10 to 90 degrees. A spreadsheet may be helpful.

c. Use the plots created in b to determine at what angle α the magnitudes of F_{Bt} and F_{Bn} are equal.

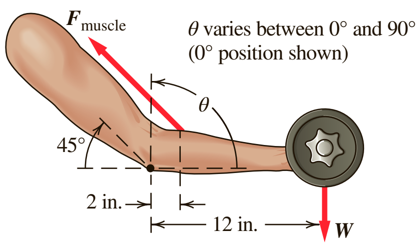
d. Use the plots created in b to determine at what angle α the tension in the cable is 120 N. Call this the limiting value of α_{limit}. If 120 N is the breaking load of the cable, should the angle α be kept to values greater than or less than α_{limit}?

5.3.39. [***] An athlete does arm curls with her arm oriented as shown. Write an expression for F_{muscle} as a function of W and θ, starting with the $\theta = 0°$ orientation shown. Show both the expression and a plot of the expression. Assume that the muscle force remains parallel to the upper arm throughout the motion and that the motion is slow enough for equilibrium to hold.



EX 5.3.39

5.3.40. [***] An athlete does arm curls with her arm oriented as shown. Write an expression for F_{muscle} as a function of W and θ, starting with the $\theta = 0°$ orientation shown. Show both the expression and a plot of the expression. Assume that the muscle force remains parallel to the upper arm throughout the motion and that the motion is slow enough for equilibrium to hold.



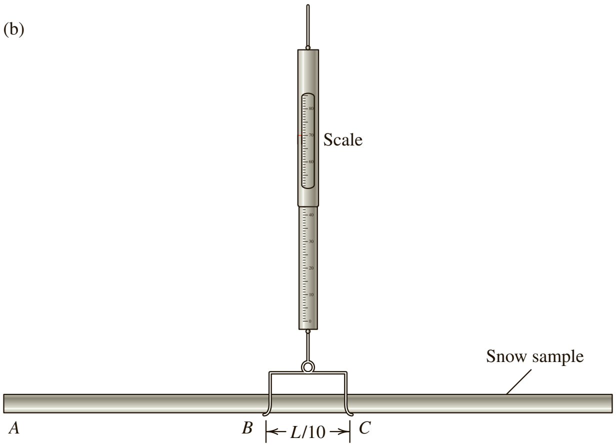
EX 5.3.40

5.3.41. [***] The scientist in the photograph is weighing a snow sample to determine its water content. He uses a tube of length L to extract a snow core and then places the snow core and tube on a set of hooks attached to a scale to weigh them. He can place the scale anywhere along the length of the tube, but if the hooks are too far off center the tube will tip and slide off the hooks. Assuming the snow and tube are of uniform density, how far can hook B be from tube end A without the tube being unbalanced and tipping?

(a)

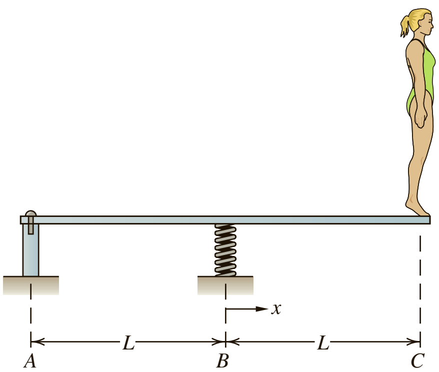
AP Photo/Rich Pedroncelli.

(b)



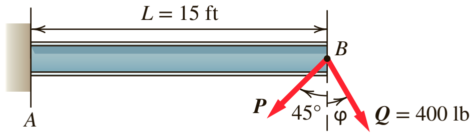
EX 5.3.41

5.3.42. [***] A diver on a diving board is shown. When the diver is between A and B, the support at A is pushing upward on the board. When the diver is between B and C, the support at A is pulling downward on the board. Assuming the diver weighs W, write an equation for the vertical force at A as a function of x, defined as the distance of the diver from support B. What is the force at A when the diver is standing over the spring? Plot the vertical force at A as a function of x.



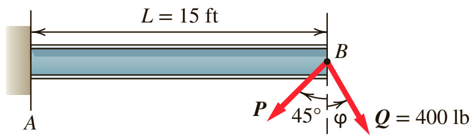
EX 5.3.42

5.3.43. [*, computer]** Loads P and Q = 400 lb are applied to a 15-ft cantilever beam at B as shown. The maximum allowable moment at A is M_A = 1500 ft-lb and the maximum allowable tension force applied to the beam by the wall at A is 400 lb. Plot P as a function of φ, and find the ranges of P and φ ($0 < \varphi < 360$) for which these conditions are satisfied. What is the maximum value of P that can be applied to the beam?



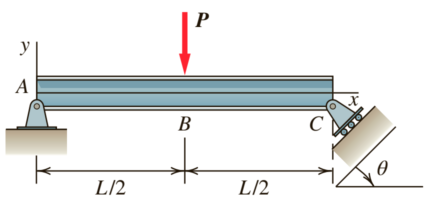
EX 5.3.43

5.3.44. [*, computer]** Loads P and Q = 400 lb are applied to a 15-ft cantilever beam at B as shown. The maximum allowable moment at A is M_A = 1500 ft-lb and the maximum allowable tension force applied to the beam by the wall at A is 400 lb. In addition, to prevent failure of the attachment at B, the maximum allowable force $P = 300$ lb. Find the ranges of P and φ ($0 < \varphi < 360$) for which these conditions are satisfied.



EX 5.3.44

5.3.45. [, computer]** The roller support at C sits on a plane inclined at an angle θ. Plot the loads A_x and A_y acting at A as a function of θ. What happens to A_x when θ = 90°? What happens to the system physically when θ = 90°?



EX 5.3.45

5.4 EQUILIBRIUM APPLIED TO FOUR SPECIAL CASES

Learning Objectives

- **Recognize when a system can be modeled as a particle, a two-force member, or a three-force member.**
- **Apply a structured analytical procedure to calculate the equilibrium of a particle, a two-force member, a three-force member, or a frictionless pulley.**

In our discussion of planar systems we have applied the three planar equilibrium equations (5.5A)–(5.5C) to a variety of situations. As we progress to analyzing more complex systems, in some instances we can make use of unique characteristics of subsystems to simplify our analysis. We now highlight four commonly found elements:

- The *particle*
- The *two-force member*
- The *three-force member*
- The *frictionless pulley*

By defining each of these elements as "the system" and applying the equilibrium equations, we gain insights that can simplify our static analysis more generally.

5.4.1 Particle Equilibrium

Learning Objectives

- **Recognize when a system can be modeled as a particle.**
- **Apply a structured analytical procedure to calculate equilibrium of a particle.**

Some situations in static analysis can be modeled as a particle subjected to forces. Importantly, modeling a situation as a particle can simplify our analysis. For the purposes of analysis, a **particle** represents an object whose size and shape have negligible effect on the response of the object to loads. Under these circumstances, we can assume that the mass (if significant) of the object is concentrated at a point. A particle, by definition, can only be subjected to concurrent forces, and the point of concurrency is the point that represents the particle.

To see that concurrency of forces is necessary for a particle to be in equilibrium, consider the situation in **Figure 5.4.1*a***, where the horizontal force $\boldsymbol{F}_2$ is not concurrent with $\boldsymbol{W}$ and $\boldsymbol{F}_1$. Now we apply the moment equilibrium equation (5.5C) to this particle system:

$$\sum M_{z@A} = 0 = (d)(F_2)$$

For this expression to be true, either d is zero, F_2 is zero, or both d and F_2 are zero. But F_2 being equal to zero is not possible if the particle is in equilibrium, since $\boldsymbol{F}_2$ serves to balance the x component of $\boldsymbol{F}_1$. Therefore we are forced to conclude that d must be zero for the particle system to be in equilibrium. This is the same as saying that $\boldsymbol{F}_2$ must be concurrent with $\boldsymbol{W}$ and $\boldsymbol{F}_1$ for the particle to be in equilibrium (as shown in **Figure 5.4.1*b***).

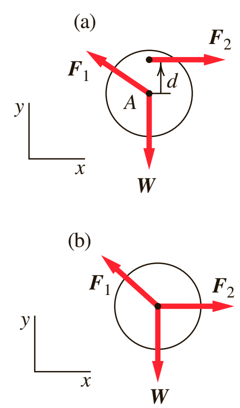


Figure 5.4.1 (*a*) A particle cannot be in equilibrium if the offset d is not zero. (*b*) Concurrent forces act on a particle in equilibrium.

In summary:

- If an object is modeled as a particle, all of the forces acting on it are concurrent.
- We need only consider the force equilibrium condition, because the moment equilibrium condition is automatically satisfied.

Check out the following example of an application of this material.
- **Example 5.4.1 Analyzing a Planar Truss Connection as a Particle**

EXAMPLE 5.4.1

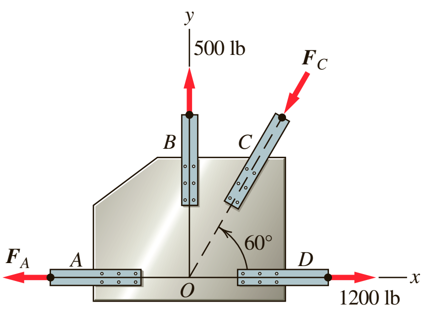


Figure 1 Four forces acting on a gusset plate.

The gusset plate shown in **Figure 1** connects members of a planar truss that is in equilibrium. The forces acting on members *B* and *D* are known. Assume the weights of the members and gusset plate are negligible compared to the applied forces. Find the magnitude of the forces $\boldsymbol{F}_A$ and $\boldsymbol{F}_C$ acting on members *A* and *C*.

Solution Strategy We are asked to find **the forces acting on members *A* and *C*.** Because all the forces acting on the gusset are concurrent (meaning their lines of action intersect at a single point), we will be able to model the gusset as a particle at *O*.

Goals Calculate the magnitudes of the forces $\boldsymbol{F}_A$ and $\boldsymbol{F}_C$ acting on members *A* and *C*.

Given Connection geometry, magnitudes and directions of forces $\boldsymbol{F}_D$ and $\boldsymbol{F}_B$.

Assumptions
- Planar system.
- Weights of members and gusset plate are negligible compared to applied forces.
- Point of concurrency of lines of action of truss members allows *O* to be modeled as a particle.
- System is in *equilibrium.*

Free-Body Diagram The free-body diagram of particle *O* is shown in **Figure 2**.

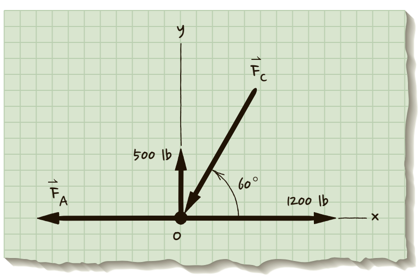


Figure 2 Free-body diagram of gusset plate, modeled as a particle.

Unknowns: F_A, F_C.

Equilibrium Equations When modeling a particle, we need only consider force equilibrium equations.

$$\sum F_x = 0\ (\rightarrow +) \Rightarrow -F_A - F_C\cos 60° + 1200\text{ lb} = 0 \quad (1)$$

$$\sum F_y = 0\ (\uparrow +) \Rightarrow -F_C\sin 60° + 500\text{ lb} = 0 \quad (2)$$

Solve Since (2) has only one unknown, we solve (2) first for F_C = 577 lb, then substitute this into (1) to find that F_A = 912 lb.

Since both F_A and F_C are positive, the forces act in the directions shown on the free-body diagram.

$$F_A = 912 \text{ lb}, F_C = 577 \text{ lb}$$

Checks We use graphical force addition to check that our answers conform with the force equilibrium condition ($\sum \boldsymbol{F}_{\text{net}} = 0$). The force vectors are laid out head to tail to form a polygon. If we calculated the forces correctly, the head of the last vector must touch the tail of the first vector.

Using a piece of graph paper to establish a scale (1 square = 100 lb), and referring to our free-body diagram, we draw the 500-lb vertical force (the choice of the first force was arbitrary). Next we add the 1200-lb horizontal force, followed by $\boldsymbol{F}_C$ (at 60°). We don't know its length, but we do know that it must terminate at the horizontal, where $\boldsymbol{F}_A$ will be added to it. We now have a closed polygon (Figure 3).

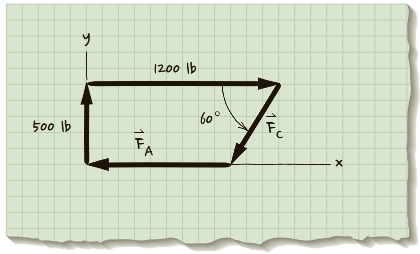


Figure 3 Graphical force addition used to check answers.

From this trapezoid, we see that the y component of $\boldsymbol{F}_C$ must be 500 lb. Therefore:

$$F_C \sin 60° = 500 \text{ lb} \Rightarrow F_C = 577 \text{ lb}$$

This is the same value for F_C that we calculated with our equilibrium equations. Checked!

The geometry of the trapezoid shows us that the length of $\boldsymbol{F}_A$ can be determined by subtracting the horizontal component of $\boldsymbol{F}_C$ from the 1200-lb force:

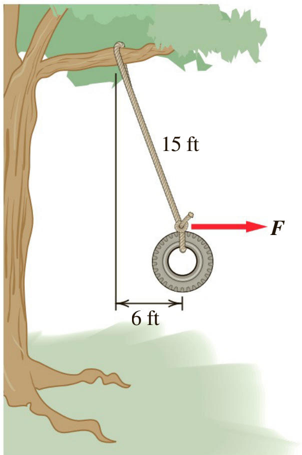
$$F_A = 1200 \text{ lb} - \underbrace{F_C}_{577 \text{ lb}} \cos 60° \Rightarrow F_A = 912 \text{ lb}$$

This is the same value for F_A that we calculated with our equilibrium equations. Checked!

Answers $F_A = 912$ lb $\quad F_C = 577$ lb

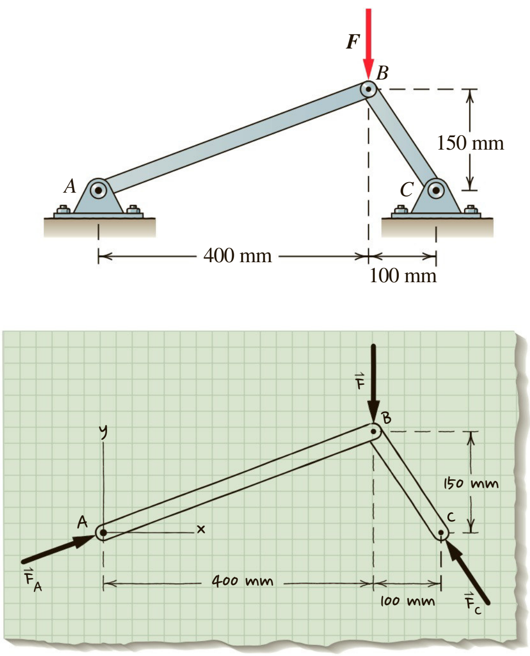
EXERCISES 5.4.1

5.4.1. [*] A child in a tire swing hanging on a 15-ft rope is pulled to the side by a horizontal force ***F***. The child and swing weigh 62 lb. Determine the tension acting on the rope and the magnitude of the force ***F***.



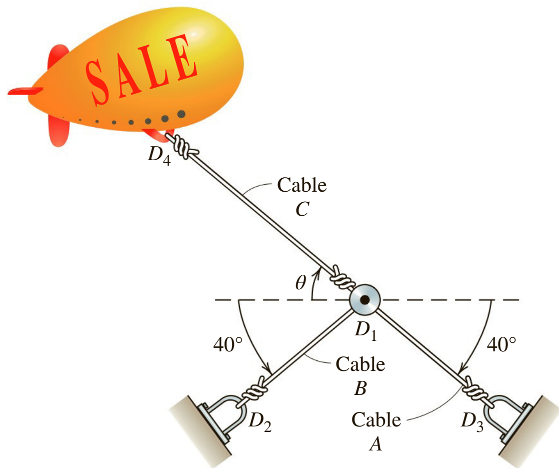
EX 5.4.1

5.4.2. [*] A force $F = 900$ N is applied to the frame at B. Using the free-body diagram of the frame, analyze point B as a particle to find the forces acting on members AB and BC.



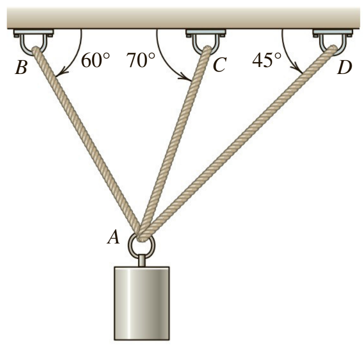
EX 5.4.2

5.4.3. [**] A car dealer advertising a sale with a helium balloon has tied cable C to a ring at D_1, which in turn is attached to the ground by cables A and B. The tension in cable A is 5 lb and the tension in cable B is 4 lb. Find the tension in cable C and the angle θ.



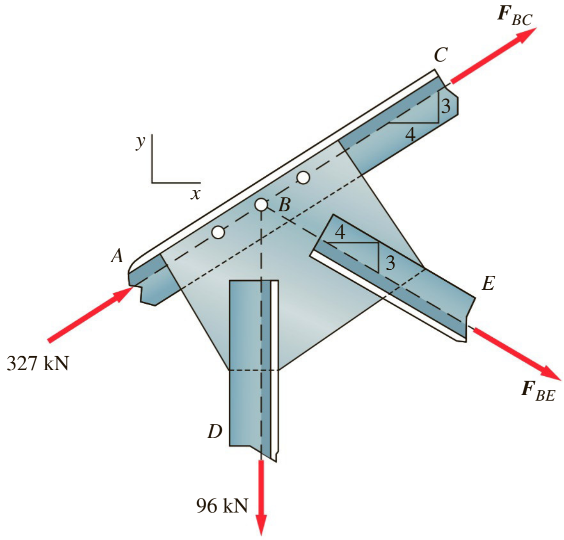
EX 5.4.3

5.4.4. [**] Three cables are tied to a ring at A to support a cylinder. The tension in cable AD is 10 lb, and in cable AC the tension is 5 lb. Determine the tension in cable AB and the weight of the cylinder.



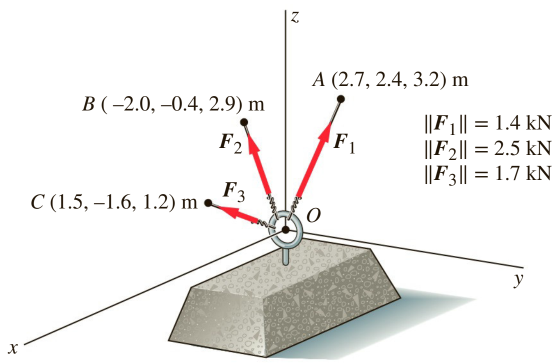
EX 5.4.4

5.4.5. [**] A free-body diagram of a roof truss connection is shown. Analyze point B as a particle and determine $\boldsymbol{F}_{BC}$ and $\boldsymbol{F}_{BE}$.



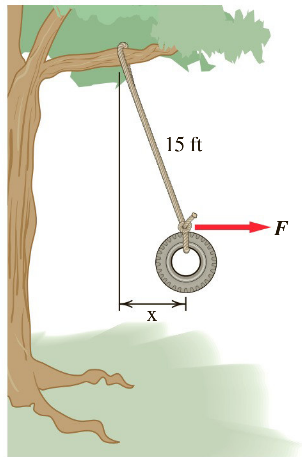
EX 5.4.5

5.4.6. [**] A concrete block is used to hold down the cables supporting a structure. Determine the minimum weight of the concrete block to keep it from lifting off the ground.



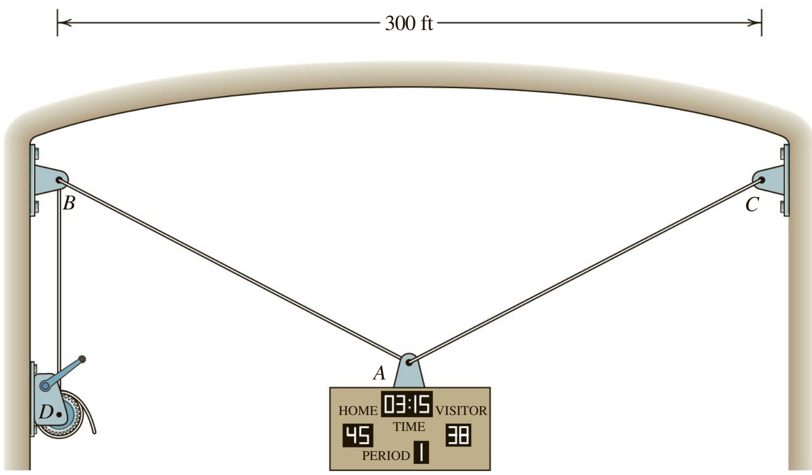
EX 5.4.6

5.4.7. [, computer]** A child is being pulled to the side in a tire swing hanging on a 15-ft rope by a horizontal force $\boldsymbol{F}$. The child and swing weigh 70 lb. Create a spreadsheet or computer program to determine the magnitude of $\boldsymbol{F}$ as the distance the swing is pulled to the side (x) varies from 0 to 8 ft.



EX 5.4.7

5.4.8. [, computer]** The 2000-lb scoreboard A is suspended above a sports arena by the cables ABD and AC. Cable segment AB and cable AC are each 160 ft long. Suppose you want to raise the scoreboard out of the way for a tennis match by using a winch at D to roll up ABD and thereby shorten cable segment AB while keeping the length of cable AC constant. (1) Plot the tension in cable segment AB as a function of its length for values of length from 142 ft to 160 ft. (2) Use your graph to estimate how much you can raise the scoreboard relative to its original position if you don't want to subject cable segment AB to a tension greater than 6000 lb.



EX 5.4.8

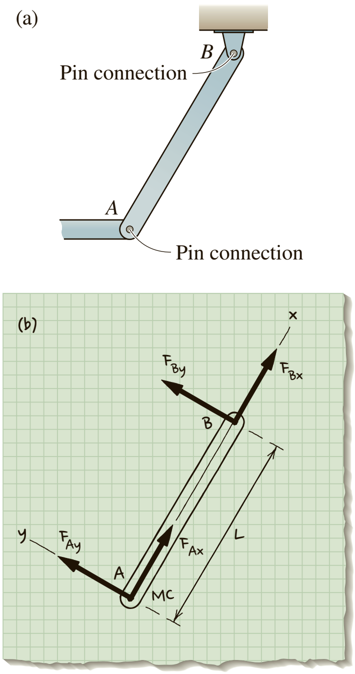


Figure 5.4.2 (*a*) A two-force member; (*b*) free-body diagram of a two- force member.

5.4.2 Two-Force Member Equilibrium

Learning Objectives

- **Recognize when a system can be modeled as a two-force member.**
- **Apply a structured analytical procedure to calculate equilibrium of a two-force member.**

A **two-force member** is a member of negligible weight with only two forces acting on it. Rod AB in **Figure 5.4.2*a*** is an example of a two-force member. A force (with two components) acts at each of the rod's pin connections, as shown in the free-body diagram in **Figure 5.4.2*b***. We can apply the equations of equilibrium (5.5) to this system:

$$\sum F_x = 0 = F_{Ax} + F_{Bx} = 0$$

$$\sum F_y = 0 = F_{Ay} + F_{By} = 0$$

$$\sum M_{z@A} = 0 = (L)(F_{By}) = 0$$

Solving these three equations, we find:

$$\mathrm{F}_{\mathrm{A}x} = -\mathrm{F}_{\mathrm{B}x}$$

$$F_{Ay} = F_{By} = 0$$

> **IMPORTANT NOTE:** These findings mean that a two-force member is in equilibrium when the forces acting on it are equal, opposite, and along the same line of action. The line of action passes through the points of application of the forces.

Two additional notes about two-force members:

1. It is common to talk about straight two-force members (or members) being in tension or in compression (**Figure 5.4.3**). A two-force member in tension wants to elongate, whereas a two-force member in compression wants to get shorter. Cables, chains, and ropes commonly act as tension-only two-force members, and columns as compression-only two-force members.
2. Two-force members are not restricted to being straight, as illustrated in **Figure 5.4.4**. What defines each of these as a two-force member is

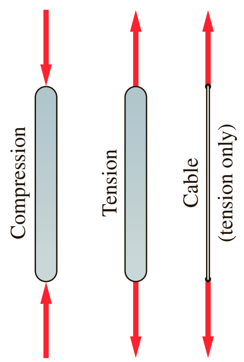


Figure 5.4.3 Two-force member in compensation and tension.

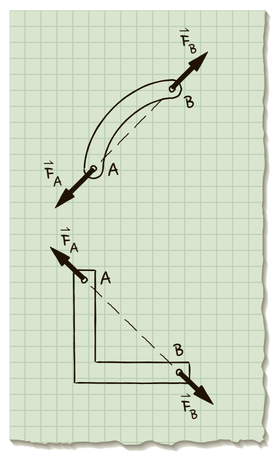


Figure 5.4.4 Example of nonstraight two-force members.

that only two forces act on the member. As with a straight member, the line of action passes through the points of application of the loads.

Check out the following example of an application of this material.
- **Example 5.4.2 Two-Force Member Analysis**

EXAMPLE 5.4.2

Consider the platform scale in **Figure 1**, and assume that the weights of the various members that make up the lever system are negligible compared to the weights on the scale. If the scale is in equilibrium, determine the force exerted by member BD on member ABC as a function of dimensions a and b, and $\mathbf{W}_C$ (the weight of the cylinder that moves along the calibrated bar).

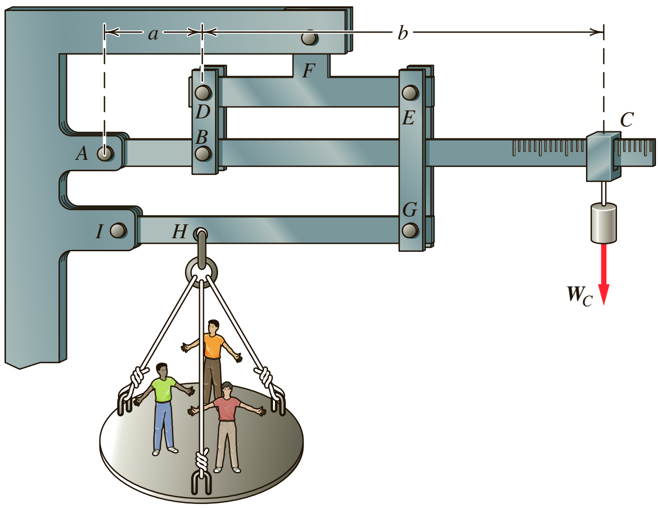


Figure 1 A platform scale—a weighty situation.

Goals Find the force exerted on member ABC by member BD.

Given
- The system's geometry.
- The scale is in equilibrium.
- The weight of the cylinder. The member weights are negligible.

Assumptions
- The connections at A, B, D, E, F, G, and I are pins. This means that each of the two parallel vertical bars joined to the horizontal members by pin connections at B and D are two-force members. This is also true for the two vertical bars joined by pin connections at E and G.
- We can model the situation as planar.

Free-Body Diagram Because we are to find the force exerted by member BD on ABC, we define our system as member ABC and draw its free-body diagram (**Figure 2**). Notice that because member BD is a two-force member, the force at B acts along the long axis of members BD.

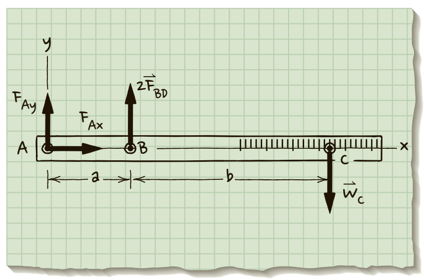


Figure 2 Free-body diagram for member ABC.

Unknowns: F_{BD}, F_{Ay}.

Equilibrium Equations Based on the planar equilibrium equations (5.5), we write:

$$\overset{+}{\rightarrow}\Sigma F_x = 0 \Rightarrow F_{Ax} = 0$$

$$\uparrow + \Sigma F_y = 0 \Rightarrow F_{Ay} + 2F_{BD} - W_C = 0$$

$$F_{Ay} = W_C - 2F_{BD} \qquad (1)$$

$$+\circlearrowleft \Sigma M_{z@A} = 0 \Rightarrow +(a)2F_{BD} - (a+b)W_C = 0$$

Solve

$$2F_{BD} = \frac{(a+b)}{a}W_C \qquad (2)$$

Checks The fact that we calculated a positive number in (2) indicates that the force that the two members BD apply to member ABC is an upward force of magnitude

$$2F_{BD} = \frac{(a+b)}{a}W_C$$

An upward force means that members BD are pulling on ABC and are therefore in tension.

We first substitute (2) into (1) to find that

$$F_{Ay} = -\frac{b}{a}W_C$$

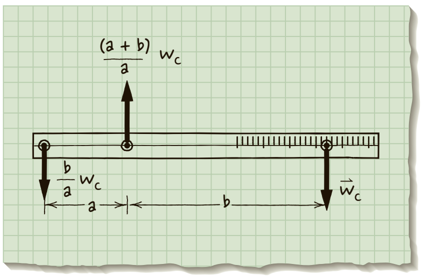


Figure 3 Free-body diagram with calculated loads.

Based on the calculated results, the final free-body diagram is as shown in **Figure 3**.

Choosing a moment center at C and using the calculated values of F_{Ay} and $2F_{BD}$ in terms of W_C, we find that

$$+\circlearrowleft \Sigma M_{z@C} = 0 \Rightarrow +(a+b)\frac{b}{a}W_C - (b)\frac{(a+b)}{a}W_C = 0$$

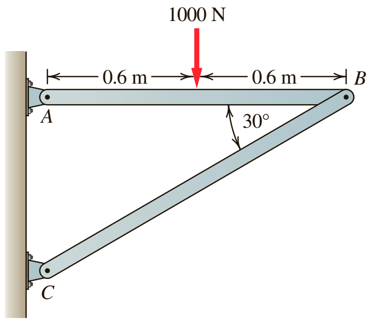
Checked!

Answers

$$2F_{BD} = \frac{(a+b)}{a}W_C$$

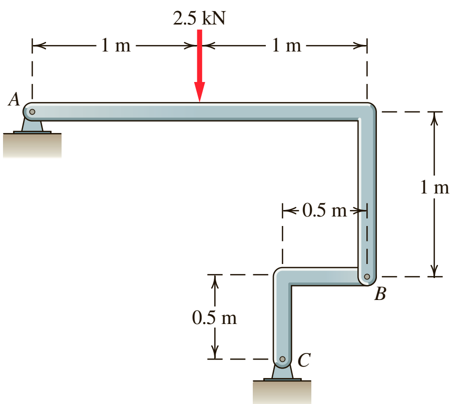
EXERCISES 5.4.2

5.4.9. [**] Determine the forces acting at *A* and *C* on frame *ABC* that is in equilibrium, noting the member *BC* acts as two-force member.



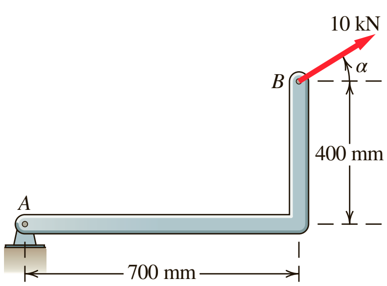
EX 5.4.9

5.4.10. [**] Determine the forces acting at *A* and *C* on frame *ABC* that is in equilibrium. And make use of the fact that member *BC* acts as a two-force member.



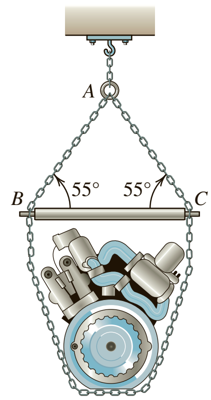
EX 5.4.10

5.4.11. [**] The L-shaped bar is loaded by a 10-kN force at *B* as shown. The weight of the bar is negligible. Note that for it to be in equilibrium, the bar must act as a two-force member. Using the required properties of a two-force member in equilibrium, determine the angle α and the loads acting on the bar at *A*.



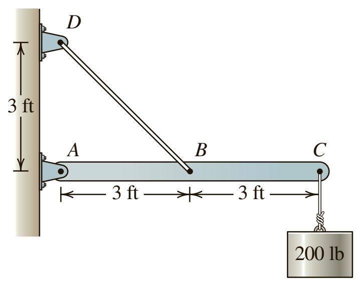
EX 5.4.11

5.4.12. [**] A 225-kg engine is suspended from a vertical chain at *A*. A second chain is wrapped around the engine and held in position by the spreader bar *BC*. Determine the force acting along the axis of the bar and the tension in chain segments *AB* and *AC*. Clearly state whether the bar is in tension or compression.



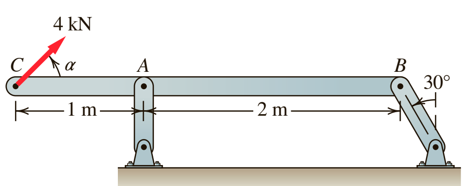
EX 5.4.12

5.4.13. [**] A weight of 200 lb hangs from a frame of negligible weight, as shown. If the frame is in equilibrium, determine the loads acting on the frame at *A* and *D*, using the fact that member *BD* can be modeled as a two-force member.



EX 5.4.13

5.4.14. [**] A force of 4 kN acts on beam *CAB* of negligible weight, as shown. Determine the angle α necessary for the beam to be in equilibrium. Also determine the loads acting on the beam at *A* and *B*.



EX 5.4.14

5.4.3 Three-Force Members in Equilibrium

Learning Objectives

- **Recognize when a system can be modeled as a three-force member.**
- **Apply a structured analytical procedure to calculate equilibrium of a three-force member.**

A **three-force member** is a member with *only* three forces acting on it. To determine the necessary relationship among these three forces, consider the situation in **Figure 5.4.5*a***. We apply the moment equilibrium equation (5.5c) to this three-force member:

$$\Sigma M_{z@A} = 0 = -(d_2)(F_{2x}) + (d_1)(F_{2y})$$

$$\frac{F_{2y}}{F_{2x}} = \frac{d_2}{d_1}$$

This latter expression tells us that if the three-force member is in equilibrium, the line of action of F_2 must go through point A, and therefore the three lines of action meet at a single point (and are concurrent at A), as shown in **Figure 5.4.5*b***.

Other examples of three-force members are shown in **Figure 5.4.6**.

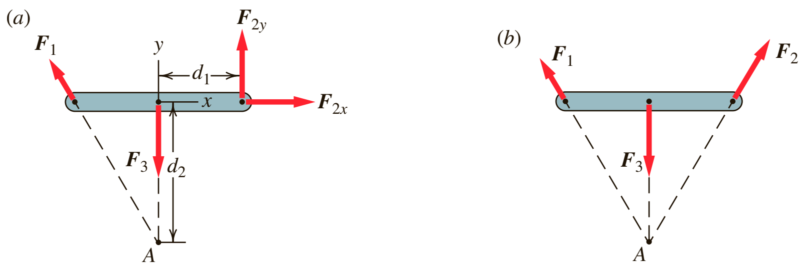


Figure 5.4.5 (*a*) A three-force member. (*b*) A three-force member in equilibrium requires that the forces be concurrent.

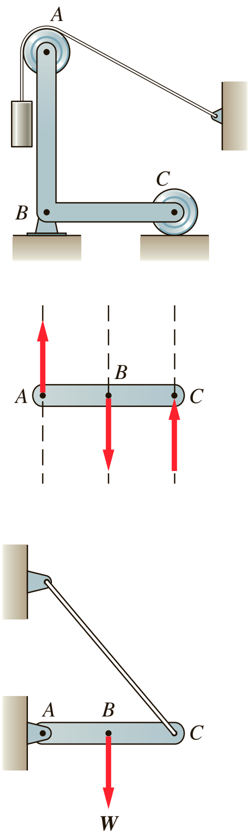


Figure 5.4.6 L-member *ABC*, beam *ABC* with parallel forces, and beam *ABC* are also three-force members.

In summary:

- If a three-force member is in equilibrium, the lines of action of the three forces are concurrent.
- In the case of a member in which the three applied forces are parallel to one another, their point of concurrency is at infinity.

Check out the following examples of applications of this material.
- **Example 5.4.3 Climbing Cam Analysis**
- **Example 5.4.4 Three-Force Member Analysis**

EXAMPLE 5.4.3

Ford is climbing a steep rock face (see Figure 1*a*). His rope is being held in a crack at the top of the cliff by a spring-loaded camming device that grips the rock (see **Figure 1*b***). If Ford is pulling on the rope with a 180-lb force, what are the loads acting on the camming device?

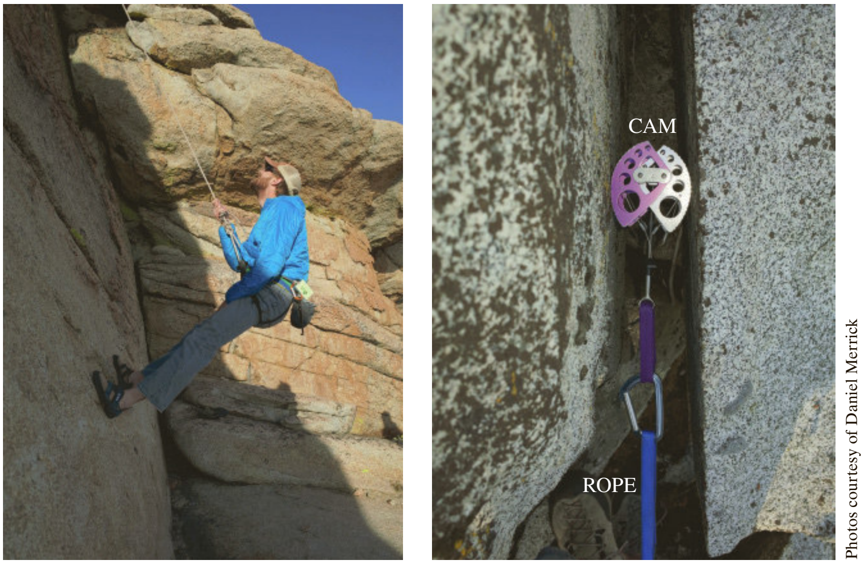


Figure 1 (*a*) Climber on rock face. (*b*) Camming device.

Solution Strategy To solve this problem we need a free-body diagram, but first we need to understand how a spring-loaded camming device works, so that we see how the loads are applied.

How Climbing Cams Work The spring-loaded camming device, also sometimes called a "friend," was invented in the 1970s. The friend consists of several cams mounted on two parallel axles that allow the device to expand and contract to fit all widths of cracks (**Figure 2**). There are many variations of this basic design, and some manufacturers also offer a climbing cam with a single axle.

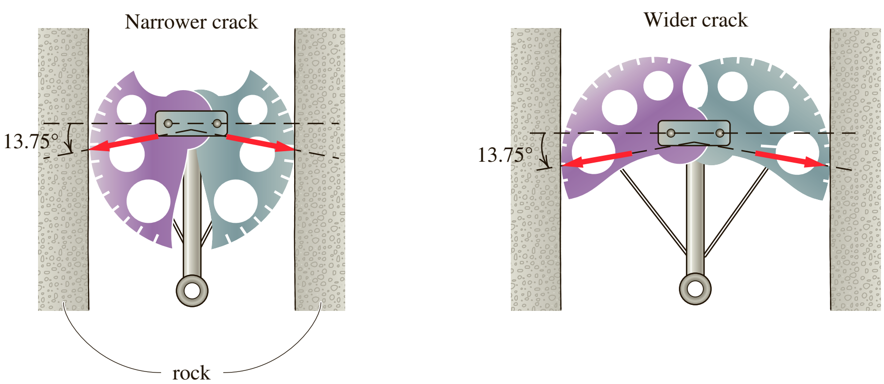


Figure 2 A camming device expands and contracts to fit all widths of cracks.

The cam shape is based on a logarithmic spiral. This is done so that the angle between the force of the cam pushing on the rock, and the rock face, is the same for any crack width. This angle is called the camming angle. The camming angle is specified by the designer to ensure that there is adequate friction to hold the cam in place.

Based on the frictional properties of most rocks and then accounting for a factor of safety, the camming angle varies between 13.25° and 16° for different manufacturers. For this device, the camming angle is 13.75°.

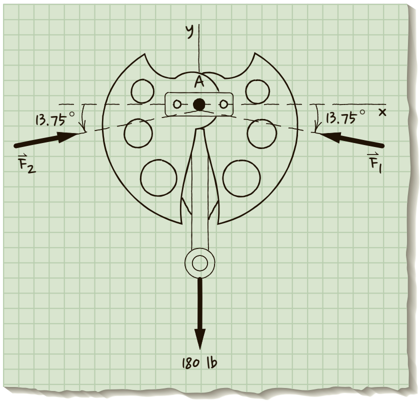


Figure 3 Free-body diagram of the camming device.

Goals Determine the magnitudes and directions of all of the loads acting on the camming device.

Given
- 180-lb load on the climbing rope
- Camming angle of 13.75°

Assumptions
- The system is in *equilibrium.*
- Model the camming device as a planar system.
- Ignore the weight of the camming device.

Free-Body Diagram We note that only three forces ($\boldsymbol{F}_1$, $\boldsymbol{F}_2$, 180 lb) act on the cam; this means that it is a three-force member. Therefore we draw the free-body diagram so that the lines of action of these three forces intersect at a single point (A), as shown in Figure 3.

Unknowns: $\boldsymbol{F}_1$, $\boldsymbol{F}_2$

Equilibrium Equations and Solve: Because the forces are concurrent at A, there is no moment acting on the system. This means that only two force-equilibrium equations are needed to solve this problem:

$$\sum F_x = 0$$
$$\sum F_y = 0$$

We can solve this problem in two ways, by writing the equations of equilibrium directly, or by drawing a force triangle. Here, we draw a force triangle to visualize the solution (Figure 4).

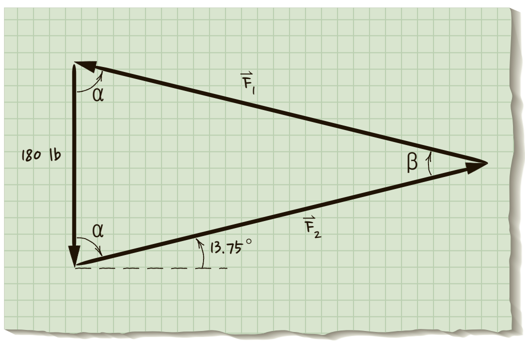


Figure 4 Force triangle.

Based on this figure, we calculate:

By the law of sines:

$$\frac{\sin\beta}{180\text{ lb}} = \frac{\sin\alpha}{F_1}$$

$$\frac{\sin 27.5°}{180\text{ lb}} = \frac{\sin 76.25°}{F_1} \Rightarrow F_1 = 379\text{ lb}$$

By the law of sines (again):

$$\frac{\sin 27.5°}{180\text{ lb}} = \frac{\sin 76.25°}{F_2} \Rightarrow F_2 = 379\text{ lb}$$

$F_1 = F_2 = 379$ lb

Checks We check our answers using Equations (5.5A) and (5.5B):

$$\begin{aligned}\sum F_x = 0\ (\rightarrow +) &\Rightarrow -F_1\cos 13.75° + F_2\cos 13.75° = 0\\ &\Rightarrow -368\text{ lb} + 368\text{ lb} = 0\\ &\Rightarrow 0 = 0 \qquad \text{(Yes!)}\\ \sum F_y = 0\ (\uparrow +) &\Rightarrow F_1\sin 13.75° + F_2\sin 13.75° - 180\text{ lb} = 0\\ &\Rightarrow 90\text{ lb} + 90\text{ lb} - 180\text{ lb} = 0\\ &\Rightarrow 0 = 0 \qquad \text{(Yes!)}\end{aligned}$$

Answers $F_1 = F_2 = 379$ lb

EXAMPLE 5.4.4

Beam AB is loaded with a 750-N load at D and supported as shown in Figure 1. Weights of various members are negligible. Find the loads acting on beam AB at A and B.

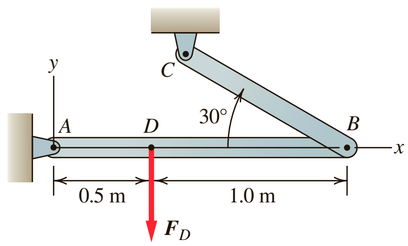


Figure 1 Beam AB is part of a frame assembly.

Solution Strategy In looking at the situation, we note the pin connections at A and B. This means that the loads at those locations will consist only of forces, and no moments. So, the loads we are being asked to find are the *forces* acting on the beam at its two ends.

Goals Calculate the forces acting on the beam at A and B, that is, F_A and F_B, respectively.

Given Dimensions of the beam, downward vertical 750-N force at D ($\boldsymbol{F}_D$), weights of members are negligible.

Assumptions
- Model the system (beam AB) as *planar*.
- Because B and C are pin-connections, member BC *acts as a two-force member*.
- The beam is in *equilibrium*.

Free-Body Diagram We note that only three forces ($\boldsymbol{F}_A$, $\boldsymbol{F}_B$, $\boldsymbol{F}_D$) act on beam AB; this means that it is a three-force member. We will

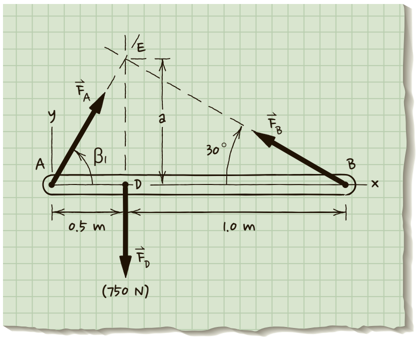


Figure 2 Free-body diagram of beam. Recognizing that the beam acts as a three-force member establishes point *E*.

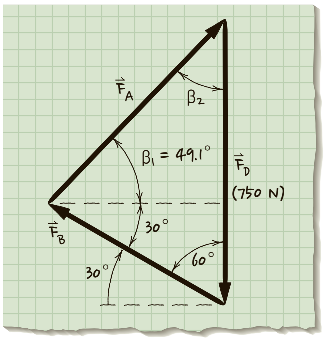


Figure 3 Graphical force addition used to check answers.

use this observation to our advantage in solving the problem. We draw the free-body diagram as shown in **Figure 2**, so that the lines of action of these three forces intersect at a single point (this is a property of a three-force member).

Unknowns: F_A, F_B, β_1

Equilibrium Equations Based on the geometry in the free-body diagram, we determine:

$$a = (1.0\text{ m})\tan 30° = 0.577\text{ m}$$

$$\tan\beta_1 = \frac{a}{0.5\text{ m}} = \frac{0.577\text{ m}}{0.5\text{ m}} \Rightarrow \beta_1 = 49.1°$$

The force equilibrium condition says that if we place the forces head to tail (as in vector addition), they should form a triangle, as depicted in **Figure 3**.

Based on this figure, we calculate:

By the law of sines:

$$\frac{\sin 79.1°}{750\text{ N}} = \frac{\sin 60°}{F_A} \Rightarrow F_A = 661\text{ N}$$

From Figure 3 $\quad \beta_2 = 40.9° \quad (=180° - 79.1° - 60°)$

By the law of sines (again):

$$\frac{\sin 79.1°}{750\text{ N}} = \frac{\sin 40.9°}{F_B} \Rightarrow F_B = 500\text{ N}$$

Solve Representing the values in vector form:

$$\boldsymbol{F}_A = \|\boldsymbol{F}_A\|\cos\beta_1\,\boldsymbol{i} + \|\boldsymbol{F}_A\|\sin\beta_1\,\boldsymbol{j} \underbrace{=}_{\substack{\|\boldsymbol{F}_A\| = 661\text{ N} \\ \beta_1 = 49.1°}} 433\text{ N}\,\boldsymbol{i} + 500\text{ N}\,\boldsymbol{j}$$

$$\boldsymbol{F}_B = \|\boldsymbol{F}_B\|\cos 30°\,\boldsymbol{i} + \|\boldsymbol{F}_B\|\sin 30°\,\boldsymbol{j} \underbrace{=}_{\|\boldsymbol{F}_B\| = 500\text{ N}} -433\text{ N}\,\boldsymbol{i} + 250\text{ N}\,\boldsymbol{j}$$

Note that the horizontal components of (the $\boldsymbol{i}$ components) of $\boldsymbol{F}_A$ and $\boldsymbol{F}_B$ are equal and opposite. Why aren't the vertical components ($\boldsymbol{j}$ components) equal and opposite?

Checks We check our answers using Equations (5.5A) and (5.5B).

$$\sum F_x = 0\ (\rightarrow +) \Rightarrow \underbrace{F_A\cos\beta_1}_{\substack{433\text{ N} \\ \text{(from above)}}} - \underbrace{F_B\cos 30°}_{\substack{433\text{ N} \\ \text{(from above)}}} = 0$$

$$\Rightarrow 433\text{ N} - 433\text{ N} = 0$$

$$\Rightarrow 0 = 0 \qquad \text{(Yes!)}$$

$$\sum F_y = 0\ (\uparrow +) \Rightarrow F_A\sin\beta_1 + F_B\sin 30° - F_D = 0$$

$$\Rightarrow 500\text{ N} + 250\text{ N} - 750\text{ N} = 0$$

$$\Rightarrow 0 = 0 \qquad \text{(Yes!)}$$

Answers

$$\boldsymbol{F}_A = 433\text{ N}\,\boldsymbol{i} + 500\text{ N}\,\boldsymbol{j}$$

$$\boldsymbol{F}_B = -433\text{ N}\,\boldsymbol{i} + 250\text{ N}\,\boldsymbol{j}$$

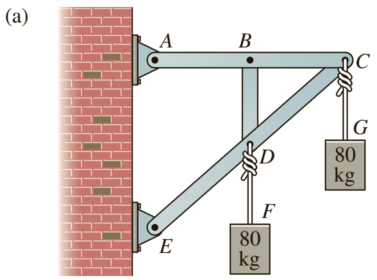
EXERCISES 5.4.3

5.4.15. [*]
For each structure shown in (a)–(c) identify members that can be classified as two-force members and members that can be classified as three-force members.

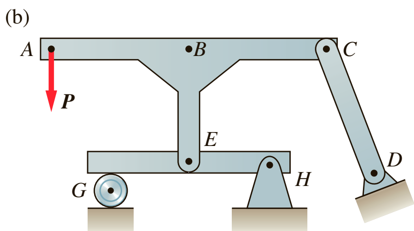
a. The pin-connected frame in (a) supports a box at C and a box at D.

b. The lever system shown in (b).

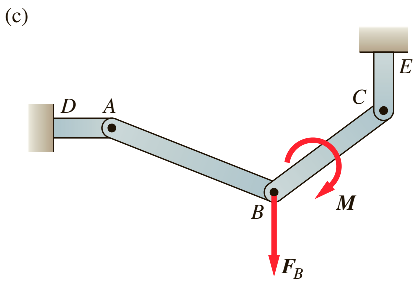
c. A moment acts on member BC and a force acts on the frame at B, as shown in (c).



EX 5.4.15a



EX 5.4.15b

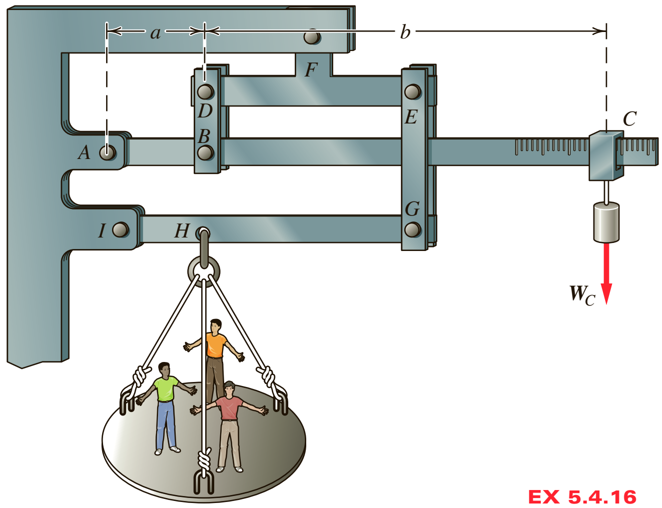


EX 5.4.15c

5.4.16. [*] For the platform scale shown identify

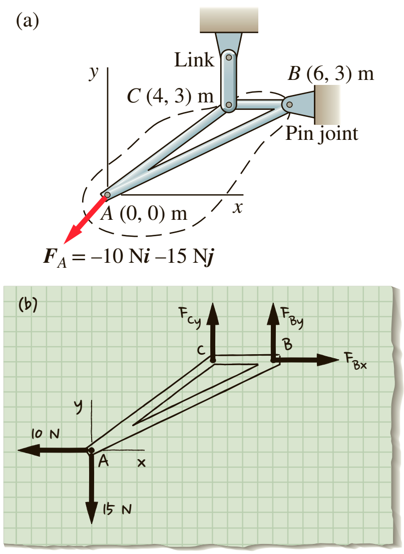
a. all of the two-force members.

b. all of the three-force members.



EX 5.4.16

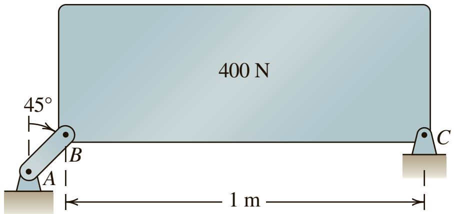
5.4.17. [*] Use graphical force addition to confirm that the system shown is a three-force member in equilibrium.



EX 5.4.17

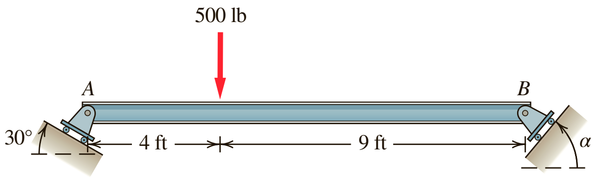
5.4.18. [*] Revisit the leaning person example in Example 5.2.1 and use graphical force addition to confirm that the person can be modeled as a three-force member in equilibrium.

5.4.19. [**] Consider the 400-N uniform rectangular plate shown. Determine the loads acting on the plate at B and C. Use the fact that the plate acts as a three-force member to check your answers.



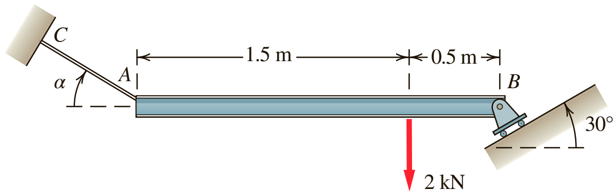
EX 5.4.19

5.4.20. [**] Consider the beam shown. Use the fact that the beam can be modeled as a three-force member to determine the the angle α and force acting on the beam at B necessary for the beam to be in equilibrium.



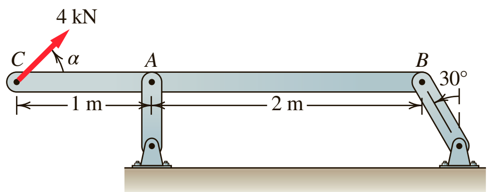
EX 5.4.20

5.4.21. [**] Consider the beam shown. Using the fact that the beam can be modeled as a three-force element to determine the the angle α and cable tension BC necessary for the beam to be in equilibrium.



EX 5.4.21

5.4.22. [**] Consider the beam shown. Use the fact that the beam can be modeled as a three-force member to determine the angle α necessary for the beam to be in equilibrium.



EX 5.4.22

5.4.4 Frictionless Pulleys in Equilibrium

Learning Objective: Apply a structured analytical procedure to calculate the equilibrium of a frictionless pulley.

A **frictionless pulley** is an element that is used to change the direction of a cable or rope. For example, in Figure 5.4.7 the rope pulls vertically on an object on the left side of the pulley, but pulls on the wall at an angle θ on the right side. A pulley is connected to its surroundings by a frictionless pin connection.

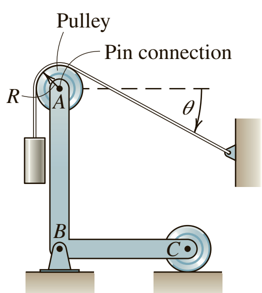


Figure 5.4.7 L-member ABC supports a pulley at A.

We isolate the pulley with radius R and draw its free-body diagram in Figure 5.4.8. We represent the unknown load at A by its x and y components F_{Ax} and F_{Ay}. The unknown rope tensions are represented by $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$.

To determine the rope tension we apply the planar moment equilibrium equation (5.5C) about point A at the center of the pulley:

$$\sum M_{z@A} = 0 = (R)(\boldsymbol{T}_1) - (R)(\boldsymbol{T}_2) = 0$$

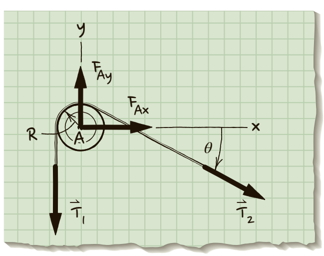


Figure 5.4.8 Free-body diagram of frictionless pulley at A.

From this equation, we find that $\|\boldsymbol{T}_1\| = \|\boldsymbol{T}_2\|$. For simplicity, let's call this T.

We have derived this important characteristic of a **frictionless pulley**:

The magnitude of the tension acting on the cable or rope remains the same as the cable or rope curves around a frictionless pulley.

This statement is not true for a pulley with friction.

Using planar equilibrium, we determine that $F_{Ax} = -T\cos\theta$ and $F_{Ay} = T + T\sin\theta$. If we calculate the resultant force, $\|\boldsymbol{F}_A\| = \sqrt{F_{Ax}^2 + F_{Ay}^2}$, and its space angle, $\cos\theta_y = F_{Ay}/F_A$, and then use the trigonometric half-angle formula, we can show that the lines of action of $\boldsymbol{F}_A$, $\boldsymbol{T}_1$, and $\boldsymbol{T}_2$ are concurrent at some point P. We can also show that $\boldsymbol{F}_A$ bisects the angle made by $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$.

The free-body diagram in **Figure 5.4.9** summarizes this finding and shows a second important characteristic of a frictionless pulley: A frictionless pulley in equilibrium is a **three-force member**.

In summary:

- The tension acting on the cable or rope is the same on both sides of a frictionless pulley.
- A frictionless pulley in equilibrium is a three-force member.

Check out the following example of an application of this material.
- **Example 5.4.5 Ideal Pulley Analysis**

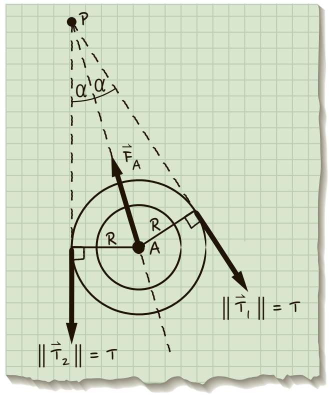


Figure 5.4.9 A frictionless pulley in equilibrium is a three-force member.

EXAMPLE 5.4.5

A man weighing 200 lb is holding a 400-lb crate with a rope and pulley system as shown in **Figure 1**. Pulley A guides the rope, but it is not attached to the wall or ceiling. (a) Find the upward lift on the man. (b) If the man moves under pulley B so that he is pulling down vertically on the rope, will he be able to hold the crate?

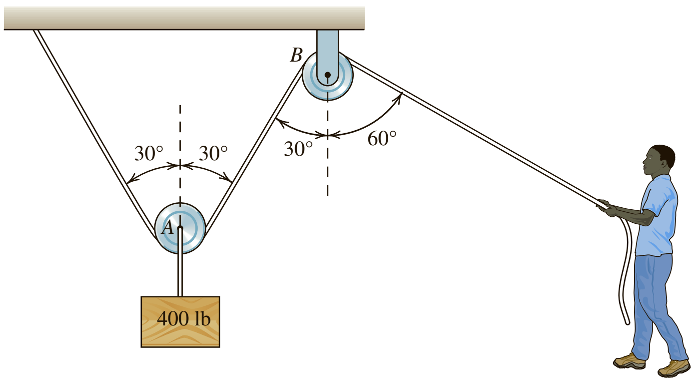


Figure 1

Strategy Overview We are being asked to find *the force pulling on the man.* To do that, we need to find the tension in the rope. We will begin by analyzing what is happening at pulley A (where we know that a 400-lb force is acting), then analyze pulley B (where the man is acting).

Goals Calculate the vertical component of the force pulling on the man and compare it to his weight.

Given Geometry of the system, weight of crate at A (400 lb), weight of the man (200 lb).

Assumptions
- Model the system as *planar.*
- The pulleys are frictionless.
- The weight of the rope and pulleys is negligible.
- The system is in *equilibrium.*

Free-Body Diagram #1 We isolate pulley A from the rest of the system to draw the free-body diagram (**Figure 2**).

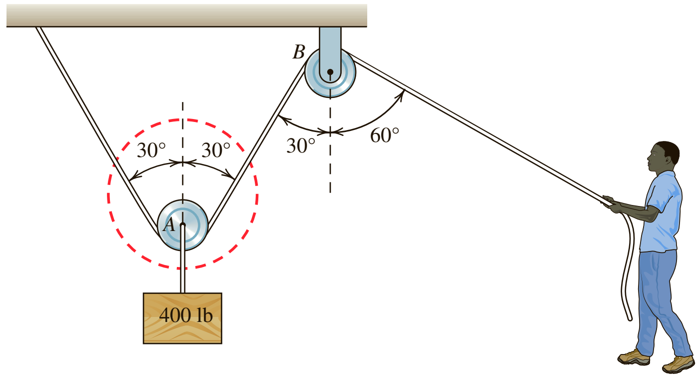


Figure 2 Isolating pulley A from the rest of the system.

Figure 3 Free-body diagram of pulley A.

When we studied frictionless pulleys, we learned that as the rope passes over the pulley it changes direction but the tension remains constant. We use this important characteristic when we draw the free-body diagram of pulley A (**Figure 3**).

Note that we chose pulley A as our starting point because $\boldsymbol{T}$ is the only unknown on this free body. Draw a free-body diagram of pulley B and convince yourself that if you started at B, you would need information from pulley A to solve for any unknowns.

Unknowns: $\boldsymbol{T}$

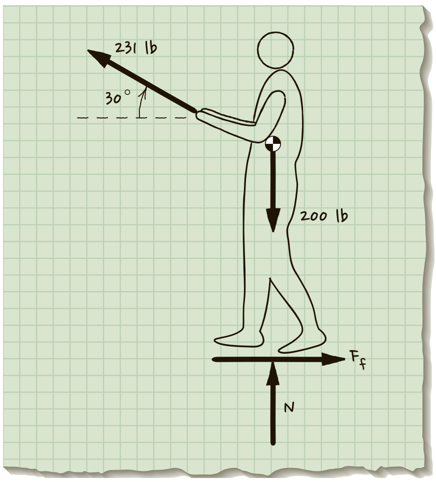


Figure 4 Free-body diagram of the man.

Equilibrium Equations For the free-body diagram of pulley A,

$$\uparrow + \sum F_y = T\cos 30° + T\ \cos 30° - 400 \text{ lb} = 0$$

$$T = \frac{400 \text{ lb}}{2\cos 30°}$$

$$T = 231 \text{ lb}$$

Free-Body Diagram #2 To determine the impact of the rope tension on the man, we need to draw a free-body diagram of the man, as shown in **Figure 4**.

Solve The vertical component of the tension is

$$T_y = \|\mathbf{T}\| \sin 30° = 115 \text{ lb}$$

This is the answer to part (a).

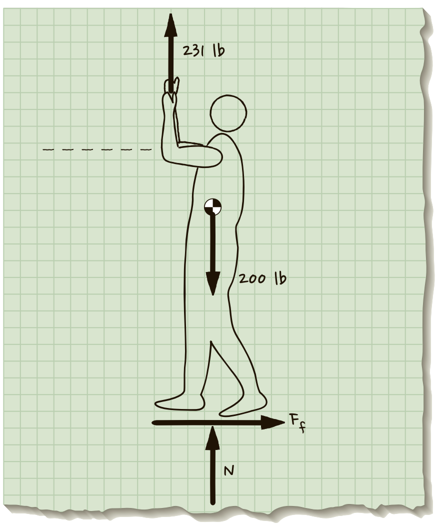


Figure 5

Free-Body Diagram #3 and Solve Now let's explore what happens to the man if he moves under pulley B so that he is pulling down vertically on the rope (**Figure 5**). When he moves, the direction of the rope tension changes, but the magnitude does not.

Looking at the free-body diagram (**Figure 5**), we see that the upward force (231 lb) will be greater than the man's 200-lb weight and he will be lifted off the ground!

Checks For this particular problem we can't identify a set of independent equations to check the tension force. We can use a force triangle (**Figure 6**) to review part (a) of the problem graphically, that is, the forces acting on pulley A.

Using the law of sines,

$$\frac{400\text{ lb}}{\sin 120°} = \frac{T}{\sin 30°}$$
$$T = 231\text{ lb} \qquad \text{(Yes!)}$$

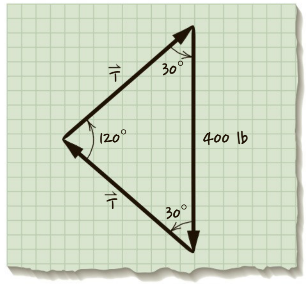


Figure 6 Force triangle.

Answers

(a) $\|\mathbf{T}\| = 231$ lb

(b) The 200-lb man would be lifted off the ground if he attempted to pull the rope from a position underneath pulley B.

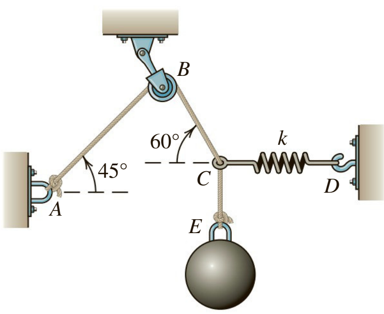
EXERCISES 5.4.4

5.4.23. [**] The sphere has a mass of 10 kg and is supported as shown. The cord ABC passes over a frictionless pulley at B and passes through a frictionless ring at C. If the sphere is in equilibrium,

a. determine the tension in the cord.

b. determine the loads acting on the pin at B supporting the pulley wheel.

c. determine the magnitude of the force pulling on the hook at D.

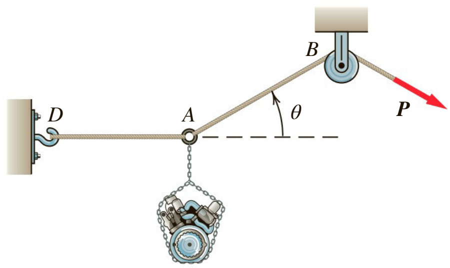


EX 5.4.23

5.4.24. [**] Consider the engine with mass of 225 kg. The system of three cables attached to a ring at A is in equilibrium.

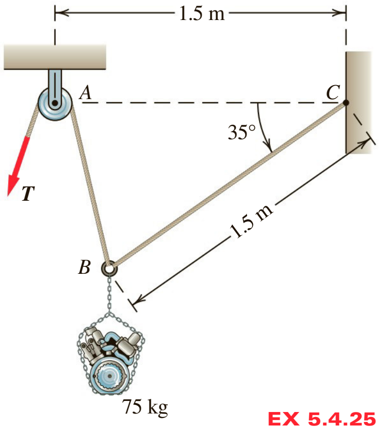
a. Determine the tension in in cables AB and AD, as a function of θ.

b. Cable AB wraps around a frictionless pulley at B. What angle θ will result in $P = 3.5$ kN for equilibrium to hold?



EX 5.4.24

5.4.25. [**] An engine is lifted with the frictionless pulley shown. If the engine and pulley are in equilibrium, what is the tension in cable AB? What is the tension in cable BC? What force T is required to hold the engine in equilibrium?

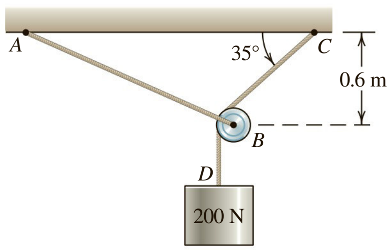


EX 5.4.25

5.4.26. **[**]** Cable *CD* passes over the small frictionless pulley *B* and holds up the metal cylinder that weighs 200 N.

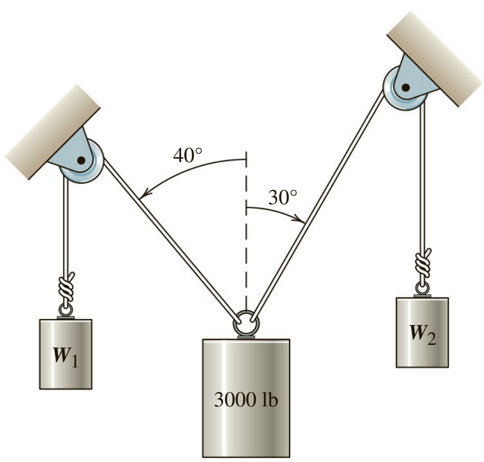
a. Determine the tension in cable *AB*.

b. Use graphical force addition to confirm that the pulley acts as a three-force member.



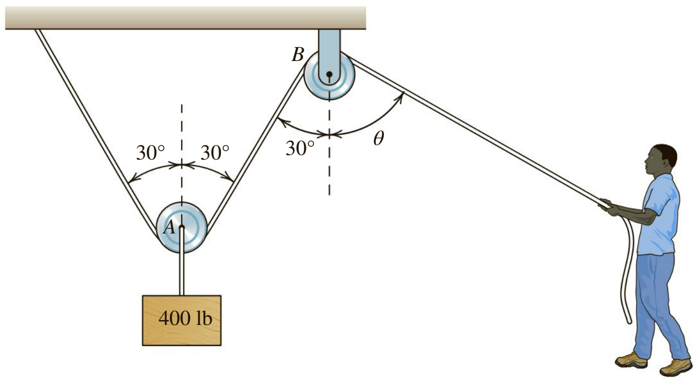
EX 5.4.26

5.4.27. **[**]** The frictionless pulley system shown is in equilibrium. Find the values of weights W_1 and W_2.



EX 5.4.27

5.4.28. **[**]** A man weighing 180 lb is holding a 400-lb crate with a rope and pulley system as shown. Pulley *A* guides the rope, but it is not attached to the wall or ceiling. As the man moves to the left or right, the angle θ and the tension on the rope change. Find the angle θ so that the man is about to be lifted off the ground. In other words, find the angle θ so that the upward force of the rope is equal to the man's weight.

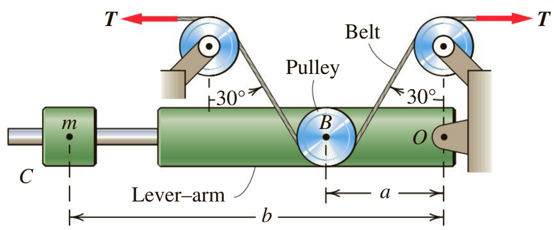


EX 5.4.28

5.4.29. **[***]** A simple belt-tensioning device is shown. The frictionless pulley is pinned to the lever-arm at *B*. Assuming that the device is in equilibrium,

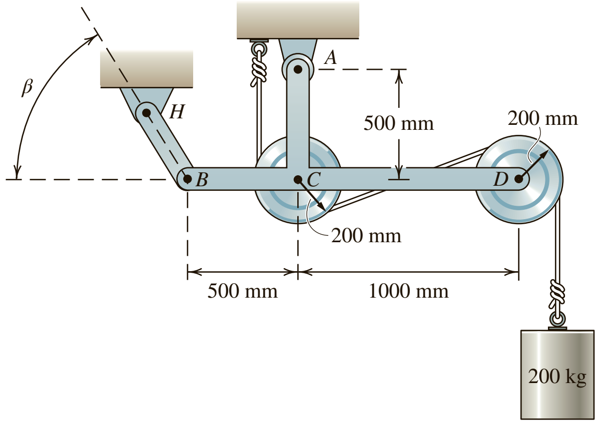
a. write an expression for tension $\boldsymbol{T}$ in the belt as a function of dimensions a and b, and the mass m of the cylinder attached at *C*.

b. determine at what ratio of (a/b) the magnitudes of the belt tension $\boldsymbol{T}$ and the weight of the mass are equal.



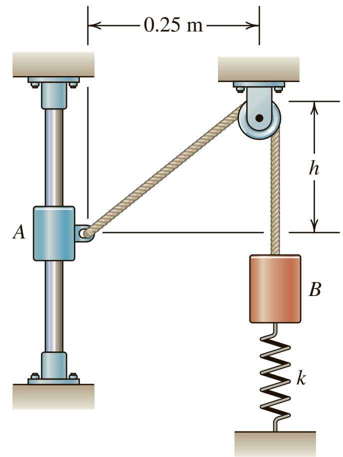
EX 5.4.29

5.4.30. **[***]** The T-shaped bar is supported by a pin connection at *A* and a short link (*HB*) at *B*. A 200-kg block is suspended from a cable that passes over frictionless pulleys *C* and *D*, which are pinned to the bar. Knowing that $\beta = 45°$ and that the bar is in equilibrium, determine the force that the link applies to the bar at *B*.



EX 5.4.30

Exercise 5.4.31. [, computer]** The collar A slides on the smooth vertical bar. Properties are given for the masses $A = 20$ kg and $B = 10$ kg, and the spring constant $k = 360$ N/m. When $h = 0.2$ m, the spring is unstretched. Determine the value of h when the system is in equilibrium.



EX 5.4.31

5.5 APPLYING THE NONPLANAR EQUILIBRIUM EQUATIONS

Learning Objective: Use the conditions of equilibrium to carry out structured static analysis of nonplanar systems.

In previous sections you have had lots of practice using the planar equilibrium equations ($\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma M_z = 0$) to determine the loads acting on particles, pulleys, two-force members, and three-force members. In fact, those three equations describe equilibrium of any system in which all the forces lie in a single plane and any moments acting on the system are about an axis perpendicular to that plane: a planar system. If the world was truly planar, this chapter would end here.

However, as you have undoubtedly experienced, much in the world is nonplanar, as it is forced in multiple directions and twisted about multiple axes. As an example, consider a sailboat being pushed and twisted by the wind on the sail, the water on the keel, and the sailor's weight (Figure 5.5.1). This is a nonplanar system.

Fortunately, everything we have developed up to now on planar equilibrium analysis still applies to finding the loads acting on a nonplanar system in equilibrium, including the *three-step engineering analysis* procedure that was laid out in Table 5.1. Things just get a bit more complex in:

1. Drawing a free-body diagram of the system that is able to capture forces and moments acting in all three directions. Sometimes an isometric view will do the trick, and sometimes two plane views (details of this were covered in Chapter 4).


gaspr13/Getty Images

Figure 5.5.1 A sailboat is a nonplanar system.

2. Based on the free-body diagram, writing and solving up to six equilibrium equations to account for forces and moments in three orthogonal directions. These equations are

Force equilibrium equations:

$$\Sigma F_x = 0 \tag{5.3A}$$
$$\Sigma F_y = 0 \tag{5.3B}$$
$$\Sigma F_z = 0 \tag{5.3C}$$

Moment equilibrium equations:

$$\Sigma M_x = 0 \tag{5.4A}$$
$$\Sigma M_y = 0 \tag{5.4B}$$
$$\Sigma M_z = 0 \tag{5.4C}$$

A sometimes tricky first step in writing the moment equilibrium equations is in wisely choosing a moment center, followed by applying cross-products ($\boldsymbol{r} \times \boldsymbol{F}$) to calculate moments, and finally actually writing the moment equilibrium equations. This process is illustrated in the examples that follow. With a little experience, you will get good at this process!

Check out the following examples of applications of this material.

- **Example 5.5.1 Analysis of a Nonplanar System with Simple Loading**
- **Example 5.5.2 Analysis of a Nonplanar System with Complex Loading**
- **Example 5.5.3 High-Wire Circus Act**
- **Example 5.5.4 Analysis of a Nonplanar System with Unknowns Other than Loads**

EXAMPLE 5.5.1 ANALYSIS OF A NONPLANAR SYSTEM WITH SIMPLE LOADING

A beam of negligible weight is fixed to a wall at A, and a cable force acts on the beam at B (**Figure 1**). A free-body diagram of the beam is given in **Figure 2**. Calculate the unknown loads acting on the beam.

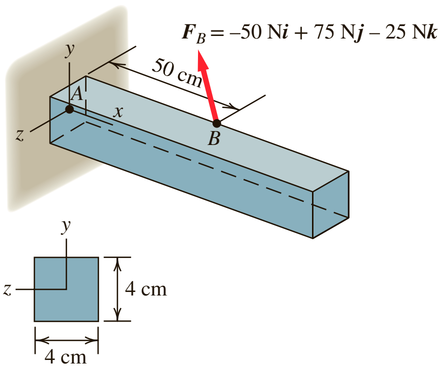


Figure 1

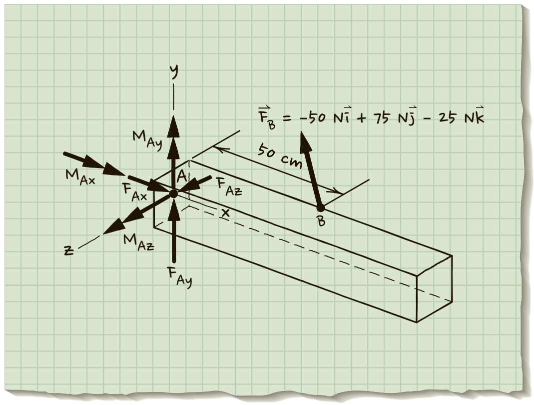


Figure 2

Solution Strategy We are asked to find *the unknown loads acting on the beam.* From the free-body diagram we note that there are six unknown loads acting on the beam where it is attached to the wall at A: three force components and three moment components. We will need to write six equilibrium equations to determine six unknowns.

Goals Calculate the three unknown force components and three unknown moment components acting on the beam.

Given Dimensions of the system, the applied load, negligible beam weight, and the free-body diagram.

Assumptions
- Nonplanar system.
- The system is in *equilibrium.*

Free-Body Diagram The free-body diagram of the beam is provided in **Figure 3**.

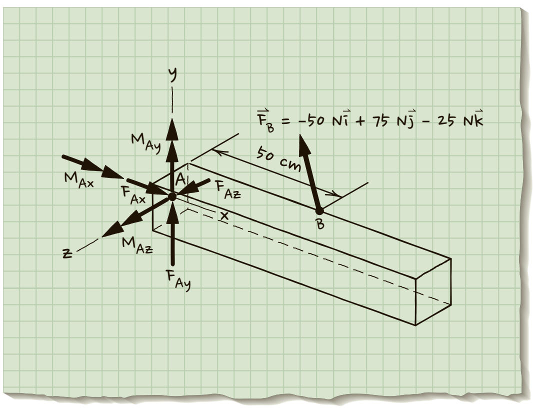


Figure 3 Free-body diagram of the system.

Unknowns: $F_{Ax}, F_{Ay}, F_{Az}, M_{Ax}, M_{Ay}, M_{Az}$.

Equilibrium Equations We start with the three force equilibrium equations. In these equations we define positive to be in the direction of the positive x, y, or z axis.

$$\sum F_x = 0 \Rightarrow F_{Ax} - 50\text{ N} = 0 \tag{1}$$

$$\sum F_y = 0 \Rightarrow F_{Ay} + 75\text{ N} = 0 \tag{2}$$

$$\sum F_z = 0 \Rightarrow F_{Az} - 25\text{ N} = 0 \tag{3}$$

Before we solve (1), (2), and (3), let's write the three moment equilibrium equations. We'll use a moment center at A, since the three unknown forces F_{Ax}, F_{Ay}, and F_{Az} all act through this point and thus have no moment arm—this will make the math easier.

Starting with the moment created by F_B, we set up the position vector and force determinant:

$$\boldsymbol{r}_{AB} \times \boldsymbol{F}_B = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 50\text{ cm} & 2\text{ cm} & -2\text{ cm} \\ -50\text{ N} & 75\text{ N} & -25\text{ N} \end{bmatrix}$$

Carrying out the determinant math, we calculate the components of the moment that F_B creates about a moment center at A.

$$\begin{aligned} \boldsymbol{r}_{AB} \times \boldsymbol{F}_B &= [(-50 + 150)\boldsymbol{i} - (-1250 - 100)\boldsymbol{j} + (3750 + 100)\boldsymbol{k}]\text{N}\cdot\text{cm} \\ &= \underbrace{100\text{ N}\cdot\text{cm}}_{M^{F_B}_{x@A}}\ \boldsymbol{i} + \underbrace{1350\text{ N}\cdot\text{cm}}_{M^{F_B}_{y@A}}\ \boldsymbol{j} + \underbrace{3850\text{ N}\cdot\text{cm}}_{M^{F_B}_{z@A}}\ \boldsymbol{k} \end{aligned}$$

Now we are ready to write the three moment equilibrium equations.

$$\sum M_{x@A} = \underbrace{100\text{ N}\cdot\text{cm}}_{M^{F_B}_{x@A}} + M_{Ax} = 0 \tag{4}$$

$$\sum M_{y@A} = \underbrace{1350\text{ N}\cdot\text{cm}}_{M^{F_B}_{y@A}} + M_{Ay} = 0 \tag{5}$$

$$\sum M_{z@A} = \underbrace{3850\text{ N}\cdot\text{cm}}_{M^{F_B}_{z@A}} + M_{Az} = 0 \tag{6}$$

Solve Solving the equations (1) through (6) we find the unknown loads:

$$\begin{aligned} F_{Ax} &= +50\text{ N} \\ F_{Ay} &= -75\text{ N} \\ F_{Az} &= +25\text{ N} \\ M_{Ax} &= -100\text{ N}\cdot\text{cm} \\ M_{Ay} &= -1350\text{ N}\cdot\text{cm} \\ M_{Az} &= -3850\text{ N}\cdot\text{cm} \end{aligned}$$

Check We illustrate two forms of checking our answers. Generally we will want to do a visual check in combination with some additional calculations.

As a *visual check*, we note that the calculated values of the force components at A are equal and opposite to the applied force at B. Since the

only forces acting on the beam system are at A and B, we expect these forces to cancel one another. And they do!

Our second check involves choosing a moment center other than A and seeing whether our answers actually result in equilibrium of the structure. We choose a moment center at B so that we can eliminate F_B from the equations and make them simpler. In this case, the position vector is from B to A to calculate the moment about B created by F_A.

$$\boldsymbol{r}_{BA} \times \boldsymbol{F}_A = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -50\text{ cm} & -2\text{ cm} & 2\text{ cm} \\ 50\text{ N} & -75\text{ N} & 25\text{ N} \end{bmatrix}$$

$$= \underbrace{100\text{ N}\cdot\text{cm}}_{M^{F_A}_{x@B}}\,\boldsymbol{i} + \underbrace{1350\text{ N}\cdot\text{cm}}_{M^{F_A}_{y@B}}\,\boldsymbol{j} + \underbrace{3850\text{ N}\cdot\text{cm}}_{M^{F_A}_{z@B}}\,\boldsymbol{k}$$

Summing the moments about the x, y, and z axes, we see that the beam is in equilibrium.

$$\sum M_{x@B} = \underbrace{100\text{ N}\cdot\text{cm}}_{M^{F_A}_{x@B}} - \underbrace{100\text{ N}\cdot\text{cm}}_{M_{Ax}} = 0 \qquad \text{(Yes!)}$$

$$\sum M_{y@B} = \underbrace{1350\text{ N}\cdot\text{cm}}_{M^{F_A}_{y@B}} - \underbrace{1350\text{ N}\cdot\text{cm}}_{M_{Ay}} = 0 \qquad \text{(Yes!)}$$

$$\sum M_{z@B} = \underbrace{3850\text{ N}\cdot\text{cm}}_{M^{F_A}_{z@B}} - \underbrace{3850\text{ N}\cdot\text{cm}}_{M_{Az}} = 0 \qquad \text{(Yes!)}$$

Answers There are multiple ways to represent the final results. Here we represent the calculated loads acting on the beam at A in vector form.

$$\boldsymbol{F}_A = 50\text{ N}\,\boldsymbol{i} - 75\text{ N}\,\boldsymbol{j} + 25\text{ N}\,\boldsymbol{k}$$
$$\boldsymbol{M}_A = -100\text{ N}\cdot\text{cm}\;\boldsymbol{i} - 1350\text{ N}\cdot\text{cm}\,\boldsymbol{j} - 3850\text{ N}\cdot\text{cm}\,\boldsymbol{k}$$

The final free-body diagram is shown in Figure 4.

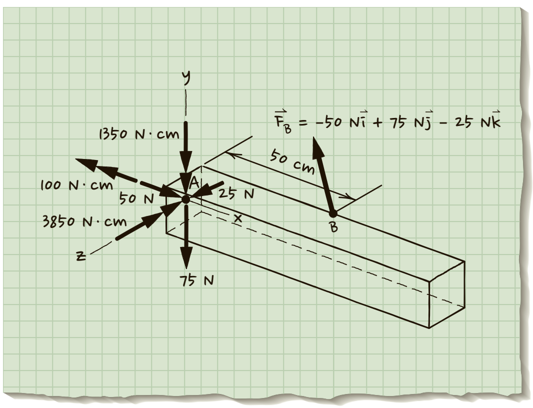


Figure 4 Free-body diagram of the system, with calculated loads shown.

EXAMPLE 5.5.2

A beam of negligible weight is fixed to a wall at A, and a cable force acts on the beam at B. In addition, a 10-N force acts at C and a 150-N·cm moment acts at D. See **Figure 1**. A free-body diagram of the beam is given (**Figure 2**). Calculate the unknown loads acting on the beam.

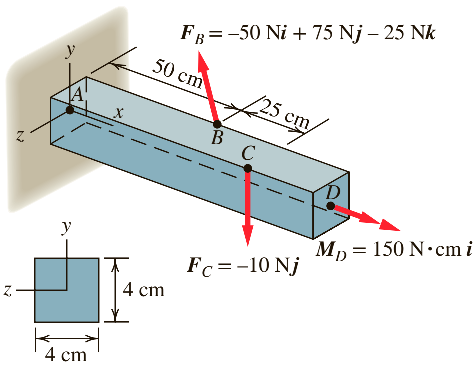


Figure 1

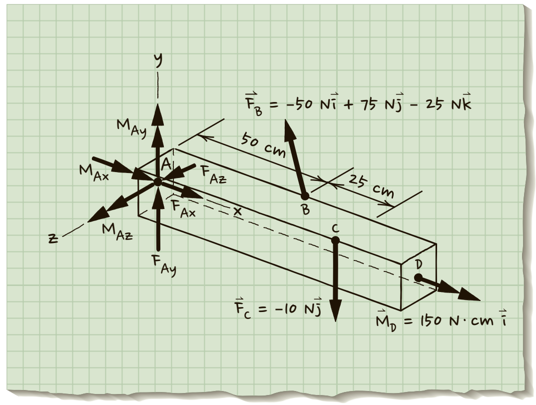


Figure 2

Solution Strategy We are asked to find *the unknown loads acting on the beam.* We note that there are six unknown loads acting on the beam where it is attached to the wall at A: three force components and three moment components. We will need to write six equilibrium equations to determine six unknowns.

Goals Calculate the three unknown force components and three unknown moment components acting on the beam.

Given Dimensions of the system, the applied loads, negligible beam weight, and the free-body diagram.

Assumptions

- Nonplanar system.
- The system is in *equilibrium.*

Free-Body Diagram The free-body diagram of the beam is shown in **Figure 3**.

Unknowns: $F_{Ax}, F_{Ay}, F_{Az}, M_{Ax}, M_{Ay}, M_{Az},$

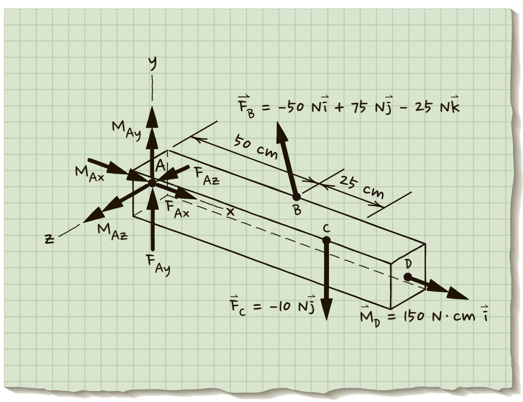


Figure 3 Free-body diagram of the system.

Equilibrium Equations We start with the three force equilibrium equations. In these equations we define positive to be in the direction of the positive x, y, or z axis.

$$\sum F_x = 0 \Rightarrow F_{Ax} - 50\text{ N} = 0 \tag{1}$$

$$\sum F_y = 0 \Rightarrow F_{Ay} + 75\text{ N} - 10\text{ N} = 0 \tag{2}$$

$$\sum F_z = 0 \Rightarrow F_{Az} - 25\text{ N} = 0 \tag{3}$$

Before we solve (1), (2), and (3), let's write the three moment equilibrium equations.

We'll use a moment center at A. Since the three unknown forces F_{Ax}, F_{Ay}, and F_{Az} all act through this point and thus have no moment arm, this will make the math easier.

Starting with the moment created by $\boldsymbol{F}_B$, we set up the position vector and force determinant:

$$\boldsymbol{r}_{AB} \times \boldsymbol{F}_B = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 50\text{ cm} & 2\text{ cm} & -2\text{ cm} \\ -50\text{ N} & 75\text{ N} & -25\text{ N} \end{bmatrix}$$

Carrying out the determinant math, we calculate the components of the moment that $\boldsymbol{F}_B$ creates about a moment center at A.

$$\begin{aligned} M^{F_B}_{@A} &= \boldsymbol{r}_{AB} \times \boldsymbol{F}_B = [(-50 + 150)\boldsymbol{i} - (-1250 - 100)\boldsymbol{j} + (3750 + 100)\boldsymbol{k}]\text{ N}\cdot\text{cm} \\ &= \underbrace{100\text{ N}\cdot\text{cm}}_{M^{F_B}_{x@A}}\ \boldsymbol{i} + \underbrace{1350\text{ N}\cdot\text{cm}}_{M^{F_B}_{y@A}}\ \boldsymbol{j} + \underbrace{3850\text{ N}\cdot\text{cm}}_{M^{F_B}_{z@A}}\ \boldsymbol{k} \end{aligned}$$

We repeat this calculation for the moment created by $\boldsymbol{F}_C$ about the moment center at A.

$$M^{F_C}_{@A} = \boldsymbol{r}_{AC} \times \boldsymbol{F}_C = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 75\text{ cm} & 2\text{ cm} & 2\text{ cm} \\ 0\text{ N} & -10\text{ N} & 0\text{ N} \end{bmatrix}$$

$$M^{F_C}_{@A} = \underbrace{20\text{ N}\cdot\text{cm}}_{M^{F_C}_{x@A}}\ \boldsymbol{i} \underbrace{-750\text{ N}\cdot\text{cm}}_{M^{F_C}_{z@A}}\ \boldsymbol{k}$$

Now we are ready to write the three moment equilibrium equations using the expression for $M^{F_B}_{@A}$ and $M^{F_C}_{@A}$.

$$\sum M_{x@A} = \underbrace{100\text{ N}\cdot\text{cm}}_{M^{F_B}_{x@A}} + \underbrace{20\text{ N}\cdot\text{cm}}_{M^{F_C}_{x@A}} + \underbrace{150\text{ N}\cdot\text{cm}}_{M_D} + M_{Ax} = 0 \tag{4}$$

$$\sum M_{y@A} = \underbrace{1350\text{ N}\cdot\text{cm}}_{M^{F_B}_{y@A}} + M_{Ay} = 0 \tag{5}$$

$$\sum M_{z@A} = \underbrace{3850\text{ N}\cdot\text{cm}}_{M^{F_B}_{z@A}} \underbrace{-750\text{ N}\cdot\text{cm}}_{M^{F_C}_{z@A}} + M_{Az} = 0 \tag{6}$$

Solve Solving equations (1) through (6), we find the unknown loads:

$$F_{Ax} = +50 \text{ N}$$
$$F_{Ay} = -65 \text{ N}$$
$$F_{Az} = +25 \text{ N}$$
$$M_{Ax} = -270 \text{ N}\cdot\text{cm}$$
$$M_{Ay} = -1350 \text{ N}\cdot\text{cm}$$
$$M_{Az} = -3100 \text{ N}\cdot\text{cm}$$

Checks In this particular example we are focusing on writing and solving the six equilibrium equations, so we have not included a check. However, you should always check your answers!

Answers There are multiple ways to represent the final results. Here we represent the calculated loads acting on the beam at A in vector form.

$$\boldsymbol{F}_A = 50 \text{ N}\,\boldsymbol{i} - 65 \text{ N}\,\boldsymbol{j} + 25 \text{ N}\,\boldsymbol{k}$$
$$\boldsymbol{M}_A = -270 \text{ N}\cdot\text{cm}\ \boldsymbol{i} - 1350 \text{ N}\cdot\text{cm}\,\boldsymbol{j} - 3100 \text{ N}\cdot\text{cm}\,\boldsymbol{k}$$

The final free-body diagram is shown in Figure 4.

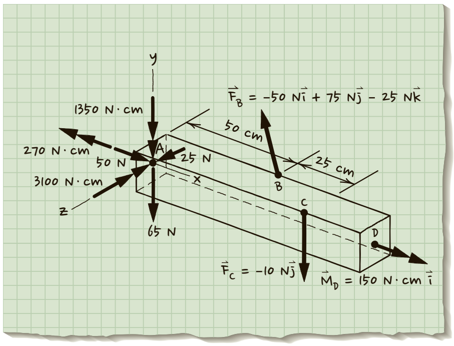


Figure 4 Free-body diagram of the system with calculated loads shown.

EXAMPLE 5.5.3

Poles AB and EF support a circus high wire and are held in the vertical position by guy wires (Figure 1). The tension in the high wire (BF) is 9 kN. The weight of the wires and poles can be ignored. Find the forces acting on guy wires BC and BD, and on pole AB at B.

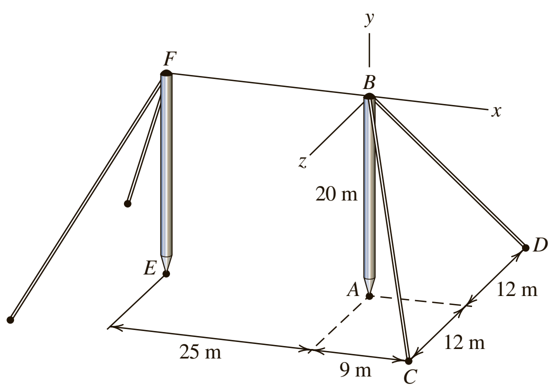


Figure 1 High wire supported by poles and guy wires.

Solution Strategy We are being asked to find *the forces acting on wires BC and BD, and on the pole AB at B (our unknowns).* In looking at the situation, we observe that there is a ball-and-socket connection at A. This means that the load at A, acting on pole AB, consists of a force (and no moment). Therefore AB is a two-force member, and there are no moments at B. This allows us to model B as a particle and find the forces acting on that particle.

Goals Calculate the tensions in BC and BD ($\boldsymbol{F}_{BC}$ and $\boldsymbol{F}_{BC}$, respectively), and the force acting on pole AB at B ($\boldsymbol{F}_{AB}$).

Given Dimensions of the system and magnitude of high-wire tension force (9 kN). Mass of wires and poles is negligible.

Assumptions

- Joint A is a ball-and-socket joint.
- Nonplanar system.
- Wires act as tension-only two-force members.
- The system is in *equilibrium.*

Free-Body Diagram A free-body diagram (**Figure 2**) of vertical pole AB with a ball-and-socket joint at end A, combined with summing moments about B, and then summing forces in the x, y, and z directions, reveals that for the pole to be in equilibrium, $F_{Ax} = F_{AZ} = F_{Bx} = F_{Bz} = 0$.

This means that pole AB is acting like a two-force member, with equal and opposite vertical forces acting on it.

We now draw a free-body diagram (**Figure 3**) of "particle" B, showing the three wire tensions (F_{BC}, F_{BD}, 9 kN), and the vertical force (F_{BA}) of the pole acting on the particle. (Notice that F_{BA} is the equal and opposite force to F_{By} in **Figure 3**).

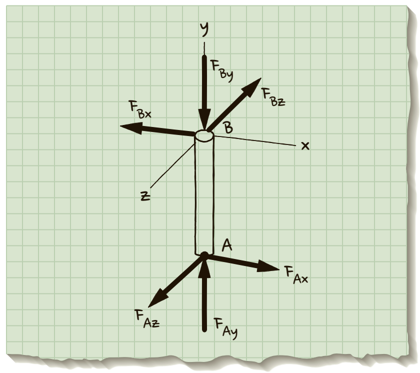


Figure 2 Free-body diagram of pole AB.

Unknowns: F_{BD}, F_{BC}, F_{BA}.

Equilibrium Equations Before we set up the equilibrium equations, we write each force in terms of rectangular components:

$$\boldsymbol{F}_{BC} = F_{BC}\boldsymbol{u}_{BC} \qquad \boldsymbol{F}_{BD} = F_{BD}\boldsymbol{u}_{BD}$$
$$\boldsymbol{F}_{BA} = F_{BA}\,\boldsymbol{j} \qquad \boldsymbol{F}_{BF} = -9\text{ kN}\,\boldsymbol{i}$$

where (from Chapter 2)

$$\boldsymbol{u}_{BC} = \frac{9\text{ m }\boldsymbol{i} - 20\text{ m }\boldsymbol{j} + 12\text{ m }\boldsymbol{k}}{\sqrt{(9\text{ m})^2 + (-20\text{ m})^2 + (12\text{ m})^2}} = \frac{9}{25}\boldsymbol{i} - \frac{20}{25}\boldsymbol{j} + \frac{12}{25}\boldsymbol{k}$$

$$\boldsymbol{u}_{BD} = \frac{9\text{ m }\boldsymbol{i} - 20\text{ m }\boldsymbol{j} - 12\text{ m }\boldsymbol{k}}{\sqrt{(9\text{ m})^2 + (-20\text{ m})^2 + (-12\text{ m})^2}} = \frac{9}{25}\boldsymbol{i} - \frac{20}{25}\boldsymbol{j} - \frac{12}{25}\boldsymbol{k}$$

Now we write the equilibrium equations based on the free-body diagram. Because B acts like a particle, we need only consider the force equations:

$$\sum F_x = 0(\rightarrow +) \Rightarrow \frac{9}{25}F_{BC} + \frac{9}{25}F_{BD} - 9\text{ kN} = 0 \qquad (1)$$

$$\sum F_y = 0(\uparrow +) \Rightarrow +F_{BA} - \frac{20}{25}F_{BC} - \frac{20}{25}F_{BD} = 0 \qquad (2)$$

$$\sum F_z = 0\text{ (out of page +)} \Rightarrow \frac{12}{25}F_{BC} - \frac{12}{25}F_{BD} = 0 \qquad (3)$$

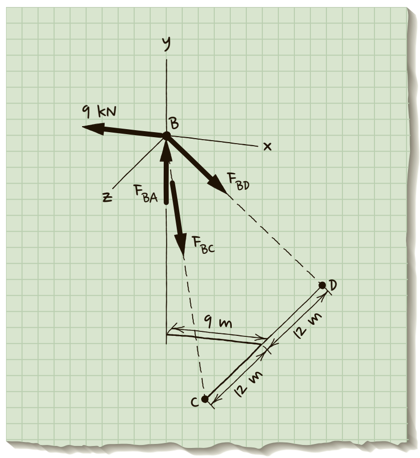


Figure 3 Free-body diagram of B.

Solve Equation (3) tells us that

$$F_{BC} = F_{BD} \tag{4}$$

We substitute (4) into (1) to find F_{BC} and F_{BD}:

$$\frac{9}{25}F_{BC} + \frac{9}{25}F_{BC} - 9\text{ kN} = 0$$
$$F_{BC} = 12.5\text{ kN} = F_{BD}$$

Substituting (5) into (2), we determine F_{BA}:

$$-F_{BA} - \frac{20}{25}(12.5\text{ kN}) - \frac{20}{25}(12.5\text{ kN}) - 9\text{ kN} = 0$$
$$F_{BA} = 20\text{ kN}$$

$F_{BC} = 12.5$ kN, $F_{BD} = 12.5$ kN, $F_{BA} = 20$ kN

Checks It makes sense that F_{BC} and F_{BD} are both positive; this indicates that the forces are in the direction assumed on the free-body diagram, which is tension in the guy wires (we would have been suspicious of our answers had either of them been calculated to be negative). Also, because of symmetry, it makes sense that F_{BC} and F_{BD} have the same magnitude (12.5 kN). Finally, the guy wires pulling down on top of the pole results in the pole being compressed, which causes the pole to push up on point B; this is consistent with $F_{BA} = 20$ kN as a positive answer, because we assumed it pointed upward.

Answers $F_{BC} = 12.5$ kN, $F_{BD} = 12.5$ kN, $F_{BA} = 20$ kN

EXAMPLE 5.5.4

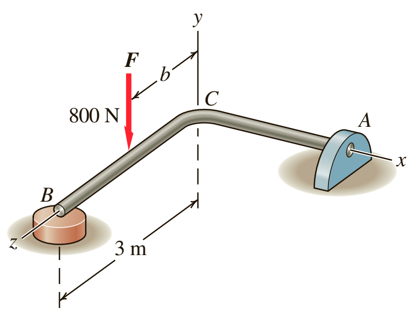


Figure 1

The L-shaped bar shown in Figure 1 is supported by a thrust bearing at A and rests on a smooth horizontal surface at B. An 800-N load $\boldsymbol{F}$ is applied at a distance b from point C. If the largest force the surface at B can support is 600 N, what is the maximum value of dimension b?

Strategy Overview We need to find the load at B in terms of the unknown length b. Once we have developed this relationship, we can solve for b so that the normal force at B does not exceed 600 N.

Goals Find the largest value of dimension b such that the normal force at B is at most 600 N.

Given Geometry/dimensions of the L-bar and the applied 800-N force. The surface at B is smooth, and the support at A consists of a thrust bearing.

Assumptions

- Ignore the weight of the L-bar (because we are not given any information about it).
- The surface at B is frictionless, because it is smooth.
- We must model the L-bar as a nonplanar system.

Free-Body Diagram We isolate the L-shaped bar from its supports to draw the free-body diagram (Figure 2). Because there is only a single bearing attached to the system, the loads at A consist of forces and moments; see Table 4.2.

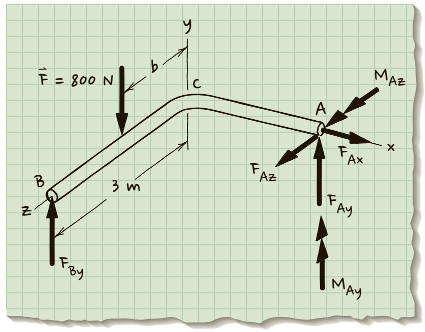


Figure 2 Free-body diagram of the L-bar.

Equilibrium Equations For the L-bar to be in mechanical equilibrium, all six equations must hold. If we sum the moments about the x axis through A, the equilibrium equation will contain only one unknown load.

Based on Equation (5.4A),

$$\sum M_{x@A} = bF - (3.0\,\text{m})F_{By} = 0$$

Solve Solving, we find F_{By} in terms of b:

$$F_{By} = \frac{b}{3.0\text{ m}}F$$

With $F = 800$ N, this becomes

$$F_{By} = \frac{b}{3.0\text{ m}}(800\text{ N})$$

Our constraint is $F_{By} \le 600$ N, and we use this constraint to solve for the maximum value of b:

$$F_{By} = \frac{b}{3.0\text{ m}}(800\text{ N}) \le 600\text{ N} \;\rightarrow b \le 2.25\text{ m}$$

Therefore the largest allowable value of b is 2.25 m. With $b \le 2.25$ m, we have $F_{By} \le 600$ N.

NOTE: Although this is a nonplanar problem, we did not need to write all six equilibrium equations in order to answer the question posed in this example.

Checks We check our answers by confirming that moment equilibrium holds at a moment center other than A. Arbitrarily, we choose B.

$$\Sigma M_{x@B} = 0 \rightarrow -(3.0\,\text{m} - b)F + (3.0\,\text{m})F_{Ay} = 0 \tag{1}$$

We find F_{Ay} by applying force equilibrium in the y direction:

$$\uparrow + \Sigma F_y = F_{Ay} + F_{By} - F = 0$$

$$F_{Ay} = F - \frac{b}{3.0\text{ m}}F$$

$$F_{Ay} = F\left(1 - \frac{b}{3.0\text{ m}}\right)$$

Substituting into Equation (1), we find that $0 = 0$. This confirms our calculations.

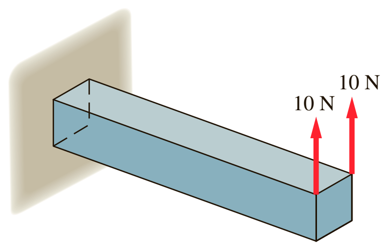
We should also check that our answer makes intuitive sense. Does it seem reasonable that F_{By} increases as b increases? Does it make sense that we need to limit b in order to put a 600-N limit on F_{By}?

Answers $b \leq 2.25$ m

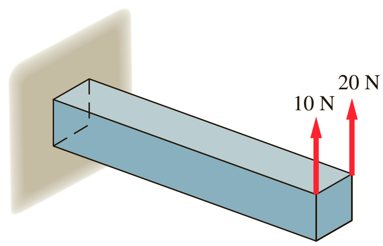
EXERCISES 5.5

5.5.1. [*] A cantilever beam is fixed at end A and loads are applied at end B. For the six configurations of loaded shown determine whether the system (defined as the beam) is planar or nonplanar, and how many of the six equilibrium equations would be required to find all non-zero loads acting on end A.

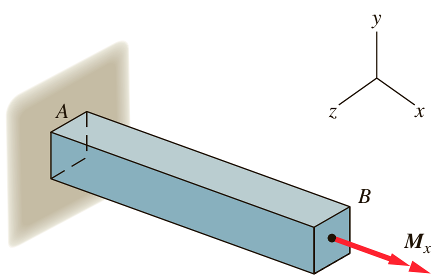
(b)



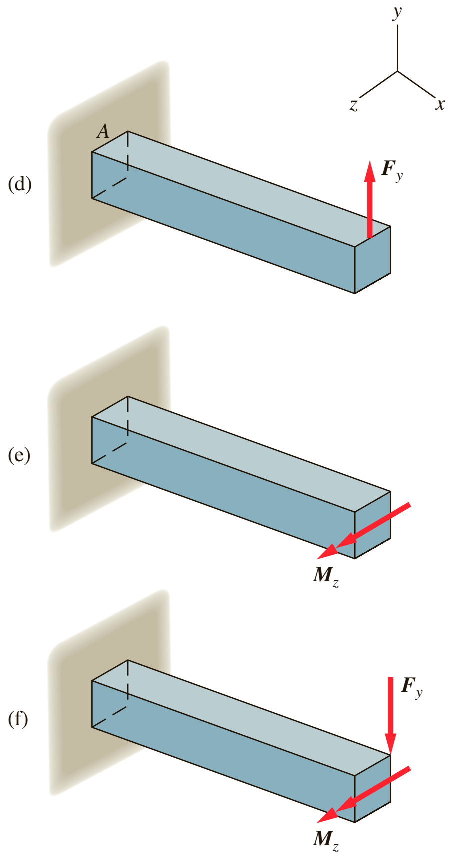
(c)



(a)

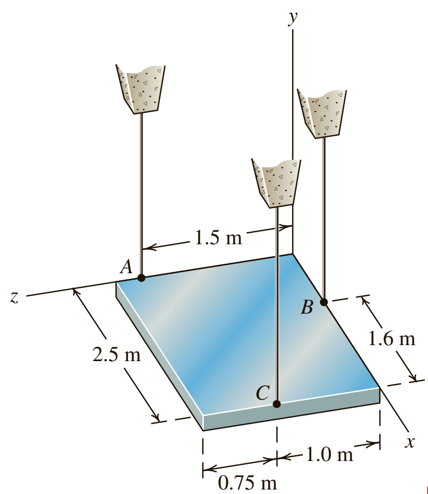


EX 5.5.1



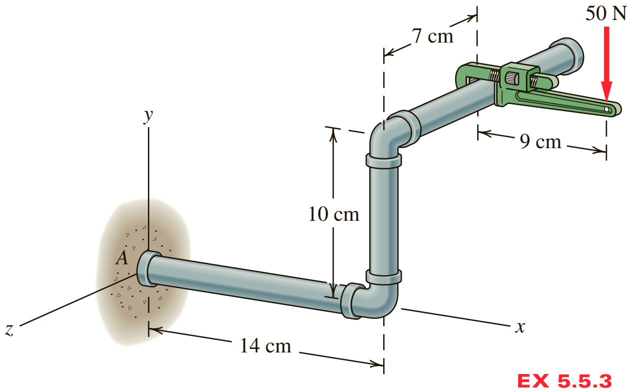
EX 5.5.1

5.5.2. [*] The rectangular plate of uniform thickness has a mass of 500 kg. Determine the tensions in the three support cables if the plate is in equilibrium.



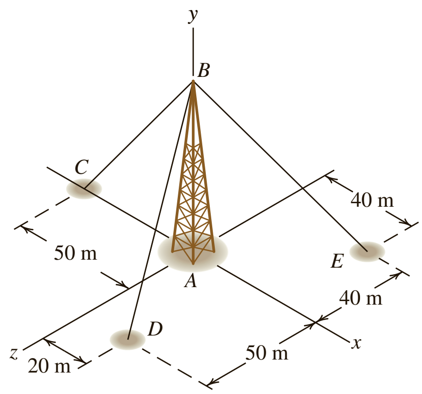
EX 5.5.2

5.5.3. [*] A 50-N force is applied to the pipe wrench attached to the pipe system shown. If the pipe is in equilibrium, determine the loads acting on the pipe at support A.



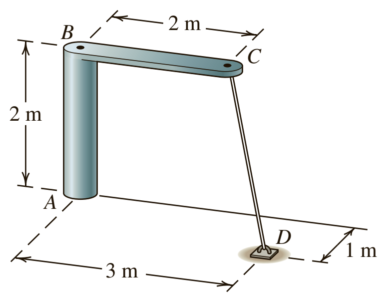
EX 5.5.3

5.5.4. [*] A tower 70 m tall is tethered by three cables, as shown. If the tension in each of the cables is 4000 N and the tower is in equilibrium, what loads act on the tower at its base?



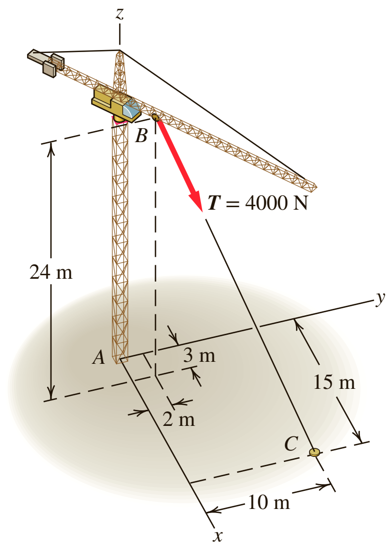
EX 5.5.4

5.5.5. [*] The bracket ABC is tethered as shown with cable CD. The tension in the cable is 750 N. If the weight of the bracket is negligible and the bracket is in equilibrium, what loads act on the bracket at A?



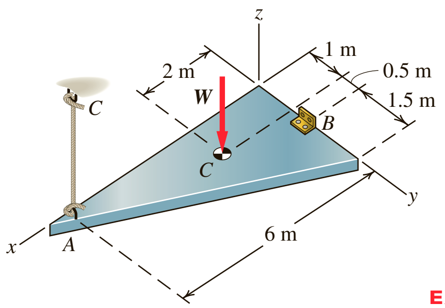
EX 5.5.5

5.5.6. [*] A tower crane is fixed at the ground at A as shown. Assume that the weight of the crane is negligible. If the tower crane is in equilibrium, determine the loads acting at its base due to the 4000-N cable tension.



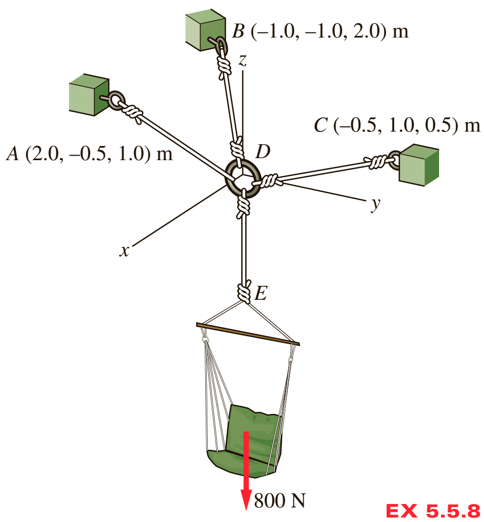
EX 5.5.6

5.5.7. [*] A triangular plate is supported by a cable at A and a hinge at B. Its weight of 400 N acts at the plate's center of gravity, as shown. If the plate is in equilibrium, determine the tension in the cable and the loads acting on the plate at the hinge.



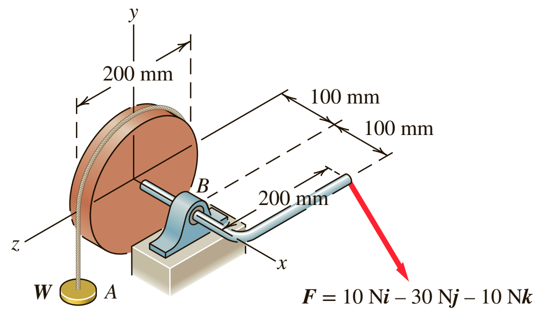
EX 5.5.7

5.5.8. [*] A hanging chair is suspended, as shown, and holds a person weighing 800 N. Determine the tension in cables DA, DB, and DC if the chair is in equilibrium.



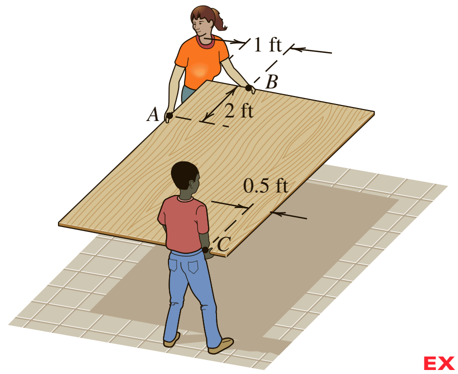
EX 5.5.8

5.5.9. [*] A pulley is used to lift a weight W. The shaft of the pulley is supported by a thrust bearing, as shown. Ignore the weights of the pulley and the shaft. If the shaft is in equilibrium, determine the loads acting on the shaft at B. Also determine the value of W.



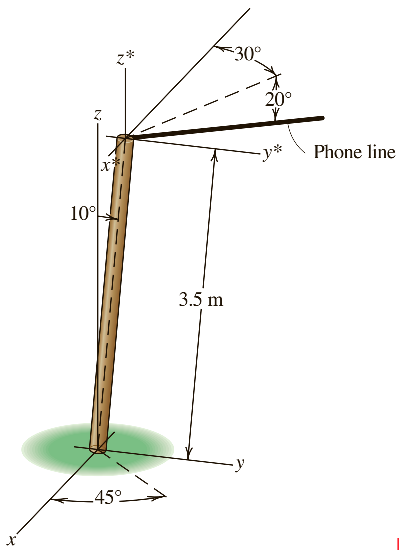
EX 5.5.9

5.5.10. [*] Two workers are carrying a 4-ft by 8-ft siding panel by grabbing the panel at the points shown. The panel weighs 80 lb. If the panel is in equilibrium, what loads must each worker apply to the panel?



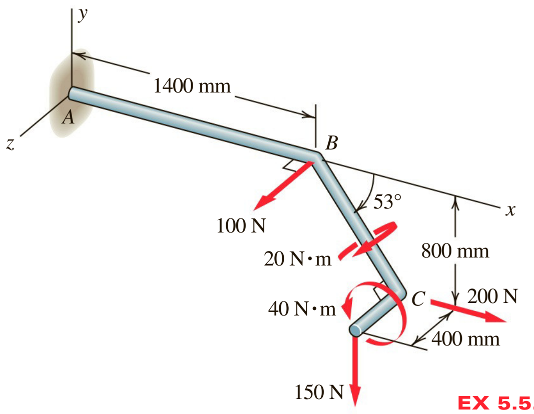
EX 5.5.10

5.5.11. [**] An above-ground phone line is attached to a 3.5-m long utility pole. It was hit by a snow plow and now leans at the 10° angle shown. The utility pole has a mass of 15 kg/m, and the tension in the phone line is 450 N. If the utility pole is in equilibrium, what loads must act on the base of the pole?



EX 5.5.11

5.5.12. [**] A bent bar is welded to the wall at end A and carries the loads shown. Determine the loads acting on the bar at end A, when it is in equilibrium.

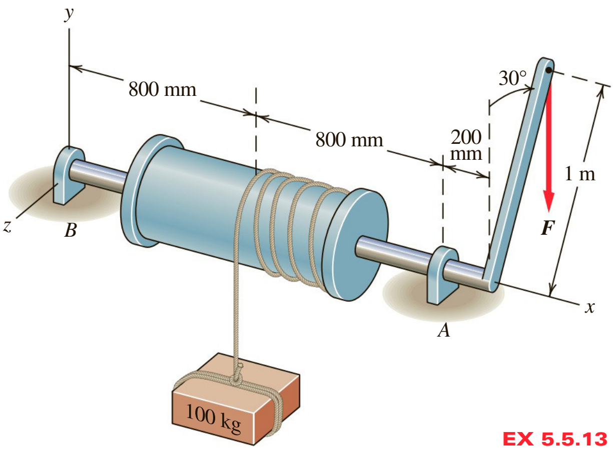


EX 5.5.12

5.5.13. [**] The winch assembly is holding a 100-kg packet in equilibrium, as shown. The drum of the winch has a diameter of 400 mm. Determine

a. the force, $\boldsymbol{F}$

b. the loads acting on the winch assembly at the journal bearings, A and B

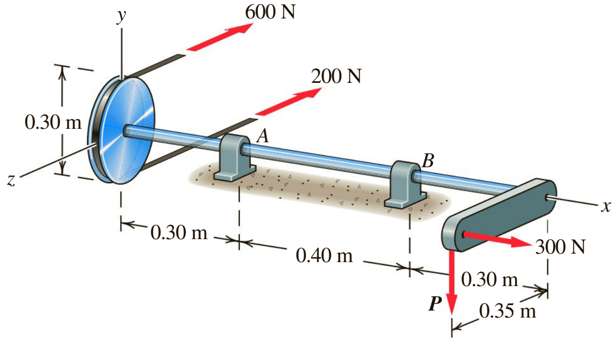


EX 5.5.13

5.5.14. [**] A shaft is loaded through a pulley and a lever that are fixed to the shaft as shown. Friction between the belt and pulley prevents the belt from slipping. The support at A is a journal bearing, and the support at B is a thrust bearing. Determine

a. the force $\boldsymbol{P}$ required for equilibrium

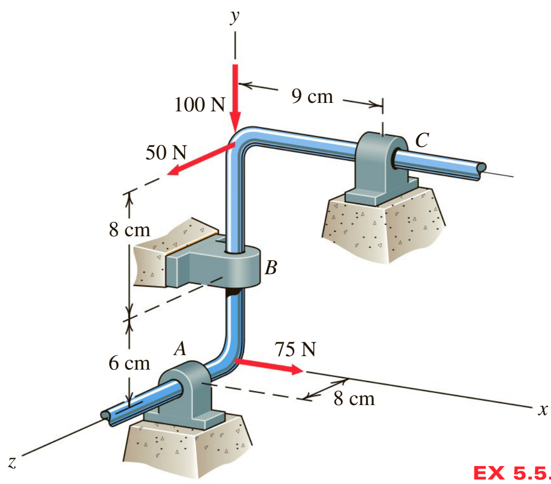
b. the loads acting on the shaft at supports at A and at B



EX 5.5.14

5.5.15. [**] The bent bar shown is supported with three well-aligned journal bearings. If the bar is in equilibrium, determine the loads acting on the bar at the supports

a. at A **b.** at B **c.** at C

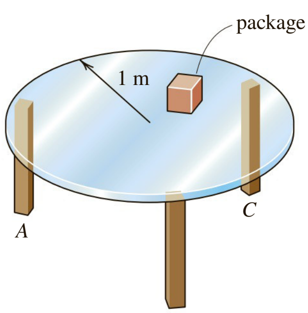


EX 5.5.15

5.5.16. [**] A package is placed on the three-legged circular table at location (R, θ), as shown. The magnitudes of the forces the ground applies to the legs are known to be $\|\boldsymbol{F}_A\| = 110$ N, $\|\boldsymbol{F}_B\| = 140$ N, and $\|\boldsymbol{F}_C\| = 130$ N. If the mass of the table is 30 kg and the table is in equilibrium, determine

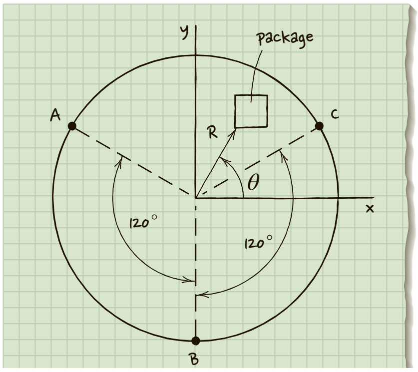
a. the mass of the package

b. the values of R and θ. In addition, describe the location of the package with a sketch of the tabletop



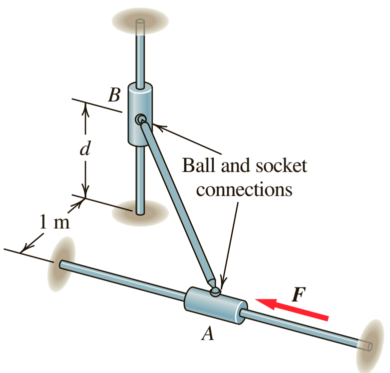
(a)

EX 5.5.16a



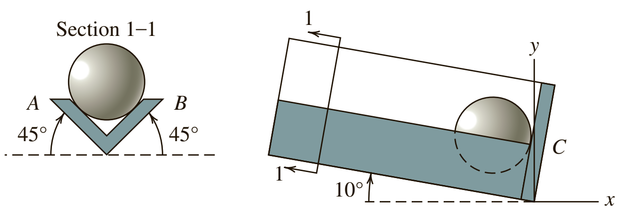
EX 5.5.16b

5.5.17. [**] Ends A and B of the 5-kg rigid bar are attached by lightweight collars that may slide over the smooth fixed rods. The center of mass of the bar is at its midpoint. The length of the bar is 1.5 m. Determine the horizontal force $\boldsymbol{F}$ applied to collar A that will result in static equilibrium as a function of the distance, d ($0.5 \leq d \leq 1.5$ cm).



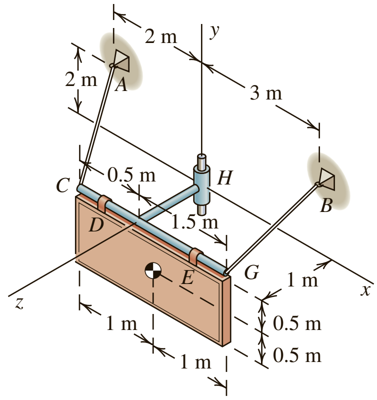
EX 5.5.17

5.5.18. [**] A 5-kg steel sphere rests between the 45° grooves A and B of the 10° incline, and against a vertical wall at C as shown. If all three surfaces of contact are smooth, determine the loads acting on the surface of the sphere.



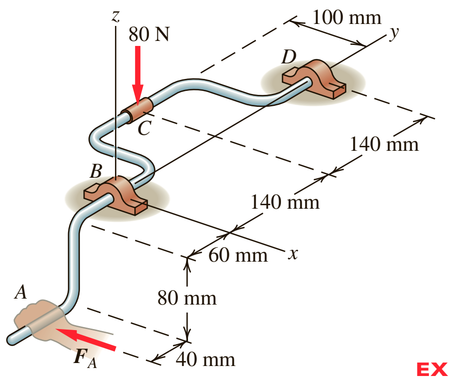
EX 5.5.18

5.5.19. [**] Cables and a collar joint support a sign of weight W (2000 N) as shown. If the sign is in equilibrium, determine the tensions in the cables and the loads at the collar joint.



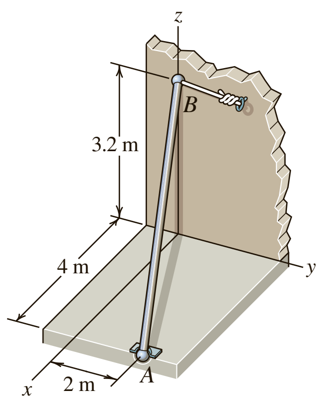
EX 5.5.19

5.5.20. [**] A crankshaft is supported by a journal bearing at B and a thrust bearing at D as shown. Ignore the weight of the crank. If the crankshaft is in equilibrium, determine the loads acting on the crankshaft at B and D. The force at A is $\boldsymbol{F}_A = -100\text{ N}\,\boldsymbol{i} - 5\text{ N}\,\boldsymbol{j}$.



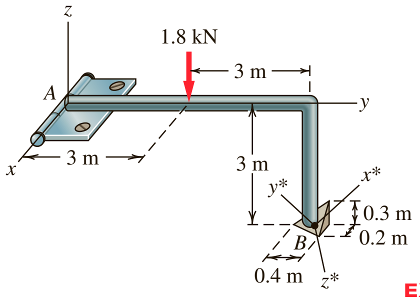
EX 5.5.20

5.5.21. [**] The uniform shaft is supported by a ball-and-socket connection at A in the horizontal floor. The ball end B rests against the smooth vertical wall as shown. The shaft weighs 1500 N. If the shaft is in equilibrium, what loads act on the shaft at A and at B?



EX 5.5.21

5.5.22. [**] The bar is supported at A by a hinge, and at B it rests against a smooth surface. If the bar is in equilibrium, what loads act on the bar at A ? What loads act at B?

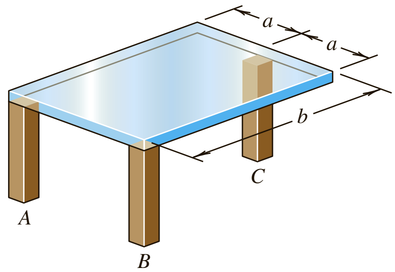


EX 5.5.22

5.5.23. [**] Three vertical legs support a uniform rectangular table of weight W. Determine

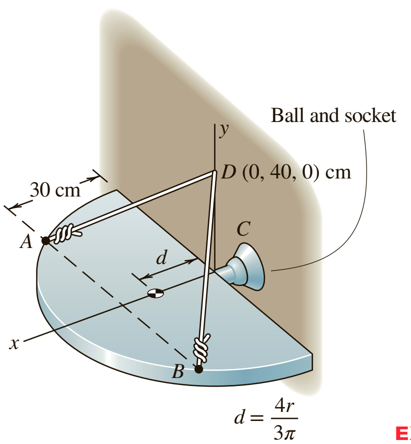
a. the loads the floor applies to the legs

b. the smallest vertical force $\boldsymbol{F}$ that can be applied to the tabletop to cause the table to tip over



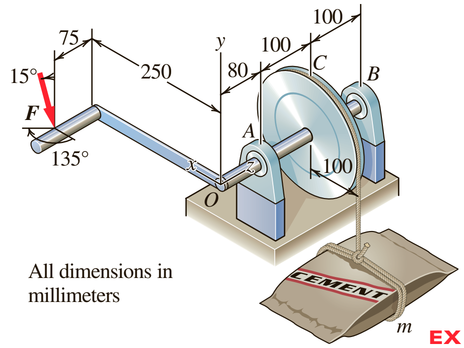
EX 5.5.23

5.5.24. [**] Consider the semi-circular plate that has a radius of 40 cm. The plate weighs 100 N with center of gravity at $(d, 0, 0)$. Determine the tension in the cables at A and B and the loads acting on the plate at C (a ball-and-socket connection).



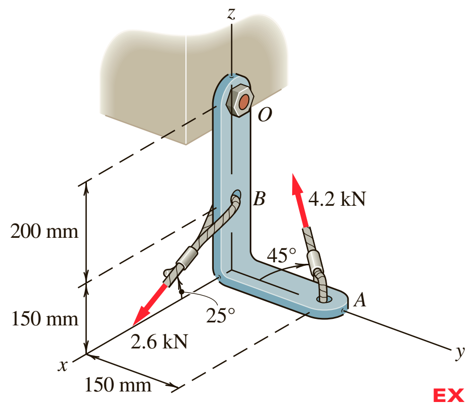
EX 5.5.24

5.5.25. [**] A hand-operated hoist is used to raise 240-N bags of dry cement onto a mixing platform, as shown. If the hoist is in equilibrium, what loads act on the shaft at the journal bearing at A and the thrust bearing at B?



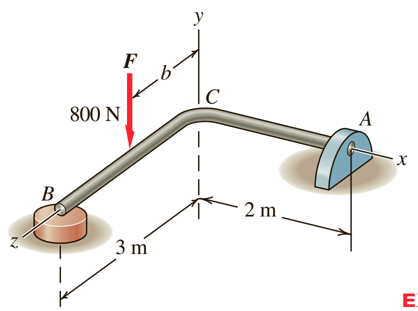
EX 5.5.25

5.5.26. [**] A bracket is bolted to a structure at O. Cables load the bracket, as shown. If the bracket is equilibrium and the weight of the bracket is negligible, what loads act on the bracket at O?



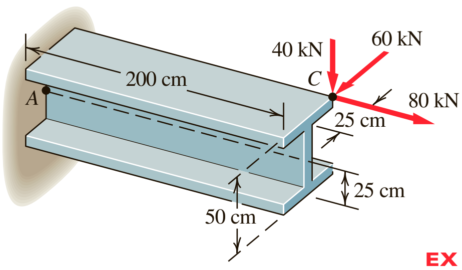
EX 5.5.26

5.5.27. [**] Consider the L-shaped bar shown. The bar is uniform along its length, and has a total weight of 500 N. It is supported by a thrust bearing at A and rests on a smooth horizontal surface at B. A load $\boldsymbol{F}$ of 800 N is applied at a distance b from point C. If the maximum load that the surface at B can support is 600 N, what is the maximum value of the dimension b? Compare the answer with Example 5.5.4.



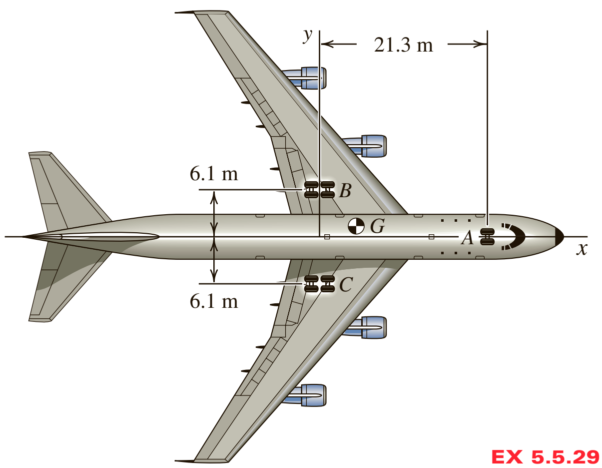
EX 5.5.27

5.5.28. [**] Point C on the top flange of the I-beam is subjected to the loads shown. Determine the loads acting on the I-beam at the support at A (which is on the axis of symmetry of the I-beam).



EX 5.5.28

5.5.29. [**] An airplane rests on the ground, as shown. Its landing gear wheels are at points A, B, and C. Its center of gravity G is at (3.0, 0.2, –4.6) m, as shown. The airplane weighs 1.56×10^6 N. What are the magnitudes of the normal forces acting on the landing gear assemblies at A, B, and C?



EX 5.5.29

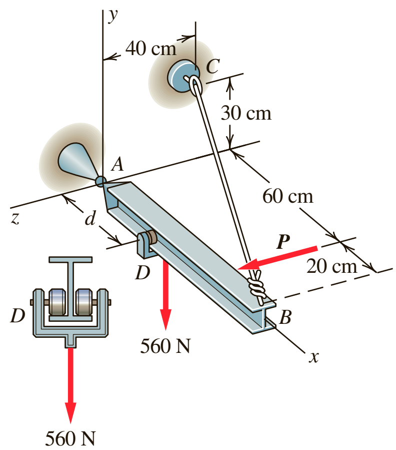
5.5.30. [***] An I-beam is supported by a ball-and-socket joint at A and a rope BC. A horizontal force, $\boldsymbol{P}$, is applied as shown. In addition, a 560-N load is suspended from a movable support at D. The uniform beam is 80 cm long and weighs 225 N.

a. Create a graph of T_{BC}, the tension in the rope, as a function of the distance d ($0 \leq d \leq 80$ cm).

b. On the same graph, plot the magnitude of the force $\boldsymbol{P}$ required to keep the beam horizontal (i.e., aligned with the y axis) as a function of the distance d ($0 \leq d \leq 80$ cm).

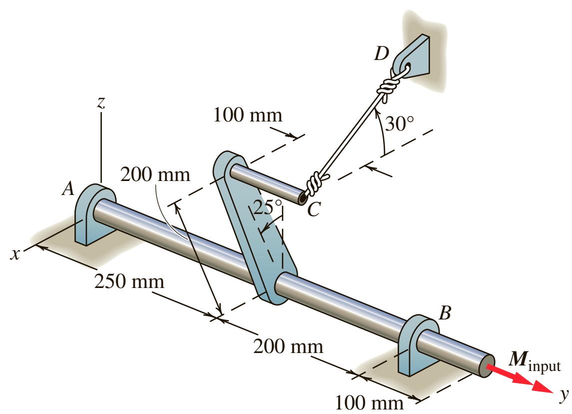
c. If the tension in the rope is not to exceed 1200 N (otherwise it will break), what limit should be specified for d?

d. Come up with two ideas of how the movement of the movable support might be limited so as not to exceed the limit you specified in **c**. Present your idea as sketches.



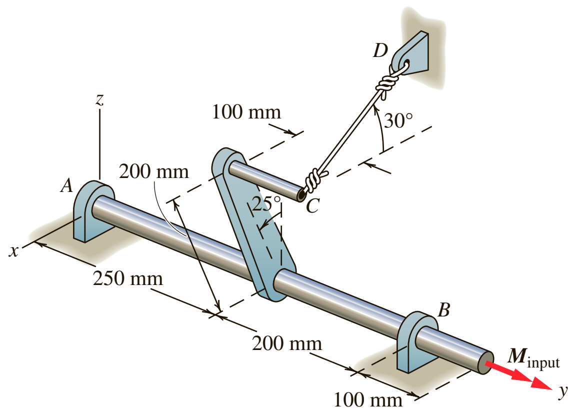
EX 5.5.30

5.5.31. [***] A shaft assembly is used to support an input moment M_{input} as shown. It is held by a journal bearing at A, a thrust bearing at B, and a cable attached at C. If $M_{input} = 100$ N·m, determine the loads acting on the shaft assembly at the bearings and at C when the shaft is in equilibrium. The cable lies in a plane parallel to the xz plane, and the bearings are properly aligned on the shaft.



EX 5.5.31

5.5.32. **[***]** If the cable used to hold the shaft will fail at a tension force greater than 300 N, determine the maximum allowable input moment.

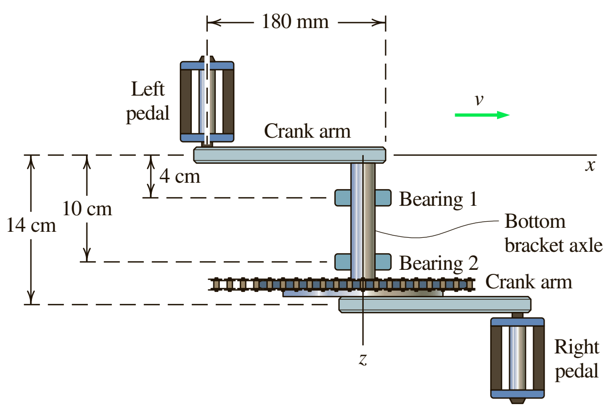


EX 5.5.32

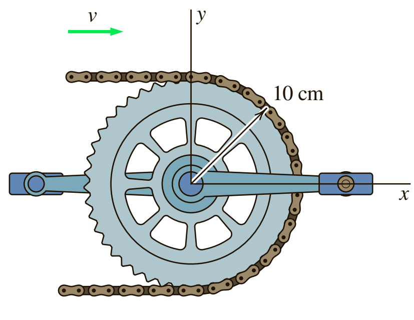
5.5.33. **[***]** A bicycle is moving at a constant velocity of 15 mph. The foot force on the right pedal is 5 N $\boldsymbol{i}$ – 80 N $\boldsymbol{j}$ and on the left pedal is –5 N $\boldsymbol{j}$. Determine

a. the loads the ball bearings apply to the bottom bracket axle for this pedal configuration

b. the tension force in the chain



(a) Top view



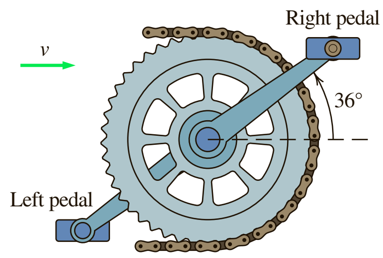
(b) Side view

EX 5.5.33

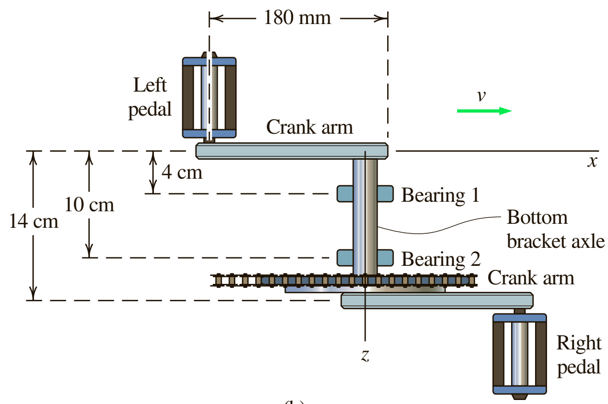
5.5.34. **[***]** A bicycle is moving at a constant velocity of 15 mph. The foot force on the right pedal is 5 N $\boldsymbol{i}$ – 30 N $\boldsymbol{j}$ and on the left pedal is 5 N $\boldsymbol{i}$ – 5 N $\boldsymbol{j}$ when in the position shown (a). The dimensions of the bottom bracket are as shown in (b). Determine

a. the loads the ball bearings apply to the bottom bracket axle

b. the tension force in the chain



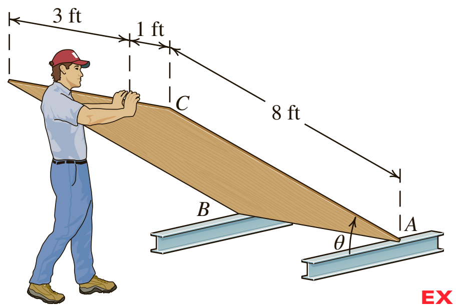
(a)



(b)

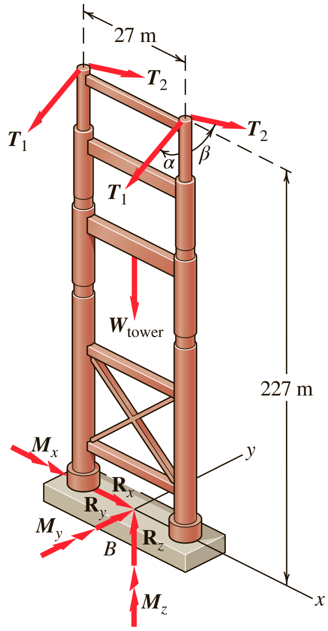
EX 5.5.34

5.5.35. **[***]** The 4-ft by 8-ft 50-lb plywood panel shown is being tilted into place by a construction worker. Corners A and B of the panel are resting on treads and the construction worker is pushing perpendicular to the panel at a distance of 1 ft from corner C. Find the horizontal force at A as a function of θ. At what tilt angle will the plywood most likely slip at A?



EX 5.5.35

5.5.36. [***] When Joseph Strauss designed the piers that serve as the foundations for the towers of the Golden Gate Bridge, he determined the forces that the piers would have to support. To perform a simplified version of his analysis we start by drawing the free-body diagram of the north tower as shown. Assume that the tension acting on the main cable as it passes over the tower is $T_1 = 253.7$ MN and $T_2 = 265.4$ MN, the tower weight is 196 MN $\alpha = 68.8°$, and $\beta = 63.0°$. Using this information find the loads (reactions) acting on the north tower at its base at B.



EX 5.5.36

5.6 ZOOMING IN ON SUBSYSTEMS

Learning Objective: Apply the equations of equilibrium to subsystems within larger systems.

A free-body diagram describes both the loads acting on a system and the geometry of the system. We use it as the basis for writing the equilibrium equations for a system that is balanced. For such a system, the conditions of equilibrium hold for the system as a whole, and *they must hold for all portions of the system*. In other words, if we zoom in on portions of a system that is in equilibrium, the conditions of equilibrium must be true for each portion.

You might well ask, "Why would I be interested in zooming in on portions of a system and checking for equilibrium?" One reason is that sometimes the loads of interest in confirming a design are internal to the system you have defined. By zooming in and defining a new, smaller system, you can have these internal loads become external loads acting on your newly defined system.

This idea is illustrated in Figure 5.6.1. When we look at the person and ladder as the system, the system external forces consist of the weights of the person and the ladder, and the forces of the wall and the ground acting on the ladder. When we separate the person from the ladder and define the ladder as the system, the system external forces consist of the weight of the ladder and the forces of the wall and the ground acting on the ladder as well as the forces of the person stepping on and holding onto the ladder. When we look at the person as the system, the only external forces are his weight and the contact forces of the ladder acting on him.

Another reason for zooming in is that sometimes when applying the conditions of equilibrium to a large system there are not enough equations to find the unknowns. If parts of a system are not rigidly attached to one another, you may be able to zoom in on portions of the system to generate additional independent equilibrium equations. We will revisit this idea in Chapters 8 and 9, where we talk about trusses, frames, and machines.

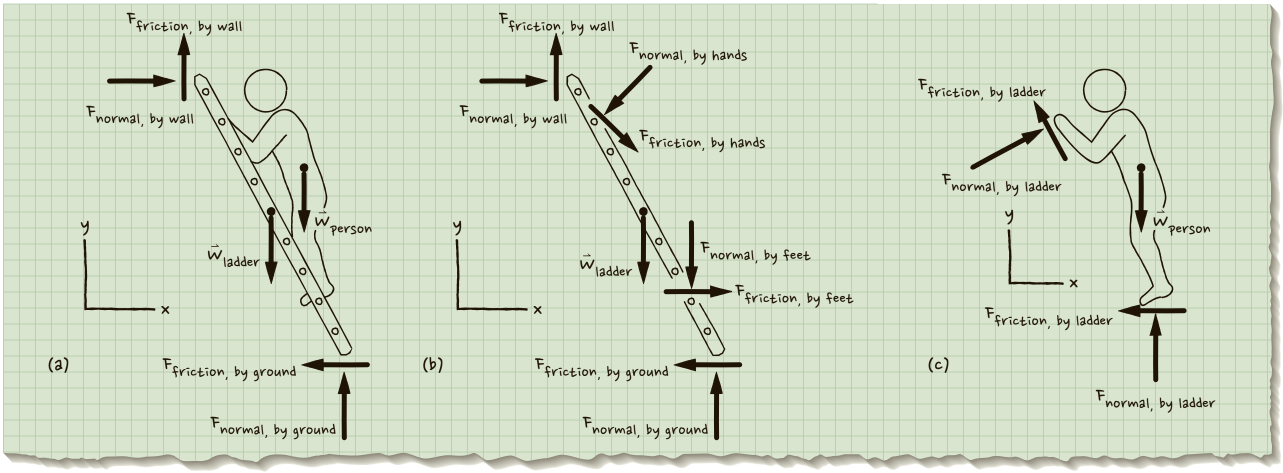


Figure 5.6.1 (a) Person–ladder free-body diagram; (b) ladder free-body diagram; (c) person free-body diagram.

Check out the following examples of applications of this material.

- **Example 5.6.1 Analysis of a Toggle Clamp**
- **Example 5.6.2 Analysis of a Pulley System**

EXAMPLE 5.6.1

A toggle clamp holds sample F (**Figure 1**). Determine the loads acting on the clamp at A in terms of the input force $\boldsymbol{P}$ applied to the handle of the toggle clamp.

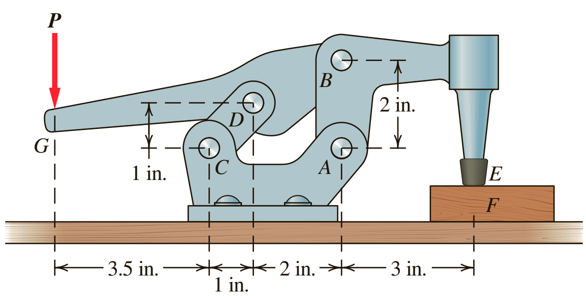


Figure 1 A toggle clamp.

Solution Strategy If we draw a free-body diagram of the whole toggle clamp (see **Figure 2**) and write the three planar equilibrium equations, we find that the three equations contain more than three unknowns (F_C, F_{Ax}, F_{Ay}, and F_E). Therefore, our overall strategy is to start with a

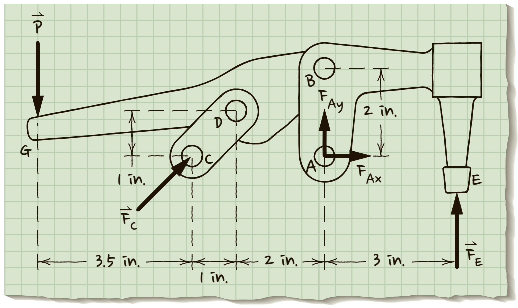


Figure 2 Free-body diagram of the entire toggle clamp.

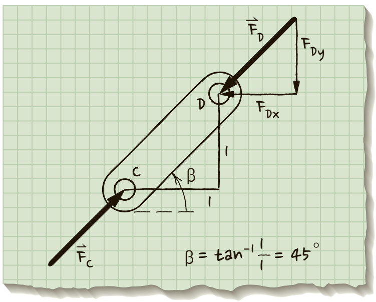


Figure 3 Free-body diagram of the two-force member *CD*, which forms an angle β with the *x* axis.

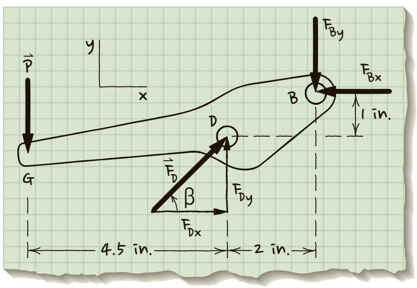


Figure 4 Free-body diagram of toggle clamp handle.

free-body of the handle and relate the input force $\boldsymbol{P}$ to the forces at pin *B*. Then we will work with a free-body diagram of member *ABE* and relate the forces at pin *B* to the forces acting at *A*. This two-step process, of zooming in first on the handle, then on member *ABE*, will allow us to relate the input force to the loads acting at *A*.

Goals Find the loads acting on the toggle clamp at *A*, expressed in terms of $\boldsymbol{P}$.

Given The configuration of the toggle clamp and the applied load on handle of toggle clamp.

Assumptions

- Model the system as *planar.*
- The system is in equilibrium.
- Ignore the weight of various members that make up the toggle clamp compared to the load.
- The pin connections *A*, *B*, *C*, and *D* are all frictionless,
- The clamping force at *E* and the input force $\boldsymbol{P}$ are purely vertical.

Free-Body Diagram #1 We draw free-body diagrams of the three members that make up the toggle clamp, beginning with member *CD* (**Figure 3**). We note that member *CD* is a two-force member. This means we know that $F_C = F_D$ and that both F_D and F_C are at an angle of β to the horizontal.

We use this information when we zoom in and isolate the handle (**Figure 4**). Because member *CD* is a two-force member, we can write

$$\boldsymbol{F}_D = \|\boldsymbol{F}_D\|\cos\beta\,\boldsymbol{i} + \|\boldsymbol{F}_D\|\sin\beta\,\boldsymbol{j}$$

where β is the angle of member *CD*, as shown in **Figure 4**.

Equilibrium Equations #1 We use the free-body diagram of *GDB* to formulate the planar equilibrium equations for the handle:

$$\text{(5.5A)} \qquad \sum F_x = 0 \;\Rightarrow\; F_{Dx} - F_{Bx} = 0$$

$$\Rightarrow \|\boldsymbol{F}_D\|\cos\beta - F_{Bx} = 0$$

$$\Rightarrow F_{Bx} = \|\boldsymbol{F}_D\|\cos\beta \qquad (1)$$

(5.5B)
$$\sum F_y = 0 \Rightarrow F_{Dy} - F_{By} - P = 0$$
$$F_{By} = F_{Dy} - P$$
$$F_{By} = \|\boldsymbol{F}_D\| \sin\beta - P \tag{2}$$

(5.5C)
$$\sum M_{z@D} = 0$$
$$(4.5\,\text{in.})(P) + (1.0\,\text{in.})(F_{Bx}) - (2.0\,\text{in.})(F_{By}) = 0 \tag{3}$$

We substitute (1) and (2) into (3) and find that

$$6.5P - \|\boldsymbol{F}_D\|(2\sin\beta - \cos\beta) = 0$$
$$\|\boldsymbol{F}_D\| = \frac{6.5}{2\sin\beta - \cos\beta}P \tag{4}$$

We substitute (4) into (1) and (2) to express the force components at B in terms of the input force $\boldsymbol{P}$:

From (1),
$$F_{Bx} = \frac{6.5\cos\beta}{2\sin\beta - \cos\beta}P \tag{1$'$}$$

From (2),
$$F_{By} = P\left(\frac{6.5\sin\beta}{2\sin\beta - \cos\beta} - 1\right) \tag{2$'$}$$

Free-Body Diagram #2 Next we zoom in and isolate member ABE and draw its free-body diagram (Figure 5). When we draw the free-body diagram of member ABE, we draw forces at B that are equal and opposite to those acting on GDB.

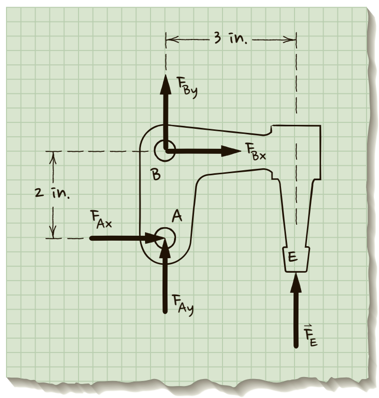


Figure 5 Free-body diagram of member ABE.

Equilibrium Equations #2 and Solve This allows us to relate $\boldsymbol{F}_B$ to $\boldsymbol{F}_A$. Based on (5.5C),

$$\sum M_{z@A} = 0 \Rightarrow (3.0\,\text{in.})(F_E) - (2.0\,\text{in.})(F_{Bx}) = 0$$
$$F_E = \frac{2}{3}F_{Bx}$$
$$F_E = \frac{13\cos\beta}{6\sin\beta - 3\cos\beta}P \tag{5}$$

Based on (5.5A),
$$\sum F_x = 0 \Rightarrow F_{Ax} + F_{Bx} = 0$$

And substituting F_{Bx} from (1$'$),
$$F_{Ax} = \frac{-6.5\cos\beta}{2\sin\beta - \cos\beta}P$$

Based on (5.5B),
$$\sum F_y = 0 \Rightarrow F_{Ay} + F_{By} + F_E = 0$$
$$F_{Ay} = -(F_{By} + F_E)$$

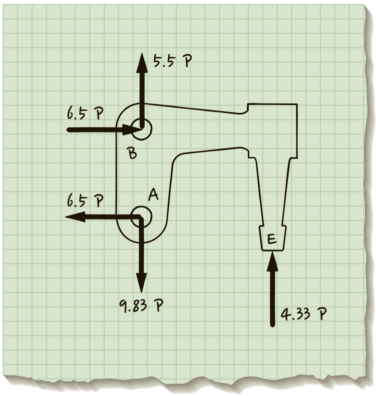


Figure 6 Free-body diagram of ABE with calculated forces.

Substituting for F_{By} and F_E from (2′) and (5),

$$F_{Ay} = -\frac{13.5 \sin\beta + 16 \cos\beta}{6 \sin\beta - 3 \cos\beta} P$$

From the dimensions of the toggle shown in **Figure 4**, we know that $\beta = 45°$. Therefore,

$$\Rightarrow F_{Ax} = -6.5P, F_{Ay} = -9.83P$$

Though not asked for in the problem statement, with a little more effort, we can find out that $F_{Bx} = 6.5P$, $F_{By} = 5.5P$, and $F_E = 4.33P$.

Checks As a check we use the free-body diagram of member ABE with the calculated forces (**Figure 6**). In this diagram we see that the downward force at A balances the upward forces at B and E, and that the x-direction forces are balanced at A and B. Summing the moments about a moment center at A will show that there is also moment balance on ABE.

Answers

$$F_{Ax} = -6.5P, F_{Ay} = -9.83P$$

The total force on the pin at A is

$$\sqrt{(-6.5P)^2 + (-9.83P)^2} = 11.8P$$

EXAMPLE 5.6.2

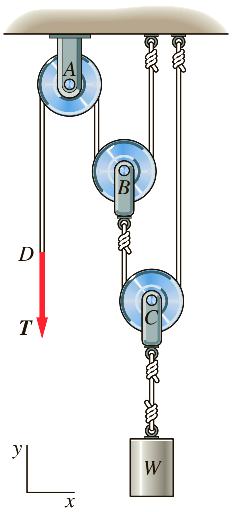


Figure 1 A cable-and-pulley system.

A cable-and-pulley system is used to support an object of weight W as shown in **Figure 1**. Each pulley is free to rotate. One cable runs continuously over pulleys A and B, and the other is continuous over pulley C. Determine the ratio W/T, where T is the force you need to apply at D to hold the object in the position shown.

Solution Strategy If we try to isolate the three pulleys as an entire system, we will find that we have to solve for too many unknowns with the available equilibrium equations. Instead, we will zoom in on each of the pulleys and draw a free-body diagram so that we have a set of free-body diagrams for each subsystem.

Goals Find the ratio W/T to hold the object in position.

Given The configuration of the pulleys.

Because the pulleys are free to rotate, we can model them as frictionless; this means that the magnitude of the tension T in the cable that goes from D and around pulleys A and B is constant. Similarly, the tension T_C in the cable that goes from B around pulley C is also constant.

Assumptions

- Model the system as *planar*.
- The system is in equilibrium (because nothing is moving).
- Ignore the weight of pulleys and cables.

Free-Body Diagrams Our plan is first to zoom in and isolate the pulley at C with the attached weight W (**Figure 2**). This will allow us to relate the tension T_C in the cable around pulley C to the weight W.

Next we zoom in and isolate the pulley at B (**Figure 3**); this will allow us to relate T_C to the tension T in the cable around pulleys A and B.

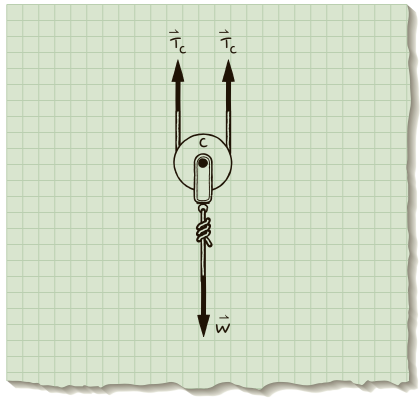


Figure 2 Free-body diagram of pulley C.

Equilibrium Equations We first use the free-body diagram of pulley C to formulate the equilibrium equations and solve for T_C in terms of W.

(5.5B) $$\sum F_y = 0 \Rightarrow T_C + T_C - W = 0$$

$$T_C = \frac{W}{2} \tag{1}$$

Now we formulate the equilibrium equations for the planar system associated with the free-body diagram of pulley B:

(5.5B) $$\sum F_y = 0 \Rightarrow -T_C + T + T = 0$$

$$T_C = 2T \tag{2}$$

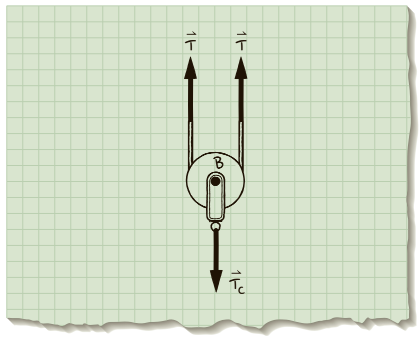


Figure 3 Free-body diagram of pulley B.

Because we are able to find T_C in terms of T by using just equilibrium in the y direction (5.5B), there is no need to write out the other two planar equilibrium equations (5.5A) and (5.5C).

Solve By equating (1) and (2), we find that

$$\frac{W}{T} = 4$$

This indicates that an object weighing W requires a force T that is only ¼ of W for the system to be in equilibrium. An alternative interpretation is that with a force of T you can hold up an object that weighs $4T$.

Checks Consider the pulley subsystem defined in **Figure 4**, which includes calculated values of T (= W/4) and T_C (= W/2). If we sum the forces in the y direction, we find that the three upward cable forces in the figure just balance the downward force of the weight W. So, it is in equilibrium.

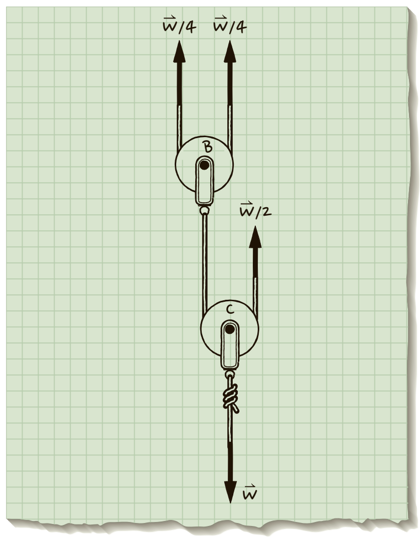


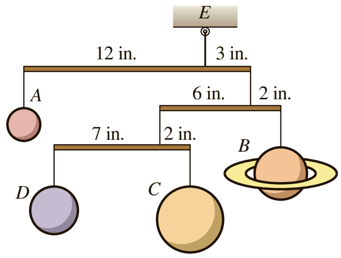
Figure 4 Free-body diagram of pulley subsystem with calculated forces.

Answers

$$\frac{W}{T} = 4$$

EXERCISES 5.6

5.6.1. **[*]** A mobile hangs from the ceiling at E from a cord. If the mobile is in equilibrium, what loads act on the mobile at E if the weight of "planet" B is 6 oz?

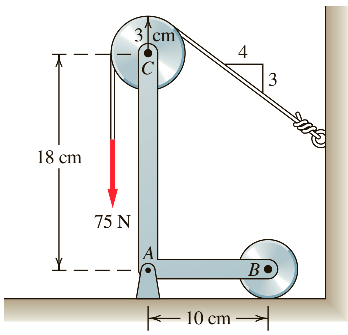


EX 5.6.1

5.6.2. **[*]** For the 75-N load supported by the L-bracket, frictionless pulley, and cable shown determine

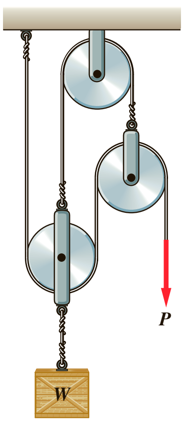
a. the loads acting on the L-bracket at supports A and B.

b. the force exerted on the bracket by the pin connection at C.



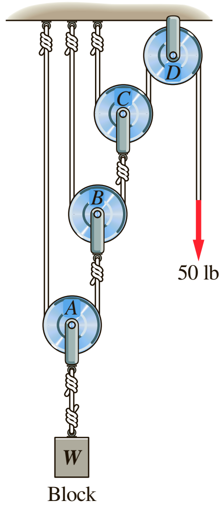
EX 5.6.2

5.6.3. **[*]** The pulley system lifting a block of weight W is in equilibrium. Determine the ratio W/P for the pulley system.



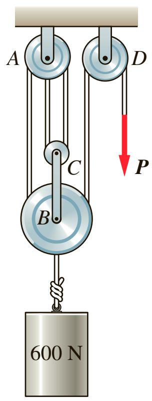
EX 5.6.3

5.6.4. **[*]** A block is supported by a system of pulleys, as shown. Knowing the system is in equilibrium and assuming the pulleys are frictionless and massless, determine the weight of the block.



EX 5.6.4

5.6.5. **[*]** A pulley system consisting of a rope and pulleys A, B, C, and D is being used to lift a 600-N load as shown. Find the force $\boldsymbol{P}$ in the rope.



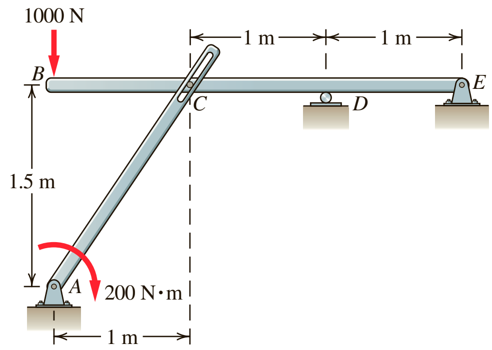
EX 5.6.5

5.6.6. **[*]** Consider the two-member frame shown. Member AC is connected to member $BCDE$ by a slider connection at C. Determine

a. the loads acting on the frame at A, D, and E.

b. the loads of member $BCDE$ acting on member AC.

c. the loads of member AC acting on member $BCDE$.



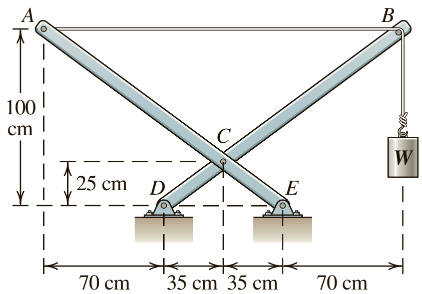
EX 5.6.6

5.6.7. [*] Consider the two-member frame. Member AE is connected at C to member BD by a pin connection. The cylinder weighs 200 N. Determine

a. the loads acting on the frame at D and E.

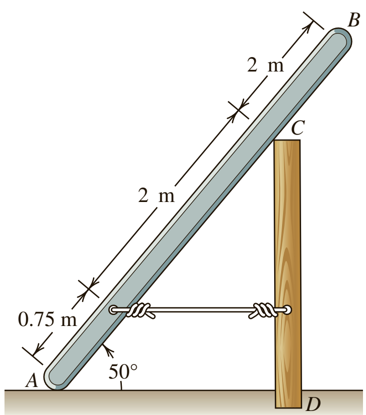
b. the loads of member AE acting on member BD at C.

c. the loads of member BD acting on member AE at C.



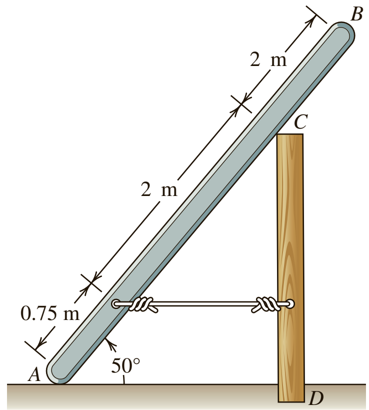
EX 5.6.7

5.6.8. [*] A bar AB weighing 1000 N is supported by a post (CD) and a cable as shown. The post weighs 250 N. Assume that all surfaces are smooth. Determine the tension in the cable and the forces acting on the bar at the contacting surfaces if the bar is in equilibrium.



EX 5.6.8

5.6.9. [*] A bar AB weighing 1000 N is supported by a post (CD) embedded in the ground and a cable as shown. The post weighs 250 N. Assume that all surfaces are smooth. Determine the loads acting on the post at D.



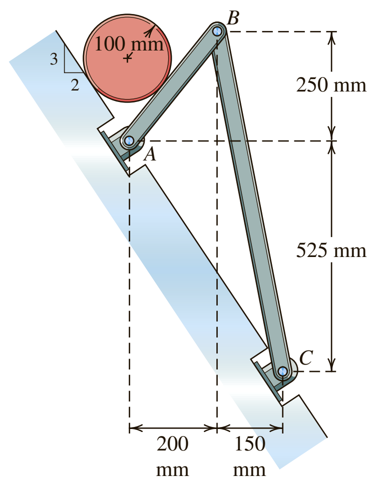
EX 5.6.9

5.6.10. [*] A two-member frame supports a 90-kg cylinder, as shown. Assume that the weight of the frame is negligible and all surfaces are smooth.

a. Determine the forces exerted on the cylinder by the contacting surfaces.

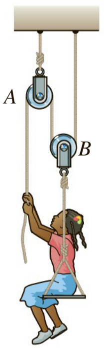
b. Determine the loads acting on the frame at supports A and C.

c. Comment on whether member AB and/or BC is a two-force member. Comment on whether the cylinder is a three-force member.



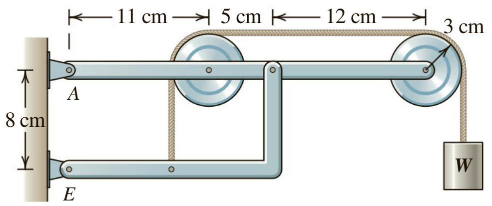
EX 5.6.10

5.6.11. [**] A child with a mass of 40 kg supports herself with the pulley-rope system shown. Assuming that the weight of the seat-bottom is negligible, how much force must she exert on the rope to be in equilibrium? Would this answer increase, decrease, or remain the same if the weight of the seat-bottom is not negligible?



EX 5.6.11

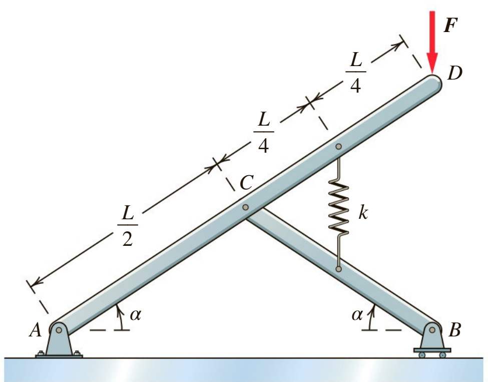
5.6.12. [**] Determine the loads acting on the frame at A and at E. W is 400 N.



EX 5.6.12

5.6.13. [**] Consider the frame shown. The force at D is vertical and of magnitude F. Determine

a. the loads acting on the frame at A and B.
b. the loads of member ACD acting on member CB.
c. the loads of member CD acting on member ACD.
d. the tension or compression in the spring.

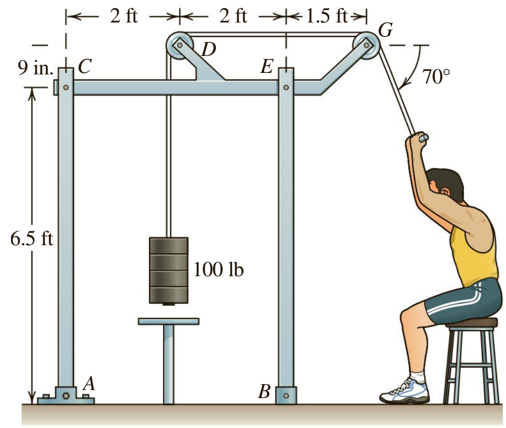


EX 5.6.13

5.6.14. [**] Consider the exercise frame in equilibrium. Determine

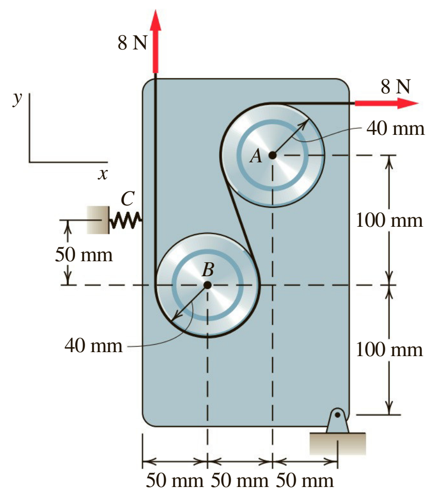
a. the loads acting on the frame at A and B if the frame is bolted to the floor at A.

b. the loads acting on the member CEG at C, D, E, and G.



EX 5.6.14

5.6.15. [**] The tape passes around frictionless pulleys A and B. If the system is oriented in a horizontal plane, determine the force exerted by the spring at point C. If it is desirable that the system not deflect more than 0.5 mm at point C, what is the minimum spring stiffness you would specify for the spring at C? Would the required spring stiffness decrease, increase, or remain the same if the tape guide was oriented in a vertical plane (with gravity acting in the $-y$ direction)?



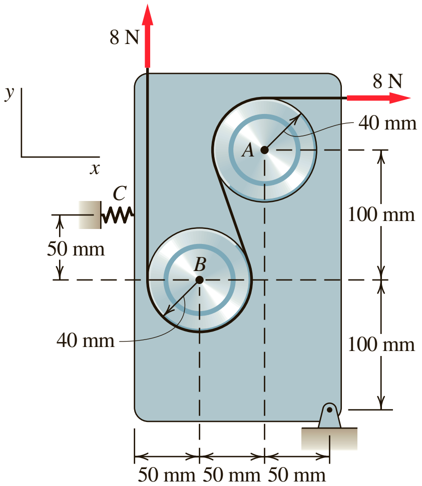
EX 5.6.15

5.6.16. [*] For the tape unit shown,

a. determine the loads at A that the tape unit applies to pulley A.

b. determine the loads at B that the tape unit applies to pulley B.

c. Pulleys A and B are attached to the tape unit by pin connections. Based on your results from a and b, would you expect the pin at A or B to fail first? Why?

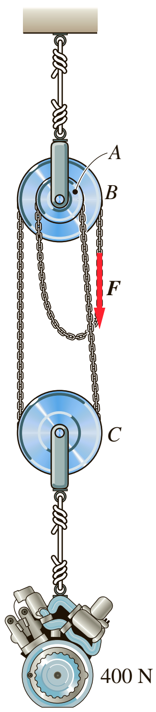


EX 5.6.16

5.6.17. [**] The chain hoist shown is used to hold an engine block weighing 400 N in equilibrium. The continuous chain is prevented from slipping by teeth around the circumference of each pulley. Pulleys A and B are connected and rotate as a unit, with radii of 7 cm and 9 cm, respectively.

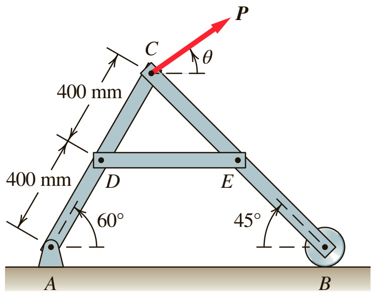
a. Determine the magnitude of the force $\boldsymbol{F}$ required to hold the engine.

b. What is the mechanical advantage of the chain hoist?



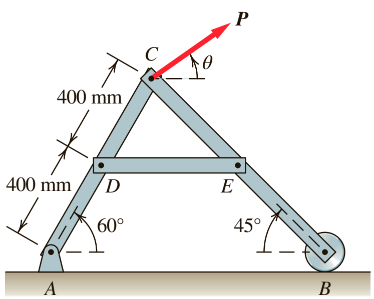
EX 5.6.17

5.6.18. [**] A pin-connected three-bar frame is loaded and supported as shown. The magnitude of the load $\boldsymbol{P}$ is 500 N. If member DE will fail at a tension force greater than 300 N and a compression force greater than 200 N, what is the range(s) of acceptable values of θ that can be carried by the frame?



EX 5.6.18

5.6.19. [**] A pin-connected three-bar frame is loaded and supported as shown, with $\theta = 40°$. If member DE will fail at a tension force greater than 300 N and a compression force greater than 200 N, what is the largest acceptable value of force $\boldsymbol{P}$ that can be safely carried by the frame?

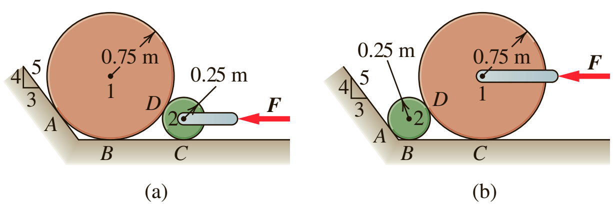


EX 5.6.19

5.6.20. [***] Disks 1 and 2 in (a) have weights of 200 N and 100 N, respectively.

a. Determine the maximum horizontal force $\boldsymbol{F}$ that can be applied to Disk 2 without causing Disk 1 to move up the incline. (Hint: What is the value of the normal force at B at the instant $\boldsymbol{F}$ gets large enough to cause Disk 1 to move up the incline?) Confirm that the line of action of the net force acting on Disk 2 goes through the center of Disk 1.

b. When the positions of the disks are reversed, as shown in (b), is the maximum horizontal force that can be applied to Disk 1 without causing Disk 1 or Disk 2 to move up the incline greater than, the same as, or less than what it was in (a)? (First use your intuition to reason through an answer, then confirm with calculations, and see (b).)



EX 5.6.20

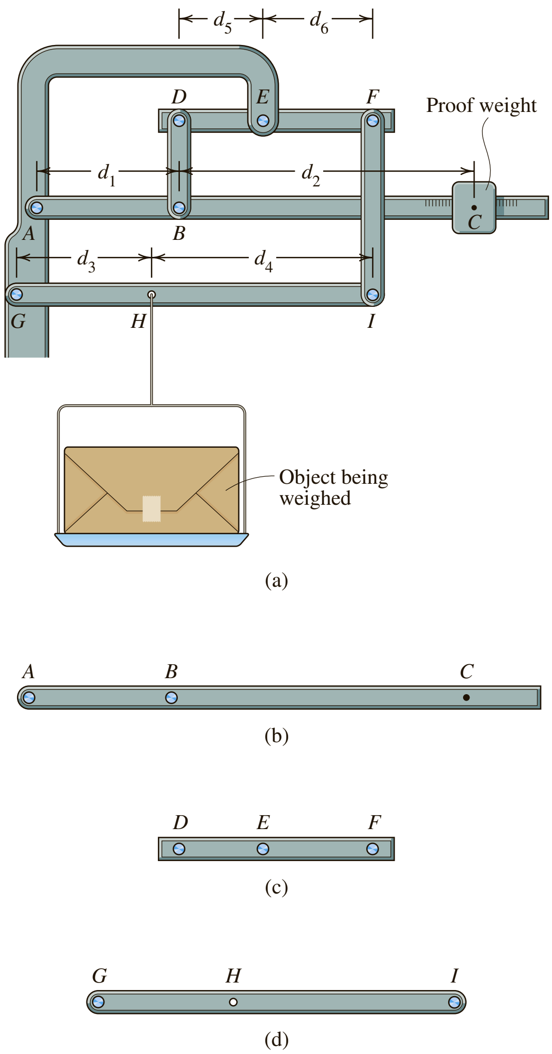
5.6.21. **[***]** A platform scale weighs objects using a combination of levers, as shown in (a). With the object to be weighed hanging from the scale at H, the proof weight C is slid along member ABC until the member is horizontal. The weight is then read off a calibrated scale on member ABC. The weight of the proof is 1 lb. Consider that the scale is in equilibrium for a particular object.

a. Define the system as shown in (b) and draw its free-body diagram.

b. Define the system as shown in (c) and draw its free-body diagram. (Be consistent with the free-body diagram in part **a**.)

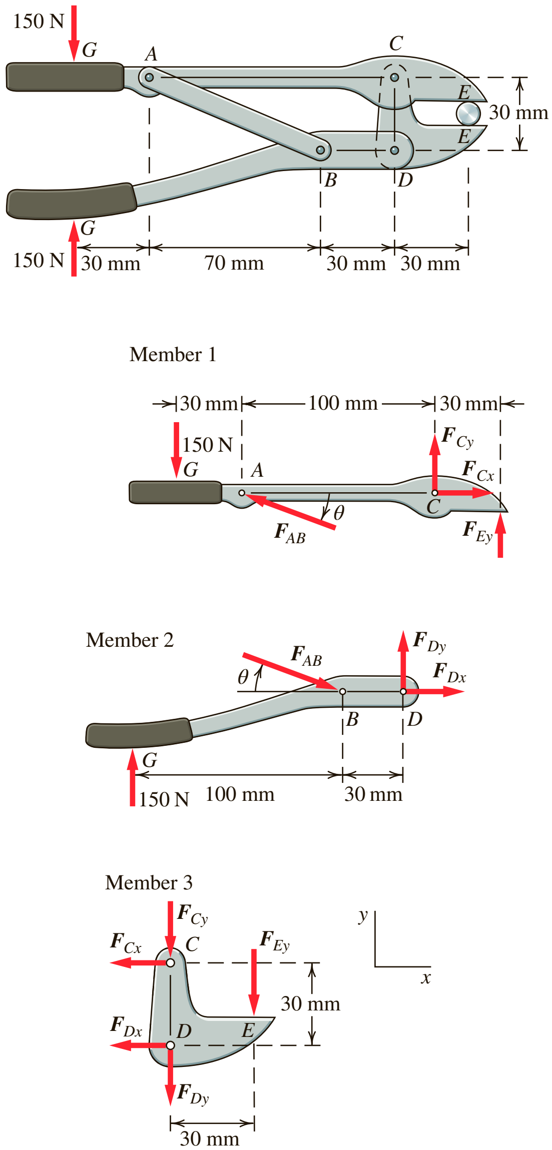
c. Define the system as shown in (d) and draw its free-body diagram. (Be consistent with the free-body diagrams in parts **a**. and **b**.)

d. If $d_1 = 1$ in., $d_2 = 19$ in., $d_3 = 5$ in., $d_4 = 20$ in., $d_5 = 16$ in., and $d_6 = 2$ in., what is the weight of the object?



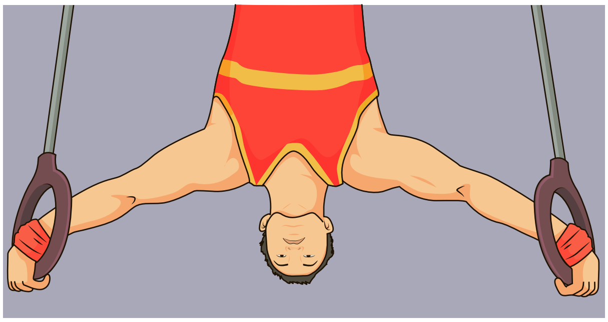
EX 5.6.21

5.6.22. **[***]** Consider the pair of pliers shown. Determine the ratio of $\|\boldsymbol{F}_E\|/\|\boldsymbol{F}_G\|$ where the magnitude of $\boldsymbol{F}_G$ is 150 N. (*Hint:* First consider equilibrium of Member 1, then of Member 2, then of Member 3.)



EX 5.6.22

5.6.23. **[***]** The figure shows a gymnast executing an inverted iron cross. This maneuver is an excellent example of an athlete in mechanical equilibrium who is subjecting his shoulders to extraordinary forces and moments. Calculating these loads will allow us to minimize injuries and develop training regimens that better prepare elite athletes for international competition. The gymnast's arms make an angle of approximately 27° with the horizontal, while the ropes make an angle of 82° with the horizontal.



EX 5.6.23

a. Draw a free-body diagram of the gymnast's arm and determine the loads acting at his shoulder. (*Hint:* In order to calculate the forces and moments at the gymnast's shoulder, we need to draw a free-body diagram of one of his arms.) If we take a section through the shoulder joint we have to replace the torso with the forces and moments that it exerts on the arm. Similarly, we have to replace the ring with the forces it exerts on the hand (it is reasonable to assume that the rings exert negligible moments on the gymnast's hands). A quick inspection of the free-body diagram should show you that you have more unknowns than you have equations. There are a few ways to calculate the force in the rope, but the most straightforward technique is to draw a free-body diagram of the whole gymnast given that he weighs approximately 154 lb.

b. The magnitude of a moment is a difficult quantity to understand intuitively. Consider the moment exerted on the gymnast's shoulder, as found in **a**. How much force would you have to apply to a standard crescent wrench to generate the same moment? State all assumptions.

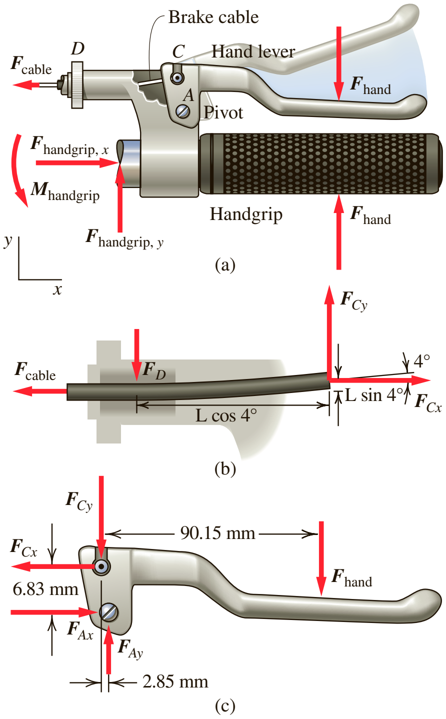
5.6.24. [***] Consider the bicycle brake hand lever shown in (a). The givens/assumptions are that all forces are coplanar and 2-dimensional, static analysis is acceptable, and the two forces labeled F_{hand} have the same line of action.

a. Based on the free-body diagram of the cable housing subassembly shown in (b) of the brake cable where it moves through the housing at D, write expressions for F_D, F_{Cy}, and F_{Cx} in terms of F_{cable}. Force F_D is the force that the metal housing of the brake applies to the cable housing.

b. Based on the free-body diagram of the brake lever arm in (c), write an expression for F_{hand} in terms of F_{cable}. (Use the results from **a**, as necessary.)

c. If F_{hand} = 150 N, what is the value of F_{cable}? (Use results as appropriate from **b**.)

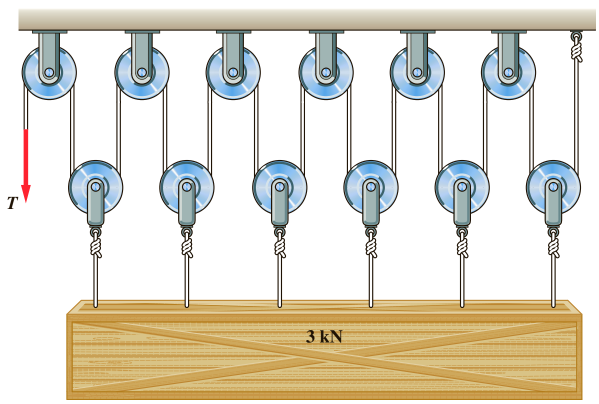
d. What is the mechanical advantage of the system (i.e., the ratio of F_{cable}/F_{hand})?



EX 5.6.24

5.6.25. [***] Leonardo Da Vinci developed a machine to lift heavy objects consisting of a series of pulleys similar to the one shown here. Assuming all of the pulleys are frictionless and the rope is vertical between the upper and lower pulleys, what is the tension $\boldsymbol{T}$ pulling on the rope to hold the 3-kN crate in place?

Find $\boldsymbol{T}$ if the locations of the pulleys are modified so that the rope segments are all at 30° from vertical.



EX 5.6.25

5.7 DETERMINATE, INDETERMINATE, AND UNDERCONSTRAINED SYSTEMS

Learning Objective: Define and identify statically determinate, statically indeterminate, and underconstrained systems.

In the examples we have considered up to now, there were as many unknown loads as there were independent equilibrium equations. We refer to these problems as **statically determinate** because we were able to determine the unknowns using only the equations of equilibrium.

As a general guideline, a nonplanar system with six unknown loads acting on it is a candidate for being a statically determinate system, as is a planar system with three unknowns. But this guideline is not hard-and-fast as we will see when we study trusses in Chapter 8 and frames in Chapter 9.

We now look at two types of systems in which the equations of equilibrium are not sufficient for finding the unknowns. These system types are referred to as **statically indeterminate systems** and **underconstrained systems**.

Statically indeterminate systems are systems in which the number of unknown internal or external loads exceeds the number of equilibrium equations. Here we explore the case when there are more supports than are necessary for the system to be in equilibrium. In other words, the system has an excess, or redundancy, of supports. In the chapters on trusses, frames and machines we explore the case of excess internal unknowns acting on subsystems such as truss members. Examples of systems having a redundancy of supports are shown in **Figure 5.7.1*a***. We could remove a support in each example and create a determinate system that is still balanced under the loads, as is done in **Figure 5.7.1*b***. The systems in **Figure 5.7.1*b*** still represent systems in equilibrium. The

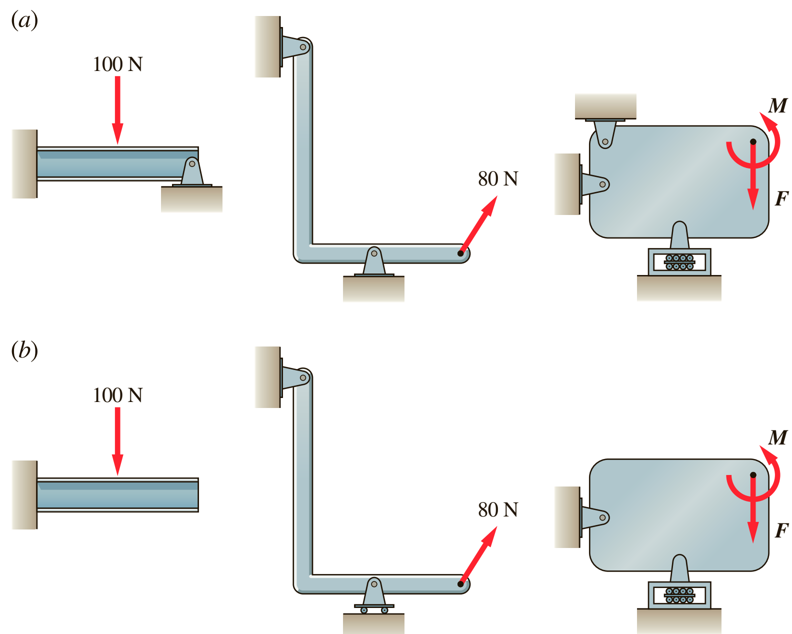


Figure 5.7.1 (*a*) Examples of systems that are statically indeterminate; (*b*) systems from (*a*) modified so as to be statically determinate.

supports that can be removed and still have a stable system are referred to as **redundant supports**. It is important to note that free-body diagrams of the indeterminate systems in **Figure 5.7.1*a*** are different from those of the determinate systems in **Figure 5.7.1*b***.

While it is possible to reduce the number of unknown loads acting on any statically indeterminate system by using the conditions of equilibrium, it is not possible to uniquely determine all the loads acting on the system. The additional equation(s) needed for determining redundant support load(s) come from a description of how the system *deforms* under the loads. This deformation is generally very small for engineered systems, so it does not violate the rigid-body assumption we have been making throughout this text.

Underconstrained systems are such that the supports and/or members are insufficient to keep the system balanced and stable. The conditions of equilibrium, as stated in (5.1) and (5.2), do not hold for an underconstrained system.

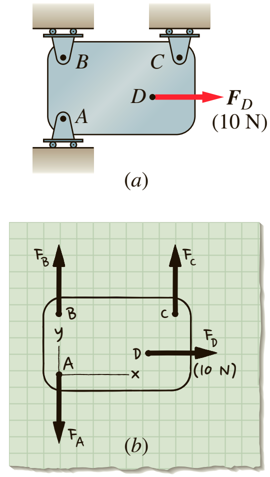


Figure 5.7.2 (*a*) An example of an underconstrained system; (*b*) its associated free-body diagram.

Figure 5.7.2 shows an underconstrained planar system, where there are three unknowns and it would appear that there are three equations with which to find the unknowns. However, one of the equations of equilibrium is an unachievable because there is no way for the net force to be zero with a nonzero applied load acting at D as we now illustrate. Assuming the system is planar, we write the first planar equilibrium equation as

$$\Sigma F_x = F_D = 0$$

This equation is cannot be satisfied if F_D is nonzero. Since F_D is 10 N in **Figure 5.7.2**, a net force acts on the system in the x direction causing the system to move and accelerate rightward. Remember, an underconstrained system will move.

Several additional examples of underconstrained systems are presented in Figure 5.7.3. Because $\boldsymbol{F}_C$ in **Figure 5.7.3*a*** has a horizontal component, the beam is underconstrained and moves to the right. Another way to check this is by noting that the beam is a three-force member but the lines of action of the three forces acting on it ($\boldsymbol{F}_A$, $\boldsymbol{F}_C$, $\boldsymbol{F}_B$) do not intersect at a common point. In **Figure 5.7.3*b***, it might appear that, because the supports at A and B have been angled relative to their position in (*a*), the system is properly constrained. The lines of action of the three forces still do not intersect at a common point, however. As another example of an underconstrained system, consider the plate in **Figure 5.7.3*c***. Because the line of action of the link B intersects point A (where the other two links are attached), the force at B cannot counter the moment created by F_D about a moment center at A. The system is again underconstrained.

Figuring out whether or not a system is statically determinate is done in the early stages of setting up a problem. As you gain experience in solving equilibrium problems, you will find that you can also do some checking for determinacy as you go about setting up the free-body diagram. You may also identify the problem as not being statically determinate as you solve the conditions of equilibrium. Regardless of when in the analysis process you notice that the system is not statically determinate, make sure to document your findings.

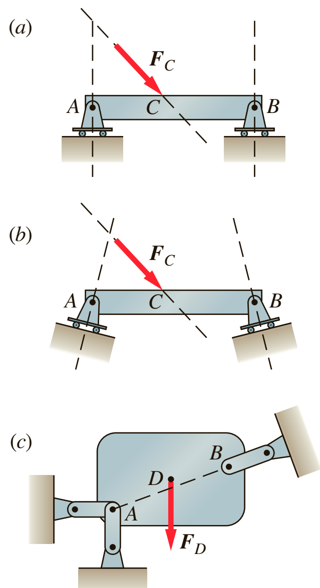


Figure 5.7.3 Examples of underconstrained systems.

Check out the following example of an application of this material.

- **Example 5.7.1 Identify Status of a Structure**

EXAMPLE 5.7.1

Identify each structure in **Figures 1*a*–3*a*** as statically determinate, statically indeterminate, or underconstrained. Associated free-body diagrams are shown in **Figures 1*b*–3*b***.

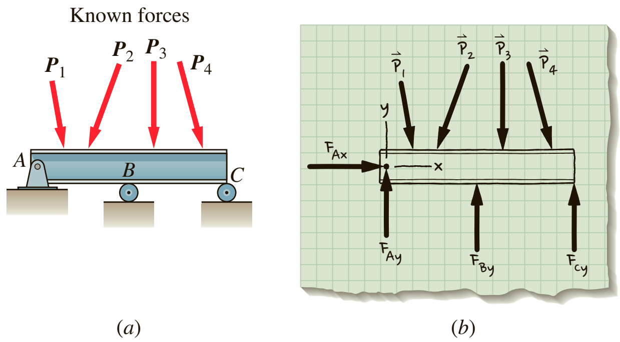


Figure 1 A beam.

(a) The beam in **Figure 1** is *statically indeterminate* because there are four unknown loads (at the supports), but there are only three equations for planar equilibrium. The number of equilibrium equations is insufficient for finding the unknown loads. Removing F_{Ay}, or F_{By} or F_{Cy} would make the system statically determinate. Removing F_{Ax} would cause the system to be underconstrained.

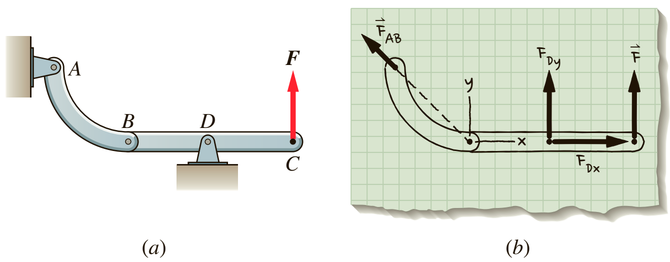
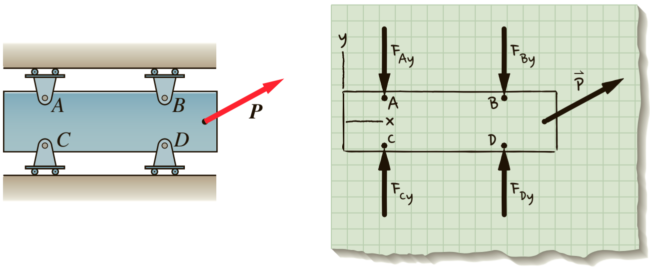


Figure 2 A system composed of two members.

(b) System $ABCD$ in **Figure 2** is *statically determinate* because there are three unknown forces (one at A and two at D)—remember that because member AB is a two-force member, there is only one unknown force at A. By using the equations for planar equilibrium we can find the forces. Note that if we had not recognized that member AB is a two-force member, there would be four unknown forces acting on the system's boundary; an additional linearly independent equilibrium equation could be generated by looking within the system.

(c) The system in **Figure 3**, which may represent a crane rail, is *undercon-strained.* A force applied as shown will start the system moving to the right.

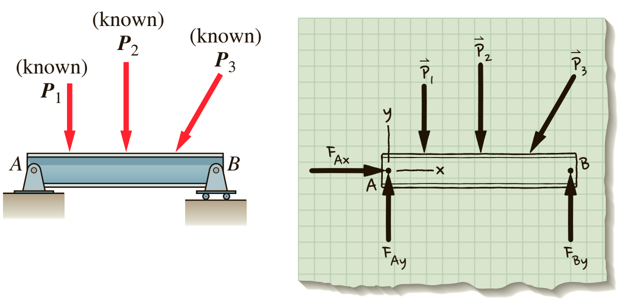


(*a*) (*b*)

Figure 3 An undercontrained beam.

EXERCISES 5.7

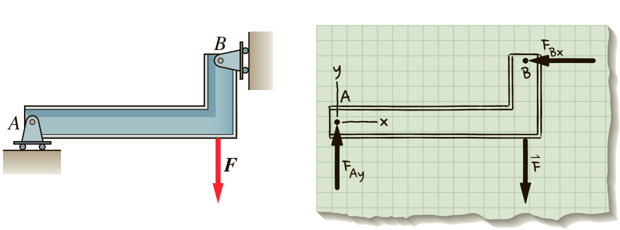
5.7.1. [*] Identify whether the structure in (*a*) is statically determinate, statically indeterminate, or underconstrained. The associated free-body diagrams is shown in (*b*).



(*a*) (*b*)

EX 5.7.1

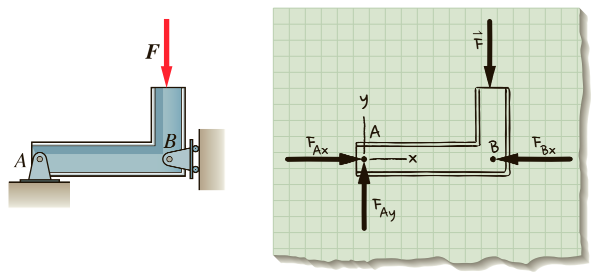
5.7.2. [*] Identify whether the structure in (*a*) is statically determinate, statically indeterminate, or underconstrained. The associated free-body diagrams is shown in (*b*).



(*a*) (*b*)

EX 5.7.2

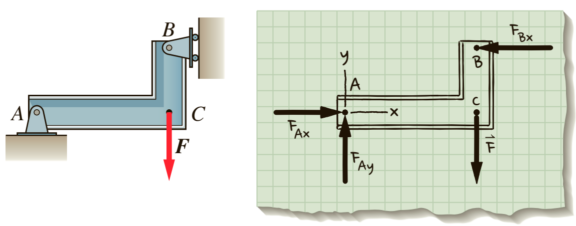
5.7.3. [*] Identify whether the structure in (*a*) is statically determinate, statically indeterminate, or underconstrained. The associated free-body diagrams is shown in (*b*).



(*a*) (*b*)

EX 5.7.3

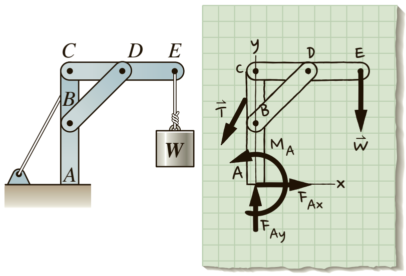
5.7.4. [*] Identify whether the structure in (*a*) is statically determinate, statically indeterminate, or underconstrained. The associated free-body diagrams is shown in (*b*).



(*a*) (*b*)

EX 5.7.4

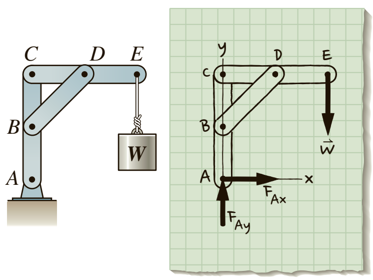
5.7.5. [*] Identify whether the structure in (*a*) is statically determinate, statically indeterminate, or underconstrained. The associated free-body diagrams is shown in (*b*).



(*a*) (*b*)

EX 5.7.5

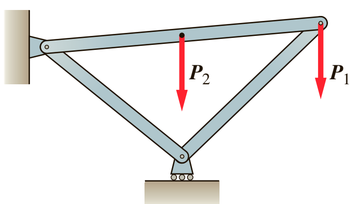
5.7.6. [*] Identify whether the structure in (*a*) is statically determinate, statically indeterminate, or underconstrained. The associated free-body diagrams is shown in (*b*).



(*a*) (*b*)

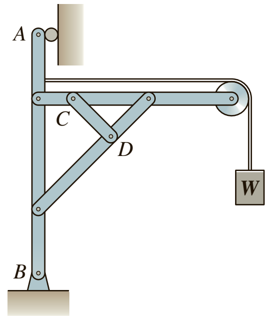
EX 5.7.6

5.7.7. [*] Identify whether the structure shown is statically determinate, statically indeterminate, or underconstrained. Isolate the structure from the supports and draw a free-body to help you determine the answer.



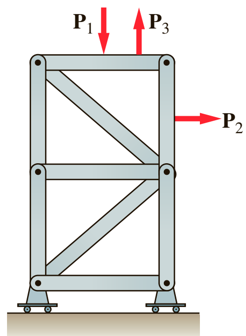
EX 5.7.7

5.7.8. [*] Identify whether the structure shown is statically determinate, statically indeterminate, or underconstrained. Isolate the structure from the supports and draw a free-body diagram to help you determine the answer.



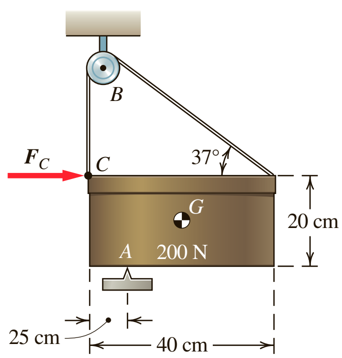
EX 5.7.8

5.7.9. [*] Identify whether the structure shown is statically determinate, statically indeterminate, or underconstrained. Isolate the structure from the supports and draw a free-body diagram to help you determine the answer.



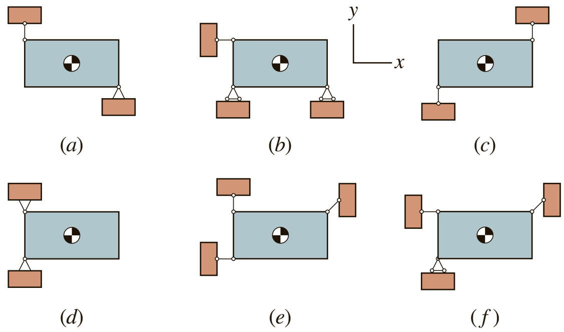
EX 5.7.9

5.7.10. [*] The box shown is supported by a knife edge at point A and by a cord that passes over the frictionless pulley, B. The weight of the box is 200 N, and its center of mass is at the geometric center, G. Identify whether the system, defined as the box, is statically determinate, statically indeterminate, or underconstrained. Isolate the box from the supports and draw a free-body diagram to help you determine the answer.



EX 5.7.10

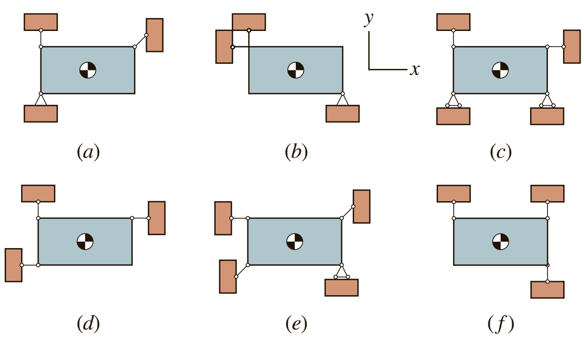
5.7.11. [*] A uniform rectangular plate weighs W, with its center of gravity as shown. Gravity acts in the $-y$ direction. For each situation classify the plate (taken as the system) as determinate, indeterminate, or underconstrained.



EX 5.7.11

Situation	Statically Determinate	Statically Indeterminate	Underconstrained
A			
B			
C			
D			
E			
F			

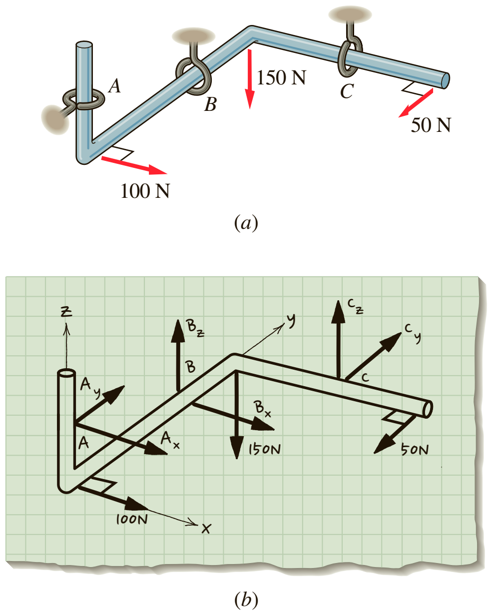
5.7.12. [*] A uniform rectangular plate weighs W, with its center of gravity as shown. Gravity acts in the $-y$ direction. For each situation classify the plate (taken as the system) as determinate, indeterminate, or underconstrained.



EX 5.7.12

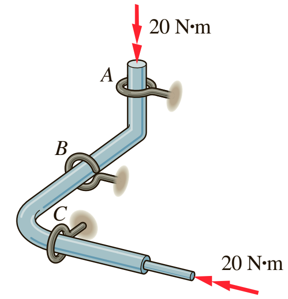
Situation	Statically Determinate	Statically Indeterminate	Underconstrained
A			
B			
C			
D			
E			
F			

5.7.13. [*] The bent bar is supported by three smooth eyebolts as shown in (*a*). Identify whether the system, defined as the bar, is statically determinate, statically indeterminate, or underconstrained. The associated free-body diagrams is shown in (*b*).

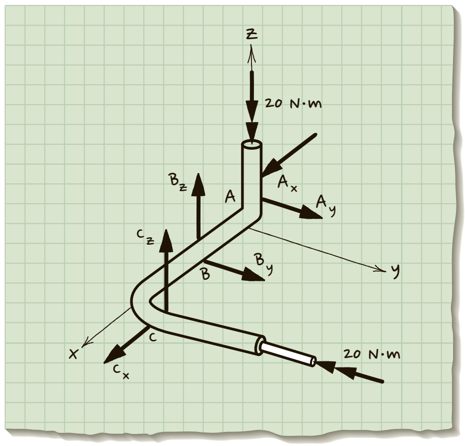


EX 5.7.13

5.7.14. [*] A rigid bent tube guides a flexible shaft that transmits a 20 N·m torque, as shown in (*a*). The tube is supported by the smooth eyebolts at A, B, and C. Identify whether the system, defined as the tube, is statically determinate, statically indeterminate, or underconstrained. The associated free-body diagram is shown in (*b*).



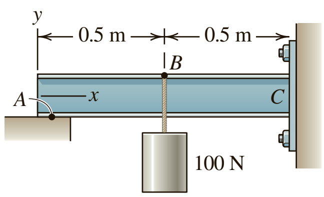
(*a*)



(*b*)

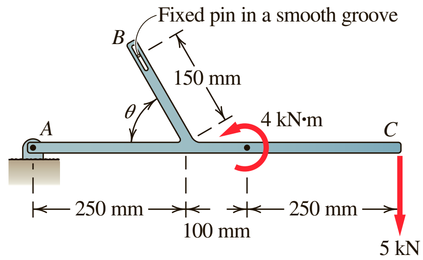
EX 5.7.14

5.7.15. [*] The beam of uniform weight is fixed at C and rests against a smooth block at A. In addition, a 100 N weight hangs from point B. The beam is statically indeterminate. Identify how the boundary connections could be changed to make the system determinate. There is more than one correct answer.



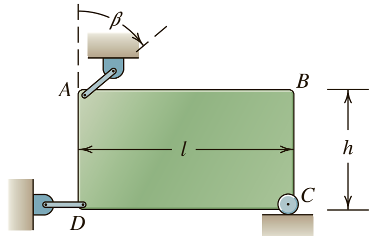
EX 5.7.15

5.7.16. [**] A 5 kN force and a 4 kN·m moment are applied to the massless forked bar as shown. The pin at B applies a force perpendicular to the groove. Formulate an equation to determine the loads acting on the bar at A and B as a function of the angle θ. Using this equation determine any values of θ for which the bar is underconstrained.



EX 5.7.16

5.7.17. [***] The plate of weight W shown is supported by links at A and D and a roller at C. For $l = 2h$, determine the value of the angle $\beta(-180° \leq \beta \leq 180°)$ for the short link at A that results in the plate in being an underconstrained system.



EX 5.7.17

5.8 JUST THE FACTS

This chapter introduces the concepts and equations of mechanical equilibrium and applies them to analyzing planar and nonplanar static systems. It covers the second half of a structured engineering analysis procedure: writing the **equilibrium equations, solving** equations carrying out **checks**, and summarizing **answers** (Table 5.1). This coverage assumes that a free-body diagram of a system has been drawn.

Conditions of Mechanical Equilibrium

When a system is in **mechanical equilibrium** (where equilibrium is a state of balance), the loads drawn on the free-body diagram have particular relationships to one another. These relationships can be represented mathematically as follows:

$$\boldsymbol{F}_{\text{net}} = \sum^{\substack{\text{All forces}\\ \text{acting}\\ \text{on system}}} \boldsymbol{F} = 0 \quad \textbf{force equilibrium condition} \tag{5.1}$$

$$\boldsymbol{M}_{\text{net}} = \sum^{\substack{\text{All moments}\\ \text{acting}\\ \text{on system}}} \boldsymbol{M} = 0 \quad \textbf{moment equilibrium condition} \tag{5.2}$$

Equations (5.1) and (5.2) are necessary and sufficient conditions for a system to be in mechanical equilibrium. These two conditions say that the vector sum of the forces (5.1) and moments (5.2) acting on the system must add to zero.

The Equilibrium Equations

It is often more convenient to work with conditions (5.1) and (5.2) in terms of vector components. We can rewrite (5.1) as the force equilibrium equations (which ensure **force balance**) and (5.2) as the moment equilibrium equations (which ensure **moment balance**):

$$\sum F_x = 0 \tag{5.3A}$$

$$\sum F_y = 0 \tag{5.3B}$$

$$\sum F_z = 0 \tag{5.3C}$$

$$\sum M_x = 0 \tag{5.4A}$$

$$\sum M_y = 0 \tag{5.4B}$$

$$\sum M_z = 0 \tag{5.4C}$$

The **force equilibrium equations** (5.3) and the **moment equilibrium equations** (5.4) are true for any system in mechanical equilibrium (i.e., a balanced system).

Any system that is in equilibrium must satisfy all six equilibrium equations. Three of these equations, however, are automatically satisfied for a planar system. The three **planar equilibrium equations** are

$$\sum F_x = 0 \tag{5.5A}$$

$$\sum F_y = 0 \tag{5.5B}$$

$$\sum M_z = 0 \tag{5.5C}$$

The first equation (5.5A) says that we add up all the x force components and they must sum to zero if the system is balanced. Similar interpretations follow for (5.5B) and (5.5C). Application of these equations allows us to solve for the loads acting on systems that are in equilibrium.

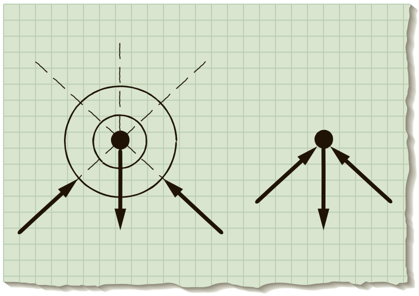


Figure 5.8.1 A particle.

Equilibrium Applied to Four Special Cases

In applying the equations of equilibrium to a system, it is useful to identify whether the system can be represented as a particle or involves two-force or three-force members or frictionless pulleys.

A **particle** is an object whose size and shape have negligible effect on the response of the object to forces. Under these circumstances, the mass (if significant) of the object can be assumed to be concentrated at a point. A particle, by definition, can only be subjected to concurrent forces; the point of concurrency is the point that represents the particle (Figure 5.8.1).

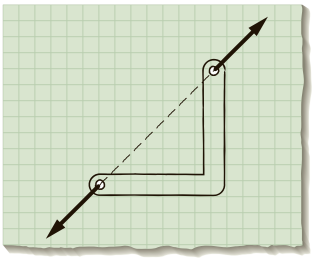


Figure 5.8.2 A two-force member.

A **two-force member** is an element of negligible weight with only two forces acting on it. A two-force member is in equilibrium when the forces acting on it are equal, opposite, and along the same line of action. The line of action passes through the points of application of the two forces (Figure 5.8.2).

A **three-force member** is an element with only three forces acting on it. If a three-force member is in equilibrium, the lines of action of the three forces are concurrent (Figure 5.8.3).

A **frictionless pulley** is an element that is used to change the direction of a cable or rope. The tension in the cable or rope is the same on both sides of a frictionless pulley (Figure 5.8.4).

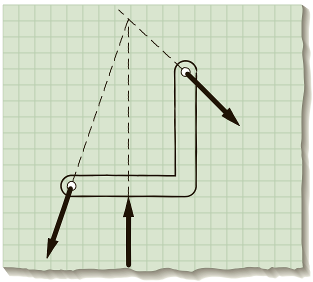


Figure 5.8.3 A three-force member.

Zooming in on Subsystems

For any system that is in mechanical equilibrium, the conditions of equilibrium hold for not only the entire system, but for all of its **subsystems**. The property of equilibrium is useful as we analyze a system and its parts by drawing a series of consistent free-body diagrams and writing (and solving) the associated equilibrium equations.

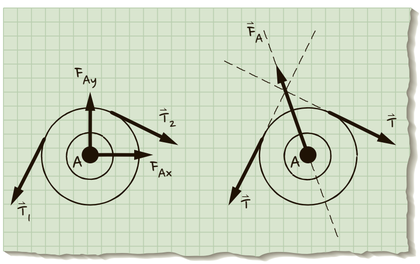


Figure 5.8.4 Frictionless pulley.

Determinate, Indeterminate, and Underconstrained Systems

A **statically determinate system** is one that can be analyzed using only the equations of equilibrium. A **statically indeterminate system** is one in which there are more unknown internal or external loads than can be determined with the equilibrium equations. A typical example is a system with more supports than required for equilibrium. An **underconstrained system** is one in which the supports and/or members are insufficient to keep the system balanced and stable.

SYSTEM ANALYSIS (SA) EXERCISES

SA5.1 Bracing Against Moving Loads

During the construction of the Reynolds Coliseum in 1948 the contractor realized that he could speed up the placing of concrete for the floor slab by creating temporary trestles consisting of 40-cm-wide wooden planks able to support the 20-ton trucks carrying the "flowable" concrete to the place where it is needed. At that time this was an ingenious approach since concrete pump trucks that can be seen today deploying a concrete pipe with a foldable boom were not invented until the 1960s, as illustrated in **Figure SA5.1.1**.

As shown in **Figure SA5.1.2** the concrete floor of the Reynolds Coliseum was supported by joists, I-beams, and columns. Assume that you were working for the contractor at that time and were asked to figure out what the maximum *additional* load would be on the columns due to the concrete trucks. You are also to recommend to the contractor how to position the temporary trestles. The steps that follow will enable you to make a recommendation.

(a) Draw the free-body diagram for each of the three planks under the front and rear tires (**Figure SA5.1.3*a***). Assume that the distance between the front axle and the middle of the rear axles of a concrete truck is 6.0 m while the tandem axle spread is 1.5 m (distance between the two rear axles). The length of the planks laid on top of the joists is 4 m, as shown in **Figure SA5.1.3*a***.

If the rear tires combined support 66% of the total weight of the truck (20 ton), determine all of the loads acting on each of the three planks. You recognize that plank 3 is a statically indeterminate system, so you ask Mat, the design engineer responsible for the Reynolds Coliseum, how you should deal with finding the loads acting on it since you have not yet had a course in mechanics of materials. After consulting a handbook[1] and writing down some calculations, Mat says that the vertical force between the joist and plank at G is 11.3% of the vertical force between the joist and plank at F.

(b) Draw a free-body diagram of each of the 8.2-m-long joists A–G (as shown in **Figure SA5.1.3*b***), assuming that the pair of planks is positioned in the center of the joist span and that each joist can be modeled as if supported by a pin connection at one end ($y = 8.2$ m)

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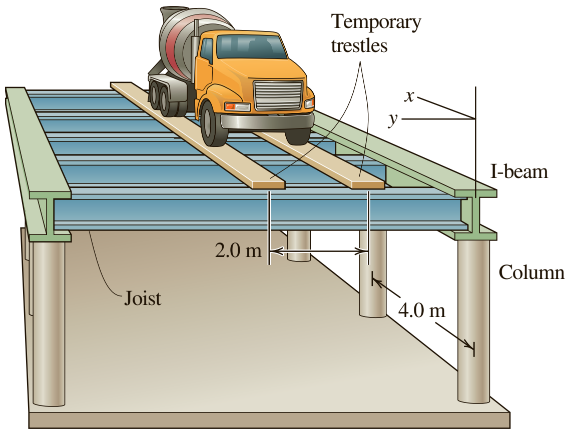
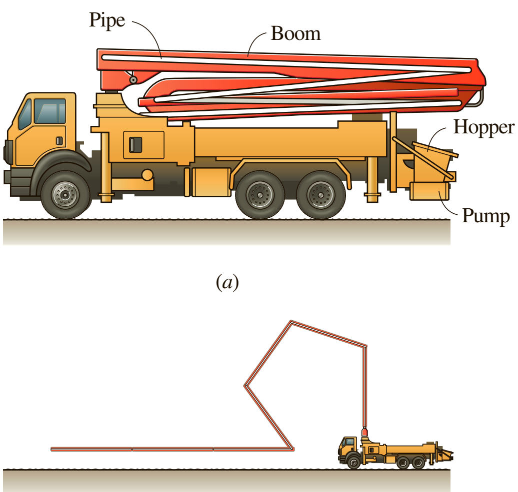


Figure SA5.1.2 Floor structure of Reynolds Coliseum (CCD = Center-to-Center Distance)

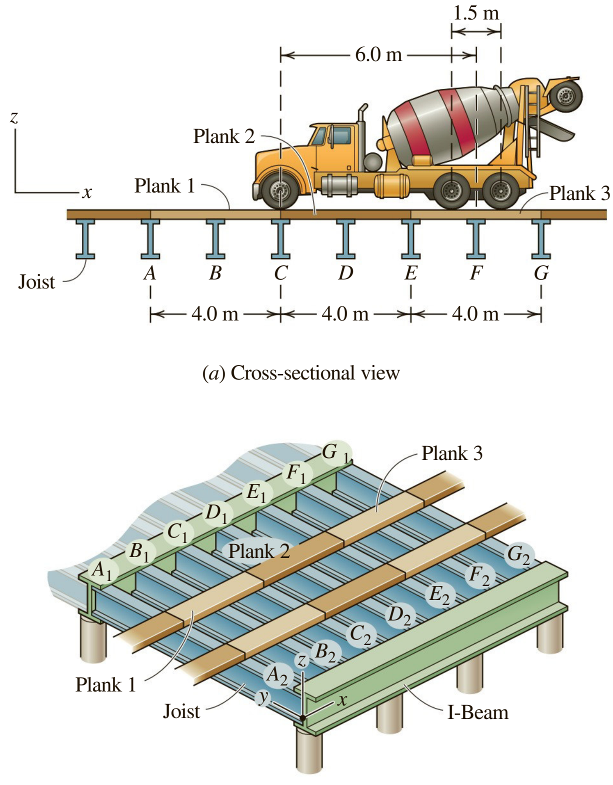


(*a*)

(*b*)

Figure SA5.1.1 Concrete pump trucks have not changed much since the 1960s

[1]Mat used the book *Formulas for Stress and Strain* by Roark and Young (McGraw-Hill, 1975). He modeled plank 3 between joists F and G as a beam that is fixed at one end (F) and on a roller at the other end (G) to come up with the 11.3% number. See page 97, Example 1c in Roark and Young.



(*a*) Cross-sectional view

(*b*) Isometric view

Figure SA5.1.3 Configuration of I-beams, planks, and joists

and a rocker at the other end ($y = 0.0$ m). This is truck Position 2, as shown in **Figure SA5.1.4**. Calculate all the loads acting on each of the joists.

(c) For the truck in Position 2 with its front tire at $x = +4.0$ m (see **Figure SA5.1.5**), what is the additional load on Columns I, II, and III along $y = 8.2$ m due to the truck?

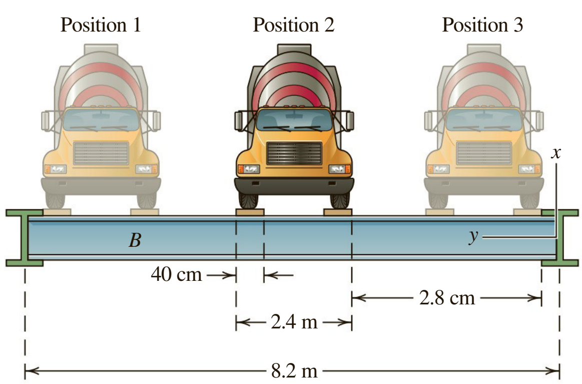


Figure SA5.1.4 Dimensions for center truck joists and beams

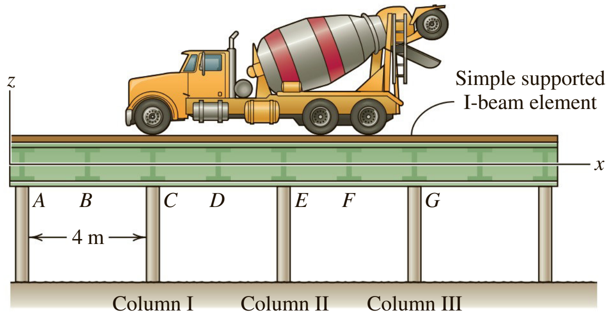


Figure SA5.1.5 Row of columns in a cross-sectional view

(d) Before being able to answer the question "Will the columns be able to carry the extra load from the 20-ton truck when in Position 2?" you find out from Mat that:

- In addition to the truck load acting on the columns, there is the weight of the formwork for the concrete and the joints. This can be modeled as a distributed load of approximately 18 tons/m onto the I-beams, as shown in **Figure SA5.1.6**.
- He designed the columns for a maximum load of 289 tons each. However, because the concrete is not hardened enough to accept his design load he will allow only 50% of 289 tons being used during construction.

Now answer the question, "Will the columns be able to carry the extra load from the 20-ton truck when in Position 2?"

(e) **Figure SA5.1.4** indicates that the truck can't always stay in the center of the joists. Since the truck is not allowed to drive over an I-beam, the worst loading case arises when the 40-cm-wide temporary trestles are positioned immediately next to its upper flange (Positions 1 and 3 in **Figure SA5.1.4**). Repeat steps (c) and (d) for Positions 1 and 3. Summarize your findings as a recommendation on how to position the temporary trestles to minimize loadings on the columns. Support your recommendation with your calculations.

(f) Repeat steps (c), (d), and (e) for alternate column spans of 6 m and 8 m.

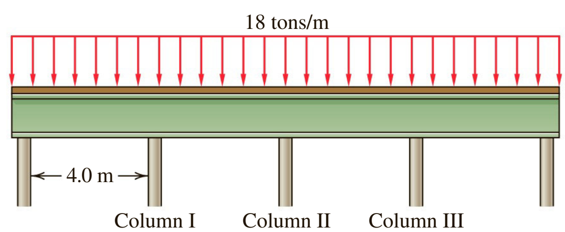


Figure SA5.1.6 Simplified loading diagram for a center I-beam

SA5.2 Keeping the Scoreboard in the Air

George's questions related to the score board inside the coliseum in **SA2.3** required the use of some basic principles of force-vectors. What you have learned since Chapter 2 has prepared you for some forensic engineering that you need to analyze an unexpected accident. As you watch the game, the tie cable between C and D suddenly breaks. The rigging arrangement for the score board is shown in **Figure SA5.2.1**.

(a) What will you observe after the cable breaks (assuming that everything else stays intact)? Describe in words and diagrams.

(b) What were the forces in the tie cable DC and suspension cables DG and DH before the accident? Recall that the score board weighs 4500 N.

(c) How large are F_{DG} and F_{DH} after the accident? Before the accident, points C and D were located 1.8 m between their respective winches.

(d) What do you recommend should be the sequence of actions to repair the rigging? In particular, would you lower the board to the floor for repair, or should it be fixed up in the air without moving it? Are there special risks before and during the repair?

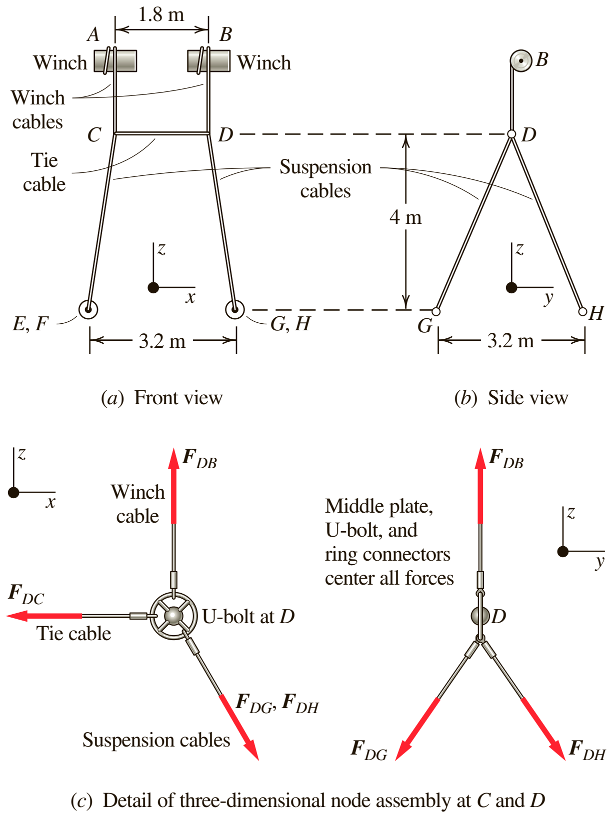


Figure SA5.2.1 Rigging arrangement for board

SA5.3 Will the Chair Flip?

The basketball game of the Wolfpack against UNC Chapel Hill is underway (as it was in **SA4.1**). In **SA4.1** Sierra was worried about the size of the hinge pin. Now she is concerned that the chair the woman is sitting in might flip over. The chair is shown in **Figure SA5.3.1**.

(a) Assuming that the mass of the chair is 5 kg with its center of mass 10 cm to the left of hinge, how close to the front of the edge of the seat can the woman sit and not have the chair flip over when 30% of her weight is supported by her legs? Recall that the mass of the woman is 61 kg.

(b) How will the answer to (a) change when Bill, who sits in the chair behind the woman, grabs the back of the chair and puts his total weight onto it? Bill's mass is 80 kg.

(c) The Wolfpack made another basket, and the woman jumps from her seat while Bill still hangs onto the chair. What happens to the chair?

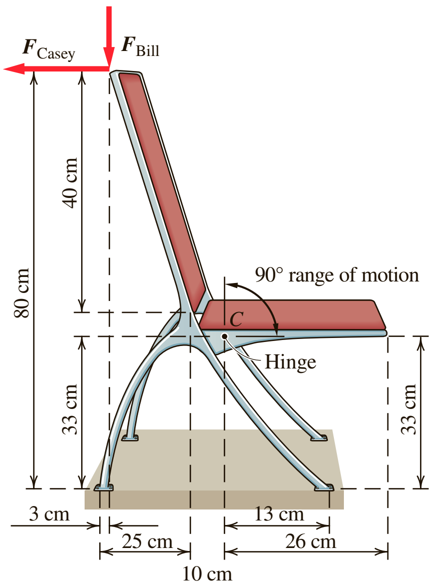


Figure SA5.3.1 Dimensions of movable chair

(d) Will your answer in (a) change when Casey, Bill's girlfriend, grabs the top of the back support and pulls herself out of her chair using a force of approximately 150 N? (Bill is no longer leaning on the chair with his 80-kg mass when Casey does this.)

(e) Suggest at least two changes to the chair design that would further reduce the risk of its flipping over.

SA5.4 Analysis of a System in Various Configurations

Materials needed: one wire clothes hanger, rubber band, paper clip, a weight (a candy bar is suggested), a scale (ruler).

Configure the hanger, rubber band, paper clip, and weight in each of the positions shown in Figure SA5.4.1. (*Note:* Some of the positions may not be achievable if the hanger is underconstrained.) In Positions 1–3 the hanger is oriented vertically, and in Positions 4–6 it is oriented horizontally. Draw a free-body diagram for the hanger in each position, noting whether it represents a statically determinate, indeterminate, or underconstrained system. Take any measurements necessary to make the free-body diagrams complete. Clearly state your assumptions.

For those positions that are statically determinate or indeterminate, find the loads acting on the hanger. For any positions that are statically indeterminate, you will not be able to find all of the loads, but will be able to reduce the number of unknown loads.

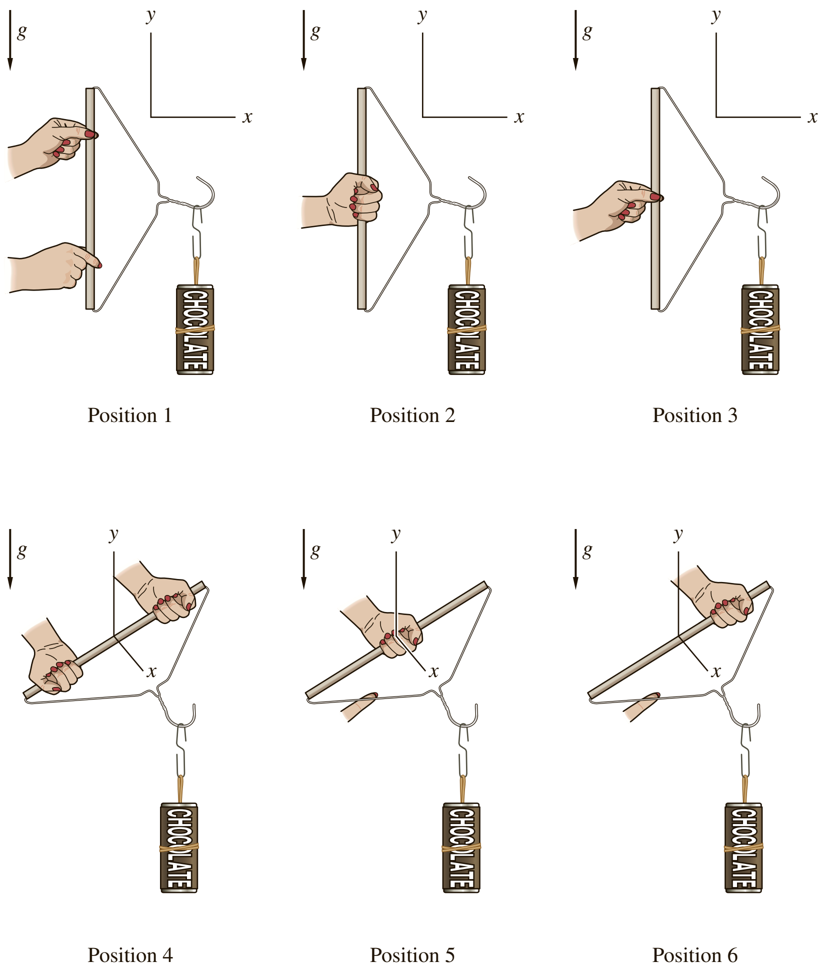


Figure SA5.4.1 Various hanger positions

SA5.5 Arm Strength

Using only a bathroom scale and a fixed surface (such as the edge of a kitchen or laboratory counter, or a push-bar door handle) design a procedure for finding the maximum force a person can exert in the upward direction with a single arm (**Figure SA5.5.1**). Present your procedure in a manner that a person not taking statics could easily follow. The procedure could be a combination of text and figures.

Follow the procedure you developed to determine the maximum upward force that you can deliver. Compare this number with the data presented in **SA2.1**.

Find 3 or 4 other students and have them follow your procedure to determine their maximum arm forces. Present their maximum arm force data in a table that also includes the average maximum force.

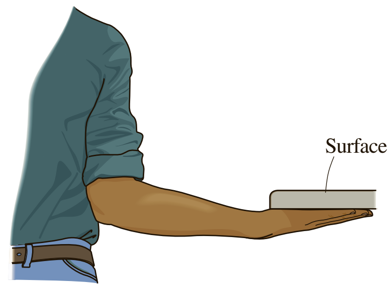


Figure SA5.5.1 Configuration of arm pushing on surface

SA5.6 Ancient Siege Engines

Ancient siege engines provided military commanders with the ability to engage an enemy from a distance; essentially it was the artillery of the armies past. Unfortunately, what is known of these medieval siege engines is limited to crude artist renditions and manuscript references. Hence, the hypothetical analysis of siege engines is still intriguing and challenging. Let us analyze one of the simplest forms of a siege engine, the catapult, as shown in **Figure SA5.6.1**. The siege engine fires a missile using the energy gained from a dropping counterweight and the advantage of a lever arm, the throwing arm. Once the throwing arm is cocked it is held in place by a vertical cable and supported by pin A (**Figure SA5.6.1**). Pin A rests on two identical frames, as shown in the overhead view in **Figure SA5.6.2**.

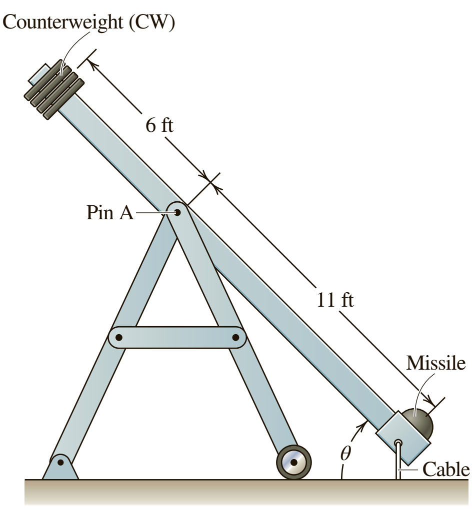


Figure SA5.6.1 Catapult

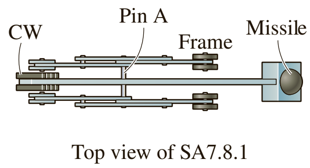


Figure SA5.6.2 Overhead view

Assume that the counterweight (CW) is 400 lb and the missile weight 50 lb. The throwing arm is generally uniform in density and shape with a weight of 100 lb, centered along its 17-foot length.

(a) Calculate the cable tension (T).

(b) Assume that the cable is no longer vertical but is attached to the throwing arm at angle $(90° - \theta)$, as shown in **Figure SA5.6.3**. Calculate the loads (reactions) at pin A and the tension in the cable when $\theta = 50°$

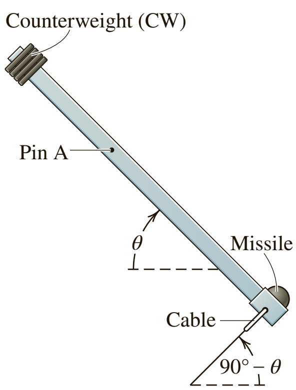


Figure SA5.6.3 Modified throwing arm.

SA5.7 Ancient Siege Engines—Other Design Ideas

(a) In loading the catapult of **Figure SA5.7.1*a***, a pulley system is used to lower the throwing arm and raise the counterweight (CW). Calculate the force P required to begin lowering the throwing arm when θ starts at 50°. Assume the counterweight is 400 lb.

(b) How does force P vary as the angle θ varies? Show the relationship with calculations.

(c) Consider the alternate two-pulley configuration (both frictionless and weightless), shown in **Figure SA5.7.1*b***. Calculate which system two-pulley configuration is the most efficient in lowering the throwing arm and raising the counterweight.

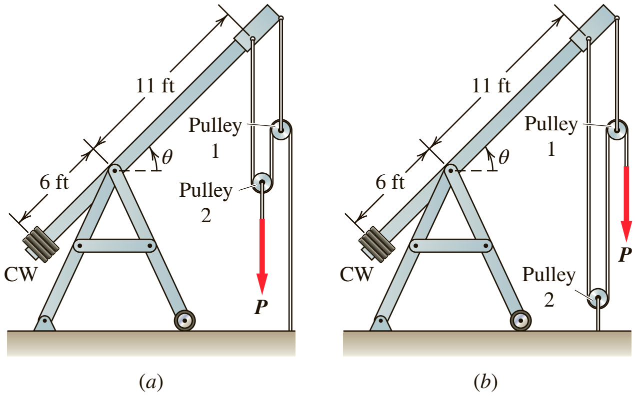


Figure SA5.7.1 (a) Pulley System 1; (b) Pulley System 2

SA5.8 Evaluation of a Lattice Boom Crane

Link-Belt is a company that makes lattice boom crawler cranes such as the 50-ton, LS-108H model, shown in **Figure SA5.8.1**. The design and analysis of these mechanical systems require all areas of engineering mechanics.

(a) Draw a complete free-body diagram of the crane for a hypothetical static load (LOAD) as shown. Consider the crane weight to be a composite system made up of the tractor weight (TRC) and the counterweight (CW). Write three equilibrium equations for the crane at the moment of impending tip about the track front (point O).

(b) If the TRC is 3000 lb and the CW is 20,000 lb, what is the maximum load the crane can lift and maintain in static equilibrium when θ is 40 degrees?

(c) How does the value of θ affect the results in (b)?

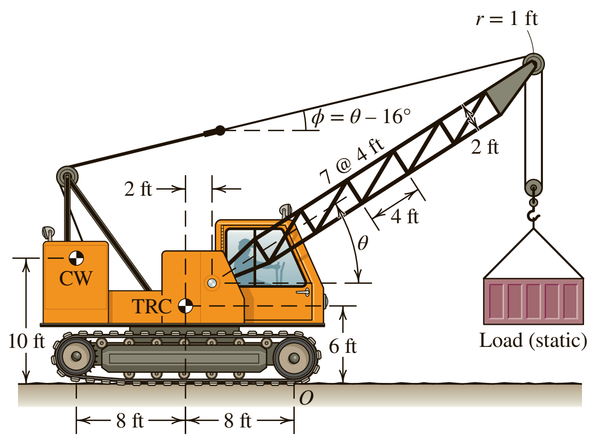


Figure SA5.8.1 Simplified crane system

CHAPTER 6

DISTRIBUTED FORCE

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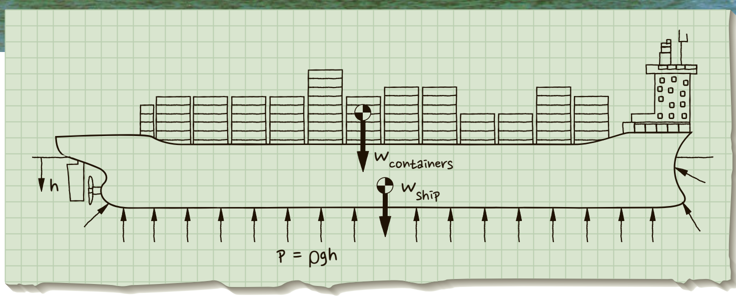


We have studied a variety of systems and the conditions under which they are in mechanical equilibrium. The forces we considered were acting either at a point on the boundary of a system or at the system's center of mass. This chapter illustrates how to represent a **distributed force** by an equivalent force that acts at a single point. We present procedures for including gravitational forces as well as distributed forces acting on the boundary of a system, along a line or over an area.

On the free-body diagram shown here the total weight of the shipping containers is represented at the center of gravity of the stack of containers. Similarly the total weight of the ship is represented at its center of gravity. The free-body diagram also shows pressure acting on the hull of the ship where it is in contact with the water.

LEARNING OBJECTIVES

On completion of this chapter, you will be able to:

- **Calculate the center of gravity and center of mass for simple and composite volumes, and the centroid for simple and composite areas and volumes.** (6.1)
- **Represent a distributed line or area load by an equivalent point force, and use the equivalent point force in static analysis.** (6.2)
- **Perform static analysis for situations involving distributed hydrostatic pressure and buoyancy forces.** (6.3)
- **Calculate the moment of inertia of a simple or composite area.** (6.4)

6.1 CENTER OF MASS, CENTER OF GRAVITY, AND THE CENTROID

Learning Objective: Calculate the center of gravity and center of mass for simple and composite volumes, and the centroid for simple and composite areas and volumes.

Distributed forces are forces that act over a line, area, or volume, rather than at a single point (**Figure 6.1.1**). We can represent distributed forces by an equivalent point force. The procedures presented in this chapter involve first summing all the individual forces exerted at various points on the line, area, or volume to find the *total force* acting and then locating the point of application of this total force such that the moment it creates is equivalent to the resultant moment created by all the individual forces. You may recognize this as the concept of **equivalent loads**, introduced in Section 3.5 and illustrated in **Figure 6.1.2**. This approach has the advantage that it simplifies the equilibrium equations in a static analysis, and minimizes the number of integrals we need to perform.

In systems that we have studied up to this point where weight of the system was important, the location of the center of mass of the system was always given. In this section we present how to locate the center of mass (as well as the center of gravity, the centroid of a volume, and the centroid of an area). Knowing how to locate the center of mass is an important part of carrying out static analysis since you will generally not be told its location in advance.

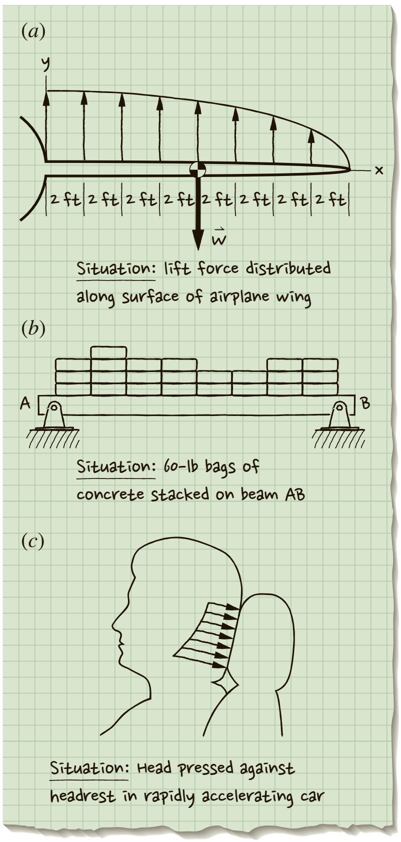


Figure 6.1.1 Examples of distributed forces.

Volumes

The particles that make up an object are distributed throughout the object's volume. The individual particle masses summed together are the **total mass** of the object. To simplify the modeling, in equilibrium analysis we generally work with the total mass of the object. We locate this total mass at a point so that the object behaves in a manner that is equivalent to the way the distributed particles, acting in concert, behave. This location in space is referred to as **center of mass**. We now outline how to find the total mass. Then, using the concept of equivalent loads, we find the location of the center of mass.

1. *Total mass M of an object*. The total mass is

$$M = \int_{\text{volume}} \rho \, dV \text{ (general case)} \tag{6.1}$$

where ρ is the object's density in mass/volume, $\rho\, dV$ is the mass of a volume element dV of the object, and integration involves integration throughout the object's volume.

If this total mass M is acted on by gravity, the associated weight W (the gravitational force) of the object is equal to Mg, where g is the gravitational acceleration, as discussed in Section 2.2. Denoting the weight of an infinitesimally small volume of the object as $dW = \rho g\, dV$, we rewrite (6.1) in terms of weight W as

$$\boxed{Mg = W = \int_{\text{volume}} dW = \int_{\text{volume}} \rho g\, dV} \tag{6.2A}$$

Alternatively, (6.2A) can be rewritten in terms of specific weight γ (where $\gamma = \rho g$ is weight per unit volume).

$$\boxed{Mg = W = \int_{\text{volume}} dW = \int_{\text{volume}} \gamma\, dV} \tag{6.2B}$$

Values of density and specific weight of commonly used engineering materials are presented in Appendix B.

2. *Location of center of mass*. To find the location of the center of mass of any object we apply the concept of equivalent loads. More specifically, we locate the object's total mass M such that M placed at that location creates a moment equivalent to the resultant moment created by the masses of individual particles. First, assume that the object of mass M located at X_M, Y_M, Z_M in **Figure 6.1.3*a*** is acted on by a uniform gravity field in the negative z direction. We can find X_M by requiring that the moment about the y axis created by Mg must be equal to the sum of the moments created by the distributed weights dW $(= \rho g\, dV)$ (**Figure 6.1.3*b***):

$$\underbrace{(X_M Mg)}_{\substack{\text{moment}\\\text{created}\\\text{by } Mg}} = \underbrace{\int_{\text{volume}} x\rho g\, dV}_{\substack{\text{moment created}\\\text{by distributed}\\\text{weights}}}$$

$$X_M = \frac{\displaystyle\int_{\text{volume}} x\rho g\, dV}{Mg} \underbrace{=}_{\substack{\text{for constant gravity,}\\\text{expression can}\\\text{be simplified}}} \frac{\displaystyle\int_{\text{volume}} x\rho\, dV}{M} \tag{6.3A}$$

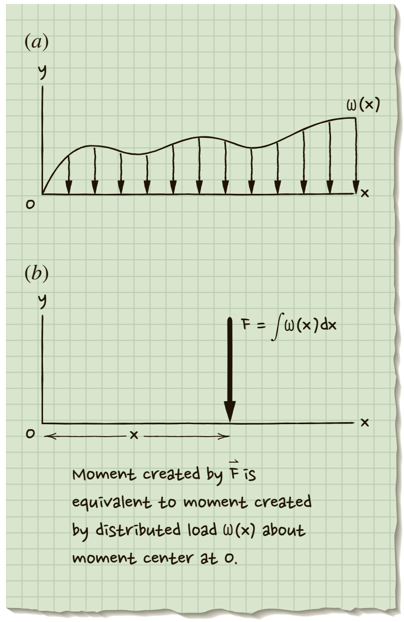


Figure 6.1.2 (*a*) A distributed load acting along an x axis; (*b*) the equivalent point force is located such that it creates the same moment as the distributed load $\omega(x)$ about moment center O.

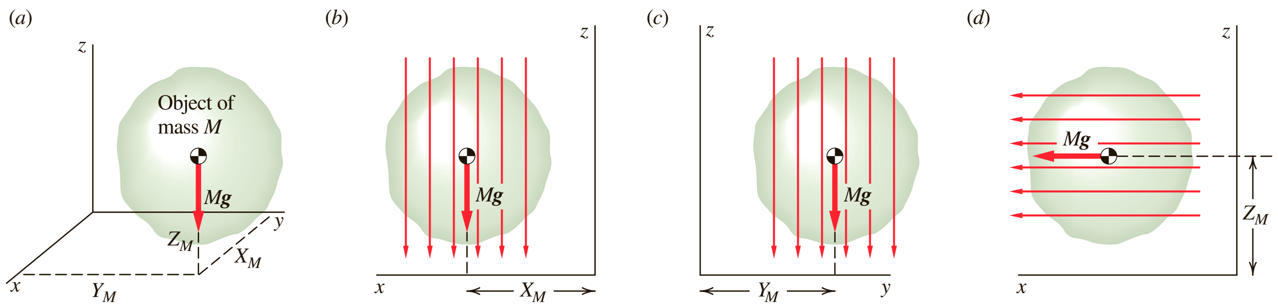


Figure 6.1.3 (*a*) Object of mass M with center of mass located at X_M, Y_M, and Z_M; (*b*) finding X_M; (*c*) finding Y_M; (*d*) finding Z_M.

Similarly, by equivalent moments about the x axis (**Figure 6.1.3*c***) we determine

$$Y_M = \underbrace{\frac{\int_{\text{volume}} y\rho g\, dV}{Mg} = }_{\substack{\text{for constant gravity,}\\ \text{expression can}\\ \text{be simplified}}} \frac{\int_{\text{volume}} y\rho\, dV}{M} \qquad (6.3B)$$

Finally, if we consider a uniform gravity field in the x direction (**Figure 6.1.3*d***) and require equivalent moments about the y axis, we determine

$$Z_M = \underbrace{\frac{\int_{\text{volume}} z\rho g\, dV}{Mg} = }_{\substack{\text{for constant gravity,}\\ \text{expression can}\\ \text{be simplified}}} \frac{\int_{\text{volume}} z\rho\, dV}{M} \qquad (6.3C)$$

In summary, the location of the center of mass of an object in a uniform gravitational field is:

$$X_M = \frac{\int_{\text{volume}} x\rho\, dV}{M}; \quad Y_M = \frac{\int_{\text{volume}} y\rho\, dV}{M}; \quad Z_M = \frac{\int_{\text{volume}} z\rho\, dV}{M} \qquad (6.4)$$

In many problems the integration required in (6.4) for finding the center of mass may be simplified by a prudent choice of reference axes. Symmetry provides an important clue in locating reference axes. Whenever there exists a line or plane of symmetry in an object of uniform density, the center of mass will be along the line or in the plane. The coordinate axes should be aligned with the line or plane. **Figure 6.1.4** shows examples of objects with lines or planes of symmetry.

3. *Location of center of gravity*. If we can treat the gravity field as uniform (which is approximately true for small objects on Earth), the center of mass (as defined in (6.4)) is also the location of the **center of gravity**, defined as the point in an object where we represent the total

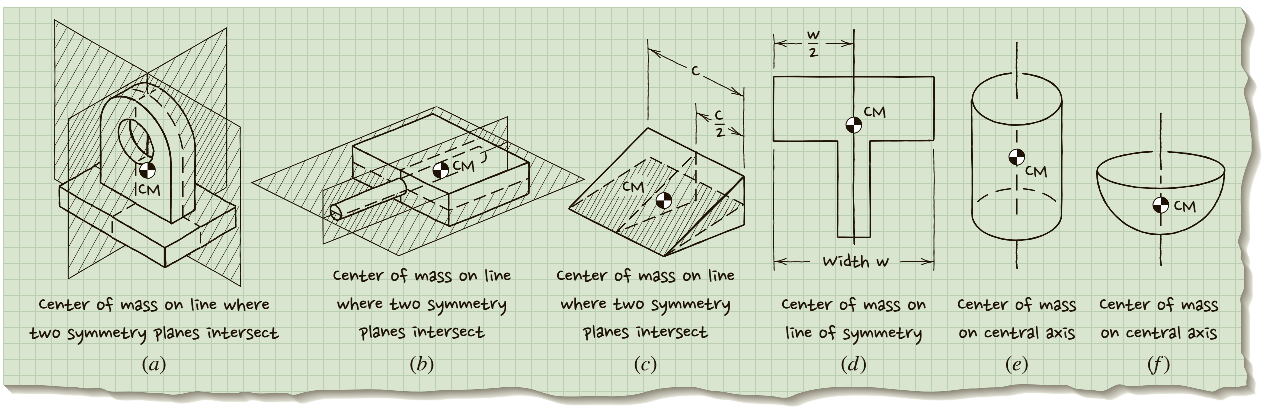


Figure 6.1.4 The center of mass of a volume lies on the planes and/or lines of symmetry.

weight of the object in order to treat the object's weight as a single point force (location X_G, Y_G, Z_G).

4. *Location of centroid*. The coordinates described by (6.4) locate the **centroid of a volume** if the volume is **homogeneous** (meaning that it is composed of a material of uniform density). For a general shape the centroid is located at

$$X_C = \frac{\int_{\text{volume}} x\,dV}{V_{tot}}; \quad Y_C = \frac{\int_{\text{volume}} y\,dV}{V_{tot}}; \quad Z_C = \frac{\int_{\text{volume}} z\,dV}{V_{tot}} \tag{6.5}$$

The locations of the centroids of several **standard volumes** are presented in Appendix C. The centroid is also the location of the center of mass and center of gravity if the volume is homogeneous. For these standard volumes, you can use Appendix C to locate the center of gravity as a labor-saving alternative to carrying out the integration called for in (6.4) and (6.5).

If we can decompose a composite volume into one made up of N standard volumes, we can use knowledge of the location of the centers of gravity of the N standard volumes to find the location of the center of gravity (X_G, Y_G, Z_G) of the composite volume (**Figure 6.1.5*a***). Call W_i the weight of an individual volume, and call X_{iG}, Y_{iG}, Z_{iG} the location of its center of gravity (**Figure 6.1.5*b***). Requiring that the moment of the composite volume be equal to the sum of the individual moments, we write

$$X_G \sum_{i=1}^{N} W_i = \sum_{i=1}^{N} W_i X_{iG};$$

$$Y_G \sum_{i=1}^{N} W_i = \sum_{i=1}^{N} W_i Y_{iG};$$

$$Z_G \sum_{i=1}^{N} W_i = \sum_{i=1}^{N} W_i Z_{iG} \tag{6.6}$$

Each term to the left of the equal sign is the moment created by the composite volume, and each term to the right is the summation of the moments created by the individual volumes. The total weight of the composite volume is $W_{\text{tot}} = \sum_{i=1}^{N} W_i$.

Substituting W_{tot} for each $\sum_{i=1}^{N} W_i$ term in (6.6) and solving for X_G, Y_G, and Z_G we have

$$X_G = \frac{\sum_{i=1}^{N} W_i X_{iG}}{W_{\text{tot}}}; \quad Y_G = \frac{\sum_{i=1}^{N} W_i Y_{iG}}{W_{\text{tot}}}; \quad Z_G = \frac{\sum_{i=1}^{N} W_i Z_{iG}}{W_{\text{tot}}} \tag{6.7A}$$

Equation (6.7A) gives the location of the **center of gravity of the composite volume**.

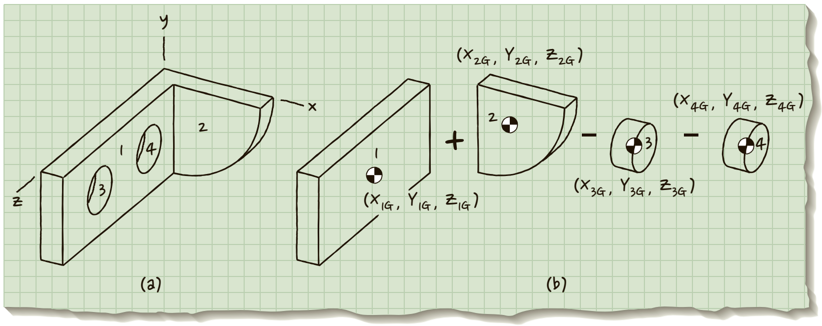


Figure 6.1.5 A volume decomposed into four separate volumes. Because volumes 3 and 4 are holes, they are negative volume.

In a similar manner, we can write the coordinates of the **center of mass of a composite volume** as

$$X_M = \frac{\sum_{i=1}^{N} M_i X_{iM}}{M_{\text{tot}}}; \quad Y_M = \frac{\sum_{i=1}^{N} M_i Y_{iM}}{M_{\text{tot}}}; \quad Z_M = \frac{\sum_{i=1}^{N} M_i Z_{iM}}{M_{\text{tot}}} \tag{6.7B}$$

where M_i is the mass of an individual volume and M_{tot} is the total mass of the composite volume. Furthermore, the coordinates of the **centroid of the composite volume** are

$$X_C = \frac{\sum_{i=1}^{N} V_i X_{iC}}{V_{\text{tot}}}; \quad Y_C = \frac{\sum_{i=1}^{N} V_i Y_{iC}}{V_{\text{tot}}}; \quad Z_C = \frac{\sum_{i=1}^{N} V_i Z_{iC}}{V_{\text{tot}}} \tag{6.7C}$$

where V_i is the volume of an element of the composite volume and V_{tot} is the total volume. In a constant gravitational field, (6.7A) and (6.7B) yield the same coordinates. Furthermore, if the composite volume is homogeneous, (6.7A), (6.7B), and (6.7C) yield identical coordinates.

Areas

An **extruded homogeneous volume** is a volume with a constant cross section as one moves along an axis; **Figure 6.1.6*a*** shows an example of an extruded volume with a constant cross section along the z axis. Two of the coordinates of the center of gravity of a homogeneous extruded volume lie in the cross section of the volume at the centroid of a plane area (**Figure 6.1.6*b***). Using the idea of equivalent loads we find the centroid of this plane area.

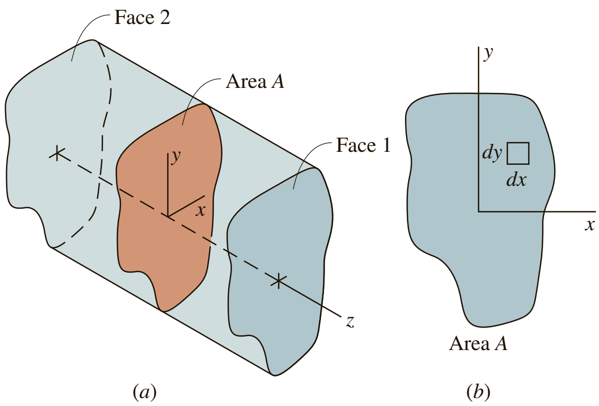


Figure 6.1.6 (*a*) A homogeneous extruded volume with constant cross section of area A; (*b*) the cross section A.

1. *Total area* A_{tot}. The total area is

$$A_{\text{tot}} = \int_{\text{area}}\int dx\,dy \tag{6.8}$$

2. *Location of centroid*. To find the location of the centroid of a plane area we apply the concept of equivalent loads by imagining that the area represents a thin sheet of constant thickness t, made up of a uniform material of density ρ (**Figure 6.1.7*a***). If gravity acts in the negative y direction, we can write

$$\underbrace{X_C W_{\text{tot}}}_{\substack{\text{moment about } z \text{ axis} \\ \text{produced by total} \\ \text{weight } W_{\text{tot}}}} = \underbrace{\int_{\text{area}}\int x\,dW}_{\substack{\text{total moment about} \\ z \text{ axis produced by} \\ \text{distributed weight}}}$$

$$X_C(t\rho g A_{\text{tot}}) = \int_{\text{area}}\int x(t\rho g\,dx\,dy) \tag{6.9}$$

where A_{tot} is defined in (6.8). Since t, g, and ρ are constants, they cancel from both sides of the equation. With rearranging we are left with

$$\boxed{X_C = \frac{\displaystyle\int_{\text{area}}\int x\,dx\,dy}{A_{\text{tot}}}} \tag{6.10A}$$

In a similar manner, if we consider gravity to act in the negative x direction (**Figure 6.1.7*b***) we determine

$$\boxed{Y_C = \frac{\displaystyle\int_{\text{area}}\int y\,dx\,dy}{A_{\text{tot}}}} \tag{6.10B}$$

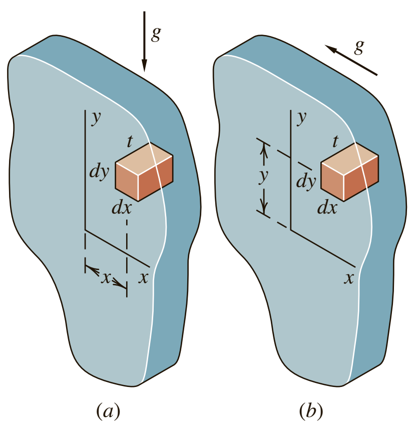


Figure 6.1.7 (*a*) Finding X_C of the centroid; (*b*) finding Y_C of the centroid.

Equations (6.10A) and (6.10B) define two of the coordinates of the center of gravity of a homogeneous extruded volume. The coordinate Z_C is at the mid-plane between the two faces of the extruded volume (i.e., midway between Face 1 and Face 2 in **Figure 6.1.6**). In addition, (6.10A) and (6.10B) define the coordinates (X_C, Y_C) of the centroid of any plane area.

The integrals in the numerator of (6.10A) and (6.10B) are called the **first area integrals** and are one of a family of integrals used to describe the properties of areas.*

The locations of centroids of several standard areas are presented in Appendix C. For these standard areas, you can use Appendix C to locate the centroid of an area as a labor-saving alternative to carrying out the integration called for in (6.8) and (6.10).

If we can decompose a composite area into one made up of N standard areas, we can use knowledge of the location of the centroids of the N standard areas to find the location of the centroid (X_C, Y_C) of the composite area (**Figure 6.1.8**). Call A_i the area of an individual area, and call X_{iC}, Y_{iC} the location of its centroid. By requiring that the moments be equivalent we find

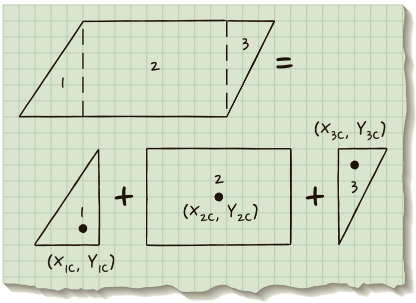


Figure 6.1.8 An area decomposed into three standard areas.

$$X_C \sum_{i=1}^{N} A_i = \sum_{i=1}^{N} A_i X_{iC}; \quad Y_C \sum_{i=1}^{N} A_i = \sum_{i=1}^{N} A_i Y_{iC} \tag{6.11}$$

Each term to the left of the equal sign reflects the moment created by the composite area, and each term to the right reflects the summation of the moments created by the individual areas. The total area of the composite area is $A_{\text{tot}} = \Sigma_{i=1}^{N} A_i$.

Substituting A_{tot} for each $\Sigma_{i=1}^{N} A_i$ term in (6.11) and solving for X_C, Y_C we have

*The family of integrals related to area are:

Area integral (area)

$$A = \int_{\text{area}} dA \tag{6.8}$$

First area integrals

$$\int_{\text{area}} x\,dA; \quad \int_{\text{area}} y\,dA \quad \text{(as used in (6.10A) and (6.10B))}$$

Second area integrals

$$I_x = \int_{\text{area}} y^2\,dA$$

$$I_y = \int_{\text{area}} x^2\,dA$$

Second area integrals are often referred to as **moments of inertia of the area** or **area moments of inertia**. The terminology "moment of inertia" is a misnomer (since no inertial concepts are involved). The second area integrals reflect the distribution of the area relative to coordinate axes in the plane of the area. We will use them in Chapter 10 when we analyze beams. Details on their calculation are presented in Section 6.4.

$$X_C = \frac{\sum_{i=1}^{N} A_i X_{iC}}{A_{tot}}; \qquad Y_C = \frac{\sum_{i=1}^{N} A_i Y_{iC}}{A_{tot}} \tag{6.12}$$

Equation (6.12) gives the location of the centroid of the composite area.

IMPORTANT NOTE! Don't confuse centroid and center of mass. The centroid is a property of the area (or volume) and is independent of the density of the material. Two otherwise identical volumes, one made of a homogenous material and the other made of material with linearly varying density, will have the same centroid. However, the locations of the centers of mass (and centers of gravity) for the two volumes WILL differ from each other.

Check out the following examples of applications of this material.

- **Example 6.1.1 Centroid of a Volume**
- **Example 6.1.2 Center of Mass with Variable Density**
- **Example 6.1.3 Locating the Centroid of a Composite Volume**
- **Example 6.1.4 Finding the Centroid of an Area**
- **Example 6.1.5 Center of Mass of a Composite Assembly**
- **Example 6.1.6 Centroid of a Built-up Section**

EXAMPLE 6.1.1

Figure 1 shows a right circular cone of height h and base radius r. Show that the location of the centroid is $X_C = 0$, $Y_C = 3h/4$, $Z_C = 0$ as indicated in Appendix C.

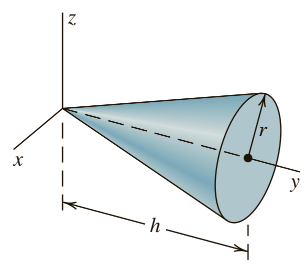


Figure 1 A right circular cone of height h and radius r.

Goal Find the coordinates of the centroid of a right circular cone.

Given Dimensions of the cone and a coordinate system.

Assume No assumptions needed.

Draw We draw an infinitesimal element dV at a distance y from the origin to use in integration of (6.5) (Figure 2).

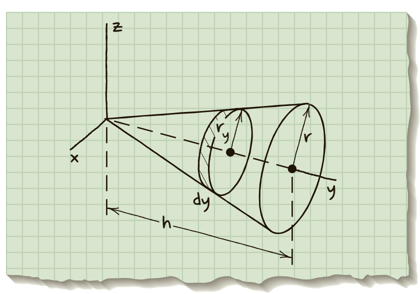


Figure 2 An infinitesimal element dV at a distance y from the origin.

Formulate Equations and Solve From symmetry we know that the centroid must lie on the y axis, as defined by $X_C = 0$, $Z_C = 0$. To use (6.5) to find Y_C we first calculate the volume of the cone. Our infinitesimal slice of cone at a distance y from the origin has a volume:

$$dV = \pi(r_y)^2 dy \tag{1}$$

We use similar triangles to solve for r_y as a function of y, r, and h

$$\frac{r_y}{y} = \frac{r}{h}$$

$$r_y = \frac{r}{h} y \tag{2}$$

We substitute (2) into (1) and integrate to find the volume of the cone:

$$V = \int dV = \int_0^h \pi r_y^2 \, dy = \int_0^h \pi \frac{r^2}{h^2} y^2 \, dy = \frac{\pi r^2}{h^2} \frac{y^3}{3} \Big]_0^h = \frac{\pi r^2 h}{3}$$

We now solve for Y_C:

$$Y_C = \frac{\int_{\text{volume}} y \, dV}{V} = \frac{\int_0^h y\left(\pi \frac{r^2}{h^2} y^2\right) dy}{V} = \frac{\frac{\pi r^2}{h^2} \frac{y^4}{4} \Big]_0^h}{V}$$

$$Y_C = \frac{\frac{\pi r^2 h^2}{4}}{\frac{\pi r^2 h}{3}} = \boxed{\frac{3}{4} h}$$

Check The result $X_C = 0$, $Y_C = 3h/4$, $Z_C = 0$ agrees with the location of the centroid of a cone, as shown in Appendix C.

EXAMPLE 6.1.2

A right circular cone (**Figure 1**) is made from a material of variable density. The density varies linearly from $3\rho_o$ at the point to ρ_o at the base. Find the location of the center of mass of the cone.

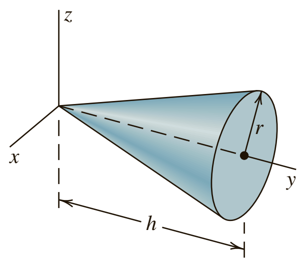


Figure 1 A right circular cone made of a material of linearly varying density.

Goal Find the x, y, and z coordinates of the center of mass of a right circular cone with linearly varying density.

Given Dimensions of the cone, linearly varying density, and a coordinate system.

Assume No assumptions needed.

Draw As in Example 6.1.1, we look at a slice of width dy at a distance y from the origin to find the volume dV (Example 6.1.1, Figure 2).

Formulate Equations and Solve From symmetry we know that the center of mass must lie on the y axis as defined by $X_M = 0$, $Z_M = 0$.

To find Y_M using (6.4), we must first develop a linear relationship between density and location in the cone. The form of the relationship will be $\rho(y) = my + b$. The constants m and b are found by imposing the boundary conditions

$$\rho(0) = 3\rho_o \quad \text{and} \quad \rho(h) = \rho_o$$

resulting in the relationship

$$\rho(y) = 3\rho_o - \frac{2\rho_o}{h} y \tag{1}$$

Furthermore, recall from Example 6.1.1 that

$$dV = \pi \frac{r^2}{h^2} y^2 \, dy \tag{2}$$

Substituting (1) and (2) into (6.1) and integrating to find the mass of the cone,

$$M = \int \rho(y)\,dV = \int_0^h \left(3\rho_o - \frac{2\rho_o}{h}y\right)\pi\frac{r^2}{h^2}y^2\,dy = \frac{\rho_o\pi r^2}{h^2}\int_0^h \left(3y^2 - \frac{2}{h}y^3\right)dy$$

$$M = \frac{\rho_o\pi r^2}{h^2}\left[\frac{3y^3}{3} - \frac{2y^4}{4h}\right]_0^h = \frac{\rho_o\pi r^2 h}{2}$$

Intermediate Check If the cone were of a constant density ρ_0, the mass of the cone would be $\rho_o V_{\text{cone}} = \rho_o\pi r^2 h/3$, as found from Appendix C. As expected, this mass is smaller than the mass we calculated with the density varying from ρ_o to $3\rho_o$. We present this as an intermediate check that our calculations are moving along in the right direction.

We now solve (6.4) for Y_M:

$$Y_M = \frac{\int_{\text{volume}} y\rho(y)\,dV}{M} = \frac{\int_0^h y\left(3\rho_0 - \frac{2\rho_0}{h}y\right)\left(\pi\frac{r^2}{h^2}y^2\right)dy}{M}$$

$$= \frac{\frac{\rho_o\pi r^2}{h^2}\int_0^h\left(3y^3 - \frac{2}{h}y^4\right)dy}{M} = \frac{\frac{\rho_o\pi r^2}{h^2}\left[\frac{3y^4}{4} - \frac{2y^5}{5h}\right]_0^h}{M}$$

$$Y_M = \frac{\frac{7\rho_o\pi r^2 h^2}{20}}{\frac{\rho_o\pi r^2 h}{2}} = \boxed{\frac{7}{10}h}$$

Check If the density were constant, the center of mass would coincide with the centroid of the cone ($3h/4$). Because the density varies linearly, with large densities toward the tip of the cone, the center of mass is closer to the tip ($7h/10$) than for the constant density case, $7h/10 < 3h/4$.

EXAMPLE 6.1.3

Figure 1 shows a concrete anchorage from a suspension bridge with a pedestrian archway formed by a semicircle of 25 ft radius at a height of 50 ft from the base. The specific gravity of concrete is $\gamma_c = 150$ lb/ft^3. Find the x, y, and z coordinates of the centroid of the anchorage.

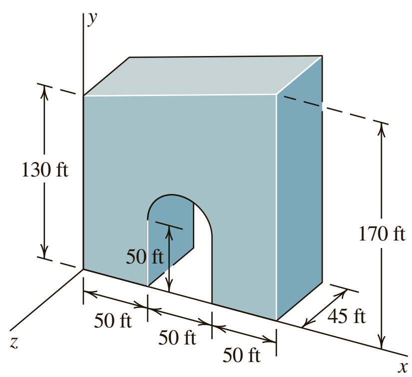


Figure 1 Dimensions of a concrete bridge anchorage.

Goal Find the coordinates of the centroid of the anchorage.

Given Dimensions of the anchorage and the specific weight of concrete.

Assume The concrete has the same specific gravity throughout the anchorage.

Draw We decompose the anchorage into standard volumes and subtract the archway from the anchorage, as in **Figure 2**.

Formulate Equations and Solve The x, y, and z coordinates of a composite volume centroid are given by (6.7C) as

$$X_C = \frac{\sum_{i=1}^{N} V_i X_i C}{V_{\text{tot}}}; \quad Y_C = \frac{\sum_{i=1}^{N} V_i Y_{iC}}{V_{\text{tot}}}; \quad Z_C = \frac{\sum_{i=1}^{N} V_i Z_{iC}}{V_{\text{tot}}} \tag{1}$$

To apply (6.7C) we begin by calculating the volume of the four volumes shown in **Figure 2**:

$$V_1 = (l_1)(h_1)(d) = (150\,\text{ft})(130\,\text{ft})(45\,\text{ft}) = 877{,}500\,\text{ft}^3$$

$$V_2 = \frac{b_2 h_2}{2} d = \frac{(150\,\text{ft})\,40\,\text{ft}}{2} 45\,\text{ft} = 135{,}000\,\text{ft}^3$$

$$V_3 = -(l_3)(h_3)(d) = -(50\,\text{ft})(50\,\text{ft})(45\,\text{ft}) = -112{,}500\,\text{ft}^3$$

$$V_4 = -\frac{\pi r_4^2}{2} d = -\frac{\pi(25\,\text{ft})^2}{2} 45\,\text{ft} = -44{,}180\,\text{ft}^3$$

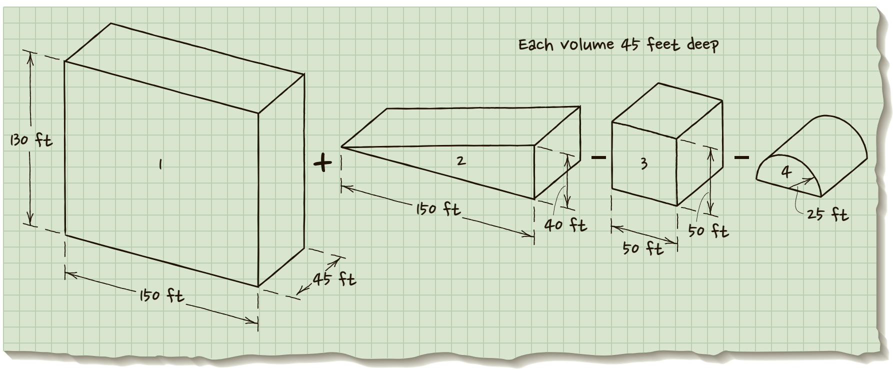


Figure 2 Decompose the anchorage into standard volumes.

Notice that volumes 3 and 4 are negative since they represent removal of volume, whereas volumes 1 and 2 represent addition of volume to the overall object.

From symmetry we know that the centroid of the anchorage is on its mid-plane defined by $Z_C = -22.5\text{ft}$. We need only calculate the x and y coordinates of the centroids, referring to Appendix C as needed.

$$X_{C1} = \frac{150\,\text{ft}}{2} = 75\,\text{ft}; \qquad Y_{C1} = \frac{130\,\text{ft}}{2} = 65\,\text{ft}$$

$$X_{C2} = \frac{2}{3} 150\,\text{ft} = 100\,\text{ft}; \qquad Y_{C2} = 130\,\text{ft} + \frac{1}{3} 40\,\text{ft} = 143.3\,\text{ft}$$

$$X_{C3} = 50\,\text{ft} + \frac{50\,\text{ft}}{2} = 75\,\text{ft}; \quad Y_{C3} = \frac{50\,\text{ft}}{2} = 25\,\text{ft}$$

$$X_{C4} = 50\,\text{ft} + \frac{50\,\text{ft}}{2} = 75\,\text{ft}; \quad Y_{C4} = 50 + \frac{4(25\,\text{ft})}{3\pi} = 60.6\,\text{ft}$$

It is convenient to summarize the data on volumes and centroid locations in a table:

Component	Vol (ft^3)	X_C (ft)	Y_C (ft)	$X_C V$ (ft^4)	$Y_C V$ (ft^4)
1	877,500	75	65	65.81×10^6	57.04×10^6
2	135,000	100	143.3	13.50×10^6	19.35×10^6
3	–112,500	75	25	-8.44×10^6	-2.81×10^6
4	–44,180	75	60.6	-3.31×10^6	-2.68×10^6
Σ	**855,820**			$\mathbf{67.56} \times 10^6$	$\mathbf{70.90} \times 10^6$

Using (1) and reading data from the last row of the table we calculate the coordinates of the center of gravity of the anchorage:

$$X_C = \frac{\sum_{i=1}^{N} V_i X_{iC}}{V_{\text{tot}}} = \frac{67.56 \times 10^6 \,\text{ft}^4}{855{,}820 \,\text{ft}^3} = \boxed{78.9 \,\text{ft}}$$

$$Y_C = \frac{\sum_{i=1}^{N} V_i Y_{iC}}{V_{\text{tot}}} = \frac{70.90 \times 10^6 \,\text{ft}^4}{855{,}820 \,\text{ft}^3} = \boxed{82.8 \,\text{ft}}$$

Check One check would be to redo the calculations with respect to a different origin. However, perhaps the best check is to see if the answers look reasonable. X_C is to the right of the center of the anchorage, as we would expect since there is more mass on the right. Y_C is in the upper portion of the anchorage, which we would expect because of the mass removed by the archway.

EXAMPLE 6.1.4

Figure 1*a* shows a homogeneous extruded volume, with cross-sectional area shown in detail in **Figure 1*b***. Determine the centroid (X_C, Y_C) and the area of the shaded cross-section.

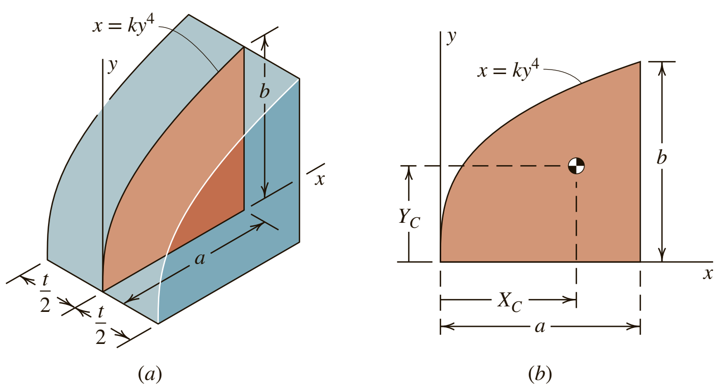


Figure 1 (*a*) A homogeneous extruded volume; (*b*) the cross section of the volume.

Goal Find the centroid and area of the shaded cross-section.

Given Information about the geometry and boundaries of the shaded cross-section, and a coordinate system.

Assume No assumptions needed.

Draw No drawings needed initially.

Formulate Equations and Solve One boundary of the cross-section is described by the curve $x = ky^4$. The value of k can be determined from evaluating $x = ky^4$ at a known point on the curve. At $x = a$, $y = b$, giving $a = kb^4$ and $k = a/b^4$. This results in

$$y = \frac{bx^{1/4}}{a^{1/4}}$$

We use (6.10A) and (6.10B) to find the centroid, and must first calculate the total area. Figure 2*a* shows the element dA, which is $y(x)$ tall and dx wide.

$$A_{\text{tot}} = \int_{\text{area}} dA = \int y(x)\,dx$$

$$A_{\text{tot}} = \int_0^a \frac{b}{a^{1/4}} x^{1/4}\,dx = \frac{b}{a^{1/4}}\left[\frac{4}{5}x^{5/4}\right]_0^a = \frac{4}{5}ab$$

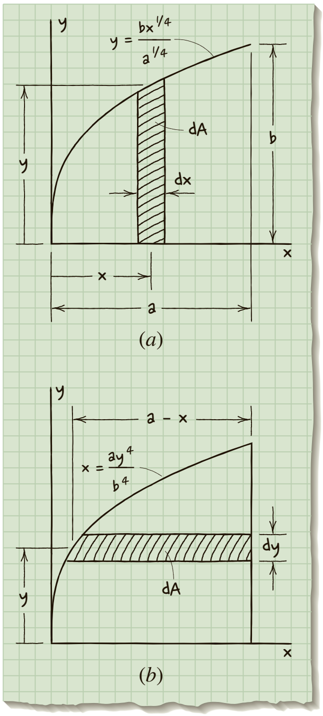


Figure 2 (*a*) The element dA is $y(x)$ tall and dx wide; (*b*) the element dA is a–x long and dy tall.

Based on (6.10A) and using the differential area dA shown in **Figure 2*a***, we compute X_C:

$$X_C = \frac{\int xy(x)\,dx}{A_{\text{tot}}} = \frac{\int_0^a x\left(\frac{b}{a^{1/4}}x^{1/4}\right)dx}{A_{\text{tot}}} = \frac{4b}{8a^{1/4}}a^{9/4}$$

$$X_C = \frac{\frac{b}{a^{1/4}}\left[\frac{4}{9}x^{9/4}\right]_0^a}{A_{\text{tot}}} = \frac{\frac{4b}{9a^{1/4}}a^{9/4}}{\frac{4}{5}ab} = \boxed{\frac{5}{9}a}$$

To compute Y_C, we slice the cross sectional area as shown in **Figure 2*b*** and create $dA = [a - x(y)]dy$. Based on (6.10B) we compute Y_C:

$$Y_C = \frac{\int y[a - x(y)]\,dy}{A_{\text{tot}}} = \frac{\int_0^b y\left(a - \frac{a}{b^4}y^4\right)dy}{A_{\text{tot}}}$$

$$Y_C = \frac{\left[\frac{ay^2}{2} - \frac{a}{b^4}\frac{1}{6}y^6\right]_0^6}{A_{\text{tot}}} = \frac{\frac{ab^2}{2} - \frac{a}{6b^4}b^6}{\frac{4}{5}ab} = \boxed{\frac{5}{12}b}$$

Check If the area were a rectangle the centroid would be located at $X_C = a/2$ and $Y_C = b/2$. Because area is missing from the upper left portion of the shape, we expect $X_C > a/2$ and $Y_C < b/2$. They are. This is a check that the answer is reasonable.

Note By finding the centroid of an area in this example, we were able to find two of the coordinates of the centroid of an extruded volume. As we will see in detail in the next section, being able to find the centroid of an area also enables us to use a single point force to model a distributed force acting on the boundary of a system. For example, if a uniform pressure p acts on a horizontal surface as shown in **Figure 3*a***, we could determine the total equivalent load and point of application. The total equivalent load F_T is

$$F_T = pA_T = p(4/5\,ab)$$

The total force would act at the centroid of the area $X_C = (5/9)a$ and $Y_C = (5/12)b$ (**Figure 3*b***).

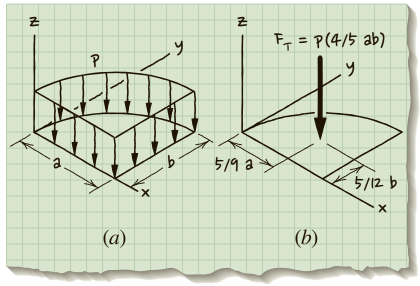


Figure 3 (*a*) A uniform pressure acts over an area; (*b*) the pressure is modeled by a point force at the centroid of the area.

EXAMPLE 6.1.5

Consider the assembly in **Figure 1*a***. The vertical face is made from sheet metal with a mass per unit area of 22 kg/m^2. The horizontal base is also sheet metal but has a mass per unit area of 45 kg/m^2. The aluminum shaft has a density of 2.71 Mg/m^3. Determine the x, y, and z coordinates of the center of mass of the assembly.

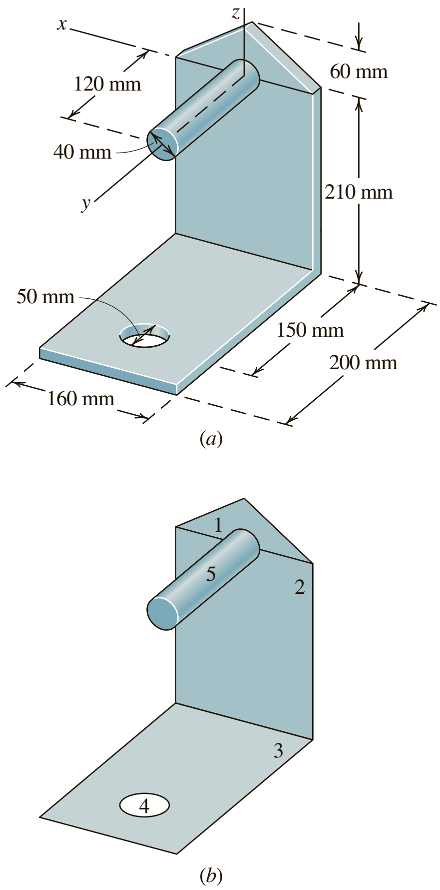


Figure 1 (*a*) An assembly composed of standard volumes; (*b*) the standard volumes are represented as areas along their midplanes.

Goal Find the coordinates of the center of mass of the assembly.

Given Assembly dimensions, material properties, and a coordinate system.

Assume We assume that the material properties are uniform (homogeneous) within the vertical face, horizontal base, and aluminum shaft. Because they are made of very thin sheet metal we can present the vertical face and horizontal base as areas along their midplanes, as shown in **Figure 1*b***.

Draw We first decompose the assembly into standard areas and volumes. (**Figure 2**).

Formulate Equations and Solve We use (6.7B) to find the center of mass of the assembly. We begin by using Appendix C to find volumes and areas of standard shapes so that we can calculate the mass of each part, noting that M_4 is negative because it is a cutout:

$$M_1 = \left(22\frac{\text{kg}}{\text{m}^2}\right)\frac{b_1h_1}{2} = \left(22\frac{\text{kg}}{\text{m}^2}\right)\frac{(0.160\,\text{m})(0.060\,\text{m})}{2} = 0.106\,\text{kg}$$

$$M_2 = \left(22\frac{\text{kg}}{\text{m}^2}\right)b_2h_2 = \left(22\frac{\text{kg}}{\text{m}^2}\right)(0.160\,\text{m})(0.210\,\text{m}) = 0.739\,\text{kg}$$

$$M_3 = \left(45\frac{\text{kg}}{\text{m}^2}\right)b_3h_3 = \left(45\frac{\text{kg}}{\text{m}^2}\right)(0.160\,\text{m})(0.200\,\text{m}) = 1.44\,\text{kg}$$

$$M_4 = \left(-45\frac{\text{kg}}{\text{m}^2}\right)\pi r_4^2 = \left(-45\frac{\text{kg}}{\text{m}^2}\right)\pi(0.025\,\text{m})^2 = -0.088\,\text{kg}$$

$$M_5 = \left(2.71\frac{\text{Mg}}{\text{m}^3}\right)\pi r_5^2 L_5 = \left(2710\frac{\text{kg}}{\text{m}^3}\right)\pi(0.020\,\text{m})^2(0.120\,\text{m}) = 0.409\,\text{kg}$$

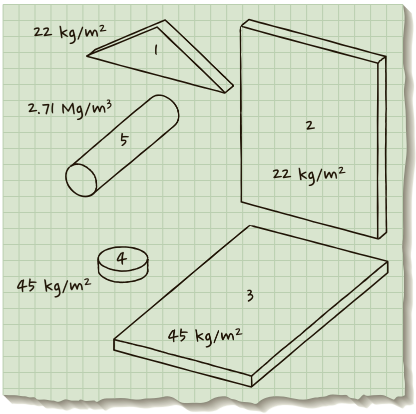


Figure 2 Decompose the assembly into standard shapes.

Because the plane defined by $x = 0$ is a plane of symmetry, we know that the center of mass occurs on that plane, meaning $X_M = 0$ (see **Figure 1**). We therefore need only calculate the y and z coordinates of the center of mass. We calculate the coordinates of the center of mass of each individual part based on coordinate axes as located in **Figure 1**, using Appendix C as a reference:

$$Y_{M1} = 0 \text{ (by inspection)}; \qquad Z_{M1} = \frac{1}{3}(60.0\text{ mm}) = 20.0\text{ mm}$$

$$Y_{M2} = 0 \text{ (by inspection)}; \qquad Z_{M2} = \frac{-210.0\text{ mm}}{2} = -105.0\text{ mm}$$

$$Y_{M3} = \frac{200.0\text{ mm}}{2} = 100.0\text{ mm}; \qquad Z_{M3} = -210.0\text{ mm (by inspection)}$$

$$Y_{M4} = 150.0\text{ mm (by inspection)}; \qquad Z_{M4} = -210.0\text{ mm (by inspection)}$$

$$Y_{M5} = \frac{120.0\text{ mm}}{2} = 60.0\text{ mm}; \qquad Z_{M5} = 0\text{ mm (by inspection)}$$

We then organize this information into a table:

Component	Mass [kg]	Y_M [mm]	Z_M [mm]	MY_M [kg mm]	MZ_M [kg mm]
1	0.106	0	20.0	0	2.11
2	0.739	0	–105.0	0	–77.6
3	1.44	100.0	–210.0	144.0	–302.4
4	–0.088	150.0	–210.0	–13.25	18.55
5	0.409	60.0	0	24.5	0
Σ	**2.61**			**155.3**	**–359.3**

Finally, we compute the location of the center of mass of the assembly.

$$X_M = \boxed{0 \text{ (by symmetry)}}$$

$$Y_M = \frac{\sum_{i=1}^{N} M_i Y_i M_i}{M} = \frac{155.3\text{ kg mm}}{2.61\text{ kg}} = \boxed{59.6\text{ mm}}$$

$$Z_M = \frac{\sum_{i=1}^{N} M_i Z_i M_i}{M} = \frac{-359.3\text{ kg mm}}{2.61\text{ kg}} = \boxed{-137.9\text{ mm}}$$

Comment We note that the location of the center of mass is a point in space that is not on the assembly. This is not unusual. For example, a symmetric object with a hole in the middle such as a washer or a pipe also has a mass center that is not on the object. We also note that the center of gravity of this assembly has the same location as its center of mass (as long as the gravity field is uniform).

Check We check if the answer is reasonable. From the sixth column of the table, we see that the base has the most influence on the location of Z_M, pulling it far down below the origin. This makes sense because although the aluminum shaft has a large mass, its center of mass is located at $z = 0$, and thus does not contribute to the summation in column six. What can you say about the calculations for Y_M?

EXAMPLE 6.1.6

A beam used in a three-story building is made from three standard steel sections welded together (**Figure 1**). A wide flange section (W18 × 76) is welded to a channel section (C10 × 15.3) at the bottom and a 1/2- in.-thick, 12-in.-wide plate at the top. Determine the coordinates of the centroid of the built-up section (area).

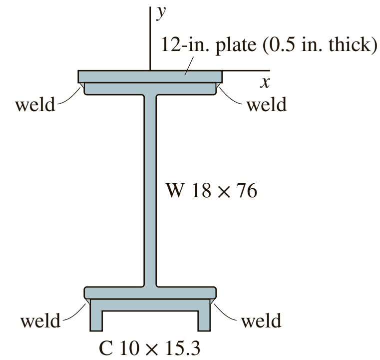


Figure 1 Steel beam built up from a wide flange section, a channel and a plate.

Goal Find the location of the centroid of the built-up structural section.

Given Dimensions of the steel plate and the specifications of the standard steel sections.

Assume The welds are so small they can be ignored in the centroid calculation.

Draw We look up the dimensions of the standard steel sections in the steel manual[†] and draw the individual pieces with their dimensions (**Figure 2**).

Formulate Equations and Solve Symmetry tells us that the centroid lies on the y axis ($X_C = 0$). We only need to find the y coordinate of the centroid.

We calculate the total area of the built-up section,

$$A_{\text{tot}} = A_{C1} + A_{C2} + A_{C3}$$

$$A_{\text{tot}} = 4.49 \text{ in.}^2 + 22.3 \text{ in.}^2 + 6.0 \text{ in.}^2 = 32.79 \text{ in.}^2$$

and the y coordinate of the centroid of each piece

$$Y_{C1} = -0.5 \text{ in.} - 18.21 \text{ in.} - 0.634 \text{ in.} = -19.334 \text{ in.}$$

$$Y_{C2} = -0.5 \text{ in.} - \frac{18.21 \text{ in.}}{2} = -9.605 \text{ in.}$$

$$Y_{C3} = -0.25 \text{ in.}$$

We substitute into (6.12) to get

$$Y_C = \frac{\sum_{i=1}^{3} A_i Y_{iC}}{A_{\text{tot}}} = \frac{\begin{array}{l}(4.49 \text{ in.}^2(-19.334 \text{ in.}))\\ +(22.3 \text{ in.}^2(-9.605 \text{ in.}))\\ +(6.0 \text{ in.}^2(-0.25 \text{ in.}))\end{array}}{32.79 \text{ in.}^2} = \frac{-302.5 \text{ in.}^3}{32.79 \text{ in.}^2}$$

$$Y_C = -9.23 \text{ in.}$$

Thus the centroid is located at $X_C = 0.0$ in. and $Y_C = -9.23$ in.

Check This answer is reasonable. If the built-up section were doubly symmetric, the centroid would be at the mid-height of the wide flange section. Since the plate on the top has a larger area than the channel on the bottom, the centroid is above the mid-height of the wide flange, which is what we found with $Y_C = -9.23$ in.

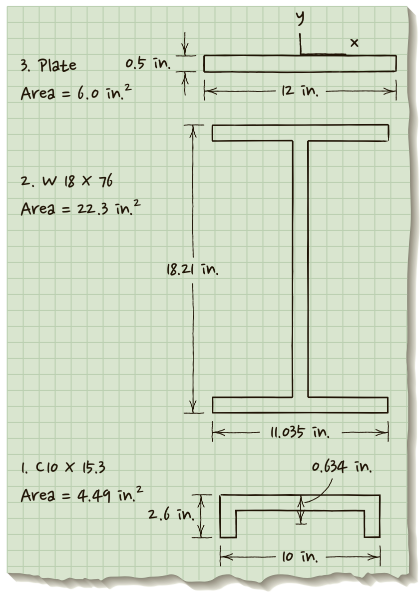
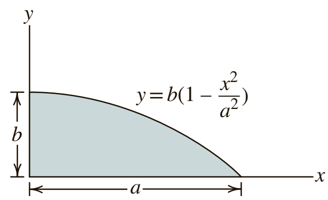


Figure 2 The dimensions and areas of standard steel shapes.

[†]American Institute of Steel Construction (2011), *Steel Construction Manual*, 14th edition. Chicago, Illinois.

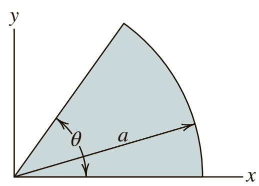
EXERCISES 6.1

6.1.1. **[*]** Calculate the area of the shaded region and locate the centroid. Present your answer in terms of a scale drawing of the shaded region.



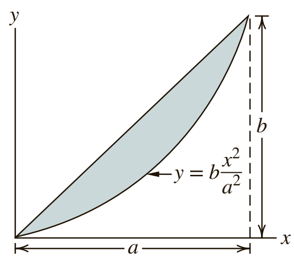
EX 6.1.1

6.1.2. **[*]** Calculate the area of the shaded region and locate the centroid. Present your answer in terms of a scale drawing of the shaded region.



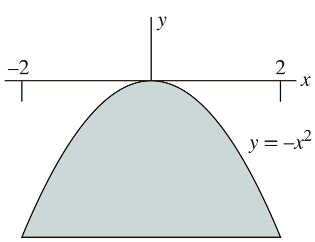
EX 6.1.2

6.1.3. **[*]** Calculate the area of the shaded region and locate the centroid. Present your answer in terms of a scale drawing of the shaded region.



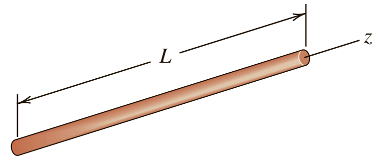
EX 6.1.3

6.1.4. **[*]** Use integration to calculate the area of the parabolic region and locate the centroid. Compare your answer with tabulated values in Appendix C.



EX 6.1.4

6.1.5. **[*]** Determine the location of the center of mass of the rod if its mass per unit length varies along its length according to $m = m_0(1 - z/4L)$. Compare your answer to the case of a homogeneous rod of mass per unit length equal to m_0.



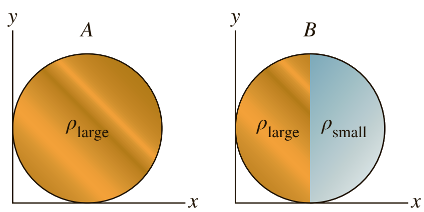
EX 6.1.5

6.1.6. **[*]** The round disk of radius R, labeled A, is made of a homogenous material with density ρ_{large}. The disk labeled B is the same size, but its left half has density ρ_{large} and its right half has density ρ_{small}.

a. Describe the location of the center of mass A in comparison to B

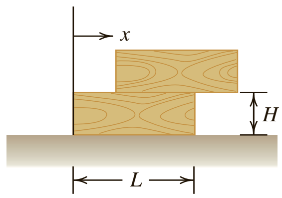
b. Describe the location of the centroid A in comparison to B

Explain your reasoning.



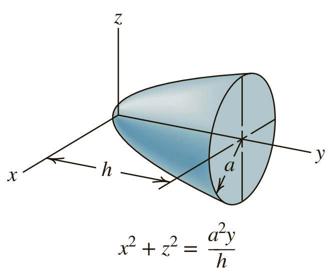
EX 6.1.6

6.1.7. **[*]** Two identical blocks with dimensions $L \times H \times B$ each weigh W. Determine the distance x that the top block can be pushed to the right before it tips. Explain the reasoning behind your answer.



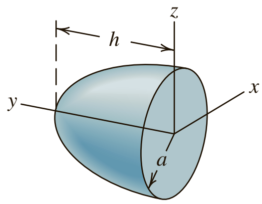
EX 6.1.7

6.1.8. **[*]** Determine the location of the centroid of the paraboloid of revolution shown.



EX 6.1.8

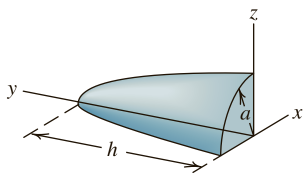
6.1.9. [*] Determine the location of the centroid of the ellipsoid of revolution shown.



$$x^2 + z^2 + \left(\frac{ay}{h}\right)^2 = a^2$$

EX 6.1.9

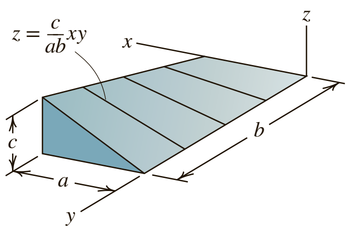
6.1.10. [*] The quarter paraboloid is homogeneous with material of density ρ. Determine its mass and the location of the center of mass.



$$x^2 + z^2 = a^2\left(1 - \frac{y}{h}\right)$$

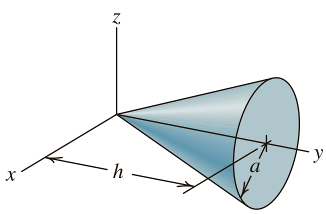
EX 6.1.10

6.1.11. [*] The solid shown is homogeneous with material density ρ. Determine its mass and the location of the center of mass.



EX 6.1.11

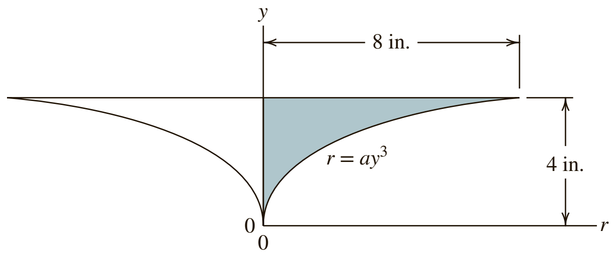
6.1.12. [*] The density in the cone varies from $3\rho_o$ at the point to $6\rho_o$ at the base. Determine the mass and the location of the center of mass of the cone.



$$x^2 + z^2 = \left(\frac{ay}{h}\right)^2$$

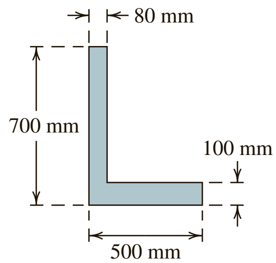
EX 6.1.12

6.1.13. [*] The object's volume is determined by revolving the shaded area through 360° about the y axis. It is homogeneous with material of density ρ. Find the location of the center of mass.



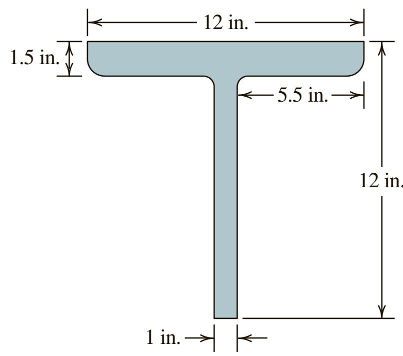
EX 6.1.13

6.1.14. [*] The L-shaped steel plate is 10 mm thick. Determine its mass and the location of its center of mass and show the results on a scale drawing of the plate.



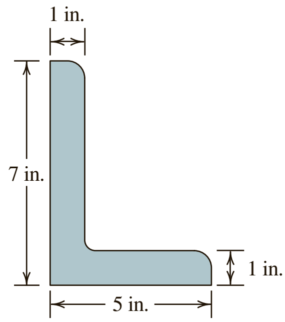
EX 6.1.14

6.1.15. [*] Calculate the area and locate the centroid of the T-section. It is reasonable to ignore the fillets (the rounding of the interior and exterior corners).



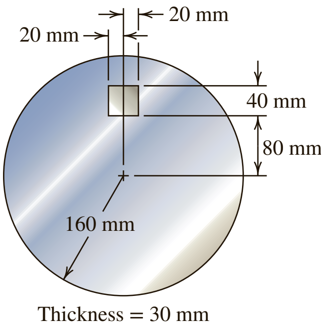
EX 6.1.15

6.1.16. [*] Calculate the area of the angle section. Also locate its centroid. It is reasonable to ignore the fillets (the rounding of the interior and exterior corners).



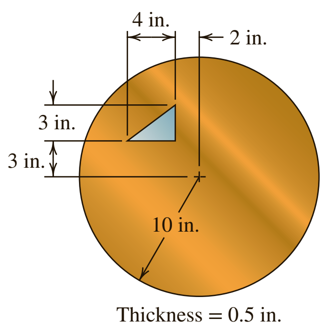
EX 6.1.16

6.1.17. [*] The 30-mm thick steel disk has an aluminum insert whose faces are flush with the faces of the disk. Calculate the area of the steel surface, excluding the area of the aluminum insert. Also locate the centroid of the steel portion.



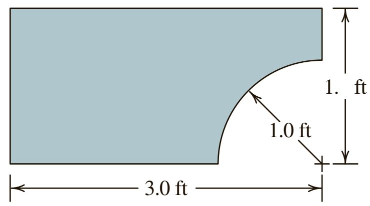
EX 6.1.17

6.1.18. [*] The 0.50-in. thick copper disk has a glass insert whose faces are flush with the faces of the disk. Calculate the area of the copper surface, excluding the area of the glass insert. Also locate the centroid of the copper portion.



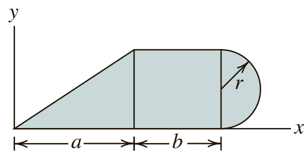
EX 6.1.18

6.1.19. [*] Calculate the area and locate the centroid of the shaded region.



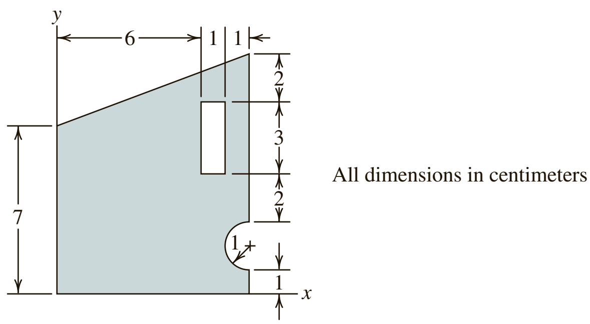
EX 6.1.19

6.1.20. [*] Calculate the area and locate the centroid of the shaded region.



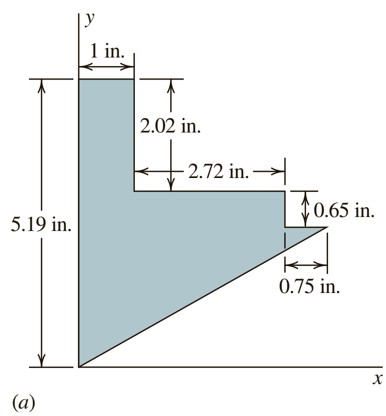
EX 6.1.20

6.1.21. [*] Calculate the area and locate the centroid of the shaded region.



EX 6.1.21

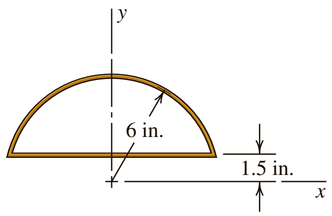
6.1.22. [*] A scale model of a B2 bomber is constructed for testing purposes. A simplified diagram of one of the wings, which is made out of a 1/8-in.-thick sheet of plastic, is shown here. Calculate the centroid of the model's wing relative to the given coordinate system.



(*a*)

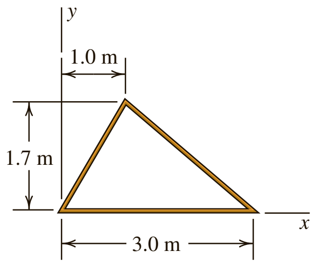
EX 6.1.22

6.1.23. [*] Copper wire 1/16 in. in diameter is bent into the semicircular configuration shown. Determine its mass and the location of its center of mass. Material densities are found in Appendix B.



EX 6.1.23

6.1.24. [*] Aluminum wire 6 mm in diameter is bent into the triangular confirmation shown. Determine its mass and the location of its center of mass. Material densities are found in Appendix B.

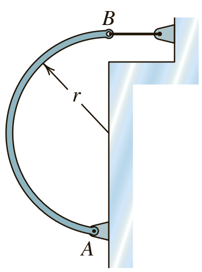


EX 6.1.24

6.1.25. [*] Consider the uniform semicircular rod of weight W and radius r. It is supported at A by a pin and is tethered by a horizontal cable at B. Determine

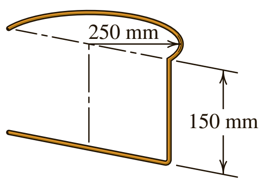
a. the center of mass of the semicircular rod

b. the loads acting on the rod at A and at B



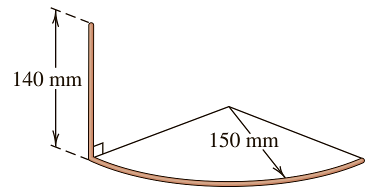
EX 6.1.25

6.1.26. [*] An aluminum wire 4 mm in diameter is bent into the confirmation shown. Determine the mass and the location of the center of mass of configuration. Material densities are found in Appendix B.



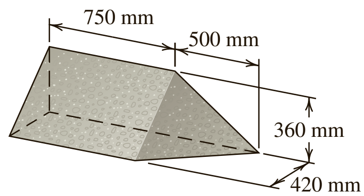
EX 6.1.26

6.1.27. [*] A copper wire is bent into the configuration shown. The diameter of the wire is 4 mm. Assuming that its density is 8900 kg/m^3 determine its mass and the location of its center of mass.



EX 6.1.27

6.1.28. [*] The object shown is made of concrete. Determine its mass and the location of its center of mass. Material densities are found in Appendix B.

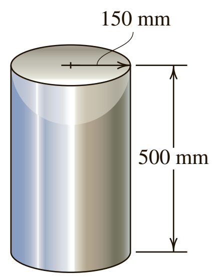


EX 6.1.28

6.1.29. [*] An object consists of a steel cylinder with a hemispherical cavity at the top, and the cavity is filled with aluminum. Material densities are found in Appendix B.

a. Determine the mass of the object and the location of its center of mass.

b. If the cavity is filled with steel instead of aluminum, determine the mass of the object and the location of its center of mass.



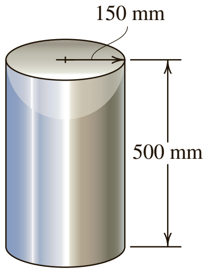
EX 6.1.29

6.1.30. [*] An object consists of an aluminum cylinder with a hemispherical cavity at the top, and the cavity is filled with steel. Material densities are found in Appendix B. Determine:

a. the mass of the object

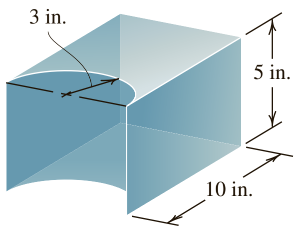
b. the location of its center of mass

c. the weight of the object



EX 6.1.30

6.1.31. [*] Calculate the volume and the location of the centroid of the solid. If the solid is made of aluminum, what is its weight?

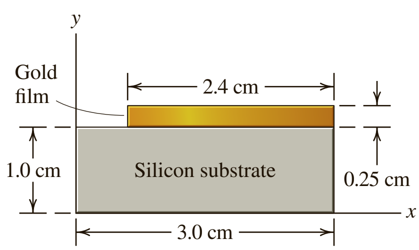


EX 6.1.31

6.1.32. [*] The Defense Department commissioned a computer manufacturer to produce for its navigational computers a mother board that can survive high-impact forces. The engineers tried explosively welding a highly conductive gold film (0.25 cm thick) to a silicon substrate. Before they test the strength of the interface they want to find the centroid of the component. Assuming that gold has a density of 19.302 g/cm^3, silicon has a density of 2.33 g/cm^3, and the component has a uniform width of 1.00 cm (in the z direction), calculate

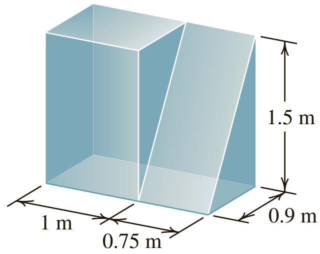
a. its centroid relative to the coordinate system shown

b. its center of mass relative to the coordinate system shown



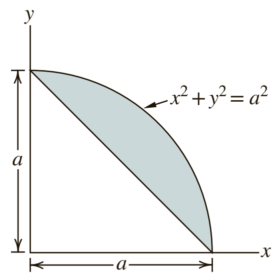
EX 6.1.32

6.1.33. [*] Calculate the volume and the location of the centroid of the glass solid. What is its weight?



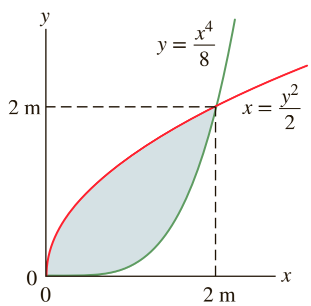
EX 6.1.33

6.1.34. [**] Calculate the area of the shaded region and locate the centroid. Present your answer in terms of a scale drawing of the shaded region.



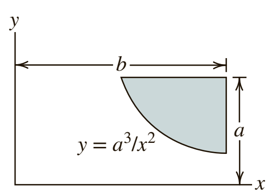
EX 6.1.34

6.1.35. [**] Calculate the area of the shaded region between the two curves and locate the centroid. Present your answer in terms of a scale drawing of the shaded region.



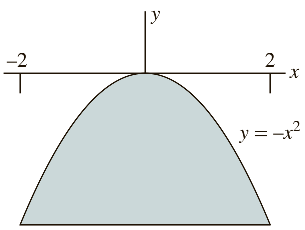
EX 6.1.35

6.1.36. [**] Calculate the area of the shaded region and locate the centroid. Present your answer in terms of a scale drawing of the shaded region.



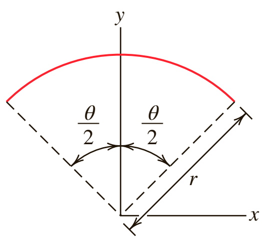
EX 6.1.36

6.1.37. [**] If the mass per unit area for the parabolic area shown varies as $m(y) = m_o(1 - y)$, find the location of the center of mass. Compare your answer to the location of the centroid, which can be found in Appendix C.



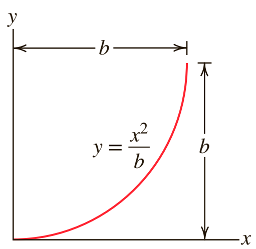
EX 6.1.37

6.1.38. [**] Locate the centroid of the circular arc.



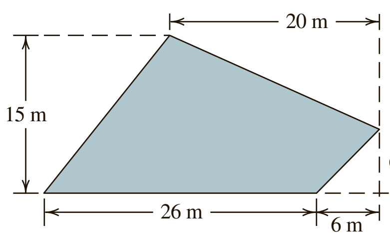
EX 6.1.38

6.1.39. [**] Locate the centroid of the parabolic arc.



EX 6.1.39

6.1.40. [**] Calculate the area and locate the centroid of the shaded region.

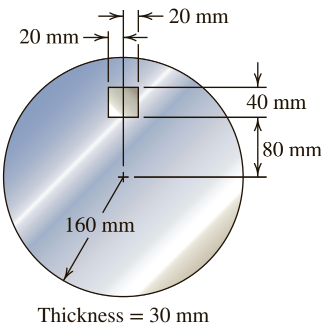


EX 6.1.40

6.1.41. [**] The 30-mm thick steel disk has an aluminum insert whose faces are flush with the faces of the disk. Determine

a. the mass of the disk with insert and the location of its center of mass

b. the centroid of the disk with insert

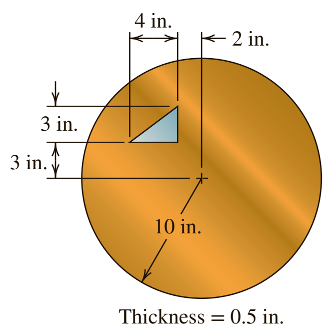


EX 6.1.41

6.1.42. [**] The 0.50-in. thick copper disk has a glass insert whose faces are flush with the faces of the disk. Determine

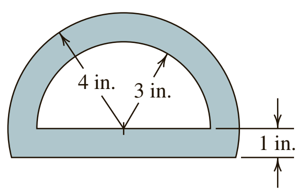
a. the mass of the disk with insert and the location of its center of mass

b. the centroid of the disk with insert



EX 6.1.42

6.1.43. [**] The aluminum plate shown is 0.25 in. thick. Determine its mass and the location of its center of mass and show the results on a scale drawing of the plate.



EX 6.1.43

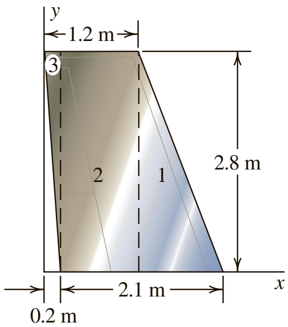
6.1.44. **[**]** The figure shows the dimensions of the rear stabilizer of a commercial aircraft. It is 1.5 cm thick (dimension in the z direction)

a. Find its centroid.

b. If the rear stabilizer is constructed of steel ($\rho = 7830$ kg/m^3), what is its mass? Where is its center of mass?

c. A mechanical engineer wants to divide the stabilizer into three sections as shown and replace one or more of them with a carbon fiber-resin composite ($\rho = 1400$ kg/m^3). What is the weight savings if section 3 is replaced with a composite material of the same dimensions? Where is the new position of the center of mass?

d. What is the weight savings if both sections 2 and 3 are replaced with the carbon fiber-resin composite material? Where is the new position of the center of mass?



EX 6.1.44

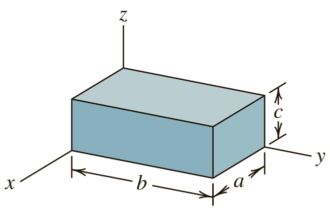
6.1.45. **[**]** The density of the rectangular volume shown varies according to

$$\rho = \rho_0\left(1 + \frac{x}{a}\frac{y}{b}\right)$$

Determine

a. the mass of the volume

b. the x, y, and z coordinates of the center of mass



EX 6.1.45

6.1.46. **[**]** The density of the rectangular volume shown varies according to

$$\rho = \rho_0\left(1 + \frac{x}{a}\frac{y}{b}\right)$$

Describe the location of the center of mass (CM) relative to the centroid (C).

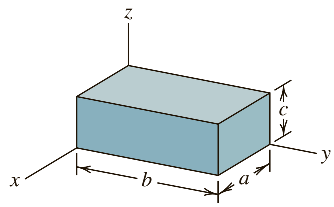
a. The CM and C have the same coordinates

b. The x, y, and z coordinates of CM are larger than those of C

c. The x, y, and z coordinates of CM are smaller than those of C

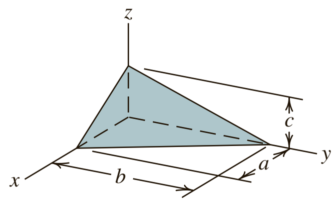
d. The x and y coordinates of CM are larger than those of C, but both have the same z coordinate

e. The x and y coordinates of CM are smaller than those of C, but both have the same z coordinate



EX 6.1.46

6.1.47. **[**]** The orthogonal tetrahedron shown is homogeneous with material density ρ. Determine its mass and the location of the center of mass.



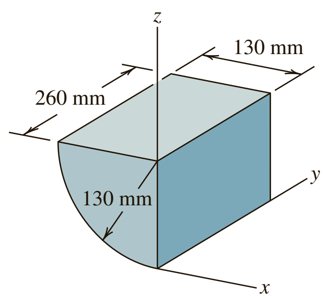
$$z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$$

EX 6.1.47

6.1.48. **[**]** A solid (quadrant of a cylinder) is shown. It is homogeneous. Find the x, y, and z coordinates of the center of mass by

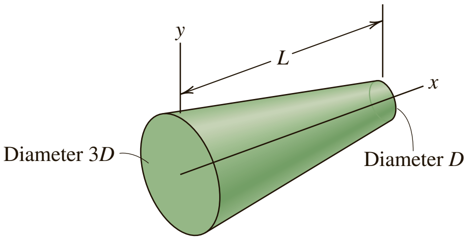
a. using integration (*Hint*: Rewrite the mass center equations in terms of cylindrical coordinates.)

b. using Appendix C



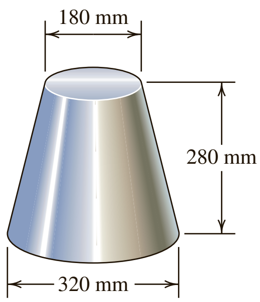
EX 6.1.48

6.1.49. **[**]** The diameter of the larger end of the tapered wood dowel of length L shown is three times the diameter of the smaller end. Determine the x coordinate of the center of mass.



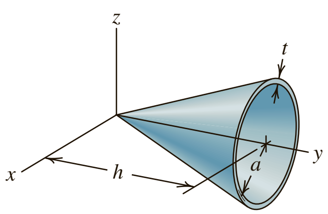
EX 6.1.49

6.1.50. **[**]** Calculate the volume and the location of the centroid of the steel solid. What is its weight?



EX 6.1.50

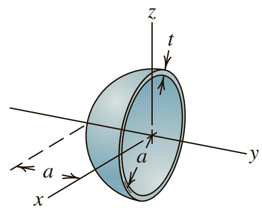
6.1.51. **[**]** The conical shell is homogeneous with material of density ρ and is of uniform thickness t. Determine its mass and the location of the center of mass.



$$x^2 + z^2 = \left(\frac{ay}{h}\right)^2$$

EX 6.1.51

6.1.52. **[**]** The hemispherical shell is homogeneous with material of density ρ and is of uniform thickness t. Determine its mass and the location of the center of mass.

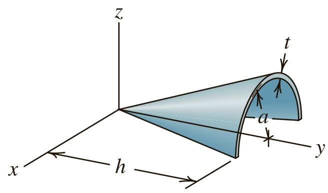


$$x^2 + y^2 + z^2 = a^2$$

EX 6.1.52

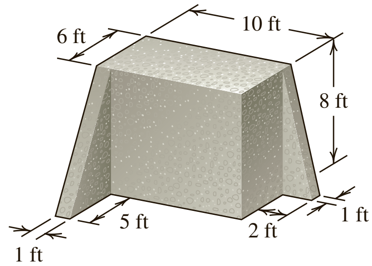
6.1.53. **[**]** Consider a semiconical shell of thickness t and mean radius a. The shell is homogeneous, with material density ρ. Determine the mass and location of the center of mass by

a. using integration
b. using information in Appendix C



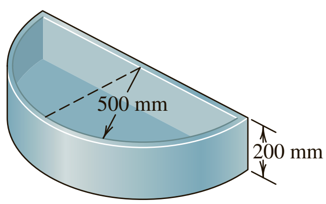
EX 6.1.53

6.1.54. **[**]** Calculate the volume and the location of the centroid of the solid shown. If the solid is made of concrete, what is its weight?



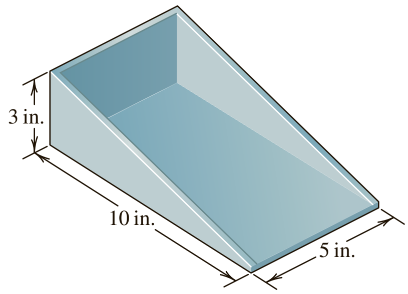
EX 6.1.54

6.1.55. **[**]** The container shown was created from bent sheet steel 2 mm thick. Determine the container's mass and the location of its mass center.



EX 6.1.55

6.1.56. [**] The container shown was created from bent sheet aluminum, 0.25 in. thick. Determine the container's mass and the location of its mass center.

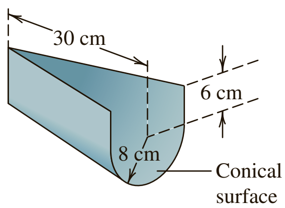


EX 6.1.56

6.1.57. [**] A sheet of 2 mm-thick aluminum has been bent into the shape shown.

a. Determine its mass and the location of its center of mass. The density of aluminum is 2690 kg/m^3.

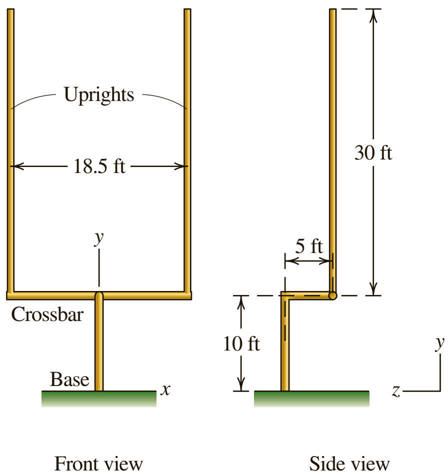
b. It is desired to make the shape out of sheet steel with the same mass as was found in **a**. What thickness of sheet steel should be specified? The density of steel is 7830 kg/m^3.



EX 6.1.57

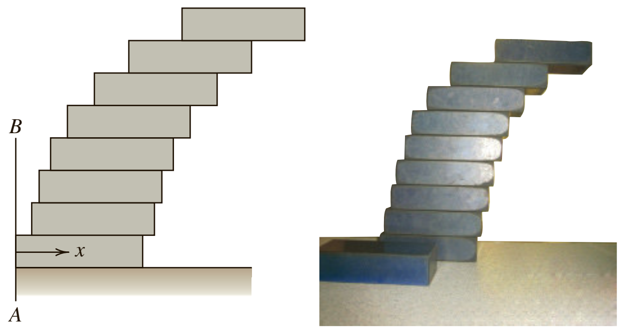
6.1.58. [**] An NFL goal post consists of the base, crossbar, and uprights. The base rises 10 ft (the bottom 6 ft are usually padded) and extends forward 5 ft. The crossbar is 18.5 ft wide, and the uprights rise 30 ft above the crossbar. The base and crossbar are each approximately 6 in. in diameter, and the uprights are 4 in. in diameter.

If the goal post weighs 900 lb and all parts are made of the same material, estimate the goal post's center of gravity relative to the bottom of its base.



EX 6.1.58

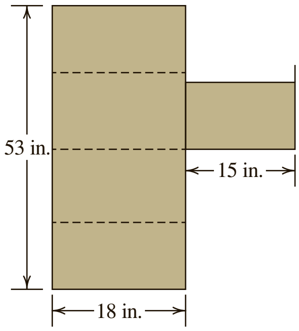
6.1.59. [, computer]** The structure shown is a false arch, which looks like an arch but doesn't have all of the structural characteristics of an arch. This type of construction was used by the Mayans and other civilizations throughout history. Each block is as far to the right as it can be without tipping. Define the distance from line *A*-*B* to the left end of a block as x. Assuming all of the blocks are identical in size and weight, the arch is eight blocks high, and the bottom layer is positioned at $x = 0$, create a spreadsheet model to determine the geometry of the arch in terms of x at each level.



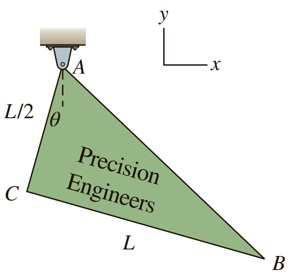
EX 6.1.59

6.1.60. [, computer]** Change the weights of the blocks in **EX6.1.59** so that the top block weighs W, the next block weighs $2W$, and the blocks get progressively heavier so that the bottom block weighs $8W$. Do you think the false arch would be more or less curved than in Exercise 6.1.59? Check your hypothesis using the spreadsheet you developed in Exercise 6.1.59.

6.1.61. [**] A four-tier filing cabinet weighs 300 lb when empty. An empty drawer weighs 25 lb, and a drawer filled with files weighs 110 lb. Drawers can be pulled out until they extend 15 inches beyond the front of the cabinet. If all drawers are full, how many drawers can be pulled to their full extent at the same time without the filing cabinet tipping over? (Assume weights act at the center of mass of objects)



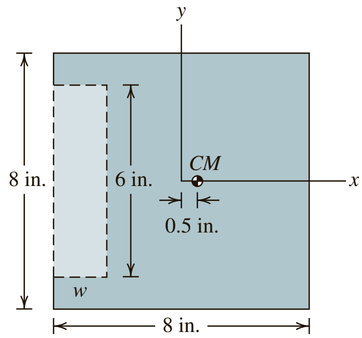
EX 6.1.61



EX 6.1.62

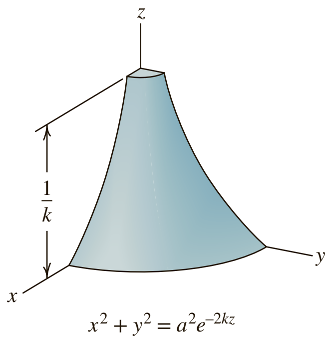
6.1.62. [***] The designers of this sign made a poor choice by using a pin support at A so that the sign is free to rotate in the x-y plane. If the sign is in equilibrium, determine magnitude of the angle θ. Where along side AB should the designer move the support so that side BC is horizontal (precisely)?

6.1.63. [***] A machinist fabricates a part by milling a notch out of an 8 in. by 8 in. metal plate. The notch is 6 in. high, w wide, and symmetric about the x axis. The specifications state that the center of mass of the plate can be no farther to the right than $x = 0.5$ in. What is the largest possible value of w?



EX 6.1.63

6.1.64. [***] The volume is homogeneous with material density ρ. Determine its mass and the location of the center of mass.

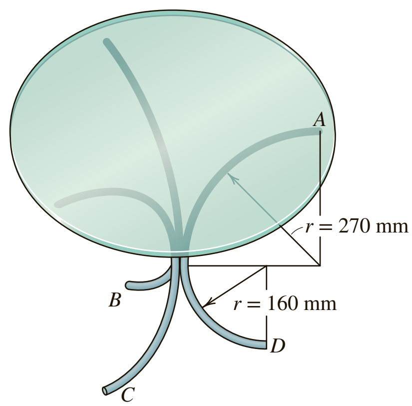


EX 6.1.64

6.1.65. [***] Consider the glass-topped patio table. Its three equally spaced legs are made of aluminum tubing with an outside diameter of 24 mm and cross-sectional area of 150 mm^2. Its glass tabletop has diameter of 600 mm and is 10 mm thick. The densities of aluminum and glass are 2690 kg/m^3 and 2190 kg/m^3, respectively.

a. Find the total mass of the table and its center of mass.

b. Now imagine that a book with a mass of 2 kg is placed at point A. Determine the forces of the floor acting on the legs at B, C, and D.



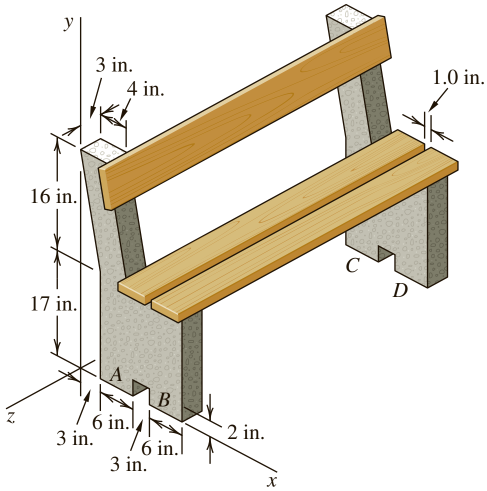
EX 6.1.65

6.1.66. **[***]** Consider the park bench shown. Its ends are made of concrete with specific weight of 0.084 lb/in^3. Its back and seat are constructed from wooden planks, each measuring $1.5 \times 7 \times 60$ in. with specific weight of 0.017 lb/in^3.

a. Find the total weight of the bench and the coordinates of its center of gravity.

b. What forces act on the concrete ends of the bench at A, B, C, and D?

c. If a child weighing 60 lb sits at either end of the bench, what forces act on the concrete ends of the bench at A, B, C, and D?



EX 6.1.66

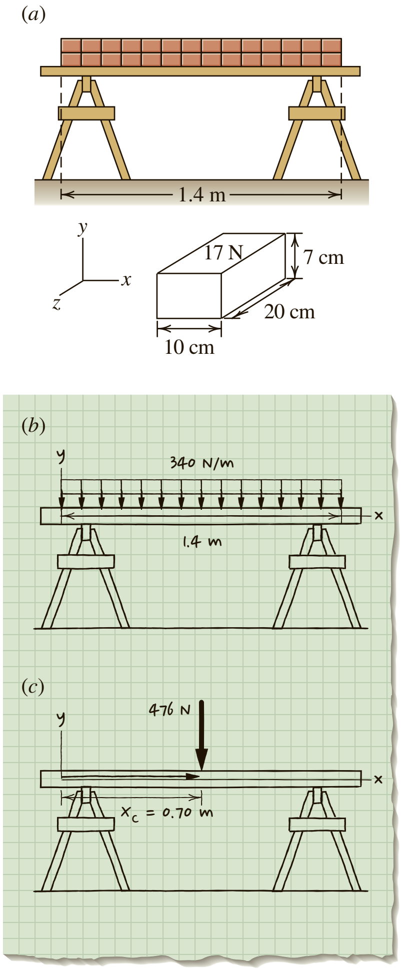


Figure 6.2.1 (*a*) Twenty-eight bricks stacked as shown, along with detail of brick; (*b*) the distributed force of the bricks acting on the plank; (*c*) distributed load represented as a single point force.

6.2 DISTRIBUTED FORCE ACTING ON A BOUNDARY

Learning Objective: Represent a distributed line or area load by an equivalent point force, and use the equivalent point force in static analysis.

We now consider how to represent distributed forces acting on a boundary by an equivalent single point force. First, we'll examine distributed loads with widths so small compared to the object they are acting on that they can be considered one-dimensional and acting along a line. We'll learn how to determine the equivalent force acting along that line and discover that the point at which it acts can be defined with only one coordinate. Then we consider a pressure acting over an area, which is two-dimensional, and therefore we'll need two coordinates to locate the point at which the total equivalent force acts.

Distributed Force Along a Line and Its Centroid

Consider a plank with two rows of bricks stacked over a span of 1.4 m, as shown in **Figure 6.2.1*a***. If each brick weighs 17 N, we can represent the load created by the weight of the bricks on the plank as a distributed line load of 340 N/m (= 17 N/brick × 28 bricks/1.4 m) over the 1.4-m span, as shown in **Figure 6.2.1*b***. This is referred to as a uniformly distributed **line load**. Notice that the units of a line load are force/length to reflect that the force is distributed over a line. This is in contrast to the forces acting on boundaries that we have considered prior to this chapter which were all point forces (and had units of force).

To replace this distributed line load with a single point force, we first find the total force. We then place this total force as a point force at the location where it creates the same equivalent moment as the distributed line load. For the uniformly distributed line load under consideration,

the total force is 17 N/brick × 28 bricks = 476 N, and it seems reasonable that this total force should be placed as a single point force at the center of the distribution (see **Figure 6.2.1*c***). We could figure this out mathematically by finding first the total force represented by the distributed force and then the location at which the total force acts:

1. *Total force.* This force is found by integrating the distributed force over the span:

$$\text{Total force in } y \text{ direction} = F_y = \int_{\text{span}} \omega\, dx \tag{6.13}$$

where ω is the distributed line load oriented in the y direction and has dimensions force/length, the integrand ($\omega\ dx$) is the force acting on segment dx, and "span" refers to the length over which ω is distributed (Figure 6.2.2*a*).

2. Location. The point at which the equivalent total force acts is called the **centroid of a line load** and is found by requiring that the moment produced by the total force be the same as the total moment produced by the distributed load (in doing this we are creating a moment that is equivalent to the one created by the distributed force). We can write this requirement as

$$\underbrace{X_C F_y}_{\substack{\text{moment}\\ \text{produced by}\\ \text{the total } y\\ \text{force}}} = \underbrace{\int_{\text{span}} x(\omega\, dx)}_{\substack{\text{total moments}\\ \text{produced by}\\ \text{distributed load}}} \tag{6.14}$$

where X_C is the location of the centroid measured from a moment center. The integrand represents the moment created by the total force, as illustrated in **Figure 6.2.2*b***. By integrating over the span, we find the total moment created by the distributed load.

Equation (6.14) solved for X_C is

$$X_C = \frac{\int_{\text{span}} x(\omega\, dx)}{Fy} \underbrace{=}_{\substack{\text{substituting in}\\ \text{from (6.13) for } F_y}} \frac{\int_{\text{span}} x(\omega\, dx)}{\int_{\text{span}} (\omega\, dx)} \tag{6.15}$$

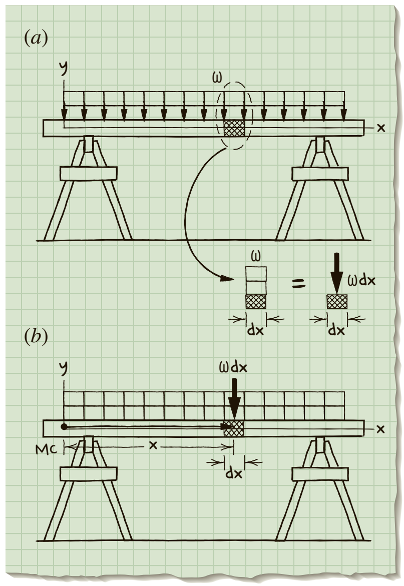


Figure 6.2.2 (*a*) The force represented by ($\omega\ dx$); (*b*) the moment created by ($\omega\ dx$) at moment center (MC).

Using (6.13) and (6.15) for the uniform line load in **Figure 6.2.1*a***, we find

$$\text{Total force in } y \text{ direction} = F_y = \int_0^{1.4\text{ m}} -340\text{ N/m}\, dx = -476\text{ N}$$

(the minus sign indicates that the force is in the negative y direction) and

$$X_C = \frac{\int_0^{1.4\text{ m}} x(-340\text{ N/m}\, dx)}{-476\text{ N}} = \frac{-\frac{x^2}{2}340\text{ N/m}\Big]_0^{1.4\text{ m}}}{-476\text{ N}} = 0.70\text{ m}$$

Equations (6.13) and (6.15) are also valid for a line load that is not distributed uniformly. Consider the brick stacking shown in **Figure 6.2.3*a***. This stacking can be approximated by a linear distribution of the form $\omega = \beta_1 x$, where β_1 equals 1821 N/m^2, and $x = 0$ is as indicated. The distribution $\omega = \beta_1 x$ spans $0 < x < 1.4$ m. If this distribution is inserted into (6.13) and (6.15), we find that the magnitude of the total force is 1.78×10^3 N, with the force located at $X_C = 0.93$ m (**Figure 6.2.3*b***).

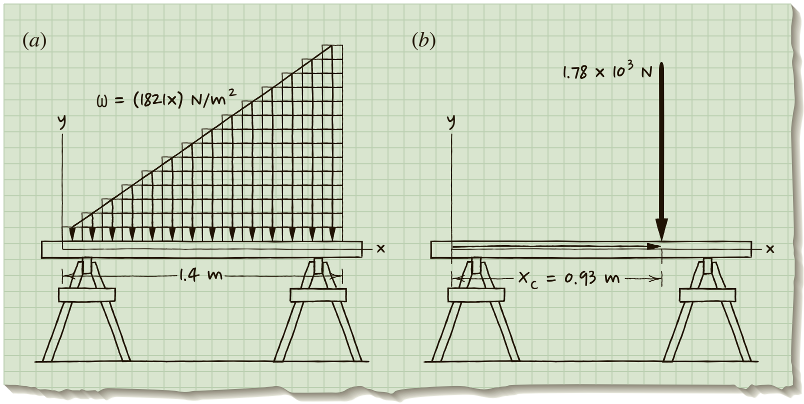


Figure 6.2.3 (*a*) Bricks stacked in a triangular pattern on top of the plank; (*b*) distributed load represented as a single point force.

Centroid of a Line Load Composed of Standard Shapes

If a line load distribution is a **standard line load distribution** (one that can be described by a simple geometric shape), the data in Appendix C can be used to locate the centroid. For example, consider the triangularly distributed force shown in **Figure 6.2.4*a***. From the properties of a triangle summarized in Appendix C, we are able to determine that

$$F_y = \frac{bh}{2} = \frac{40\,\text{cm}(-70\,\text{N/cm})}{2} = -1400\,\text{N} \qquad \text{and}$$

$$X_C = \frac{2b}{3} = \frac{2(40\,\text{cm})}{3} = 26.7\,\text{cm}$$

Alternately we could carry out the integration in (6.13) and (6.15) to find the same answers (but this would involve more work!).

If a distributed force can be decomposed into standard distributions, we can use the centroid locations of the standard distributions as the basis for finding the centroid of the composite distribution. For example, the distribution in **Figure 6.2.5*a*** can be approximated by the four distributions shown in **Figure 6.2.5*b***. The total force of the distribution is the sum of the total forces of the standard distributions. If the composite

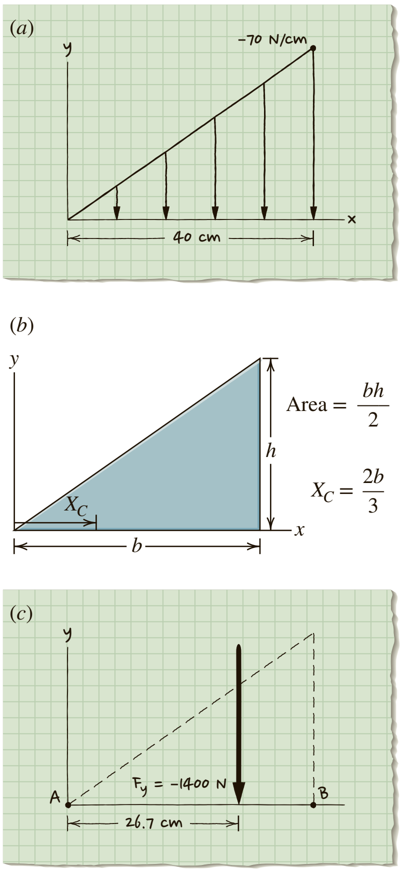


Figure 6.2.4 (*a*) Triangular line load distribution; (*b*) Appendix C data on right triangles; (*c*) distributed line load represented as a single point force.

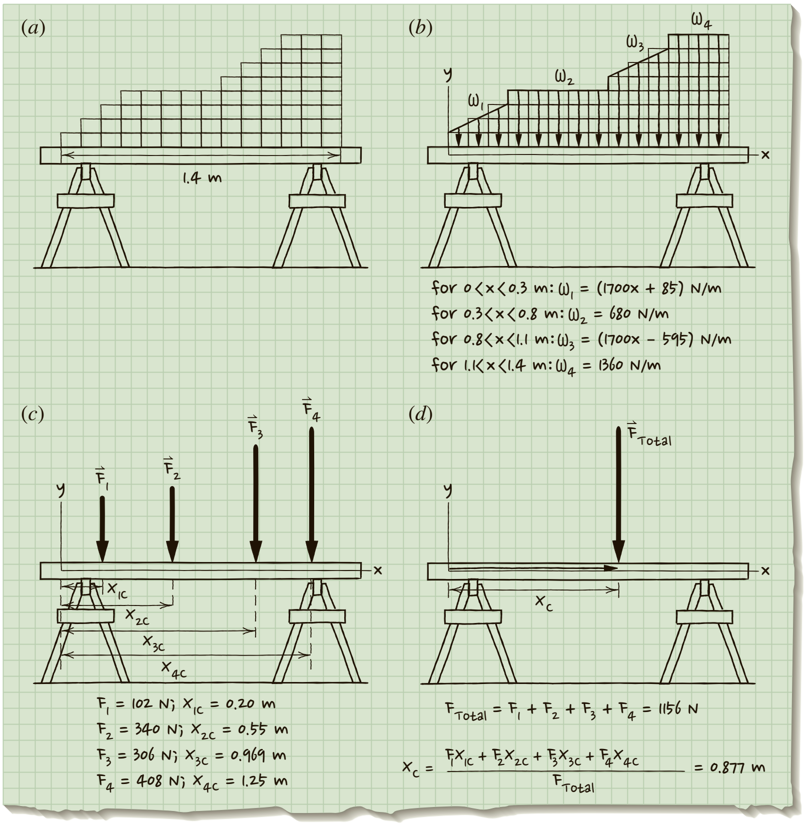


Figure 6.2.5 (*a*) Bricks stacked on a plank; (*b*) bricks represented as multiple line loads; (*c*) line loads represented as point forces; (*d*) single point force that represents the multiple line loads.

distribution consists of N standard distributions, we can write the magnitude of the total force F_{total} as

$$F_{\text{total}} = \sum_{i=1}^{N} \underbrace{F_i}_{\substack{\text{force of each}\\ \text{standard}\\ \text{distribution}}} \tag{6.16}$$

where F_i is the total force associated with distribution i. The location (X_C) of the centroid of the composite distribution is then found by finding the equivalent moment as

$$\underbrace{\underbrace{X_C}_{\substack{\text{centroid of}\\ \text{composite line}\\ \text{distribution}}} \underbrace{\sum_{i=1}^{N} F_i}_{\text{total force}}}_{\substack{\text{moment created}\\ \text{by total force}}} = \underbrace{\sum_{i=1}^{N} \underbrace{F_i X_{iC}}_{\substack{\text{force and centroid}\\ \text{of each standard}\\ \text{distribution}}}}_{\substack{\text{sum of moments}\\ \text{created by forces of}\\ \text{standard distributions}}}$$

or by substituting from (6.16) and solving for X_C, we have:

$$X_C = \frac{\sum_{i=1}^{N} F_i X_{iC}}{F_{\text{total}}} \tag{6.17}$$

where N is the number of standard distributions and X_{iC} and F_i are the centroid and total load associated with the ith standard distribution (**Figure 6.2.5*c***). The application of (6.17) is illustrated in **Figure 6.2.5*d*** and in Example 6.2.5.

Distributed Force Over an Area and Its Centroid

Consider a piece of plywood stacked with 32 bricks over an area measuring 0.80 m by 0.40 m, as shown in **Figure 6.2.6*a***. If each brick weighs 17 N, we can represent the weight of the bricks acting on the plywood as 544-N force distributed over the 0.32-m^2 (= 0.8 m × 0.4 m) area (**Figure 6.2.6*b***), so that the distributed force is 544 N/0.32 m^2 = 1700 N/m^2. This is referred to as a **uniform pressure load**, where uniform means uniformly distributed.

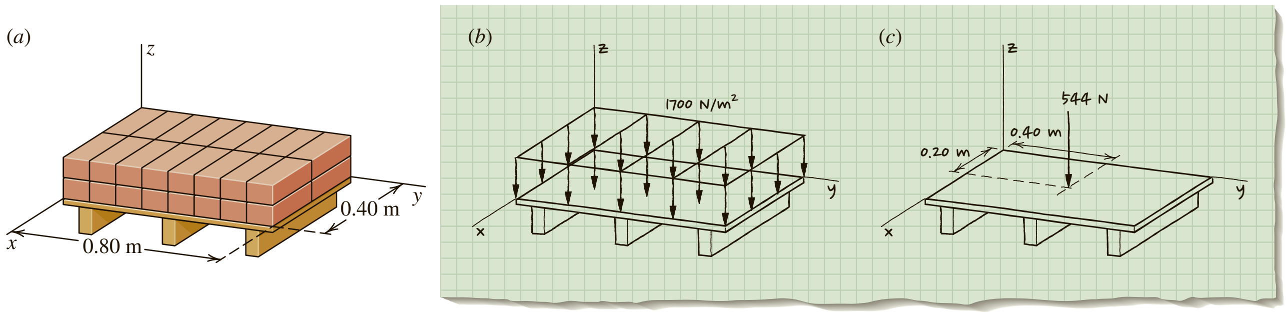


Figure 6.2.6 (*a*) Thirty-two bricks stacked over an area; (*b*) bricks represented as a uniform pressure applied to the area; (*c*) pressure represented as a single point force.

To represent this pressure load by a single point force, we first need to find the *total force*. We place this total force as a point force at the *location* where the total moment it creates on the system is the same as the moment created by the pressure load. For the uniform pressure load under consideration, the total load is 544 N, and it would be placed, considering symmetry, as a point load at the center of the distribution (**Figure 6.2.6*c***). We could figure this out mathematically by finding first the total force represented by the pressure load and then the location at which the total force acts.

1. *Total force.* This force is found by integrating the load over the surface area:

$$\text{Total force in } z \text{ direction} = F_z = \iint\limits_{\substack{\text{surface}\\\text{area}}} p\,dx\,dy \tag{6.18}$$

where p is the pressure load in dimensions of force/area, the integrand $p\;dx\;dy$ is the force acting on the small area $dx\;dy$, and "surface area" in the limit refers to the total area over which the pressure acts (**Figure 6.2.7*a***). Notice that a double integral is needed because we are integrating over an area.

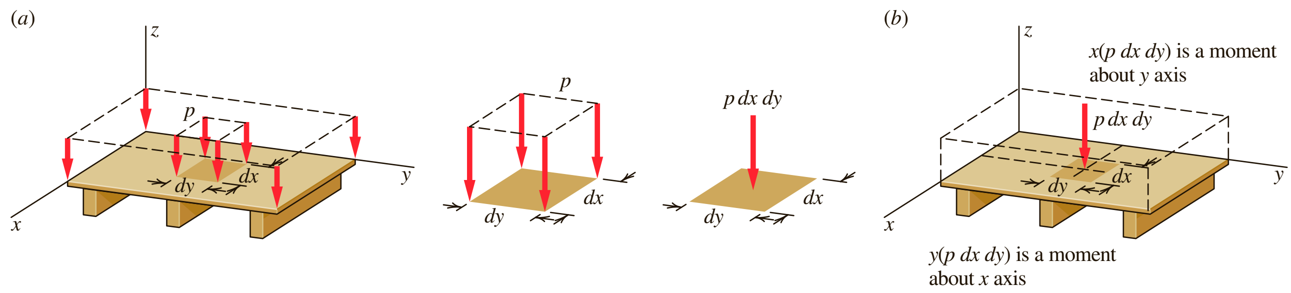


Figure 6.2.7 (*a*) Force over area $dx\,dy$ represented as pressure (p) or force $p\,dx\,dy$; (*b*) force over area $dx\,dy$ creates moments about x and y axes.

2. *Location.* This location at (X_C, Y_C) is called the **center of pressure** (or pressure center) and is found by requiring that the moment the total force produces is the same as the moment produced by the pressure load. The moment will have a component about the x axis and one about the y axis:

Moment about the x axis:

$$\underbrace{Y_C F_z}_{\substack{\text{moment}\\ \text{about } x \text{ axis}\\ \text{produced by}\\ \text{total } z \text{ force}}} = \underbrace{\iint\limits_{\substack{\text{surface}\\ \text{area}}} y(p\,dx\,dy)}_{\substack{\text{total moment about}\\ x \text{ axis produced}\\ \text{by pressure load}}} \tag{6.19A}$$

where Y_C is the distance in the y direction to the moment center and the integrand represents the sum of the infinitesimal moments about the x axis created by the infinitesimal forces $p\,dx\,dy$, as illustrated in **Figure 6.2.7*b***.

Moment about the y axis:

$$\underbrace{X_C F_z}_{\substack{\text{moment}\\ \text{about } y \text{ axis}\\ \text{produced by}\\ \text{total } z \text{ force}}} = \underbrace{\iint\limits_{\substack{\text{surface}\\ \text{area}}} x(p\,dx\,dy)}_{\substack{\text{total moment about}\\ y \text{ axis produced}\\ \text{by pressure load}}} \tag{6.19B}$$

where X_C is the distance in the x direction to the moment center and the integrand represents the sum of the infinitesimal moments about the y axis created by the infinitesimal forces $p\,dx\,dy$, as illustrated in **Figure 6.2.7*b***.

Equations (6.19A) and (6.19B) can be solved for Y_C and X_C, respectively.

$$Y_C = \frac{\iint\limits_{\substack{\text{surface}\\ \text{area}}} y\,p\,dx\,dy}{F_z} \underbrace{=}_{\substack{\text{substituting in}\\ \text{from (6.18)}}} \frac{\iint\limits_{\substack{\text{surface}\\ \text{area}}} y\,p\,dx\,dy}{\iint\limits_{\substack{\text{surface}\\ \text{area}}} p\,dx\,dy} \tag{6.20A}$$

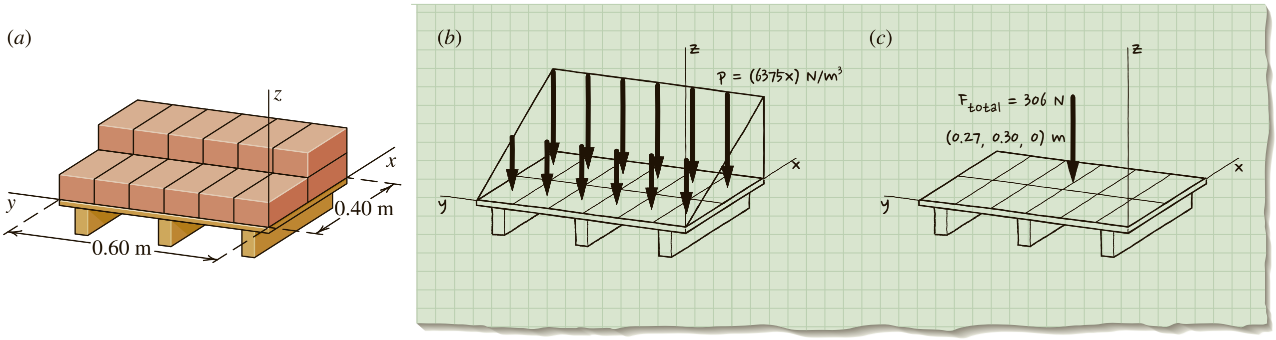


Figure 6.2.8 (*a*) Stepped brick stack; (*b*) approximate pressure distribution representing stepped brick stacking; (*c*) single point force that represents stepped stacking.

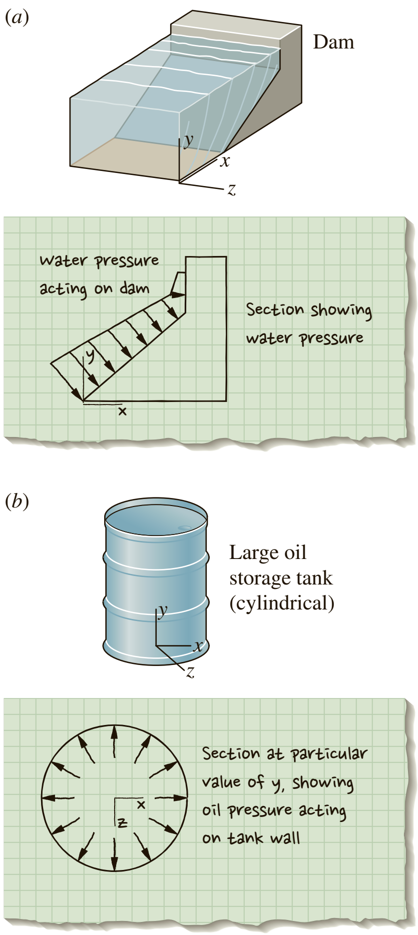


Figure 6.2.9 Examples of pressure acting on areas not aligned with *xyz* coordinate axes.

$$X_C = \frac{\displaystyle\iint_{\substack{\text{surface}\\\text{area}}} x\,p\,dx\,dy}{F_z} \underbrace{=}_{\substack{\text{substituting in}\\\text{from (6.18)}}} \frac{\displaystyle\iint_{\substack{\text{surface}\\\text{area}}} x\,p\,dx\,dy}{\displaystyle\iint_{\substack{\text{surface}\\\text{area}}} p\,dx\,dy} \tag{6.20B}$$

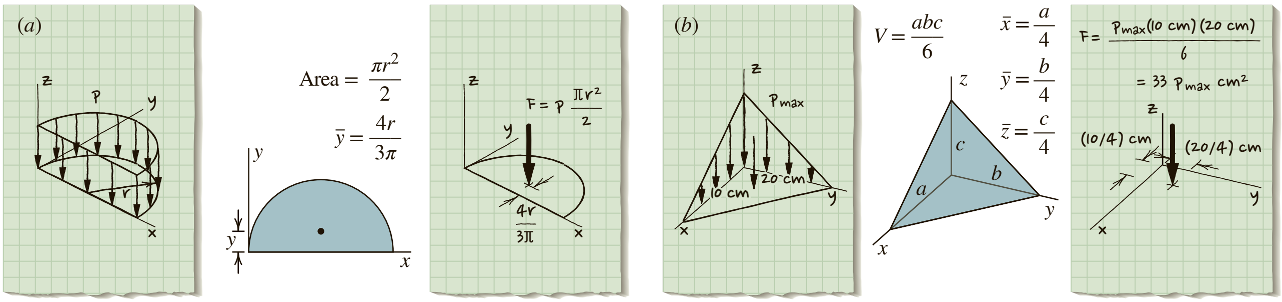
The coordinates (X_C, Y_C) found with (6.20A) and (6.20B) are the location of the center of pressure.

Equations (6.18) and (6.20) are also valid for a nonuniform pressure load. Consider the brick stacking shown in **Figure 6.2.8*a***. This stacking can be approximated by a linear distribution of the form $p = \beta_2 x$, where β_2 equals 6375 N/m^3 with the $x = 0$, $y = 0$ location as indicated in **Figure 6.2.8*b***. If this pressure distribution is inserted into (6.18) and (6.20), we find that the total load of the 18 bricks is 306 N and the point at which this total force acts is $X_C = 0.27$ m, $Y_C = 0.30$ m (see **Figure 6.2.8*c***).

The examples in **Figures 6.2.6** and **6.2.8** have the pressure aligned with one of the coordinate axes. Situations in which this is not the case are illustrated in **Figure 6.2.9**. To find the equivalent point force and its location, we must carefully select the orientation of the coordinate system. For example, to simplify calculations, we may want to define an x^*y^* coordinate system with x^* parallel to the sloped surface of the dam in **Figure 6.2.9*a***.

Finding the location of the center of pressure can be simplified if basic geometric shapes are involved. If, for example, a uniform pressure acts over a standard rectangular or triangular area, the information in Appendix C on area and centroid is useful (**Figure 6.2.10*a***). The total force is simply the product of the magnitude of the uniform pressure and the area, and is located at the centroid of the area. If, on the other hand, a nonuniform pressure forms a standard volume, the information in Appendix C on volume and centroid is useful (**Figure 6.2.10*b***).

If we can decompose a composite pressure distribution into one made up of N standard distributions, we can use knowledge of the location of pressure centers of the N standard pressure distributions to find the pressure center of the composite pressure distribution. This works for both uniform pressure distributions and for nonuniform pressure distributions.



(Appendix A.2) (Appendix A.3)

Figure 6.2.10 (*a*) A uniform pressure *p* acting over a semicircular area; (*b*) nonuniform pressure represented as single point force.

Check out the following examples of applications of this material.

- **Example 6.2.1 Using Integration to Find Total Force**
- **Example 6.2.2 Inclined Beam with Nonuniform Distribution**
- **Example 6.2.3 Beam Subjected to Polynomial Load Distribution**
- **Example 6.2.4 Using Properties of Standard Shapes to Find Total Force**
- **Example 6.2.5 Centroid of Distribution Composed of Standard Line Loads**
- **Example 6.2.6 Calculating Center of Pressure of a Pressure Distribution**
- **Example 6.2.7 Pressure on a Rectangular Water Gate**

EXAMPLE 6.2.1

A beam is subjected to a complex distributed load that can be divided into three standard line loads as shown in Figure 1. For each line load segment, determine the total force and the location of its centroid.

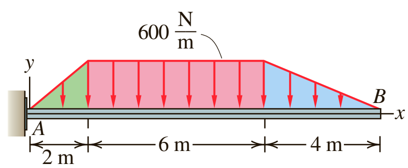


Figure 1 A cantilever beam subjected to a distributed load made up of standard line loads.

Goal For each of the three standard line load distributions find the total force and the location of the centroid.

Given Dimensions of the beam; type of support at *A*; information about the shape of the distributed load.

Assume System is planar, an upward force is positive and the weight of the beam is negligible and therefore does not contribute to the loads acting on the beam.

Draw We decompose the complex distributed load into three standard distributed loads to facilitate the analysis (Figure 2).

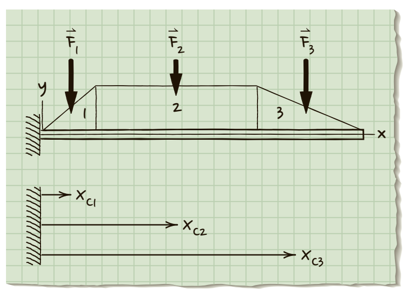


Figure 2 Each standard line load is represented by an equivalent total force at the centroid of the standard shape.

Formulate Equations and Solve For each line load we use (6.13) and (6.15) to find the equivalent total force and its location. We explore an alternative approach in Example 6.2.4 in which we use Appendix C for determining the total force and its location.

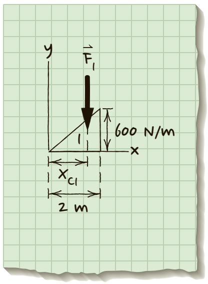


Figure 3 Segment 1 ($0 < x < 2$ m).

Segment 1 ($0 < x < 2$ m) (**Figure 3**): The distributed load ω_1 is described by

$$\omega_1 = \frac{-600\,\text{N/m}}{2\,\text{m}}x = -300\frac{\text{N}}{\text{m}^2}x$$

(with x given in meters). Substituting ω_1 into (6.13) results in

$$F_1 = \int_0^{2\text{ m}}\left(-300\frac{\text{N}}{\text{m}^2}x\right)dx = \left.\frac{-300\frac{\text{N}}{\text{m}^2}x^2}{2}\right]_0^{2\text{ m}} = \boxed{-600\text{ N}}$$

Substituting ω_1 into (6.15) gives:

$$X_{C1} = \frac{\int_0^{2\text{ m}} x\left(-300\frac{\text{N}}{\text{m}^2}x\right)dx}{F_1} = \left.\frac{-300\frac{\text{N}}{\text{m}^2}x^3}{3F_1}\right]_0^{2\text{ m}} = \left.\frac{-300\frac{\text{N}}{\text{m}^2}x^3}{3(-600\text{ N})}\right]_0^{2\text{ m}} = \boxed{\frac{4}{3}\text{ m}}$$

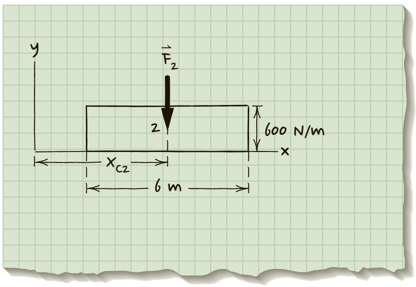


Figure 4 Segment 2 ($2\text{ m} < x < 8$ m).

Segment 2 ($2\text{ m} < x < 8$ m) (**Figure 4**): The distributed load ω_2 is described by $\omega_2 = -600$ N/m. Substituting ω_2 into (6.13):

$$F_2 = \int_{2\text{ m}}^{8\text{ m}}\left(-600\frac{\text{N}}{\text{m}}\right)dx = -600\frac{\text{N}}{\text{m}}x\Big]_{2\text{ m}}^{8\text{ m}} = \boxed{-3600\text{ N}}$$

Substituting ω_2 into (6.15):

$$X_{C2} = \frac{\int_{2\text{ m}}^{8\text{ m}} x\left(-600\frac{\text{N}}{\text{m}}\right)dx}{F_2} = \left.\frac{-600\frac{\text{N}}{\text{m}}x^2}{2F_2}\right]_{2\text{ m}}^{8\text{ m}} = \left.\frac{-600\frac{\text{N}}{\text{m}}x^2}{2(-3600\,\text{N})}\right]_{2\text{ m}}^{8\text{ m}} = \boxed{5\text{ m}}$$

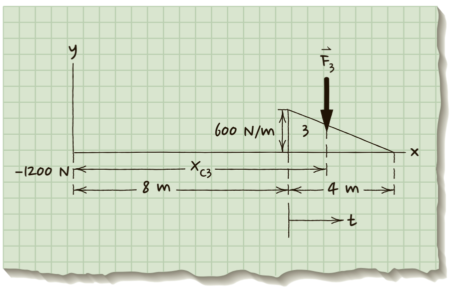


Figure 5 Segment 3 ($8\text{ m} < x < 12$ m).

Segment 3 ($8\text{ m} < x < 12$ m) (**Figure 5**): We define a temporary axis t with its origin at $x = 8$. This serves to simplify the integration by changing the limits of integration to 0 m to 4 m from 8 m to 12 m. The distributed load ω_3 is then described by

$$\omega_3 = -600\frac{\text{N}}{\text{m}} + \frac{600\frac{\text{N}}{\text{m}}}{4\,\text{m}}t = -600\frac{\text{N}}{\text{m}} + 150\frac{\text{N}}{\text{m}^2}t$$

Substituting ω_3 into (6.13):

$$F_3 = \int_{0\text{ m}}^{4\text{ m}}\left(-600\frac{\text{N}}{\text{m}} + 150\frac{\text{N}}{\text{m}^2}t\right)dt = -600\frac{\text{N}}{\text{m}}t + \left.\frac{150\frac{\text{N}}{\text{m}^2}t^2}{2}\right]_{0\text{ m}}^{4\text{ m}} = \boxed{-1200\text{ N}}$$

Substituting ω_3 into (6.15):

$$T_C = \frac{\int_{0\text{ m}}^{4\text{ m}} t\left(-600\frac{\text{N}}{\text{m}} + 150\frac{\text{N}}{\text{m}^2}t\right)dt}{F_3} = \left.\frac{-600\frac{\text{N}}{\text{m}}\frac{t^2}{2} + 150\frac{\text{N}}{\text{m}^2}\frac{t^3}{3}}{-1200\text{ N}}\right]_{0\text{ m}}^{4\text{ m}} = \frac{4}{3}\text{ m}$$

$$X_{C3} = 8\text{ m} + T_C = 8\text{ m} + \frac{4}{3}\text{ m} = \boxed{\frac{28}{3}\text{ m}}$$

Check The total force and centroid for each standard load can be checked by comparing with calculations using Appendix C (see Example 6.2.4).

EXAMPLE 6.2.2

The inclined beam in **Figure 1** is subjected to the vertical force distribution shown. The value of the load distribution at B is ω_0 (force units per horizontal length unit). Determine the loads acting on the beam at supports A and B.

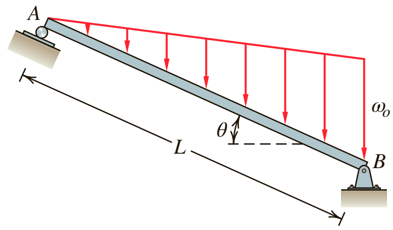


Figure 1 Inclined beam subjected to vertical distributed load.

Goal Calculate the loads acting on the beam at A and B.

Given Length and orientation of beam, the types of supports at A and B, and information to describe the distributed load.

Assume The system is planar, upward force is positive, and that the weight of the beam is negligible.

Draw A free-body diagram of the beam is shown in **Figure 2**. We have established two coordinate systems, one aligned with the horizontal and vertical (xy), which we will use for calculating F, the other along the slope of the inclined beam (x^* y^*), which makes calculating the forces at A and B less complex.

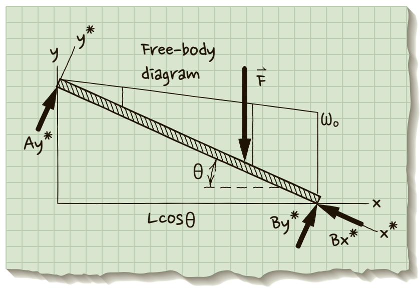


Figure 2 Free-body diagram of beam.

Formulate Equations and Solve First we find the total equivalent force and its location. Next we apply equilibrium conditions to find the loads acting at A and B. We use two approaches. In the first we use (6.13) and (6.15) to find the equivalent total force and its location. In the second approach we use the information in Appendix C for a standard line load distribution.

Approach 1: We begin by determining $\omega(x)$. Using a linear distribution with unknown constants k_0 and k_1 to represent the load distribution:

$$\omega(x) = k_o + k_1 x$$

and applying the boundary conditions $\omega(0) = 0$ and $\omega(L\cos\theta) = -\omega_o$ to determine the constants results in

$$\omega(0) = k_o + k_1(0) = 0 \qquad k_o = 0$$

$$\omega(L\cos\theta) = k_o + k_1(L\cos\theta) = -\omega_o \qquad k_1 = -\omega_0/(L\cos\theta)$$

Therefore, $\omega(x) = \dfrac{-\omega_o}{L\cos\theta}x$

Using (6.13) to find the total force,

$$F = \int_0^{L\cos\theta} \omega(x)\,dx = \int_0^{L\cos\theta} \frac{-\omega_o}{L\cos\theta}x\,dx = \frac{-\omega_o}{L\cos\theta}\frac{x^2}{2}\Bigg]_0^{L\cos\theta}$$

$$= \frac{-\omega_o}{2}\frac{L^2\cos^2\theta}{L\cos\theta} = \frac{-\omega_o L\cos\theta}{2} \qquad (1)$$

Using (6.15) to find the centroid,

$$X_C = \frac{\int_0^{L\cos\theta} \omega(x)\,x\,dx}{F} = \frac{\int_0^{L\cos\theta} \frac{-\omega_o}{L\cos\theta} x^2 dx}{F} = \frac{\frac{-\omega_o}{L\cos\theta}\frac{x^3}{3}\Big]_0^{L\cos\theta}}{F} = \frac{\frac{-\omega_o L^2\cos^2\theta}{3}}{\frac{-\omega_o L\cos\theta}{2}} = \frac{2}{3}L\cos\theta$$

Approach 2: We use Appendix C to find the total equivalent force and its location. Because ω_0 is given in force units per horizontal length, we must use the horizontal length ($L\cos\theta$) as the base of our triangle.

$$F = \frac{bh}{2} = \frac{(L\cos\theta)(-\omega_o)}{2} = \frac{-\omega_o L\cos\theta}{2}$$

The centroid is located at 2/3 the distance from the vertex:

$$X_C = \frac{2}{3}L\cos\theta$$

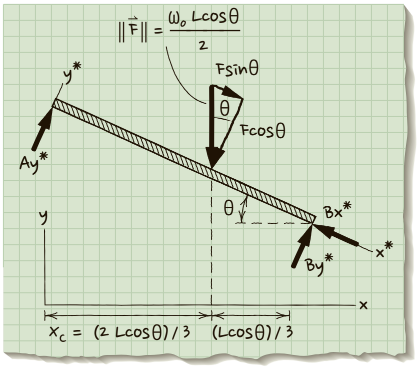


Figure 3 The total force F is located at 2/3 $L\cos\theta$ to the right of A.

These values, the same as we calculated using the integral approach, are shown in **Figure 3**, which is also a free-body diagram of the beam.

To find the loads at supports A and B, we refer to the free-body diagram and use a coordinate system that has its x^* axis along the beam to apply equilibrium equations parallel and perpendicular to the beam.

$$\sum F_{x^*} = 0 = F\sin\theta - B_{x^*}$$

We substitute for F from (1), noting from **Figure 3** that $F\sin\theta$ is in the *positive* x^* direction. There are two sign conventions in play here, one that says negative loads act downward, and a second that we are using to sum forces in the x^* direction.

$$B_{x^*} = F\sin\theta = \left(\frac{\omega_o L\cos\theta}{2}\right)\sin\theta = \frac{\omega_0 L\cos\theta\sin\theta}{2} \qquad (2)$$

Using B as the moment center gives

$$\sum M_{z@B} = 0(\curvearrowleft +) = -A_{y^*}L + F\cos\theta(L/$$

$$A_{y^*} = \frac{F\cos\theta}{3} = \frac{\omega_o L\cos\theta}{2}\frac{\cos\theta}{3} = \boxed{\frac{\omega_o L\cos^2\theta}{6}} \qquad (3)$$

$$\sum F_{y^*} = 0 = A_{y^*} + B_{y^*} - F\cos\theta$$

$$B_{y^*} = F\cos\theta - A_{y^*}$$

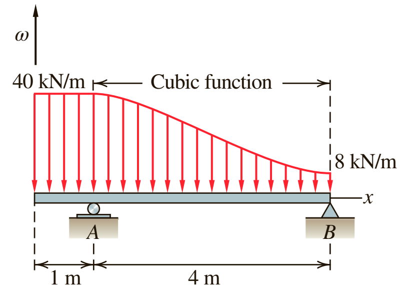
Substituting for F from (1) and for A_{y^*} from (3) gives

$$B_{y^*} = \frac{\omega_o L\cos\theta}{2}\cos\theta - \frac{\omega_o L\cos\theta}{6} = \boxed{\frac{\omega_o L\cos^2\theta}{3}} \qquad (4)$$

Check The solution can be checked by summing the moments about A, and plugging in the results. In addition, note that B_{y^*} is twice as big as A_{y^*}; this is as expected since F is located two times closer to B than to A.

EXAMPLE 6.2.3

A beam is subjected to the complex load shown in **Figure 1**. The given cubic polynomial load extends between the 40-kN/m load and the 8-kN/m load. Determine the loads at supports A and B.



Cubic:

$\omega(x) = k_0 + k_1 x + k_2 x^2 + k_2 x^3$

Slope = 0 at $x = 1$ m and $x = 5$ m

Figure 1 Load distribution made up of a standard line load and a load distributed according to a polynomial.

Goal Find the loads at supports A and B.

Given Dimensions of beam; types of supports at A and B; information about magnitude and shape of the load distribution.

Assume The system is planar, the weight of the beam is negligible, and a positive load acts upward.

Draw We decompose the distributed load into two segments representing the uniform load and the cubic load (**Figure 2**).

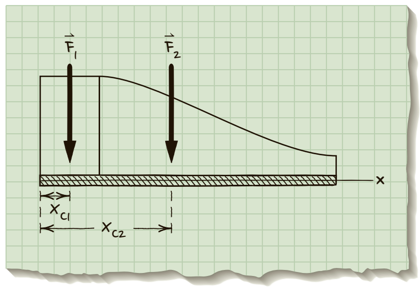


Figure 2 Divide the load into two segments.

Formulate Equations and Solve

Segment 1 (rectangle) (**Figure 3**):

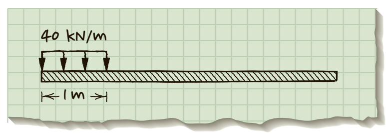


Figure 3 Segment 1 (rectangular).

$$F_1 = (1\,\text{m})(-40\,\text{kN/m}) = -40.0\,\text{kN}$$

The point at which the total force F_1 is applied is

$$X_{C1} = \frac{1\,\text{m}}{2} = 0.50\,\text{m}$$

Segment 2 (cubic line load) (**Figure 4**): We first solve for the coefficients of the cubic polynomial using the boundary conditions given for the distributed load. We define a new variable t to simplify the calculations:

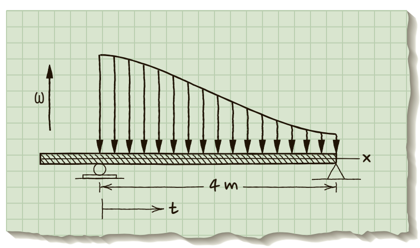


Figure 4 Segment 2 (cubic).

$$\omega(t) = k_0 + k_1 t + k_2 t^2 + k_3 t^3$$

$$\frac{d\omega}{dt}(t) = k_1 + 2k_2 t + 3k_3 t^2$$

Plug the conditions into the polynomial and its first derivative:

$$\text{At } t = 0\text{ m} \rightarrow \omega = -40\ \text{kN/m} \quad \text{and} \quad \frac{d\omega}{dt} = 0$$

$$k_0 = -40 \qquad (1)$$

$$k_1 = 0 \qquad (2)$$

$$\text{At } t = 4\text{ m} \rightarrow \omega = -8\ \text{kN/m} \quad \text{and} \quad \frac{d\omega}{dt} = 0$$

$$-40 + 16k_2 + 64k_3 = -8 \qquad (3)$$

$$8k_2 + 48k_3 = 0 \qquad (4)$$

Solving equation (4) and substituting into (3) gives:

$$k_2 = -6k_3$$

$$-40 + 16(-6k_3) + 64k_3 = -8$$

$$k_3 = -1 \quad \text{and} \quad k_2 = -6$$

Therefore, for segment 2:

$$\omega(t) = -40 + 6t^2 - t^3$$

We now calculate F_2 and X_{C2} for segment 2 using (6.13) and (6.15):

$$F_2 = \int_{0\text{ m}}^{4\text{ m}} \omega(t)\,dt = \int_{0\text{ m}}^{4\text{ m}} (-40 + 6t^2 - t^3)\,dt$$

$$F_2 = \left[-40t + 2t^3 + \frac{t^4}{4}\right]_{0\text{ m}}^{4\text{ m}} = -96.0\text{ kN}$$

$$T_{C2} = \frac{\int_{0\text{ m}}^{4\text{ m}} t\omega(t)\,dt}{F_2} = \frac{\int_{0\text{ m}}^{4\text{ m}} t(-40 + 6t^2 - t^3)\,dt}{-96.0\text{ kN}}$$

$$T_{C2} = \frac{\left.\frac{-40}{2}t^2 + \frac{6}{4}x^4 - \frac{1}{5}x^5\right]_{0\text{ m}}^{4\text{ m}}}{-96.0\text{ kN}} = \frac{-140.8\text{ kN}\cdot\text{m}}{-96.0\text{ kN}} = 1.47\text{ m}$$

$$X_{C2} = T_{C2} + 1.00\text{ m} = 2.47\text{ m}$$

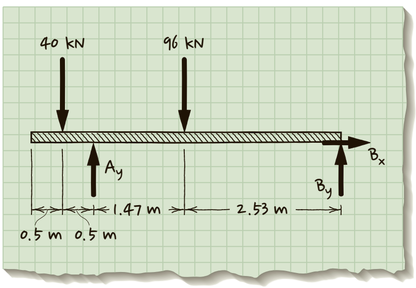


Figure 5 Free-body diagram of beam *AB*.

We now draw a free-body diagram of the beam with the two total equivalent loads so that we have a clear picture of the dimensions to use in the equilibrium equations (**Figure 5**).

$$\sum F_x = 0(\rightarrow +) = B_x = \boxed{0\text{ kN}}$$

$$\sum M_{z@A} = 0(\curvearrowleft +) = B_y(4\text{ m}) + 40\text{ kN}(0.5\text{ m}) - 96\text{ kN}(1.47\text{ m})$$

$$B_y = \frac{[(-40)(0.5) + (96)(1.47)]\text{ kN}\cdot\text{m}}{4\text{ m}} = \frac{120.8}{4}\text{kN} = \boxed{30.2\text{ kN}}$$

$$\sum F_y = 0(\uparrow +) = A_y + B_y - 40\text{ kN} - 96\text{ kN} = 0$$

$$A_y + 136\text{ kN} - 30.2\text{ kN} = \boxed{105.8\text{ kN}}$$

Check We check the results by summing the moments about *B*.

$$\sum M_{z@B} = 0(\curvearrowleft +) = 40.0\text{ kN}(4.5\text{ m}) - 105.8\text{ kN}(4\text{ m}) + 96.0\text{ kN}(2.53\text{ m}) = -0.32\text{ kN}\cdot\text{m}$$

We note that the moments don't sum to exactly zero; instead, there is a small residual of 0.32 kN · m. This is due to round-off error because we carried only three significant digits in our solution. This residual is quite small, about 0.2% of the moments we are calculating, so we can accept our solution as correct.

While we could carry more significant digits and reduce the residual, it is probably not justified. In most engineering applications there are many sources of uncertainty and error such as the magnitude of the loads, the properties of the materials, and the exact dimensions of the system; thus we tolerate some inaccuracy in our solutions. Acceptable tolerance levels are often specified by the system designer. Tolerances for the space shuttle would be much smaller than those for a desk chair.

EXAMPLE 6.2.4

Reexamine the beam of Example 6.2.1 (**Figure 1**). For each line load segment, use Appendix C to determine the total force and the location of its centroid. Also determine the loads acting on the beam at A, assuming the beam is in equilibrium.

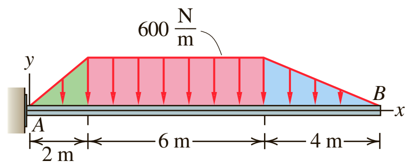


Figure 1 A cantilever beam subjected to a distributed load made up of standard line loads.

Goal For each of the three standard line load distributions find the total force and the location of the centroid. In addition, determine the loads acting at support A using static analysis.

Given Dimensions of the beam; type of support at A; information about the shape of the distributed load.

Assume System is planar, an upward force is positive, and the weight of the beam is negligible.

Draw As in Example 6.2.1, we first decompose the complex distributed load into three standard distributed loads to facilitate the analysis (**Figures 2, 3 and 4**). Later we will need to draw a free-body diagram of the beam.

Formulate Equations and Solve (a) For each line load we use we use Appendix C to find the equivalent total force and its location. We then use the conditions of equilibrium to determine the loads acting at A.

Segment 1: Triangular line load. The total force is the area of the triangle ($A = bh/2$, with $b = 2$ m and $h = -600$ N/m:

$$F_1 = (2\text{ m})(-600\text{ N/m})/2 = \boxed{-600\text{ N}}$$

The location of the total force is 2/3 of the distance from the triangle vertex, as shown in **Figure 2**.

$$X_{C1} = \frac{2}{3}(b) = \frac{2}{3}(2) = \boxed{\frac{4}{3}\text{ m}}$$

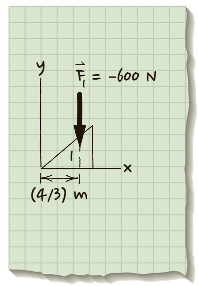


Figure 2 Segment 1: Triangular: line load.

Segment 2: Rectangular line load. The total force is the area of the rectangle ($A = bh$), with $b = 6$ m and $h = -600$ N/m:

$$F_2 = (6\text{ m})(-600\text{ N/m}) = \boxed{-3600\text{ N}}$$

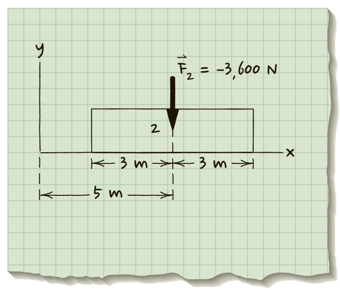


Figure 3 Segment 2: Rectangular line load.

The location of the total force is 1/2 of the distance along edge b, as shown in **Figure 3**. We must include the 2 m from the left end of the beam to the edge of the rectangle when calculating the location of X_{C2}.

$$X_{C2} = 2\text{m} + \frac{6}{2}\text{m} = \boxed{5\text{ m}}$$

Segment 3: Triangular line load. The total force is the area of the triangle:

$$F_3 = (4\text{ m})(-600\text{ N/m})/2 = F_3 = \boxed{-1200\text{ N}}$$

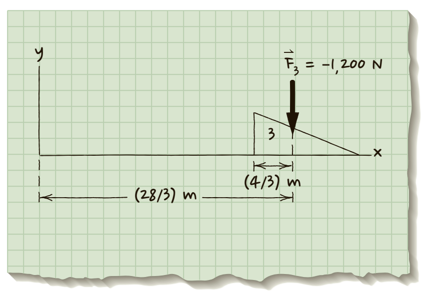


Figure 4 Segment 3: Triangular line load.

The location of this force is the 8 m from the left end of the beam to the triangle plus 1/3 of the distance from the large side of the triangle, as shown in **Figure 4**.

$$X_{C3} = 8\text{ m} + \frac{1}{3}(4\text{ m}) = \boxed{\frac{28}{3}\text{ m}}$$

This approach to finding the forces and their locations results in the same answer as we found using (6.13) and (6.15) in Example 6.2.1, and requires a much simpler set of calculations. Therefore, if it is reasonable to model a distributed load as one of the standard distributions in Appendix C, by all means do so.

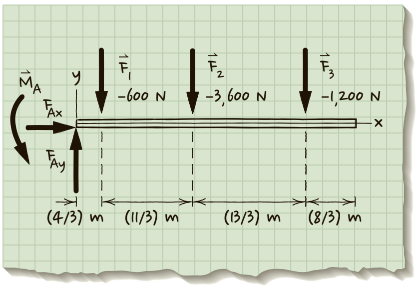


Figure 5 Free-body diagram of Beam *AB*.

(b) To find the loads at support A we start by drawing a free-body diagram (**Figure 5**) with the distributed load modeled by the equivalent total forces we calculated previously.

When we set up the equations for planar equilibrium to find the unknown loads at A, we arbitrarily choose A as the moment center. We could choose B as the moment center and obtain the same result.

Based on (5.5A), (5.5B), and (5.5C):

$$\sum F_x = 0(\rightarrow +)$$

$$\boxed{F_{Ax} = 0}$$

$$\sum M_{Z@A} = 0(\curvearrowleft +)$$

$$(-600\,\text{N})\left(\frac{4}{3}\text{ m}\right) - (3600\,\text{N})(5\,\text{m}) - (1200\,\text{N})\left(\frac{28}{3}\text{ m}\right) + M_A = 0$$

$$\boxed{M_A = 30{,}000\,\text{N}\cdot\text{m}}$$

$$\sum F_y = 0(\uparrow +)$$

$$F_{Ay} - 600\,\text{N} - 3600\,\text{N} - 1200\,\text{N} = 0$$

$$\boxed{F_{Ay} = 5400\,\text{N}}$$

Check To check our solution we use (5.5C) to sum moments about a different moment center. For example

$$\sum M_{Z@B} = 0(\curvearrowleft +)$$

$$M_A - F_{Ay}(12\,\text{m}) + (600\,\text{N})\left(\frac{32}{3}\text{ m}\right) + (3600\,\text{N})(7\,\text{m}) + (1200\,\text{N})\left(\frac{8}{3}\text{ m}\right) = 0$$

$$30{,}000\,\text{N}\cdot\text{m} - (5400\,\text{N})(12\,\text{m}) + (600\,\text{N})\left(\frac{32}{3}\text{ m}\right) + (3600\,\text{N})(7\,\text{m}) + (1200\,\text{N})\left(\frac{8}{3}\text{ m}\right) = 0$$

$$0 = 0$$

Yes, the beam is in equilibrium.

EXAMPLE 6.2.5

A beam is subjected to the loads represented by the load diagram in Figure 1. Determine the total force acting on the beam and locate its line of action with respect to support A.

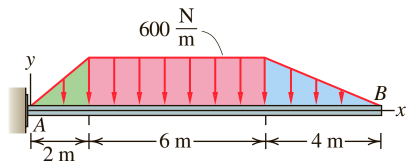


Figure 1 This distributed load can be represented by a single force at its centroid.

Goal Calculate the total equivalent force acting on the beam and determine the point at which it acts.

Given Information about the type of support at A, and the geometry of the beam and distributed load.

Assume The system is planar, the weight of the beam is negligible, and upward force is positive.

Draw From Example 6.2.1 we have the magnitudes and locations of the total equivalent loads for the three different segments (Figure 2*a*).

Formulate Equations and Solve For each standard line load distribution, we find the total equivalent force and its location. This step was completed in Examples 6.2.1 and 6.2.4. We use these results in (6.16A) and (6.16B) to find the total equivalent force and its location for this complex load distribution. From **Figure 2*a***,

$$F_1 = -600\,\text{N},\ X_{C1} = 4/3\ \text{m}$$
$$F_2 = -3600\,\text{N},\ X_{C2} = 5\ \text{m}$$
$$F_3 = -1200\,\text{N},\ X_{C3} = 28/3\ \text{m}$$

We use these values in (6.16A) and (6.16B) to find F_{tot} and X_C:

$$F_{\text{total}} = \sum_{i=1}^{3} F_i = -600\,\text{N} - 3600\,\text{N} - 1200\,\text{N} = \boxed{-5400\,\text{N}}$$

$$X_C = \frac{\sum_{i=1}^{3} F_i X_{Ci}}{F_{\text{total}}} = \frac{-600\,\text{N}\left(\frac{4}{3}\text{m}\right) - 3600\,\text{N}(5\,\text{m}) - 1200\,\text{N}\left(\frac{28}{3}\text{m}\right)}{-5400\,\text{N}} = 5.56\,\text{m}$$

$$X_C = 5.56\,\text{m}$$

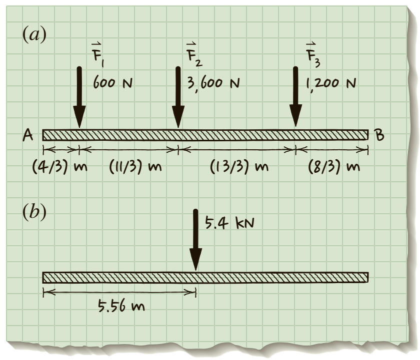


Figure 2 (*a*) Three concentrated loads at the centroids of the standard line loads, (*b*) a single equivalent load at the centroid of the distribution.

Check The equivalent total force is shown in **Figure 2*b***. It is reasonable that the location of the equivalent load should be in the left half of the beam. The load is not symmetric with respect to the centerline of the beam, and a larger portion of the load sits on the left half.

We could check our answer by calculating the centroid of the distribution measured with respect to B.

EXAMPLE 6.2.6

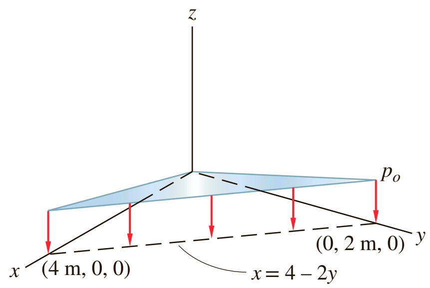


Figure 1 Linearly varying pressure distribution.

The area shown in **Figure 1** is bounded by the y and x axes and the line $x = 4 - 2y$ m. The area is acted upon by a pressure distribution that varies linearly from the origin of the axes to a maximum value of p_o (N/m^2) at $x = 4 - 2y$ m. Find the total equivalent force F_T and the location at which it acts.

Goal Find the total equivalent force acting on the area and the center of pressure.

Given Parameters of a linearly varying pressure distribution and the bounds of the triangular area over which it acts.

Assume No assumptions required.

Draw No drawings required.

Formulate Equations and Solve We use (6.18) to find the total force. We describe the pressure distribution by the linear equation

$$p(x, y) = p_0\left(\frac{y}{2} + \frac{x}{4}\right)\text{N/m}^2$$

First integrating with respect to x and then with respect to y we find F_T.

$$F_T = \int_{\text{surface}}\int_{\text{area}} p(x, y)\,dx\,dy = \int_0^2\int_0^{4-2y} P_0\left(\frac{y}{2} + \frac{x}{4}\right)dx\,dy$$

$$F_T = p_o\int_0^2\left[\frac{xy}{2} + \frac{x^2}{8}\right]_0^{4-2y} dy = p_o\int_0^2\left(2 - \frac{y^2}{2}\right)dy = p_o\left[2y - \frac{y^3}{6}\right]_0^2 = \boxed{\frac{8p_o}{3}\text{N}}$$

We use (6.20A) and (6.20B) to find the centroid of the pressure distribution. Based on (6.20A) we write

$$Y_C = \frac{\displaystyle\int_{\text{surface area}}\int y\,p(x, y)\,dx\,dy}{F_T} = \frac{\displaystyle\int_0^2\int_0^{4-2y} P_0 y\left(\frac{y}{2} + \frac{x}{4}\right)dx\,dy}{F_T}$$

$$= \frac{p_o}{F_T}\int_0^2\left[\frac{y^2x}{2} + \frac{yx^2}{8}\right]_0^{4-2y} dy = \frac{p_0}{F_T}\int_0^2\left(\frac{-y^3}{2} + 2y\right)dy$$

$$= \frac{p_o}{F_T}\left[\frac{-y^4}{8} + y^2\right]_0^2 = \frac{2p_0}{F_T} = \frac{2p_o}{8p_o/3} = \boxed{\frac{3}{4}\text{m}}$$

Based on (6.20B) we write

$$X_C = \frac{\displaystyle\int_{\text{surface area}}\int x\,p(y, x)\,dx\,dy}{F_T} = \frac{\displaystyle\int_0^2\int_0^{4-2y} P_0 x\left(\frac{y}{2} + \frac{x}{4}\right)dx\,dy}{F_T}$$

$$= \frac{p_o}{F_T}\int_0^2\left[\frac{yx^2}{4} + \frac{x^3}{12}\right]_0^{4-2y} dy = \frac{p_0}{F_T}\int_0^2\left(\frac{y^3}{3} - 4y + \frac{16}{3}\right)dy$$

$$= \frac{p_o}{F_T}\left[\frac{y^4}{12} - 2y^2 + \frac{16y}{3}\right]_0^2 = \frac{4p_0}{F_T} = \frac{4p_o}{8p_o/3} = \boxed{\frac{3}{2}\text{m}}$$

Check We check to see whether our results seem reasonable by comparing them to those for a constant pressure p_o acting over the triangular area, which we can quickly calculate from Appendix C. If the pressure were constant p_o over the triangular area, the total force would be $4p_o$ and we know that the total force for the linearly varying pressure must be smaller. The total load of $8p_o/3$ seems reasonable for our linearly varying pressure, since it is a little more than half of $4p_o$.

For the constant pressure p_o the center of pressure is located at (x = 4/3 m, y = 2/3 m). It seems reasonable that the center of pressure for the linearly varying pressure (x = 3/2 m, y = 3/4 m) would be farther from the origin of the coordinate system, since the load is distributed so that a larger portion of the load is away from the origin.

EXAMPLE 6.2.7

Figure 1 shows a cross section through a rectangular gate that is $h = 30$ ft high and $w = 8$ ft wide (where the width dimension is perpendicular to the yz plane). The gate is subjected to a load generated by fresh water stored to a depth of $d = 25$ ft in a reservoir behind the gate. The pressure that the water applies to the gate varies linearly, as indicated in **Figure 2**. Determine the magnitude of the total force $\boldsymbol{F}$ exerted on the gate by the water and its location with respect to the hinge.

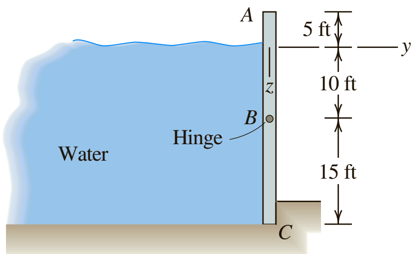


Figure 1 A rectangular gate controls the height of water in a reservoir.

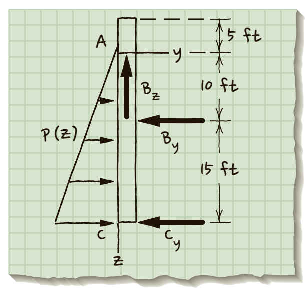


Figure 2 The water pressure applied to the gate increases linearly with depth.

Goal Find the magnitude of the total force of the water on the gate and the location of the centroid of the pressure distribution with respect to the hinge.

Given Information about the geometry of the rectangular water gate and the depth of the fresh water.

Assume The weight of the gate is negligible (we aren't given a weight), the hinge is frictionless, the fluid is static, and the system is in equilibrium.

Draw Based on the information given in the problem and our assumptions, we draw a free-body diagram (**Figure 2**) of the gate. We have selected positive z acting downward, meaning that the x axis acts into the page, and locate the origin of the coordinate system at the top of the water. B_y and B_z are the forces of the hinge acting on the gate, and C_y is the force due to the sill pushing on the bottom of the gate.

Formulate Equations and Solve The pressure distribution varies according to

$$p(z) = \gamma_{\text{wat}} z$$

The specific weight of water, $\gamma_{\text{wat}} = \rho g = 62.4$ lb/ft^3. Therefore, $p(z) = 62.4z$ in units of lb/ft^2.

From (6.18), the total force of the water acting on the gate is

$$F = \int_0^{25\text{ ft}}\int_0^{8\text{ ft}} p(z)\,dx\,dz \int_0^{25\text{ ft}}\int_0^{8\text{ ft}}\left(62.4\frac{\text{lb}}{\text{ft}^3}\right)z\,dx\,dz = \int_0^{25\text{ ft}}\left(62.4\frac{\text{lb}}{\text{ft}^3}\right)[xz]_0^{8\text{ft}}\,dz$$

$$F = \left(62.4\frac{\text{lb}}{\text{ft}^3}\right)(8\text{ ft})\int_0^{25\text{ ft}} z\,dz = \left(499.2\frac{\text{lb}}{\text{ft}^2}\right)\left[\frac{z^2}{2}\right]_0^{25\text{ ft}} = 156{,}000\text{ lb}$$

$$F = 156\,\text{kip}$$

We now calculate Z_C, the distance to the pressure center, by rewriting (6.20B) in terms of z, instead of y.

$$Z_C = \frac{\int_0^{25\text{ ft}}\int_0^{8\text{ ft}} zp(z)\,dx\,dz}{F} = \frac{\int_0^{25\text{ ft}}\int_0^{8\text{ ft}}\left(62.4\frac{\text{lb}}{\text{ft}^3}\right)z^2\,dx\,dz}{F}$$

$$Z_C = \frac{\left(62.4\frac{\text{lb}}{\text{ft}^3}\right)(8\text{ ft})\int_0^{25\text{ ft}} z^2\,dz}{F} = \frac{\left(499.2\frac{\text{lb}}{\text{ft}^2}\right)\left[\frac{z^3}{3}\right]_0^{25\text{ ft}}}{F}$$

$$= \frac{2{,}600{,}000\,\text{lb}\cdot\text{ft}}{156{,}000\,\text{lb}} = 16.67\,\text{ft}$$

Z_C is measured with respect to the top of the water. We find Z_{hinge} (the distance between the hinge and the centroid) using

$$\begin{aligned} Z_{\text{hinge}} &= Z_C - \text{distance from top of water to hinge} \\ &= 16.67\,\text{ft} - 10\,\text{ft} = 6.67\,\text{ft} \end{aligned}$$

A free-body diagram of the gate is shown in **Figure 3**.

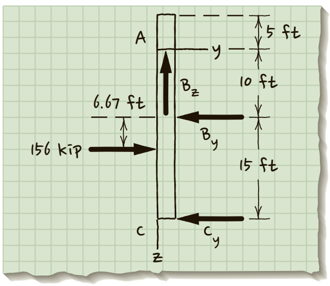


Figure 3 A free-body diagram of the gate shows the total force of the water acting at the center of pressure.

Check We can use Appendix C to check our results because the pressure distribution can be modeled as a standard line load distribution. We multiply the pressure distribution by the width of the gate to calculate the force per unit length along the height of the gate.

$$\omega_{\max} = (62.4\,\text{lb/ft}^3)(25\,\text{ft})(8\,\text{ft}) = 12{,}480\,\text{lb/ft}$$

Based on the triangular distribution in Appendix C, we find

$$F = \frac{(25\,\text{ft})(12{,}480\,\text{lb/ft})}{2} = 156{,}000\,\text{lb}$$

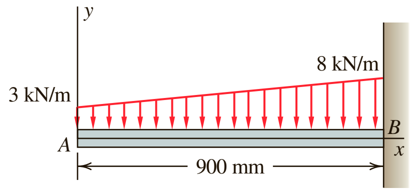
$$Z_C = \frac{2(25\,\text{ft})}{3} = 16.67\,\text{ft}$$

$$Z_{\text{hinge}} = 16.67\,\text{ft} - 10\,\text{ft} = 6.67\,\text{ft}$$

Yes, our answer checks!

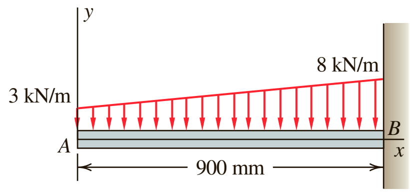
EXERCISES 6.2

6.2.1. [*] A line load acts on the top of beam *AB*, as shown. Use integration to determine the point force and its location (centroid) that are equivalent to the line load.



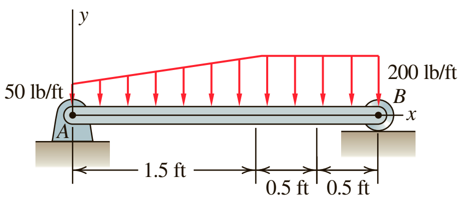
EX 6.2.1

6.2.2. [*] Use the information in Appendix C to determine the point load and its location that are equivalent to the line load acting on the beam.



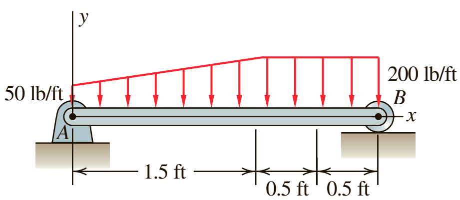
EX 6.2.2

6.2.3. [*] A line load acts on the top of beam *AB*, as shown. Use integration to determine an equivalent point load and its location.



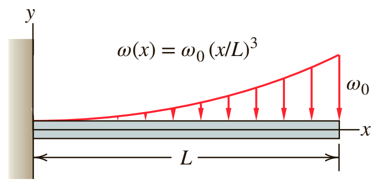
EX 6.2.3

6.2.4. [*] Use the information in Appendix C to determine the point load and its location that are equivalent to the line load acting on the beam.



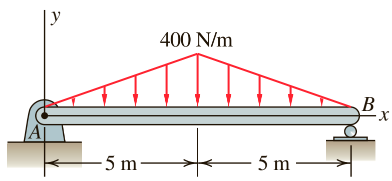
EX 6.2.4

6.2.5. [*] Use integration to determine the point load and its location (centroid) that are equivalent to the line load shown.



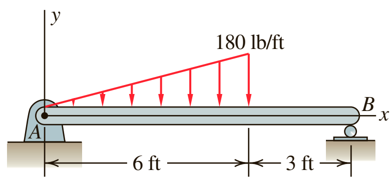
EX 6.2.5

6.2.6. [*] Use integration to determine an equivalent point load for the line load and its location.



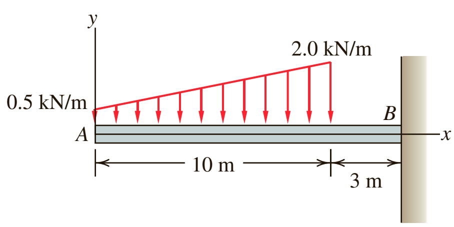
EX 6.2.6

6.2.7. [*] Use integration to determine the point load and its location (centroid) that are equivalent to the line load acting on the beam.



EX 6.2.7

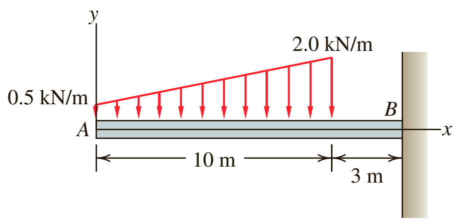
6.2.8. [*] Use integration to determine the point load and its location (centroid) that are equivalent to the line load acting on the beam.



EX 6.2.8

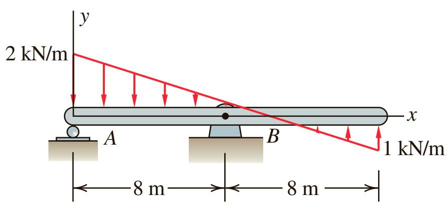
6.2.9. [*] a. Use the information in Appendix C to determine the point load and its location (centroid) that are equivalent to the line load.

b. Determine the loads acting on the beam at the wall if the beam is in equilibrium. Assume that the weight of the beam is negligible.



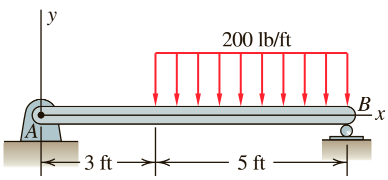
EX 6.2.9

6.2.10. [*] Use integration to determine the point load and its location (centroid) that are equivalent to the line load acting on the beam.



EX 6.2.10

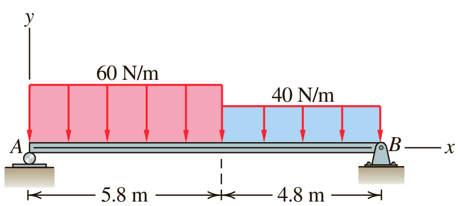
6.2.11. [*] The load acting along the top of the beam between 3 ft < x < 8 ft is a uniform line load. Use integration to determine the point force and its location (centroid) that are equivalent to the line load.



EX 6.2.11

6.2.12. [*] a. Use the information in Appendix C to determine an equivalent point load for the line load and its location.

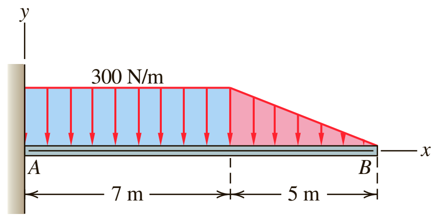
b. Determine the loads acting on the beam at A and B if the beam is in equilibrium. Assume that the weight of the beam is negligible.



EX 6.2.12

6.2.13. [*] a. Use the information in Appendix C to determine the point load and its location that are equivalent to the line load acting on the beam.

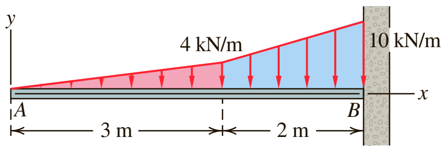
b. Determine the loads acting on the beam at the wall if the beam is in equilibrium. Assume that the weight of the beam is negligible.



EX 6.2.13

6.2.14. [*] a. Use the information in Appendix C to determine the point load and its location that are equivalent to the line load acting on the beam.

b. Determine the loads acting on the beam at the wall if the beam is in equilibrium. Assume that the weight of the beam is negligible.

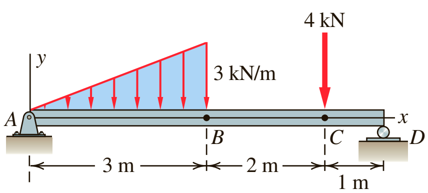


EX 6.2.14

6.2.15. [*] A linear varying line load acts along the beam between $0 < x < 3$ m. A 4-kN point force acts at C ($x = 5$ m), as shown. Determine

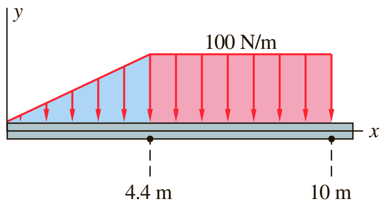
a. the point force and its location that are equivalent to the combined distributed force and point force at C

b. the loads acting on the beam at the supports if the beam is in equilibrium (Assume that the weight of the beam is negligible.)



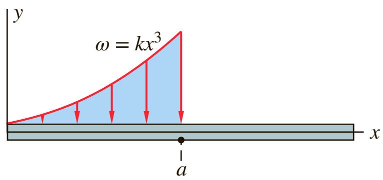
EX 6.2.15

6.2.16. [*] A line load acts on the top of a horizontal surface between $0 < x < 10$ m. Determine the location and magnitude of the point force that is equivalent to the line load. Use information provided in Appendix C.



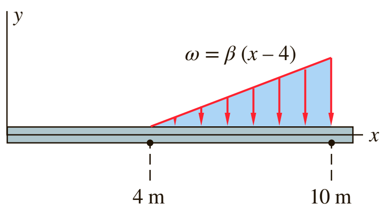
EX 6.2.16

6.2.17. [*] A line load applied to the top of a horizontal surface is described by $\omega = kx^3$, where $k = 100\ \text{N/m}^4$. Determine the point force and its location (centroid) that are equivalent to the line load.



EX 6.2.17

6.2.18. [*] A line load acting on the top of a horizontal surface between $4\ \text{m} < x < 10\ \text{m}$ is described by $\omega = \beta\,(x - 4)$, where $\beta = 100\ \text{N/m}^2$. Determine the magnitude and location of the point force that is equivalent to the line load.

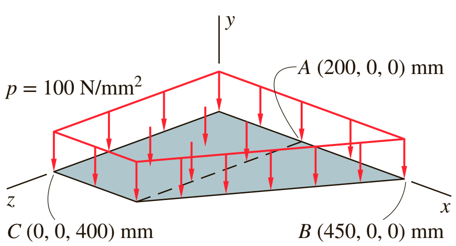


EX 6.2.18

6.2.19. [*] A uniform pressure (p) acts on a horizontal surface as shown.

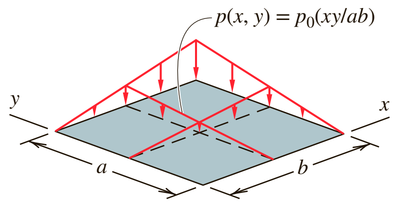
a. Use integration to find the total (equivalent) force and the center of pressure.

b. Use the data in Appendix C to find the equivalent force and center of pressure.



EX 6.2.19

6.2.20. [*] Determine the point force and center of pressure for the pressure distribution shown. Use integration.

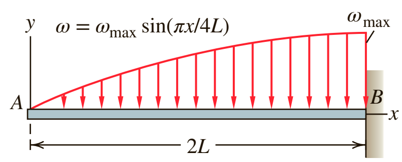


EX 6.2.20

6.2.21. [**] The sine-shaped line load acts on the beam shown.

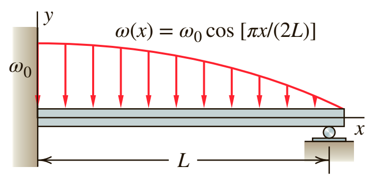
a. Find the magnitude and location of the equivalent point force.

b. Determine the loads acting on the beam at B if the beam is in equilibrium.



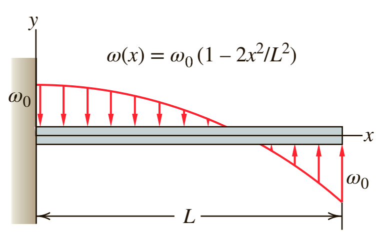
EX 6.2.21

6.2.22. [**] Use integration to determine the point load and its location (centroid) that are equivalent to the line load shown.



EX 6.2.22

6.2.23. [**] Use integration to determine the magnitude and location of the point load and that is equivalent to the line load applied to the cantilever beam.

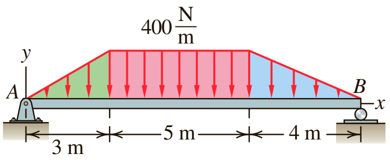


EX 6.2.23

6.2.24. [**] The line loads shown act on a beam.

a. Find the magnitude and location of the equivalent point force.

b. Determine the loads acting on the beam at A and B if the beam is in equilibrium.

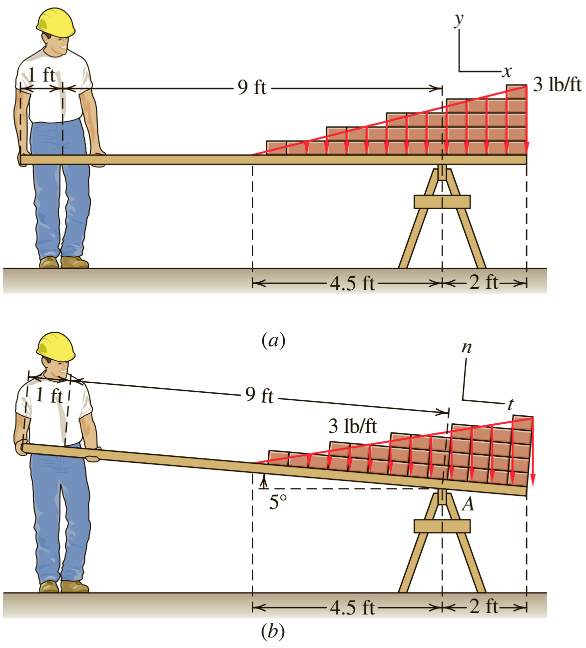


EX 6.2.24

6.2.25. [**] A carpenter holds a 2×12 board that weighs 6 lb/linear foot. The bricks piled on the 2×12 result in an additional triangular distributed load. Assume that the force the carpenter applies with his hands can be represented by a single force F_c at 1 ft from the left end.

a. Determine the vertical force that the carpenter must apply to the 2×12 for there to be equilibrium for the position in **Figure *a***. Is this a reasonable force for the carpenter to apply?

b. When the 2×12 is in the position shown in **Figure *b*** does the magnitude of F_c increase, decrease, or remain the same as in (a)? Find the loads acting on the board at A.

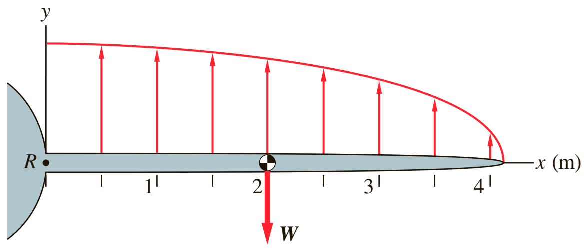


EX 6.2.25

6.2.26. [**] The lift forces on the airplane's wing are represented by eight forces. The magnitude of each force is given in terms of its x position on the wing by $200\sqrt{[1-(x/17)^2]}$ N. The weight of the wing W is 1600 N, and the wing has a width of 1 m.

a. Find the point force and its location that are equivalent to the eight forces shown.

b. Determine the loads acting on the wing at the root R if the wing is in equilibrium.



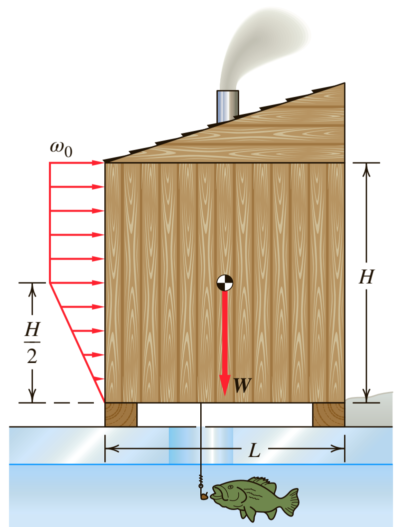
EX 6.2.26

6.2.27. [**] An ice house is sitting on runners on the ice covering Lake Sissabagema in northern Minnesota and the wind is gusting out of the north at speeds up to 40 mph.

a. The wind load is approximated by the distributed load shown, with ω_0 = 40 lb/ft. Assume that L = 7 ft, H = 8 ft, and the force due to gravity, W, is 800 lb (a crate, a portable television, a small stove, and three fishermen). Calculate the total force that the wind exerts on the ice house and the loads acting on the runners.

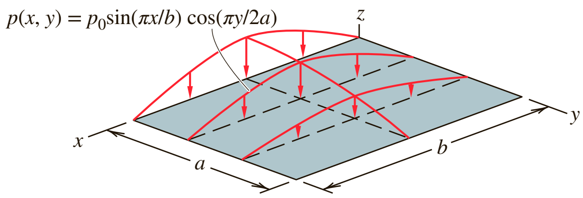
b. How do the forces calculated in **a** change if instead a uniform wind load of ω_0 = 40 lb/ft acts over the height of the ice house?

c. If the wind load is uniformly distributed over the entire side of the ice house, how strong does it have to be to cause the ice house to tip?



EX 6.2.27

6.2.28. [**] Determine the point force and its location (center of pressure) for the pressure distribution shown. Use integration.

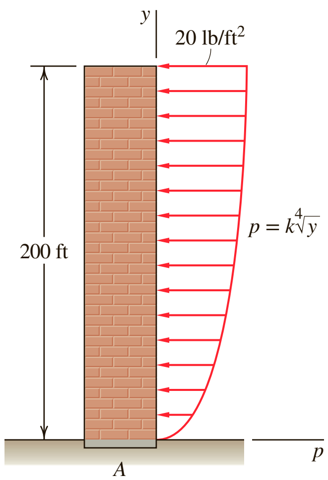


EX 6.2.28

6.2.29. [**] In a wind study, engineers have approximated the wind pressure distribution on the windward wall of a 200-ft building with the function shown. The width of the wall is 300 ft. (into the page).

a. Determine the loads acting on the wall at its base due to the wind load.

b. If the moment acting on the wall at A may not be greater than 8×10^4 k · ft, suggest two changes that might be made to the design to meet this requirement.

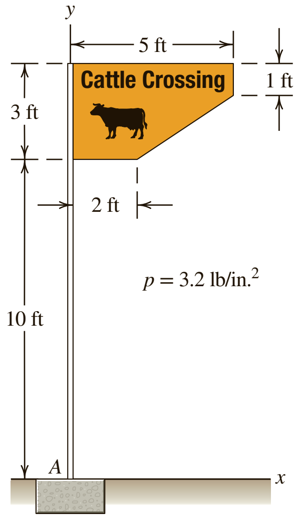


EX 6.2.29

6.2.30. [**] A uniform wind pressure distribution acts on the cattle crossing sign. The sign is in equilibrium and the pole is fixed into the ground at A.

a. Ignoring any wind acting on the pole, determine the equivalent point force and its location.

b. Ignoring the weight of the sign and pole, what loads act on the pole at A.



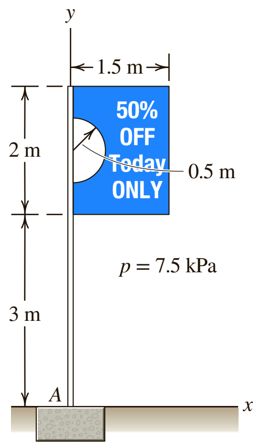
EX 6.2.30

6.2.31. [**] A uniform wind pressure distribution acts on the shaded area of the sign. The sign is in equilibrium and its pole is fixed into the ground at A.

a. Ignoring any wind acting on the pole, determine the equivalent point force and its location (center of pressure).

b. Ignoring the weight of the sign and the pole, what loads act on the pole at A?

c. If the sign is reversed so that the solid side is attached to the pole and the cutout is on the right, determine if the loads acting on the pole at A would increase, decrease, or remain the same.

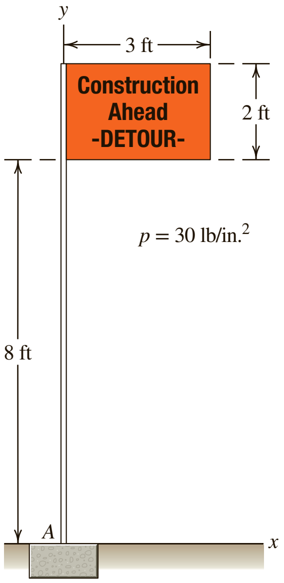


EX 6.2.31

6.2.32. [**] A uniform wind pressure distribution acts on the construction sign. The sign is in equilibrium and the pole is fixed into the ground at A.

a. Determine the equivalent point force and its location (center of pressure).

b. The pole will fail if the twisting moment about the y axis exceeds 50 k · ft, what is the maximum wind pressure that can be exerted on the sign?

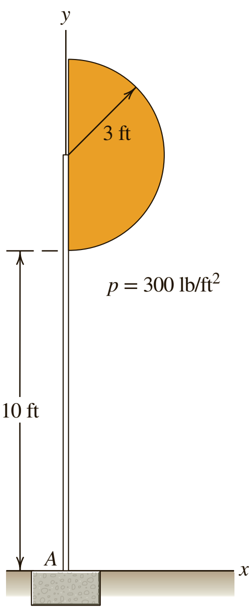


EX 6.2.32

6.2.33. [**] A uniform wind pressure distribution acts on the semicircular sign. The sign is in equilibrium and its pole is fixed into the ground at A.

a. Ignoring any wind acting on the pole, determine the magnitude and location of the point force equivalent to the wind pressure.

b. Ignoring the weight of the sign and the pole, what loads act on the pole at A?



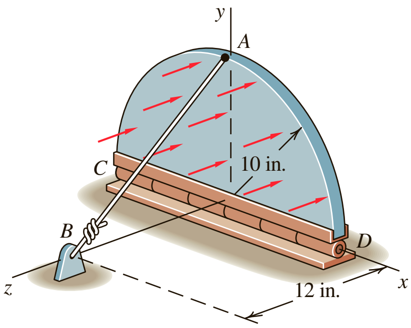
EX 6.2.33

c. If the sign is 1/8 in.-thick and made of steel, what gravity loads act on the pole at A? Compare the magnitudes of the loads produced by the wind with those produced by the sign weight.

6.2.34. [**] A semicircular plate is supported in a wind tunnel by a hinge along CD and by a cable running from A to B. The plate is made of steel, and the horizontal wind pressure acting on the plate is 40 psi. Determine

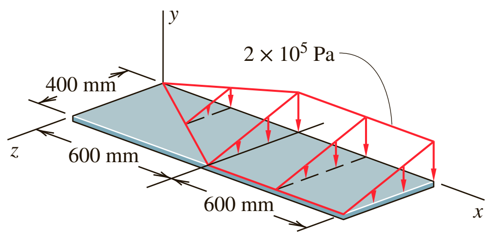
a. center of pressure and the point force that is equivalent to the wind pressure

b. the tension in the cable and the loads acting at the hinge assuming the plate is in equilibrium



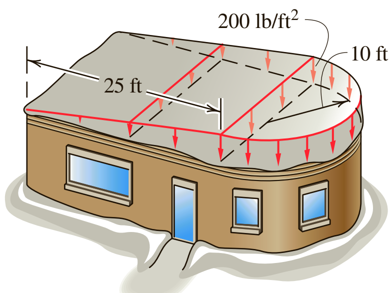
EX 6.2.34

6.2.35. [**] The pressure distribution on a rectangular plate is as shown. Determine the magnitude and location of the point force that is equivalent to this distribution.



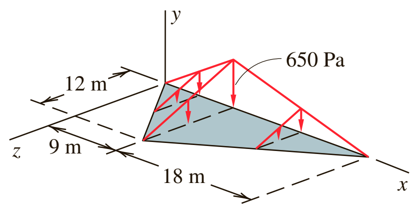
EX 6.2.35

6.2.36. [**] The figure shows the distribution of the snow load acting on a roof. Determine the equivalent point force and its location (center of pressure).



EX 6.2.36

6.2.37. **[**]** The snow load on a roof is as shown. Determine the point force and its location (center of pressure) that are equivalent to the distributed snow load.

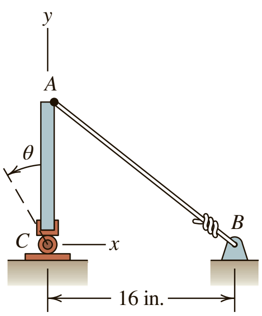


EX 6.2.37

6.2.38. **[***]** The semicircular steel plate is 0.25 in. thick. The wind pressure on the plate is 40 psi. The plate rotates from the vertical at an angle θ such that $0 < \theta < 45°$. Write

a. expressions in terms of θ for the magniude and location of the equivalent point force

b. expressions in terms of θ for the tension in the cable when the plate is in equilibrium (Present your answer as equations and as a plot of tension versus angle θ.)

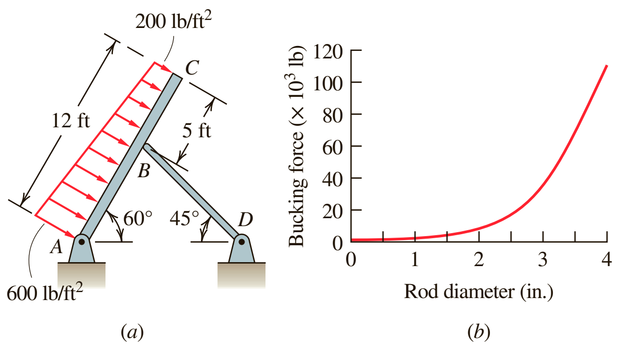


EX 6.2.38

6.2.39. **[***]** The rectangular plate *ABC* has a width of 3 ft (perpendicular to the plane of the page).

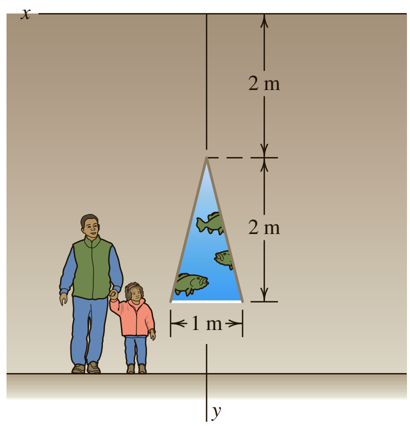
a. Determine the compressive force in rod *BD*.

b. Buckling is a form of failure of long slender members loaded in compression. Figure ***b*** shows the compressive force that causes 8.57-ft steel rods of various diameters to buckle. Based on the force found in **a**, what is the minimum diameter that should be specified for rod *BD* to ensure that it will not buckle? Explain your reasoning.



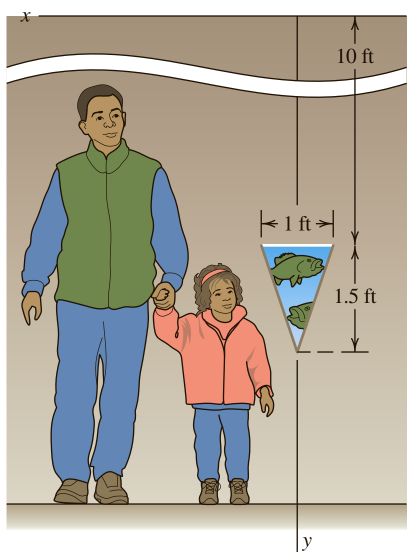
EX 6.2.39

6.2.40. **[***]** Water pressure acts on the vertical freshwater aquarium window shown. If water pressure varies as a linear function of the depth (measured by y in the figure), determine the center of pressure and the point force that is equivalent to the water pressure acting on the window. Your answer should include a scale drawing of the window, showing the point force and its location.



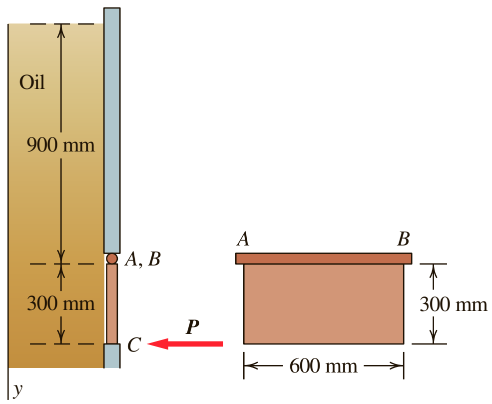
EX 6.2.40

6.2.41. **[***]** Water pressure acts on the vertical freshwater aquarium window shown. If water pressure varies as a linear function of the depth (measured by y in the figure), determine the center of pressure and the point force that is equivalent to the water pressure acting on the window. Your answer should include a scale drawing of the window, showing the point force and its location.



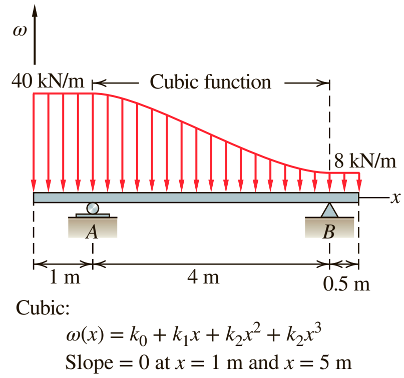
EX 6.2.41

6.2.42. [***] A flat plate is used as an access port in an oil tank. It is hinged along AB and is held in place by force $\boldsymbol{P}$ acting at C. Oil pressure p varies as a linear function of depth (measured by y in the figure); the relationship is of the form $p = \rho_{oil} g y$, where ρ_{oil} is the density of oil and is 800 kg/m^3. Determine the force $\boldsymbol{P}$ applied at C required to keep the plate in place.



EX 6.2.42

6.2.43. [*, computer]** For the loading shown, the cubic polynomial load extends between the 40-kN/m load and the 8-kN/m load. Determine the loads at supports A and B. You may want to use a computer to solve the four simultaneous equations you generate.



EX 6.2.43

6.3 HYDROSTATIC PRESSURE

Learning Objective: Perform static analysis for situations involving distributed hydrostatic pressure and buoyancy forces.

If you have ever been at the bottom of a swimming pool and had to clear your ears or if you have ever used a pressure regulator attached to a scuba tank to swim the ocean depths, you have had personal experience with hydrostatic pressure. The pressure within a static fluid is called the **hydrostatic pressure** when the fluid is a liquid and **aerostatic pressure** when the fluid is a gas. The pressure is in units of force/area. Hydrostatic pressure and aerostatic pressure are two examples of distributed pressure loads, as discussed in Section 6.2. These loads, if deemed significant by the analyst, must be included in static analysis of a system. Now we consider how hydrostatic pressure changes with depth.

If a liquid at rest acts at the boundary of a system, the pressure exerted by the hydrostatic pressure is normal to the boundary and oriented so as to push on the boundary. Furthermore, at any particular location within the fluid, the magnitude of the pressure acting at that location is the same in all directions, as we illustrate in Example 6.3.1. Even though the magnitude of the pressure acting at a particular location in a liquid is the same in all directions, the magnitude of the pressure is not the same everywhere in the liquid. In a liquid at rest we find that the hydrostatic pressure at depth h (Figure 6.3.1) is given by the expression:

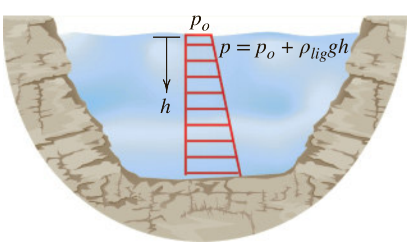


Figure 6.3.1 Hydrostatic pressure increases with distance below the surface and acts perpendicular to the boundary surface.

$$p = p_o + \rho_{liq} g h \tag{6.21A}$$

where p_o is the atmospheric pressure acting on the top surface of the liquid, ρ_{liq} is the density of the liquid, and g is the gravitational constant. The quantity ρg is called the specific weight of the liquid. Equation (6.21A) says that *hydrostatic pressure increases with depth in a linear*

manner and is valid when density (or specific weight) is constant everywhere in the liquid. The derivation of (6.21A) is shown in Example 6.3.2. Equation (6.21A) can be rewritten in terms of specific weight γ_{liq} (where $\gamma_{\text{liq}} = \rho_{\text{liq}}\, g$ is weight per unit volume) as

$$p = p_o + \gamma_{\text{liq}} h \tag{6.21B}$$

Values of density and specific weight for water are given in Appendix B.

The pressure p in (6.21A) and (6.21B) is referred to as the **absolute pressure**. Sometimes fluid pressure is measured with instruments that read pressure at depth relative to atmospheric pressure. This relative pressure is called **gage pressure** and is expressed as

$$P_{\text{gage}} = \rho_{\text{liq}} gh \tag{6.22A}$$

or in terms of specific weight as

$$p = \gamma_{\text{liq}} h \tag{6.22B}$$

Note that the term on the right of (6.22A) and (6.22B) is also the rightmost term in (6.21A) and (6.21B). Thus (6.21A) tells us that the absolute pressure at depth h in any fluid is atmospheric pressure p_o plus the gage pressure ρgh at that depth.

For fluid pressure acting on a flat surface of any shape, regardless of orientation, we determine the equivalent point force by multiplying the surface area *in contact with the water* by the pressure acting on that surface at its centroid.

We write this as

$$F_{\text{liq}} = \rho_{\text{liq}} g h_{\text{cent}} A \tag{6.23}$$

where h_{cent} is the distance from the fluid surface to the centroid of A, the surface area in contact with the water (**Figure 6.3.2**). We place the total equivalent load at the **center of pressure** (as defined in Section 6.2). The center of pressure is at the centroid of the hydrostatic pressure distribution acting on the surface, as shown in **Figure 6.3.3**. Note that depending on the orientation of the surface, the center of pressure (CP) and the centroid (C) of the surface area are not necessarily at the same location. If the submerged surface is horizontal, the center of pressure and centroid coincide.

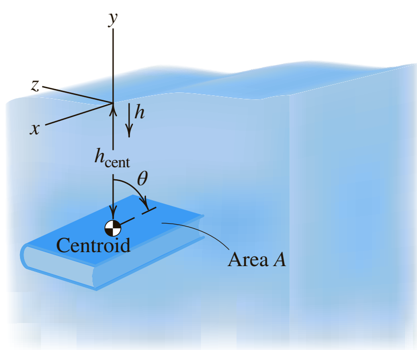


Figure 6.3.2 To calculate the total force acting on a submerged flat surface, we multiply the pressure at the centroid by the surface area.

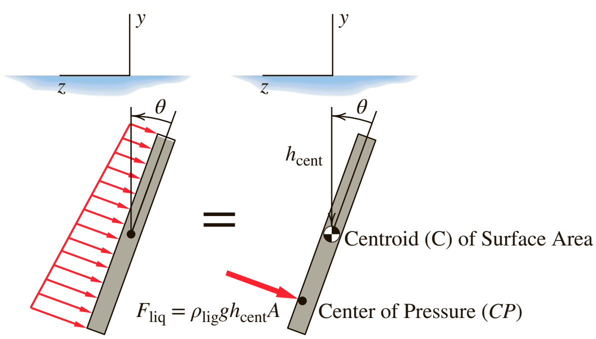


Figure 6.3.3 Equivalent point force acts at center of pressure.

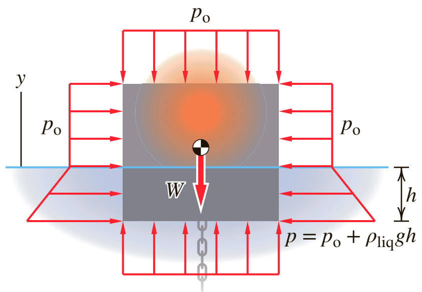


Figure 6.3.4 Pressure distributions acting on a partially submerged buoy (block).

Buoyancy

Buoyancy governs whether an object sinks or floats in a fluid. The discovery of this principle is credited to Archimedes. To illustrate the concept, we create a free-body diagram of a block partially submerged in a liquid (**Figure 6.3.4**). Acting on the block are atmospheric pressure, hydrostatic pressure, and gravity, as shown. If the block is in mechanical equilibrium, we can write the force equilibrium equation in the y direction as

$$\sum F_y = -W + (p_o + \rho_{liq}gh)A - p_oA = 0$$
$$-W + \rho_{liq}ghA = 0$$

where ρ_{liq} is the density of the fluid. The product hA is the volume of the submerged portion of the block, V_{subm}. Therefore

$$\sum F_y = -W + \underbrace{\rho_{liq}gV_{subm}}_{F_{buoy}} = 0 \tag{6.24}$$

The term $\rho_{liq}gV_{subm}$ is an upward force acting on the object and is commonly referred to as the **buoyancy force**. The buoyancy force is always directed upward, and we shall use the notation F_{buoy} to represent it. The centroid of V_{subm} (the volume of the submerged portion of the object) is known as the **center of buoyancy**. The buoyancy force passes through the center of buoyancy. The center of buoyancy, along with the center of gravity of the object, play a key role in determining stability of partially submerged objects, as illustrated in **Figure 6.3.5**.

Equation (6.24) says that if the weight of an object is balanced by the buoyancy force, the object is in mechanical equilibrium. Being in equilibrium, the object will move neither up nor down in the fluid. Alternately, if the weight and buoyancy force do not balance each other, the object either sinks or rises in the fluid:

$$F_{buoy} < W;\ \text{object sinks} \tag{6.25A}$$

$$F_{buoy} > W;\ \text{object rises} \tag{6.25B}$$

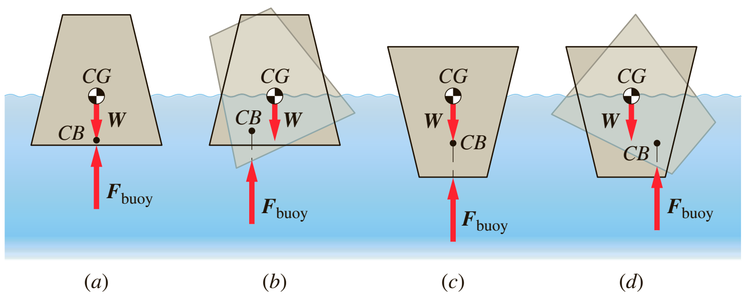


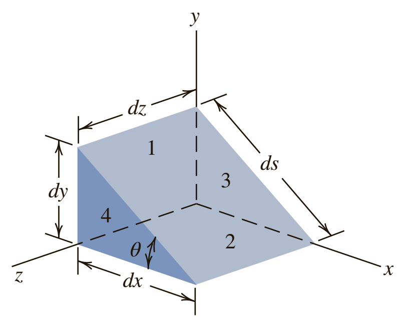
Figure 6.3.5 (*a*) Bottom-heavy volume: W aligned with $\boldsymbol{F}_{buoy}$, floating volume is stable; (*b*) bottom-heavy volume tipped: W and $\boldsymbol{F}_{buoy}$ form a couple that works to return volume to upright position (stable); (*c*) top-heavy volume: W aligned with $\boldsymbol{F}_{buoy}$, floating volume is stable; (*d*) top-heavy volume tipped: W and $\boldsymbol{F}_{buoy}$ form a couple that does not work to return volume to upright position (unstable).

Check out the following examples of applications of this material.

- **Example 6.3.1 Proof of Nondirectionality of Fluid Pressure**
- **Example 6.3.2 Proof that Hydrostatic Pressure Increases Linearly with Depth**
- **Example 6.3.3 Hydrostatic Pressure on Vertical Reservoir Gate**
- **Example 6.3.4 Hydrostatic Pressure on Sloped Gate**
- **Example 6.3.5 Pressure Distribution over a Curved Surface**
- **Example 6.3.6 Center of Buoyancy and Stability**

EXAMPLE 6.3.1

Figure 1 shows an arbitrary infinitesimal triangular prism of liquid taken from an arbitrary point in a liquid at rest. The faces of the prism are numbered as follows: face 1 is parallel to the yz plane, face 2 is on the bottom, face 3 is on the top, and face 4 is the triangle at the front of the prism running parallel to the xy plane. Prove that the pressure exerted on the prism by the surrounding liquid is the same in all directions.



θ = Arbitrary direction

Figure 1 Infinitesimal triangular prism of liquid at an arbitrary point in a liquid at rest.

Goal Prove for a liquid at rest that the pressure at an arbitrary point is the same in all directions.

Given The fluid is at rest; the liquid prism is taken from an arbitrary point; the dimensions and angle θ are as shown in **Figure 1**.

Assume The prism represents a particle since it is infinitesimal and the particle is in equilibrium because it is at rest. The weight of the particle can be ignored. The pressure on each of the faces is different (p_1, p_2, p_3, p_4) and the pressure on any face acts perpendicular to that face.

Formulate Equations and Solve Because the particle is in equilibrium we know the forces must sum to zero. Since the pressure must be perpendicular to any surface, the only two surfaces with pressure in the x direction are 1 and 3. Based on equilibrium equation (5.3A) we write:

$$\sum F_x = 0(\rightarrow +) = p_1\ dy\ dz - p_3(ds\ \sin\theta)dz$$

$$p_1\ dy = p_3\ ds\ \sin\theta \qquad (1)$$

The only two surfaces with pressure in the y direction are 2 and 3. Based on equilibrium equation (5.3B) we write:

$$\sum F_y = 0(\uparrow +) = p_2\ dx\,dz - p_3\ ds\,dz\cos\theta$$

$$p_2\,dx = p_3\ ds\cos\theta \qquad (2)$$

From **Figure 1** we see that

$$ds\ \cos\theta = dx \qquad (3)$$

$$ds\ \sin\theta = dy \qquad (4)$$

Substituting (4) into (1) and (3) into (2) we get

$$p_1\ dy = p_3\ dy$$

$$p_2\ dx = p_3\ dx$$

$$p_1 = p_3 = p_2$$

If we were to sum the forces in the z direction the analysis would show that

$$p_4 = p_3$$

Combining our results we can conclude that $\boxed{p_1 = p_2 = p_3 = p_4}$

EXAMPLE 6.3.2

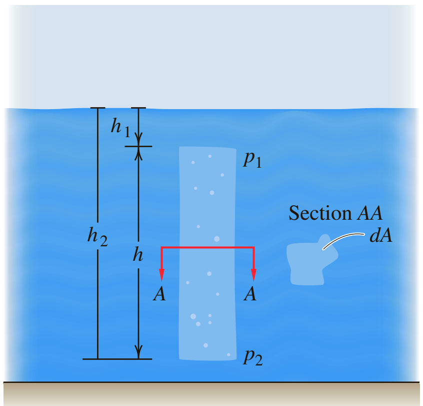


Figure 1 A column of motionless liquid subjected to pressure p_1 at the top and p_2 at the bottom.

Figure 1 shows a vertical column of motionless liquid that is part of a larger body of liquid of constant density ρ_{liq}. The column, of height h, extends from depth h_1 to depth h_2, and has a cross-sectional area dA. The pressure is p_1 at depth h_1 and p_2 at depth h_2. Use this column to prove that hydrostatic pressure increases linearly as a function of depth h in a homogeneous motionless liquid.

Goal Prove that hydrostatic pressure increases linearly as a function of depth.

Given The dimensions of an arbitrary column of liquid; the liquid is motionless and of constant density.

Assume No assumptions necessary.

Draw We draw a free-body diagram of the column of liquid, keeping in mind that in a liquid at rest, the pressures must act perpendicular to any surface (**Figure 2**).

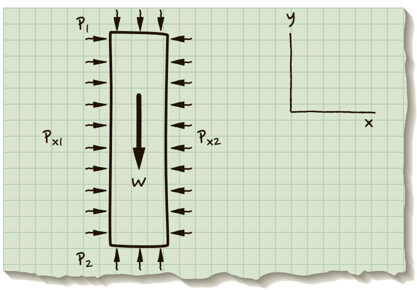


Figure 2 Hydrostatic pressure acts perpendicular to each surface of the column.

Formulate Equations and Solve We apply equilibrium equations to the column of fluid in the vertical direction.

$$\sum F_y = 0(\uparrow +)$$

$$p_2\, dA - p_1\, dA - W = 0 \tag{1}$$

where W is the weight of the fluid in the column, which is given by

$$W = \rho_{\text{liq}} g V_{tot} = \rho_{\text{liq}} gh\, dA \tag{2}$$

Substituting (2) into (1) gives

$$p_2\, dA - p_1\, dA - \rho_{\text{liq}} gh\, dA = 0$$

Solving for p_2 gives

$$\boxed{p_2 = p_1 + \rho_{\text{liq}} gh} \tag{3}$$

This is the equation of a line representing p_2 as a function of depth, with intercept p_1 and slope $\rho_{\text{liq}}\, g$ (**Figure 3**). Therefore we can conclude that the pressure varies linearly with depth.

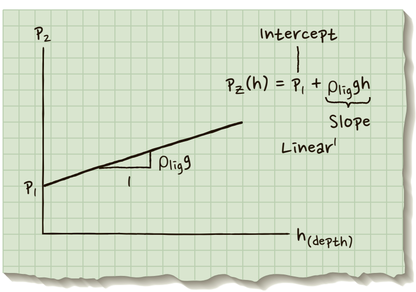


Figure 3 The relationship between p_2 and depth h is linear.

Note: Equation (6.21A) is a special case of (3), when p_1 is located at the surface of the liquid and is therefore p_o, the atmospheric pressure.

EXAMPLE 6.3.3

Figure 1*a* shows a cross section through a rectangular gate that is 8 m high and 3 m wide (where the width dimension w is perpendicular to the plane of the page). The gate blocks the end of a freshwater channel and opens automatically when the water reaches a certain depth. It opens by rotating, as shown in **Figure 1*b***. Determine the depth d at which the gate just begins to open.

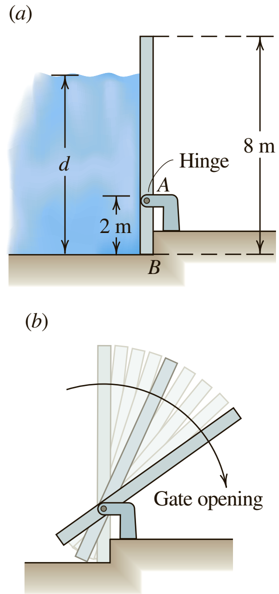


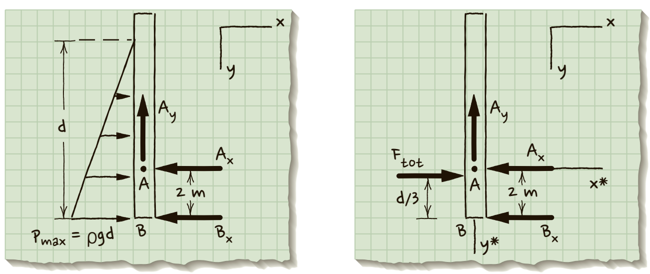
Figure 1 When water in a reservoir reaches a certain depth d, the gate opens.

Goal Find the depth of water that causes the gate to open.

Given Information about the dimensions of the rectangular water gate and the location of the hinge. The gate holds back fresh water (ρ = 1000 kg/m^3).

Assume The weight of the gate is negligible, the hinge is frictionless, the fluid is static, and the system is in equilibrium. We treat the system as planar because no fluid pressure acts on the sides of the gate.

Draw Included on the free-body diagram of the gate are the hydrostatic gage pressure (based on (6.22)), the forces at the frictionless hinge, and the force of the sill at B pushing on the bottom of the gate (**Figure 2*a***). We then represent the hydrostatic pressure by an equivalent total force at the center of pressure (**Figure 2*b***).



(*a*) (*b*)

Figure 2 Free-body diagrams of gate: (*a*) showing triangular hydrostatic pressure distribution; (*b*) showing total equivalent force due to hydrostatic pressure.

Comment Because we have chosen to work with gage pressure (as opposed to absolute pressure from (6.21)), we do not need to include the atmospheric pressure on the back side of the gate. If we had chosen to work with absolute pressure, the free-body diagram of the gate would look as in **Figure 3**.

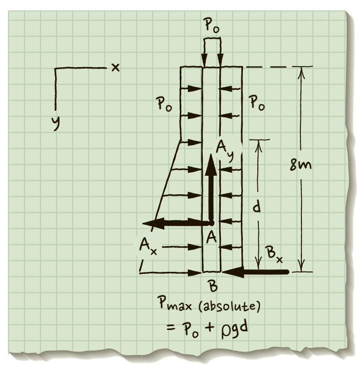


Figure 3 Free-body diagram drawn in terms of absolute pressure.

Formulate Equations and Solve The pressure distribution is triangular, and we use the properties of standard shapes in Appendix C to find the magnitude and location of the total hydrostatic gage force

$$F_{tot} = \frac{p_{max} dw}{2} = \frac{(\rho g d)dw}{2} = \frac{\rho g d^2 w}{2}$$

at a distance $d/3$ from the bottom of the gate. Just as the gate is about to open, the force B_x must be zero. To find B_x, we sum the moment about

the z axis, using the hinge at A as the moment center. To simplify the calculation, we define an x^*y^* coordinate system with its origin at the hinge (**Figure 2*b***).

$$\sum M_{z@A} = 0(\curvearrowleft +)$$

$$F_{\text{tot}}\, y^* - B_x(2\text{ m}) = 0$$

$$y^* = \frac{B_x(2\text{ m})}{F_{\text{tot}}}$$

Setting $B_x = 0$ gives $y^* = 0$. This tells us that just as the gate is about to open, the total force F_{tot} must act through the hinge, which is located 2 m from the base of the gate. Therefore when the gate is about to open, the location of the center of pressure is

$$\frac{d}{3} = 2\text{ m}$$

The gate opens when $d = 6\text{ m}$.

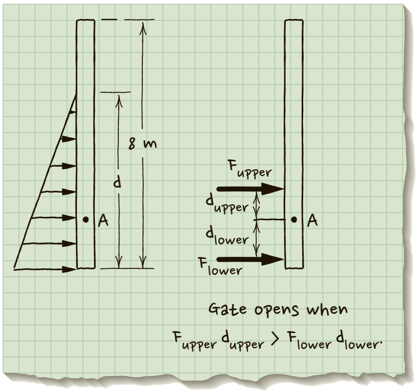


Figure 4 The gate opens when the moments due to the hydrostatic pressure above and below the hinge just balance.

Check We can check the result using an alternative approach to finding the depth at which the gate opens. Recognize that the gate opens when the moment created by the portion of the hydrostatic gage pressure above the hinge (which wants to open the gate) just equals the moment created by the hydrostatic gage pressure below the hinge (which wants to close the gate) (**Figure 4**). At larger depths, the opening moment will be greater than the closing moment, so the gate will open. This approach is a lot more work, because it means finding the total force and centroid for two separate distributed loads (as summarized in **Figure 4**).

EXAMPLE 6.3.4

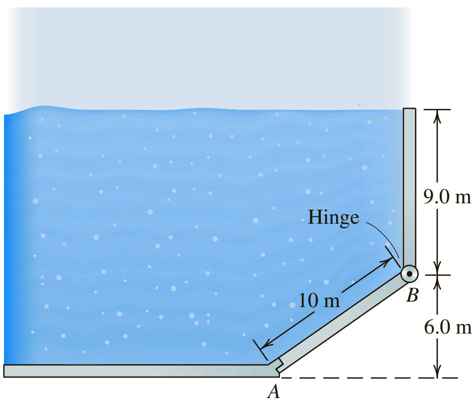


Figure 1 The sloped gate retains water in the tank.

Figure 1 shows a cross section through a tank with a sloped rectangular gate that is 10 m long and 3 m wide (width is measured into the page). The gate is hinged along its top edge and held closed by a force acting on its bottom edge at A. Friction in the hinge and the weight of the gate can be neglected. If the tank is holding water, find the total load on the gate and the magnitude of the force acting on the bottom edge of the gate.

Goal Find the total load on the gate due to the liquid and the value of $\boldsymbol{F}_A$ needed to keep the gate closed.

Given The dimensions of the rectangular gate and the depth of water. The gate is held closed by a force at A. We can ignore the weight of the gate and any friction at the hinge.

Assume Since no water acts on the sides of the gate, the system can be treated as planar. The surface the gate rests against at A is frictionless, requiring $\boldsymbol{F}_A$ to be perpendicular to the gate. Also, we assume that the liquid is static and the system is in equilibrium.

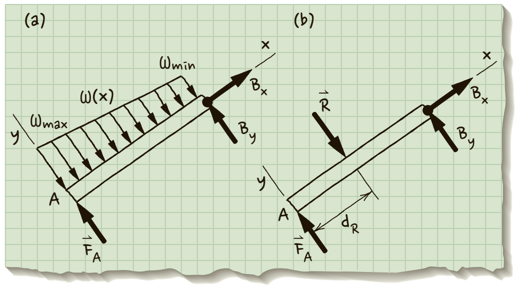


Figure 2 Free-body diagrams of gate: (*a*) showing trapezoidal hydrostatic pressure distribution; (*b*) showing total equivalent load acting at center of pressure.

Draw We draw a free-body diagram of the sloped gate with the pressure load (represented $\omega(x)$ in units of force/length) and include the loads at the supports (**Figure 2*a***). Because we will work with gage pressure (6.22), we do not need to include the atmospheric pressure that acts on the back side of the gate. We then represent the distributed load by an equivalent point force $\boldsymbol{R}$ at a distance d_R from A (**Figure 2*b***).

Formulate Equations and Solve We recognize the load distribution as a line load made up of standard line load distributions, so we use Appendix C to calculate the total load due to the liquid pressure and its location. Since the pressure distribution is a trapezoid, we will break it into a rectangle and a triangle.

The magnitude of the force/unit length at point B is the hydrostatic gage pressure at B multiplied by the width of the gate:

$$\omega_{\min} = \rho_{\text{liq}} g h_B w = (1000\ \text{kg/m}^3)(9.81\,\text{m/s}^2)(9.0\ \text{m})(3\ \text{m})$$

$$\omega_{\min} = 264.9\ \text{kN/m}$$

The magnitude of the force/unit length at point A is the hydrostatic gage pressure at A multiplied by the width of the gate:

$$\omega_{\max} = \rho_{\text{liq}} g h_A w = (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(15\ \text{m})(3\ \text{m})$$

$$\omega_{\max} = 441.5\ \text{kN/m}$$

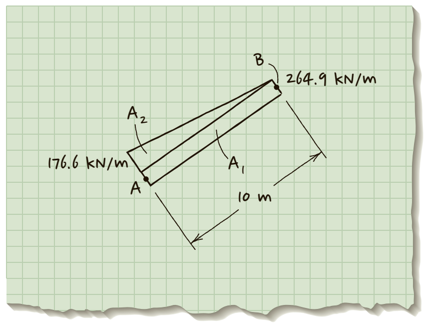


Figure 3 The trapezoidal load distribution is divided into a triangle and a rectangle.

Figure 3 shows how we divide the load into two standard line load distributions: a rectangle (A_1) of height 264.9 kN/m and a triangle (A_2) of height 441.5 − 264.9 = 176.6 kN/m. We then calculate the total force for each standard line load and locate them at their centers of pressure as shown in **Figure 4**.

Using Appendix C we find that

$$F_1 = (10\ \text{m})(264.9\ \text{kN/m}) = 2649\ \text{kN, and } d_1 = 10/2 = 5.0\ \text{m}$$

$$F_2 = (10\ \text{m})(176.6\ \text{kN/m})/2 = 883\ \text{kN, and } d_2 = 10/3 = 3.33\ \text{m}$$

Forces F_1 and F_2, and dimensions d_1 and d_2 are shown in **Figure 4**.

Using (6.16), we find the magnitude of the total force on the gate due to the hydrostatic gage pressure:

$$R = F_1 + F_2 = 2649\ \text{kN} + 883\ \text{kN}$$

$$\boxed{R = 3532\ \text{kN}}$$

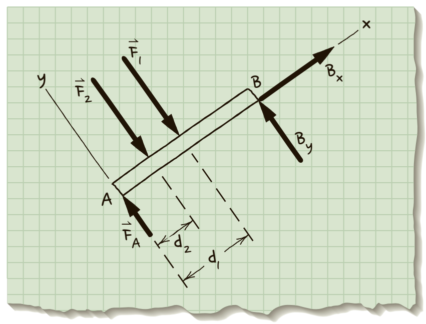


Figure 4 Place the total equivalent load for each standard line load at its centroid.

To find the magnitude of force $\boldsymbol{F}_A$, we use the equilibrium equation (5.5C), selecting the moment center at B:

$$\sum M_{Z@B} = 0(\curvearrowleft +)$$

$$2649\,\text{kN}(10.0\,\text{m} - 5.0\,\text{m}) + 883\,\text{kN}(10.0\,\text{m} - 3.33\,\text{m}) - F_A(10\,\text{m}) = 0$$

$$\boxed{F_A = 1913\ \text{kN}}$$

Check One check of the total force R and its location would be to solve the problem using an alternative approach, such as integrating the pressure distribution. We can check equilibrium by summing moments about a point other than B.

EXAMPLE 6.3.5

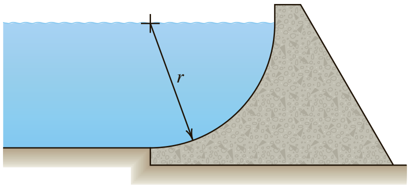


Figure 1 A seawall with a cylindrical face of radius r.

Figure 1 shows a cross section through a 100-ft-long concrete seawall. The face of the seawall is a quarter circle with a radius of 25 ft and is subjected to a load generated by the pressure of the seawater (γ = 64.3 lb/ft^3). Determine the magnitude of the total force $\boldsymbol{F}$ exerted by the seawater on the seawall and the center of pressure.

Goal Find the magnitude of the total force $\boldsymbol{F}$ represented by hydrostatic gage pressure acting on the seawall and its location (center of pressure).

Given Information about the geometry of the seawall the specific gravity of seawater.

Assume The water is at rest, the system is in equilibrium, and we can treat the system as planar.

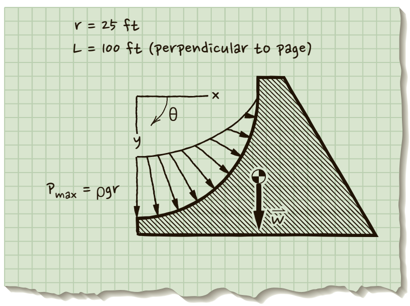


Figure 2 Water pressure acts perpendicular to the surface.

Draw Based on the information given in the problem and our assumptions, we draw the water acting on the seawall (**Figure 2**). We define y positive downward. We chose to work with gage pressure; therefore we don't draw atmospheric pressure on the back side of the seawall.

Formulate Equations and Solve Because hydrostatic pressure must always act perpendicular to a surface, the direction of the pressure varies along the height of the wall. As shown in **Figure 2**, the pressure acts horizontally at the top of the wall and vertically at the bottom. To account for this we need to integrate the horizontal components and the vertical components separately and combine them to calculate the total force $\|\boldsymbol{F}\| = \sqrt{F_x^2 + F_y^2}$:

At any point on the seawall the differential force dF is the pressure at that point multiplied by the differential area dA:

$$dF = p(x, y)\,dA$$

Breaking dF into its rectangular components gives

$$dF_x = p(x, y)\,dA\cos\theta \tag{1}$$

$$dF_y = p(x, y)\,dA\sin\theta \tag{2}$$

We represent the hydrostatic gage pressure in terms of cylindrical coordinate system so that ultimately we can integrate (1) and (2) with respect to θ. Based on (6.22A) (written in terms of y)

$$p(x, y) = \rho g y = \rho g r\sin\theta \tag{3}$$

We define a differential element $r\,d\theta$ along the length of the seawall (L) as

$$dA = L(r\,d\theta) \tag{4}$$

We substitute (3) and (4) into (1) and (2) to get

$$dF_x = \rho g r\sin\theta(Lr\,d\theta)\cos\theta \tag{5}$$

$$dF_y = \rho g r\sin\theta(Lr\,d\theta)\sin\theta \tag{6}$$

and integrate (5) and (6) from 0 to $\pi/2$ to find the total force in the horizontal and vertical directions:

$$F_x = \int_0^{\pi/2} \rho g L r^2 \cos\theta \sin\theta \, d\theta = -\rho g L r^2 \left[\frac{\cos^2\theta}{2}\right]_0^{\pi/2} = \frac{\rho g L r^2}{2}$$

$$F_y = \int_0^{\pi/2} \rho g L r^2 \sin^2\theta \, d\theta = \rho g L r^2 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = \frac{\rho g L r^2 \pi}{4}$$

The total force is found from:

$$\|\boldsymbol{F}\| = \sqrt{F_x^2 + F_y^2} = \sqrt{\left(\frac{\rho g L r^2}{2}\right)^2 + \left(\frac{\rho g L r^2 \pi}{4}\right)^2} = \frac{\rho g L r^2}{2}\sqrt{1 + \frac{\pi^2}{4}}$$

Substituting numerical values gives

$$\|\boldsymbol{F}\| = \frac{\left(64.3\frac{\text{lb}}{\text{ft}^3}\right)(100 \text{ ft})(25 \text{ ft})^2}{2}\sqrt{1 + \frac{\pi^2}{4}} = \boxed{3740 \text{ kip}}$$

The total force $\boldsymbol{F}$ acts perpendicular to the quarter-circle surface at angle θ_{CP}, as shown in Figure 3. The center of pressure (CP) will then be at

$$X_{CP} = r \cos\theta_c, \quad Y_{CP} = r \sin\theta_c$$

and θ_{CP} is defined in terms of F_y and F_x as

$$\theta_{CP} = \tan^{-1}\frac{F_y}{F_x} = \tan^{-1}\frac{\frac{\rho g L r^2 \pi}{4}}{\frac{\rho g L r^2}{2}} = \tan^{-1}\frac{\pi}{2} = 57.5° \tag{7}$$

Therefore

$$X_{CP} = r \cos\theta_{CP} = 25 \text{ ft} \cos(57.5°) = \boxed{13.4 \text{ ft}}$$

$$Y_{CP} = r \sin\theta_{CP} = 25 \text{ ft} \sin(57.5°) = \boxed{21.1 \text{ ft}}$$

Note: We avoided integrating to find the center of pressure, because we were able to identify the relationship in (7) between the force components and their resultant, which must be perpendicular to the quarter-circle surface.

Check We could check our result by using another approach to solve this problem. This approach, which is based on calculating forces acting on projected areas, is explored in **Exercise 6.3.20**.

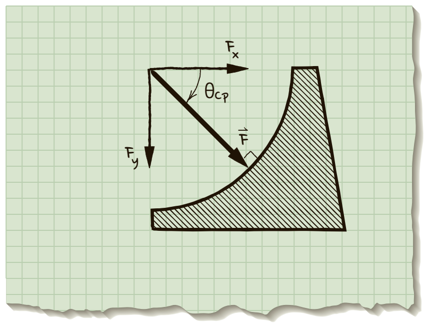


Figure 3 The resultant force acts perpendicular to the wall at the center of pressure.

EXAMPLE 6.3.6

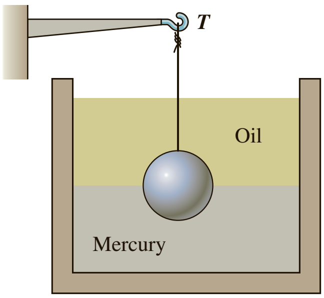


Figure 1 A steel ball suspended in oil and mercury.

A solid steel ball of radius $r = 50.0$ mm and specific weight $\gamma_s = 76.8$ kN/m^3 is suspended by a thin wire and lowered into a tank containing oil and mercury ($\gamma_{oil} = 7.80$ kN/m^3, $\gamma_{Hg} = 133$ kN/m^3) until it is submerged half in the oil and half in the mercury (**Figure 1**). Determine the tension T in the wire.

Goal Find the tension in the wire that is holding up the ball.

Given Properties of the steel ball and the liquid it is suspended in.

Assume The specific weights are uniform and the ball is at rest (in equilibrium).

Draw The hydrostatic forces acting in the horizontal direction sum to zero, so we have not included them on the free-body diagram of the steel ball (**Figure 2**).

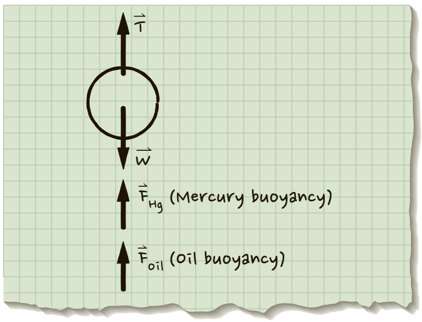


Figure 2 Free-body diagram of steel ball.

Formulate Equations and Solve We apply equilibrium in the vertical direction:

$$\sum F_y = 0(\uparrow +) = T + F_{oil} + F_{Hg} - W$$
$$T = W - F_{oil} - F_{Hg} \tag{1}$$

where

$$W = \gamma_{ball} V_{ball} = \gamma_s \frac{4}{3}\pi r^3 = \frac{4}{3}(76.8\,\text{kN/m}^3)\pi(0.0500\text{ m})^3 = 40.21\text{ N}$$

$$F_{oil} = V_{ball\ submerged}(\gamma_{oil}) = \frac{1}{2}V_{ball}(\gamma_{oil})$$
$$= \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)\gamma_{oil} = \frac{1}{2}\frac{4}{3}\pi(0.0500\text{ m})^3(7.80\text{ kN/m}^3) = 2.042\text{ N}$$

$$F_{Hg} = V_{ball\ submerged}(\gamma_{Hg}) = \frac{1}{2}V_{ball}(\gamma_{Hg})$$
$$= \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)\gamma_{Hg} = \frac{1}{2}\frac{4}{3}\pi(0.0500\text{ m})^3(133\,\text{kN/m}^3) = 34.82\text{ N}$$

Find T using (1) and the calculated values of F_{oil} and F_{Hg}:

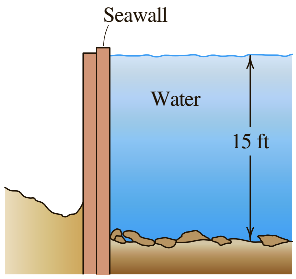
$$T = W - F_{oil} - F_{Hg} = 40.21\text{ N} - 2.042\text{ N} - 34.82\text{ N} = \boxed{3.35\text{ N}}$$

If T were negative, it would mean that the wire is in compression. A thin wire would not support a compression force, so it would buckle and the steel ball would rise. A downward force would therefore be needed to sink the steel ball halfway into each liquid.

Check Other than redoing the calculations, there is no good check for this problem.

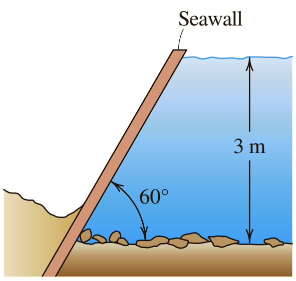
EXERCISES 6.3

6.3.1. [*] Determine the point force and its location (center of pressure) that are equivalent to the hydrostatic pressure acting on the seawall. The seawater is 2.5% denser than fresh water. The width of the seawall (dimension into the page) is 20 ft. Your answer should include a scale drawing of the seawall, showing the point force and its location.



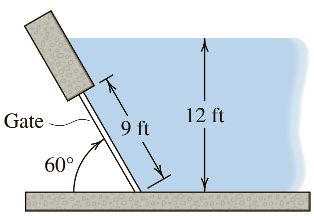
EX 6.3.1

6.3.2. [*] Determine the point force and its location (center of pressure) that are equivalent to the hydrostatic pressure acting on the seawall. The seawater is 2.5% denser than fresh water. The width of the seawall (dimension into the page) is 6 m. Your answer should include a scale drawing of the seawall, showing the point force and its location.



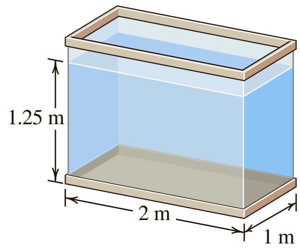
EX 6.3.2

6.3.3. [*] Determine the point force and its location (center of pressure) that are equivalent to the hydrostatic pressure acting on the gate. The width of the gate (dimension into the page) is 2 ft, and the water is fresh water. Your answer should include a scale drawing of the gate, showing the point force and its location.



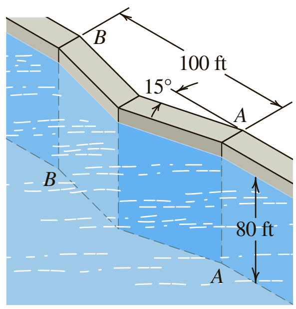
EX 6.3.3

6.3.4. [*] The freshwater fish tank holds water to a depth of 1.25 m. For each face, including the bottom, determine the point force and its location (center of pressure) that are equivalent to hydrostatic pressure. Your answer should include a scale drawing of the fish tank, showing the point forces and their locations.



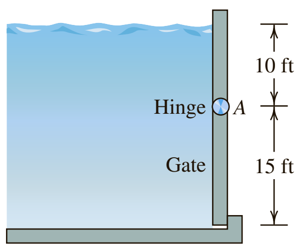
EX 6.3.4

6.3.5. [*] The two-gate structure shown is used to retain fresh water. The gates span a 100-ft opening, are hinged along *A*-*A* and *B*-*B*, and when closed, each is rotated 15° back from the vertical plane. If the gates are to be opened when the water is 80 ft deep, determine the moment that must be exerted by the motors at each hinge.



EX 6.3.5

6.3.6. [*] The rectangular gate has a width of 8 ft (dimension into the page) and rotates about a hinge at *A*. Determine the point force and its location (center of pressure) that are equivalent to the water pressure acting on the gate. Express the location with respect to *A*.

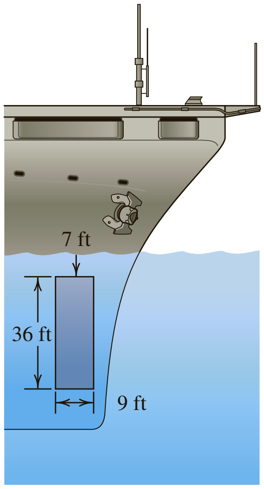


EX 6.3.6

6.3.7. [*] A rectangular vertical plate in the hull of an aircraft carrier is submerged with its top edge 7 ft below the water surface and its bottom edge 36 ft below the top. The plate is 9 ft wide.

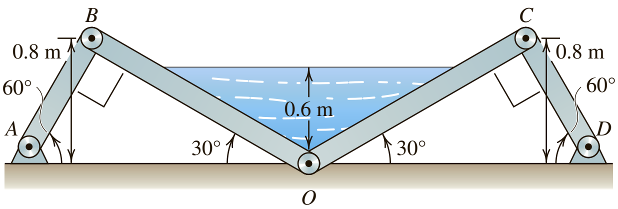
a. Determine the magnitude of the normal pressure force that the sea water ($\gamma = 64.0$ lb/ft^3) exerts on the plate.

b. Locate the center of pressure on the plate. State your answer with respect to the lower left corner of the plate.



EX 6.3.7

6.3.8. [*] A V-shaped freshwater trough, shown in cross section, is built from two steel plates hinged at O. The sides are supported by struts hinged at each end and spaced every 3 m. Determine the compressive force F in each strut. Neglect the weight of the members.



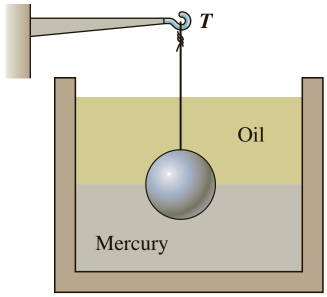
EX 6.3.8

6.3.9. [*] A steel sphere of radius $r = 30$ mm and of specific weight $\gamma_s = 76.8$ kN/m^3 is suspended by a thin wire.

a. When the sphere hangs from the thin wire in air, what is the tension T in the wire?

b. The sphere is lowered into a tank containing oil of specific weight $\gamma_{\text{oil}} = 7.8$ kN/m^3 and mercury of specific weight $\gamma_{\text{Hg}} = 133$ kN/m^3, until it is two-thirds submerged in oil and one-third submerged in mercury. Determine the tension T in the wire.

c. Express your answer in **b** as a percentage of your answer in **a**.

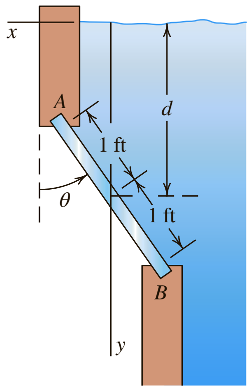


EX 6.3.9

6.3.10. [**] The center of the rectangular window in a freshwater aquarium is at a distance d below the surface of the water. The horizontal width of the window is 2 ft.

a. Write expressions (as functions of θ and d) for the point force and its location (center of pressure) that are equivalent to the water pressure acting on the window.

b. The seal around the perimeter of the window will leak at pressures greater than 60 psi, and the window will break if a force perpendicular to its surface greater than 5000 lb is applied. What is the maximum depth d that the window should be installed below the surface of the water to ensure that it will work adequately?



EX 6.3.10

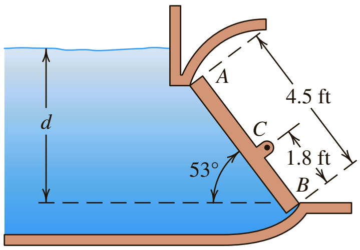
6.3.11. [**] The automatic gate valve AB consists of a 4.5 ft × 3.0 ft rectangular plate that pivots about a horizontal hinge that runs through C (into the page). The valve is part of a dam holding a freshwater reservoir. To open, the valve rotates in a clockwise direction about the hinge at C.

a. Neglecting the weight of the gate valve, determine the depth d in the reservoir at which the gate valve will begin to open.

b. If the weight of the plate is included in the analysis, will the depth of water necessary to open the valve increase, decrease, or remain the same relative to the answer in **a**? (Don't do any calculations, just reason through how various moments act on the valve.)

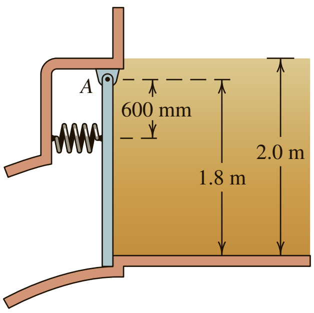
c. The rectangular plate is made of steel and is 0.5 in. thick. Determine the depth d in the reservoir at which the gate valve will begin to open.

d. Compare your answers from **a**, **b**, and **c**.



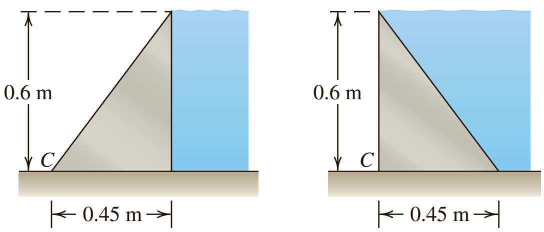
EX 6.3.11

6.3.12. [**] The 750-mm-wide gate valve pivots about a horizontal shaft at A. It is held in the closed position by a preloaded spring. The density of the oil in the tank behind the gate is 800 kg/m^3. Determine the preload in the spring for which the gate will open when $d = 2.0$ m. Does your answer indicate that the spring is preloaded in tension or compression?



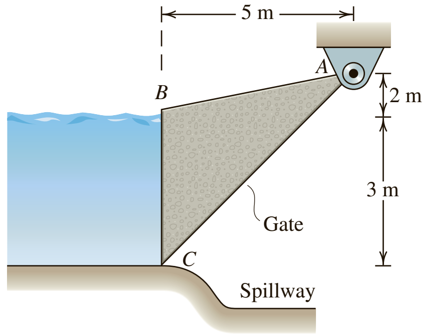
EX 6.3.12

6.3.13. [**] Two 1-m wide (into the page) aluminum model dams are placed in a tank with fresh water in the orientations shown. By evaluating the overturning of each dam about point C, determine which orientation is more stable. (The specific weight of aluminum is approximately 28 kN/m^3.)



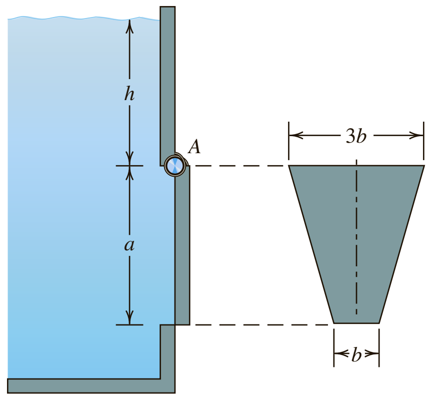
EX 6.3.13

6.3.14. [**] The water gate whose cross section is shown has a uniform density of 4000 kg/m^3. The gate is 5 m wide. If the gate is in equilibrium, determine the loads acting on the gate at A and C. Assume that the contact at C is frictionless.



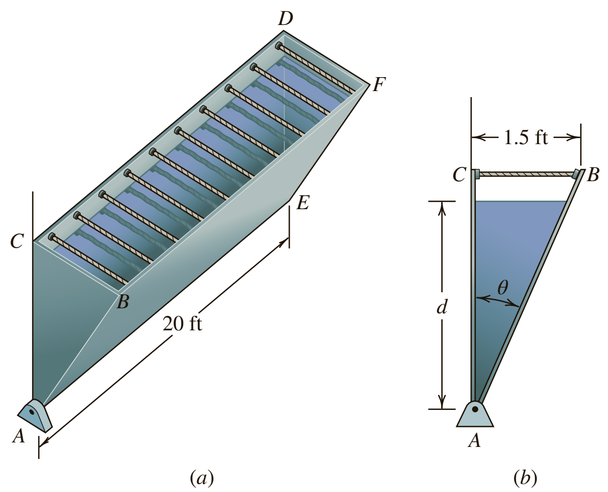
EX 6.3.14

6.3.15. [**] A tank filled with liquid of density ρ is sealed with a trapezoidal flat plate. The plate is attached at the upper edge A of the trapezoid with a hinge and torsional spring. Determine the moment M the spring must exert to hold the gate in a closed position against the pressure of the liquid.



EX 6.3.15

6.3.16. [**] The triangular water trough consists of panel $ABEF$, which is supported by a horizontal hinge along AE. In addition, 10 horizontal cables, each 1.5 ft long, support



EX 6.3.16

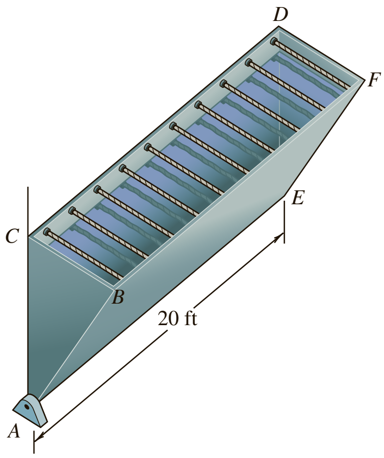
the panel along CD and are equally spaced along the 20 ft length of the trough. If the tension in any one cable is to be limited to 60 lb, what is the maximum depth d that the water trough should be filled?

6.3.17. [**] The triangular water trough consists of panel $ABEF$, which is supported by a horizontal hinge along AE as shown in **Figure *a***. In addition, 10 horizontal cables support the panel along CD and are equally spaced along the 20 ft length of the trough. The cable length of the trough is adjustable, as shown in Figure *b*.

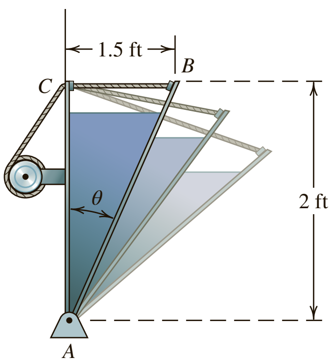
a. How does the volume in the trough usable for storing water vary with the angle θ? Present your answer as an equation and as a plot of volume versus angle θ. What value of θ results in the maximum water storage (call this θ_{max})?

b. How does the tension in any one of the cables vary with the angle θ? (Assume that in any given position the trough is filled just to the point of overflowing.) Present your answer as an equation and as a plot of tension versus angle θ.

c. If you wish to limit the tension in any one cable to 100 lb at θ_{max} (as found in **a**), how many cables should be used to support the trough?



(*a*)



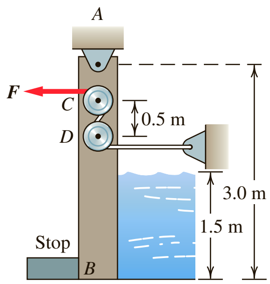
(*b*)

EX 6.3.17

6.3.18. [**] Gate AB is 3 m tall and 1 m wide (dimension perpendicular to page) and holds back water of depth 1.5 m. The density of water is 1000 kg/m^3.

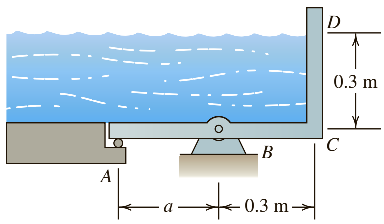
a. What is the minimum force F that must be applied to start the gate swinging open (i.e., for the bottom edge of the gate to just move away from the stop at B)? The pulleys at C and D are each 0.4 m in diameter and their centers are spaced 0.5 m apart. Assume that the pin joint at A is frictionless and that when the gate rests against the stop, there is normal contact.

b. If the average student can pull with a force of 600 N, how many students will be needed to just get the gate to swing open?



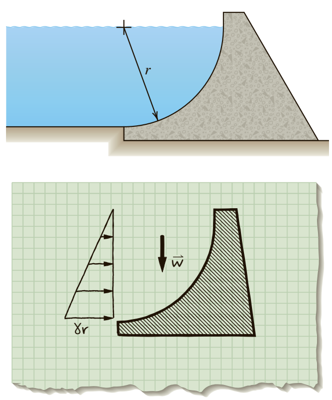
EX 6.3.18

6.3.19. [**] The end of the freshwater channel consists of a plate $ABCD$ that is hinged at B and is 0.4 m wide (dimension perpendicular to page). Determine the length a for which the force acting on the plate at A is zero. Neglect the weight of the plate.



EX 6.3.19

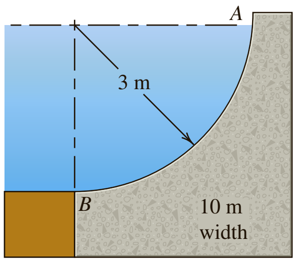
6.3.20. [**] The magnitude of the total force $\boldsymbol{F}$ exerted by the seawater on the seawall was calculated in Example 6.3.5. Show that if you calculate $\boldsymbol{F}$ acting on the wall using the free-body diagram shown, you will arrive at the same



EX 6.3.20

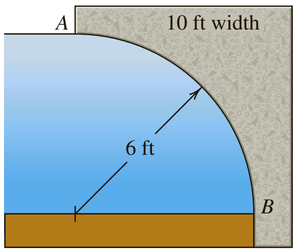
result. W is the weight of water acting on the horizontal projected area, and the horizontal pressure distribution acts on the vertical projected area.

6.3.21. [**] Determine the point force and its location (center of pressure) that are equivalent to the water pressure acting on the curved surface AB.



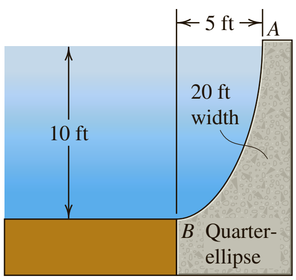
EX 6.3.21

6.3.22. [**] Determine the point force and its location (center of pressure) that are equivalent to the water pressure acting on the curved surface AB.



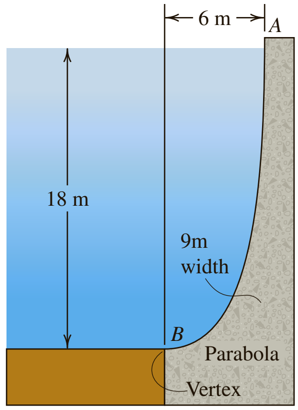
EX 6.3.22

6.3.23. [**] Determine the point force and its location (center of pressure) that are equivalent to the water pressure acting on the curved surface AB.



EX 6.3.23

6.3.24. [**] Determine the point force and its location center of pressure) that are equivalent to the water pressure acting on the curved surface AB.

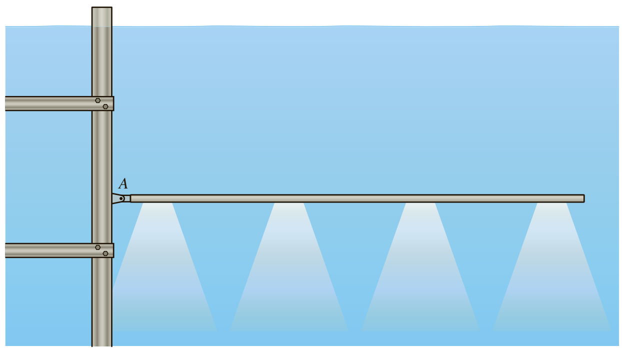


EX 6.3.24

6.3.25. [**] A cylindrical waterproof light fixture is used by divers doing underwater dock repair in Lake Michigan. It is 1.5 m long with a radius of 40 mm, and has a uniform density of 500 kg/m^3. It is pinned to the dock to keep it in place, as shown. To maintain a horizontal orientation (best for downward lighting), additional weight must be added to the fixture.

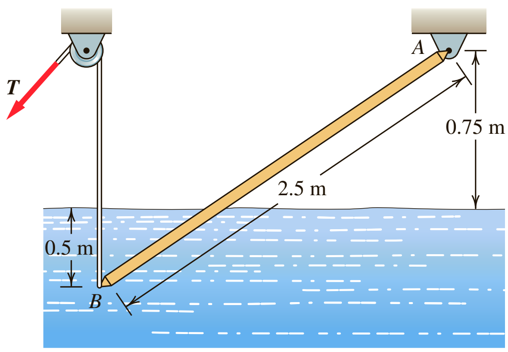
a. What is the minimum additional weight needed? Where should it be attached to the fixture?

b. Would the required additional weight be less, more, or the same if the water were seawater rather than fresh water?



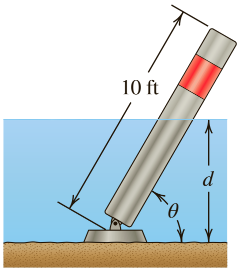
EX 6.3.25

6.3.26. [**] The uniform 31-kg pole of 100-mm diameter is hinged at A and can be lifted from the water with a vertical rope and pulley. When its lower end is immersed in fresh water to a depth of 0.5 m, determine the tension T in the rope.



EX 6.3.26

6.3.27. [**] A 10-ft-long buoy is tied to the bottom of a shallow lake. It is of uniform specific weight of 37 lb/ft^3. What is the minimum lake depth $d_{\min}$ required for the buoy not to lean more than 45° from the vertical?

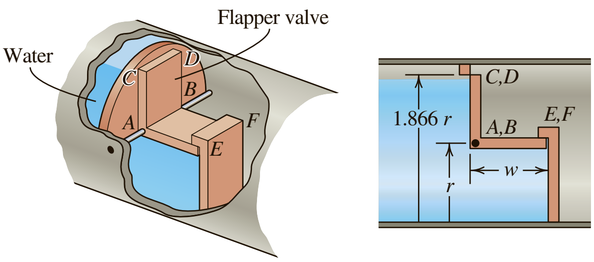


EX 6.3.27

6.3.28. [**] A flapper valve installed in a pipe allows water to flow along the pipe when the water level in the pipe rises to 1.866r (where r is the radius of the pipe). The valve and pipe are shown.

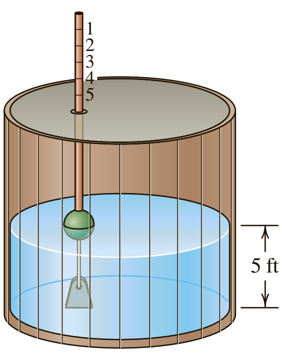
a. For this valve to work, what is the minimum length w of leg $ABEF$? Express the result in terms of r.

b. If $r = 1.0$ m, make a scale drawing of the valve, showing all dimensions.



EX 6.3.28

6.3.29. [**] A mountain community uses a large oak tank for water storage, and wants to monitor the water level in the tank when it is less than 5 ft to ration water if necessary. The float gauge that has been designed so that as the float moves up and down with water level, the scale that protrudes through the top of the tank can be read. The float is a styrofoam sphere with radius of 0.5 ft, and the maximum depth of water in the tank is 12 ft. What is the maximum tension in the cable that tethers the float to the bottom? If the cable's breaking strength is 100 lb, what is the safety factor on the cable? In this case, safety factor is defined as the ratio of strength/tension.

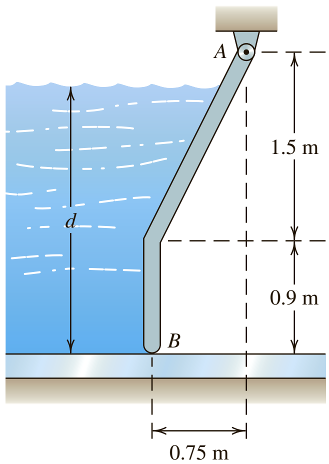


EX 6.3.29

6.3.30. [***] Gate AB dams a 2-m-wide water channel (dimension perpendicular to page) and is supported by hinges along its top edge A.

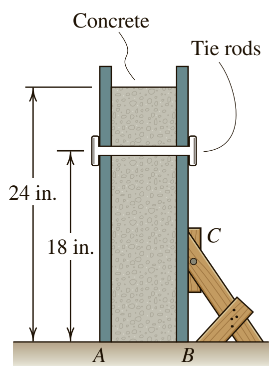
a. If the floor of the channel is frictionless, determine the forces on the gate at A and B when $d = 2.4$ m.

b. What is the minimum force that must be applied to just get the gate to open? Where would this force be applied?



EX 6.3.30

6.3.31. [***] Upright panels and tie rods are used for concrete formwork. The cross section of a 30-ft-long segment of formwork is shown. Thirty tie rods are equally spaced along the 30-ft segment. Determine the tension in each tie rod and the loads acting on the form at C when the concrete is in a liquid state. The specific weight of liquid concrete is approximately 150 lb/ft^3. Assume that the lower edges of the upright panels (A and B) can be modeled as hinges.



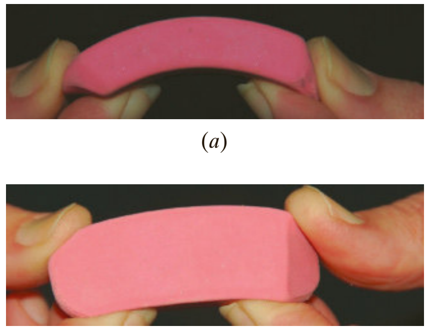
EX 6.3.31

6.4 AREA MOMENT OF INERTIA

Learning Objective: Calculate the moment of inertia of a simple or composite area.

When designing and analyzing systems and structures, engineers need to consider not only the distribution of the loads, but also the distribution of the cross-sectional areas of the components and members. The distribution of the area of the cross section of a member affects its resistance to bending (also called stiffness), how large the internal stresses are, and whether it may be susceptible to failure by buckling.

Try this experiment. Hold an eraser in your hands so that the wide face is horizontal, and bend it about a horizontal axis perpendicular to the vertical face (**Figure 6.4.1*a***). Now turn the eraser on its side and bend it about the horizontal axis (**Figure 6.4.1*b***). You should feel that the second case (the eraser on its side) is more resistant to bending and you must apply more force. This difference in resistance to bending is related to the moment of inertia of the eraser's cross-section. A larger moment of inertia causes more bending resistance. We will discover in this section that **moment of inertia**[‡] of an area is a descriptor of the distribution of the area relative to its centroid.



(*a*)

(*b*)

Jeffrey R. Koseff

Figure 6.4.1 An eraser bent about two different axes (*a*) axis perpendicular to the smaller dimension; (*b*) axis perpendicular to larger dimension.

Consider the area shown in **Figure 6.4.2**. By definition, the moment of inertia of this area about the x axis is

$$I_x = \int_{\text{area}} y^2\, dA \tag{6.26A}$$

and about the y axis is

$$I_y = \int_{\text{area}} x^2\, dA \tag{6.26B}$$

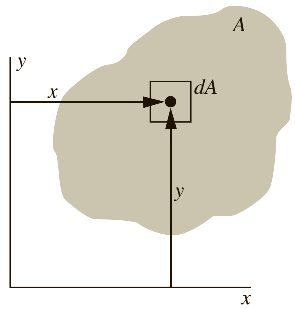


Figure 6.4.2 The moment of inertia about an axis is a function of the square of the distance of dA from the axis.

The unit of the moment of inertia of area is the fourth power of length.

The moment of inertia of an area I was first introduced in a footnote in section 6.1 as one of the "family of integrals" related to area. The area moment of inertia integrals in (6.26A) and (6.26B) are also called the **second area integrals**. Other members of the "family" are **first area integrals**[§], used to find the centroid of an area (x_C, y_C), and the **area integral** ($A = \int_{\text{area}} dA$), used to find the area.

The integrals in (6.26A) and (6.26B) indicate that the moment of inertia of an area about an axis is a function of the square of the distance of the area from that axis. Let's explore this relationship by calculating I_x for a square and a rectangle of equal area (**Figure 6.4.3**).

[‡]The terminology "moment of inertia" is a misnomer, since no inertial concepts are involved.

[§]First area integrals are

$$Q_x = \int_{\text{area}} x\, dA;\; Q_y = \int_{\text{area}} y\, dA$$

and form the numerator in (6.10A) and (6.10B) in the equation of the centroid of an area.

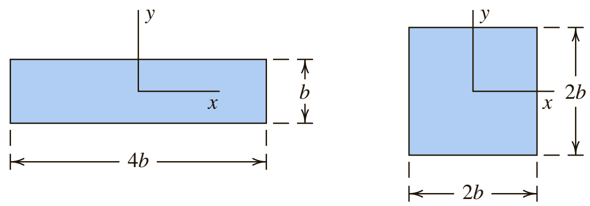


Figure 6.4.3 A rectangle and square, both with areas equal to $4b^2$.

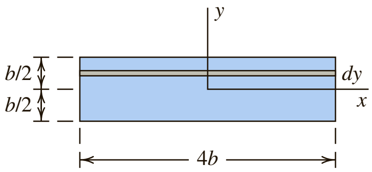


Figure 6.4.4 Represent the differential area as a sliver across the width of the rectangle: $dA = 4b\,dy$.

First we find I_x for the rectangle. By taking a slice across the rectangle and defining the differential element $dA = 4b\ dy$ (**Figure 6.4.4**), we simplify the integration so that we only need to integrate with respect to y. Equation (6.26A) becomes

$$I_{x\text{ rectangle}} = \int_{\text{area}} y^2\, dA = \int_{-b/2}^{b/2} y^2\, 4b\, dy = \left[\frac{4}{3} b y^3\right]_{-b/2}^{b/2} = \frac{b^4}{3}$$

For I_x of the square, we also define a differential element that spans the width of the square so that $dA = 2b\,dy$:

$$I_{x\text{ square}} = \int_{\text{area}} y^2\, dA = \int_{-b}^{b} y^2\ 2b\, dy = \left[\frac{2}{3} b y^3\right]_{-b}^{b} = \frac{4b^4}{3}$$

Although the square and rectangle have the same area, I_x of the square is *four times* larger than I_x of the rectangle. This is due to the *distribution* of the area. The entire area of the rectangle lies within a distance of $\pm b/2$ from the x axis. In contrast, half the area of the square lies at distances greater than $b/2$ from the x axis.

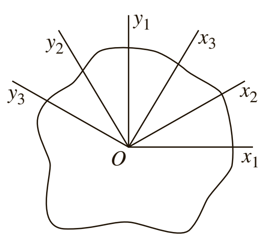
The importance of the distribution of area is again illustrated by calculating I_y for the rectangle. We use (6.26B) and define a differential sliver dx wide that extends from the bottom to the top of the rectangle ($dA = b\ dx$):

$$I_{y\text{ rectangle}} = \int_{\text{area}} x^2\, dA = \int_{-2b}^{2b} x^2\, b\, dx = \left[\frac{bx^3}{3}\right]_{-2b}^{2b} = \frac{16b^4}{3}$$

Note that $I_{y\text{ rectangle}}$ is 16 times larger than $I_{x\text{ rectangle}}$ for a rectangle of these dimensions.

We now understand why it took more force to bend an eraser when turned on its side. The cross section of a typical eraser is 20 mm wide and 10 mm high, meaning that I_y is about four times I_x. Thus it takes a larger applied moment to bend the eraser when turned on its side.

If the area is a standard shape (e.g. circle, rectangle), you can generally find the values of the moment of inertia in a reference table such as **Table C.1**. Tabulated moments of inertia about axes through the centroid (I_{xC} and I_{yC}) and about axes through the edges of the areas (I_x and I_y) are based on the application of (6.26A) and (6.26B).



$I_{x_1} + I_{y_1} = I_{x_2} + I_{y_2} = \bullet\bullet\bullet = \text{constant}$

Figure 6.4.5 For a given origin, the sum $(I_x + I_y)$ is a constant.

Though not proven here, the sum $(I_x + I_y)$ is a constant, independent of the orientation of axes (**Figure 6.4.5**). This sum is called the **polar**

moment of inertia and is represented as I_O. Analogous to the area moment of inertia, the polar moment of inertia expresses the distribution of the area relative to an axis through point O perpendicular to the area, and is used to evaluate an object's resistance to torsion (twisting).

To calculate the polar moment of inertia for the arbitrary shape in Figure 6.4.6, we define a differential area dA at a distance r from O. The polar moment of inertia is then:

$$I_O = I_x + I_y = \int_{\text{area}} r^2\, dA \tag{6.27}$$

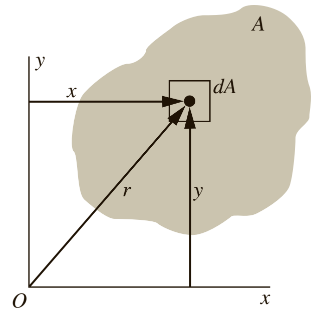


Figure 6.4.6 Calculating the polar moment of inertia about O.

Furthermore, there is one particular orientation of the axes for which the value of I_x or I_y is a maximum and the other is a minimum. These values are called the **principal moments of inertia** and the corresponding axes are called the **principal axes**.

Parallel-Axis Theorem

Consider that the moments of inertia have been calculated relative to axes located at the area's centroid; call these values I_{xC} and I_{yC}. The **parallel axis theorem** allows us to use these values of I_{xC} and I_{yC} to find the moments of inertia of the area relative to any other parallel axes by

$$I_x = I_{x_C} + A(y_1)^2 \tag{6.28A}$$

$$I_y = I_{y_C} + A(x_1)^2 \tag{6.28B}$$

where x_1 and y_1 are the distances between the centroidal axis and the parallel axis, as depicted in Figure 6.4.7. Equations (6.28A) and (6.28B) can be used to find I_x and I_y based on known values of I_{x_C} and I_{y_C}, for example the tabulated values from **Table C.1**. Alternately, if I_x and I_y are known, these equations can be rearranged to calculate I_{x_C} and I_{y_C}. The parallel axis theorem is frequently used when calculating the moment of inertia of composite areas.

From (6.28A) and (6.28B) we see that the moment of inertia increases as the axis is moved farther from the centroid. For any orientation of axes, the moment of inertia about axes through the centroid are the smallest values of I_x and I_y.

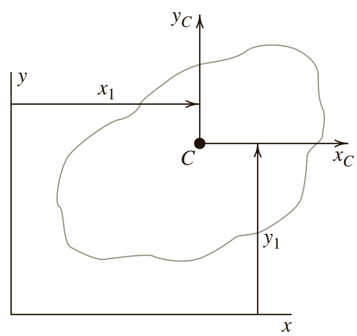


Figure 6.4.7 x_1 and y_1 indicate the distance from the xy axes to the axes acting through the centroid of the area.

Moment of Inertia of a Composite Area

The moment of inertia of an area that is composed of distinct parts can be found by determining I for each of the parts relative to the same axis, and then summing them together. Equations (6.26A) and (6.26B) for calculating the moment of inertia of an area apply to both positive and negative areas. In the case of a negative area (i.e., a hole), a negative sign is used in calculating the moment of inertia.

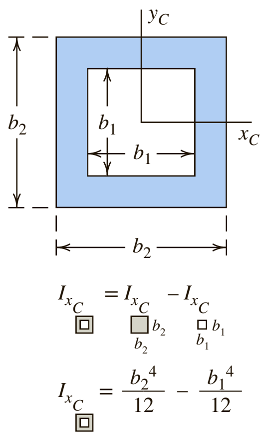


Figure 6.4.8 Calculating the moment of inertia of a composite area.

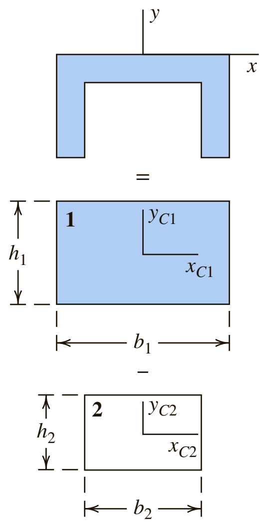


Figure 6.4.9 The centroids of the parts do not coincide with the x axis for determining I_x of the composite section.

Figure 6.4.8 illustrates the calculation of I_{x_C} for a composite section consisting of a square with a square hole. The centroid of each of the squares, as well as the composite square, coincides with the origin of the $x_C y_C$ axes. We look up I_{x_C} for a rectangle in **Table C.1**:

$$I_{x_C} = \frac{bh^3}{12}$$

Therefore the moment of inertia about the centroidal x axis of the composite area is

$$I_{x_C} = I_{2x_C} - I_{1x_C} = \frac{b_2^4}{12} - \frac{b_1^4}{12}$$

The calculation is more complex if the centroids of the parts of the composite area do not coincide with the axis about which we want to find the moment of inertia. For example, in **Figure 6.4.9** we want to find I_x relative the x axis at the top of the channel. The channel can be decomposed into two rectangles. The centroid of rectangle 1 is $h_1/2$ below the x axis and the centroid of rectangle 2 is $h_1 - h_2/2$ below the x axis. We must apply the parallel axis theorem to each rectangle to find the moment of inertia about the *same* x axis (I_{1x} and I_{2x}), and then sum them. The resulting calculations are presented here:

$$I_{1x} = I_{x_{C1}} + A_1 y_1^2 = \frac{b_1 h_1^3}{12} + b_1 h_1 \left(\frac{h_1}{2}\right)^2 = \frac{b_1 h_1^3}{3}$$

$$I_{2x} = I_{x_{C2}} + A_2 y_2^2 = \frac{b_2 h_2^3}{12} + b_2 h_2 \left(h_1 - \frac{h_2}{2}\right)^2$$

$$I_x = I_{1x} - I_{2x} = \frac{b_1 h_1^3}{3} - \frac{b_2 h_2^3}{12} - b_2 h_2 \left(h_1 - \frac{h_2}{2}\right)^2$$

Radius of Gyration

The moments of inertia of an area are related to another property of the area, the **radius of gyration**. The radius of gyration of an area A about the x axis (r_x) and about the y axis (r_y) has units of length, and is defined to be

$$r_x = \sqrt{\frac{I_x}{A}} \tag{6.29A}$$

$$r_y = \sqrt{\frac{I_y}{A}} \tag{6.29B}$$

A physical interpretation of the radius of gyration is the distance from axis at which we can concentrate the entire area and still have the same I as the original area. Engineers use the radius of gyration in the design and analysis of structural members, in particular to determine if a member subjected to axial compression loads is too slender and thus may fail due to buckling.

Check out the following examples of applications of this material.
- **Example 6.4.1 Moment of Inertia Using Integration**
- **Example 6.4.2 Moment of Inertia Using Parallel Axis Theorem**
- **Example 6.4.3 Moment of Inertia of a Composite Area**

EXAMPLE 6.4.1

The area in **Figure 1** is bounded on the left by the y axis, on the right by a parabola ($y = bx^2/a^2$) and on the top by the line $y = b$. Determine the moments of inertia about the x and y axes (I_x and I_y).

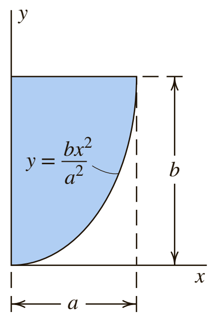


Figure 1 Area formed by a parabola, a horizontal line, and the y axis.

Goal Find I_x and I_y for the defined shape.

Given Equations and parameters that define the area, and a coordinate system.

Assume No assumptions needed.

Draw We draw two infinitesimal elements dA needed to integrate (6.26A) and (6.26B). The first represents dA at a distance y from the x axis (**Figure 2**) and the second dA at a distance x from the y axis (**Figure 3**).

Formulate Equations and Solve We integrate (6.26A) to determine I_x and (6.26B) to determine I_y. By defining dA as a horizontal or vertical slice we reduce the double integral in (6.26A) and (6.2B) to a single integral.

Referring to **Figure 2** we develop an expression for dA and integrate (6.26A):

$$dA = x\,dy = \frac{a}{\sqrt{b}}\sqrt{y}\,dy$$

$$I_x = \int y^2\,dA = \int_0^b (y^2)\frac{a}{\sqrt{b}}\sqrt{y}\,dy = \frac{a}{\sqrt{b}}\int_0^b y^{5/2}\,dy = \frac{a}{\sqrt{b}}\frac{2}{7}y^{7/2}\Big]_0^b = \boxed{\frac{2}{7}ab^3}$$

Referring to **Figure 3** we develop an expression for dA and integrate (6.26B):

$$dA = (b - y)\,dx = \left(b - \frac{bx^2}{a^2}\right)dx$$

$$I_y = \int x^2\,dA = \int_0^a x^2\left(b - \frac{bx^2}{a^2}\right)dx = \frac{bx^3}{3} - \frac{b}{a^2}\frac{x^5}{5}\Big]_0^a = \boxed{\frac{2}{15}a^3b}$$

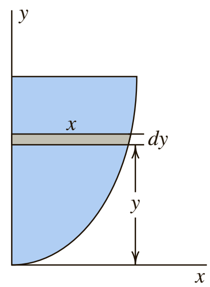


Figure 2 An infinitesimal element dA at a distance y from the x axis.

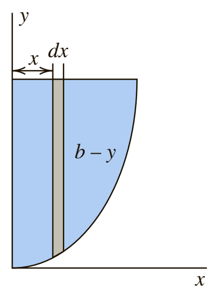


Figure 3 An infinitesimal element dA at a distance x from the y axis.

Check It is always a good idea to check the units. The units of I_x and I_y are correct because they are length^4 (ab^3 and a^3b). We can check if our results are reasonable by comparing with the moment of inertia of a similar standard shape. We choose a right triangle of height b and base a, with its point at the origin (see Example 6.4.2). Because the triangle's area is a bit smaller than the parabolic shape, we would expect the triangle's I_x and I_y to be smaller. $I_{x\,\text{triangle}} = 0.25ab^3 < I_{x\,\text{parabola}} = 0.29ab^3$ and $I_{y\,\text{triangle}} = 0.08a^3b < I_{y\,\text{parabola}} = 0.13a^3b$. Our answers look reasonable.

EXAMPLE 6.4.2

Calculate I_x, I_y, r_x and r_y for the right triangle shown in **Figure 1**.

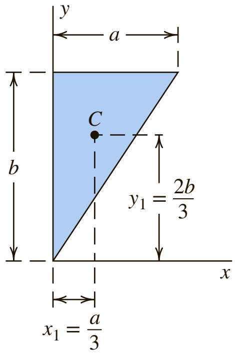


Figure 1 Right triangle with centroid at C.

Goal Find the (a) moment of inertia and (b) the radius of gyration about the x and y axes for the defined right triangle.

Given Dimensions of the triangle, location of its centroid, and a coordinate system.

Assume No assumptions needed.

Draw No drawing needed.

Formulate Equations and Solve (a) A triangle is a standard shape whose properties are tabulated in Appendix C. From Table C.1 we find the area of the triangle and its moment of inertia about x and y axes through the centroid.

$$A = \frac{ab}{2}$$

$$I_{x_C} = \frac{ab^3}{36}$$

$$I_{y_C} = \frac{a^3b}{36}$$

We then use the parallel axis theorem (6.28A) to determine the moment of inertia about the x axis. The x axis is at distance $y_1 = 2b/3$ from the centroidal x axis.

$$I_x = I_{x_C} + Ay_1^2 = \frac{ab^3}{36} + \frac{ab}{2}\left(\frac{2b}{3}\right)^2 = \boxed{\frac{ab^3}{4}}$$

Similarly, we use (6.28B) to determine the moment of inertia about the y axis, which is at distance $x_1 = a/3$ from the centroidal y axis.

$$I_y = I_{y_C} + Ax_1^2 = \frac{a^3b}{36} + \frac{ab}{2}\left(\frac{a}{3}\right)^2 = \boxed{\frac{a^3b}{12}}$$

(b) We use the results from (a), (6.29A) and (6.29B) to determine the radii of gyration.

$$r_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{ab^3/4}{ab/2}} = \sqrt{\frac{3b^2}{2}} = \boxed{b\sqrt{\frac{3}{2}}}$$

$$r_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{a^3b/12}{ab/2}} = \sqrt{\frac{a^2}{6}} = \boxed{\frac{a}{\sqrt{6}}}$$

Check I_y is tabulated in Table C.1, and our result agrees with the table. We could use (6.26A) and integrate to check I_x.

EXAMPLE 6.4.3

Calculate I_x and I_y for the T-section with respect to the centroidal axes shown in Figure 1.

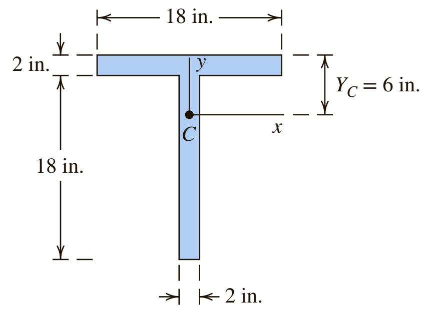


Figure 1 T-section with centroid at C.

Goal Find I_x and I_y for the defined T-section.

Given Dimensions of the T-section, location of its centroid, and a coordinate system.

Assume No assumptions needed.

Draw We divide the T-section into two rectangles and identify the centroid for each one (Figure 2).

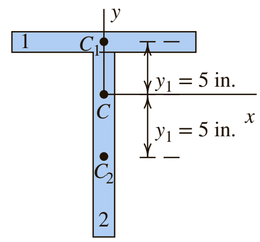


Figure 2 The composite section is divided into two rectangles.

Formulate Equations and Solve We find the moment of inertia of the composite section by finding I_x and I_y for each rectangle and summing them.

$$I_x = I_{1x} + I_{2x} \qquad \text{and} \qquad I_y = I_{1y} + I_{2y}$$

Table C.1 gives the moment of inertia for a rectangle about axes through its centroid. For each rectangle we need to use the parallel axis theorem (6.28A) to find the moment of inertia about the x axis.

$$I_{ix} = I_{x_{Ci}} + Ay_{1i}^2$$

Since the y axis of the T-section coincides with the centroidal y axes for both of the rectangles we do not need the parallel axis theorem for I_y. We use a tabular format for our calculations to keep the calculations organized:

Rectangle	Area in.^{2}	I_{x_C} in.^{4}	y_1 in.	Ay_1^2 in.^{4}	I_{y_C} in.^{4}
1	36.0	12.0	5.0	900	972
2	36.0	972	5.0	900	12.0
Σ		984		1,800	984

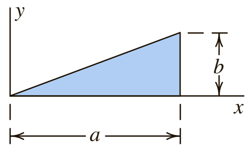
$$I_x = \sum I_{ix} = \sum I_{x_{Ci}} + \sum Ay_{1i^2} = 984 \text{ in.}^4 + 1{,}800 \text{ in.}^4 = \boxed{2{,}784 \text{ in.}^4}$$

$$I_y = \sum I_{iy} = \boxed{984 \text{ in.}^4}$$

Check There is no easy way to check our solution, but we could use (6.26A) and (6.26B) and integrate to find I_x and I_y.

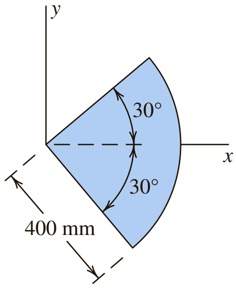
EXERCISES 6.4

6.4.1. **[*]** Use integration to evaluate the moments of inertia of area I_x and I_y of the triangle shown.



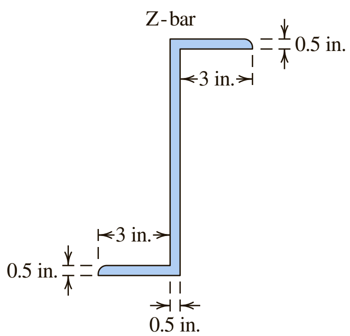
EX 6.4.1

6.4.2. **[*]** Use integration to evaluate the moments of inertia of area I_x and I_y of the circular sector shown.



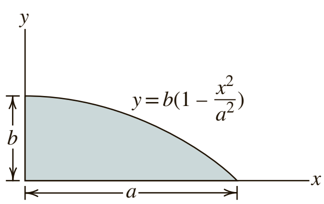
EX 6.4.2

6.4.3. **[*]** Use integration to evaluate the moments of inertia of area I_{xC} and I_{yC} of the Z-bar shown. Ignore the rounded edges on the cross-section.



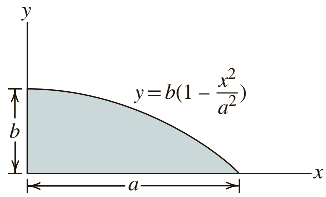
EX 6.4.3

6.4.4. **[*]** With respect to the axes shown, calculate I_x of the shaded region.



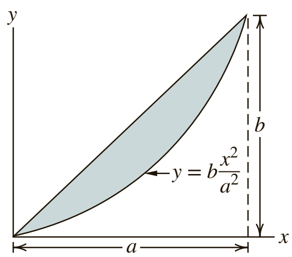
EX 6.4.4

6.4.5. **[*]** With respect to the axes shown, calculate I_y of the shaded region.



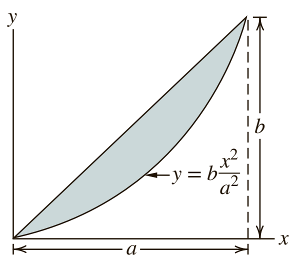
EX 6.4.5

6.4.6. **[*]** With respect to the axes shown, calculate I_x of the shaded region.



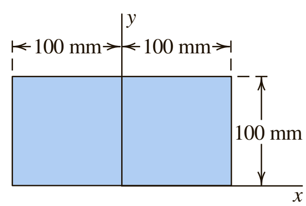
EX 6.4.6

6.4.7. **[*]** With respect to the axes shown, calculate I_y of the shaded region.



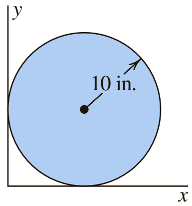
EX 6.4.7

6.4.8. **[*]** Determine I_x and I_y of the rectangular area shown using the properties in **Table C.1**.



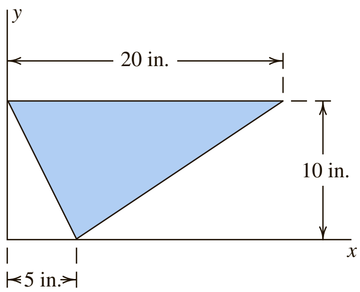
EX 6.4.8

6.4.9. [*] With respect to the axes shown, calculate the moments of inertia of area I_x and I_y of the circular area using the properties in **Table C.1**.



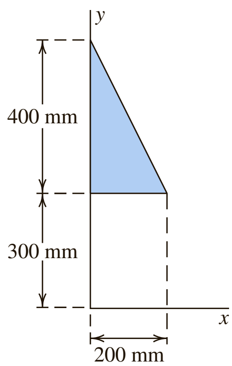
EX 6.4.9

6.4.10. [*] With respect to the axes shown, calculate the moments of inertia of area I_x and I_y of the triangular area using the properties in **Table C.1**.



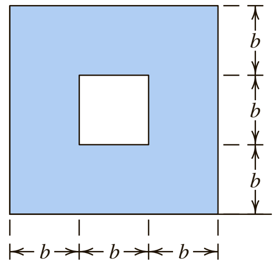
EX 6.4.10

6.4.11. [*] For the axes shown, determine I_x and I_y of the triangular shaded area using the properties in **Table C.1**.



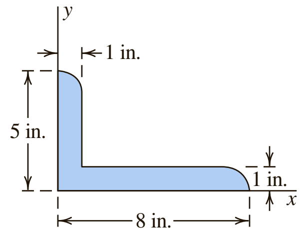
EX 6.4.11

6.4.12. [*] Determine the moments of inertia with respect to axes through the centroid I_{xC} and I_{yC} of the shaded area shown.



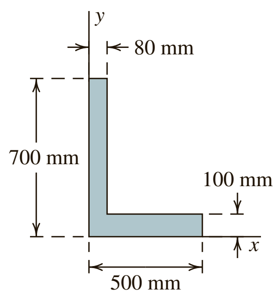
EX 6.4.12

6.4.13. [*] For the axes shown, determine the moments of inertia of area I_x and I_y of the unequal leg angle section. Ignore the effect of the rounded edges.



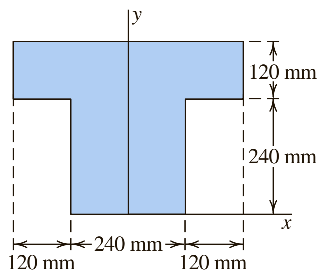
EX 6.4.13

6.4.14. [*] For the axes shown, determine the moments of inertia of area I_x and I_y of the composite area.



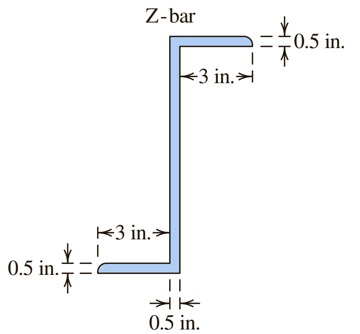
EX 6.4.14

6.4.15. [*] For the axes shown, determine the moments of inertia of area I_x and I_y of the T-shaped area.



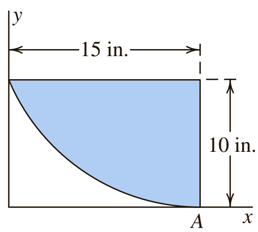
EX 6.4.15

6.4.16. [*] Using information from Appendix C, determine I_{xc} and I_{yc} of the composite area. Ignore the effects of the rounded ends.



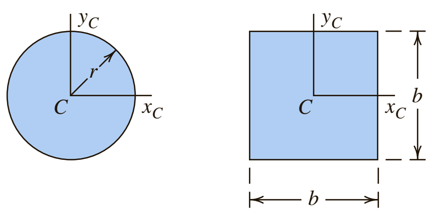
EX 6.4.16

6.4.17. [**] Use integration to evaluate the moments of inertia of area I_x and I_y of the parabolic section with its vertex at A.



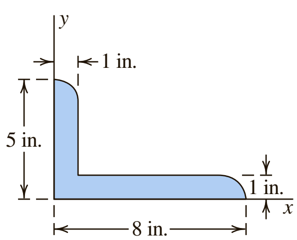
EX 6.4.17

6.4.18. [**] The circle and the square shown have the same area. Find moments of inertia I_{xc} and I_{yc} of the circle and the square in terms of r.



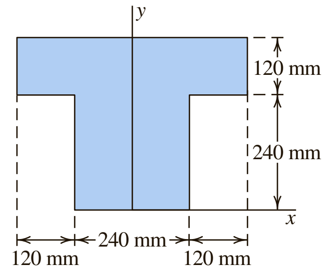
EX 6.4.18

6.4.19. [**] Determine the location of the centroid and the moments of inertia of area I_{xc} and I_{yc} of the unequal leg angle section. Should your answer be smaller or larger than if you calculated I_x and I_y about the xy axes shown? Ignore the effect of the rounded edges.



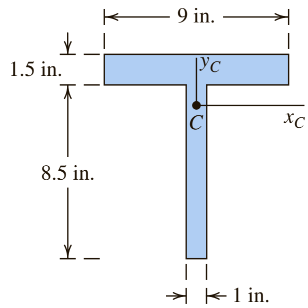
EX 6.4.19

6.4.20. [**] Determine the location of the centroid and the moments of inertia of area I_{xc} and I_{yc} of the T-shaped area shown.



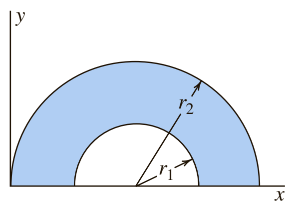
EX 6.4.20

6.4.21. [**] Determine the location of the centroid and the moments of inertia of area I_{xc} and I_{yc} of the T-shaped area shown.



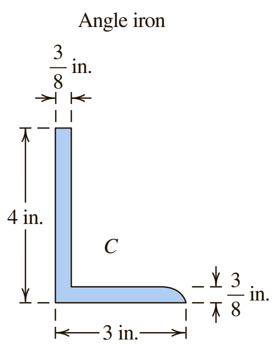
EX 6.4.21

6.4.22. [**] For the axes shown, determine the moments of inertia of area I_x and I_y of the composite area.



EX 6.4.22

6.4.23. [**] Determine the location of the centroid C and the moments of inertia of area I_{xc} and I_{yc} of the composite area shown. Take into account the quarter circle on the end of the horizontal leg.

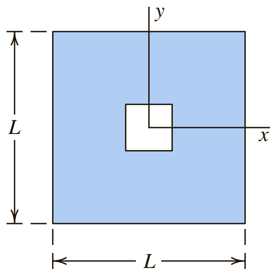


EX 6.4.23

6.4.24. [, computer]** The figure shows the cross section of a square tube of outer dimension L. Vary the dimension of the square hole from 0 to L and calculate the moment of inertia about the x axis I_x and area A. Define I_o as the moment of inertia of the solid square and A_o as the area of the solid square. Create two plots

a. I_x ($\%I_o$) versus hole dimension ($\%L$)
b. I_x ($\%I_o$) versus area A ($\%A_o$)

When $I_x = 60\%I_o$, what is the hole dimension? Using I_x as an indicator of member strength and A as an indicator of cost, describe whether reducing the area by using a hollow section is or is not a cost effective strategy.

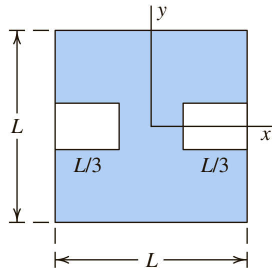


EX 6.4.24

6.4.25. [, computer]** The wide-flange section shown is created by removing two notches from a solid square of dimension $L \times L$. Vary the height of the notch from 0 to L and calculate the moment of inertia about the x axis I_x and area A. Define I_o as the moment of inertia of the solid square and A_o as the area of the solid square.

a. Plot I_x ($\%I_o$) versus notch height ($\%L$). At what notch height does $I_x = 80\%I_o$? How large is A at this notch height? Using I_x as an indicator of member strength and A as an indicator of cost, describe whether reducing the area by using a notched section is or is not a cost effective strategy.

b. Change the width of each notch to $5L/12$, and perform the same I_x calculations. Add an additional I_x ($\%I_o$) versus notch height ($\%L$) line to your plot. How does changing the width of the notch affect I_x?



EX 6.4.25

6.5 JUST THE FACTS

Center of Mass, Center of Gravity, and the Centroid

The individual particle masses summed together are the **total mass** of the object. The total mass is

$$M = \int_{\text{volume}} \rho\, dV \text{ (general case)} \tag{6.1}$$

where ρ is the object's density in mass/volume, $\rho\, dV$ is the mass of a volume element dV of the object, and integration is throughout the object's volume.

The weight of the object is given by

$$Mg = W = \int_{\text{volume}} dW = \int_{\text{volume}} \rho g\, dV \tag{6.2A}$$

Alternately, (6.2A) can be rewritten in terms of **specific weight** γ, (where γ is weight per unit volume as is ρg) as

$$Mg = W = \int_{\text{volume}} dW = \int_{\text{volume}} \gamma\, dV \tag{6.2B}$$

The **center of mass** of an object in a uniform gravitational field is at location

$$X_M = \frac{\int_{\text{volume}} x\rho\, dV}{M}; \quad Y_M = \frac{\int_{\text{volume}} y\rho\, dV}{M}; \quad Z_M = \frac{\int_{\text{volume}} z\rho\, dV}{M} \tag{6.4}$$

If we can treat the gravity field as uniform and parallel (which is approximately true for small objects on Earth), the center of mass (as defined in (6.4)) is also the location of the **center of gravity**.

The **centroid of a volume** is the location in a volume composed of a **homogeneous** material (meaning that it is uniform throughout the volume) and is at

$$X_C = \frac{\int_{\text{volume}} x\, dV}{V_{total}}; \quad Y_C = \frac{\int_{\text{volume}} y\, dV}{V_{total}}; \quad Z_C = \frac{\int_{\text{volume}} z\, dV}{V_{total}} \tag{6.5}$$

The locations of the centroid of several standard volumes are presented in Appendix C. The centroid is also the location of the center of mass and center of gravity if the volume is of uniform density.

If we decompose a composite volume into one made up of N standard volumes, we can use knowledge of the location of the centers of gravity of the N standard volumes to find the location of the center of gravity (X_G, Y_G, Z_G) of the composite volume. Call W_i the weight of an individual volume, and call X_{iG}, Y_{iG}, Z_{iG} the location of its center of gravity. Therefore the location of the **center of gravity of the composite volume** is

$$X_G = \frac{\sum_{i=1}^{N} W_i X_{iG}}{W_{\text{tot}}}; \quad Y_G = \frac{\sum_{i=1}^{N} W_i Y_{iG}}{W_{\text{tot}}}; \quad Z_G = \frac{\sum_{i=1}^{N} W_i Z_{iG}}{W_{\text{tot}}} \tag{6.7A}$$

Similarly, the **center of mass of a composite volume** is

$$X_M = \frac{\sum_{i=1}^{N} M_i X_{iM}}{M_{\text{tot}}}; \quad Y_M = \frac{\sum_{i=1}^{N} M_i Y_{iM}}{M_{\text{tot}}}; \quad Z_M = \frac{\sum_{i=1}^{N} M_i Z_{iM}}{M_{\text{tot}}} \tag{6.7B}$$

and the **centroid of the composite volume** is at

$$X_C = \frac{\sum_{i=1}^{N} V_i X_{iC}}{V_{\text{tot}}}; \quad Y_C = \frac{\sum_{i=1}^{N} V_i Y_{iC}}{V_{\text{tot}}}; \quad Z_C = \frac{\sum_{i=1}^{N} V_i Z_{iC}}{V_{\text{tot}}} \tag{6.7C}$$

The **centroid of an area**, which is its geometric center, is at

$$X_C = \frac{\int\limits_{\text{area}} x\,dx\,dy}{A_{\text{tot}}} \tag{6.10A}$$

$$Y_C = \frac{\int\limits_{\text{area}} y\,dx\,dy}{A_{\text{tot}}} \tag{6.10B}$$

where

$$A_{\text{tot}} = \int\limits_{\text{area}} dx\,dy \tag{6.8}$$

Appendix C presents the locations of centroids of several standard areas.

If we decompose a composite area into one made up of N standard areas, we can use knowledge of the locations of the centroids of the N standard areas to find the location of the centroid (X_C, Y_C) of the composite area by

$$X_C = \frac{\sum_{i=1}^{N} A_i X_{iC}}{A_{\text{tot}}}; \quad Y_C = \frac{\sum_{i=1}^{N} A_i Y_{iC}}{A_{\text{tot}}} \tag{6.12}$$

Distributed Force Acting on a Boundary

The single point force equivalent to a distributed line load is

$$\text{Total force in } y \text{ direction} = F_y = \int\limits_{\text{span}} \omega\,dx \tag{6.13}$$

where ω is the distributed line load oriented in the y direction and has dimension force/length, and "span" refers to the length over which ω is distributed. The point at which the single point force acts is called the **centroid of a line load** and is at

$$X_C = \frac{\int\limits_{\text{span}} x(\omega\,dx)}{F_y} \underbrace{=}_{\substack{\text{substituting in}\\ \text{from (6.13) for } F_y}} \frac{\int\limits_{\text{span}} x(\omega\,dx)}{\int\limits_{\text{span}} (\omega\,dx)} \tag{6.15}$$

If a line load distribution is a **standard line load distribution** (which is one that can be described as a simple geometric shape), the data in Appendix C can be used to locate the centroid. If a distributed force can be decomposed into N standard distributions, we can use the centroid locations of the standard distributions as the basis for finding the

centroid of the composite distribution. The magnitude of the total force $\boldsymbol{F}_{\text{total}}$ is

$$F_{\text{total}} = \sum_{i=1}^{N} \underbrace{F_i}_{\substack{\text{force of each}\\ \text{standard}\\ \text{distribution}}} \tag{6.16}$$

where F_i is the total force associated with distribution i. The location of the centroid of the composite distribution is then at

$$\boxed{X_C = \frac{\sum_{i=1}^{N} F_i X_{iC}}{F_{\text{total}}}} \tag{6.17}$$

where X_{iC} and F_i are the centroid and total load associated with the ith standard distribution.

A **pressure load** is a force distributed over an area. The single point force that is equivalent to a pressure load acting on the boundary of a system is

$$\boxed{\text{Total force in } z \text{ direction} = F_z = \iint\limits_{\substack{\text{surface}\\ \text{area}}} p\,dx\,dy} \tag{6.18}$$

where p is the pressure load in dimensions of force/area, the integrand $p\,dx\,dy$ is the force acting on the small area $dx\,dy$, and "surface area" in the limit refers to the total area over which the pressure acts. This location at (X_C, Y_C) of this single point force is called the **center of pressure** (or **pressure center**) and is at

$$Y_C = \frac{\iint\limits_{\substack{\text{surface}\\ \text{area}}} y\,p\,dx\,dy}{F_z} \underbrace{=}_{\substack{\text{substituting}\\ \text{in from (6.17)}}} \frac{\iint\limits_{\substack{\text{surface}\\ \text{area}}} y\,p\,dx\,dy}{\iint\limits_{\substack{\text{surface}\\ \text{area}}} p\,dx\,dy} \tag{6.20A}$$

$$X_C = \frac{\iint\limits_{\substack{\text{surface}\\ \text{area}}} x\,p\,dx\,dy}{F_z} \underbrace{=}_{\substack{\text{substituting}\\ \text{in from (6.17)}}} \frac{\iint\limits_{\substack{\text{surface}\\ \text{area}}} x\,p\,dx\,dy}{\iint\limits_{\substack{\text{surface}\\ \text{area}}} p\,dx\,dy} \tag{6.20B}$$

If we decompose a composite pressure distribution into one made up of N standard distributions, we can use knowledge of the locations of pressure centers of the N standard pressure distributions to find the pressure center of the composite pressure distribution. This works for both uniform pressure distributions and for nonuniform pressure distributions.

Hydrostatic Pressure

If a liquid acts at the boundary of a system, the pressure exerted by the hydrostatic pressure is normal to the boundary and oriented so as to push on the boundary. The hydrostatic pressure at depth h is given by the expression:

$$\boxed{p = p_o + \rho_{\text{liq}} g h} \tag{6.21A}$$

where p_o is the atmospheric pressure acting on the top surface of the liquid, ρ_{liq} is the density of the liquid, and g is the gravitational acceleration. The quantity ρg is called the specific weight of the liquid.

The pressure p in (6.21A) is referred to as the **absolute pressure**. Sometimes fluid pressure is measured with instruments that read pressure at depth relative to atmospheric pressure. This relative pressure is called **gage pressure** and is expressed as

$$\boxed{p_{\text{gage}} = \rho_{\text{liq}} g h} \tag{6.22A}$$

Buoyancy force is an upward (opposite gravity) force that is produced by a surrounding fluid acting on a fully or partially submerged object. The buoyancy force is defined as

$$F_{\text{buoy}} = \rho_{\text{liq}} g V_{\text{subm}}$$

where ρ_{liq} is the density of the liquid and V_{subm} is the volume of liquid displaced by a solid object. If the weight of an object is balanced by the buoyancy force, the object is in mechanical equilibrium. Being in equilibrium, the object will move neither up nor down in the fluid. Alternately, if the weight and buoyancy force do not balance each other, the object either sinks or rises in the liquid:

$$F_{\text{buoy}} < W; \quad \text{object sinks} \tag{6.25A}$$

$$F_{\text{buoy}} > W; \quad \text{object rises} \tag{6.25B}$$

Area Moment of Inertia

The **moment of inertia** is a property of an area that describes the distribution of that area relative to a specified axis. The moment of inertia of an area has units of $(\text{length})^4$ and is defined about the x axis and y axis as

$$\boxed{I_x = \int_{\text{area}} y^2 \, dA} \tag{6.26A}$$

$$\boxed{I_y = \int_{\text{area}} x^2 \, dA} \tag{6.26B}$$

The moments of inertia of selected standard shapes are found in Appendix C.

The area moment of inertia integrals in (6.26A) and (6.26B) are also called the **second area integrals**. Equations (6.26A) and (6.26B) apply to both positive and negative areas. In the case of a negative area (i.e., a hole), a negative sign is used in front of the integrals.

The sum $(I_x + I_y)$ is a constant, independent of the orientation of axes. This sum is called the **polar moment of inertia**. Analogous to the area moment of inertia, the polar moment of inertia expresses the distribution of the area relative to an axis through point O perpendicular to the area. The polar moment of inertia is:

$$\boxed{I_O = I_x + I_y = \int_{\text{area}} r^2\, dA} \tag{6.27}$$

There is one particular orientation of the axes for which the value of I_x or I_y is a maximum and the other is a minimum. These values are called the **principal moments of inertia** and the corresponding axes are called the **principal axes**.

The **parallel axis theorem** relates moments of inertia about the centroidal axes (I_{xc} and I_{yc}) to moments of inertia of the area relative to any other parallel axes by

$$\boxed{I_x = I_{x_C} + A(y_1)^2} \tag{6.28A}$$

$$\boxed{I_y = I_{y_C} + A(x_1)^2} \tag{6.28B}$$

where x_1 and y_1 are the distances between the centroidal axes and the parallel axes. Equations (6.28A) and (6.28B) can be used to find I_x and I_y based on known values of I_{x_C} and I_{y_C}, for example the tabulated values from **Table C.1**. Alternately, if I_x and I_y are known, these equations can be rearranged to calculate I_{x_C} and I_{y_C}.

The moment of inertia of an area that is composed of distinct parts can be found by determining I for each of the parts relative to the same axis, and then summing them together.

The **radius of gyration** is the distance from axis at which we can concentrate the entire area and still have the same I as the original area. The radius of gyration of an area A about the x axis (r_x) and about the y axis (r_y) has units of length, and is defined to be

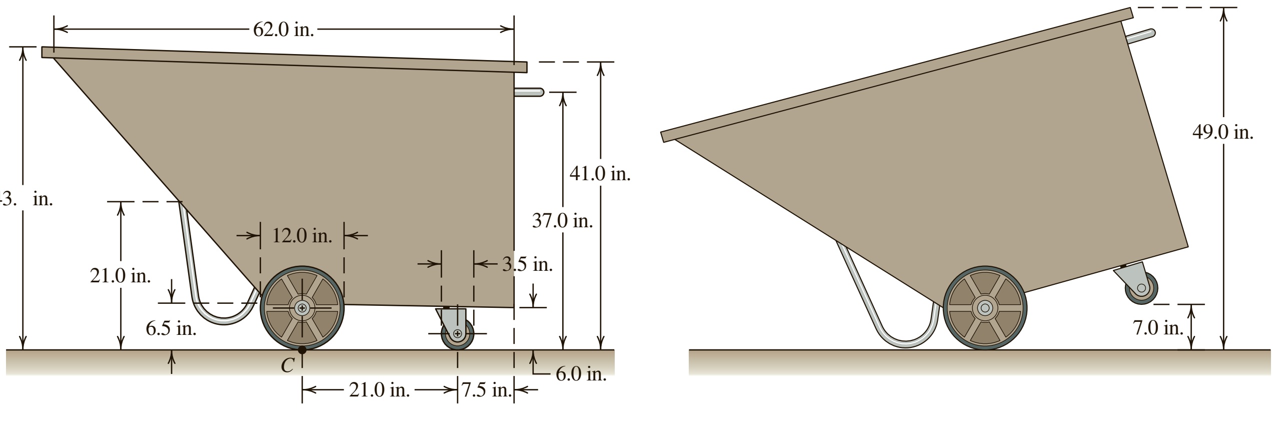
$$\boxed{r_x = \sqrt{\frac{I_x}{A}}} \tag{6.29A}$$

$$\boxed{r_y = \sqrt{\frac{I_y}{A}}} \tag{6.29B}$$

SYSTEM ANALYSIS (SA) EXERCISES

SA6.1 What Does It Take to Empty the Trash?

Consider the trash cart shown in **Figure SA6.1.1** (this cart should look familiar as it was first introduced in **SA4.2**). For the loading in **Figure SA6.1.2** address the following:



(*a*) (*b*)

Figure SA6.1.1 Main cart positions with dimensions: (*a*) cart in normal position; (*b*) ready to be dumped.

(a) Create the free-body diagram for the filled cart shown in **Figure SA6.1.2.** Assume that the weight of the cart is negligible compared to the load.

(b) Find the location of the center of mass of the trash in the bin relative to a coordinate system with its origin at *C* (the contact point of the front wheel and the ground).

(c) How much force does the janitor need to apply at the cart handle so the small wheels just come off the ground? If 64 pounds is considered to be a safe lifting load for a healthy adult male[1] is the janitor likely to injure his back by just lifting the small wheels off the ground?

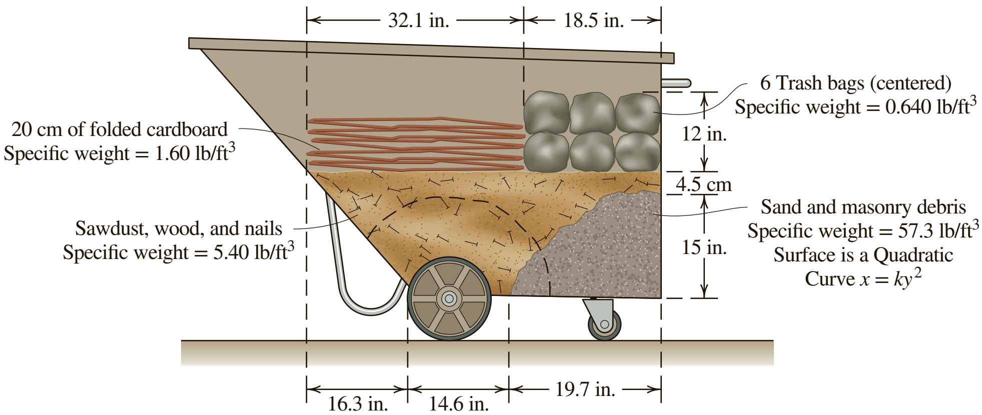


Figure SA6.1.2 Distribution of loads inside the 29-in.-wide cart.

[1]What constitutes a safe lifting load for a healthy male depends on such factors as posture and how many times the load is to be lifted. The issue of what constitutes a safe lifting load is not resolved at all by OSHA, NIOSH, and ILO because of the complexity of spine/muscle interaction. More information can be found at https://www.osha.gov/SLTC/etools/electricalcontractors/materials/heavy.html

(d) How much would the force change if the sand and masonry debris had been loaded toward the front end of the bin? (Assume a slightly modified quadratic curve that is flipped as indicated with a dashed line.)

(e) In order to empty the cart, the janitor needs to bring it into a dump position where the metal strut touches the floor as indicated in **Figure SA6.1.1*b***). Without further calculations, plot the approximate lifting force for either (c) or (d) from the beginning until the material starts sliding out of the bin.

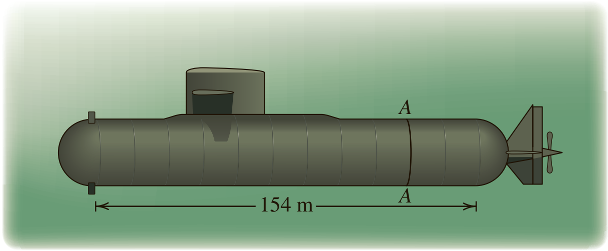
(f) Finally, draw the free-body diagram for the dump position for load distribution (b) when the two main wheels still rest on the floor. How much does the janitor have to lift at this point? What additional force will it take to begin dumping (i.e., to get the main wheels off the floor)?

SA6.2 Ballast in Submarines[2]

Consider an idealized model of a submarine in **Figure SA6.2.1**. We assume that it is a cylinder with length $L = 154$ m, outer radius $r_o = 7.2$ m, and an inner radius $r = 6.0$ m. The space in between can be filled with air or water to make the submarine rise or float.

(a) If the equipment, supplies, and people in the interior of the submarine have an average density of 1285 kg/m^3, what fraction of the ballast chamber must be filled with air to maintain neutral buoyancy?

(b) For most submarines the ballast chamber is divided into separate spaces so that if one is damaged, the entire submarine does not sink. If the ballast chamber of the vessel described above is divided into 12 equally spaced chambers along the length of the submarine, how many can fail before the submarine will be unable to maintain neutral buoyancy?



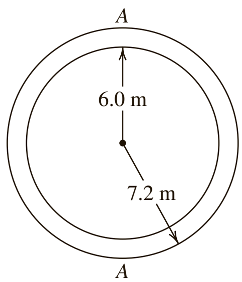


Figure SA6.2.1 The Kursk was one of the larger submarines in the Russian Navy. It was approximately 154 m long, 182 m tall, and 9 m wide, and was powered by two nuclear-powered turbines. When submerged, it displaced 18,300 tons of water.

SA6.3 How to Remove the Packages[3]

(a) The board shown in **Figure SA6.3.1*a*** is 20 m long and has a mass of 3 kg. On the front of each cube is its mass in kilograms. The goal is to remove the cubes without causing the board to tip over. Draw a freebody diagram of the board and use equilibrium conditions to determine in what order the cubes must be removed. Describe your approach mathematically and in writing.

(b) Repeat (a) for the board layout in **Figure SA6.3.1*b***.

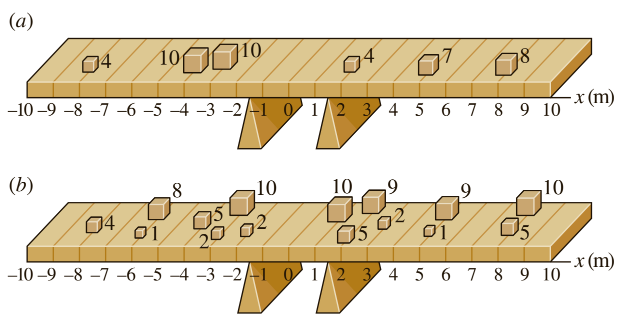


Figure SA6.3.1 A pathological board game that can be solved with statics.

[2]Adapted by Eric Nauman from an article by Jim Collins for the U.S. Airways Attache, March 2001, illustration by Nigel Holms.

[3]Adapted by Eric Nauman from No Tipping by Dennis E. Shasha in the April 2001 Issue of *Scientific American*.

SA6.4 The Freedom Ship

As of this printing, the Freedom Ship has not been completed, but it has been featured on the Discovery Channel's "Engineering the Impossible." It's projected cost is over \$9 billion and, if it is ever built, it will be approximately 1 mile long, 750 feet wide, and nearly 25 stories (~340 ft.) tall. It was planned to be a floating city continuously circling the globe, visiting most of the earth's inhabited coastal regions every two years. **Figure SA6.4.1** is a concept drawing of the ship.

Perhaps the most impressive statistic is the proposed weight of the ship, approximately 2.7–3.0 million tons. It will be able to house nearly 50,000 passengers with a crew of 15,000. Your goal is to figure out how deep this boat will sit in the water. A very simple analysis you might say. But it is a good starting point. The most difficult aspect is the actual construction. It is quite likely that completely new manufacturing techniques will have to be developed before the ship ever leaves port.

First, recall Archimedes' principle: the upward acting buoyancy force acting on a submerged (or partially submerged) object is equal to the weight of the water that is displaced (density of water = ~1000 kg/m^3). A relatively simple free-body diagram then yields an expression for the depth that the ship sits in the water in terms of its weight and the overall dimensions of the ship.

When faced with such large numbers, it is useful to do a few simple calculations to develop some intuition for the problem. How much higher or lower will the boat sit if you decrease or increase its overall weight by 10%? What would the change be if you lengthened or shortened the overall dimensions by 10%? This is a simple example of a sensitivity analysis and, in addition to helping an engineer better understand the problem, it may alert him or her to a previously unconsidered design parameter or problem constraint.

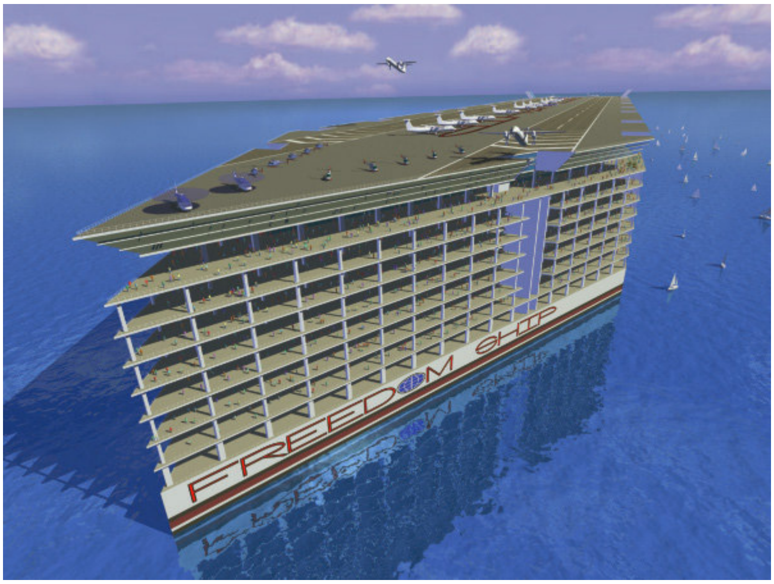


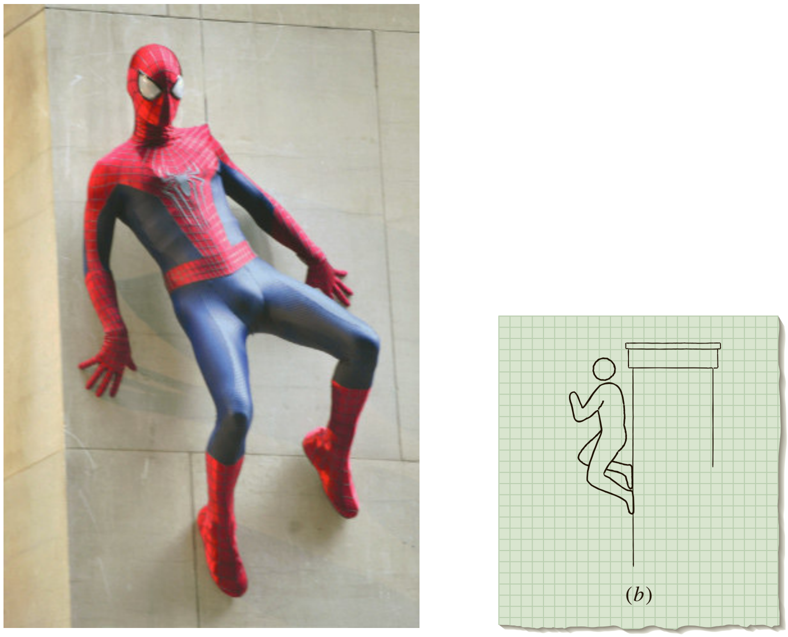
Figure SA6.4.1 A computer-generated overhead image of the proposed Freedom Ship (www.freedomship.com).

SA6.5 Center of Mass Calculations

As we all know, Spider-Man is able to stick to virtually any surface. In **Figure SA6.5.1**, he is contacting the wall with only his feet. Given that Peter Parker is 5 ft, 10 in. tall and weighs approximately 170 lb, calculate his center of gravity and estimate the reaction forces exerted on his feet.

At first glance, this is a very involved problem and a number of assumptions need to be made in order to obtain a good approximation. As always, start with a free-body diagram of Spider-Man, assuming that the center of gravity of each body part occurs at its midpoint. **Figure SA6.5.1*b*** may be helpful. Then think very carefully about the kinds of reactions that may be supported by our superhero's feet. If it helps, you can assume that the loads are evenly distributed between each foot.

The data in **Tables SA6.5.1** and **SA6.5.2** will be useful.



(a)

Figure SA6.5.1 (*a*) Spider-Man sticking to a wall; (*b*) a simplified two-dimensional view

Table SA6.5.1 Mean segment weights expressed as percentages of total body weight

Segment	Weight for Males	Weight for Females
Head and trunk	55.10	53.20
Total arm	5.77	4.97
Thigh	10.50	11.75
Shank and foot	6.18	6.68

Adapted from Williams and Lissner, *Biomechanics of Human Motion* (Philadelphia: W. B. Saunders, 1962).

Table SA6.5.2 Mean segment lengths expressed as percentages of total body height

Segment	Length for Males	Length for Females
Head and trunk	40.75	39.75
Total arm	32.9	33.3
Thigh	23.2	24.9
Shank and foot	28.95	29.95

Adapted from Williams and Lissner, *Biomechanics of Human Motion* (Philadelphia: W. B. Saunders, 1962).

SA6.6 Fighter Jet Design

The design of a high-performance jet, especially one that will see combat, is an extremely long process that requires input from teams of mechanical, electrical, chemical, and biomedical engineers. This problem will help you get started and give you an idea of some of the many design considerations involved. Choose one of the planes from **Table SA6.6.1** and use the information provided in the table and in **Figure SA6.6.1** to estimate the forces and moments that keep the wing attached to the fuselage. Suggested steps are as follows:

Step 1: Choose an operating mass and calculate the weight of the aircraft.

Step 2: Assume that all the lift force that keeps the plane in the air is generated by the wings. Let's further assume that the pressure is distributed evenly over the entire underside of the wing. In that case, the lift force acts through the centroid of the wing. Estimate the location of the wing's centroid. If the plane is moving at constant velocity, what is the pressure exerted on the underside of the wing?

Step 3: If we assume that the wing is constructed from a uniform piece of steel (7830 kg/m^3) 2 cm thick, what is the gravitational force exerted on the wing?

Step 4: Assume that the wing is cantilevered to the fuselage. Calculate the forces and moments that act on the wing at the fuselage. Keep in mind that there are several ways to do this. Does your answer reflect a best-case scenario, a worst-case scenario, or a relatively accurate estimate?

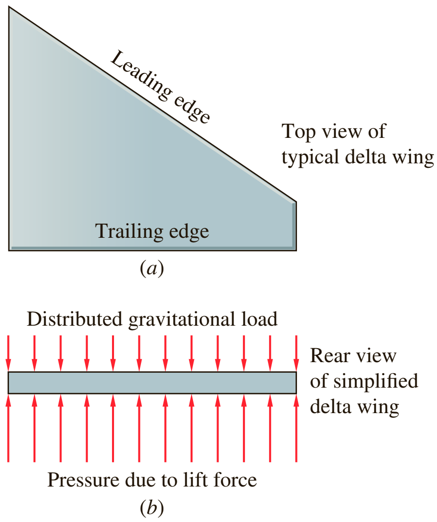


Figure SA6.6.1 A delta wing design commonly used on military fighter jets (e.g., the F-15E Eagle)

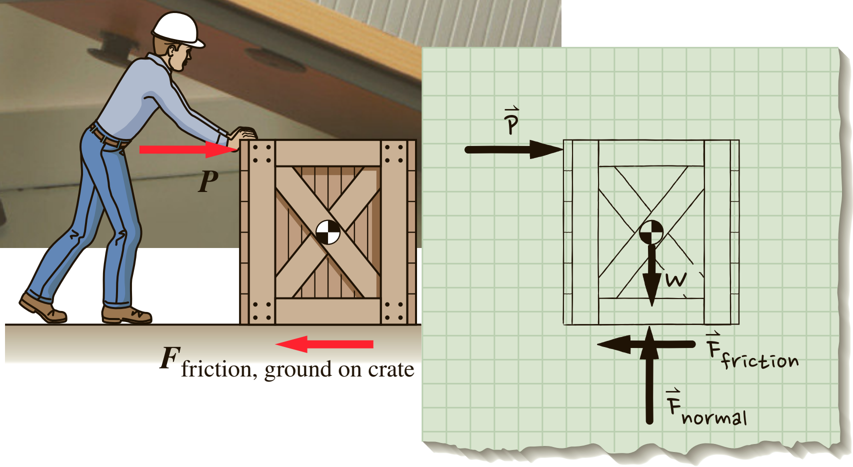
Table SA6.6.1 **Aircraft Specifications**

Aircraft	Wingspan (m)	Length (m)	Mass (kg) Empty	Normal	Maximum	Powerplant (kN)
Mirage 2000 C	9.13	14.36	7,500	10,680	17,000	1 × 64
F15-E Eagle	13.05	19.43	14,379		36,741	2 × 65
F/A 18 C Hornet	11.43	17.07	10,455	16,652	25,401	2 × 71
F 14A Tomcat	19.54	19.10	18,191	26,632	74,349	2 × 93
F 104 Starfighter	6.68	16.69	6,387	9,840	13,054	1 × 44
YF 22 Lightning II	13.11	19.56	13,608		26,308	2 × 156
F-16 Fighting Falcon	9.45	15.03	8,663	9,791	19,187	1 × 123
MIG 29 Fulcrum	11.36	17.32		15,300	19,700	2 × 49
A-10A Thunderbolt II	17.53	16.26	9,771	14,865	22,680	2 × 40
Sukhoi Su-37	15.16	22.18		26,000	34,000	2 × 130
Tornado ADV	13.91	18.68	14,502		27,986	2 × 40
B-LB Lancer Bomber	41.67	44.81	87,091		216,365	4 × 65
AV-8B Harrier	9.25	14.12	6,336	10,410	14,061 (STO) 8,596 (VTO)	1 × 106
SR-71 Blackbird	16.94	31.17			53,049	2 × 133

Adapted from *The Encyclopedia of World Military Aircraft*, edited by David Donald and Jon Lake (New York: Barnes & Noble, 2000).

DRY FRICTION AND ROLLING RESISTANCE

Getty Images



As we saw in Chapter 2, friction force is the resistance caused by interactions between the surfaces of objects when attempting to slide one surface across another. Friction force is oriented *parallel* to the two surfaces in a direction opposite the direction of (pending) sliding. The presence of friction may be critical to the operation of some devices (e.g., bicycle brakes) or be detrimental to others (e.g., journal bearings). Regardless, if friction is significant, its presence needs to be included in equilibrium analysis.

Rolling resistance is another force that may need to be accounted for in equilibrium analysis of rotating systems. You have experienced this burdensome force if you have ridden a bicycle with underinflated tires.

In this chapter we describe the nature of dry friction and rolling resistance, and outline how to include their presence in equilibrium analysis.

LEARNING OBJECTIVES

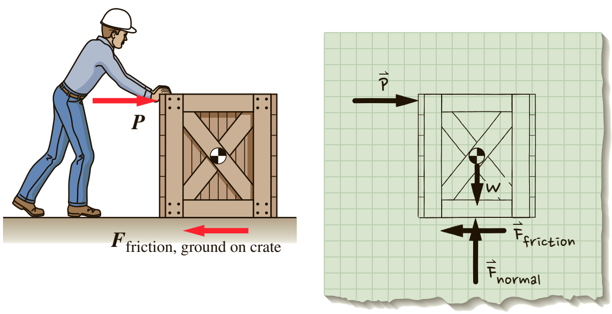
On completion of this chapter, you will be able to:

- **Describe the Coulomb friction model for static (non-moving) and kinetic (moving) situations.** (7.1)
- **Analyze static equilibrium problems that include dry friction.** (7.2)
- **Analyze static equilibrium problems that include rolling resistance.** (7.3)

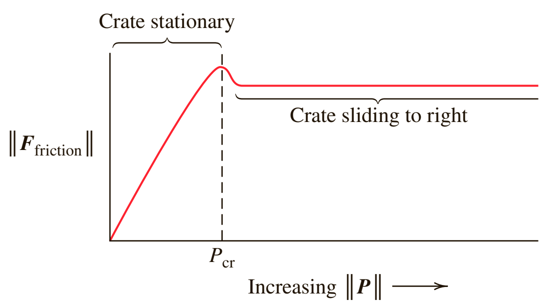
7.1 COULOMB FRICTION MODEL

Learning Objective: Describe the Coulomb friction model for static (non-moving) and kinetic (moving) situations.

We have all experienced friction when pushing something across the floor. For example, if you push on a crate of weight $\boldsymbol{W}$ as it rests on the ground, as in **Figure 7.1.1*a***, the friction force exerted by the ground on the crate is in the direction opposite to the sliding direction. An equal and opposite friction force (not shown in **Figure 7.1.1*a***) is exerted by the crate on the ground (per Newton's third law). Friction is a complicated phenomenon that is not fully understood and is an area of continuing research.[1] Current thinking is that friction results from microscopic irregularities called asperities present in all surfaces and from molecular



(*a*)



(*b*)

Figure 7.1.1 (*a*) Person pushing a crate and the crate's free-body diagram; (*b*) friction force $\boldsymbol{F}_{\text{friction}}$ versus push force $\boldsymbol{P}$.

[1]The area of work involved with studying friction and wear characteristics of material surfaces is called *tribology*.

attraction, as depicted in **Figure 7.1.2**. On a macroscopic scale, friction between surfaces results in the behavior shown in **Figure 7.1.1*b***; an increase in the magnitude of force $\boldsymbol{P}$ is balanced by an increase in the friction force ($\boldsymbol{F}_{\text{friction}}$) up to a maximum value (P_{cr}) and the crate remains stationary. An attempt to increase the magnitude of force $\boldsymbol{P}$ beyond P_{cr} results in the crate sliding to the right.

We can generalize the behavior in **Figure 7.1.1*b*** to say that the friction force ($\boldsymbol{F}_{\text{friction}}$) is always parallel to the contacting surfaces of the two objects and is directed so as to oppose their relative motion. Furthermore, the friction force is related to and limited by the normal contact force ($\boldsymbol{F}_{\text{normal}}$) and the characteristics (e.g., smoothness) of the objects' surfaces, it is perpendicular to the normal force, and normal contact force *must be* present in order for friction force to be present (but not vice versa). We can model the behavior of contacting surfaces of many systems with the elementary friction law proposed by Charles-Augustin de Coulumb in 1781. The **Coulumb friction law** states:

1. If the two solid objects remain stationary relative to one another (i.e., the two objects do not slide relative to one another), the friction force is such that

$$\|\boldsymbol{F}_{\text{friction}}\| < \mu_s \|\boldsymbol{F}_{\text{normal}}\| \tag{7.1}$$

where μ_s is called the **coefficient of static friction**. Sample values of μ_s are given in **Table 7.1** and should only be used if values from experiments on the actual contacting surfaces are not available. For specific cases the values in **Table 7.1** may be incorrect by more than 100%.

2. If the friction force is such that

$$\|\boldsymbol{F}_{\text{friction}}\| = \mu_s \|\boldsymbol{F}_{\text{normal}}\| \tag{7.2}$$

the objects are in a state of impending sliding relative to one another. Therefore, the product $\mu_s \|\boldsymbol{F}_{\text{normal}}\|$ is the largest magnitude of friction

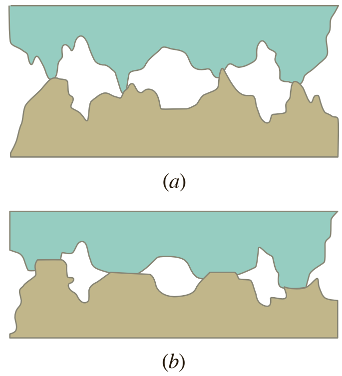


Figure 7.1.2 (*a*) Contacting surfaces touch at peaks of asperities; (*b*) Under load, contact area grows as asperities collapse.

Table 7.1 Sample Coefficients of Friction*

Materials	μ_s	μ_k
Mild steel on mild steel	0.74	0.57
Aluminum on mild steel	0.61	0.47
Copper on mild steel	0.53	0.36
Cast iron on cast iron	1.10	0.15
Brake material on cast iron	0.40	0.30
Leather on cast iron	0.60	0.56
Rubber on metal	0.40	0.30
Rubber on wood	0.40	0.30
Rubber on pavement	0.90	0.80
Leather on oak	0.61	0.52
Glass on nickel	0.78	0.56

*Eugene A. Avallone and Theodore Baumeister III, *Marks' Standard Handbook for Mechanical Engineers*, 10th Edition (McGraw-Hill Publishers, 1996).

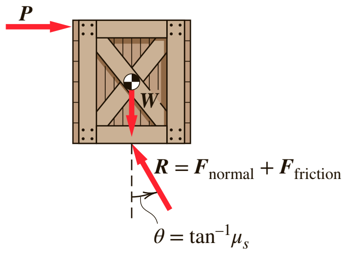


Figure 7.1.3 The resultant force $\boldsymbol{R}$ when friction is at its limit of $\mu_s\|\boldsymbol{F}_{\text{normal}}\|$.

force that can exist at the contacting surfaces without sliding occurring. When this maximum friction force has been developed, the angle θ between $\boldsymbol{F}_{\text{normal}}$ and the resultant $\boldsymbol{R}$ $(= \boldsymbol{F}_{\text{normal}} + \boldsymbol{F}_{\text{friction}})$ is called the **angle of friction**, and $\theta = \tan^{-1}\mu_s$ (**Figure 7.1.3**).

3. If the two contacting surfaces slide relative to one another, the friction force is given by

$$\|\boldsymbol{F}_{\text{friction}}\| = \mu_k \|\boldsymbol{F}_{\text{normal}}\| \tag{7.3}$$

where μ_k is the **coefficient of kinetic friction**. In general, for given contacting surfaces, $\mu_k < \mu_s$ (which is consistent with the behavior depicted in **Figure 7.1.1*b***). Sample values of μ_k are given in **Table 7.1**.

You might be wondering how to use the Coulomb friction law in static analysis. We illustrate its use by considering the crate sitting on the ground in **Figure 7.1.1**. For example, say that you are interested in

- *Finding* $\boldsymbol{F}_{\text{friction}}$ *for a given value of* $\boldsymbol{P}$ *if the crate is in equilibrium.* You would apply the conditions of equilibrium to calculate the size of $\boldsymbol{F}_{\text{friction}}$, followed by using (7.1) as a check; if the calculated value of $\|\boldsymbol{F}_{\text{friction}}\|$ is less than or equal to $\mu_s\|\boldsymbol{F}_{\text{normal}}\|$, the crate is stationary relative to the ground. If, on the other hand, the calculated value of $\|\boldsymbol{F}_{\text{friction}}\|$ is greater than $\mu_s\|\boldsymbol{F}_{\text{normal}}\|$, the crate is not stationary relative to the ground and will slide (and accelerate) to the right. This means that equilibrium is not possible with the given values of $\boldsymbol{P}$ and $\boldsymbol{W}$ (weight of the crate) and the character of the contacting surfaces (as indicated in the value of μ_s).
- *Finding the maximum allowable magnitude of* $\boldsymbol{P}$ *such that the crate will not slide.* You would apply the conditions of equilibrium to calculate the magnitude of $\boldsymbol{P}$ when $\|\boldsymbol{F}_{\text{friction}}\|$ is equal to $\mu_s\|\boldsymbol{F}_{\text{normal}}\|$ (the state of impending sliding represented by (7.2)).
- *Finding the magnitude of* $\boldsymbol{P}$ *such that the crate moves to the right at a constant speed.* You would apply the conditions of equilibrium to calculate the magnitude of $\boldsymbol{P}$ when $\|\boldsymbol{F}_{\text{friction}}\|$ is equal to $\mu_k\|\boldsymbol{F}_{\text{normal}}\|$ as indicated in (7.3).

Coulomb friction is active and present in many mechanical systems. It enables a bicycle to move forward. Where the rear wheel pushes backward on the ground, the ground pushes forward on the bicycle wheel, as depicted in **Figure 7.1.4**. This backward-forward force pair is the friction between the wheel and the ground. Without dry friction between the tire and the ground, a bicycle would not function.

Coulomb friction is also responsible for keeping tension in the main cables of the Golden Gate Bridge. Friction keeps the anchorages in place, as depicted in **Figure 7.1.5**. The greater the friction coefficient, the less the necessary weight of the anchorages.

Keep in mind that our discussion here focuses on dry friction. When a fluid film separates the surfaces, the contact is fully lubricated, and principles from fluid mechanics—which are outside the scope of this text—are required to describe the interaction of one object relative to the other.

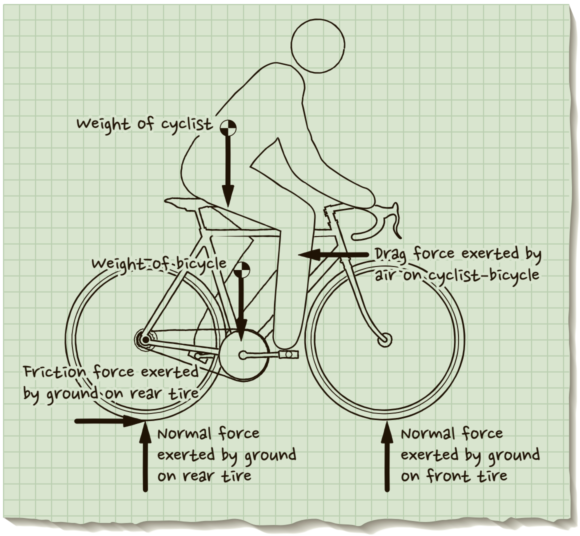


Figure 7.1.4 The friction force on the rear wheel pushes a bicycle forward.

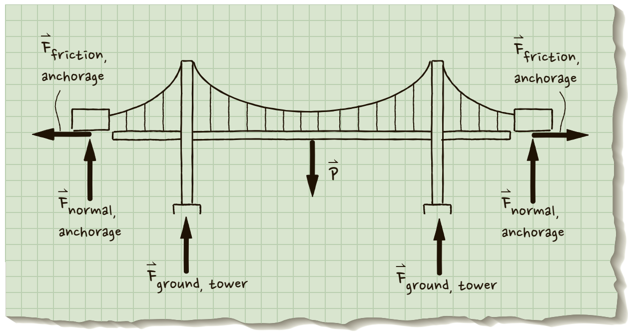


Figure 7.1.5 Friction keeps the bridge anchorages in place.

Check out the following example of an applications of this material.
- **Example 7.1.1 Dry Friction-Sliding or Tipping**

EXAMPLE 7.1.1

A person attempts to slide a 200-lb cabinet across the floor by pushing with a force of 70 lb at 3 feet above the base of the cabinet as shown in Figure 1. The coefficient of static friction between the cabinet and the floor is $\mu_s = 0.37$. Determine if the cabinet will slide or tip over.

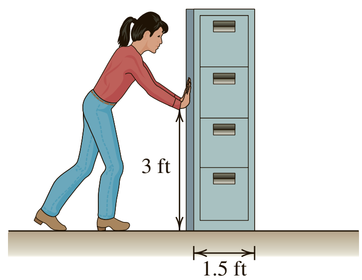


Figure 1 A person attempts to slide a cabinet across the floor.

Goal Determine if the forces applied to a cabinet will cause it to slide or tip over.

Given Information about relevant dimensions, the magnitude of the applied force, and μ_s.

Assume The system can be treated as planar.

Draw We draw a free-body diagram of the cabinet (Figure 2), placing its weight at the center of gravity and the normal force at the center of the base.

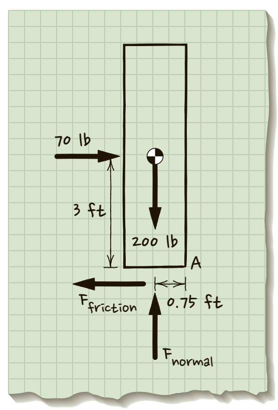


Figure 2 Free body diagram of cabinet.

Formulate Equations and Solve We first analyze for sliding.

$$\sum F_y(\uparrow +) = 0 = F_{\text{normal}} - 200\text{ lb} \quad \Rightarrow \quad F_{\text{normal}} = 200\text{ lb}$$

$$\sum F_x(\rightarrow +) = 0 = 70\text{ lb} - F_{\text{friction}} \quad \Rightarrow \quad F_{\text{friction}} = 70\text{ lb}$$

According to (7.2) impending sliding occurs when $\|\boldsymbol{F}_{\text{friction}}\| = \mu_s \|\boldsymbol{F}_{\text{normal}}\|$. Checking this condition reveals

$$\mu_s F_{\text{normal}} = (0.37)(200\text{ lb}) = 74\text{ lb}$$

$$F_{\text{friction}} = 70\text{ lb} < \mu_s F_{\text{normal}} = 74\text{ lb}$$

Since F_{friction} is less than 74 lb, the cabinet will not slide.

Now we analyze for tipping. The cabinet will tip about corner A, and when it tips $\boldsymbol{F}_{\text{normal}}$ will have moved to point A because A will be the only part of the cabinet touching the floor. Choosing A as the moment center, we check if the overturning moment is greater than the righting moment.

$$\sum M_{z@A}(\curvearrowleft +) \quad \Rightarrow \quad M_{\text{overturning}} = -(70\,\text{lb})(3\,\text{ft}) = -210\,\text{ft}\cdot\text{lb}$$

$$\sum M_{z@A}(\curvearrowleft +) \quad \Rightarrow \quad M_{\text{righting}} = (200\,\text{lb})(0.75\,\text{ft}) = 150\,\text{ft}\cdot\text{lb}$$

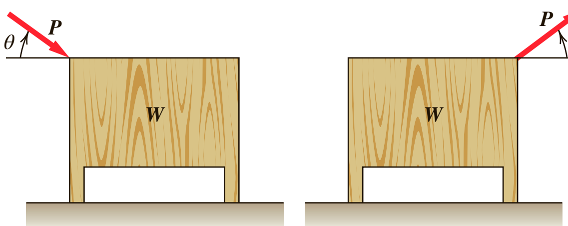
$$M_{\text{overturning}} > M_{\text{righting}}$$

We conclude that tipping will occur because the overturning moment is larger than the righting moment.

Check There is no good independent check of the solution. In this case, the check should consist of reviewing equations to ensure that all signs and calculations are correct.

EXERCISES 7.1

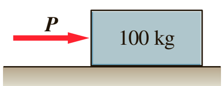
7.1.1. **[*]** Two students are moving desks around a classroom. Student (a) thinks pushing the desk will require less force. Student (b) thinks it will require less force to pull on it. Who is right? Explain your reasoning.



(a) (b)

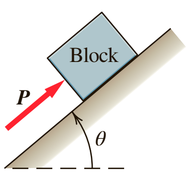
EX 7.1.1

7.1.2. **[*]** What is the magnitude of the horizontal force $\boldsymbol{P}$ that must be exerted on the 100-kg block to cause the block to move if the coefficient of static friction is 0.25?



EX 7.1.2

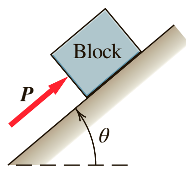
7.1.3. **[*]** What is the maximum angle θ for which the block of mass m will not slide down the incline if the coefficient of static friction is 0.30 and $\|\boldsymbol{P}\| = 0$?



EX 7.1.3

7.1.4. **[*]** If the static and kinetic coefficients of friction are 0.35 and 0.20, respectively, and $\|\boldsymbol{P}\| = 0$ determine the friction force acting on the block if

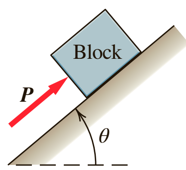
a. $\theta = 10°$ **b.** $\theta = 25°$



EX 7.1.4

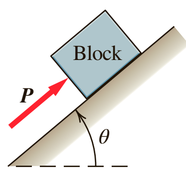
7.1.5. **[*]** If the static and kinetic coefficients of friction are 0.60 and 0.45, respectively, and the angle of the incline is 15°, determine the friction force acting on the block if

a. $P = 10$ N **b.** $P = 60$ N



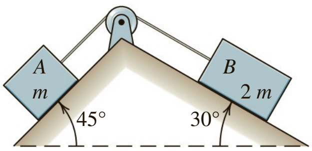
EX 7.1.5

7.1.6. **[*]** If the static coefficient of friction between the block and the incline is 0.60 and the angle of the incline is 15°, determine the range of values of the force P for which the block will not slide up or down the incline.



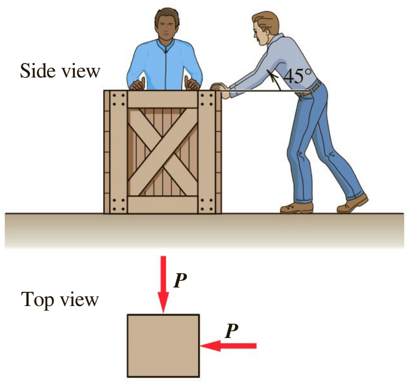
EX 7.1.6

7.1.7. [**] The coefficient of static friction between block A and its incline is 0.25. What must the minimum coefficient of static friction between block B and its incline be if the blocks are in equilibrium? The mass of block B is twice that of block A. If the coefficient of friction is less than this minimum, in which direction will the blocks slide?



EX 7.1.7

7.1.8. [**] Two people attempt to push an 80-lb box across the floor. They are pushing at right angles to one another and both are pushing downward with force P at 45° to the horizontal. If the coefficient of static friction is 0.25, determine the minimum force P that each person must apply to the box to move it.

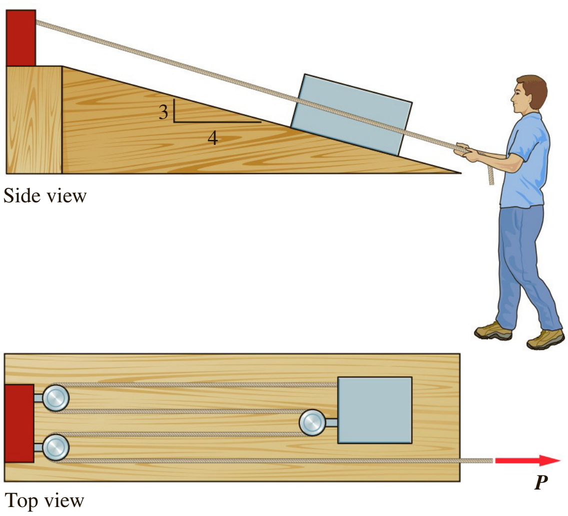


EX 7.1.8

7.1.9. [**] A worker pulls a 400-N box up a ramp using a system consisting of a rope and three frictionless pulleys, with the rope parallel to the slope, as shown. The static and kinetic friction coefficients are 0.4 and 0.35, respectively. Determine

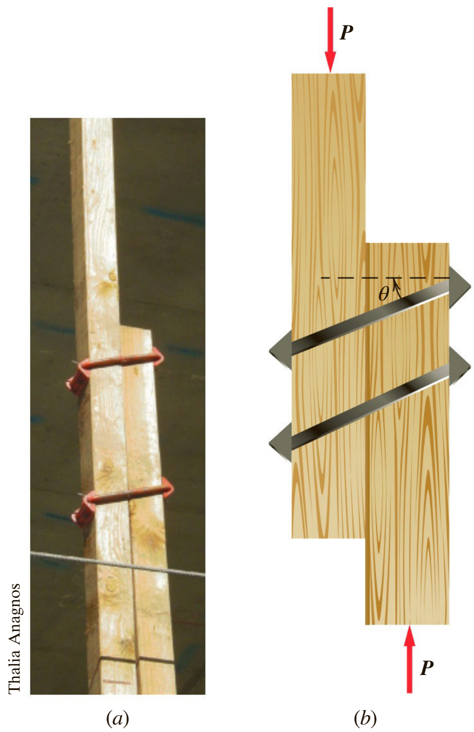
a. the force P_s that the worker needs to apply to the rope to get the box moving

b. the force P_k the worker needs to apply to slide the box up the slope.



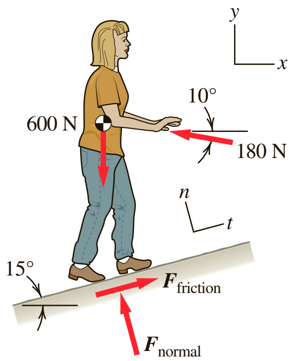
EX 7.1.9

7.1.10. [**] Shore clamps, as shown in (a), are used to clamp temporary shoring in a construction project. Each clamp consists of a rectangular collar that can carry a maximum tension of 7000 lb, and two pivotal plates with grooves to grip the wood. Assuming $\mu_s = 0.40$ between the two pieces of lumber, and $\theta = 20°$, determine the maximum load $\boldsymbol{P}$ that can be applied without the lumber slipping. To simplify the problem represent the two clamps by a single clamp with twice the capacity.



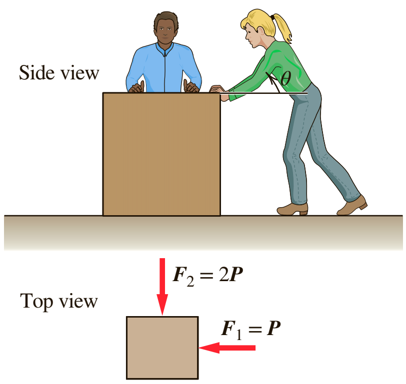
EX 7.1.10

7.1.11. [**] A person who weighs 600 N is pushing a recycling container up a 15° incline. The forces acting on the person are shown. Determine is the value of the ratio $F_{\text{friction}}/F_{\text{normal}}$. (Note: The coefficient of static friction between the woman's shoes and the incline must be at least as great as this ratio if her feet do not slip as she walks up the incline.)



EX 7.1.11

7.1.12. [***] Two people attempt to push a 200-lb box across the floor ($\mu_s = 0.25$). They are pushing with force F_1 and F_2 at right angles to one another and each is pushing downward at θ to the horizontal, as shown. Each person is capable of applying a maximum force of 70 lb and if the vertical component of the force applied to the box by either person is greater than 25 lb the box will crush. Determine the maximum allowable angle θ of F_1 and F_2 such that the box starts moving and it is not crushed.



EX 7.1.12

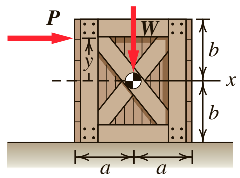
7.1.13. [*, computer]** The applied force $\boldsymbol{P}$ is located at a distance y above the center of mass of the storage unit. Consider whether the storage unit of weight W will slide, tip, or not move.

a. Write an expression for y at the point of application of $\boldsymbol{P}$ at which the unit will tip instead of slide. This expression will be a function of dimensions a and b, and the coefficient of static friction.

b. Create a plot of y versus coefficient of sliding friction when $a = 1$ m and $b = 1$ m, clearly marking the regions of sliding, no movement, and tipping.

c. Create a plot of y versus coefficient of sliding friction when $a = 2$ cm and $b = 1$ cm, clearly marking the regions of sliding, no movement, and tipping.

d. Compare the plots in **b** and **c**. Comment on how they compare and why they look the way they do.

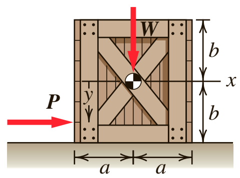


EX 7.1.13

7.1.14. [*, computer]** The applied force $\boldsymbol{P}$ is located at a distance y below the center of mass of the storage unit. Consider whether the storage unit of weight W will slide, tip, or not move.

a. Write an expression for y at the point of application of $\boldsymbol{P}$ at which the unit will tip instead of slide. This expression will be a function of dimensions a and b, and the coefficient of static friction

b. Create a plot of y versus coefficient of sliding friction when $a = 1$ m and $b = 1$ m. What does this plot tell you about the tendency of the unit to slide versus tip over?

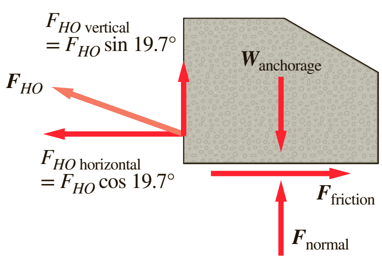


EX 7.1.14

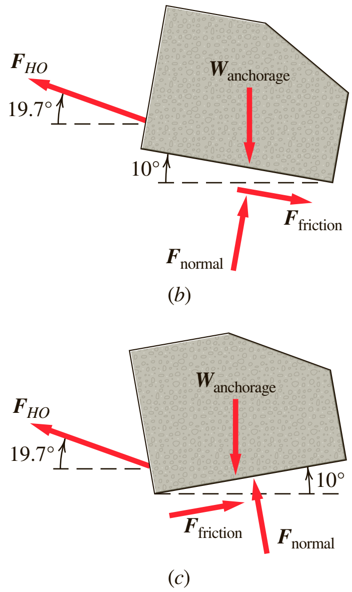
7.1.15. [*, computer]** When we performed our analysis of the anchorage of the Golden Gate Bridge in Appendix E, we drew the free-body diagram shown in (a). Based on having determined the force in member HO to be 251.2 MN and assuming a coefficient of static friction of $\mu_{static} = 0.6$, we found that the anchorage had to weigh 479 MN to prevent both uplift and sliding.

a. If the anchorage were on an incline of 10° as shown in (b), considering prevention of both uplift and sliding, what would the required weight of the anchorage be?

b. If the anchorage were tilted as shown in (c), would it have to be heavier or lighter than if it were on level ground? Explain your reasoning.



(a)



EX 7.1.15

c. Write a general equation for the friction force needed to prevent sliding, as a function of the angle of incline. Use Excel, MATLAB, or another program to make a plot of friction force versus angle of incline.

d. Write a general equation for the weight of the anchorage that controls the design. Use Excel, MATLAB, or another program to make a plot of anchorage weight versus angle of incline.

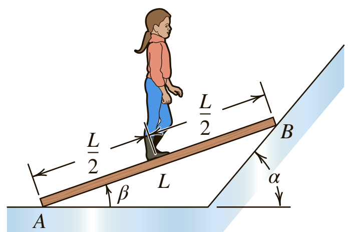
7.1.16. [*, computer]** A child of weight W balances on a beam of length L. Plane A is smooth and B is rough. If the beam and child are in equilibrium and the weight of the beam is negligible

a. Determine the friction (F_{Bf}) and the normal force (F_{Bn}) at B in terms of L, W, α, and β. Present your answers as equations.

b. If $L = 3$ m, $W = 400$ N, and $\beta = 25$ degrees, create plots of F_{Bf} and F_{Bn} as α varies from 10 to 90 degrees.

c. If the plank will slide along the plane at B when $F_{Bf} = 0.5\, F_{Bn}$, what is the maximum value of α such that the plank will not slide?

d. If you wanted the plank setup to be in equilibrium at values of α greater than what you determined in **c**, what changes could you recommend to the setup (name at least two changes)?



EX 7.1.16

7.2 FRICTION IN STATIC ANALYSIS: WEDGES, BELTS, AND JOURNAL BEARINGS

Learning Objective: Analyze static equilibrium problems that include dry friction.

We now present the analysis of two simple systems that operate because of dry friction—wedges and belts. We also describe a system component that functions better without friction, though in reality friction is often present—the journal bearing.

Wedges

A wedge is a simple machine used to make adjustments in the position of one object relative to another. Wedges can also be used to apply a large force. For example, the wedge in Figure 7.2.1 can be used to raise the block. To calculate the maximum force $\boldsymbol{P}$ required to raise the block we apply equilibrium conditions when the wedge is just about to move to the right (and therefore begins to raise the block). When the wedge is just about to move, we can describe the relationship between friction

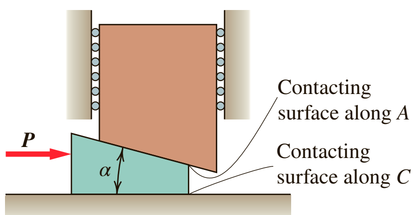


Figure 7.2.1. A wedge used to raise a block. Support is such that the surface at the wall is frictionless.

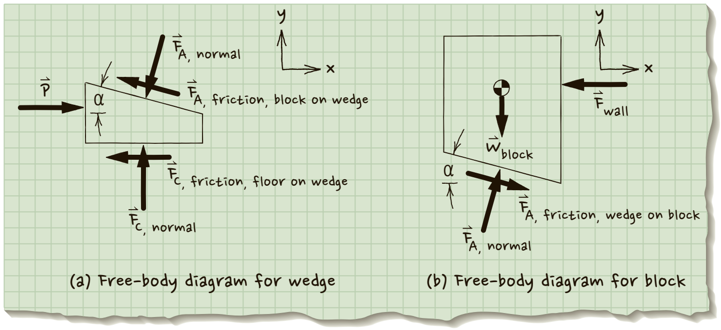


Figure 7.2.2. (*a*) Free-body diagram of wedge; (*b*) free-body diagram of block.

and normal force using (7.2). More specifically, for the contacting surfaces along A and B (**Figure 7.2.2*a***) we write

$$\left\|\boldsymbol{F}_{A,\text{friction}}\right\| = \mu_s \left\|\boldsymbol{F}_{A,\text{normal}}\right\| \tag{7.4A}$$

$$\left\|\boldsymbol{F}_{C,\text{friction}}\right\| = \mu_s \left\|\boldsymbol{F}_{C,\text{normal}}\right\| \tag{7.4B}$$

The application of equilibrium to the block (**Figure 7.2.2*b***) then allows us to write

$$\sum F_y = -W + F_{A,\text{fiction}} \cos\alpha - \mu_s F_{A,\text{normal}} \sin\alpha = 0 \tag{7.5}$$

which can be solved for $F_{A,\text{normal}}$:

$$F_{A,\text{normal}} = \frac{W}{\cos\alpha - \mu_s \sin\alpha} \tag{7.6}$$

Application of equilibrium equations to the wedge (assuming that its mass is very small) (**Figure 7.2.2*a***) then allows us to write

$$\sum F_x = P - \mu_s F_{C,\text{normal}} - \mu_s F_{A,\text{normal}} \cos\alpha - F_{A,\text{normal}} \sin\alpha = 0 \tag{7.7A}$$

$$\sum F_y = F_{C,\text{normal}} + \mu_s F_{A,\text{normal}} \sin\alpha - F_{A,\text{normal}} \cos\alpha = 0 \tag{7.7B}$$

Equations (7.7A) and (7.7B) in combination with (7.6) can be solved for P:

$$P = \frac{W}{\cos\alpha - \mu_s \sin\alpha}\left[(1-\mu_s^2)\sin\alpha + 2\mu_s \cos\alpha\right] \tag{7.8}$$

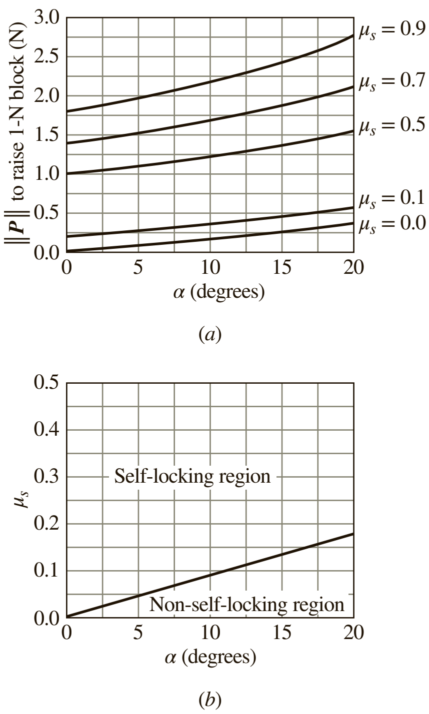


Figure 7.2.3 (*a*) The force $\boldsymbol{P}$ necessary to lift a 1-N block for various coefficients of static friction; (*b*) conditions under which a wedge-block system is self-locking.

Figure 7.2.3*a* shows plots of (7.8) for various values of static friction and illustrates that the force $\boldsymbol{P}$ required to lift a 1-N block increases with the static friction coefficient and with the wedge angle. Notice that for a small coefficient of static friction and wedge angle, $\|\boldsymbol{P}\| < 1$ (e.g., $\mu_s = 0.1$, $\alpha = 10°$, $\|\boldsymbol{P}\| = 0.38$), whereas for a larger coefficient of static friction and

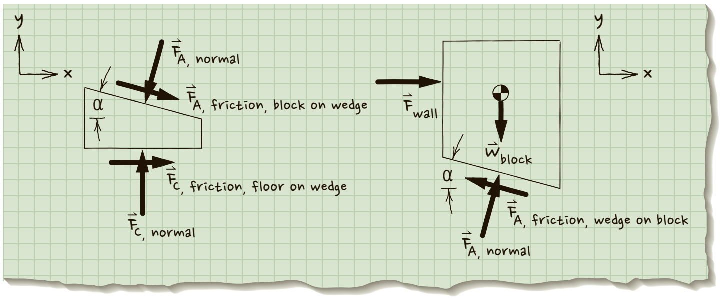


Figure 7.2.4. Free-body diagrams of wedge and block with no force $\boldsymbol{P}$ applied.

wedge angle $\|\boldsymbol{P}\| > 1$ (e.g., $\mu_s = 0.7$, $\alpha = 20°$, $\|\boldsymbol{P}\| = 2.10$). This means that the proper selection of coefficient of static friction and wedge angle will allow lifting of a large weight ($\boldsymbol{W}$) with a (relatively) small force $\boldsymbol{P}$.

Under a certain condition, the wedge is **self-locking**. This means that the wedge will remain in place even when there is no applied force $\boldsymbol{P}$. This might be advantageous if you need to keep an object in a raised position for an extended period of time. We can determine the condition under which a wedge–block system will be self-locking by imposing equilibrium on the block's and wedge's free-body diagrams in **Figure 7.2.4** (notice how these free-body diagrams are different from the ones in **Figure 7.2.2**).

For the block we write:

$$\sum F_y = -W + F_{A,\text{normal}} \cos\alpha + \mu_s F_{A,\text{normal}} \sin\alpha = 0 \qquad (7.9\text{A})$$

For the wedge we write

$$\sum F_x = \mu_s F_{C,\text{normal}} + \mu_s F_{A,\text{normal}} \cos\alpha - F_{A,\text{normal}} \sin\alpha = 0 \quad (7.9\text{B})$$

$$\sum F_y = F_{C,\text{normal}} - \mu_s F_{A,\text{normal}} \sin\alpha - F_{A,\text{normal}} \cos\alpha = 0 \quad (7.9\text{C})$$

From (7.9A)–(7.9C) we find that for the wedge to be self-locking when

$$\tan\alpha \le \frac{2\mu_s}{\left(1 - \mu_s^2\right)} \qquad (7.10)$$

Figure 7.2.3*b* shows a plot of wedge angle (α) versus static coefficient of friction (as determined from (7.10)). Convince yourself that the correct regions of this plot have been marked as the "self-locking region" and the "non-selflocking region."

Belts

Belts are used to connect mechanical components to one another, often with the intention of transferring power from one component to another. For example, a rubber belt connects a pulley on the crankshaft of an automobile engine to a pulley on the water pump (**Figure 7.2.5**). A belt operates because of friction between it and the pulleys that it connects. To

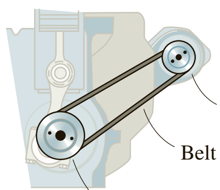


Figure 7.2.5. A belt connects the crankshaft and the water pump on an automobile engine.

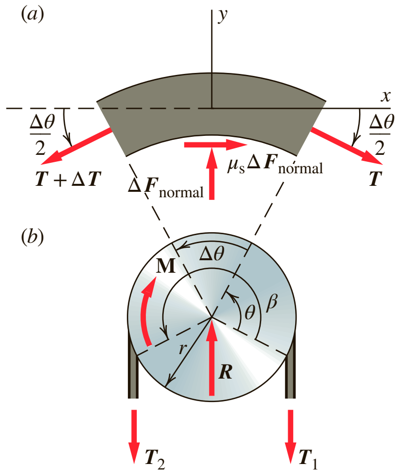


Figure 7.2.6. (*a*) free-body diagram of belt segment; (*b*) free-body diagram of pulley and belt.

calculate the condition under which a belt will not slip on the pulley, we consider equilibrium of a portion of the belt, as shown in **Figure 7.2.6*a***:

$$\sum F_x = -(T + \Delta T)\cos\frac{\Delta\theta}{2} + T\cos\frac{\Delta\theta}{2} + \mu_s \Delta F_{\text{normal}} = 0 \quad (7.11\text{A})$$

$$\sum F_y = -(T + \Delta T)\sin\frac{\Delta\theta}{2} - T\sin\frac{\Delta\theta}{2} + \Delta F_{\text{normal}} = 0 \quad (7.11\text{B})$$

If $\Delta\theta$ is small, we can write

$$\sin\frac{\Delta\theta}{2} \approx \frac{\Delta\theta}{2}; \qquad \cos\frac{\Delta\theta}{2} \approx 1 \quad (7.12)$$

Substituting (7.12) into (7.11A) and (7.11B), and recognizing that in the limit each Δ term can be written as a differential, we have

$$dT = \mu_s dF_{\text{normal}} \quad (7.13\text{A})$$

$$Td\theta = dF_{\text{normal}} \quad (7.13\text{B})$$

Combining (7.13A) and (7.13B) we write

$$\frac{dT}{T} = \mu_s\, d\theta \quad (7.14)$$

which we then integrate between the limits of T_1 to T_2 (the tensions at the ends of the belt—see **Figure 7.2.6*b***) and 0 to β (wrap angle of belt, in radians, **Figure 7.2.6*b***):

$$\int_{T_1}^{T_2} \frac{dT}{T} = \int_0^{\beta} \mu_s\, d\theta \quad (7.15)$$

$$\ln\frac{T_2}{T_1} = \mu_s \beta \quad (7.16\text{A})$$

Equation (7.16A) can also be presented as

$$e^{\mu_s \beta} = \frac{T_2}{T_1} \quad (7.16\text{B})$$

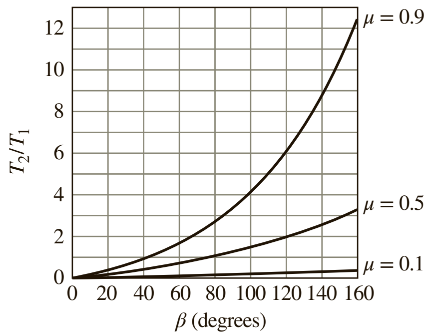


Figure 7.2.7. The ratio T_2/T_1 versus belt wrap angle (β).

Figure 7.2.7 plots this expression and shows that the greater the belt wrap (larger values of β), the greater the ratio of T_2/T_1 before belt slipping occurs. Equations (7.16A) and (7.16B) (as well as **Figure 7.2.7**) also work for cables wrapped around drums.

Journal Bearings

A machine may involve rotating or translating members, or both. An example of a rotating system is a gear train (**Figure 7.2.8**). As long as each member in a machine is rotating about an axis of fixed orientation at a constant angular velocity, the conditions of mechanical equilibrium developed in Chapter 5 apply. Dry friction between shafts and bearings increases the moment necessary to turn the the shafts. Therefore, in rotating systems one generally aims to reduce friction between shafts and bearings to a very small value (ideally to $\mu_s = 0$); this can be done with the use of lubricants and/or more sophisticated bearing systems.

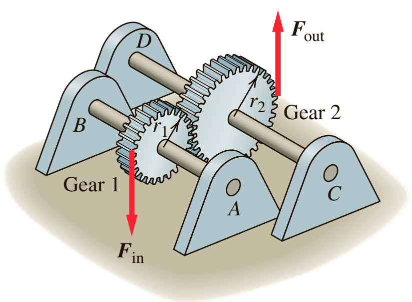


Figure 7.2.8. A gear train, with journal bearings at *A*, *B*, *C* and *D*.

When machines contain shafts that are held by dry or only partially lubricated journal bearings, the analysis may need to account for the presence of **journal bearing friction**. Consider the journal bearing shown in **Figure 7.2.9** with bearing load $\boldsymbol{L}$. The force $\boldsymbol{L}$ is the radial force to be supported by the bearing. If the bearing is well lubricated, the friction is negligible and then the bearing support force $\boldsymbol{F}$ is aligned with $\boldsymbol{L}$ (**Figure 7.2.10*a***). In contrast, if the bearing is dry or only partially lubricated, friction between the shaft (which is rotating clockwise) and the bearing sleeve results in the shaft "riding up" the inner surface of sleeve just to the point of slipping (**Figures 7.2.10*b*** and ***c***). At the contact point A we consider the components of $\boldsymbol{F}$ in the normal and tangential directions, with F_{tangent} being the friction force (**Figures 7.2.10*c***). For a shaft of radius r, the friction force F_{tangent} creates a moment about a moment center at O of

$$M_O = (r \bullet F_{\text{tangent}}) \tag{7.17}$$

in the direction *opposite* the clockwise rotation of the shaft. We can write the friction force when there is impending slip in terms of the normal force using Coulomb's friction law as $F_{\text{friction}} = F_{\text{tangent}} = \mu F_{\text{normal}}$, where μ is the coefficient of friction. If the angle θ in **Figure 7.2.10** is small, then F_{normal} is approximately equal to the magnitude of the bearing support force $\boldsymbol{F}$. Therefore, we write (7.17) as

$$M_O = r \bullet \mu F \tag{7.18}$$

In most engineering design situations, one would work to minimize M_O by making μ as small as is practical. For a shaft that is stationary relative to a journal bearing, the coefficient of static friction (μ_s) should be used in (7.18). If the shaft is rotating relative to the journal bearing, the coefficient of kinetic friction (μ_k) should be used in (7.18). Sample values of μ_s and μ_k are listed in **Table 7.1**. Generally, $\mu_s > \mu_k$.

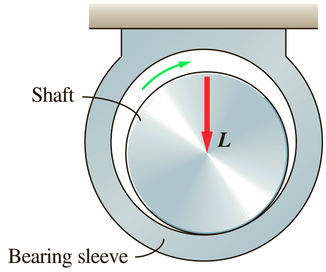


Figure 7.2.9. A journal bearing consists of a shaft within a bearing sleeve. The sleeve is fixed and the shaft rotates within the sleeve.

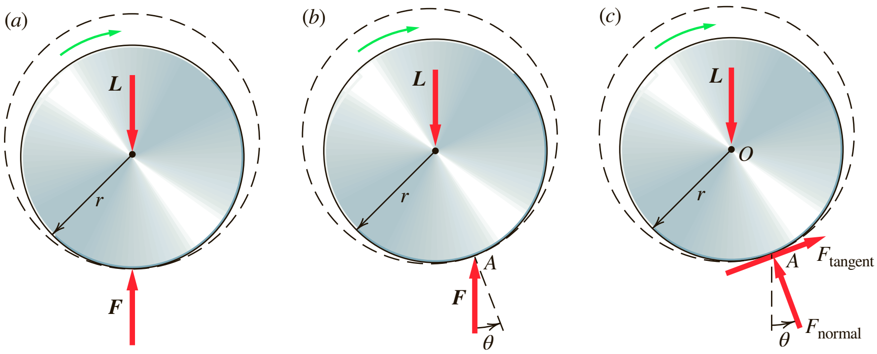


Figure 7.2.10. (*a*) A stationary or frictionless shaft; (*b*) shaft rotates in clockwise direction (force $\boldsymbol{F}$ is sleeve acting on shaft); (*c*) force $\boldsymbol{F}$ shown in terms of its components.

Check out the following example of an applications of this material.
- **Example 7.2.1 Analysis of a Pulley System with Bearing Friction**

EXAMPLE 7.2.1

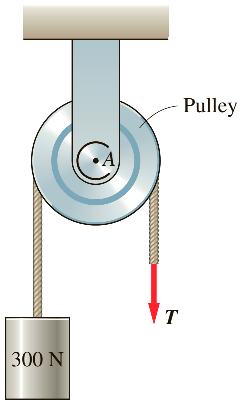


Figure 1 A pulley with bearing friction.

The system in **Figure 1** consists of a pulley that fits loosely over a shaft. The shaft is rigidly supported at its ends and does not turn. The diameter of the wheel is 150 mm, and that of the shaft is 20 mm. Between the shaft and the pulley the coefficient of static friction is 0.40, and the coefficient of kinetic friction is 0.35.

(a) Determine the minimum tension in the rope to hold the 300-N block in a stationary position.

(b) Determine the minimum tension in the rope to just start raising the block. Also express this answer as a percentage of the tension found in (a).

Goal Find the tension T in the rope to keep the weight stationary (a), and to just begin raising it (b).

Given We are given the diameters of the pulley (150 mm) and the shaft (20 mm) on which it is loosely fit. In addition, we are given the coefficients of static ($\mu_s = 0.40$) and kinetic ($\mu_k = 0.35$) friction and the weight of block (300 N) being held by the pulley system.

Assume The weight of the pulley is negligible and that the rope does not slide in the pulley groove.

Draw For **(a)** First we draw the free-body diagram of the pulley in a stationary position (**Figure 2*a***). The clockwise moments created about a moment center at A by the rope tension and the bearing friction are balanced by the counterclockwise moment created by the block's weight about A. Bearing friction in the clockwise direction means that it is helping to keep the block stationary (and therefore, the required tension T is at its minimum).

Formulate Equations and Solve For **(a)** From (7.18), the moment created by the bearing friction is $M = r \bullet \mu F$ where F is the magnitude of the force supported by the bearing and r is the radius of the shaft. For the situation in **Figure 2*a***, the force supported by the

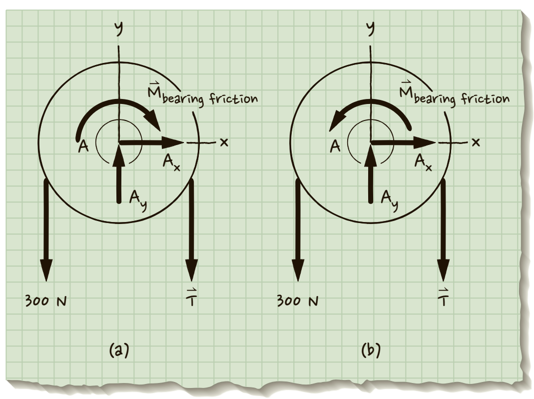


Figure 2 Free-body diagrams: (*a*) stationary pulley; (*b*) just-about-to-move-upward pulley.

bearing is A_y, and it is appropriate for us to use the static coefficient of friction μ_s (because the pulley is not turning relative to the shaft for part (a)). Therefore,

$$M_{\text{bearing friction}} = r_{\text{shaft}} \bullet \mu_s A_y \tag{1}$$

Now we find the value of A_y by using equilibrium:

$$\sum F_y(\uparrow +) = A_y - 300\text{ N} - T = 0$$

$$A_y = 300\text{ N} + T \tag{2}$$

Substituting (2) into (1), we find that the magnitude of the moment created by the bearing friction is

$$M_{\text{bearing friction}} = r_{\text{shaft}} \bullet \mu_s(300\text{ N} + T) \tag{3}$$

Based on equilibrium and the free-body diagram in **Figure 2** we write:

$$\sum M_{z@A}(\curvearrowleft +) = 300\text{ N}\left(\frac{150}{2}\text{ mm}\right) - T\left(\frac{150}{2}\text{ mm}\right) - \underbrace{M_{\text{bearing friction}}}_{\text{from (3)}} = 0 \tag{4}$$

$$300\text{ N}\left(\frac{150}{2}\text{ mm}\right) - T\left(\frac{150}{2}\text{ mm}\right) - r_{\text{shaft}} \bullet \mu_s(300\text{ N} + T) = 0$$

Rearranging and substituting in for $r = 10$ mm and $\mu_s = 0.40$, we solve for the minimum tension T in the rope to hold the 300-N block in a stationary position:

$$T = 270\text{ N}$$

Draw For **(b)** we draw the free-body diagram of the pulley just as the block is about to start moving upward (**Figure 2*b***). The clockwise moment created about a moment center at A by the rope tension is balanced by counterclockwise moments created by the block's weight and the bearing friction. The moment created by the bearing friction being in the counterclockwise direction means that it is working to prevent the impending upward movement of the block.

Formulate Equations and Solve For **(b)**, as in the calculations for (a), the magnitude of the moment created by the bearing friction is

$$M_{\text{bearing friction}} = r_{\text{shaft}} \bullet \mu_s(300\text{ N} + T)$$

but its direction is in the counterclockwise direction. It is appropriate to use the static coefficient of friction because the block is just about to start moving upward.

Based on equilibrium and **Figure 2*b*** we write:

$$\sum M_{z@A}(\curvearrowleft +) = 300\text{ N}\left(\frac{150}{2}\text{ mm}\right) - T\left(\frac{150}{2}\text{ mm}\right) + M_{\text{bearing friction}} = 0 \tag{5}$$

$$300\text{ N}\left(\frac{150}{2}\text{ mm}\right) - T\left(\frac{150}{2}\text{ mm}\right) + r_{\text{shaft}} \bullet \mu_s(300\text{ N} + T) = 0 \tag{6}$$

Rearranging and substituting in for r_{shaft} and μ_s, we solve for the minimum tension T in the rope when the block is just about to start moving upward.

$$T = 334\ \text{N}$$

This is 23.8% greater than the tension required to hold the block stationary.

Check The results are summarized in Table 1.

These results say that

- more tension is required to just start raising the block. This makes sense, since the moment created by the tension must balance the moments created by the block's weight and the bearing friction.
- less tension is required to hold the block stationary. This makes sense, since the moments created by tension and bearing friction balance the moment created by the block's weight.

Table 1 Summary of Results

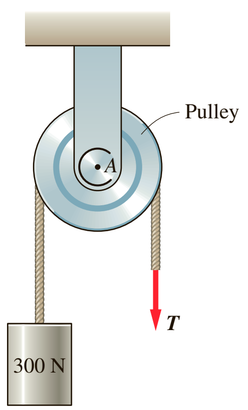
Condition	Tension
Block stationary	270 N
Block just starting to move upward	334 N

EXERCISES 7.2

7.2.1. **[*]** Consider the pulley shown and analyzed in Example 7.2.1.

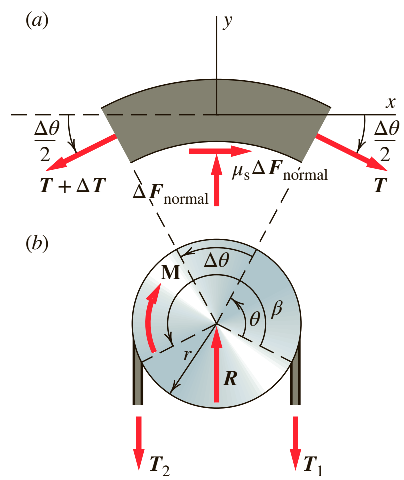
a. Will the tension in the rope to raise the block at a constant rate be greater than, less than, or equal to the 334 N to just start the block moving upward that was calculated in Example 7.2.1b? And why?

b. Will the tension in the rope to lower the block at a constant rate be greater than, less than, or equal to the 270 N to just hold the block that was calculated in Example 7.2.1a? And why?



EX 7.2.1

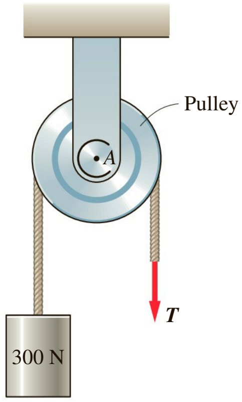
7.2.2. **[*]** The free-body diagram depicts a pulley, rotating in the clockwise direction, driving a belt. Draw the free-body diagram of the belt segment if the belt is driving the pulley in the counter clockwise direction.



EX 7.2.2

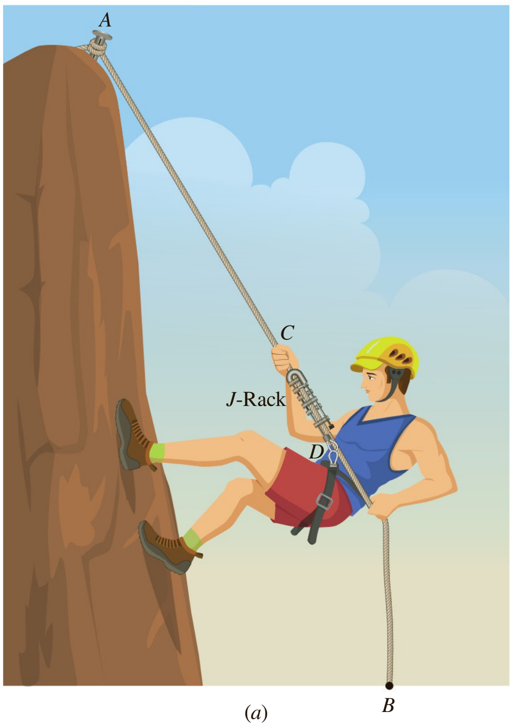
7.2.3. **[*]** Consider the pulley shown and analyzed in Example 7.2.1. If the tension pulling on the rope is less

than 270 N, will the block stay stationary, rise at a constant velocity, lower at a constant velocity, accelerate upward, or accelerate downward?



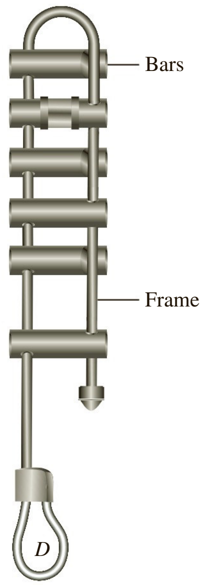
EX 7.2.3

7.2.4. [*] Rappelling is a technique to descend a steep rock face by sliding down a rope. A J-rack is a device used by rappellers to attach themselves to the descent rope. The rappeller is attached to the J-rack, and the J-rack "grabs" the rope (*a*). A basic J-rack is shown in (*b*) and (*c*). Describe how friction works to "grab" the rope *AB*.

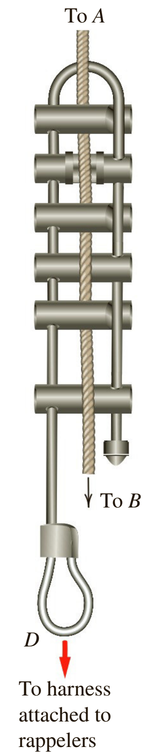


(*a*)

EX 7.2.4a

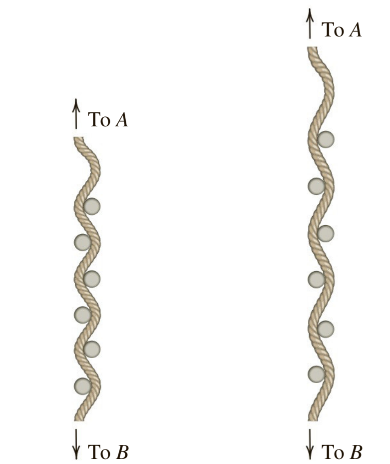


(*b*) **EX 7.2.4b**



(*c*) **EX 7.2.4c**

7.2.5. [*] By moving the bars of the J-rack farther apart or closer together, a rappeller can change her rate of descent. Will she be descending faster or slower with the bars in configuration (**1**) or configuration (**2**)? And why? (Note: the figure is a side view that shows only how the rope *AB* weaves through the bars. The frame has been omitted for clarity.)

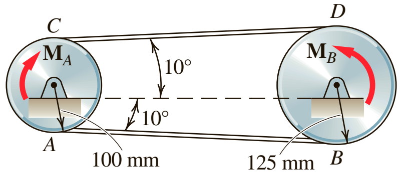


Configuration (1) Configuration (2) **EX 7.2.5**

7.2.6. [*] Draw free-body diagrams of pulleys *A* and *B*

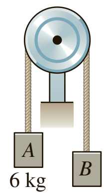
a. for the case when M_A is driving the system, clearly indicating whether T_{AB} is less than, greater than, or equal to T_{CD}.

b. for the case when M_B is driving the system, clearly indicating whether T_{AB} is less than, greater than, or equal to T_{CD}.



EX 7.2.6

7.2.7. [*] The rope connecting the 6-kg block A to block B passes over a fixed cylinder. Determine the largest and smallest masses of block B for which static equilibrium is possible if the coefficient of static friction between the pulley and the rope is 0.30.

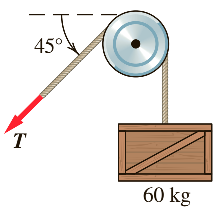


EX 7.2.7

7.2.8. [*] A cable is completely wrapped around the horizontal shaft shown. One end is attached to a 60-kg crate, and a tensile force T pulls the other. The coefficient of static friction between the shaft and the cable is 0.30. Determine

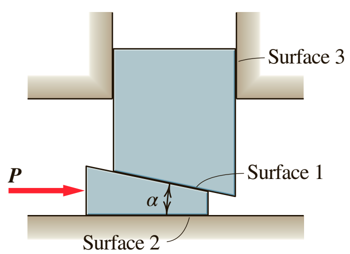
a. the minimum tension T for which the crate will not descend.

b. the maximum tension T for which the crate will not rise.



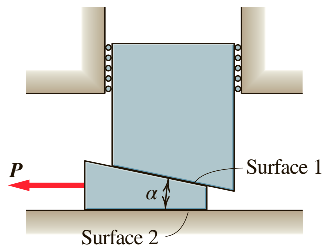
EX 7.2.8

7.2.9. [, computer]** Determine an expression for the force $\boldsymbol{P}$ as a function of wedge angle α and static coefficient μ_s required to raise the block if friction acts along surfaces 1, 2, and 3. Plot your answer and compare with equation (7.8).



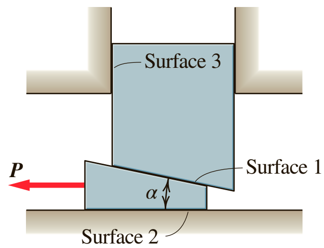
EX 7.2.9

7.2.10. [, computer]** Determine an expression for the force $\boldsymbol{P}$ as a function of wedge angle α and static coefficient μ_s required to remove the wedge if friction acts along surfaces 1 and 2. Plot your answer.



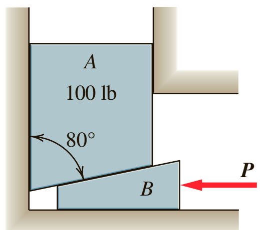
EX 7.2.10

7.2.11. [, computer]** Determine an expression for the force $\boldsymbol{P}$ as a function of wedge and angle α and static coefficient μ_s required to remove the wedge if there is friction acting along surfaces 1, 2, and 3. Plot your answer.



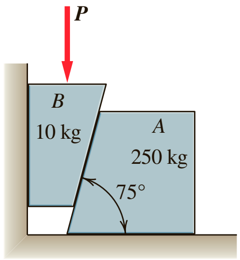
EX 7.2.11

7.2.12. [**] Determine the magnitude of the minimum horizontal force $\boldsymbol{P}$ that must be applied to wedge B to raise block A. The coefficient of friction between all surfaces is 0.15. Is the wedge self-locking?



EX 7.2.12

7.2.13. [**] Determine the magnitude of the minimum downward force $\boldsymbol{P}$ that must be applied to wedge B to move block A. The coefficient of friction between all surfaces is 0.15. Is the wedge self-locking?



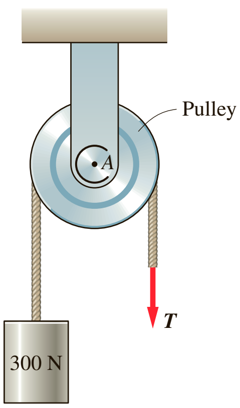
EX 7.2.13

7.2.14. [**] A pulley fits loosely over a shaft that is rigidly supported at its ends and does not turn. The diameter of the pulley wheel is 150 mm, and that of the shaft is 20 mm. The coefficient of static friction is 0.40, and the coefficient of kinetic friction is 0.35 between the shaft and the pulley.

a. Determine the tension in the rope to raise the block at a constant rate.

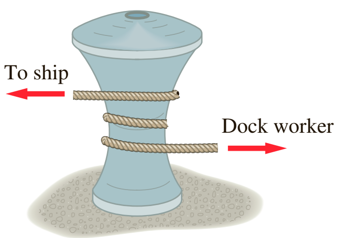
b. Determine the tension in the rope to lower the block at a constant rate.

c. Present the answers obtained in **a** and **b**, along with those found in Example 7.2.1 as a well-organized table. Also describe why the relative sizes of the forces seem reasonable.



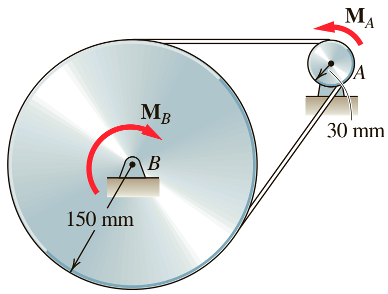
EX 7.2.14

7.2.15. [**] A ship is secured by wrapping a rope around a capstan. If a dock worker can apply a 170-N force to counteract a 7-kN force by the ship, determine the number of complete turns of the rope about the capstan required to keep the rope from slipping if the coefficient of static friction is 0.30.



EX 7.2.15

7.2.16. [**] Determine the maximum moment M_A that a motor may apply to pulley A without exceeding the maximum allowable belt tension of 200 N. Also determine the corresponding moment M_B exerted on pulley B by its drive shaft if the system is in equilibrium. The coefficient of static friction between the belt and the pulleys is 0.30.



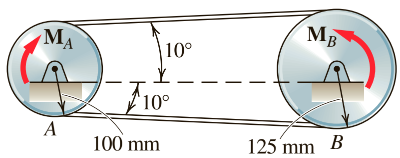
EX 7.2.16

7.2.17. [**] It is known that the maximum moment M_A that a motor can apply to pulley A without causing the belt to slip over either pulley is 20 N·m. The coefficient of static friction is 0.40. For this moment, determine

a. the tension in the belt on either side of pulley A

b. the moment M_B on pulley B for equilibrium

c. whether slip is impending at pulley A or pulley B

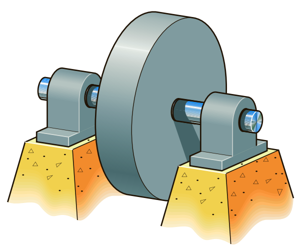


EX 7.2.17

7.2.18. [**] A grinding wheel weighing 3 lb is supported by a journal bearing at each end of the axle, as shown. The coefficients of static (μ_s) and kinetic (μ_k) friction are 0.15 and 0.10, respectively, and the diameter of the axle is 0.75 in. Determine

a. the moment required to just get the wheel–axle assembly to rotate.

b. the moment required to rotate the wheel at a constant speed.



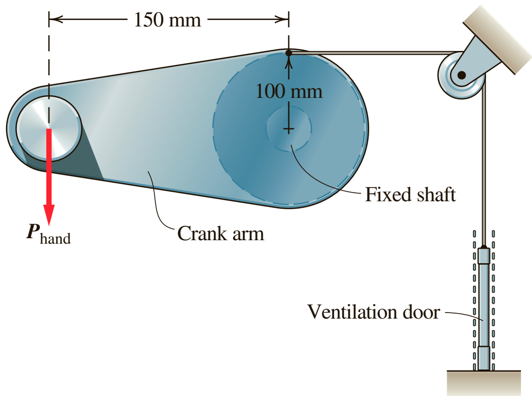
EX 7.2.18

7.2.19. [*] A ventilation door weighing 75 N is raised and lowered by force P_{hand} via a crank system. The crank arm rotates about a fixed shaft. The diameter of the fixed shaft is 100 mm, and the coefficient of friction (μ) between the crank arm and shaft is 0.15.

a. Determine the minimum force to raise the door ($P_{hand,raise}$).

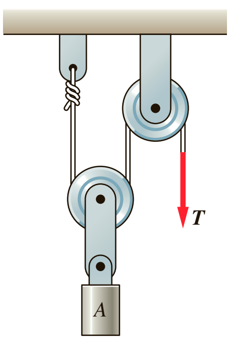
b. Determine the maximum force to lower the door ($P_{hand,lower}$).

c. When $P_{hand,lower} < P_{hand} < P_{hand,raise}$, describe the state of the door—is it moving upward, downward, or stationary?



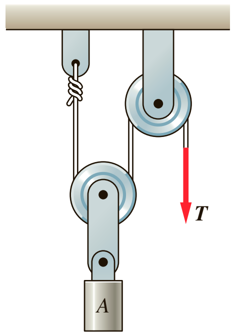
EX 7.2.19

7.2.20. [**] Each pulley in the system has a radius of 110 mm and a mass of 5 kg. They are mounted on shafts of 15-mm radius supported by journal bearings and are used to raise block A of mass 15 kg. Determine the force T that is required to raise the block at a constant rate if the bearings are frictionless. Defining mechanical advantage as the load lifted divided by the the applied force, determine the mechanical advantage (see Chapter 9).



EX 7.2.20

7.2.21. [**] Each pulley in the system has a radius of 110 mm and a mass of 5 kg. They are mounted on shafts of 15-mm radius supported by journal bearings and are used to raise block A of mass 15 kg. Determine the force T that is required to raise the block at a constant rate if the coefficient of kinetic friction between the shafts and bearings is 0.18. Defining mechanical advantage as the load lifted divided by the the applied force, determine the mechanical advantage (see Chapter 9).



EX 7.2.21

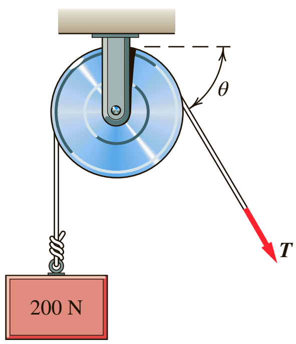
7.2.22. [**] The pulley consists of a 100-mm diameter wheel that fits loosely over a 15-mm diameter shaft. A rope passes over the pulley and is attached to a 200-N weight. The static and kinetic coefficients of friction between the pulley and the shaft are 0.35 and 0.25, respectively. The weight of the pulley is 10 N. The angle $\theta = 90°$.

a. Determine the minimum tension T in the rope needed to hold the 200-N weight in a stationary position.

b. Determine the minimum tension in the rope needed to just start raising the 200-N weight. Also express your answer as a percentage of the tension found in a.

c. Determine the tension in the rope needed to raise the 200-N weight at a constant rate. Also express your answer as a percentage of the tension found **a**.

d. Determine the tension in the rope needed to lower the 200-N weight at a constant rate. Also express your answer as a percentage of the tension found in **a**.



EX 7.2.22

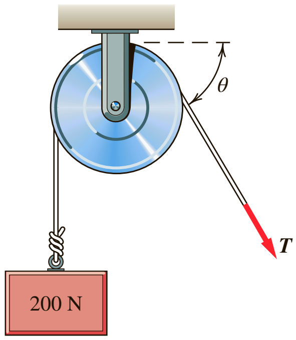
7.2.23. [**] The pulley consists of a 100-mm diameter wheel that fits loosely over a 15-mm diameter shaft. A rope passes over the pulley and is attached to a 200-N weight. The static and kinetic coefficients of friction between the pulley and the shaft are 0.35 and 0.25, respectively. The weight of the pulley is 10 N. The angle $\theta = 60°$.

a. Determine the minimum tension T in the rope needed to hold the 200-N weight in a stationary position.

b. Determine the minimum tension in the rope needed to just start raising the 200-N weight. Also express your answer as a percentage of the tension found in a.

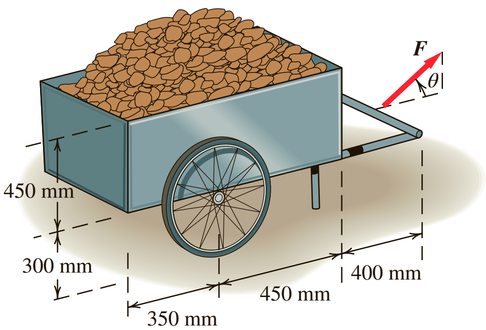
c. Determine the tension in the rope needed to raise the 200-N weight at a constant rate. Also express your answer as a percentage of the tension found in **a**.

d. Determine the tension in the rope needed to lower the 200-N weight at a constant rate. Also express your answer as a percentage of the tension found in **a**.



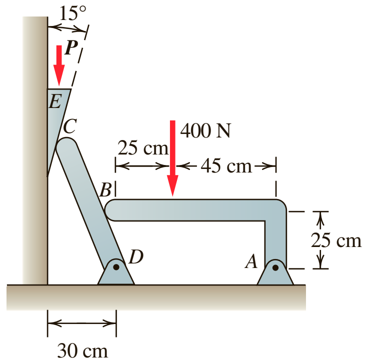
EX 7.2.23

7.2.24. [**] A two-wheel cart is used to haul yard waste along a dirt road. Determine the force ***F*** that must be applied to the handle of the cart to pull it at a constant speed. Relevant information about the cart and its load: the cart is equipped with 600-mm diameter high-pressure tires that fit loosely over a 20-mm diameter fixed axle; the coefficient of kinetic friction between the wheel and the axle is 0.20; the cart weighs 200 N and is carrying 1000 N of yard waste; and the center of mass of the cart and waste is 75 mm in front of the axle.



EX 7.2.24

7.2.25. [***] What is the minimum vertical force ***P*** applied to wedge *E* necessary to push end *C* of the bar *CD* to the right? The coefficient of static friction at all surfaces is 0.15. Assume that the pins at *A* and *D* are frictionless. If the pins at *A* and *D* are not frictionless, would the required force increase, decrease, or remain the same? (To answer this second question do not do any calculations—support your answer with qualitative reasoning.)

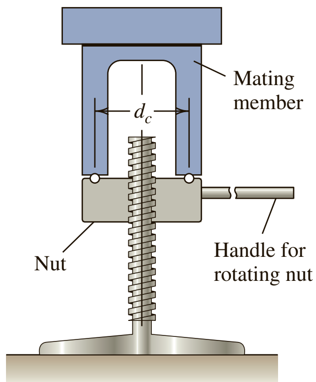


EX 7.2.25

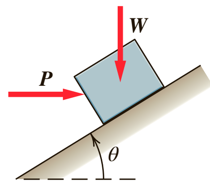
7.2.26. [*, computer]** A power screw as shown in (*a*) can be used to convert rotary motion of a nut to linear motion of the mating member. The treads on the power screw act like a ramp wound around a cylinder. The ramp therefore becomes a "spiral staircase" for the nut to move along. A simplified representation of a power screw's ramp and nut is shown in (*b*), where *P* is the force required to get the nut to move up the ramp. The weight acting on the nut, *W*, is the combined weight of the nut and the mating member.

a. Write an expression for *P* as a function of *W*, *θ*, and the coefficient of friction.

b. Create a set of plots similar to those in **Figure 7.2.3a**

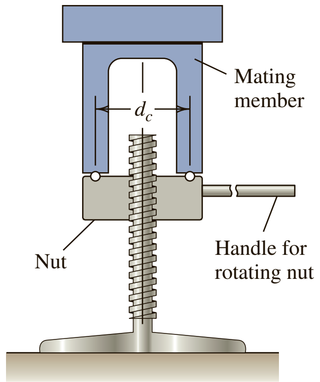


EX 7.2.26*a*

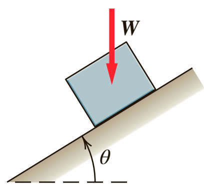


EX 7.2.26*b*

7.2.27. [*, computer]** A power screw as shown in (*a*) can be used to convert rotary motion of a nut to linear motion of the mating member. The treads on the power screw act like a ramp wound around a cylinder. The ramp therefore becomes a "spiral staircase" for the nut to move along. A simplified representation of a power screw's ramp and nut in a self-locked position is shown in (*b*), where the nut just sits on the ramp. The weight acting on the nut, *W*, is the combined weight of the nut and the mating member. Write an expression for the relationship between ramp angle *θ* versus coefficient of friction. Plot this expression, and describe its implications for selecting a power screw.



EX 7.2.27*a*



EX 7.2.27*b*

7.2.28. [***] Remote-controlled autonomous reconnaissance vehicles are commonly used in civilian and military applications. As with any other technologically advanced system many factors are considered in their design. The two hypothetical vehicles shown are nearly identical, with the primary difference being that the vehicle in Figure (a) has tracks and the one in (b) has wheels. Both axles are drive axles for the two systems.

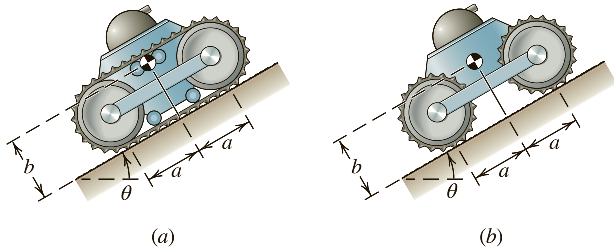
a. Draw a complete free-body diagram for each vehicle resting on the slope shown.

b. Assume dimension a to be 0.5 m and dimension b to be 0.7 m. If the coefficient of static friction is approximately 0.8 for both systems, calculate the maximum angle that each vehicle can rest on without sliding or tipping down the slope.

c. Discuss your answers to **b**. Are the answers what you expected? Comment on the assumptions in **b**. Are these assumptions realistic?

d. How does the location of the center of gravity affect the vehicle's ability to remain stable on the slope without sliding or tipping? As each dimension is changed, how does it affect the maximum angle of the slope?

e. Discuss your personal thoughts on whether wheels or tracks would be better for this system. Which would you recommend and why?



EX 7.2.28

7.3 ROLLING RESISTANCE

Learning Objective: Analyze static equilibrium problems that include rolling resistance.

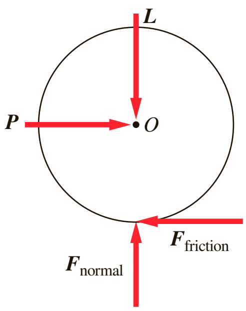


Figure 7.3.1 Perfectly rigid wheel on perfectly rigid surface, rolling to the right. According to the equilibrium conditions, if the wheel is moving to the right at a constant speed then $P = F_{\text{friction}} = 0$.

The analysis of machines that contain wheels may need to account for the presence of **rolling resistance**. You have experienced firsthand the presence of rolling resistance if you have ever attempted to ride a bicycle with underinflated tires. Consider a perfectly rigid wheel and the surface on which it is rolling (**Figure 7.3.1**). Acting on the wheel is $\boldsymbol{L}$, the weight of the wheel plus any vertical load exerted by whatever the wheel is attached to. The contact point between wheel and surface is directly below the center of the wheel, and equilibrium of moments shows us that $\boldsymbol{P}$, the force required to keep the wheel rolling, is zero when there is no rolling resistance.

In contrast, when a real wheel rolls on a surface, the wheel and surface both deform slightly, thereby creating resistance to rolling. For example, a rubber wheel rolling at constant speed on a paved road is shown in exaggerated form in **Figure 7.3.2*a*** and **7.3.2*b***. If the wheel speed is constant, equilibrium of moments about the wheel center requires that the normal contact force $\boldsymbol{F}_{\text{normal}}$, shown in **Figure 7.3.2*c*** acting at point A, must act through the wheel center as shown. By summing moments about A, we find

$$M_A = La - Pb = 0 \tag{7.19}$$

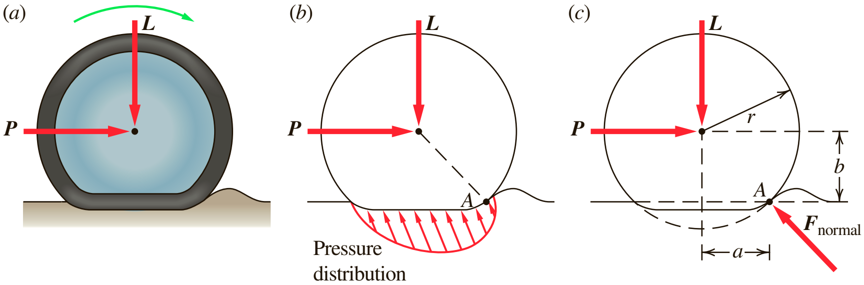


Figure 7.3.2 (*a*) Real wheel on real surface, rolling to the right; (*b*) depiction of pressure distribution between surface and wheel; (*c*) if the wheel is moving to the right at constant speed, $P = (a/r)L$.

Generally the wheel deformation involved is so small that b can be replaced by r (the radius of the wheel), and (7.19) becomes

$$P = (a/r)L \tag{7.20}$$

Remember that P is the magnitude of the force required to keep the wheel rolling because of the presence of rolling resistance.

The ratio of (a/r) is referred to as the **coefficient of rolling friction** (f_r).[2] Values of f_r reported in the literature for loaded pneumatic rubber tires that are properly inflated range from 0.01 (hard road) to 0.05 (well-packed gravel) to 0.40 (loose sand). The coefficient of rolling friction depends on the size of the wheel radius; for two identically constructed wheels, the one with a larger radius would have a smaller coefficient of rolling friction.

[2]Some texts refer to the distance a as the coefficient of rolling resistance. We go with the definition presented in Marks' *Standard Handbook for Mechanical Engineers*, 8th Edition.

Check out the following example of an applications of this material.
- **Example 7.3.1 Rolling Resistance**

EXAMPLE 7.3.1

The 100-kg steel wheel in **Figure 1** has a radius of 50 mm and rests on a ramp made of wood. At what angle θ will the wheel begin to roll down the ramp with constant velocity if the coefficient of rolling resistance is 0.02?

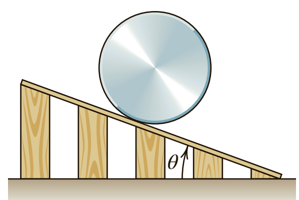


Figure 1 A wheel on a ramp.

Goal Find the ramp angle θ at which the steel wheel will begin to roll down the wooden ramp.

Given The mass and radius of the steel wheel and the coefficient of rolling resistance between the steel and the wood.

Assume The wheel will roll and not slide down the ramp.

Draw We draw the free-body diagram of the wheel (**Figure 2*a***). **Figure 2*b*** shows an alternate free-body diagram, where the weight W of the wheel is drawn in terms of its components ($W \sin\theta$ and $W \cos\theta$).

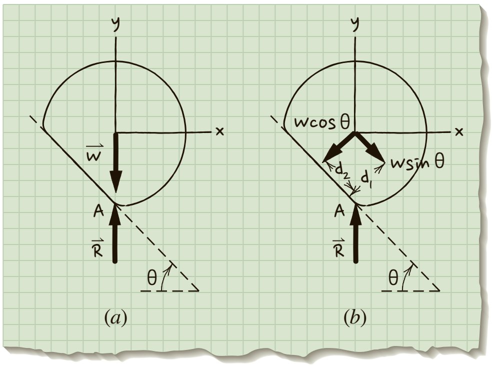


Figure 2 Free-body diagrams of wheel with wheel deformation highly exaggerated.

Formulate Equations and Solve Now we find the value of R (the force of the ramp pushing on the wheel) by enforcing equilibrium:

$$\sum F_y(\uparrow +) = -W + R = 0 \quad \Rightarrow \quad R = W \tag{1}$$

$$\sum M_{z@A}(\curvearrowleft +) = -W \sin\theta\,(d_1) + W \cos\theta\,(d_2) = 0 \tag{2}$$

where d_1 and d_2 are shown in **Figure 2*b***. For small wheel deformations $d_1 \approx r$ and $d_2 = a$ (as shown in **Figure 7.3.2*c***). Therefore (2) can be rewritten as

$$-W \sin\theta(r) + W \cos\theta(a) = 0 \tag{3}$$

Rearrange (3), to

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{a}{r} = f_r \tag{4}$$

By definition, the coefficient of rolling resistance is $f_r = a/r$. With $f_r = 0.02$ we solve (4) for θ to find $\theta = 1.14°$

$$\theta = 1.14°$$

At $\theta = 1.14°$, the steel wheel will roll down the ramp at a constant speed. (*Note:* At angles greater than 1.14°, the wheel will accelerate down the ramp. Also notice that the answer is independent of the weight of the wheel.)

Check If the wheel–ramp interface had larger rolling resistance, the value of a would increase. Substituting a larger value of a into (4) would result in a larger angle θ. This makes intuitive sense—the larger the rolling resistance, the greater the force required to roll the wheel. In this case this force is the component of the wheel's weight parallel to the ramp, and this component gets bigger with ramp angle.

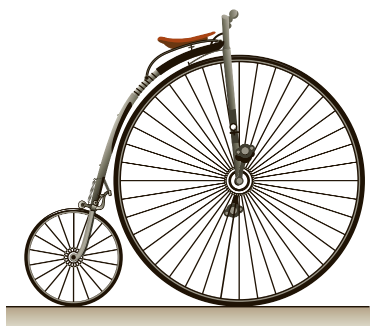
EXERCISES 7.3

7.3.1. [*] According to the Department of Energy (DOE), you can improve your fuel economy by about 3.3 percent if you keep your tires inflated properly.

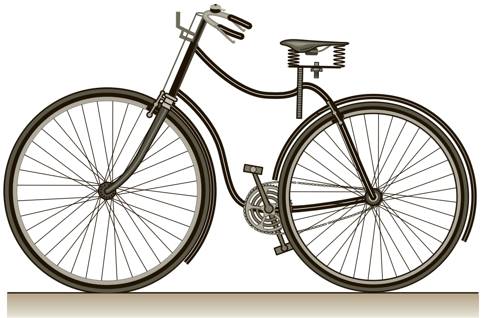
a. Using what you have learned about rolling resistance, describe why this makes sense.

b. Based on a 3.3 percent improvement in fuel economy, what would be the total fuel savings (in gallons) over the lifetime of a hypothetical vehicle?

7.3.2. [*] Consider the two bicycles popular in the second half of the 19th century—the penny-farthing (a) or ordinary cycle, and the safety bicycle (b). Assuming the same type of tires on both, which bicycle would you rather ride from the perspective of rolling resistance, and why?

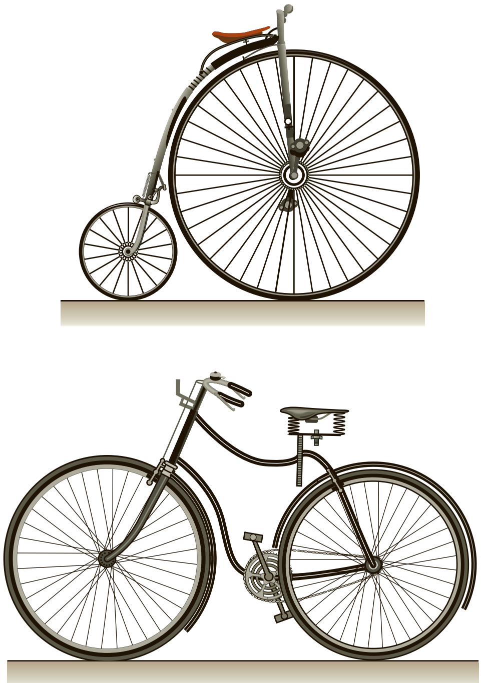


EX 7.3.2*a*



EX 7.3.2*b*

7.3.3. [*] If the diameter of the large wheel on the penny-farthing is 1.5 m, and the diameters of the two wheels on the safety bicycle are 0.6 m, calculate the ratio of the rolling resistance force on the penny-farthing to that on the safety.



EX 7.3.3

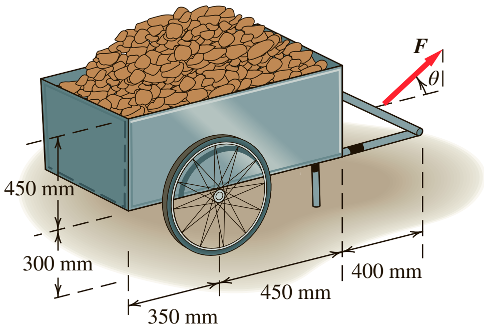
7.3.4. [*] A 2000-lb automobile has four 23-in.-diameter tires. Neglect bearing friction. Determine the horizontal force required to push the automobile

a. on level pavement, where the coefficient of rolling friction is $f_r = 0.03$.

b. on loose sand, where the coefficient of rolling friction is $fr = 0.35$.

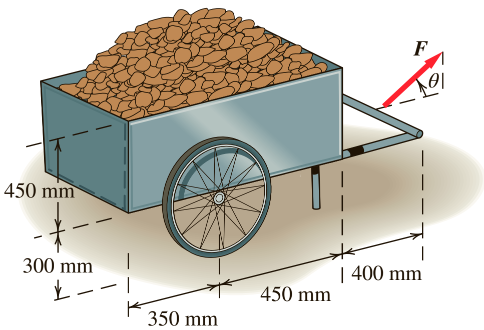
7.3.5. [*] A 1100-kg automobile is observed to roll at a constant speed down a 1° incline. The auto has 550-mm-diameter tires. Determine the coefficient of rolling friction. Neglect bearing friction.

7.3.6. [**] Consider the two-wheel cart shown. Determine the force (both horizontal and vertical components) that must be applied to the handle of the cart to pull it at a constant speed due to rolling resistance. The coefficient of rolling resistance is 0.0002. The cart is equipped with 600-mm diameter high-pressure tires that fit loosely over a 20-mm diameter fixed axle; the cart weighs 200 N and is carrying 1000 N of yard waste; and the center of mass of the cart and waste is 75 mm in front of the axle. Ignore bearing friction.



EX 7.3.6

7.3.7. [**] Consider the two-wheel cart shown. Determine the force (both horizontal and vertical components) that must be applied to the handle of the cart to pull it at a constant speed. The cart is equipped with 600-mm diameter high-pressure tires that fit loosely over a 20-mm diameter fixed axle; the coefficient of kinetic friction between the wheel and the axle is 0.30; the cart weighs 200 N and is carrying 1000 N of yard waste; and the center of mass of the cart and waste is 75 mm in front of the axle. Include the effects of rolling resistance (f_r = 0.0002) and bearing friction.



EX 7.3.7

7.4 JUST THE FACTS

In this chapter we looked at friction and rolling resistance, and how to include these factors in analysis.

Coulomb Friction Model

Friction may play a critical role in how a device operates or may be something an engineer attempts to minimize to reduce loads needed to operate a system. At a fundamental level, friction represents the relationship between two surfaces wanting to shift or slide with respect to one another. A commonly used model of this relationship is the **Coulomb friction law**. This law describes two surfaces that are pushed together (therefore there is normal contact between them). If the force trying to slide one surface relative to the other (which we call the friction force) is less than a critical value, no sliding occurs. This is described as:

$$\|\boldsymbol{F}_{\text{friction}}\| < \mu_s \|\boldsymbol{F}_{\text{normal}}\| \tag{7.1}$$

On the other hand, if the friction force is just at the level of the critical value, then there is impending sliding:

$$\|\boldsymbol{F}_{\text{friction}}\| = \mu_s \|\boldsymbol{F}_{\text{normal}}\| \tag{7.2}$$

If this impending sliding becomes actual sliding, the value of the friction force is limited to a second critical value:

$$\|\boldsymbol{F}_{\text{friction}}\| = \mu_k \|\boldsymbol{F}_{\text{normal}}\| \tag{7.3}$$

The values of **μ_s and μ_k are the static and kinetic coefficients of friction**, respectively. They depend on the materials involved, as well as their surface finish. Sample values are presented in **Table 7.1**.

Friction in Static Analysis: Wedges, Belts, and Journal Bearings

Examples of modeling friction in wedge, belt-drive, and journal bearing systems were presented in Section 7.2.

A **wedge** is a simple machine used to make adjustments in the position of one object relative to another. We can use equilibrium analysis to derive a relationship between the horizontal force applied to a wedge and its shape and coefficient of static friction.

$$P = \frac{W}{\cos\alpha - \mu_s \sin\alpha}\left[(1 - \mu_s^2)\sin\alpha + 2\mu_s\cos\alpha\right] \tag{7.8}$$

where P is the applied horizontal load, α is the angle of the wedge, μ_s is the coefficient of static friction, and W is the weight of the load acting on the wedge.

Under a certain condition, the wedge is **self-locking**. This means that the wedge will remain in place (holding up the block) even when no horizontal force is applied. A wedge is self-locking when

$$\tan \alpha \leq \frac{2\mu_s}{(1 - \mu_s^2)} \tag{7.10}$$

Belts are used to connect mechanical components to one another. A belt operates because of friction between it and the pulleys that it connects. A belt will not slip on the pulley when

$$\ln \frac{T_2}{T_1} = \mu_s \beta \tag{7.16A}$$

$$e^{\mu_s \beta} = \frac{T_2}{T_1} \tag{7.16B}$$

where β is the wrap angle of the belt (the angle, in radians, formed by the first and last spots the belt touches the pulley), T_2 is the tension on the pulling side of the belt, T_1 is the tension on the resisting side of the belt, and μ_s is the coefficient of static friction.

Journal bearing friction occurs when shafts are supported by dry or only partially lubricated journal bearings. The moment created by this friction is opposite to the rotation of the shaft and represented by

$$M_O = r \bullet \mu F \tag{7.18}$$

where M_O is the moment acting on the shaft, r is the radius of the shaft, F is the bearing support force, and μ is the coefficient of friction (static if the shaft is not rotating and kinetic if the shaft is rotating at constant velocity).

Rolling Resistance

Rolling resistance results from real wheels and real surfaces not being perfectly rigid. These real systems (that do deform) require an extra force (P) to remain in motion. We can model P as:

$$P = (a/r)L \tag{7.20}$$

Where r is the radius of the wheel, a is the coefficient of rolling resistance, and the ratio of (a/r) is called the **coefficient of rolling friction**. Values of this coefficient for well inflated tires range from 0.01 (hard surface) to 0.4 (loose sand).

CHAPTER 8

MEMBER LOADS IN TRUSSES

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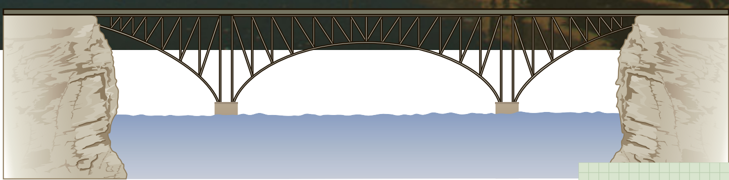
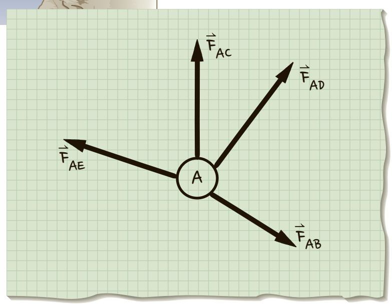




In this chapter we use equilibrium analysis to determine member loads for truss systems. We introduce several approaches to carry out the analysis and discuss situations for which each approach is most appropriate. By the end of this chapter, you will be able to systematically find the loads acting on members within a truss system.

LEARNING OBJECTIVES

On completion of this chapter, you will be able to:

- **Define and identify a truss.** (8.1)
- **Carry out equilibrium analysis of truss members using the method of joints.** (8.2)
- **Carry out equilibrium analysis of truss members using the method of sections or a combination of method of joints and method of sections.** (8.3)
- **Define and identify zero-force members in truss-structures.** (8.4)
- **Define and identify statically determinate, statically indeterminate, and unstable trusses.** (8.5)

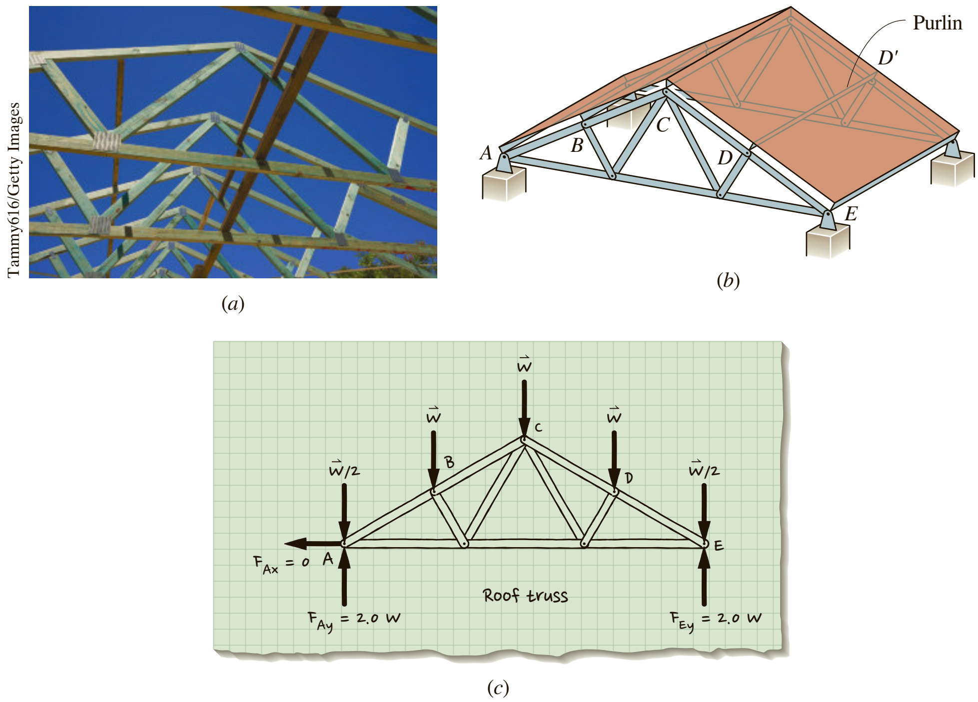
8.1 DEFINING A TRUSS

Learning Objective: Define and identify a truss.

In Section 5.6 we noted that if the conditions of equilibrium hold for a system as a whole, they must hold for all portions of the system. Consider, for example, the roof-structure shown in **Figures 8.1.1*a*** and ***b***. Defining the 11 members as the system, we can draw a free-body diagram of the system and set up and solve the equations of equilibrium to determine the loads acting on the system (**Figure 8.1.1*c***).

Now suppose that we want to find the loads acting on a member *within* the system, say *AB* or *BC*. To find these loads, we apply two key ideas:

- Loads internal to a system exist as equal and opposite pairs and are therefore self-canceling (this is Newton's third law, and we refer to the pairs as third-law force pairs). When we draw a boundary around a member (or



(*a*) (*b*) (*c*)

Figure 8.1.1 (*a*) Trusses during roof construction; (*b*) Schematic of a roof structure; (*c*) Free-body diagram and analysis of truss support forces.

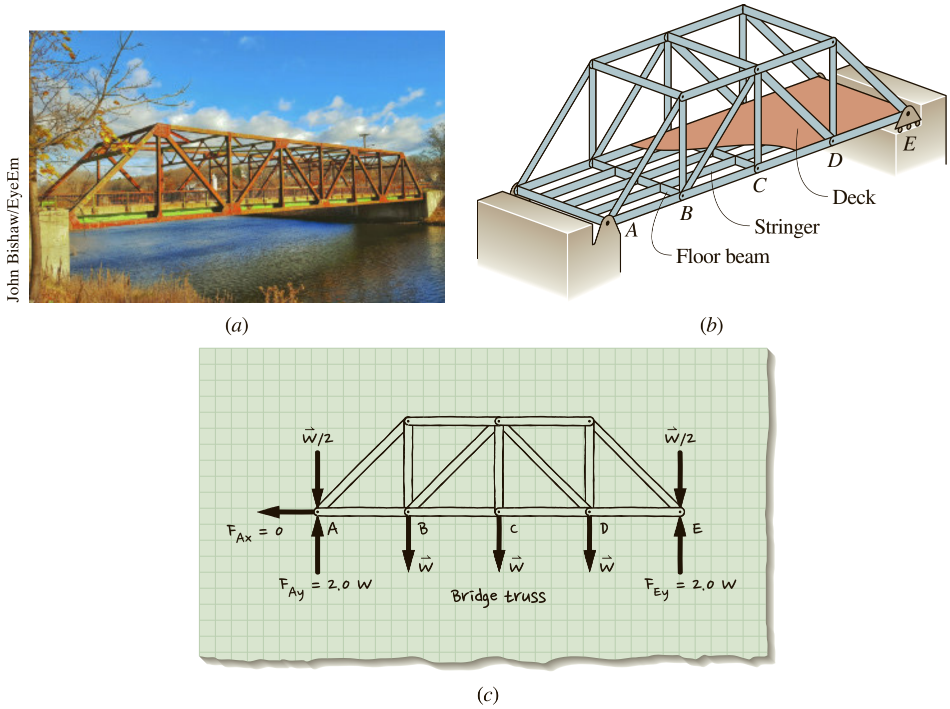


Figure 8.1.2 (*a*) An example of a truss bridge; (*b*) Schematic of a truss bridge; (*c*) Free-body diagram and analysis of truss support forces.

group of members) in a system, one component of each force pair at the boundary becomes an external force acting on the member.

- If the system as a whole is in equilibrium, then each member (or group of members) is in equilibrium. This means that equilibrium equations can be written for each member in terms of the external loads acting on that member.

In this chapter we apply these two key ideas to analyze a common engineering structure called a truss-structure (or a "truss," for short). We will follow the same basic analysis process for subsystems that was introduced in Section 5.6 to explore how the members that make up a truss are loaded.

To begin, let's get more specific about what makes a truss a truss.

A **truss** is a structural system, made up of two-force members, that is generally lightweight compared to the loads it can support. Trusses are commonly seen in bridges, roof supports, cranes, derricks, towers, and amusement park structures. For example, the truss shown in **Figure 8. 1.1** carries the weight of the roof to the walls, and the truss in **Figure 8.1.2** carries the weight of a bridge roadway to the vertical supports.

For a system to be classified as a truss, it must:

1. Consist exclusively of *straight members joined at their ends* to form a rigid frame.
2. Have joints that can be *represented as pin connections*, even though the actual joints may consist of welds, rivets, large bolts, or pins (often in conjunction with a gusset plate). We idealize these joints into the representation shown in **Figure 8.1.3**—that of a simple pin connection. In this idealized joint, the pin fits smoothly into holes at the ends of the straight members and is therefore capable of transferring

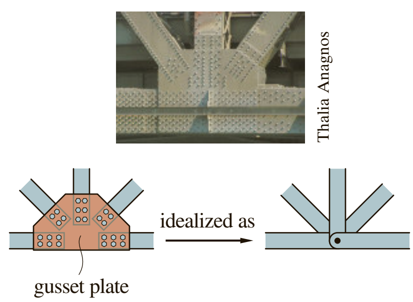


Figure 8.1.3 A connection in a truss, idealized as a pin connection.

forces between members but not capable of transferring moments. Furthermore, the centerline of each straight member is assumed to intersect the center axis of the pin (**Figure 8.1.4**).

3. Carry external *forces* (and no moment), *exclusively at the pin joints*. This requirement is not as restrictive as it might first appear. For example, in **Figure 8.1.1*b***, the roof force is transferred to purlins, then to the truss, and finally to the walls. In **Figure 8.1.2*b***, the weight of the roadway is transferred to the stringers and floor beams, then to the trusses, and finally to the vertical supports.

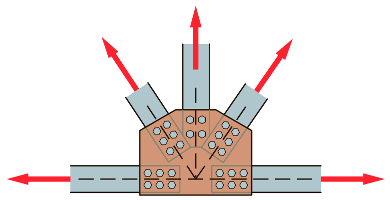


Figure 8.1.4 Forces of two-force members connected at a pin connection are concurrent.

As a consequence of these three requirements, the members that make up a truss behave as two-force members. This means that if we isolate any member from the rest of the truss and draw a free-body diagram of that member, the only way for the member to be in equilibrium is for the two forces acting on it to be along the same line of action; that is, **collinear** (as we discussed in Section 5.4.2 of Chapter 5 on two-force members).

A truss is made up *entirely* of two-force members. This is in contrast to a frame, where at least one of the members is a multiforce member. The system in **Figure 8.1.5*a*** is a truss, whereas the system in **Figure 8.1.5*b*** is a frame. We will examine frames in Chapter 9.

Trusses are classified as planar or space (i.e., nonplanar). A **planar truss** is one in which all forces and members lie in a single plane (or it is reasonable to assume that they all lie in a single plane). The basic building block of a planar truss is a triangle, composed of three two-force members connected by pins (**Figure 8.1.6*a***). The triangle is the simplest

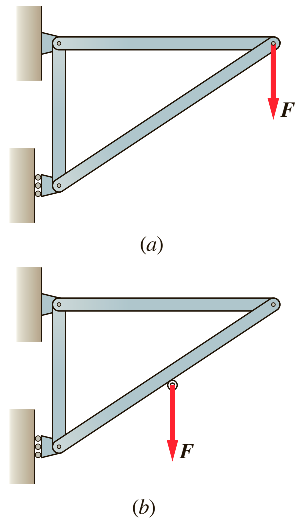


Figure 8.1.5 (*a*) Force ***F*** acts at a pin joint, so structure is a truss; (*b*) Force ***F*** acts along a member, so structure is a frame.

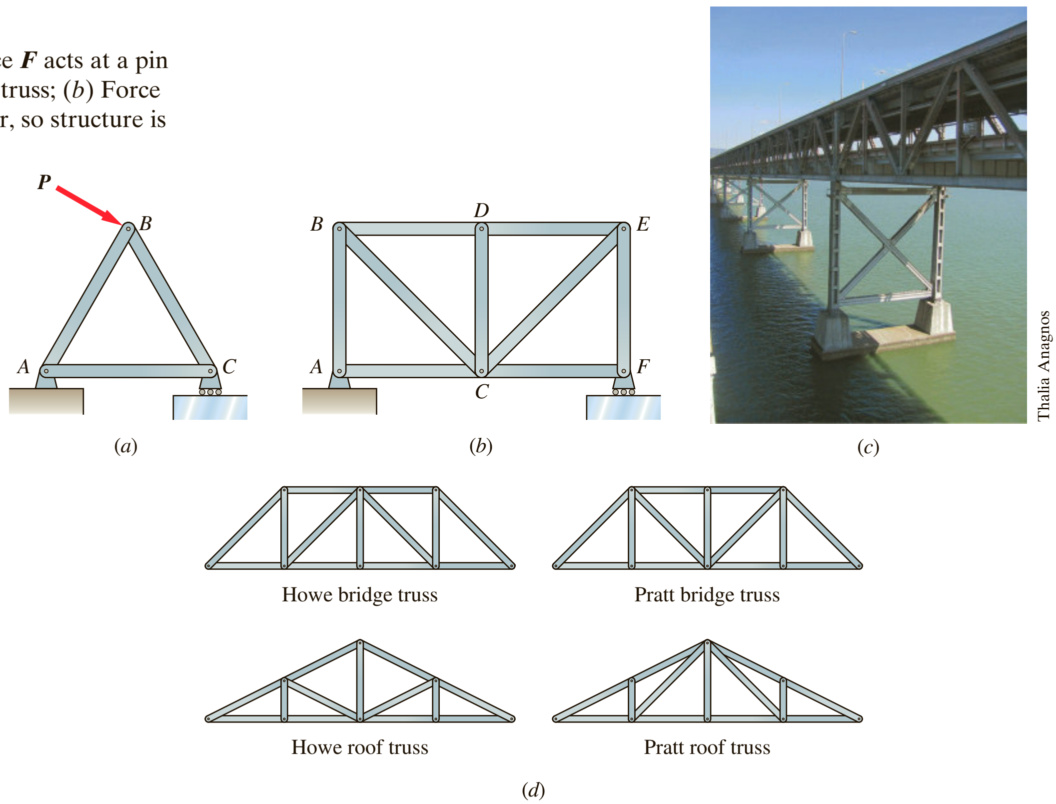
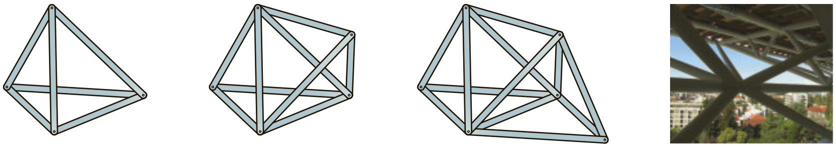


Figure 8.1.6 Examples of various planar trusses.



(a) (b) (c) (d)

Thalia Anagnos

Figure 8.1.7 Examples of space trusses of increasing complexity. In reality, a space truss rarely has true ball and socket joints.

rigid structure that can be created with two-force members; by rigid we mean that the structure is internally stable (an idea we will return to in the last section of this chapter).

Systems built up from a number of basic triangles are simple planar trusses. Various structures made up of planar trusses are illustrated in **Figures 8.1.6*b–d***. Notice that in the roof and bridge structures shown in **Figures 8.1.1** and **8.1.2**, two planar trusses have been connected by cross-pieces to distribute the force to the trusses; in this manner, planar trusses can be incorporated into three-dimensional structures.

If a truss is not planar, it is a **space truss**. The basic building block of a space truss is a tetrahedron, which is an assembly of six two-force members joined by ball-and-socket joints (**Figure 8.1.7**). We model space trusses as though the members are connected by ball-and-socket joints, but in many structures the connections are welded or bolted as shown in **Figure 8.1.7*d***.

Trusses are typically designed so that they are connected either to the ground or to a supporting structure with pin joints and/or rocker connections. A rocker or roller connection accommodates expansion or contraction of the truss as the temperature changes and as applied forces deform the members. This configuration is illustrated in **Figure 8.1.8**.

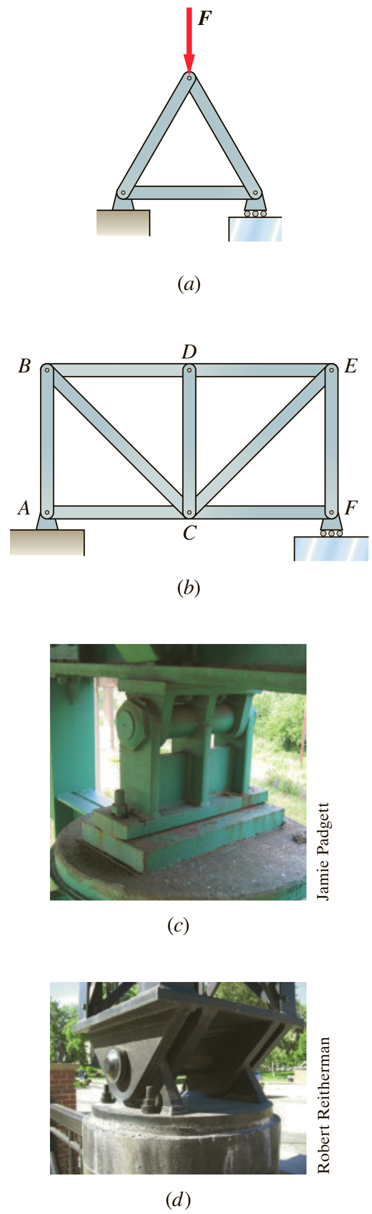


Figure 8.1.8 (*a*) and (*b*) Idealized rollers and pins as truss supports. (*c*) rocker (*d*) pin

8.2 TRUSS ANALYSIS BY METHOD OF JOINTS

Learning Objective: Carry out equilibrium analysis of truss members using the method of joints.

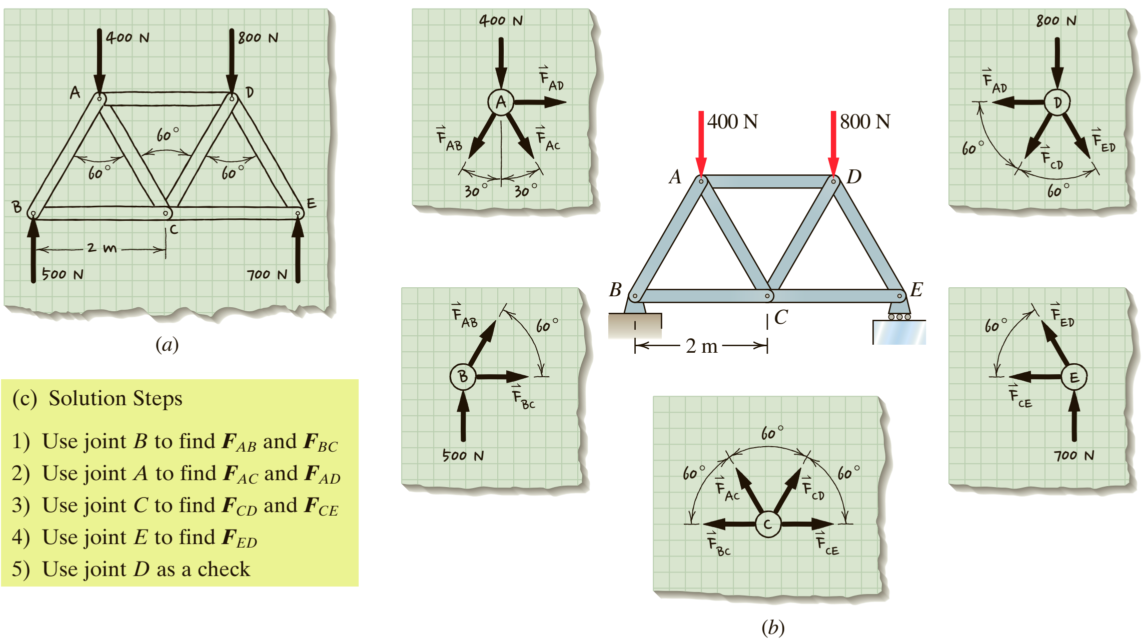
Truss analysis involves determining the force that each two-force member and pin connection must carry for given external forces. The calculated forces acting on the two-force members are then used to check for the adequacy of the cross section and material of each member. For a two-force member in tension, the check is to ensure that calculated tension is less than a critical level for tensile failure. For a two-force member in compression, the check is to ensure that the calculated compression is less than a critical level for compressive failure, including buckling. The calculated forces acting on the pins (which are shear forces) are checked to ensure adequacy of the cross section and material of each pin; an inadequate pin would undergo a shear failure.

We outline two procedures for finding the loads carried by two-force members in a truss—the **method of joints** and the **method of sections**. In general, the method of joints is preferable for determining the force acting on every member in a truss, and the method of sections is preferable for determining the forces for only a few members.

Both methods assume that the reason for analyzing the system has been defined and that information about the problem has been recorded, including whether it is reasonable to categorize the system as a planar or space truss, and whether the weight of the members is negligible relative to other forces acting on the truss. If the weight of the members is not negligible, it can be included in an approximate manner by determining the weight of each member, and placing one-half of this weight at the pins at the end of the member.

The first step in both methods is creating a free-body diagram of the entire truss, as in **Figure 8.2.1*a***, and using the equilibrium equations to find the external forces acting on the entire truss.

The next step in the method of joints is to isolate each pin connection and draw its free-body diagram (**Figure 8.2.1*b***). For each truss member connected to a pin joint, there is a third-law force pair. One component of this pair is the force exerted by the two-force member on the pin joint (or pin, for short). Draw this force on the pin free-body diagram as an arrow with its line of action along the centerline of the member. Label the force with a subscript that describes the member by the two pins connecting it to the rest of the truss and the pin (e.g., $\boldsymbol{F}_{AB,B}$ is the force of member AB, which runs between pins A and B acting on pin B. In **Figure 8.2.1** we have shortened this to $\boldsymbol{F}_{AB}$). Be sure to include any external forces acting on each pin in the pin's free-body diagram.



(c) Solution Steps

1) Use joint B to find $\boldsymbol{F}_{AB}$ and $\boldsymbol{F}_{BC}$
2) Use joint A to find $\boldsymbol{F}_{AC}$ and $\boldsymbol{F}_{AD}$
3) Use joint C to find $\boldsymbol{F}_{CD}$ and $\boldsymbol{F}_{CE}$
4) Use joint E to find $\boldsymbol{F}_{ED}$
5) Use joint D as a check

Figure 8.2.1 Method of Joints (*a*) A free-body diagram of a truss; (*b*) free-body diagrams of pins A, B, C, D, and E; (*c*) solution steps for finding member forces.

The steps of zooming-isolating-drawing are repeated for each pin. Be consistent in defining force labels and force directions. When done, you should have as many free-body diagrams as you have pins, as shown in **Figure 8.2.1*b***.

Now set up the force equilibrium equations for each pin. For a planar truss two independent force equations can be written for each pin (and for a space truss there are three). The forces produce no moment about the pin's center because their lines of action intersect at a common point, so we do not need to consider the moment equilibrium conditions. The pin is, therefore, treated as a particle.

If there are N pins, there will be $2N$ force equilibrium equations for a planar truss (and $3N$ for a space truss). If equilibrium conditions have been applied to the truss as a whole (as was done to determine the support loads in **Figure 8.2.1*a***), only $2N–3$ of the equations generated in considering the N pins for a planar truss will be linearly independent ($3N – 6$ will be linearly independent for a nonplanar truss).

We solve these equations for unknown forces. It is easiest to start at a pin that has at least one known force and at most two unknowns for a planar truss (three unknowns for a space truss)—this means that we can find the two unknowns (three for a space truss) using the associated force equilibrium equations. We proceed to the next pin that has at least one known force and at most two (three for a space truss) unknowns and solve its equilibrium equations, until all pins, and therefore all equilibrium equations, have been considered, as summarized in **Figure 8.2.1*c***.

How do we interpret whether the calculated forces indicate that members are in tension or compression? Remember that there are two components in any third-law force pair. In the case of the forces between a member and a pin, one component is that of the member acting on the pin and the other is the pin acting on the member. We show this pair for pin A and member AB in **Figure 8.2.2**. These two forces are equal and opposite, and the force of pin A acting on member AB is such that it pulls on member AB, putting the member in tension. Drawing all the forces from members acting on pins as "outward rays," as in **Figure 8.2.1**, implies that the members are in tension. If we then calculate a positive force value, it confirms that the associated two-force member is in tension, while a negative force value indicates that it is in compression.

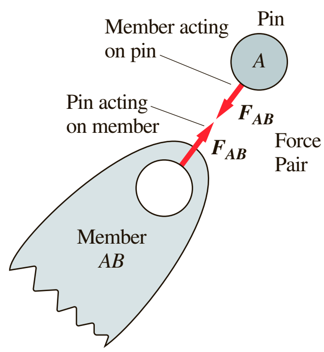


Figure 8.2.2 Third-law force pair between pin A and member AB, drawn so as to indicate member AB in tension.

> **IMPORTANT NOTE!** By drawing the force components acting on a pin as "outward rays," we can interpret a calculated positive value of force as indicating the associated two-force member is in tension, and a calculated negative value of force as indicating the associated two-force member is in compression. This will make tracking which members are in tension and which are in compression much easier.

After solving the equilibrium equations, check your results. Consider which two-force members are in tension and which are in compression—are these answers consistent with your intuition? If you can compare these force values with the capacity of the members, are you concerned about the design? Finally, present your answers in a manner that clearly indicates the forces acting on the truss and on each two-force member.

Check out the following example of an application of this material.

- **Example 8.2.1 Truss Analysis Using Method of Joints**

EXAMPLE 8.2.1

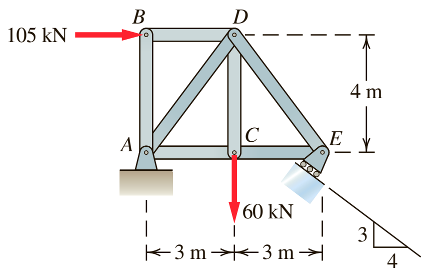


Figure 1 Planar truss supported on a pin and a roller on an inclined plane.

The planar truss shown in **Figure 1** is supported by a pin connection at A and a roller on a frictionless inclined plane at E. Use the method of joints to find the force acting on each member.

Goal Find the force supported by each member using the method of joints.

Given Information about the geometry of the truss, the supports, and the loads acting at B and C.

Assume The weight of the members is negligible.

Draw We first draw a free-body diagram of the entire truss (**Figure 2**) to find the loads at supports A and E. Then we draw a sequence of free-body diagrams for pins A, B, C, and D.

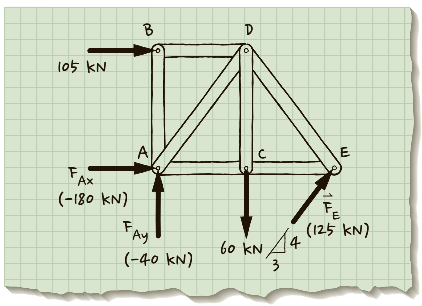


Figure 2 Free-body diagram of entire truss.

Formulate Equations and Solve Before diving in and writing equations, consider how many unknowns there are—10: (F_{Ax}, F_{Ay}, F_E, F_{AB}, F_{AC}, F_{AD}, F_{BD}, F_{CD}, F_{CE}, F_{DE}). We will therefore be writing 10 equilibrium equations to find these 10 unknowns.

For the truss as a whole we can determine the forces acting at A and E; these forces are shown on the free-body diagram of the whole truss **Figure 2** as:

$$F_{Ax} = -180\text{ kN},\qquad F_{Ay} = -40.0\text{ kN},\qquad F_E = 125\text{ kN}$$

Next we analyze each pin and solve for forces. Good candidate pins to start with are joints B and E; both have only two unknown forces. We arbitrarily choose joint B.

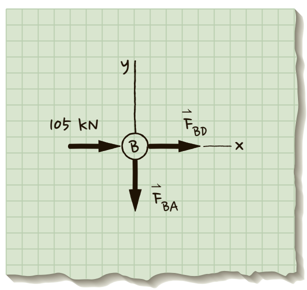


Figure 3 Free-body diagram of pin B showing known force in known direction and unknown forces in tension.

In drawing the free-body diagram of pin B, we choose to draw the unknowns F_{BC} and F_{BD} so that they are pulling on the joint. In response, the joint is pulling on the members so that they are in tension. Joint B's free-body diagram is shown in **Figure 3**.

Equilibrium at Joint B is:

$$\sum F_x(\rightarrow +) = 105\text{ kN} + F_{BD} = 0 \Rightarrow \boxed{F_{BD} = -105\text{ kN}}\quad\text{(compression)}\quad (1)$$

$$\sum F_y(\uparrow +) = -F_{BA} = 0 \Rightarrow \boxed{F_{BA} = 0\text{ kN}}\quad (2)$$

(Note that our calculation says that member BA is a zero-force member. This is consistent with Case 2, in the discussion of zero-force members.)

We next analyze pin A (**Figure 4**). In **Figure 4**, we include the support forces acting on pin A:

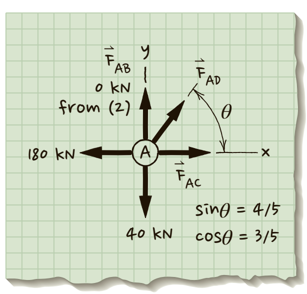


Figure 4 The free-body diagram of pin A includes the known support forces.

$$\sum F_y(\uparrow +) = -40.0\text{ kN} + F_{AD}\sin\theta = 0$$

$$F_{AD}\left(\frac{4}{5}\right) = 40.0\text{ kN} \quad\Rightarrow\quad \boxed{F_{AD} = 50.0\text{ kN}}\quad\text{(tension)}\quad (3)$$

$$\sum F_x(\rightarrow +) = -180\text{ kN} + F_{AD}\cos\theta + F_{AC} = 0$$

$$F_{AC} = 180\text{ kN} - F_{AD}\left(\frac{3}{5}\right)\quad (4)$$

Substitute (3) into (4) and solve for F_{AC}:

$$F_{AC} = 180\text{ kN} - 50.0\text{ kN}\left(\frac{3}{5}\right) \Rightarrow \boxed{F_{AC} = 150\text{ kN}} \text{ (tension)} \quad (5)$$

We now look at either pin C or pin D. We arbitrarily choose C (**Figure 5**) and substitute from (4) to get:

$$\sum F_x(\rightarrow +) = -F_{AC} + F_{CE} = 0 \Rightarrow \boxed{F_{CE} = 150\text{ kN}} \text{ (tension)}$$

$$\sum F_y(\uparrow +) = F_{CD} - 60.0\text{ kN} = 0 \Rightarrow \boxed{F_{CD} = 60.0\text{ kN}} \text{ (tension)} \quad (6)$$

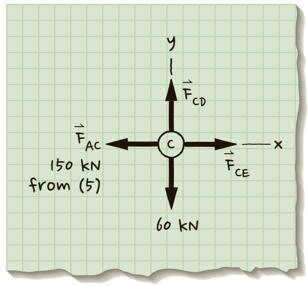


Figure 5 Free-body diagram of pin C.

Finally we consider pin D (**Figure 6**):

$$\sum F_x(\rightarrow +) = F_{BD} - F_{AD}\sin\alpha + F_{DE}\sin\alpha = 0$$

Substituting from (1) and (3),

$$105\text{ kN} - 50.0\text{ kN}\left(\frac{3}{5}\right) + F_{DE}\left(\frac{3}{5}\right) = 0$$

$$F_{DE}\left(\frac{3}{5}\right) = -105\text{ kN} + 30.0\text{ kN} = -75.0\text{ kN}$$

$$\Rightarrow \boxed{F_{DE} = -125\text{ kN}} \text{ (compression)} \quad (7)$$

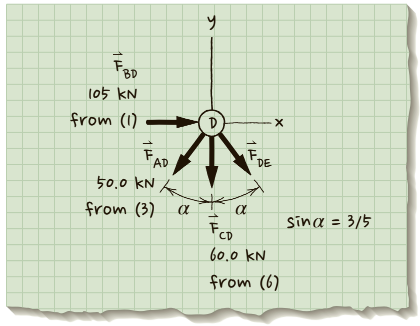


Figure 6 Free-body diagram of pin D.

Check One check on our results is to analyze pin E (**Figure 7**), for which we now know all of the forces, and confirm that E is in equilibrium. If pin E is not in equilibrium, then we have made a mistake in our analysis at one or more of the joints.

$$\sum F_x(\rightarrow +) = -150\text{ kN} + 125\text{ kN}\left(\frac{3}{5}\right) + 125\text{ kN}\left(\frac{3}{5}\right) = 0$$

$$-150\text{ kN} + 75.0\text{ kN} + 75.0\text{ kN} = 0 \Rightarrow 0 = 0$$

$$\sum F_y(\uparrow +) = 125\text{ kN}\left(\frac{4}{5}\right) - 125\text{ kN}\left(\frac{4}{5}\right) = 0 \Rightarrow 0 = 0$$

Yes, pin E is in equilibrium!

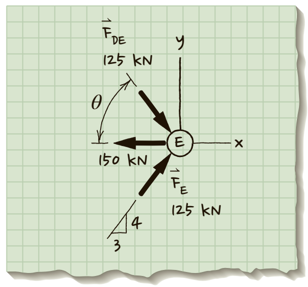
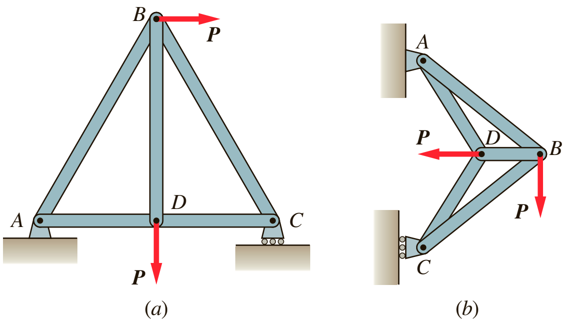


Figure 7 Free-body diagram of pin E.

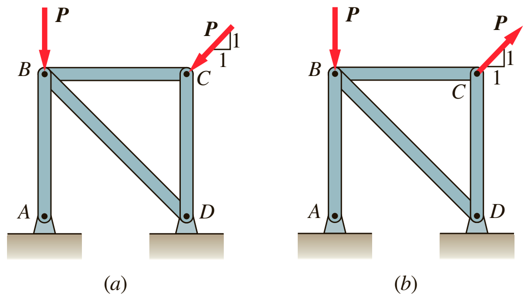
EXERCISES 8.2

8.2.1. [*] Consider the trusses shown. Assume that the trusses are in equilibrium and the member weights can be neglected. Identify which members are in tension and which are in compression.



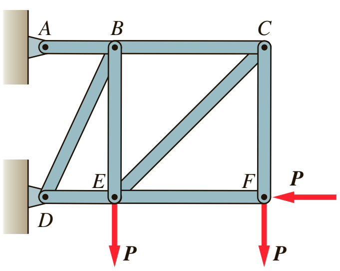
EX 8.2.1

8.2.2. [*] Consider the trusses shown. Assume that the trusses are in equilibrium and the member weights can be neglected. Identify which members are in tension and which are in compression.



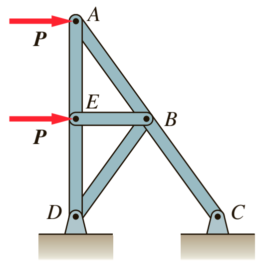
EX 8.2.2

8.2.3. [*] Consider the truss shown. Assume that the truss is in equilibrium and the member weights can be neglected. Identify which members are in tension and which are in compression.



EX 8.2.3

8.2.4. [*] Consider the truss shown. Assume that the truss is in equilibrium and the member weights can be neglected. Identify which members are in tension and which are in compression.

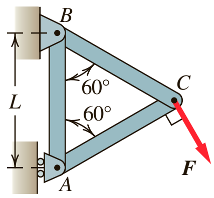


EX 8.2.4

8.2.5. [*] For the truss shown, assume member weights are negligible compared to the applied load.

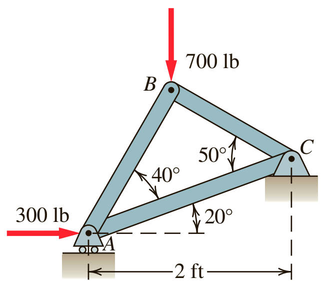
a. Determine the force in each member and state whether each member is in tension or compression.

b. If the magnitude of the tension or compression in any member is not to exceed 400 N, what is the maximum allowable magnitude of the force ***F***?



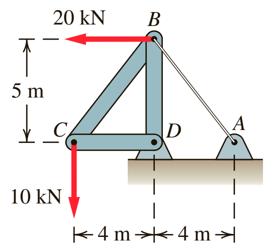
EX 8.2.5

8.2.6. [*] Determine the force in each member of the truss shown. State whether each member is in tension or compression.



EX 8.2.6

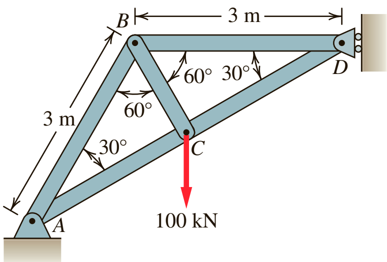
8.2.7. [*] Before you start your analysis of the truss shown, list those members you think are in tension, those you think are in compression, and those for which you are unsure whether they are in tension or compression.



EX 8.2.7

Determine the force in each member and state whether each member is in tension or compression. After you complete your analysis note where your preliminary list differs from the results.

8.2.8. [*] Determine the force in each member of the truss shown. State whether each member is in tension or compression.

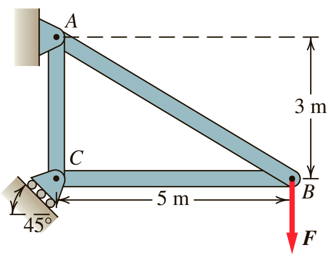


EX 8.2.8

8.2.9. [*] Consider the truss loaded as shown.

a. Determine the force in each member. State whether each member is in tension or compression.

b. If the maximum tension in any member is to be limited to 1000 N and the maximum compression to 500 N, what is the largest allowable magnitude of the force ***F***?

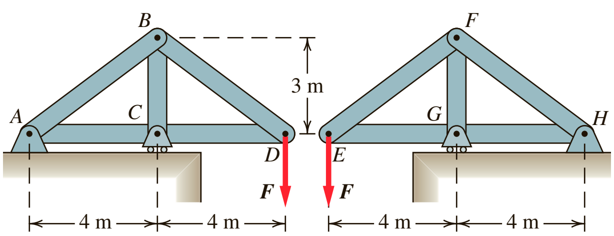


EX 8.2.9

8.2.10. [*] The trusses shown are used as part of a footbridge structure (the Forth Bridge in Scotland, built in 1890, uses a similar basic configuration of trusses). The force ***F*** represents the load that a footbridge between D and E transfers to the two trusses.

a. Determine the force in each of the five members of truss $ABCD$ in terms of F and state whether each member is in tension or compression.

b. If the magnitude of ***F*** is 20 kN, what are the values of the forces in members AB, AC, BC, BD, and CD?

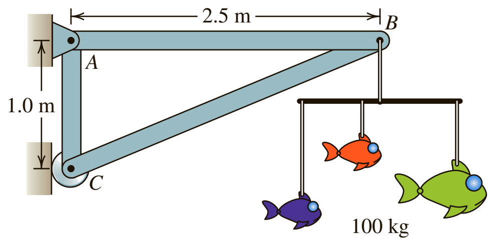


EX 8.2.10

8.2.11. [*] An art piece with mass of 100 kg is displayed at the end of a truss, as shown.

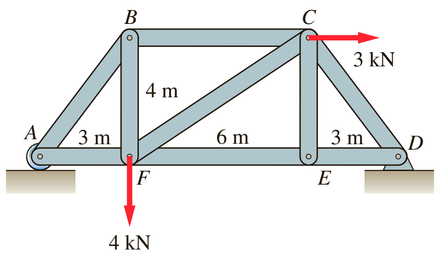
a. Find the loads acting on the truss at A and C, and find the force in each member.

b. Which member(s) could fail in a buckling mode? Which members could be replaced with cables?



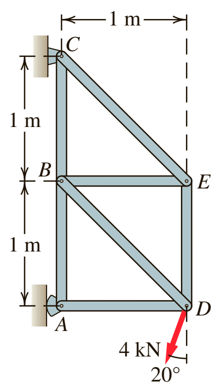
EX 8.2.11

8.2.12. [*] Determine the forces in members CD, CF, and EF of the truss using the method of joints.



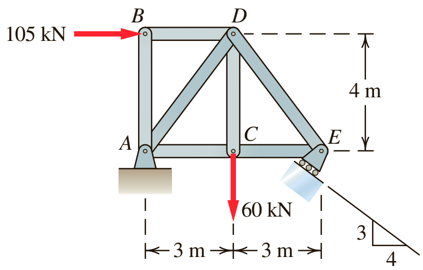
EX 8.2.12

8.2.13. [*] Determine the force in each member of the loaded truss.



EX 8.2.13

8.2.14. [*] The truss shown was analyzed in Example 8.2.1 by first enforcing equilibrium on the truss as a whole and then considering equilibrium of each joint. Instead, analyze this truss by starting with equilibrium of joints rather than of the structure as a whole and confirm that you get the same results.



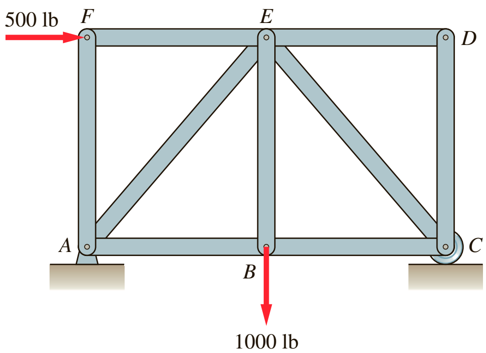
EX 8.2.14

8.2.15. [**] The truss is loaded as shown.

a. Determine all of the loads in the members.

b. Modify the supports so that support *A* is a roller and support *C* is a pin. With this new support arrangement and the same loading, calculate the force in members *AB* and *AE*.

c. Indicate if the member forces for *AB* and *AE* are the same or different for the two configurations.



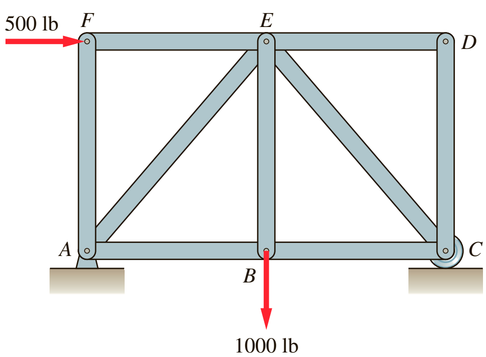
EX 8.2.15

8.2.16. [**] The truss is loaded as shown.

a. Determine all of the loads in the members.

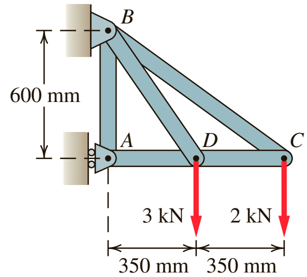
b. Move the 500-lb horizontal load from joint *F* to joint *D*. With this loading, calculate the force in members *AB*, *AE*, and *EF*.

c. Indicate if the member forces for *AB*, *AE*, and *EF* are the same or different for the two configurations.



EX 8.2.16

8.2.17. [**] For the truss loaded as shown, determine the force in each member and state whether each member is in tension or compression.

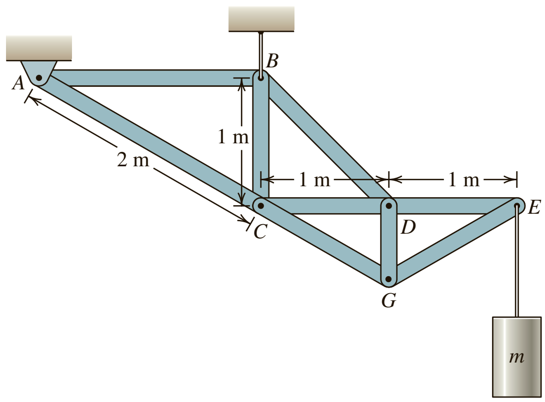


EX 8.2.17

8.2.18. [**] Consider the truss loaded as shown.

a. Determine the force in each member as a function of *m* and state whether each member is in tension or compression.

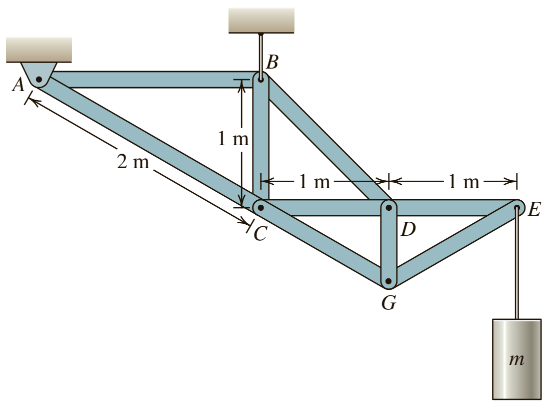
b. If the magnitude of the tension or compression in any member is not to exceed 2 kN, what is the largest mass *m* that can hang at *E*?



EX 8.2.18

8.2.19. [**] Consider the truss loaded as shown.

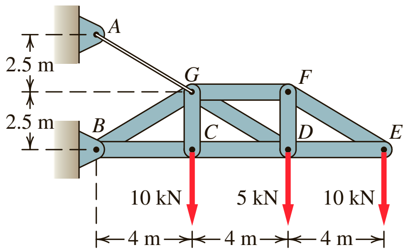
a. Determine the force in each member as a function of *m* and state whether each member is in tension or compression.



EX 8.2.19

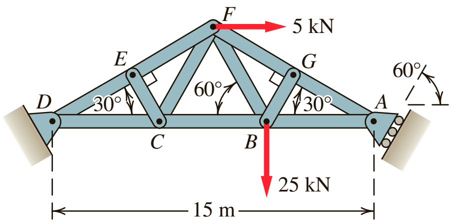
b. If the magnitude of the force in any member is limited to 2 kN in tension and 0.5 kN in compression, what is the largest mass m that can hang at E?

8.2.20. [**] For the truss loaded as shown determine the force in each member and state whether each member is in tension or compression.



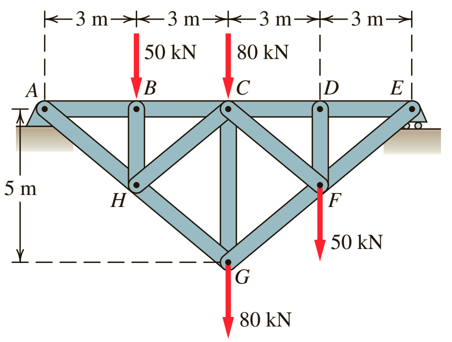
EX 8.2.20

8.2.21. [**] Determine the force in each member of the truss shown. State whether each member is in tension or compression.



EX 8.2.21

8.2.22. [**] Before you start your analysis of the truss shown, list those members you think are in tension, those you think are in compression, and those for which you are unsure whether they are in tension or compression. After you complete your analysis note where your list differs from the results. Determine the force in each member and state whether each member is in tension or compression.



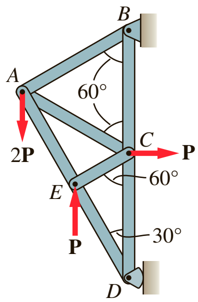
EX 8.2.22

8.2.23. [**] Consider the truss loaded as shown.

a. Determine the loads acting on the truss at B and D.

b. Determine the forces in all members as a function of P, clearly stating whether they are in tension or compression.

c. If the maximum tension in any member is limited to 1500 lb and the maximum compression to 200 lb, what is the largest allowable magnitude of the force $\boldsymbol{P}$?

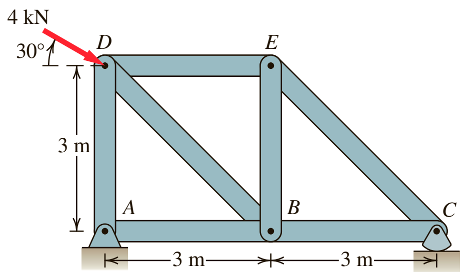


EX 8.2.23

8.2.24. [**] A force is applied to the truss at D.

a. Find the loads acting on the truss at A and C, and find the force in each member.

b. Which member(s) could buckle? Which member(s) could be replaced with cables?



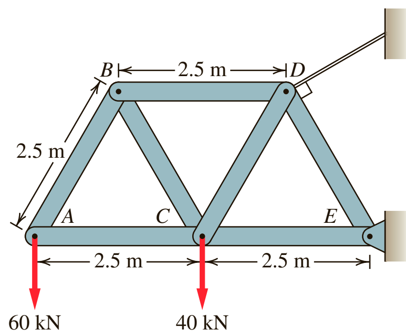
EX 8.2.24

8.2.25. [**] Consider the loaded truss, configured as equilateral triangles measuring 2.5 m on each side.

a. Find the loads acting on the truss at D and E.

b. Find the force in each member.

c. Which member(s) could fail in a buckling mode? Which member(s) could be replaced with cables?

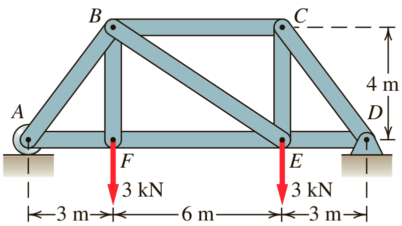


EX 8.2.25

8.2.26. [**] Forces are applied to a truss as shown.

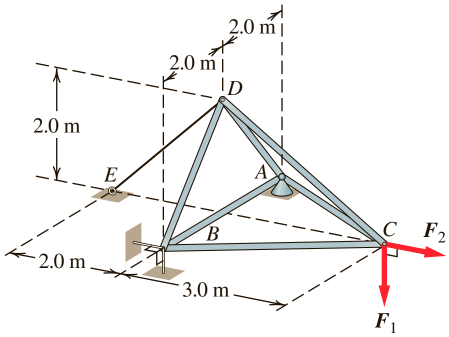
a. Find the loads acting on the truss at A and D, and find the force in each member.

b. Which member(s) could fail in a buckling mode? Which member(s) could be replaced with cables?



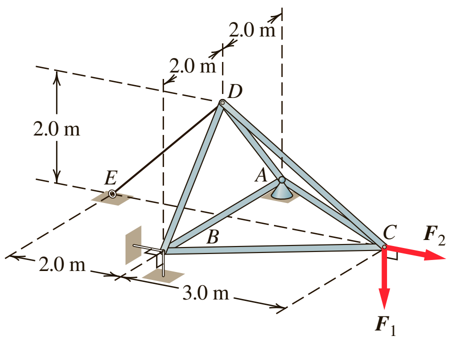
EX 8.2.26

8.2.27. [**] Consider the space truss shown. The magnitudes of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are 40 kN and 0 kN, respectively. Determine the force in each member. State whether each member is in tension or compression.



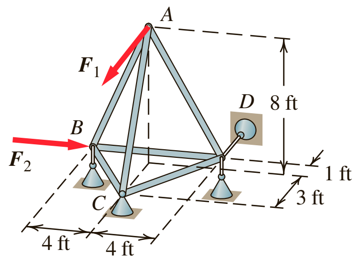
EX 8.2.27

8.2.28. [**] Determine the force in each member of the space truss if the magnitudes of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are 40 kN and 30 kN, respectively. State whether each member is in tension or compression.



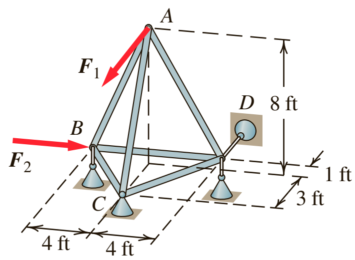
EX 8.2.28

8.2.29. [**] The magnitudes of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ acting on the space truss are 8 kip and 0 kip, respectively. Before you begin your analysis list those members you think are in tension, those you think are in compression, and those for which you are unsure whether they are in tension or compression. Determine the force in each member and state whether it is in tension or compression. After you complete the analysis note where your initial list of tension/compression members differs from the results.



EX 8.2.29

8.2.30. [**] Determine the force in each member of the space truss if the magnitudes of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are 8 kip and 4 kip, respectively. State whether each member is in tension or compression.



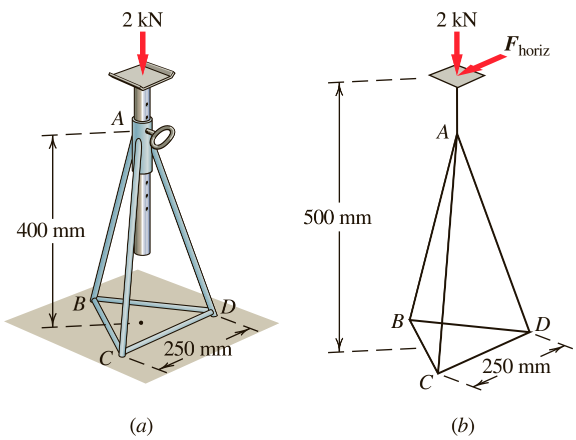
EX 8.2.30

8.2.31. [***] A force of 2 kN acts on the top of the automobile jack stand shown. The stand can be modeled as a space frame with an equilateral triangle as its base, resting on the ground at B, C, and D. Assume only vertical ground reaction forces at B, C, and D.

a. Determine the forces in members BC and AB. Based on a symmetry argument, what can you say about the forces in members CD, BD, AC, and AD?

b. Which members are susceptible to buckling? Which members would fail due to tensile forces?

c. In addition to the 2-kN vertical force, a horizontal force (F_{horiz}) acts at a height of 500 mm, as shown. At what magnitude will F_{horiz} cause the jack stand to tip over?



EX 8.2.31

8.3 TRUSS ANALYSIS BY METHOD OF SECTIONS

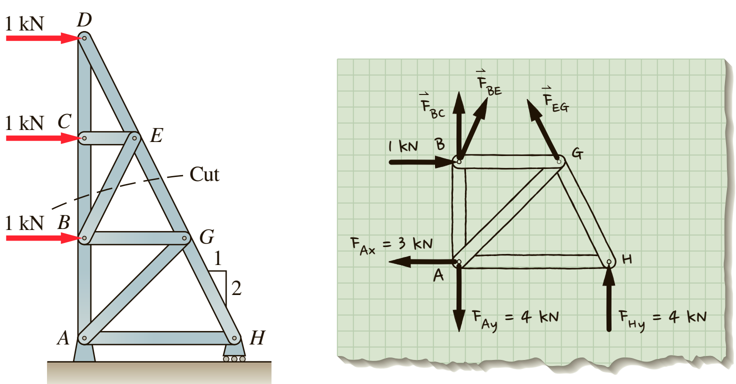
Learning Objective: Carry out equilibrium analysis of truss members using the method of sections or a combination of method of joints and method of sections.

Like the method of joints, analysis by the method of sections begins with a free-body diagram and equilibrium equations of the truss as a whole to find the forces acting on it. The next step, however, differs: we draw a second free-body diagram of a part of the truss with a strategic boundary cut through a member of interest to find the force in that member. Therefore the method of sections provides an efficient means of finding the force acting on a truss member directly because it is not necessary to proceed from pin to pin as in the method of joints.

Using the method of sections involves isolating a portion of the truss such that the boundary cuts through at most three unknown members for a planar truss or six members for a space truss (**Figure 8.3.1*a***) and drawing a free-body diagram for the isolated portion (**Figure 8.3.1*b***).

Wherever the boundary cuts a two-force member, there is a force pair. One component of this pair is the force exerted by the part of the two-force member outside of the boundary on the part of the member inside the boundary (e.g., F_{BC}, F_{BE}, and F_{EG} in **Figure 8.3.1*b***).

Draw each component on the free-body diagram as an arrow with its line of action along the centerline of the cut two-force member, as shown in **Figure 8.3.1*b***. Drawn in this manner, this force indicates tension in the cut member. If we find a negative value for this force when we solve the equilibrium equations, the member is actually in compression. Label the force using a subscript that names the pins the two-force member connects (e.g., F_{BC} represents the force in member BC that connects pin B and pin C). Using this free-body diagram, set up the equilibrium equations (three if planar, six if space) and solve for



(*a*) (*b*)

Figure 8.3.1 Method of Sections (*a*) A truss under consideration, where we want to find the forces in members *BC*, *BE*, and *EG*; (*b*) free-body diagram of a portion of the truss, based on cut boundary defined in (*a*).

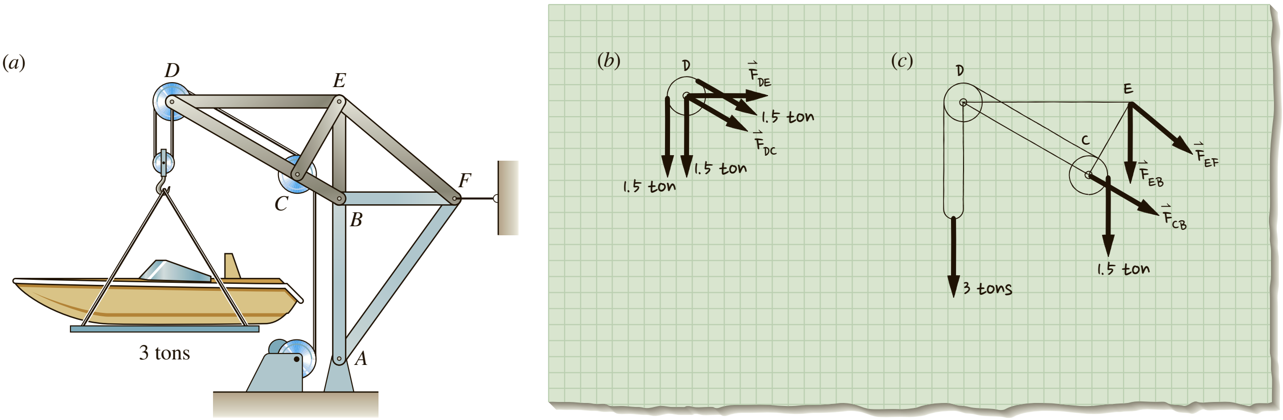


Figure 8.3.2 Consider a boat hoist comprised of trusses (*a*) Say it is of interest to find the forces in members *DE*, *DC*, *CB*, *EB*, and *EF*. (*b*) Use method of joints to find the forces in members *DE* and *DC*, and (*c*) use the method of sections to find the forces in members *CB*, *EB*, and *EF*.

the unknowns. Finally, as with the method of joints, check and present your answers.

It is possible to combine the method of joints and the method of sections. Some of the member loads in a truss can be found with one method and other member loads with the other method. Suppose, for example, that we wish to find the force in a member of a planar truss, and it is not possible to pass a boundary through the member without passing through more than three members that have unknown forces. Such a situation is shown in **Figure 8.3.2**. It may be possible, as in this situation, to first apply the method of joints at an adjacent pin joint to determine one of the unknown forces, then proceed with the method of sections. Using methods in conjunction with one another may end up involving fewer overall calculations than using a single method.

Regardless of which method you use first in a combination approach, you will ultimately determine the same member forces. The reason for considering a combined strategy is to minimize the overall computational complexity of the problem. Remember to map out your solution strategy before diving into either method.

Check out the following examples of applications of this material.

- **Example 8.3.1 Method of Sections and Wise Selection of Moment Center Location**
- **Example 8.3.2 Method of Sections and Where to Cut**
- **Example 8.3.3 Combining Method of Joints and Method of Sections**

EXAMPLE 8.3.1

Given the truss and loading shown in **Figure 1**, use the method of sections to find the forces supported by members DE and DG.

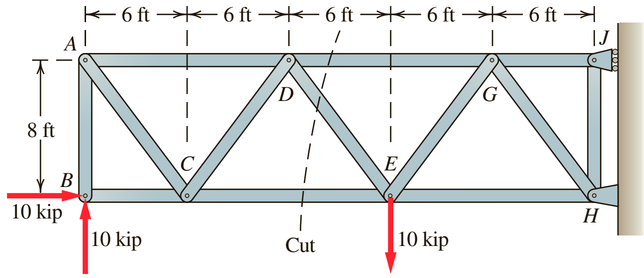


Figure 1 Planar truss.

Goal Find the forces supported by members DE and DG using the method of sections.

Given Information about the geometry of the truss and the loading on it.

Assume We can treat the system as planar and the weight of the members can be neglected.

Draw We run a boundary cut through members CE, DE, and DG to isolate a portion of the truss (**Figure 1**). Because we have three planar equilibrium equations, generally the maximum number of members (with unknown forces) we should cut is three. We can choose to analyze the portion of the truss to the left or right of the cut. We choose the left portion, because we can find the forces acting on the members DE and DG without finding the forces acting at the supports J and H (**Figure 2**). We draw the unknown forces acting on cut members CE, DE, and DG, with the assumption that the members are in tension and therefore the forces are pulling on the members.

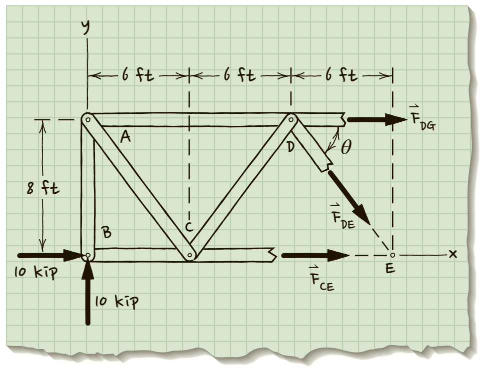


Figure 2 Free-body diagram of portion of truss to the left of the cut.

Formulate Equations and Solve We use the planar equilibrium equations for the left portion of the truss (**Figure 2**) to find the forces acting on DE and DG.

We choose E as the moment center because F_{CE} and F_{DE} act through E and therefore have no moment arm with respect to E. Thus the moment equilibrium equation will have only one unknown, F_{DG}. (Notice that the moment center does not have to be on the portion of the truss we are analyzing.)

$$\sum M_{z@E}(\curvearrowleft +) = -10\,\text{kip}(18\,\text{ft}) - F_{DG}(8\,\text{ft}) = 0$$

$$F_{DG} = -10\,\text{kip}\left(\frac{18\,\text{ft}}{8\,\text{ft}}\right) \Rightarrow F_{DG} = -22.5\,\text{kip}\,(\text{compression})$$

$$\sum F_y(\uparrow +) = 10\,\text{kip} - F_{DE}\sin\theta = 0$$

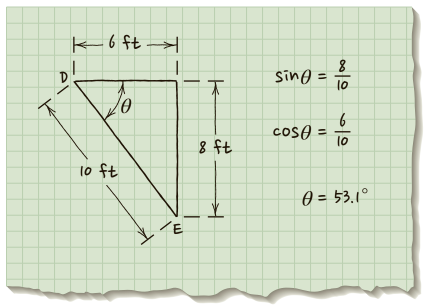


Figure 3 Determine truss angle.

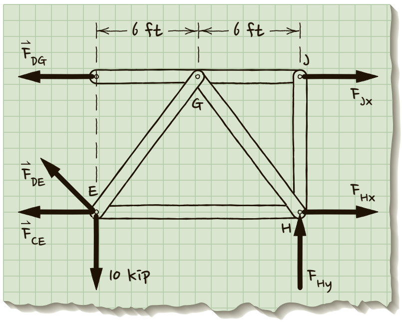


Figure 4 Free-body diagram of portion of truss to the right of the cut.

We determine the angle θ from the geometry in **Figure 3** to be 53.1°.

$$10\text{ kip} - F_{DE}\sin 53.1° = 0 \Rightarrow \boxed{F_{DE} = 12.5\text{ kip}} \quad \text{(tension)}$$

Using the method of sections, we were able to determine the forces acting on DE and DG by solving two equilibrium equations. We would have analyzed eight equations to obtain the same result using the method of joints.

Check To check our result, we can draw a free-body diagram of the portion of the truss to the right of the cut (**Figure 4**) and apply our equilibrium equations. To do this we would first need to find the loads at supports H and J.

EXAMPLE 8.3.2

A stiffening truss for a bridge deck is shown in **Figure 1**. Use the method of sections to find the force supported by member UV.

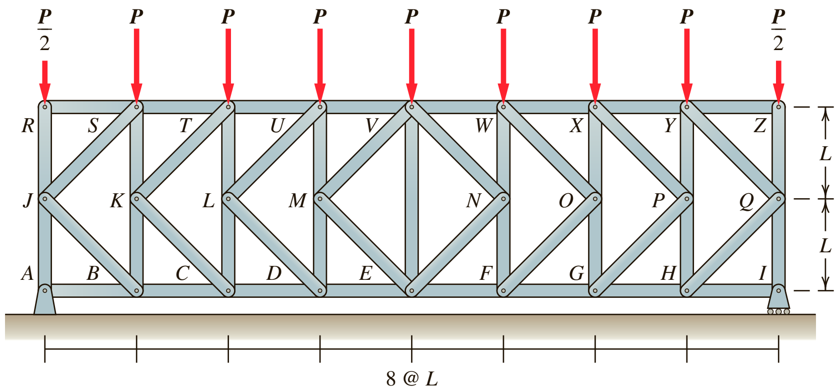


Figure 1 Stiffening truss for bridge deck.

Goal Find the force supported by member UV using the method of sections.

Given We are given information about the geometry and loading of the truss.

Assume The weight of the members can be neglected and we can treat the system as planar.

Draw We use the free-body diagram of the entire truss (**Figure 2**) to determine the forces at the supports. Since none of the applied forces are acting horizontally, F_{Ax} must be zero. Because the truss and the loading are symmetric, we know that the vertical force at each support must be equal to one-half the applied load ($F_{Ay} = F_{Iy} = 4P$). To prove the two statements we made here, we apply our planar equilibrium equations to the free-body diagram in **Figure 2** to obtain:

$$\boxed{F_{Ax} = 0, F_{Ay} = 4P, F_{Iy} = 4P}$$

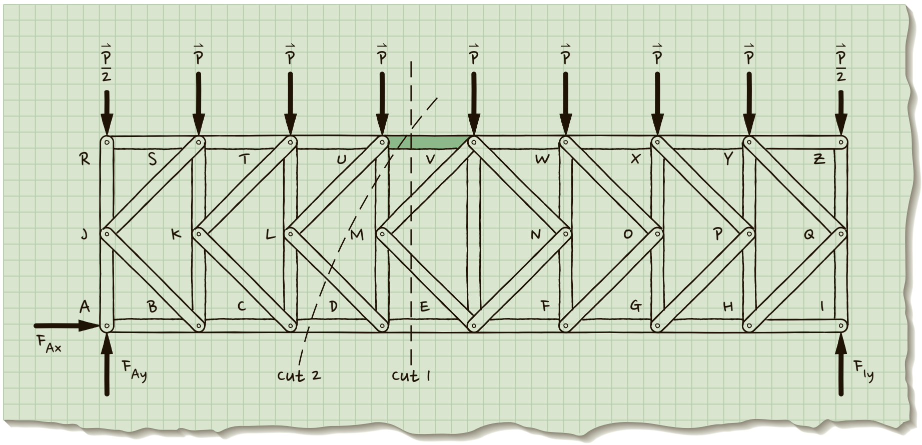


Figure 2 Free-body diagram of entire truss with member of interest shaded

When we cut a section (Cut 1) through the member of interest (UV), we have cut four members with unknown forces. We can draw a free-body diagram of a section to the left or right of this cut. We choose to analyze the left portion (**Figure 3**). Unfortunately, no matter where we place our moment center, we have at least two unknowns in the moment equilibrium equation. Similarly, if we try to sum forces in the x or y direction, we have two or more unknowns in the equilibrium equation. Therefore, Cut 1 by itself does not allow us to find the magnitude of any forces.

Instead we make a cut through CD, DL, MU, and UV (Cut 2 in **Figure 2**) and draw a free-body diagram of the left portion of the truss (**Figure 4**). We see on this free-body diagram that although we have cut four members with unknown forces, three of the member forces are concurrent at D (F_{CD}, F_{DL}, and F_{MU}). Therefore we can use the moment equilibrium equation with a moment center at D to find F_{UV}.

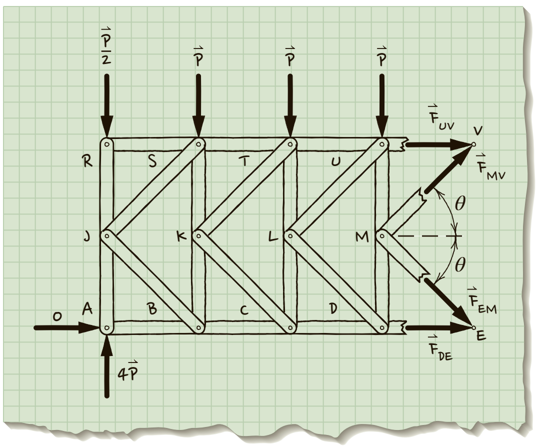


Figure 3 Free-body diagram of portion to the left of Cut 1.

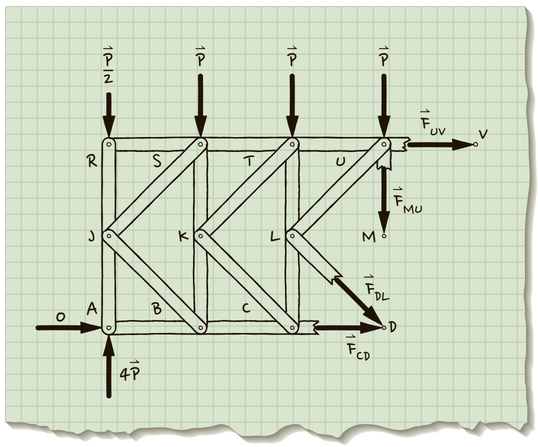


Figure 4 Free-body diagram of portion to the left of Cut 2.

Formulate Equations and Solve We apply the moment equilibrium equation to the free-body diagram in **Figure 4**:

$$\sum M_{z@D}(\curvearrowleft +) = -4P(3L) + \frac{P}{2}(3L) + P(2L) + P(L) - F_{UV}(2L) = 0$$

$$F_{UV}(2L) = -\frac{15}{2}PL \quad \Rightarrow \quad F_{UV} = -\frac{15}{4}P \qquad \text{(compression)}$$

Check We can check our results by substituting the value we have found for the unknown member force on the free-body diagram of the portion of the truss to the right of Cut 2. We then solve the moment equilibrium equation with a moment center at D for the new free-body diagram. If the moments sum to zero, member force UV is correct.

EXAMPLE 8.3.3

Given the truss and loading shown in **Figure 1**, use a combination of the method of joints and the method of sections to find the force supported by member DJ.

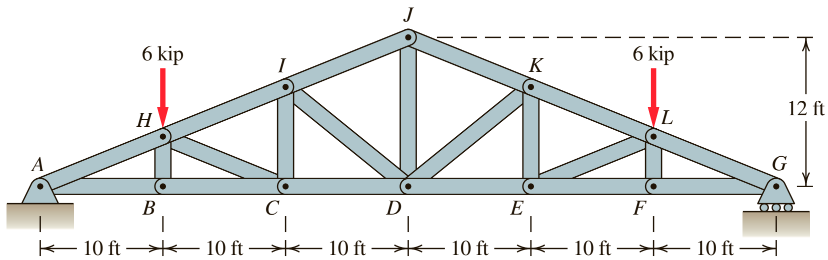


Figure 1 Roof truss loaded at joints H and L.

Goal Find the force in member DJ using a combination of the method of joints and the method of sections.

Given Information about the geometry and loading of the truss.

Assume The weight of the members can be neglected and we can treat the system as planar.

Draw We first draw a free-body diagram of the entire truss in order to find the loads at support A (**Figure 2**). ($F_{Ay} = 6$ kip, $F_{Ax} = 0$ kip, calculations not shown)

If we attempt to use just the method of sections to solve for DJ, we find that we must cut at least four members (Cut 1) when we cut through DJ (**Figure 3**). However, because three of the members are concurrent at D, we can sum the moments about D to find F_{JK}. This leaves only two unknowns at joint J. Therefore, we draw a free-body diagram of pin J (**Figure 4**) and use the method of joints to find the force acting on member DJ.

Formulate Equations and Solve We use the method of sections for the portion of the truss to the left of Cut 1 to calculate F_{JK} (**Figure 3**).

$$\sum M_{z@D}(\curvearrowleft +) = -6\,\text{kip}(30\,\text{ft}) + 6\,\text{kip}(20\,\text{ft}) - F_{JK}\cos\theta(12\,\text{ft}) = 0$$

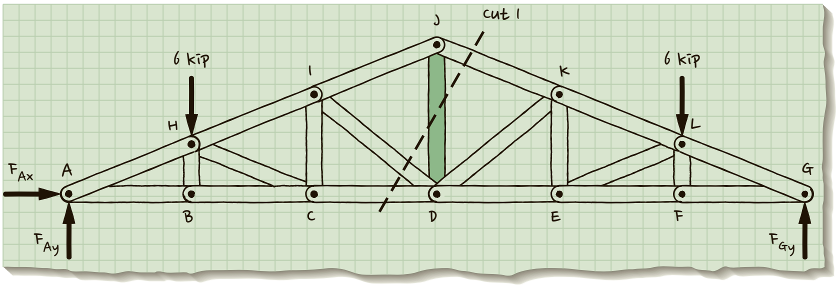


Figure 2 Free-body diagram of entire truss.

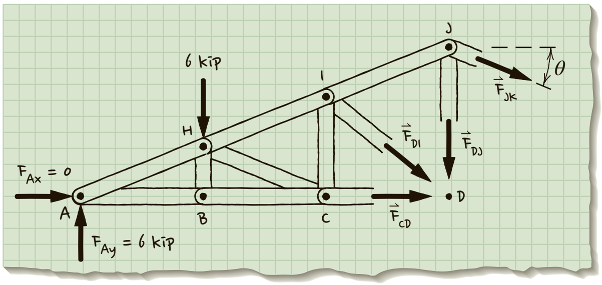


Figure 3 Free-body diagram of portion to the left of Cut 1.

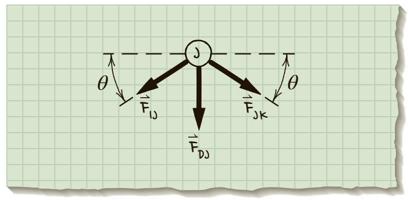


Figure 4 Free-body diagram of pin J.

Substituting in from **Figure 5** $\theta = 21.8°$, we solve for F_{JK},

$$F_{JK} = \frac{-6\text{ kip}(10\text{ ft})}{\cos(21.8°)(12\text{ ft})} = -5.39\text{ kip (compression)} \tag{1}$$

Using the method of joints at joint J, we solve for F_{DJ} (**Figure 4**):

$$\sum F_x(\rightarrow +) = -F_{IJ}\cos\theta - (5.39\text{ kip})\cos\theta = 0$$

$$F_{IJ} = -5.39\text{ kip (compression)} \tag{2}$$

$$\sum F_y(\uparrow +) = -F_{IJ}\sin\theta - F_{JK}\sin\theta - F_{DJ} = 0$$

With F_{JK} given in (1), F_{IJ} given in (2), and $\theta = 21.8°$, we find

$$-(-5.39\text{ kip})\sin 21.8° - (-5.39\text{ kip})\sin 21.8° - F_{DJ} = 0$$

$$\boxed{F_{DJ} = 4.00\text{ kip}} \quad \text{(tension)}$$

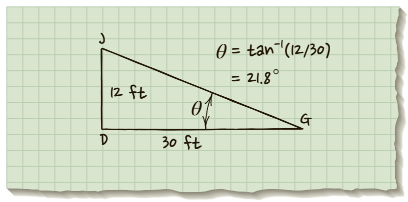


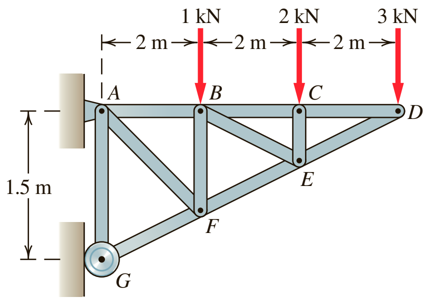
Figure 5 Truss geometry.

Check Since the truss and its loading are symmetric, we would expect our calculations to result in the same forces in members IJ and JK. We found this to be the case. Also, we should anticipate that given the loading on the truss, members IJ and JK should be in compression. This was also found to be the case.

This solution involved writing six equilibrium equations; we were able to get the solution down to so few equations by combining the method of joints and the method of sections, and by judicious selection of the moment center. Compare this solution with one using the method of joints requiring 12 equations. The conclusion? Plan your strategy before undertaking the calculations.

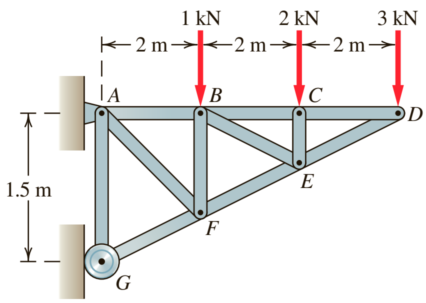
EXERCISES 8.3

8.3.1. [*] Determine the force in member AF of the loaded truss using the method of sections.



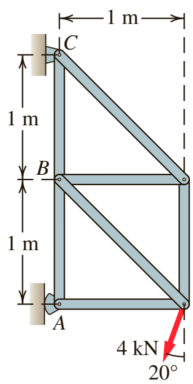
EX 8.3.1

8.3.2. [*] Determine the force in member BF of the loaded truss using the method of sections.



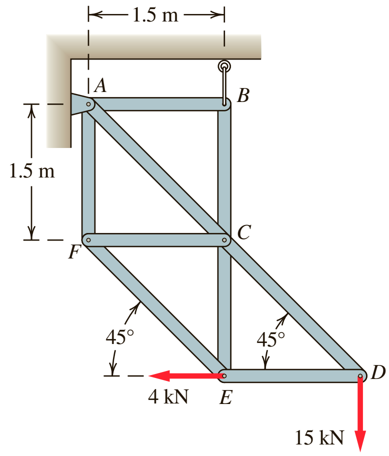
EX 8.3.2

8.3.3. [*] Using the method of sections, determine the force in member BD of the loaded truss.



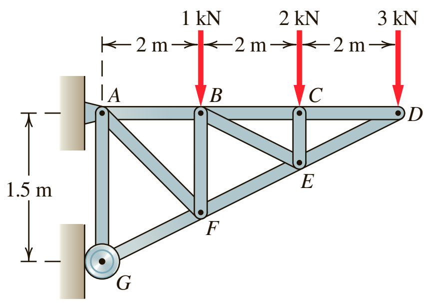
EX 8.3.3

8.3.4. [*] Using the method of sections, determine the force in members AC and AF of the loaded truss.



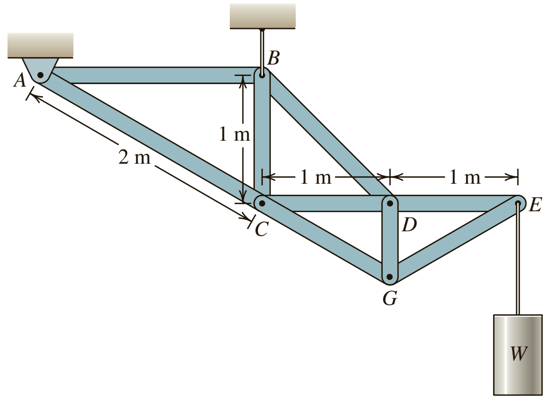
EX 8.3.4

8.3.5. [*] Determine the force in members BC, BE, and EF of the truss using the method of sections. Indicate whether each member is in tension or compression.



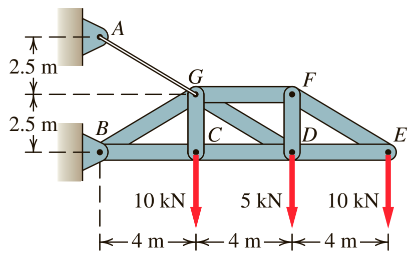
EX 8.3.5

8.3.6. [*] For $W = 6$ kN determine the force in members BC, CD, and CG of the truss using the method of sections. Indicate whether each member is in tension or compression.



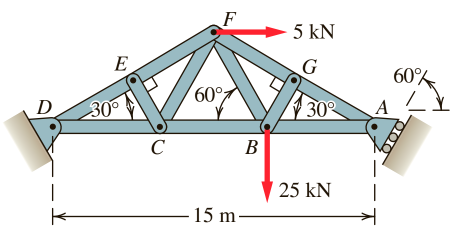
EX 8.3.6

8.3.7. [*] Determine the force in members *CD*, *DG*, and *GF* of the truss using the method of sections. Indicate whether each member is in tension or compression.



EX 8.3.7

8.3.8. [*] Determine the force in members *CB*, *CF*, and *EF* of the truss using the method of sections. Indicate whether each member is in tension or compression.

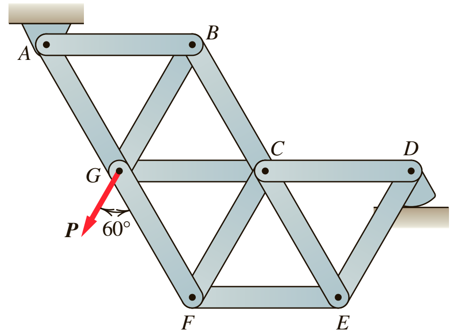


EX 8.3.8

8.3.9. [**] Consider the truss made up exclusively of equilateral triangles.

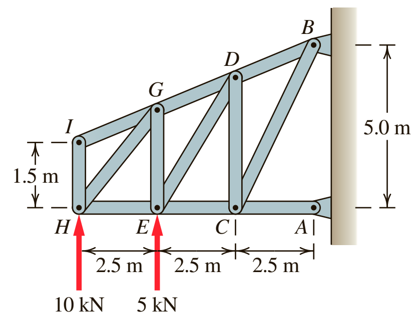
a. Using the method of sections, determine the force in *EF* as a function of ***P***.

b. If the magnitude of the force in member *EF* must be equal to or less than 1 kN, what is the largest allowable value (magnitude) of ***P***?



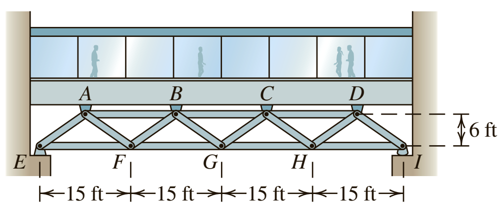
EX 8.3.9

8.3.10. [**] Determine the forces in members *CD*, *DE*, and *DG* of the truss using the method of sections.



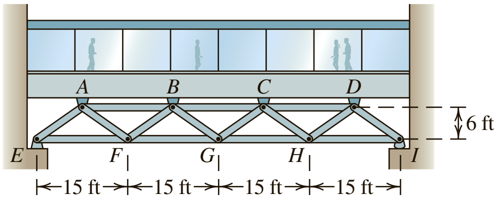
EX 8.3.10

8.3.11. [**] A walkway between two buildings is supported by a Warren truss. The walkway exerts downward vertical 12,000-lb loads at *A*, *B*, *C*, and *D*. Model the support at *E* as a pin connection and the support at *I* as a roller. Determine the forces in members *EF*, *FG*, *GH*, and *HI*, clearly noting which members are in compression and which are in tension.



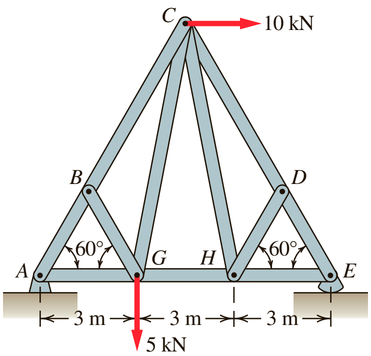
EX 8.3.11

8.3.12. [**] A walkway between two buildings is supported by a Warren truss. The walkway exerts downward vertical 12,000-lb loads at A, B, C, and D. Model the support at E as a pin connection and the support at I as a roller, Determine the forces in members AB, AF, and FG, clearly noting which members are in compression and which are in tension.



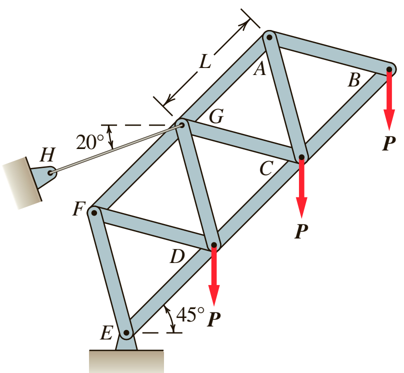
EX 8.3.12

8.3.13. [**] Determine the forces in members CG and BG of the truss.



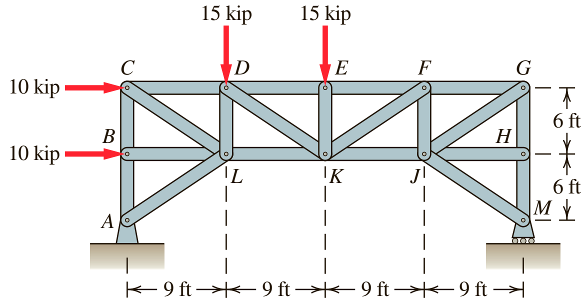
EX 8.3.13

8.3.14. [**] Cable GH holds the truss composed of equilateral triangles in the position shown. Determine the forces in members DF, DG, and FG.



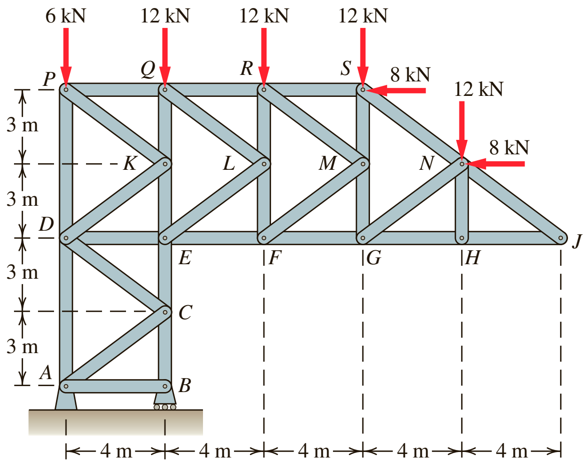
EX 8.3.14

8.3.15. [**] Determine the forces in members DK and EF of the truss shown.



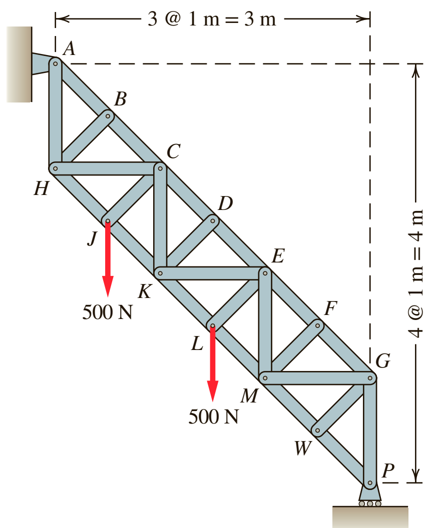
EX 8.3.15

8.3.16. [**] Determine the forces in members EF and QR of the truss shown.



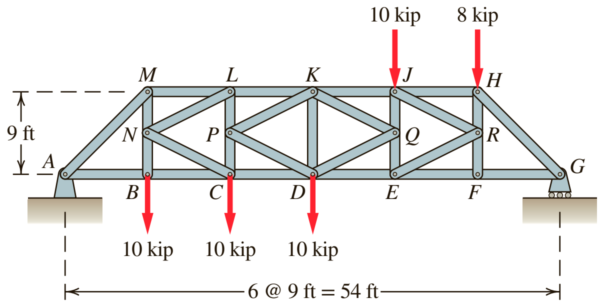
EX 8.3.16

8.3.17. [**] Determine the forces in members EL and LM of the truss shown.



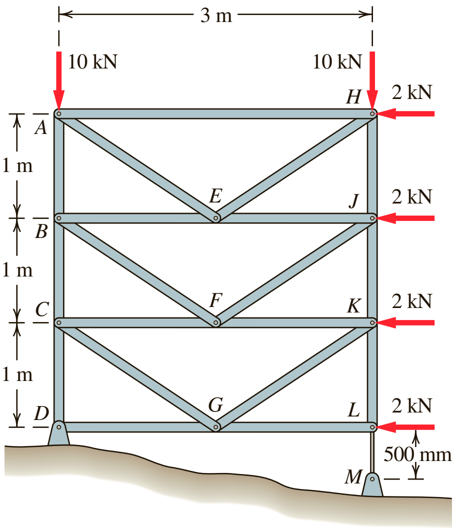
EX 8.3.17

8.3.18. [**] Determine the force in member JK of the truss shown.



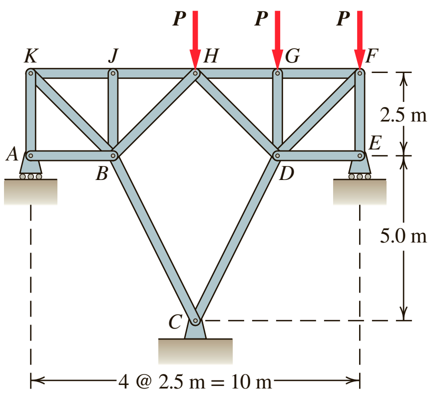
EX 8.3.18

8.3.19. [**] Determine the forces in members BC and JK of the truss shown.



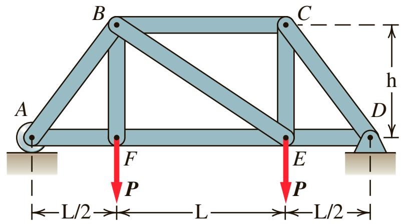
EX 8.3.19

8.3.20. [**] Determine the forces in members DC, DE, and FG of the truss for $P = 4$ kN.



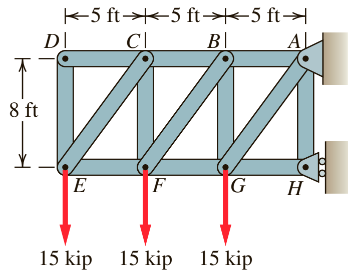
EX 8.3.20

8.3.21. [**] Examine the impact of the height of the truss on its member forces. Vary h from $0.25L$ to $3L$ and plot the non-dimensional forces F_{FE}/P, F_{BE}/P, and F_{BC}/P as a function of h. Assume these members will fail if the tension force is greater than or equal to $5P$ or if the compressive force is greater than or equal to $0.6P$. What is the smallest value of h that will satisfy these conditions? How does the answer change if the forces at F and E are acting upward?



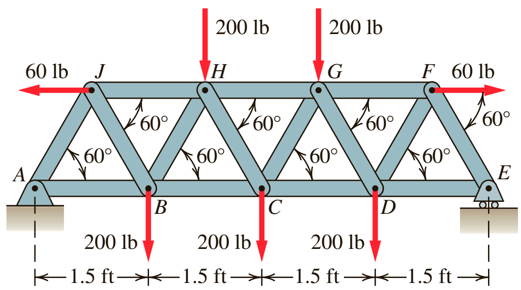
EX 8.3.21

8.3.22. [**] For the truss shown use a combination of the method of sections and the method of joints to determine the forces in BF, CF, EF, and FG.



EX 8.3.22

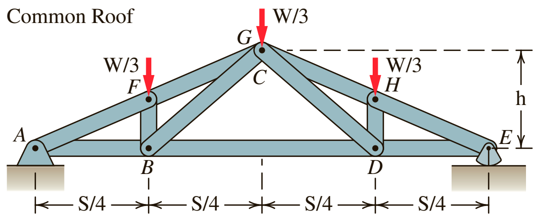
8.3.23. [**] For the truss shown use a combination of the method of sections and the method of joints to determine the forces in BC, CD, CG, CH, and GH.



EX 8.3.23

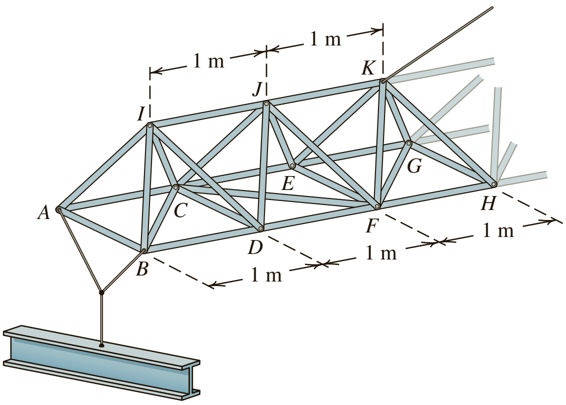
8.3.24. [***] For the model roof truss shown use the method of sections to determine the minimum height h and its associated weight W_{truss} given the following: $W = 66$ N, $S = 24$ cm, the maximum allowable compression in any member is 69 N, the maximum allowable tension in any member is 735 N. (Hint: solve for the members forces

as a function of h, then equate the maximum compression and tension forces to 69 N and 735 N, respectively.)



EX 8.3.24

8.3.25. [***] A space frame can be repeated to form a more complex structure. For example, $ABCDI$ is repeated to form a crane arm. If the mass of the object being lifted by the crane is 3000 kg, what are the forces in members AC, BC, BD, AI, and BI?

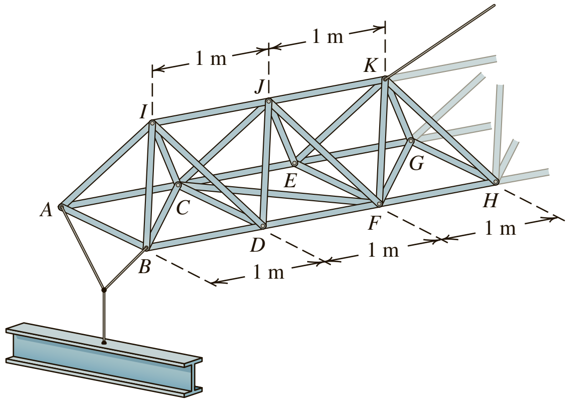


EX 8.3.25

8.3.26. [***] Consider the boom shown.

a. Determine the forces in members BI, DI, DJ, IJ, IK, BD, DF, and FH. Note which members are in tension and which are in compression.

b. Is there a pattern of compression and tension in your answers in **a**? If so, explain why the pattern does or does not make intuitive sense.



EX 8.3.26

8.4 IDENTIFYING ZERO-FORCE MEMBERS

Learning Objective: Define and identify zero-force members in truss-structures.

A truss may be subjected to a combination of **live loads** (occupants and moveable objects in or on the structure), **dead loads** (weight of the building materials and the structure itself), and **environmental loads** (created by snow, wind, earth movement, and earthquake forces).

Under certain loading conditions, a truss member may have no load acting on it. We call members that carry no load **zero-force members**. Two common cases in which zero-force members occur are described here. Identifying zero-force members early in the analysis is important, as it simplifies the analysis.

Case 1 (Two-Member Zero-Force Joint) The two truss members connected to the pin at A in Figure 8.4.1 are zero-force members. The

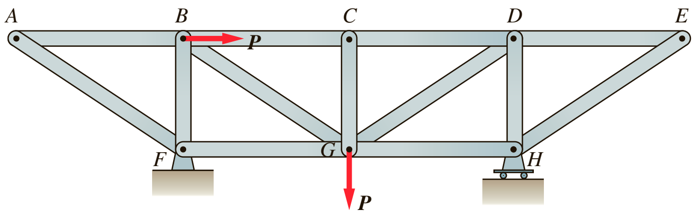


Figure 8.4.1 A truss with zero-force members.

characteristics of joint A are:

1. Only two two-force members are connected at the joint.
2. No external forces are applied to the pin joint.
3. The members connected to the joint do not have the same line of action.

When these conditions exist, then both members are zero-force members.

To prove this we analyze pin joint A from **Figure 8.4.1,** as shown in Figure 8.4.2.

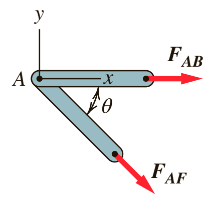


Figure 8.4.2 Members AB and AF are zero-force members.

$$\sum F_x = F_{AB} + F_{AF}\cos\theta = 0 \tag{8.1A}$$

$$\sum F_y = -F_{AF}\sin\theta = 0 \tag{8.1B}$$

Because (8.1B) must be valid for all values of θ, F_{AF} must be zero and therefore AF is a zero-force member. Substituting $F_{AF} = 0$ into (8.1A) gives

$$F_{AB} = 0$$

indicating that AB is also a zero-force member.

A similar analysis carried out at pin E would show that members ED and EH are zero-force members. Alternately, we could note that joint E meets the three conditions we outlined above about zero-force members.

Case 2 (Three-Member Zero-Force Joint) Of the three truss members connected at pin C in **Figure 8.4.1**, member CG is a zero-force member. The characteristics of joint C are:

1. Three two-force members are pinned together at the pin connection.
2. No external loads are applied to the pin connection.
3. Two of the members have the same line of action, and the third member is at an angle to that line of action.

Under these conditions, the member at an angle to the line of action of the other two is a zero-force member, and the magnitude of the forces acting on the other two members are equal.

To prove this we analyze the pin connection C in Figure 8.4.3,

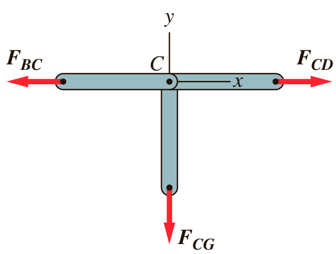


Figure 8.4.3 Member CG is a zero-force member.

$$\sum F_x = F_{CD} - F_{CB} = 0 \tag{8.2A}$$

$$\sum F_y = -F_{CG} = 0 \tag{8.2B}$$

Equation (8.2B) says that the force in member CG is zero (and therefore it is a zero-force member). Equation (8.2A) says that the forces in members CD and CB are equal and opposite.

> **IMPORTANT NOTE!** Keep in mind that zero-force members exist because of two factors: the geometry of the system and the loading. For different loading cases, members AF, AB, ED, EH, or CG in the Figure 8.4.1 truss may not be zero-force members; for this reason, one would be ill-advised to remove these members!

Check out the following example of an application of this material.
- **Example 8.4.1 Identifying Zero-Force Members**

EXAMPLE 8.4.1

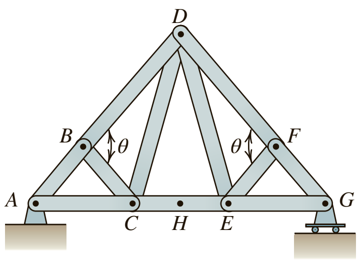


Figure 1 A planar truss structure.

Given the truss shown in **Figure 1**, for the following loading conditions, determine which members act as zero-force members, which members are in compression, and which are in tension.

Complete this exercise for when a downward vertical force ($-P\boldsymbol{j}$) acts at (a) pin D, (b) pin C, or (c) at point H.

To identify the zero-force members, we will apply common cases discussed in Section 8.4.

(a) Vertical force ($-P\boldsymbol{j}$) acts at pin D: Applying Case 2 at joints B and F, we see that BC and EF are zero-force members. Because of these two zero-force members, we can also apply Case 2 at joints C and E to conclude that members CD and DE are also zero-force members (see **Figure 2**). Once we have used Case 2 at joints B, F, C, and E to determine that BC, EF, CD, and DE are zero force members, we also can conclude from Case 2 that $F_{AB} = F_{BD}$, $F_{DF} = F_{FG}$, and $F_{AC} = F_{CE} = F_{EG}$.

Now we consider whether the other members in the truss are in tension or compression. We sketch a free-body diagram of joint G in **Figure 3**. With the reaction force F_{Gy} acting upward, the only way that the joint can be in equilibrium is if F_{GF} is the opposite direction from how it is drawn in the figure. This means that member GF is in compression, and member EG is in tension.

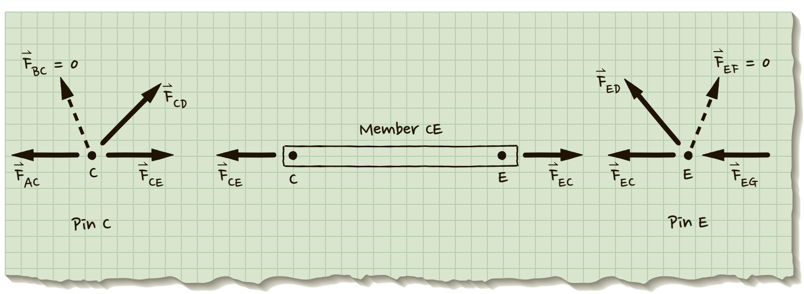


Figure 2 Free-body diagrams of member CE and pins C and E when force is applied at D.

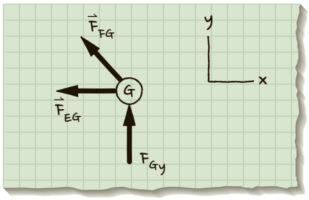


Figure 3 Free-body diagram of joint G.

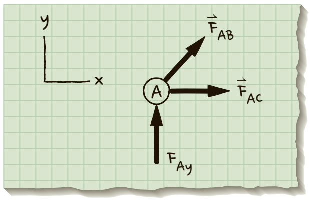


Figure 4 Free-body diagram of joint A.

We now sketch a free-body diagram of joint A in **Figure 4**. The only way that the joint can be in equilibrium is if F_{AB} is the opposite direction from how it is drawn in the figure. This means that member AB is in compression, and AC is in tension.

Since $F_{AC} = F_{CE} = F_{EG}$, member CE is also in tension.

A summary of zero-force, tension, and compression members is given in **Figure 5**.

(b) Vertical force ($-P\boldsymbol{j}$) acts at pin C: As with (a), by Case 2 applied at joints B and F, BC and EF are zero-force members. Because of zero-force member EF, we can also apply Case 2 at joint E to find that DE is a zero-force member.

Using the same reasoning as in (a), we can determine that members AB, BD, DF, and FG are in compression, and that AC and EG are in tension.

We still need to determine whether member CD is in tension or compression. Here a free-body diagram of joint C is useful, as shown in **Figure 6**. We opt to simplify this free-body diagram by omitting member BC because we know it is zero. With the applied force acting at C, the only way for the joint to be in equilibrium is if member CD is in tension.

A summary of zero-force, tension and compression members is given in **Figure 7**.

(c) Vertical force ($-P\boldsymbol{j}$) acts at point H. As with (a), by Case 2 applied at joints B and F, BC and EF are zero-force members. There are no other zero-force members in the truss. You might be tempted to say that CD and DE would also be zero-force members, based on the same reasoning used in (a). But notice that by applying the force $\boldsymbol{P}$ along member CE (and not at a joint connecting two members) member CE acts as a multiforce member. Consequently the loads at pin C from member CE (**Figure 8**) are transferred to other locations on the truss by members AC

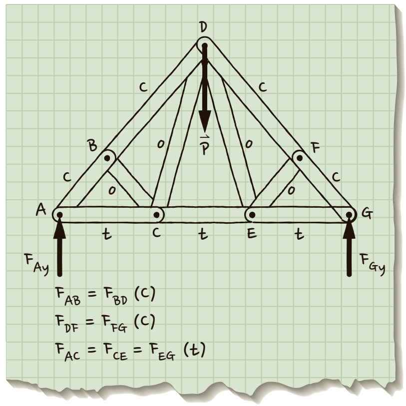


Figure 5 Summary of member loads in (a).

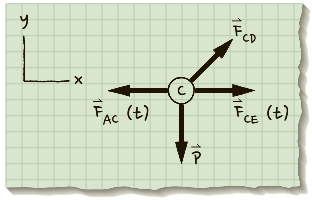


Figure 6 Free-body diagram of joint C.

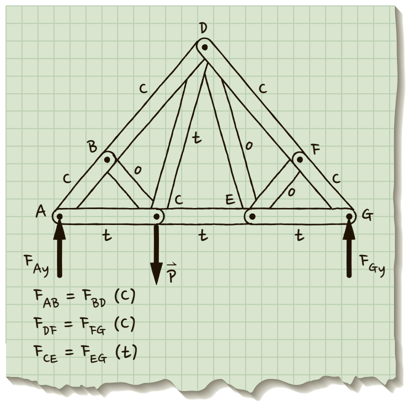


Figure 7 Summary of member loads in (b).

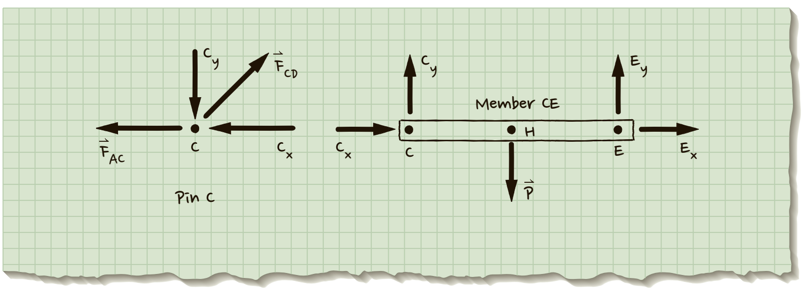


Figure 8 Free-body diagrams of member CE and pin C when force is applied at H.

and CD. Summing forces in the x and y directions at pin C in **Figure 8** shows that Cy is transferred by CD and C_x is transferred by both AC and CD. A similar situation occurs at pin E, indicating that DE and EG are both carrying loads.

A summary of zero-force, tension, and compression members is given in **Figure 9**.

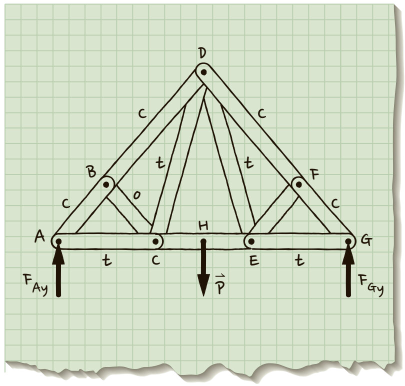
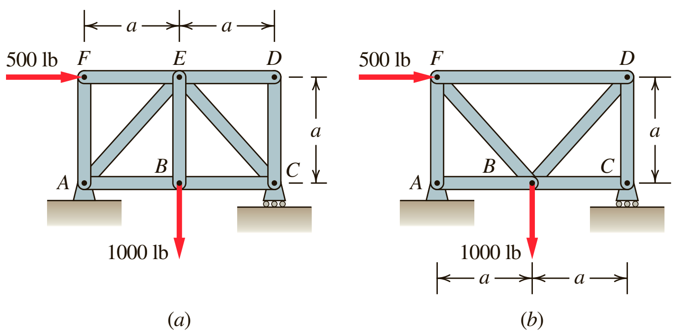


Figure 9 Summary of member loads in (c).

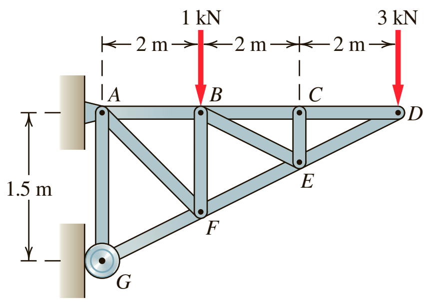
EXERCISES 8.4

8.4.1. [*] For the planar trusses shown, determine which members are in compression, which are in tension, and which are zero-force members.



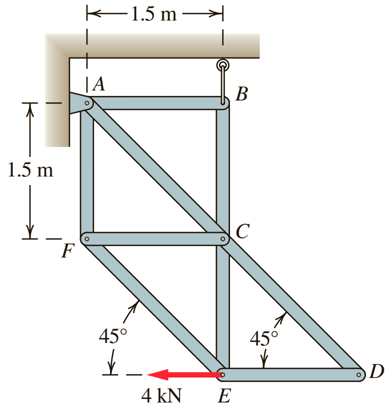
EX 8.4.1

8.4.2. [*] Determine which are zero-force members.



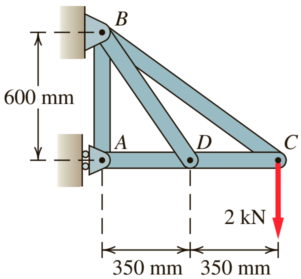
EX 8.4.2

8.4.3. [*] For the truss loaded as shown identify any zero-force members.



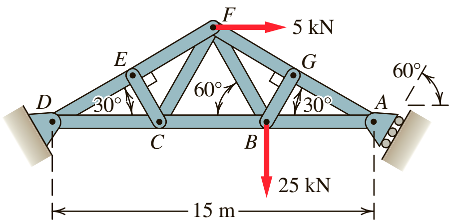
EX 8.4.3

8.4.4. [**] For the truss loaded as shown identify any zero-force members.



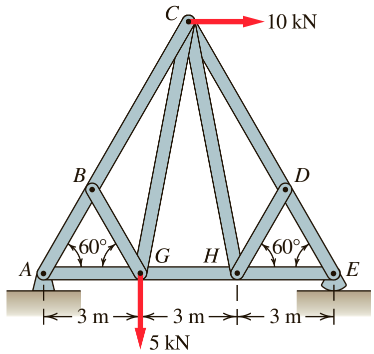
EX 8.4.4

8.4.5. [*] For the truss loaded as shown identify any zero-force members.



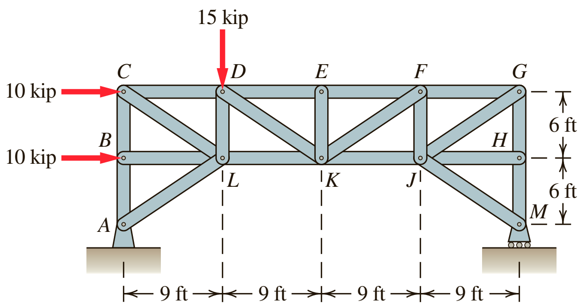
EX 8.4.5

8.4.6. [*] For the truss loaded as shown identify any zero-force members.



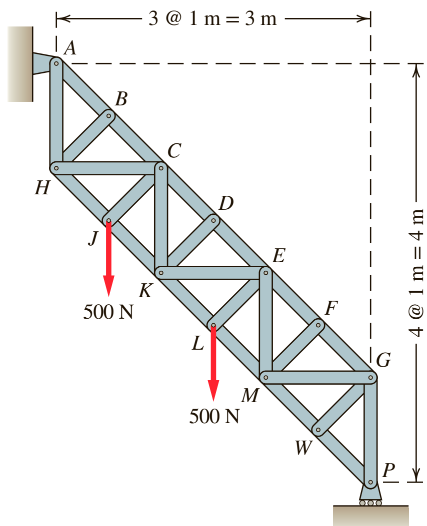
EX 8.4.6

8.4.7. [**] For the truss loaded as shown identify any zero-force members.



EX 8.4.7

8.4.8. [**] For the truss loaded as shown identify any zero-force members.



EX 8.4.8

8.5 DETERMINATE, INDETERMINATE, AND UNSTABLE TRUSSES

Learning Objective: Define and identify statically determinate, statically indeterminate, and unstable trusses.

We first considered statically determinate, statically indeterminate, and underconstrained systems in section 5.7. So far in this chapter we have considered only trusses that are statically determinate—that is, stable, with as many unknown support forces and two-force member forces as there are independent equilibrium equations. **Figure 8.5.1*a*** shows a statically determinate truss.

A truss with more support forces and/or two-force members than can be determined with equilibrium equations is categorized as statically indeterminate (**Figure 8.5.1*b***). We call the excess supports and members redundancies. The redundancies can be internal (excess truss members) or external (excess support loads). To find the forces at the supports and in the members for such a truss, we must combine mechanical equilibrium analysis with concepts from mechanics of materials.

A truss is categorized as underconstrained if there are too few supports to be stable and internally unstable if there are too few members to be stable. Notice that the truss in **Figure 8.5.1*c*** is unstable, and will in fact move because it is a mechanism. Finding the forces at the supports and in the members in any particular configuration requires concepts from dynamics and kinematics.

By comparing the number of equilibrium equations, number of two-force members, and number of support loads for a particular truss structure, we can evaluate whether the truss is statically determinate, indeterminate, or unstable. **Tables 8.1** (planar trusses) and **8.2** (nonplanar trusses) summarize details of this evaluation. Determining that a truss is statically determinate is a good idea before diving into the application of methods of joints or sections—those methods will only be sufficient for finding the support and two-force member loads if the truss is statically determinate. If the truss is statically indeterminate, concepts from both static equilibrium *and* mechanics of materials will need to be applied. If the truss is unstable, static equilibrium conditions will not be sufficient to complete the analysis.

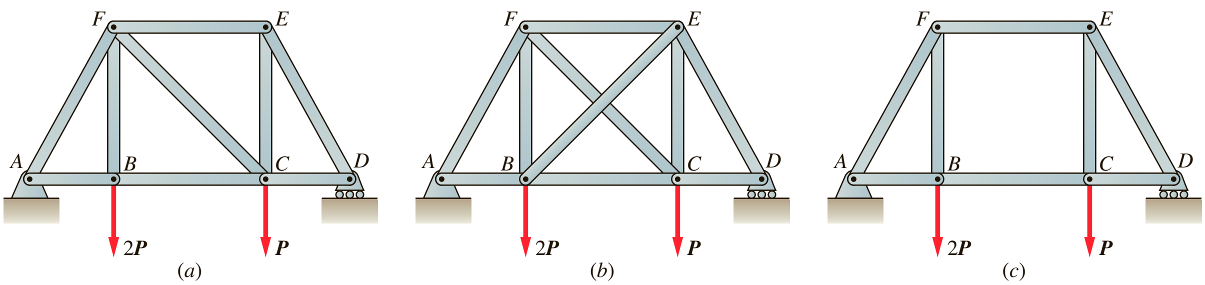


Figure 8.5.1 (*a*) Internally stable truss without redundancy that is also statically determinate; (*b*) internally stable truss with redundancy that is also statically determinate; (*c*) internally unstable truss.

Table 8.1 Check for Static Determinacy and Internal Stability in Planar Trusses

Count the number of two-force members (m), the number of joints (j), and the number of support forces (r). The quantity $2j$ represents the number of linearly independent equations and $m + r$ represents the total number of unknowns. Check to see which inequality or equality listed below holds.

$2j = m + r$	(8.3A)	The truss is **statically determinate** if all members contribute to configuration stability (**Figure 8.5.1*a***) and the equilibrium equations are sufficient to solve for member and support loads. All planar trusses that are statically determinate obey the equality $2j = m + r$; however, not all trusses that obey this equality are statically determinate. As such, the equality is a *necessary but not sufficient* condition.
$2j < m + r$	(8.3B)	The truss is **statically indeterminate**. There is an excess of members and/or support loads, and you cannot solve for unknown loads with equilibrium conditions alone (**Figure 8.5.1*b***).
$2j > m + r$	(8.3C)	Either the planar truss has too few members, causing it to be **internally unstable** or too few supports and thus is **underconstrained**. The truss may be part of an engineering device called a mechanism, and concepts from dynamics and kinematics are required to solve for the unknown loads (**Figure 8.5.1*c***).

Table 8.2 Check for Static Determinacy and Internal Stability in Space Trusses

Count the number of two-force members (m), the number of joints (j), and the number of support forces (r). The quantity $3j$ represents the number of linearly independent equations and $m + r$ represents the total number of unknowns. Check to see which inequality or equality listed below holds.

$3j = m + r$	(8.4A)	The truss is **statically determinate** if all members contribute to configuration stability and the equilibrium equations are sufficient to solve for member and support loads. All space trusses that are determinate obey the equality $3j = m + r$; however, not all trusses that obey this equality are statically determinate. As such, the equality is a *necessary but not sufficient* condition.
$3j < m + r$	(8.4B)	The truss is **statically indeterminate**. There is an excess of members and/or supports, and you cannot solve for unknown loads with equilibrium conditions alone.
$3j > m + r$	(8.4C)	Either the space truss has too few members, causing it to be **internally unstable** or too few supports and thus is **underconstrained**. The truss may be part of an engineering device called a mechanism, and concepts from dynamics and kinematics are required to solve for the unknown loads.

Check out the following examples of applications of this material.

- **Example 8.5.1 Checking the Status of Planar Trusses**
- **Example 8.5.2 Checking the Status of Space Trusses**

EXAMPLE 8.5.1

By using the process for checking internal stability in a planar truss and counting the number of joints (pins), members, and equations, determine whether each system in **Figure 1** is internally stable without redundancy, internally stable with redundancy, or internally unstable. Also, describe why the truss is externally statically determinate, statically indeterminate, or underconstrained.

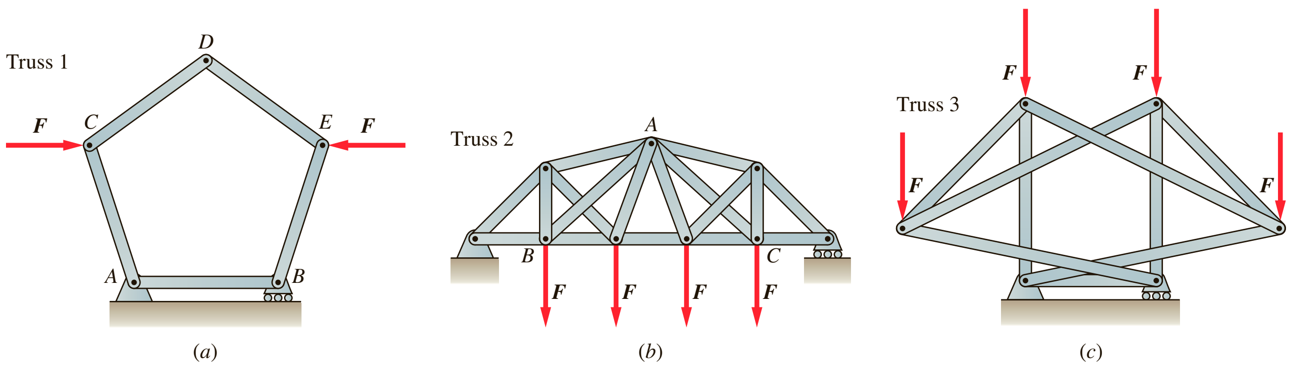


Figure 1 Example Trusses.

Truss 1 (Figure 1*a*) The pin and roller at A and B produce three unknown support forces ($r = 3$) and because this is a planar truss three equilibrium equations can be used to check the equilibrium of the truss as a whole; therefore the truss would be statically determinate *if* it were also internally stable. Using the rules defined in the process for checking internal stability of a planar truss:

Number of support forces $= r = 3$
Number of joints $= j = 5$
Number of two-force members $= m = 5$
Number of equations $= 2j = 10$

The number of unknown forces (eight) is less than the number of equations (ten), therefore the truss is *underconstrained*. The three support forces can be determined by enforcing equilibrium on the truss as a whole, making this *externally statically determinate*. However, the truss is *internally unstable* because there are too few members to prevent the truss from deforming excessively. Question: What additional support conditions and/or two-force members might be added to make this

system statically determinate? This question has multiple answers. One choice would be to add members BC and CE.

Truss 2 (Figure 1*b*)

Number of support forces = $r = 3$
Number of joints = $j = 9$
Number of two-force members = $m = 17$
Number of equations = $2j = 18$

Overall the truss is statically indeterminate with two redundancies. The truss is *internally stable with redundancy and externally statically determinate* because there are two more unknown member forces than equilibrium equations, the truss is constructed of rigid triangles, and we can determine the three support forces directly with three equilibrium equations. Question: Which members are redundant? One possible answer to this question is to remove members AB and AC.

Truss 3 (Figure 1*c*)

Number of support forces = $r = 3$
Number of joints = $j = 6$
Number of two-force members = $m = 9$
Number of equations = $2j = 12$

The truss is *internally stable without redundancy and externally statically determinate* because the number of unknowns equals the number of equations. Imagine the challenge of constructing this truss with its many overlapping members.

EXAMPLE 8.5.2

Use the process for checking internal stability in a space truss to identify each space truss as being internally stable without redundancy, internally stable with redundancy, or internally unstable. Also, describe why the truss is externally statically determinate, statically indeterminate, or underconstrained.

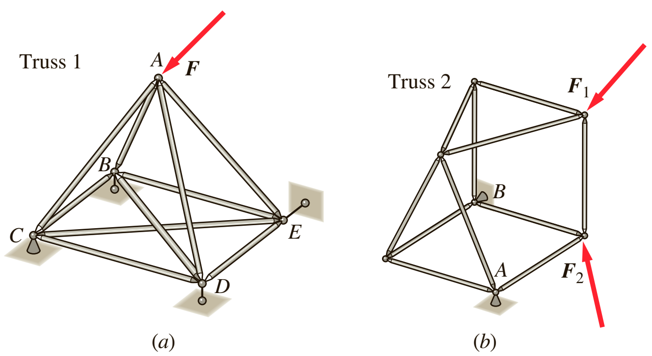


Figure 1 Example space trusses.

Truss 1 (Figure 1*a*) Using the rules defined in the process for checking internal stability of a space truss:

Number of support forces = $r = 6$
Number of joints = $j = 5$
Number of two-force members = $m = 10$
Number of equations = $3j = 15$

Overall the truss is statically indeterminate with one redundancy. The truss is *internally stable with redundancy* because we have 10 members and only nine remaining equations after we have solved for the support forces. Using the six equations of equilibrium, we can, however, solve for six loads at the supports; therefore this truss is *externally statically determinate*. Which member could be removed so that the truss would be internally stable without redundancy? A good choice would be member BD or CE, but not both.

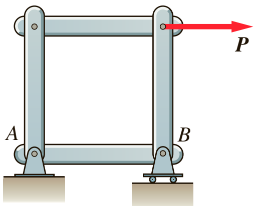
Truss 2 (Figure 1*b*)

Number of support forces = $r = 6$
Number of joints = $j = 7$
Number of two-force members = $m = 11$
Number of equations = $3j = 21$

The truss is *internally unstable* because the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ would cause the truss to deform excessively; we see this in that the number of members (eleven) is less than the number of remaining equations (fifteen) after solving for the support forces. The addition of four appropriately placed members would make the truss internally stable. *If* the truss were internally stable (which it is not), the six boundary supports would result in a structure that is externally statically determinate. But because it is internally unstable, we also conclude that it is underconstrained; the six support forces at A and B are insufficient to prevent the truss from moving.

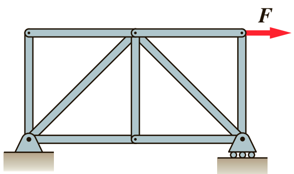
EXERCISES 8.5

8.5.1. [*] Identify each structure as statically determinate, statically indeterminate, underconstrained, internally stable without redundancy, internally stable with redundancy, or internally unstable.



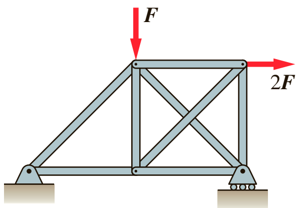
(*a*)

EX 8.5.1*a*



(*b*)

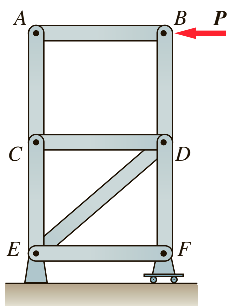
EX 8.5.1*b*



(*c*)

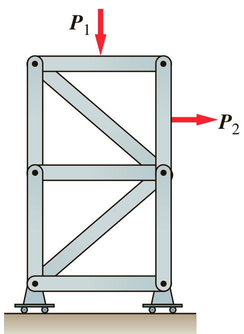
EX 8.5.1*c*

8.5.2. [*] Explain why the truss shown is unstable. Suggest two different design changes that would make the truss stable.



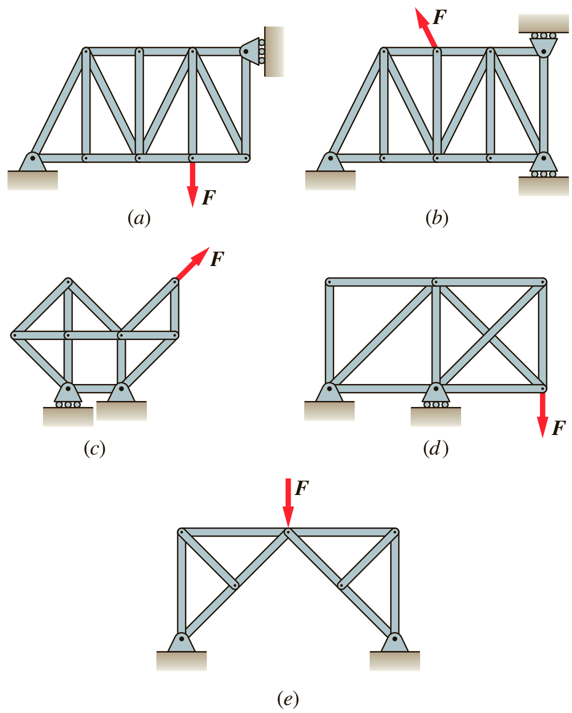
EX 8.5.2

8.5.3. [*] What happens to the structure shown when only load $\boldsymbol{P}_1$ is applied; when only $\boldsymbol{P}_2$ is applied?



EX 8.5.3

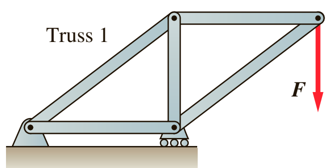
8.5.4. [*] Select the terms that describe each of the trusses: statically determinate, statically indeterminate, underconstrained, internally stable without redundancy, internally stable with redundancy, internally unstable. Also note any zero-force members.



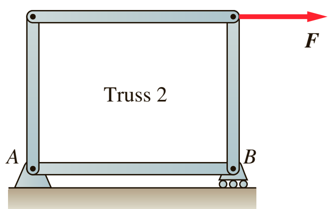
EX 8.5.4

8.5.5. [*] By using **Table 8.1** and counting the number of joints (pins), members, and equations, determine whether each system is internally stable without redundancy, internally stable with redundancy, or internally unstable. Also, describe why the truss is externally statically determinate, statically indeterminate, or underconstrained.

(*a*)

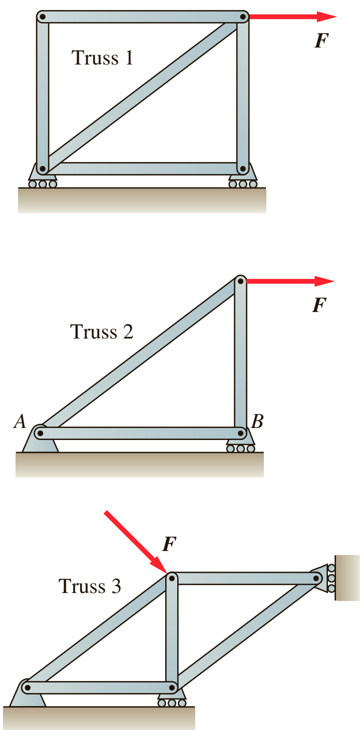


(*b*)



EX 8.5.5

8.5.6. [*] By using **Table 8.1** and counting the number of joints (pins), members, and equations, determine whether each truss is internally stable without redundancy, internally stable with redundancy, or internally unstable. Also, describe why the truss is externally statically determinate, statically indeterminate, or underconstrained.

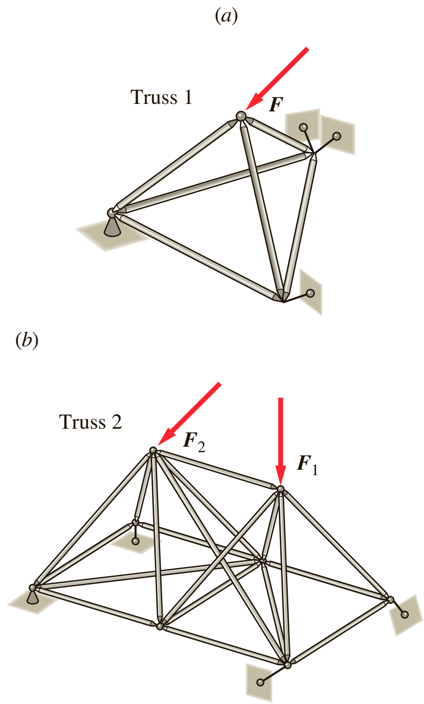


EX 8.5.6

8.5.7. [*] By using **Table 8.2** and counting the number of joints (pins), members, and equations, determine whether each system is internally stable without redundancy, internally stable with redundancy, or internally unstable. Also, describe why the truss is externally statically determinate, statically indeterminate, or underconstrained.

a. Space truss 1

b. Space truss 2



EX 8.5.7

8.6 JUST THE FACTS

Defining a Truss

A **truss** is a structural system, made up exclusively of two-force members, that is generally lightweight compared to the loads it can support. Trusses

1. consist exclusively of straight members,
2. have joints that can be represented as pin connections, and
3. carry external forces exclusively at joints.

A **planar truss** is one in which all forces and members lie in a single plane (or it is reasonable to assume that they all lie in a single plane). The basic building block of a planar truss is a triangle, composed of three two-force members connected by pins. The triangle is the simplest **rigid structure** that can be created with two-force members; by rigid we mean that the structure is internally stable. If a truss is not planar, it is a **space truss**. The basic building block of a space truss is a tetrahedron composed of two-force members.

Analyzing Trusses by the Method of Joints

Analyzing Trusses by the Method of Sections

The two procedures for finding the forces carried by two-force members in a truss are the **method of joints** and the **method of sections**, or some combination of the two procedures. Both methods assume that the reason for analyzing the system has been defined and that information about the problem has been documented. The method of joints consists of considering equilibrium of the truss as a whole (in order to find the support forces acting on the truss), and of each of the pin connections that make up the truss. The method of sections consists of considering equilibrium of the truss as a whole, and of portions of the truss with a boundary cut through the two-force members themselves.

Identifying Zero-Force Members

In undertaking either the method of joints or the method sections, it is useful to identify **zero-force members** as part of the analysis. These are two-force members that for a particular loading of the truss carry zero force. Two common cases in which zero-force members occur are

Case 1 (Two-Member Zero-Force Joint)

1. Only two two-force members are connected at the joint.
2. No external forces are applied to the pin joint.
3. The members connected to the joint do not have the same line of action.

Case 2 (Three-Member Zero-Force Joint)

1. Three two-force members are pinned together at the pin connection.
2. No external loads are applied to the pin connection.
3. Two of the members have the same line of action, and the third member is at an angle to that line of action.

Determinate, Indeterminate, and Unstable Trusses

The method of joints and the method of sections work only for trusses that are both statically determinate and internally stable without redundancy. A truss with more members and/or supports than can be determined from equilibrium equations is labeled statically indeterminate—concepts from mechanics of materials in addition to mechanical equilibrium conditions are required to determine the member loads for the system. With any fewer members or supports, the truss would deform excessively and is labeled undercontrained or internally unstable—concepts from dynamics are required to describe the member loads and motion of the system. Checks to determine whether a truss is statically determinate, statically indeterminate, or underconstrained are outlined in **Table 8.1** for planar trusses and in **Table 8.2** for space trusses.

SYSTEM ANALYSIS (SA) EXERCISES

SA8.1 The Marvelous Truss

During construction, the large field-spanning girders for the Reynolds Coliseum had to be kept vertical before the roof could be put on. This was mainly accomplished with the help of trusses, beams, and horizontal cross bracings connecting the girders. Figure SA8.1.1 presents a picture taken after the roof was put on. The construction started in 1942 but was interrupted in 1943 due to a lack of skilled workers because of World War II. When completed, at 48 m × 98 m, it represented the largest such building in the southeast. On December 2, 1949, the opening game in the new William Neal Reynolds Coliseum (named after William Neal Reynolds of Winston-Salem), NC State's Wolfpack basketball team won 67-47 over Washington.

At the same time, the trusses fulfilled other functions such as supporting electrical conduits and light fixtures. It was common at that time to create the connections between truss and beam with rivets and steel plates, as shown in Figure SA8.1.2. Today, the same connections would be welded or bolted. Figure SA8.1.3 shows a model that represents a truss and its loading conditions.

(a) Develop a free-body diagram of the truss for the case where the side load (SL) is 0 kN. Start by replacing the distributed loads on the top truss members into point loads (in the joints) and add the light fixtures. Assume that the weight of the truss elements is negligible.

State Archives of North Carolina Raleigh

Figure SA8.1.1 Trusses keep girders in place during the construction of Reynolds Coliseum during WWII.

Leonhard Bernold

Figure SA8.1.2 View of truss-beam connections.

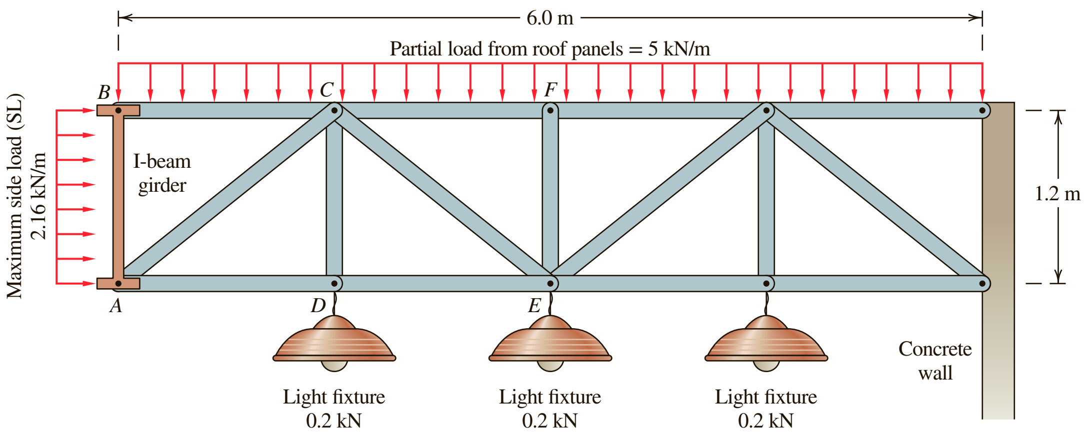


Figure SA8.1.3 Model of one truss supporting I-beam steel girder.

(b) Use hand calculations or develop a quick spreadsheet to compute the forces in members *BC*, *AC*, *CD*, *CE*, *AD*, *DE*, *CF*, and *EF*.

(c) Consider that during construction (see **Figure SA8.1.1**) a strong wind is blowing, resulting in a side load (SL = 2.16 kN/m, which represents a wind pressure of 0.24 kN/m^2 resulting from a wind of approximately 90 mph). Speculate which truss members will be impacted. Verify your speculations by revising your calculation from (b).

(d) Assume that the wind suddenly changes its direction during construction. Use the method of sections to assess the force in member *CE* if SL changes to –2.16 kN/m.

SA8.2 Designing a Bridge[5]

Scenario: Your design company has been solicited to submit a bridge design for the rural community of Hector, Arkansas. Hector is a small Ozark town in northwest Arkansas in the Arkansas River Valley. Located near Hector is the Illinois Bayou (**Figure SA8.2.1**), one of the few bayous in the world with Class II/III rapids. Prone to springtime flooding, the Bayou recently took out the bridge near Scottsville, five miles from Hector. The bridge is critical in serving the Hector area's principal agricultural interest, chicken farming. Without the bridge, the current chicken truck traffic has slowed to a standstill, leaving 500,000 chickens waiting on the other side of the bridge. The noise of clucking, coupled with the smell, is causing the citizens to demand a quick solution.

Hector has limited means to support the construction of this new bridge. Their budget is limited to $20,000, including the cost of labor. A team that delivers a bridge under cost may be awarded a bonus by the town of Hector. The bridge should be a through/overhead truss. This means that the roadbed will be laid through the truss as shown in **Figure SA8.2.2**.

Your design company submits a proposal containing initial site analysis and bridge design for consideration by the town of Hector. To assist you in the development of your proposal to Hector, a state-of-the-art bridge design simulation software is being made available to you free of charge. *Bridge Designer* is a realistic tool that allows users to rapidly design and test a variety of bridge structures. It can be downloaded at https://bridgecontest.org/resources/download/. When you use *Bridge Designer* please limit your materials to carbon steel and ensure that every member measures 140 mm × 140 mm. Additional criteria that your design must meet are listed in **Table SA8.2.1**.

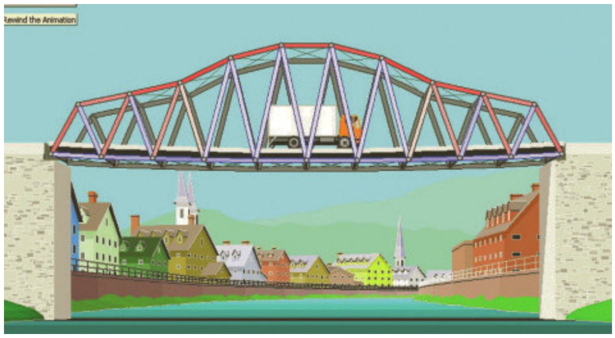


Figure SA8.2.2

John Feland

Figure SA8.2.1

Table SA8.2.1 Bridge Design Criteria

Design Criteria	Real Bridge
Span	40 meters
Width	5 meters
Maximum height	14.5 meters
Maximum load	~1.5 MN
Minimum factor of safety	1.75
Maximum cost	$20,000
Material	Carbon steel
Dimensions of cross section	140 mm × 140 mm
Max internal compressive load	Varies with length (buckling)
Max internal tensile load	4655 kN

[5] This design exercise was originally proposed by John Feland, at the time a Ph.D. candidate in Mechanical Engineering at Stanford University. John is from Hector, Arkansas, and grew up on a chicken farm.

Cost Analysis Guidance: Teams should utilize the cost analysis functions in *Bridge Designer*. The software automatically calculates the cost of your bridge. All you need to do is include the cost report from *Bridge Designer* in your final report.

Deliverable: The cover sheet for your proposal is shown in Figure SA8.2.3. Your proposal should contain an executive summary and bridge design printouts. The executive summary should be about a page in length and should briefly discuss the requirements of the bridge and how your particular design met those requirements. The bridge design printouts come from *Bridge Designer*. *Bridge Designer* allows you to print out labeled versions of your truss. In addition, you can obtain a listing of the materials used for each member and the maximum loads in each member during the load test. You should also print out a screen shot of the bridge under no truck loading and under truck loading. The printouts should consist of about 4 pages.

Presented to the citizens of Hector by:

Team: ______________________

Member 1: ______________________

Member 2: ______________________

Final estimated cost: ______________________

Number of joints: ______________________

Number of members: ______________________

Calculation to determine whether the final design is internally stable without or with redundancy (calculations go here):

Contents	**Page Number**
Summary of results (1 page maximum executive summary)	___
Bridge design printouts	___
Bridge schematic (labeled max tension and compression members)	___
Load report (with highlighted minimum factor of safety)	___
Screenshot of unloaded bridge	___
Screenshot of loaded bridge	___
Cost analysis (printed from *Bridge Designer*)	___

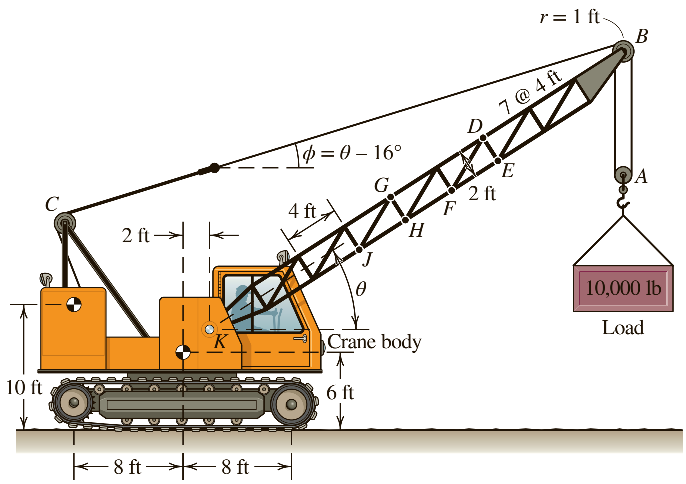
Figure SA8.2.3 Illinois Bayou Bridge proposal cover sheet.

SA8.3 Lattice Boom Crawler Crane

Link-Belt is a company that makes lattice boom crawler cranes such as the 50-ton, LS-108H model (SA8.3.1(a)). The analysis and design of these mechanical systems require knowledge of all areas of engineering mechanics.

Link Belt Construction Equipment

(a) A sample Link-Belt 50-ton crane.



(b) Simplified lattice boom crawler crane.

SA8.3.1

A simplified lattice boom crawler crane is shown in **SA8.3.1 (b)**. The analysis outlined below looks at various configurations of the crane and considers which configurations result in the most severe loadings of members of the crane.

(a) Assume that the crane is holding a 10,000-lb load in static equilibrium, as shown in **SA8.3.1(b)**. Write an expression as a function of θ for the tension in the cable that runs from the center of pulley B through pulley A, and back around pulley B to the winch at C. Assume that all pulleys are frictionless and weightless. For a reasonable range of the angle θ, at what angle is the tension maximum? What is the maximum tension?

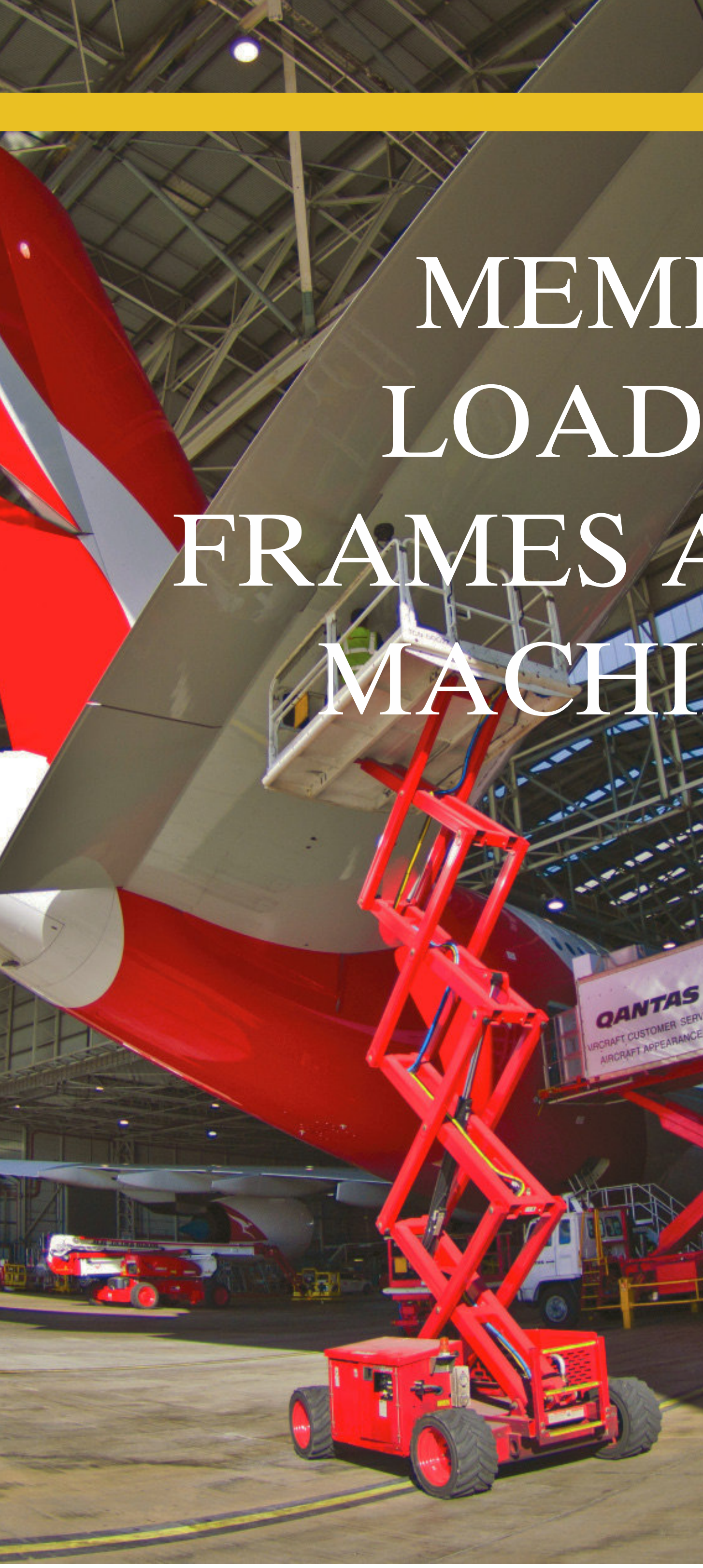
(b) Write expressions for the forces in members EF and HJ as functions of the angle θ. For a reasonable range of the angle θ, at what angle is the tension or compression in member EF largest? For a reasonable range of the angle θ, at what angle is the tension or compression in member HJ largest?

(c) Organize your findings in (a) and (b) in a table that would allow a fellow engineer to readily understand the maximum loads in the cable, and in members EF and HJ.

CHAPTER 9

MEMBER LOADS IN FRAMES AND MACHINES

In this chapter we continue our analysis of loads within systems, focusing on frames and machines. We define the difference between a truss, a frame, and a machine and show how this distinction affects the analysis. By the end of this chapter, you will be able to systematically find the loads acting on members within a frame or machine system.



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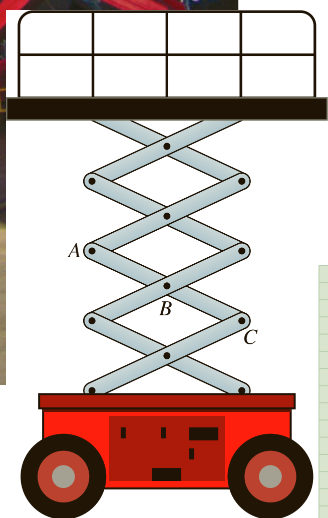
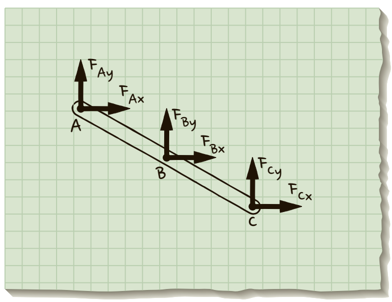




LEARNING OBJECTIVES

On completion of this chapter, you will be able to:

- **Carry out equilibrium analysis of a frame and its members.** (9.1)
- **Carry out equilibrium analysis of a machine and its members, including the effects of bearing friction and rolling resistance.** (9.2)
- **Determine whether a frame is statically determinate or indeterminate, or unstable.** (9.3)

9.1 DEFINING AND ANALYZING FRAMES

Learning Objective: Carry out equilibrium analysis of a frame and its members

A **frame** is a system designed to support loads, both forces and moments. It consists of members that are connected together by a variety of methods: welds, bolts, pins, rivets, nails, wedging, and/or glue. Frames may be made up of one member or many, but at least one member must be a **multiforce member**, which is any member that is not a two-force member. **Figure 9.1.1** shows examples of frames you may have passed on your way to class, in a workshop, at an airport, or at a construction site.

Frames can be classified as **planar** or **nonplanar**. Planar frames are ones in which all forces and members lie in a single plane (or it is reasonable to treat them as such) and all moments act about an axis perpendicular to this plane. A frame that is not planar is nonplanar. All of the frames in **Figure 9.1.1** are planar, whereas those shown in **Figure 9.1.2** are nonplanar.

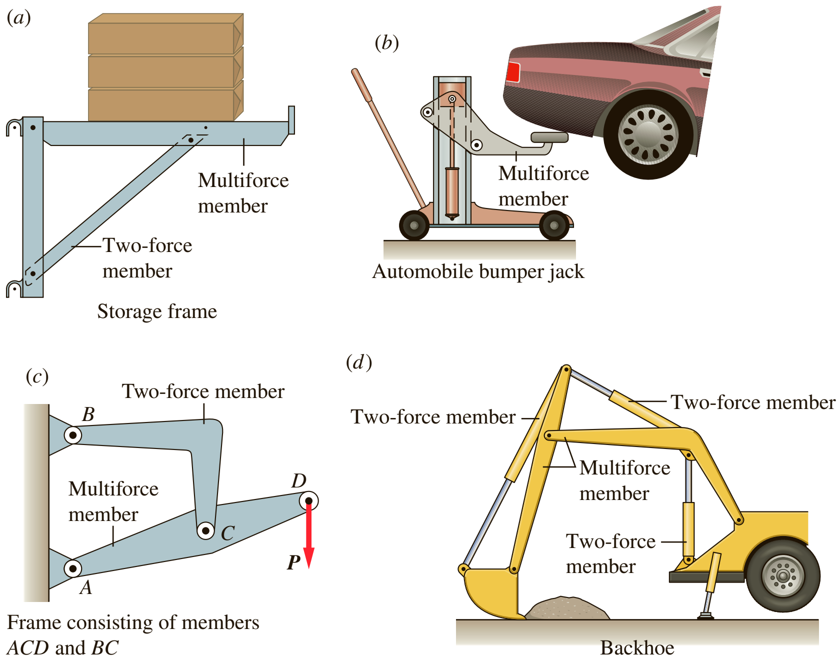


Figure 9.1.1 Examples of planar frames: (*a*) Storage frame. (*b*) Automobile jack. (*c*) Frame consisting of members *ACD* and *BC*. (*d*) Back hoe.

Classifying Structures as Frames or Trusses

A structure may change from being classified as a truss to being classified as a frame, simply by virtue of where external forces act on it. For example, in **Figure 8.1.5(*a*)**, because the external force ***F*** acts at a pin, the three members that make up the structure act as two-force members; and the structure is classified as a truss. Contrast that to the same structure in **Figure 8.1.5(*b*)** where the external force ***F*** is applied at a point along the length of one of the members–now this member acts as a multiforce member and the structure is classified as a frame.

For equilibrium analysis of both trusses and frames we will undertake the same general steps of making assumptions, defining the system, drawing free-body diagrams, writing and solving equilibrium equations, and interpreting and checking answers. But the particulars are different. If a structure can be classified as a truss, the method of joints and/or sections can be followed, as developed in Sections 8.2 and 8.3. If it is a frame, a more general approach for carrying out equilibrium analysis is needed (as described below).

Classifying structures as trusses or frames is very useful for engineers communicating with one another. If you describe a structure as a "truss" to a fellow engineer, she will picture something built up of simple elements configured as triangles. In contrast, if you say "frame," she is likely to picture members connected together such that they not only experience simple tension or compression, but also bending.

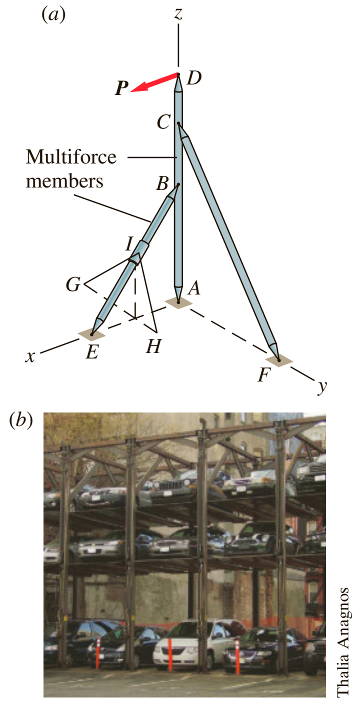


Figure 9.1.2 Examples of nonplanar frames (*a*) Tower held by members *CF* and *BE*, and guy wires *GI* and *HI*. (*b*) A frame used to park cars.

Check out the following example of an application of this material.

- **Example 9.1.1 Identify Systems as Trusses or Frames**

EXAMPLE 9.1.1

Identify each system in **Figure 1** as either a truss or frame. Justify your answer.

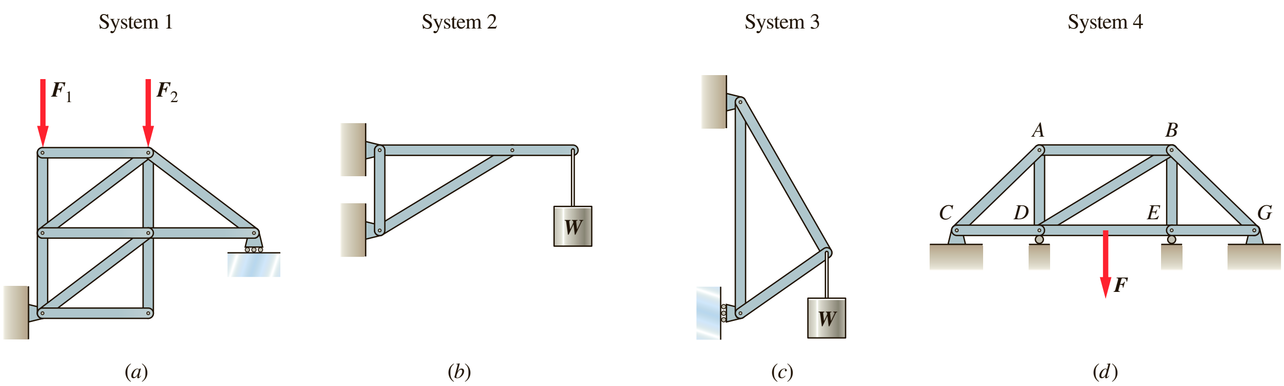


Figure 1 Four systems to be identified as trusses or frames.

System 1 (Figure 1*a*) The system is a *truss*. Assuming we ignore their weights, all of the members are two-force members because they are all connected by pins and the forces are applied at the pins. (*Question:* How could we include member weights and still do a truss analysis?)

System 2 (Figure 1*b*) The system is a *frame* because it has at least one multiforce member (the member from which the weight is hanging). (*Question:* Based on inspection, will the vertical member carry any load?)

System 3 (Figure 1*c*) The system is a *truss*. If we ignore the weights of the members, they are all two-force members.

System 4 (Figure 1*d*) The system is a *frame* because *DE* is a multiforce member. All other members are two-force members if their weights are negligible. Notice that portions of the system are trusses (*ACD* and *BEG* form trusses). (*Question:* Is there anything "odd" about the number of boundary conditions associated with this frame? We will have a lot more to say about this in the final section of this chapter.)

Determining Member Loads in Frames

Often in the analysis of frames we are concerned with finding the loads acting on the boundary of the frame along with the loads internal to the frame (acting on individual members). Suppose, for example, that the frame in **Figure 9.1.3*a*** is a design idea for a set of lifting tongs. As part of evaluating the design idea we might be interested in finding the shear force acting on the pins at *B* and *C* in **Figure 9.1.3*a*** when the frame is lifting an 800-lb crate. The calculated shear forces acting on the pins would then be compared with their shear force capacity (i.e., the force level that causes the pin to fail) to see whether the pins are sufficient for their proposed task. Or suppose that **Figure 9.1.3*b*** shows the design idea for a ventilation door. To specify the material and cross-section of two-force member *BD* so that it is large enough to not buckle, we need to find the force acting on member *BD*.

As part of carrying out an analysis of either the lifting tongs or powered door in **Figure 9.1.3** we will draw free-body diagrams of the system and parts of the system, and write and solve multiple sets of linearly independent equilibrium equations.

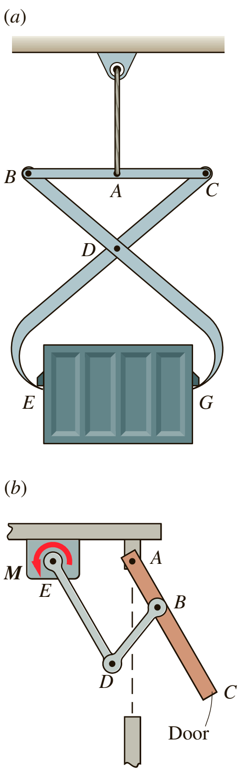


Figure 9.1.3 Examples of two frames: (*a*) lifting tongs, and (*b*) powered door.

Check out the following examples of applications of this material.

- **Example 9.1.2 Planar Frame Analysis**
- **Example 9.1.3 Finding Loads at Frame Supports**
- **Example 9.1.4 Analysis of Frame with Friction**
- **Example 9.1.5 Nonplanar Frame Analysis**

EXAMPLE 9.1.2

Three hydraulic cylinders control the motion of the backhoe arm and bucket in Figure 1. In the position shown, a 14.0-kN horizontal force is applied to the bucket at point L. Assuming the weight of the backhoe arm is negligible compared to forces acting on it, find the forces acting on pins A and G.

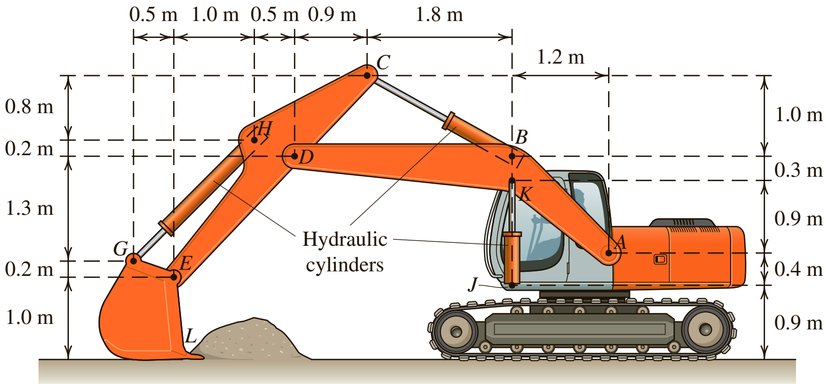


Figure 1 Backhoe scooping a load of soil.

Goal We are to find the forces acting on the pins at A and G.

Given We are given information about the geometry of the backhoe, and the load at L.

Assume Treat the system as planar because all the forces are acting in a single plane. We also assume that all of the hydraulic cylinders act as two-force members because they are pinned at both ends and have no other external forces acting on them.

Draw, Formulate Equations and Solve We draw an imaginary boundary around the whole backhoe arm, which isolates it from the cab at A and J. We draw a free-body diagram of the arm (Figure 2) realizing that since hydraulic cylinder JK is a two-force member, the

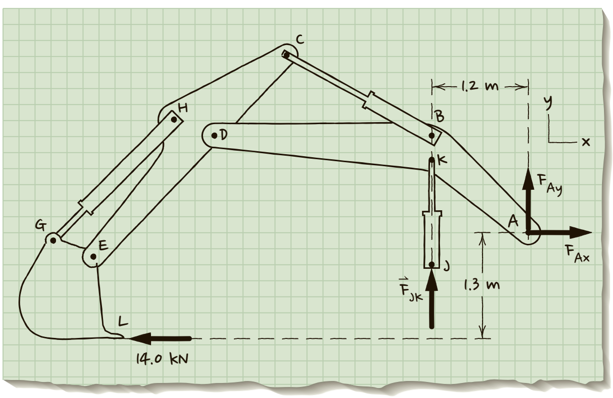


Figure 2 Free-body diagram of backhoe arm.

unknown force at J must act along the axis of the member. We represent the unknown force at the pin connection A by two mutually perpendicular forces (see **Table 4.1**). Note that there are three unknown forces on this free-body diagram, including the unknown load at A that we are looking for.

We set up the equations for planar equilibrium to find the unknown load at A.

Based on the free-body diagram in **Figure 2**, we write:

$$\sum M_{z@A}(\curvearrowleft +) = -F_{JK}(1.2\text{ m}) - 14.0\text{ kN}(1.3\text{ m}) = 0$$

$$F_{JK} = -14.0\text{ kN}\left(\frac{1.3\text{ m}}{1.2\text{ m}}\right) = -15.2\text{ kN}$$

The negative sign indicates that the force at J is acting in the opposite direction from our free-body diagram, meaning that the hydraulic cylinder is actually in tension. Now writing the force equilibrium equations:

$$\sum F_x(\rightarrow +) = -14.0\text{ kN} + F_{Ax} = 0 \Rightarrow F_{Ax} = 14.0\text{ kN}$$

$$\sum F_y(\uparrow +) = F_{JK} + F_{Ay} = 0 \Rightarrow F_{Ay} = -F_{JK} = 15.2\text{ kN}$$

Now we have all of the information to find the magnitude of $\boldsymbol{F}_A$ (the force acting on the pin at A):

$$\|\boldsymbol{F}_A\| = (F_{Ax}^2 + F_{Ay}^2)^{1/2} = [(14.0\text{ kN})^2 + (15.2\text{ kN})^2]^{1/2}$$

$$\boxed{\|\boldsymbol{F}_A\| = 20.7\text{ kN}}$$

The orientation angle α of $\boldsymbol{F}_A$ (**Figure 3**) is given by

$$\boxed{\alpha = \tan^{-1}\left(\frac{15.2\text{ kN}}{14.0\text{ kN}}\right) = 47.4°}$$

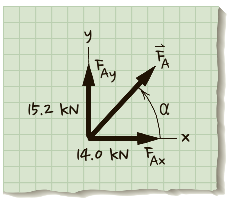


Figure 3 Orientation angle of pin force at A.

Alternately, we could give the answer in vector notation as $\boldsymbol{F}_A = 14.0\text{ kN}\boldsymbol{i} + 15.2\text{ kN}\boldsymbol{j}$, or in terms of space angles as $\|\boldsymbol{F}_A\| = 20.7\text{ kN}$, $\theta_x = 47.4°$, $\theta_y = 42.6°$, $\theta_z = 90°$.

Draw, Formulate Equations and Solve Now we consider the bucket in order to find the force acting at pin G. We isolate the bucket and draw a free-body diagram (**Figure 4**). Using the free-body diagram drawn in **Figure 4**, we set up the equations for planar equilibrium to find the load acting on hydraulic cylinder HG.

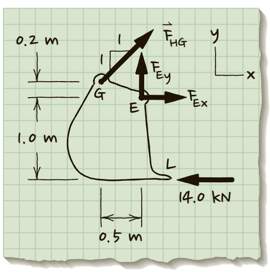


Figure 4 Free-body diagram of bucket.

$$\sum M_{z@E} = 0(\curvearrowleft +)$$

$$-F_{HG}\cos 45°(0.2\text{ m}) - F_{HG}\sin 45°(0.5\text{ m}) - 14.0\text{ kN}(1.0\text{ m}) = 0$$

$$F_{HG} = -14.0\text{ kN}\left(\frac{1.0\text{ m}}{0.495\text{ m}}\right) \Rightarrow \boxed{F_{HG} = -28.3\text{ kN}}$$

The negative sign for the F_{HG} indicates that the force is acting in the opposite direction from that on the free-body diagram so the hydraulic cylinder is in compression.

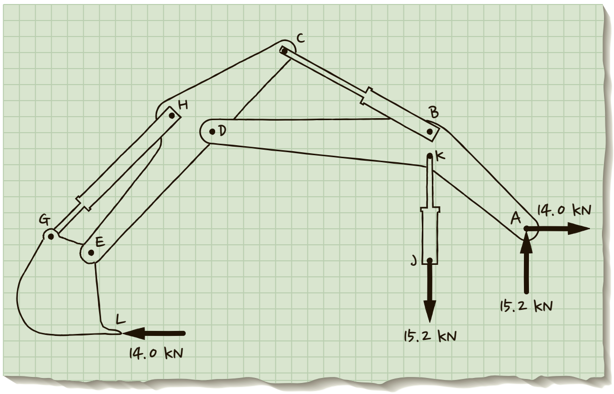


Figure 5 Forces acting on the backhoe arm.

Check It is useful to redraw the free-body diagram with the values of the loads at the supports (**Figure 5**); this shows us that the couple created by the vertical forces at J and A is balanced by the couple created by the horizontal forces at L and A.

EXAMPLE 9.1.3

The frame in **Figure 1** is subjected to the force and moment shown. Assuming the weights of the frame members are negligible, find the loads acting on the frame at the supports at A and C. Also express these loads in terms of shear forces acting on pins A and C.

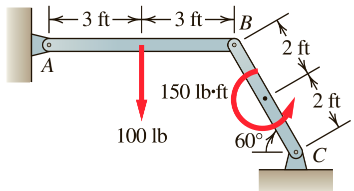


Figure 1 A force and moment act on frame ABC.

Goal Find the loads acting on the frame at the supports A and C.

Given Dimensions and support conditions for the frame as well as magnitudes and locations of loads.

Assume The system is planar because the 100-lb force acts in the plane of the frame and the moment acts about an axis perpendicular to the frame.

Draw We draw free-body diagrams for the two members, making sure that forces at B on member AB are equal and opposite to the forces at B on member BC (**Figure 2** and **Figure 3**).[1]

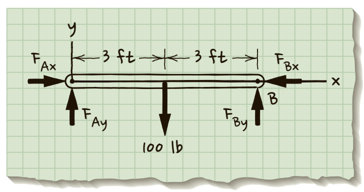


Figure 2 Free-body diagram of member AB.

Formulate Equations and Solve By taking the frame apart and analyzing each member we have six equations with which to find six unknowns (some of which are the loads at A and C).

[1]In **Figure 3** the two forces F_{Bx} and F_{By} are drawn in the opposite direction from **Figure 2**. These are the halves of the internal force pairs at B.

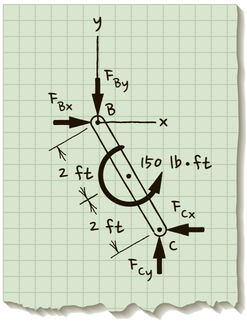


Figure 3 Free-body diagram of member *BC*.

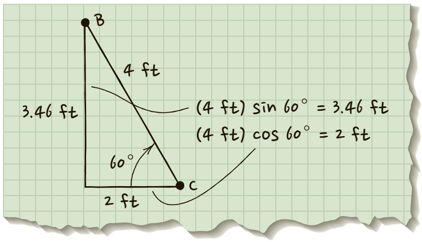


Figure 4 Geometry calculations for member *BC*.

For member *AB* (Figure 2):

$$\sum M_{z@B}(\curvearrowleft +) = -F_{Ay}(6\text{ ft}) + (100\text{ lb})(3\text{ ft}) = 0$$

$$F_{Ay} = 100\text{ lb}\left(\frac{3\text{ ft}}{6\text{ ft}}\right) \Rightarrow \boxed{F_{Ay} = 50.0\text{ lb}} \tag{1}$$

$$\sum F_x(\rightarrow +) = F_{Ax} - F_{Bx} = 0 \Rightarrow F_{Ax} = F_{Bx} \tag{2}$$

$$\sum F_y(\uparrow +) = F_{Ay} + F_{By} - 100\text{ lb} = 0$$

Substituting for F_{Ay} from (1) gives

$$F_{By} = 50.0\text{ lb} \tag{3}$$

For member *BC* (Figure 3), with geometry indicated in **Figure 4**:

$$\sum M_{z@C}(\curvearrowleft +) = 150\text{ lb·ft} + F_{By}(2\text{ ft}) - F_{Bx}(3.46\text{ ft}) = 0$$

We substitute for F_{By} from (3) to get

$$F_{Bx}(3.46\text{ ft}) = 150\text{ lb·ft} + (50.0\text{ lb})(2\text{ ft}) \Rightarrow F_{Bx} = 72.2\text{ lb} \tag{4}$$

$$\sum F_x(\rightarrow +) = F_{Bx} - F_{Cx} = 0 \Rightarrow \boxed{F_{Cx} = 72.2\text{ lb}}$$

$$\sum F_y(\uparrow +) = F_{Cy} - F_{By} = 0$$

$$F_{Cy} = F_{By} \Rightarrow \boxed{F_{Cy} = 50.0\text{ lb}}$$

Finally, we substitute (4) into (2) and solve for F_{Ax}:

$$F_{Ax} = F_{Bx} \Rightarrow \boxed{F_{Ax} = 72.2\text{ lb}}$$

The force components F_{Ax}, F_{Ay} and F_{Cx}, F_{Cy} are forces from outside the system acting on the system at pins *A* and *C* (and where the system is the frame). We combine (F_{Ax}, F_{Ay}) and (F_{Cx}, F_{Cy}) into the shear forces acting on the pins:

$$\text{Pin } A \text{ shear force} = \|\boldsymbol{F}_A\| = \sqrt{F_{Ax}^2 + F_{Ay}^2} \Rightarrow \boxed{\|\boldsymbol{F}_A\| = 87.9\text{ lb}}$$

$$\text{Pin } C \text{ shear force} = \|\boldsymbol{F}_C\| = \sqrt{F_{Cx}^2 + F_{Cy}^2} \Rightarrow \boxed{\|\boldsymbol{F}_c\| = 87.9\text{ lb}}$$

Check As a check, the moment equilibrium equation could be written with the above results for a free-body diagram of the entire frame using either *A*, *C*, or even *B* as a moment center. Another check would be to draw the results on each of the member free-body diagrams and apply the equilibrium equations, being sure to use a different moment center than was used in the above **Formulate Equations** and **Solve** step.

Note: In this solution we generated six linearly independent equations that contained the six unknowns (F_{Ax}, F_{Ay}, F_{Bx}, F_{By}, F_{Cx}, F_{Cy}). The basis for these six equations was the free-body diagrams of members *AB* (**Figure 2**) and *BC* (**Figure 3**). Alternately, we could have generated six equations based on a free-body diagram of the whole frame (not shown) and either the free-body diagram in **Figure 2** or **3**, as we do in Example 9.1.4.

EXAMPLE 9.1.4

End C of the frame in **Figure 1** rests against a rough surface.

(a) Assuming that the weight of the members can be neglected, find the loads acting on the frame at supports A and C.

(b) If the component of force acting on the frame parallel to the surface at C is less than $\mu F_{C\,\text{normal contact}}$, where the coefficient of friction $\mu = 0.6$, the frame will not slide. Based on your findings in (a), will the frame slide?

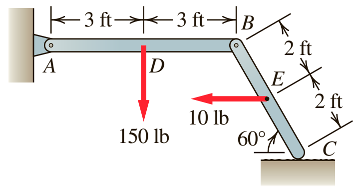


Figure 1 Friction holds the frame member in place at C.

Goal Find the loads at the supports A and C (a) and determine whether or not the frame will slide (b).

Given Information about the geometry of the structure, the loading placed on it, and the value of the coefficient of friction.

Assume The system is planar because the members lie in a single plane and all the loads act in that same plane. The frame is stationary where it rests on the ground at C (therefore, it is not sliding), which assumes that the component of force acting on the frame parallel to the surface at C is less than or equal to the product of the coefficient of friction and the normal force at C. We will check this assumption in (b).

Draw We draw the free-body diagram for the entire frame (**Figure 2**) and for member AB (**Figure 3**).

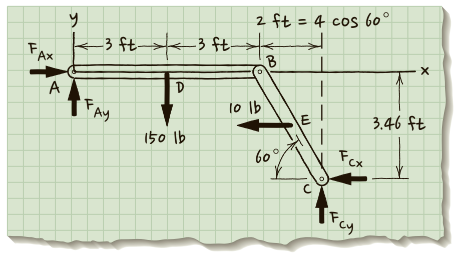


Figure 2 Free-body diagram of entire frame.

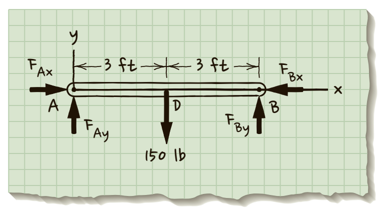


Figure 3 Free-body diagram of member AB.

Formulate Equations and Solve **(a) For member AB (Figure 3):**

$$\sum M_{z@B}(\curvearrowleft +) = -(6\text{ ft})F_{Ay} + (150\text{ lb})(3\text{ ft}) = 0$$

$$F_{Ay} = 150\text{ lb}\left(\frac{3\text{ ft}}{6\text{ ft}}\right) \Rightarrow \boxed{F_{Ay} = 75.0\text{ lb}} \qquad (1)$$

For the entire frame (Figure 2): Refer to **Figure 4** in Example 9.1.3 as an aid in calculating the moment arms:

$$\sum M_{z@C}(\curvearrowleft +) = -F_{Ay}(6\text{ ft} + 2\text{ ft}) - F_{Ax}(3.46\text{ ft})$$
$$+ 150\text{ lb}(3\text{ ft} + 2\text{ ft}) + 10\text{ lb}(2\text{ ft})\sin 60° = 0$$

We substitute for F_{Ay} from (1) to get

$$-75.0\text{ lb}(8\text{ ft}) - F_{Ax}(3.46\text{ ft}) + 150\text{ lb}(5\text{ ft}) + 10\text{ lb}(1.73\text{ ft}) = 0$$

$$\boxed{F_{Ax} = \frac{167.3\text{ lb}\cdot\text{ft}}{3.46\text{ ft}} = 48.4\text{ lb}} \qquad (2)$$

$$\sum F_x(\rightarrow +) = F_{Ax} - F_{Cx} - 10\text{ lb} = 0$$

We substitute for F_{Ax} from (2) to get

$$\boxed{F_{Cx} = 38.4\text{ lb}}$$

$$\sum F_y(\uparrow +) = F_{Ay} + F_{Cy} - 150\text{ lb} = 0$$

We substitute for F_{Ay} from (1) to get

$$F_{Cy} = 75.0 \text{ lb}$$

(b) We confirm that sliding is not predicted at C. According to the Coulomb friction law as described in detail in Chapter 7, we need to confirm: $|F_{Cx}|$ (friction force) $\leq \mu|F_{Cy}|$ (max allowable friction force without sliding).

In other words, if $|F_{Cx}| \leq \mu|F_{Cy}|$, then the support will not slide at C. Otherwise, the support will slide.

$$F_{Cx} = (38.4 \text{ lb}) < \mu F_{Cy} = (0.6)(75.0 \text{ lb}) = 45.0 \text{ lb}$$

We conclude that the frame is stationary at C and we have properly modeled the presence of friction. If we had found that $F_{Cx} > \mu F_{Cy}$ we would have concluded that the frame slides along the ground and the frame is therefore not in equilibrium (and the analysis we carried out in (a) is not valid).

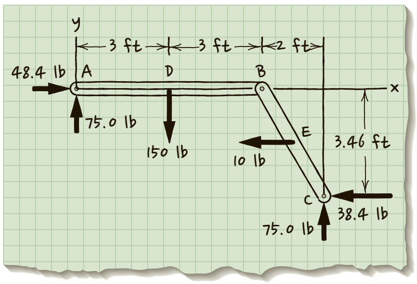


Figure 4 Calculated loads acting on frame.

Check We could check the moment equilibrium with the results above using an alternate moment center. For example, if we choose the moment center to be point D (**Figure 4**) we find

$$\sum M_{z@D}(\curvearrowleft +) = -75.0 \text{ lb}(3 \text{ ft}) - 10 \text{ lb}\left(\frac{3.46 \text{ ft}}{2}\right)$$

$$+75.0 \text{ lb}(3 \text{ ft} + 2 \text{ ft}) - 38.4 \text{ lb}(3.46 \text{ ft}) = 0$$

The left-hand side of the equation sums to -0.16 lb·ft. This is very close to zero, and we can say our answer is correct. (If we were to carry one more significant digit, using $F_{Ay} = 38.35$ lb, then the left-hand side would decrease to 0.009.)

EXAMPLE 9.1.5

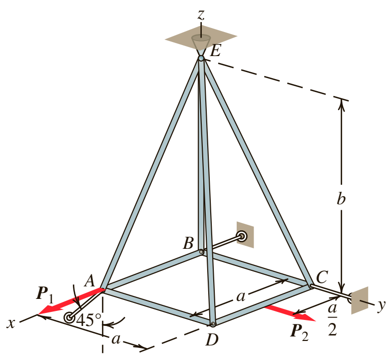


Figure 1 A nonplanar frame.

The nonplanar frame shown in **Figure 1** is anchored by a ball-and-socket joint at E. It is prevented from rotating by the links attached to the ground at joints A, B, and C. Assume the weight of the frame members can be ignored.

(a) Find an expression for the forces acting on members AB, AD, and AE.

(b) For $a = 1.0$ m, $b = 1.0$ m, $P_1 = 15.0$ kN, $P_2 = 15.0$ kN, find the forces in members AB, AD, and AE.

(c) Define the **safety factor** as the ratio: $\text{SF} = \dfrac{\text{force capacity of a member}}{\text{force in a member}}$

The **force capacity** of a member is the force at which the member fails, and is a function of the member's cross-sectional area and the material from which it is constructed. If the force capacity of members AB, AD, and AE is an axial force of 20 kN, calculate safety factors for these members for the conditions defined in (b). Confirm that this ratio is greater than unity for each of the three members (this means that failure is not predicted in the member).

Goal Find expressions for the forces acting on members AB, AD, and AE (b). Also calculate the safety factors for these members (c).

Given Information about the supports and geometry of the non-planar frame and a definition of safety factor. An axial force of 20 kN causes failure in members AB, AD, and AE.

Assume Members AB, BC, AD, AE, BE, DE, and CE act as two-force members because they are pinned at both ends and no forces act on them between the ends. Note that member CD is not a two-force member because a force P_2 acts along its length.

Draw We draw a free-body diagram of the entire frame (**Figure 2**).

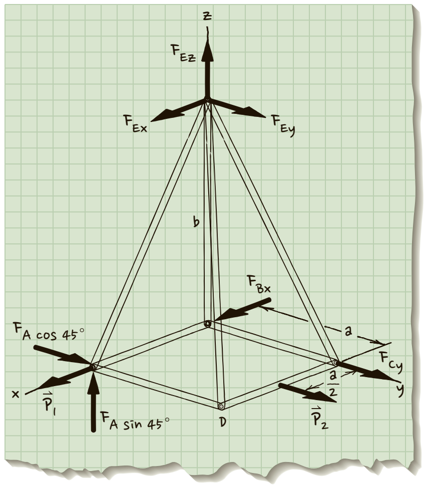


Figure 2 Free-body diagram of entire frame.

Formulate Equations and Solve To find the forces in members AB, AD, and AE, we need to first calculate the six unknown forces at the supports, as drawn in **Figure 2**. Using the six equilibrium equations we would find that:

$$F_A = -\frac{P_2}{2\cos 45°}, \quad F_{Bx} = -P_1 + \frac{a}{b}\frac{P_2}{2}, \quad F_{Cy} = -\frac{P_2}{2},$$

$$F_{Ex} = -\frac{a}{b}\frac{P_2}{2}, \quad F_{Ey} = 0, \quad F_{Ez} = \frac{P_2}{2}$$

(a) We analyze joint A to find the forces acting on members AB, AD, and AE (**Figure 3**). We have assumed that all of the members are in tension and thus pull away from the joint.

Equilibrium at joint A requires that:

$$\sum \boldsymbol{F} = \boldsymbol{P}_1 + \boldsymbol{F}_A + \boldsymbol{F}_{AB} + \boldsymbol{F}_{AD} + \boldsymbol{F}_{AE} = 0$$

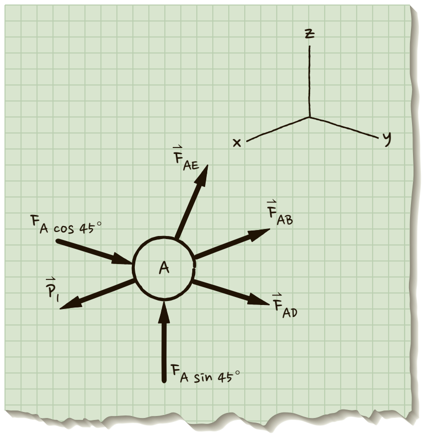


Figure 3 Free-body diagram of particle A.

where

$$\boldsymbol{P}_1 = P_1\boldsymbol{i} \qquad \boldsymbol{F}_A = -\frac{P_2}{2\boldsymbol{j}} - \frac{P_2}{2\boldsymbol{k}}$$

$$\boldsymbol{F}_{AB} = -F_{AB}i \qquad \boldsymbol{F}_{AD} = F_{AD}\boldsymbol{j}$$

$$\boldsymbol{F}_{AE} = -F_{AE}\frac{a}{\sqrt{a^2+b^2}}\boldsymbol{i} + F_{AE}\frac{b}{\sqrt{a^2+b^2}}\boldsymbol{k} \qquad \text{(Figure 4)}$$

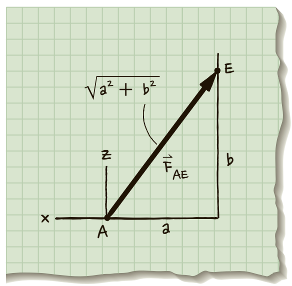


Figure 4 Geometry of member AE.

Three equations result from applying equilibrium in the $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ directions:

$$\text{In the } \boldsymbol{i} \text{ direction:} \quad P_1 - F_{AB} - F_{AE}\frac{a}{\sqrt{a^2+b^2}} = 0$$

$$\text{In the } \boldsymbol{j} \text{ direction:} \quad -\frac{P_2}{2} + F_{AD} = 0$$

$$\text{In the } \boldsymbol{k} \text{ direction:} \quad -\frac{P_2}{2} + F_{AE}\frac{b}{\sqrt{a^2+b^2}} = 0$$

We solve these for F_{AB}, F_{AD}, and F_{AE}:

$$\boxed{F_{AB} = P_1 - \frac{a}{b}\frac{P_2}{2}} \quad \boxed{F_{AD} = \frac{P_2}{2}} \quad \boxed{F_{AE} = \frac{P_2}{2}\frac{\sqrt{a^2+b^2}}{b}}$$

(b) We now consider a specific geometry and loading of the frame; namely, $a = 1.0$ m, $b = 1.0$ m, and $P_1 = P_2 = 15.0$ kN. The forces in members AB, AD, and AE are found from substituting into the results of (a) giving

$$\boxed{F_{AB} = 7.5\,\text{kN}} \quad \boxed{F_{AD} = 7.5\,\text{kN}} \quad \boxed{F_{AE} = \frac{15}{\sqrt{2}}\,\text{kN}}$$

(c) We have been told that the force capacity of each member AB, AD, and AE is 20 kN (we will assume that this is its capacity in both tension and compression). Therefore the safety factor in the given loading is:

$$(\text{Safety factor})_{AB} = \text{SF}_{AB} = \frac{20\,\text{kN}}{7.5\,\text{kN}} \Rightarrow \boxed{\text{SF}_{AB} = 2.7}$$

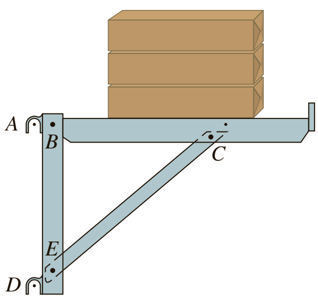
$$\text{SF}_{AD} = \frac{20\,\text{kN}}{7.5\,\text{kN}} \Rightarrow \boxed{\text{SF}_{AD} = 2.7}$$

$$\text{SF}_{AE} = \frac{20\,\text{kN}}{\frac{15}{\sqrt{2}}\,\text{kN}} \Rightarrow \boxed{\text{SF}_{AE} = 1.9}$$

This ratio is greater than unity for each member, which means failure is not predicted in any of these members.

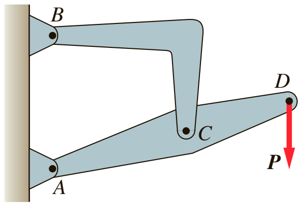
EXERCISES 9.1

9.1.1. [*] Consider the frame shown. Assume the system is in equilibrium and the members are of negligible weight. Draw a free-body diagram of the frame (as a whole), and for each of its members, taking into account the presence of two-force members where appropriate. Make sure to draw loads to properly represent Newton third-law force pairs.



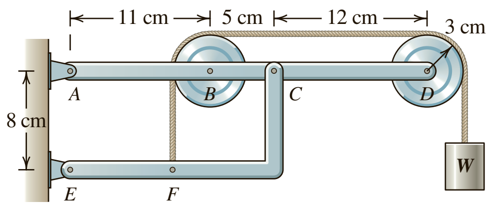
EX 9.1.1

9.1.2. [*] Consider the structural system shown. Assume the system is in equilibrium and the members are of negligible weight. Draw a free-body diagram of the structure (as a whole), and for each of its members, taking into account the presence of two-force members where appropriate. Make sure to draw loads to properly represent Newton third-law force pairs.



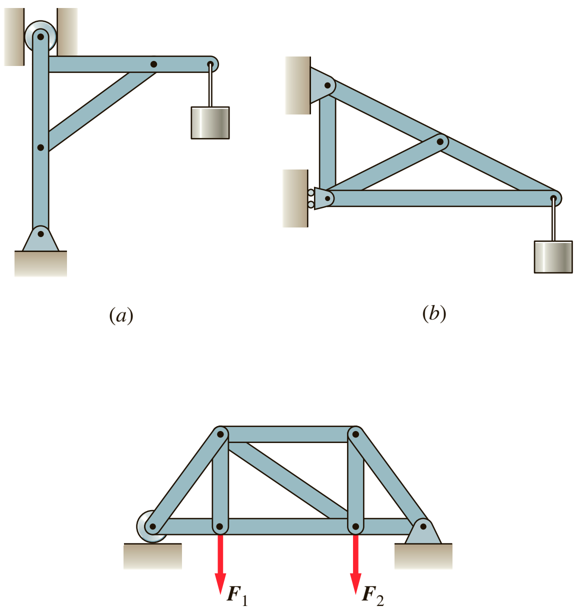
EX 9.1.2

9.1.3. [*] Consider the frame shown. Draw a free-body diagram of the frame (as a whole), and for each of its members, taking into account the presence of two-force members where appropriate. Make sure to draw loads to properly represent Newton third-law force pairs.



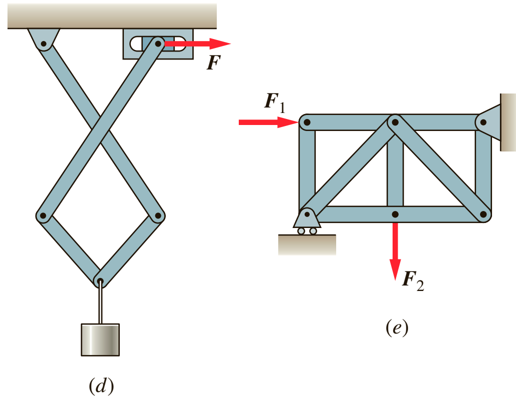
EX 9.1.3

9.1.4. [*] Classify each of the structures shown as a frame or a truss. Include your reasoning.



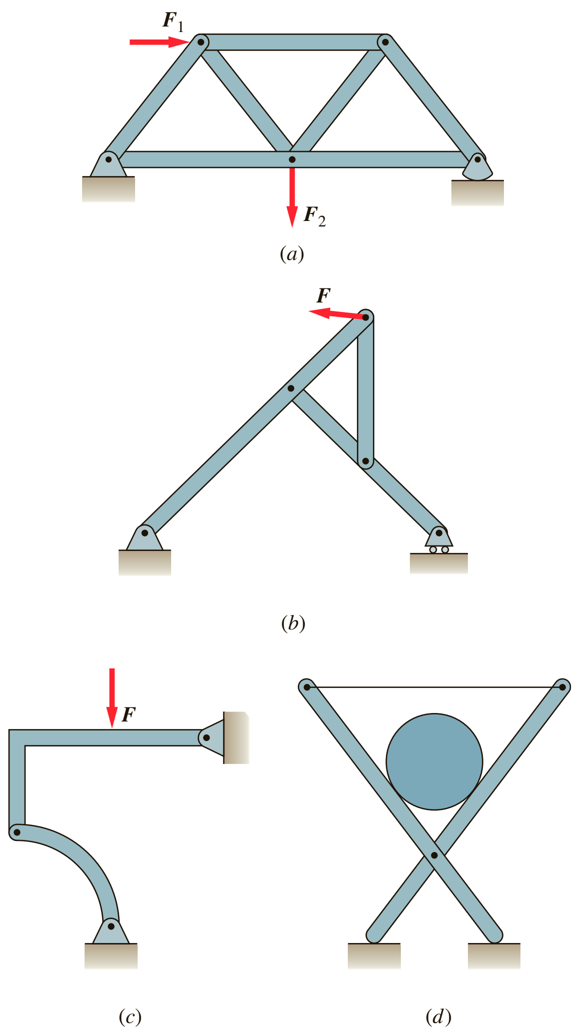
(c)

EX 9.1.4(a, b, c)



EX 9.1.4(d, e)

9.1.5. [*] Classify each of the structures shown as a frame or a truss. Include your reasoning.



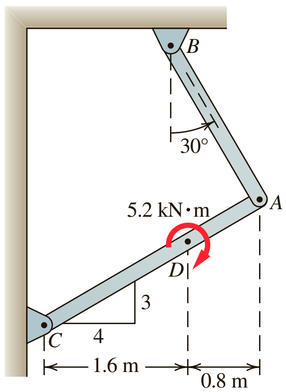
EX 9.1.5

9.1.6. [*] Why does it make sense to classify a bicycle frame as a frame and not as a truss? How would the design need to change to be classified as a truss? What would be the major disadvantage of a truss structure in this situation? Are there any advantages?

9.1.7. [*] Consider the frame with a moment applied to member AC at D.

a. Draw the free-body diagram of member AC.

b. Use the free-body diagram from **a** to determine the loads acting on member AC.

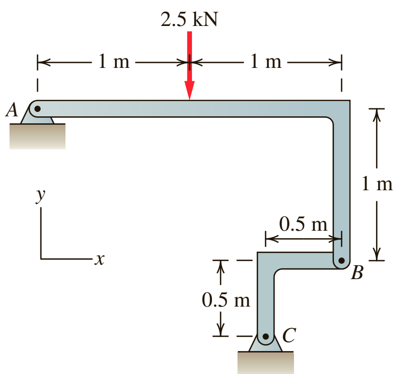


EX 9.1.7

9.1.8. [*] Consider the frame shown. Determine

a. the loads acting on the frame at A and C

b. the loads acting on member AB

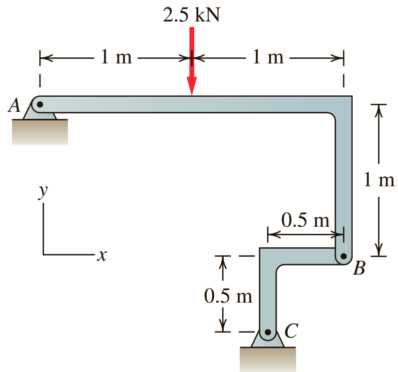


EX 9.1.8

9.1.9. [*] Consider the frame shown. Determine

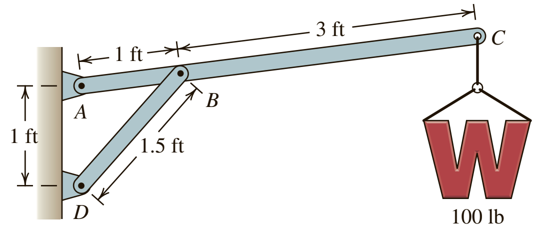
a. the loads acting on member BC

b. the shear force acting on the pin at B



EX 9.1.9

9.1.10. [*] Consider the frame shown. Determine the shear force acting on the pin at B.

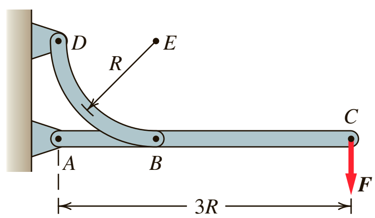


EX 9.1.10

9.1.11. [*] Consider the frame shown where member BD forms a quarter circle. Determine

a. the loads acting on the frame at A and D

b. the loads acting on member AC

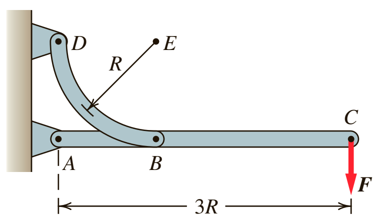


EX 9.1.11

9.1.12. [*] Consider the frame shown where member BD forms a quarter circle.

a. Determine the shear force acting on the pin at B.

b. If R = 200 mm and the value of the shear force should not exceed 500 N, what is the maximum allowable magnitude of the force $\boldsymbol{F}$?

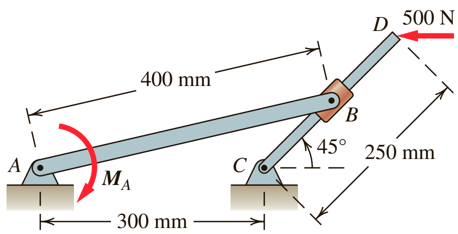


EX 9.1.12

9.1.13. [*] Consider the frame shown. Member AB is pinned at end B to a collar that may slide over the smooth bar CD. The frame is in equilibrium. Determine

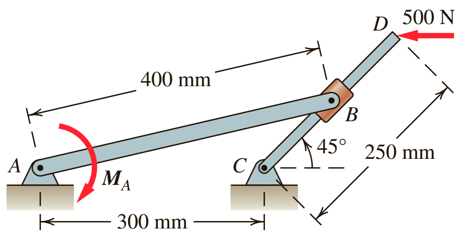
a. the magnitude of the moment at A

b. the other loads that act on the frame at A



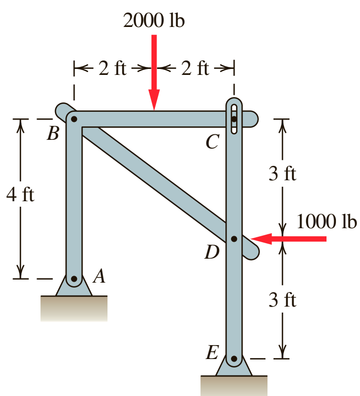
EX 9.1.13

9.1.14. [*] Consider the frame shown. Member AB is pinned at end B to a collar that may slide over the smooth bar CD. The frame is in equilibrium. Determine the loads that act on the frame at C.



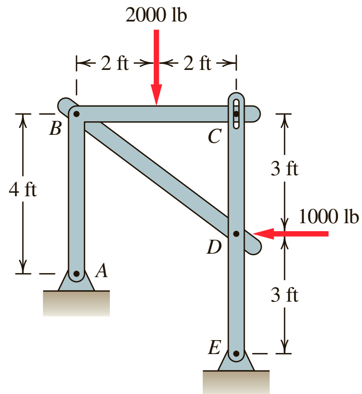
EX 9.1.14

9.1.15. [*] For the frame shown, the load at D is applied to member CDE. Determine the loads acting on members ABC and EC.



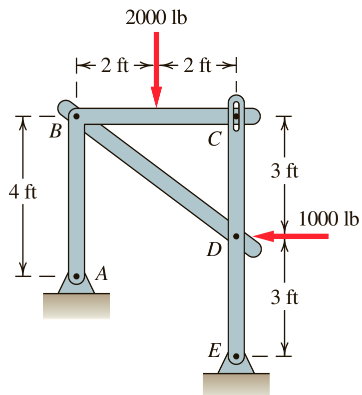
EX 9.1.15

9.1.16. [*] For the frame shown, the load at D is applied to member CDE. Determine the loads acting on the frame at A and E.



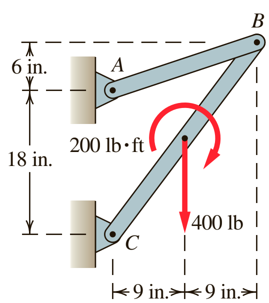
EX 9.1.16

9.1.17. [*] For the frame shown, the load at D is applied to member CDE. Determine the shear force acting on the pin at B.



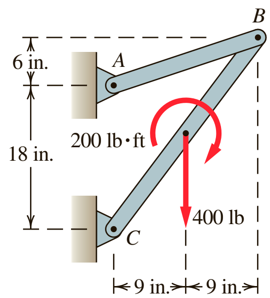
EX 9.1.17

9.1.18. [*] Consider the frame shown. Determine the loads acting on members AB and BC.



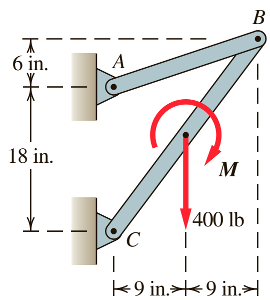
EX 9.1.18

9.1.19. [*] Consider the frame shown. Determine the loads acting on the frame at A and C.



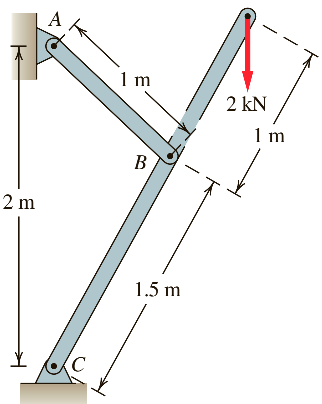
EX 9.1.19

9.1.20. [*] For the frame shown the maximum allowable shear in the pin at B is 500 lb. Determine the maximum moment M that can be applied to the frame.



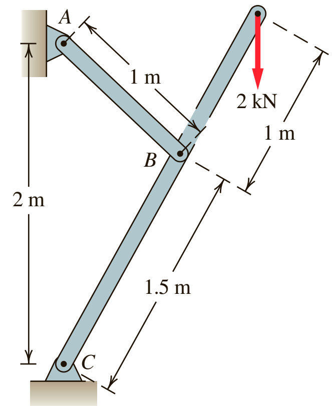
EX 9.1.20

9.1.21. [*] Consider the frame shown. Determine the loads acting on members AB and BC.



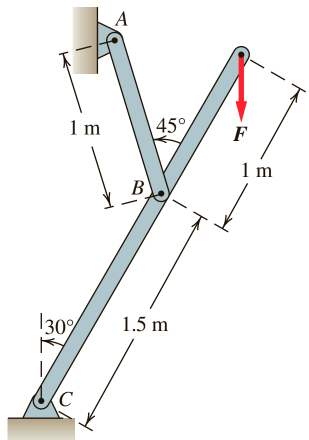
EX 9.1.21

9.1.22. [*] Consider the frame shown. Determine the loads acting on the frame at A and C.



EX 9.1.22

9.1.23. [*] The maximum allowable shear force for the pin at B is 7 kN. Determine the maximum downward force F that can be applied to the frame.

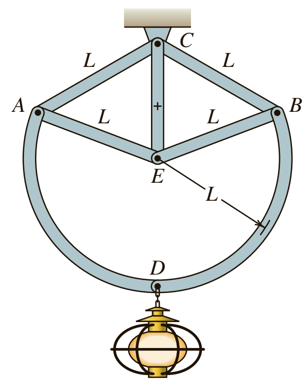


EX 9.1.23

9.1.24. [**] Consider the frame that holds up a lantern.

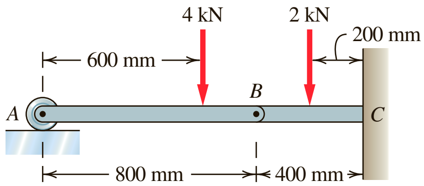
a. Determine the force in members AC, AE, and CE.

b. If the magnitude of the force (compression or tension) in members AC, AE, and CE is not to exceed 100 lb, what is the heaviest lantern that should hang from the frame?



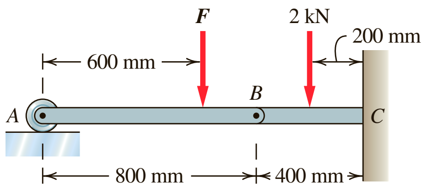
EX 9.1.24

9.1.25. [**] Consider the frame shown. Determine the loads acting on members *AB* and *BC*.



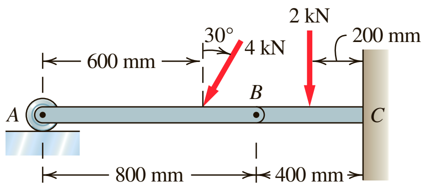
EX 9.1.25

9.1.26. [**] For the frame shown the maximum allowable support force at *A* is 2 kN and the maximum allowable moment at *C* is 2 kN·m. Determine the maximum downward force *F* that can be applied.



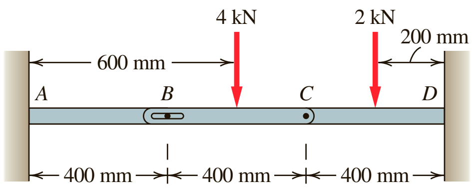
EX 9.1.26

9.1.27. [**] Consider the frame shown. Determine the shear force acting on the pin at *B*.



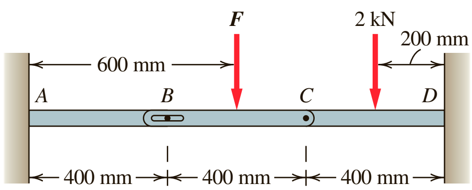
EX 9.1.27

9.1.28. [**] Consider the frame shown. Determine the loads acting on members *AB*, *BC*, and *CD*.



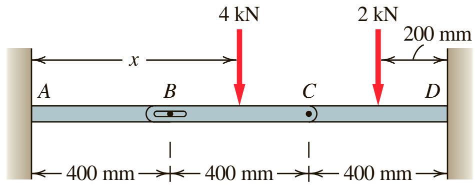
EX 9.1.28

9.1.29. [**] For the frame shown the maximum allowable moment at *A* is 1 kN·m and the maximum allowable moment at *D* is 1.5 kN·m. Determine the maximum downward force *F* that can be applied.



EX 9.1.29

9.1.30. [**] The maximum allowable shear force for the pin at *C* is 3 kN. Determine how far from *A* the 4 kN force can be placed without exceeding the capacity of the pin at *C*.

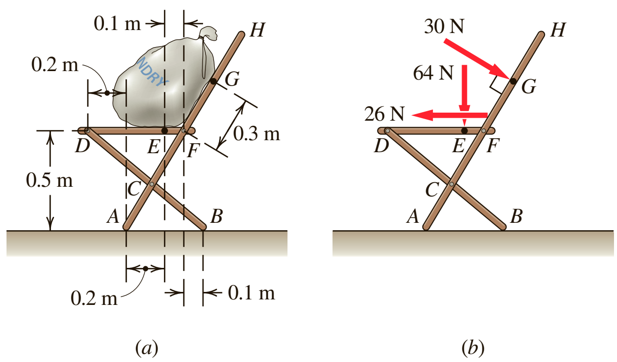


EX 9.1.30

9.1.31. [**] A bag of laundry resting on a chair shown in (*a*) applies forces to the chair as shown in (*b*). Assuming the floor is smooth, determine

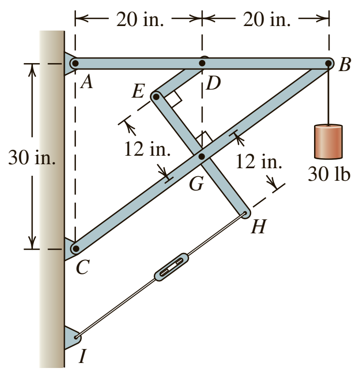
a. the loads acting on the chair at *A* and *B*

b. the shear forces acting on pins *F*, *C*, and *D*



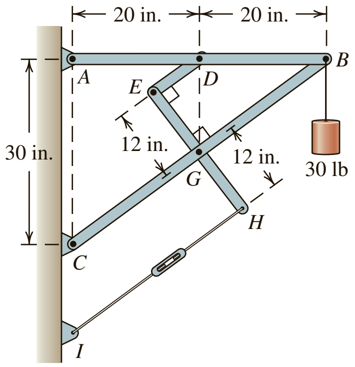
EX 9.1.31

9.1.32. [**] The tension in the cable that runs between *H* and *I* is 50 lb. Determine the loads acting on the frame at *A* and at *C*, and on member *EH*.



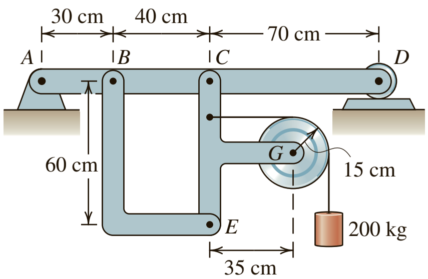
EX 9.1.32

9.1.33. [**] Consider the frame shown. If the tension in the cable that runs between *H* and *I* is 50 lb, determine the shear forces acting on the pins at *G* and *E*.



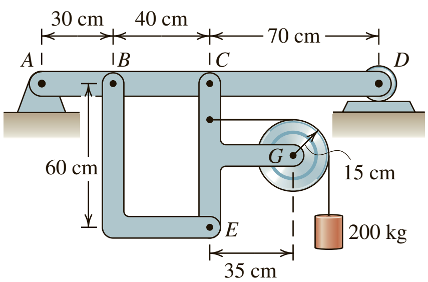
EX 9.1.33

9.1.34. [**] The frame supports the 200-kg mass in the manner shown. Determine the forces acting on each of the members, and on the frame at *A* and *D*.



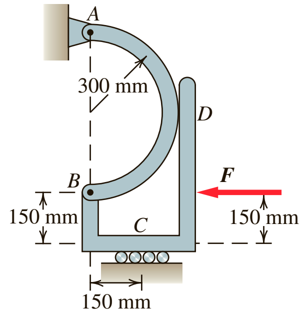
EX 9.1.34

9.1.35. [**] The frame supports the 200-kg mass in the manner shown. Determine the shear force acting on the pin at *E*.



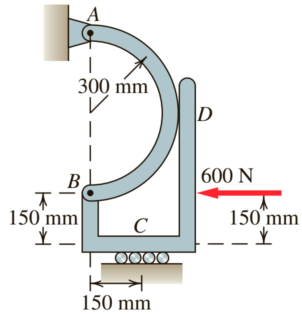
EX 9.1.35

9.1.36. [**] For the frame shown the roller at *C* can support no more than 3000 N. Determine the magnitude of ***F*** and the loads acting on the frame at *A*.



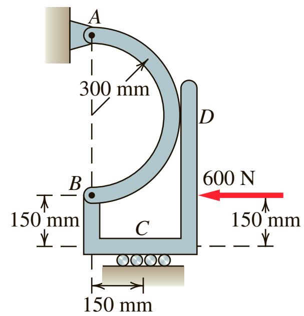
EX 9.1.36

9.1.37. [**] Consider the frame shown. Determine the loads acting on members *AB* and *BCD*.



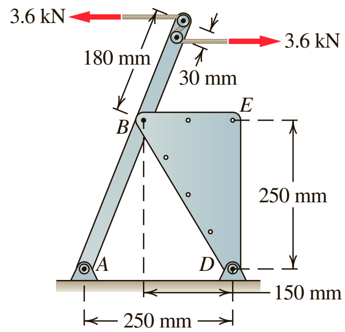
EX 9.1.37

9.1.38. [**] Consider the frame shown. Determine the shear force acting on the pin at *B*.



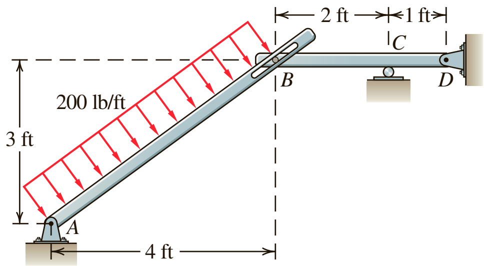
EX 9.1.38

9.1.39. [**] Lever *AB* is held in position by adjusting plate *BDE*, as shown. Determine the shear forces acting on the pins at *A*, *B*, and *D*.



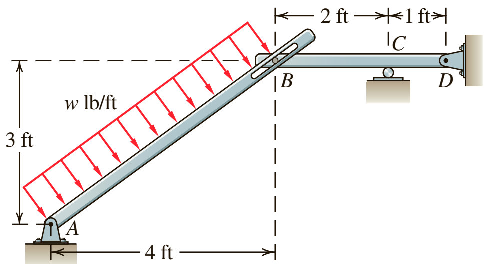
EX 9.1.39

9.1.40. [**] Consider the frame shown. Determine the loads acting on the frame at A and C.



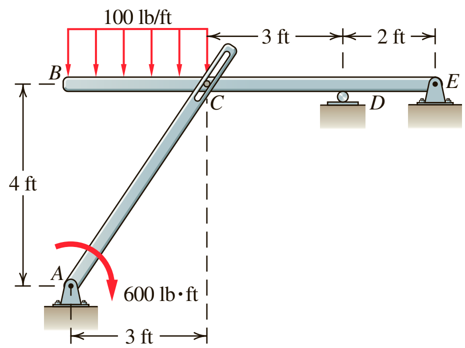
EX 9.1.40

9.1.41. [**] The pin at B can support a maximum shear force of 800 lb. Determine the maximum distributed load that can be applied to member AB.



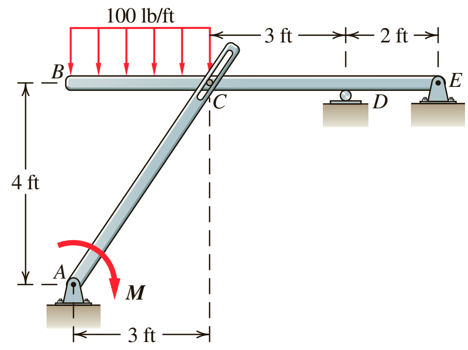
EX 9.1.41

9.1.42. [**] Consider the frame shown. Determine the loads acting on the frame at A, D, and E.



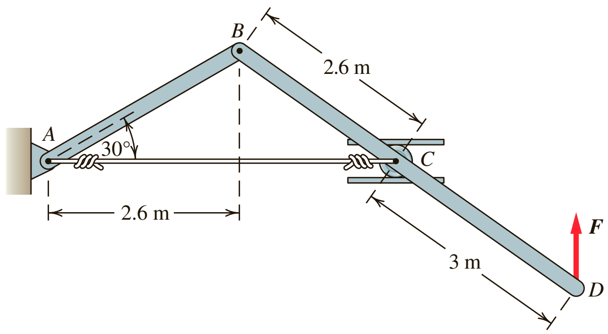
EX 9.1.42

9.1.43. [**] The pin at C can support a maximum shear force of 200 lb. Determine the maximum moment M that can be applied to member AC.



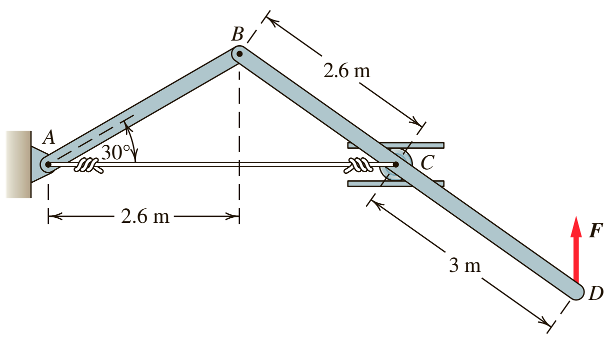
EX 9.1.43

9.1.44. [**] Consider the frame with a force $F = 200$ N applied at D. Determine the tension in the cable AC.



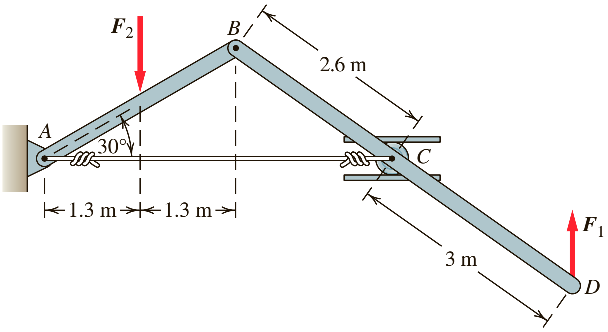
EX 9.1.44

9.1.45. [**] The maximum force cable AC can support without failing is 200 lb. Determine the maximum upward force F that can be applied at D.



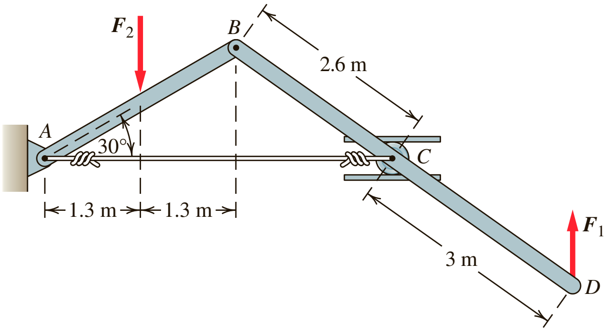
EX 9.1.45

9.1.46. [**] Consider the frame shown. The magnitudes of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are 500 N and 3000 N, respectively. Determine the tension in the cable AC.



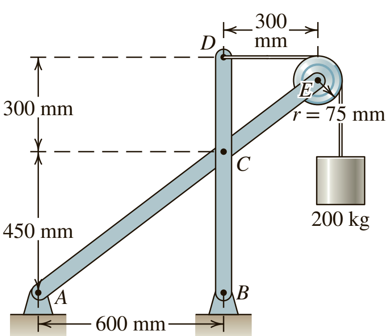
EX 9.1.46

9.1.47. [**] Consider the frame shown. The magnitudes of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are 500 N and 3000 N, respectively. Determine the shear force acting on the pin at B.



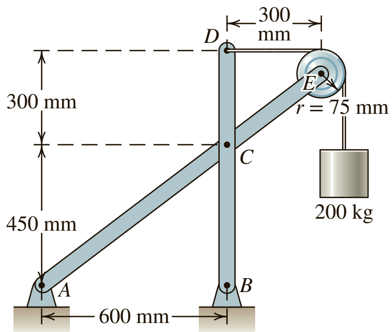
EX 9.1.47

9.1.48. [**] Consider the frame shown. Determine the loads acting on members AE and BD.



EX 9.1.48

9.1.49. [**] Consider the frame shown. Determine the loads acting on the frame at A and B.

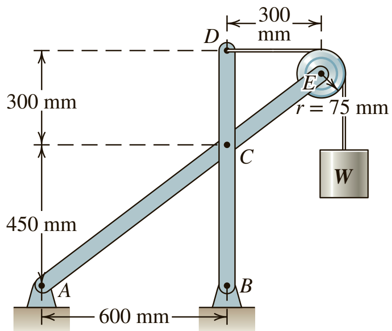


EX 9.1.49

9.1.50. [**] The maximum allowable shear on the pin at E is 2 kN.

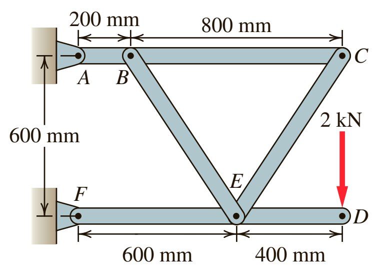
a. Determine the maximum load W that can hang from the frame.

b. For the value of W found in **a** find the shear on the pin at C.



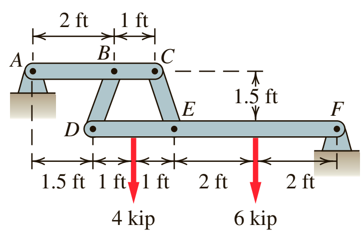
EX 9.1.50

9.1.51. [**] Consider the frame shown. AC and FD are each a single continuous member. Determine the loads acting on the frame at A and F, and the shear forces acting on the pins at B and C.



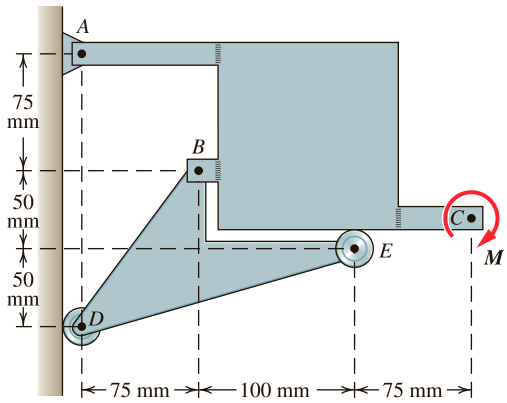
EX 9.1.51

9.1.52. [**] Consider the frame shown. Determine the loads acting on the frame at A and F, and the shear forces acting on the pins at B and C.



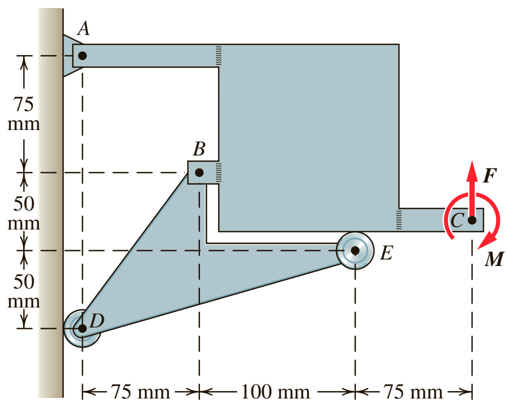
EX 9.1.52

9.1.53. **[**]** Consider the frame shown. Determine the loads acting on the frame at A and D when the magnitude of $\boldsymbol{M}$ is 75 N•m.



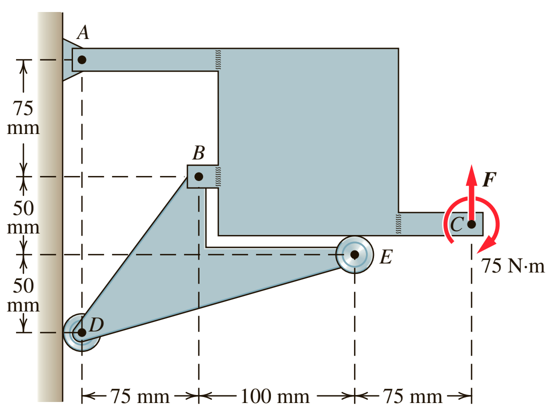
EX 9.1.53

9.1.54. **[**]** The magnitudes of $\boldsymbol{F}$ and $\boldsymbol{M}$ are 50 N and 40 N•m respectively. Determine the loads acting on members AC and BED. Also determine the shear force acting on the pin at B.



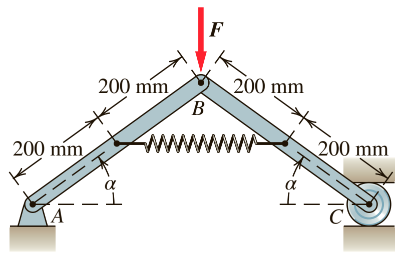
EX 9.1.54

9.1.55. **[*]** Consider the frame shown. Determine the largest magnitude of $\boldsymbol{F}$ that can be applied at C (in addition to the 75 N•m moment load), if the frame is not to collapse.



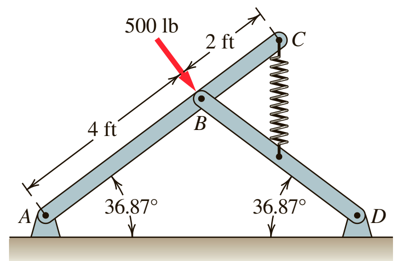
EX 9.1.55

9.1.56. **[***]** Frame ABC is in equilibrium. The spring has a stiffness of 8 kN/m and an unstretched length of 220 mm. If $\alpha = 36.87°$ in the position shown, determine the magnitude of the force $\boldsymbol{F}$.



EX 9.1.56

9.1.57. **[***]** Consider the frame shown. The spring has a stiffness of 60 lb/in. and an unstretched length of 15 in. Determine the loads acting on the frame at A and D.

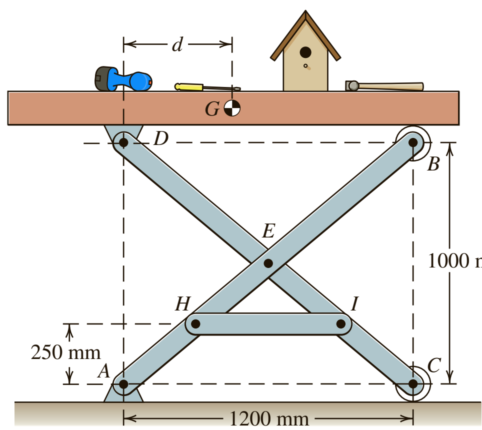


EX 9.1.57

9.1.58. **[***]** The top of a folding workbench (along with associated materials on top of the workbench) has a mass of M. The center of mass is located at a distance d from the hinge at D, as shown.

a. Write a relationship between the force in member HI and the distance d.

b. What values of d would cause the top of the workbench to tip over?



EX 9.1.58

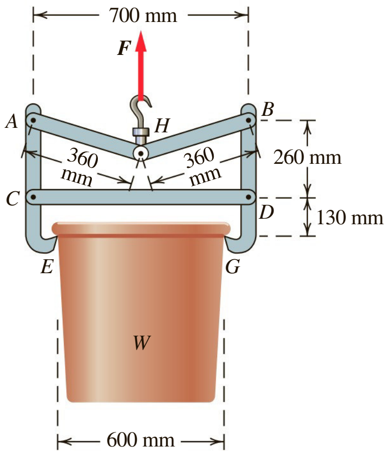
9.1.59. **[***]** The frame shown is part of a lifting device for transporting containers of weight W.

a. Determine the magnitude of the tension or compression in member DC as a function of W.

b. Determine the shear forces acting in the pins at A and C, as functions of W.

c. If the force in member DC exceeds 2000 N, member DC will fail, and if the shear force in any pin exceeds 1000 N, that pin will fail. Given this, what is the maximum container weight you would recommend lifting? Support your answer with calculations.

d. At the weight you determine in **c**, what force is exerted on the container at E and G?

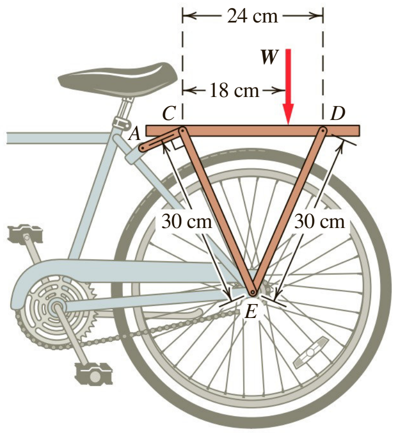


EX 9.1.59

9.1.60. [***] A typical bicycle rack is shown.

a. Determine the loads acting on the rack at A and E if the weight being carried on the rack is 160 N. (The total weight is 320 N, but you need only consider 1/2 of the design because of symmetry.) The connections at A, C, D, and E are pin connections.

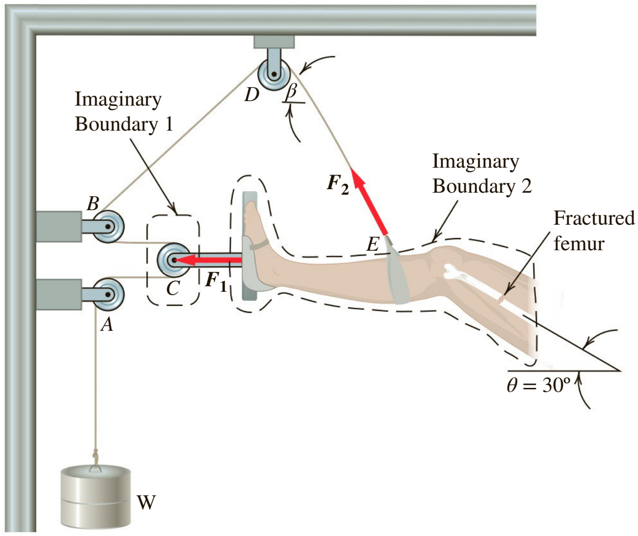
b. Buckling will occur in member CE or DE if the compressive force in either member exceeds 100 N. Are you concerned about buckling as a failure mode?



EX 9.1.60

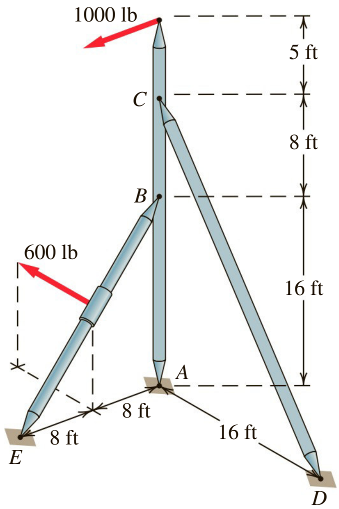
9.1.61. [***] Consider the traction set-up designed to help in healing a fractured femur. This particular device is referred to as Hamilton-Russell traction.

What weight W and angle β should the device be set at such that a force of 400 N tension is applied at the femoral (upper leg) fracture site? HINTS: 1) the two boundaries in the figure isolate potential free-body diagrams; 2) The femur can be treated as a two-force member.



EX 9.1.61

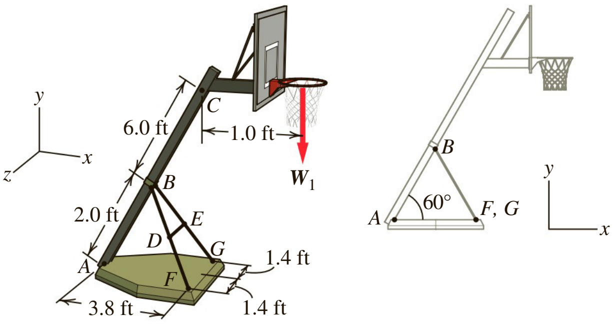
9.1.62. [***] Determine the loads acting on the frame at A, D, and E. Treat each connection as a ball and socket joint.



EX 9.1.62

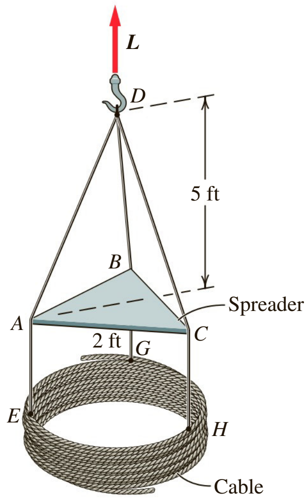
9.1.63. [***] A portable basketball goal is shown. A hinge with its axis aligned with the z direction attaches members BF and BG to post ABC. Horizontal member

DE is located midway along BF and BG. Members BD, BG, and DE are connected by ball and socket joints, and A is connected to the base plate by a ball and socket. Post ABC weighs 100 lb and the backstop and net 40 lb (shown as W_1 in the figure). Determine the force in member DE.



EX 9.1.63

9.1.64. [***] The cabling system holds up the 900-lb cable-spool and 45-lb triangular spreader attached to the cables at A, B, and C. Determine the tensions in all segments of the cables.



EX 9.1.64

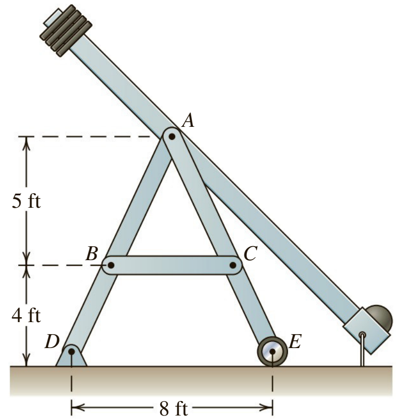
9.1.65. [***] Ancient siege engines provided military commanders with the ability to engage an enemy from a distance; essentially they were the artillery of the armies past. What is known of these medieval siege engines is limited to crude artist renditions and manuscript references so, the hypothetical analysis of siege engines is intriguing and challenging. Let us analyze one of the simplest forms of siege engine, the catapult, as shown in **Figure (*a*)**. This siege engine fires a missile using the energy gained from dropping a weight and the advantage of a lever arm.

a. If the lever arm applies the 600 lb and 800 lb loads to the pin at A, as shown in **Figure (*b*)**, calculate the loads acting on the frame at D and E.

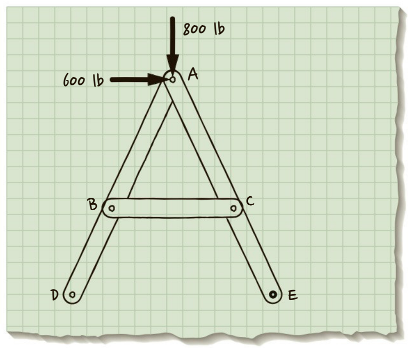
b. Calculate the shear forces acting on the pins at A, B, and C.

c. Determine the loads acting on members AD, AE, and BC. Based on these results, which member would you be most concerned about failing, and why?

d. If the boundary connection at E were changed from a roller to a pin connection, how would your calculations in (a) and (b) change? (Describe in words and with sketches.)



(a)



(b)

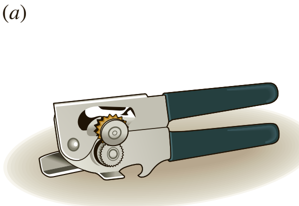
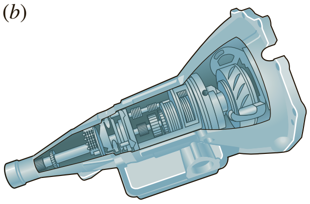
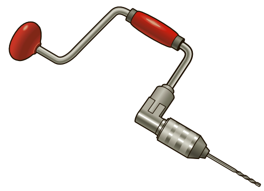
EX 9.1.65

9.2 DEFINING AND ANALYZING MACHINES

Learning Objective: Carry out equilibrium analysis of a machine and its members, including the effects of bearing friction and rolling resistance.

A **machine** is a system designed to change the direction and/or magnitude of loads or motion. In considering machines, we often think in terms of the load *into* the system, the load *out of* the system, and the ratio of output to input. **Table 9.1** illustrates some other familiar examples of machines.

Table 9.1 Some Common Machines

Manual Can Opener: Multiplies Force	Car Transmission: Multiplies Force or Torque	Balance Scale: Equalizes Moments	Hand Mixer or Drill: Multiplies Rotational Speed
(*a*) 	(*b*) 	(*c*) 	(*d*) 

Like frames, machines support loads. In fact, many machines can be classified as frames because they are assemblies of various members (including at least one multiforce member). However, because the primary purpose of a machine is to modify loads or motion, machines are a distinct classification.

Finding Member Loads in a Machine

The basic analysis of a machine in mechanical equilibrium is identical to that of a frame in terms of analyzing separate members by drawing free-body diagrams, setting up the equilibrium equations, and then using these equations to find the unknown loads. Often the analysis of a machine also involves the calculation of the ratio of output load to input load (referred to as the **mechanical advantage**) or output motion to input motion. Depending on the machine and its function, the input-to-output load ratio of concern might be a ratio of forces or of moments.

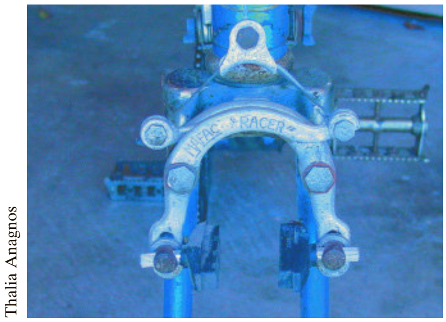
Sometimes the biggest challenge in analyzing a machine is understanding how it works. For this reason, you should always take the time to gain this understanding *before* diving into calculations (this same advice holds for analysis of any system!).

Check out the following examples of applications of this material.

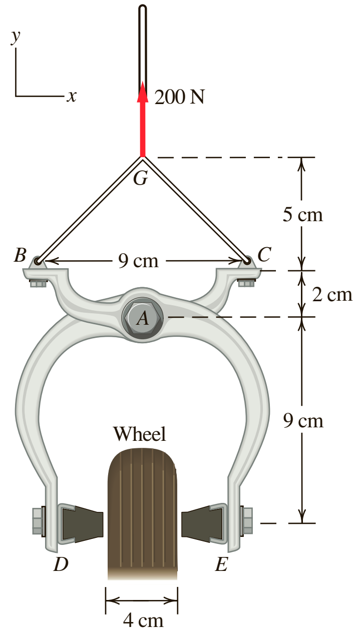
- **Example 9.2.1 Analysis of a Bicycle Brake**
- **Example 9.2.2 Analysis of a Toggle Clamp**

EXAMPLE 9.2.1

A bicycle rider squeezes the brake lever on the handlebars, which pulls on a cable that is indirectly attached to the brake calipers. The cable pulls up at G on a wire attached to the calipers at B and C, causing the two calipers to rotate about pin A and engage with the wheel (**Figure 1**). A normal force and friction force (acting perpendicular to the xy plane) are generated at both D and E between the brake pad and the wheel rim. Find the normal force due to the brake pad acting on the wheel rim at E when the cable is pulling upward with a force of 200 N on the center-pull bicycle brake at G. Also calculate the mechanical advantage of the caliper system. Ignore the force generated at A by the spring (not shown in **Figure 1*a***) that restores the brake to its open position when the brake lever is released.



(*a*)



(*b*)

Figure 1 Center pull caliper brakes.

Goal Find the normal force acting on the wheel rim at E and the mechanical advantage of this brake system.

Given A coordinate system, along with information about how the brake calipers work, their associated geometry, and the force pulling on wire BGC at G.

Assume The system is planar and in equilibrium (with the brake pads engaged with the wheel). Component weights are negligible, pin A is frictionless, and the connection at G between the cable and the wire is frictionless.

Draw We draw a free-body diagram of the connection at G (**Figure 2**) to determine the tensile force acting on wire BGC (T_{BGC}). We also need a free-body diagram of the caliper. Fortunately brake symmetry allows us to analyze just one of the calipers. We draw a free-body diagram of caliper BAE (**Figure 3**) and apply the planar equilibrium equations to determine F_{normal}, the normal force acting on the brake pad at E. F_{normal} acting on the brake pad is equal and opposite to the normal force acting on the wheel rim.

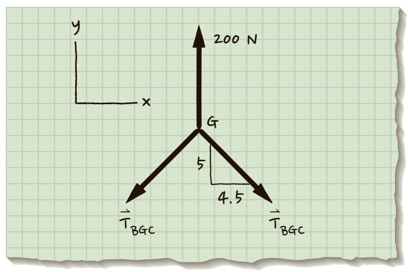


Figure 2 Free-body diagram of cable and wire connection at G.

Formulate Equations and Solve First we find T_{BGC}, the wire tension pulling on the caliper at B. The wire can slide freely through the connection at G because there is no friction, therefore the tension acting

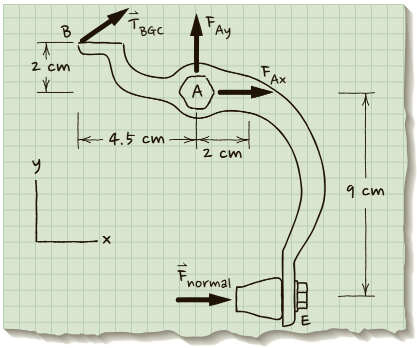


Figure 3 Free-body diagram of caliper *BAE*.

on the wire to the left of the connection is equal to that acting on the right portion.

Based on **Figure 2** we write:

$$\sum F_y(\uparrow +) = -T_{BGCy} - T_{BGCy} + 200 \text{ N} = 0$$

Using the geometry of the wire to calculate the y component, we get:

$$-T_{BGC}\left(\frac{5}{6.73}\right) - T_{BGC}\left(\frac{5}{6.73}\right) + 200 \text{ N} = 0$$

$$T_{BGC} = 134.5 \text{ N} \tag{1}$$

We now analyze caliper BAE (**Figure 3**) to find F_{normal}:

$$\sum M_{z@A}(\circlearrowleft +) = -T_{BGCy}(4.5 \text{ cm}) - T_{BGCx}(2 \text{ cm}) + F_{\text{normal}}(9 \text{ cm}) = 0$$

Recognizing that $T_{BGCy} = \frac{5}{6.73}T_{BGC}$, $T_{BGCx} = \frac{4.5}{6.73}T_{BGC}$ and that $T_{BGC} = 134.5$ N (from (1)) we get:

$$-134.5 \text{ N}\left(\frac{5}{6.73}\right)(4.5 \text{ cm}) - 134.5 \text{ N}\left(\frac{4.5}{6.73}\right)(2 \text{ cm}) + F_{\text{normal}}(9 \text{ cm}) = 0 \Rightarrow \boxed{F_{\text{normal}} = 70.0 \text{ N}}$$

The mechanical advantage of the caliper system is the ratio $F_{\text{output}}/F_{\text{input}}$. If we consider that the caliper system is made up of two calipers, the total "output" force is 2×70 N $= 140$ N. The "input" force is 200 N. Therefore the mechanical advantage of the caliper system is:

$$\boxed{\frac{140 \text{ N}}{200 \text{ N}} = 0.70}$$

You may be surprised that this number is less than one. Remember–this is the mechanical advantage of *just* the brake calipers. The mechanical advantage of the whole brake system would need to include both the caliper and the brake lever.

Check To check our result, we reanalyze the problem by first finding F_{Ay} and F_{Ax} and then F_{normal}.

Based on **Figure 3**:

$$\sum F_y(\uparrow +) = T_{BGCy} + F_{Ay} = 0 \quad \Rightarrow \quad F_{Ay} = -T_{BGC}\left(\frac{5}{6.73}\right) = -100 \text{ N}$$

$$\sum M_{z@E}(\circlearrowleft +) = -T_{BGCy}(6.5 \text{ cm}) - T_{BGCx}(11 \text{ cm}) - F_{Ay}(2 \text{ cm}) - F_{Ax}(9 \text{ cm}) = 0$$

$$-134.5 \text{ N}\left(\frac{5}{6.73}\right)(6.5 \text{ cm}) - 134.5 \text{ N}\left(\frac{4.5}{6.73}\right)(11 \text{ cm}) - (-100 \text{ N})(2 \text{ cm}) - F_{Ax}(9 \text{ cm}) = 0 \quad \Rightarrow \quad F_{Ax} = -160.0 \text{ N}$$

$$\sum F_x(\rightarrow +) = T_{BGCx} + F_{Ax} + F_{\text{normal}} = 0$$

$$134.5 \text{ N}\left(\frac{4.5}{6.73}\right) + (-160.0 \text{ N}) + F_{\text{normal}} = 0 \quad \Rightarrow \quad F_{\text{normal}} = 70.0 \text{ N}$$

We get the same result!

EXAMPLE 9.2.2

A toggle clamp holds workpiece F (Figure 1)

(a) Assuming the weight of the clamp members can be ignored, determine the mechanical advantage of this toggle clamp, defined as the ratio of the output force at the workpiece to $\boldsymbol{P}$.

(b) For the dimensions shown in the accompanying list, what is the mechanical advantage of this toggle clamp?

Dimensions for Toggle Clamp
$a = 1$ in.
$b = 4.5$ in.
$c = 1.5$ in.
$d = 4$ in.
$e = 3$ in.
β is such that $\cos\beta = 4/5$, $\sin\beta = 3/5$

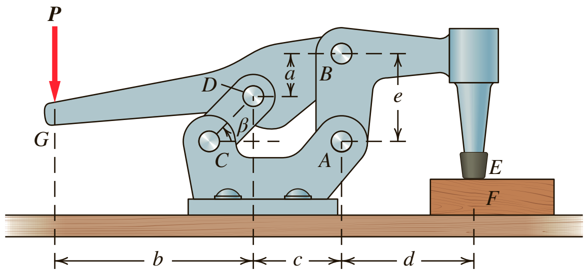


Figure 1 Toggle clamp acting on workpiece at E.

Goal Find the force at E ($\boldsymbol{F}_E$) in terms of $\boldsymbol{P}$, expressed as mechanical advantage. Also calculate the mechanical advantage for a particular geometry.

Given The configuration of the toggle clamp and its key dimensions.

Assume The system is planar, the pin connections A, B, C, D are all frictionless, the clamping force at E is purely vertical, as is the input force $\boldsymbol{P}$. Member CD is a two-force member (because it has pin connections at its ends and is loaded by force through its ends).

Draw We draw free-body diagrams of individual parts of the toggle clamp, recognizing that $\boldsymbol{F}_D$ is at angle β with the horizontal because member CD is a two-force member (Figure 2 and Figure 3).

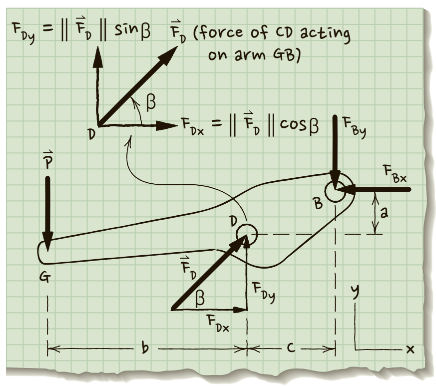


Figure 2 Free-body diagram of handle.

Formulate Equations and Solve (a) We first isolate the handle (**Figure 2**); doing this allows us to relate $\boldsymbol{F}_B$ to $\boldsymbol{P}$. Next we isolate member ABE (**Figure 3**); doing this allows us to relate $\boldsymbol{F}_B$ to $\boldsymbol{F}_E$. Finally, we relate $\boldsymbol{F}_E$ to $\boldsymbol{P}$.

We can write $\boldsymbol{F}_D = F_{Dx}\,\boldsymbol{i} + F_{Dy}\,\boldsymbol{j}$. Furthermore, because member CD is a two-force member, we can write $F_{Dx} = \|\boldsymbol{F}_D\|\cos\beta$ and $F_{Dy} = \|\boldsymbol{F}_D\|\sin\beta$.

Now we formulate the equilibrium equations for the planar system in **Figure 2**.

$$\sum M_{z@B}(\curvearrowleft +) = P(b+c) - \|\boldsymbol{F}_D\|\sin\beta(c) + \|\boldsymbol{F}_D\|\cos\beta(a) = 0$$

$$\|\boldsymbol{F}_D\| = \frac{b+c}{c\sin\beta - a\cos\beta}P \tag{1}$$

$$\sum F_x(\rightarrow +) = -F_{Bx} + \|\boldsymbol{F}_D\|\cos\beta = 0 \quad \Rightarrow \quad F_{Bx} = \|\boldsymbol{F}_D\|\cos\beta \tag{2}$$

$$\sum F_y(\uparrow +) = -F_{By} + \|\boldsymbol{F}_D\|\sin\beta - P = 0 \quad \Rightarrow \quad F_{By} = \|\boldsymbol{F}_D\|\sin\beta - P \tag{3}$$

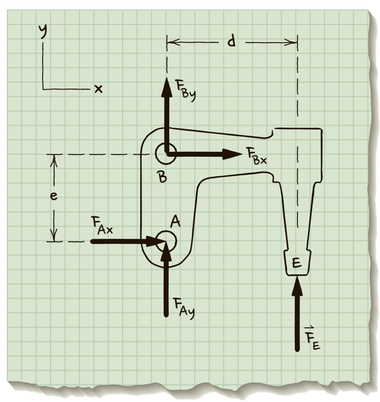


Figure 3 Free-body diagram of member ABE.

We substitute (1) into (2) and (3) to express the force components at B in terms of the input force $\boldsymbol{P}$:

From (2) $$F_{Bx} = \frac{(b+c)\cos\beta}{c\sin\beta - a\cos\beta}P \qquad (2^*)$$

From (3) $$F_{By} = \left(\frac{(b+c)\sin\beta}{c\sin\beta - a\cos\beta} - 1\right)P \qquad (3^*)$$

Moment equilibrium for member ABE (**Figure 3**) gives:

$$\sum M_{z@A}(\curvearrowleft +) = -F_{Bx}e + F_E d = 0$$

$$F_E = \frac{e}{d}F_{Bx} \underset{\substack{\text{substituting} \\ \text{from } (2^*)}}{\Rightarrow} F_E = \left(\frac{e}{d}\right)\frac{(b+c)\cos\beta}{c\sin\beta - a\cos\beta}P \qquad (4)$$

Rearranging (4) gives us the ratio we are interested in:

$$\boxed{\frac{F_E}{P} = \left(\frac{e}{d}\right)\frac{(b+c)\cos\beta}{c\sin\beta - a\cos\beta}}$$

We are asked to calculate the ratio for the particular geometry given. Substituting the values of a, b, c, d and e we find that

The mechanical advantage of the toggle clamp is 36 to 1.

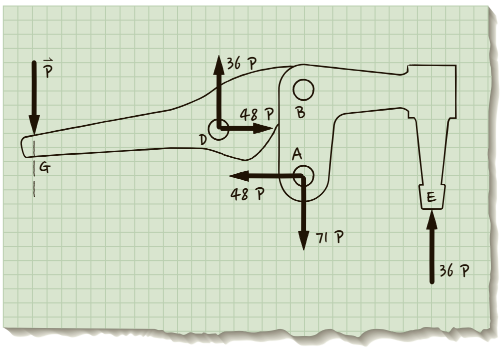


Figure 4 Free-body diagram of the toggle clamp.

Check To check our result, we consider a free-body diagram of the whole toggle clamp, as shown in **Figure 4**, with calculated force values (based on the dimensions given) shown. Notice that F_{Dx} and F_{Ax} balance one another. In addition, if we sum the moments around point A we find:

$$\sum M_{z@A}(\curvearrowleft +) = P(6\text{ in.}) - 36P(1.5\text{ in.}) + 48P(2\text{ in.}) + 36P(4\text{ in.}) = 0$$

Therefore our calculated values show that the toggle clamp as a whole is in equilibrium.

Rotating Machines

Moving machine members may be rotating, translating, or both. This is in contrast to frames and trusses, which are generally stationary. As long as each member in a machine is rotating about an axis of fixed orientation at a constant angular velocity, the conditions of mechanical equilibrium developed in Chapter 5 apply.

When we model a system as an *idealized* machine, we assume that all associated bearings are frictionless. In addition, any wheels in the machine are assumed to be perfectly rigid and therefore cause no rolling resistance. In real machines, bearing friction and rolling resistance do exist, and their presence may sometimes be significant. In Chapter 7, techniques for incorporating the presence of bearing friction and rolling resistance in equilibrium analysis were developed. We note that the presence of bearing friction may be important when the bearing is dry or only partially lubricated and is modeled as a moment about the center of the shaft that is in the

direction *opposite* the shaft's rotation, as detailed in Equation (7.18). Rolling resistance becomes significant when wheels and the ground they roll along are not perfectly rigid, thus requiring additional force for the wheel to be propelled forward, as detailed in Equation (7.20).

Machine Efficiency

Mechanical efficiency is defined as the ratio of a machine's output-to-input with friction and rolling resistance, divided by the output-to-input without friction. Friction and rolling resistance are "losses" that reduce a machine's ability to affect multiplication of input and output loads or speeds. Typically efficiency is expressed as a percentage, so 100 percent mechanical efficiency is a system with no frictional or rolling resistance losses.

Check out the following examples of applications of this material.
- **Example 9.2.3 Analysis of a Frictionless Gear Train**
- **Example 9.2.4 Analysis of a Gear Train with Friction**

EXAMPLE 9.2.3

A single-stage gear train is shown in **Figure 1**. The input force $\boldsymbol{F}_{\text{in}}$ causes Shaft 1 to rotate counterclockwise. Gear 1 meshes with Gear 2 and causes Shaft 2 to rotate clockwise and exert the output force $\boldsymbol{F}_{\text{out}}$ on an output device (not shown). Each gear is mounted on a shaft of 5-mm radius supported by journal bearings, as shown. The gear train is rotating at a constant rate. The gear diameters are $r_1 = 25$ mm and $r_2 = 35$ mm.

(a) Determine the ratio of output M_{out} to input M_{in} moments in terms of the diameters of various gears and shafts when the journal bearings are frictionless. This ratio is the mechanical advantage of this gear train. Also find the ratio for the particular case when $r_1 = 25$ mm and $r_2 = 35$ mm.

(b) Find the forces acting on the four journal bearings in terms of F_{in}.

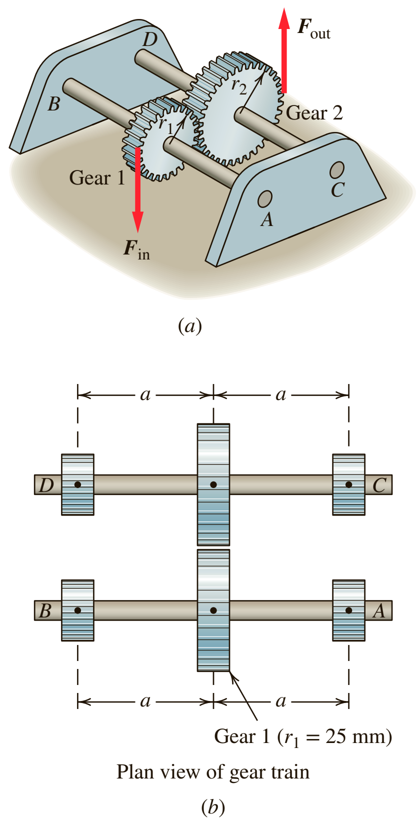


Figure 1 (*a*) A single-stage gear train (*b*) plan view of gear train.

Goal Find a relationship between the output and input moments when the gear train is in equilibrium. Based on **Figure 1**, the requested ratio is $M_{\text{out}}/M_{\text{in}} = r_2F_{\text{out}}/r_1F_{\text{in}}$. In addition, find the loads at the journal bearings at A, B, C, and D in terms of F_{in}.

Given The geometry of the gear train and that the bearings are frictionless. The input is via Gear 1, and the output is via Gear 2.

Assume The weight of various members that make up the gear train can be ignored and the journal bearings are frictionless (we will check this assumption in Example 9.2.4). Because there must be a boundary connection to prevent the shafts from sliding in the x direction, and we

[2]Gear trains are used in rotating systems to either increase or decrease moment or to decrease or increase rotational speed. In an automotive transmission, which is an example of a gear train, the moment from the engine (the input into the transmission) is increased and the rotational speed is decreased relative to the drive shaft (the output from the transmission). Often the input and output moments acting around a shaft are referred to as "torques."

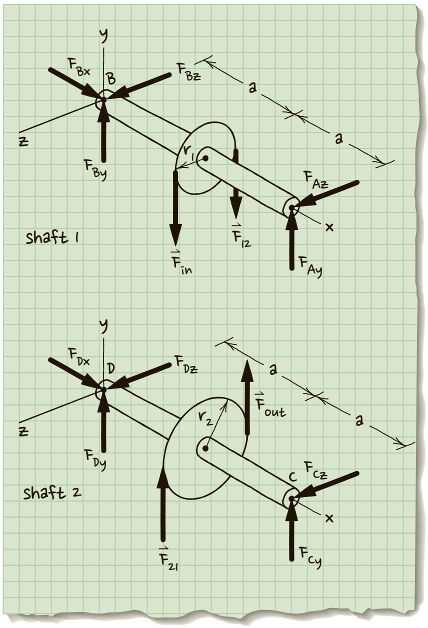


Figure 2 Free-body diagrams of Shaft 1 and Shaft 2.

are not told whether any of the bearings acts as a thrust bearing (**Table 4.2**), we arbitrarily assume that bearings B and D act as thrust bearings (and will check this assumption in our analysis).

Draw We draw free-body diagrams for Shaft 1 and Shaft 2 in **Figure 2**. Notice that a third-law force pair acts between Gears 1 and 2.

Formulate Equations and Solve From the free-body diagram of Shaft 1 we can write:

$$M_{\text{in}} = (r_1)(F_{\text{in}})$$

$$\sum M_x(\curvearrowleft +) = M_{\text{in}} - (r_1)(F_{12}) = 0$$

$$F_{12} = \frac{M_{\text{in}}}{r_1}$$

Similarly, from the free-body diagram of Shaft 2 we can write:

$$M_{\text{out}} = (r_2)(F_{\text{out}})$$

$$\sum M_x(\curvearrowleft +) = M_{\text{out}} - (r_2)(F_{21}) = 0$$

$$F_{21} = \frac{M_{\text{out}}}{r_2}$$

From Newton's third law, we know that $F_{12} = F_{21}$. Therefore, we can write:

$$M_{\text{out}} = r_2 \frac{M_{\text{in}}}{r_1}$$

Substituting in for $r_1 = 25$ mm and $r_2 = 35$ mm, we find that the requested ratio of $M_{\text{out}}/M_{\text{in}}$ is 1.4.

The mechanical advantage of the gear train is 1.4

Now we go about finding the forces acting at the bearings at A, B, C, and D, as requested in part **(b)** of this example.

We find the forces at A and B based on the free-body diagram for Shaft 1 by writing:

$$\sum F_x(\rightarrow +) = 0 \quad \Rightarrow \quad \boxed{F_{Ax} = 0} \tag{1}$$

$$\sum M_{x@B}(\curvearrowleft +) = F_{\text{in}} r_1 - F_{12} r_1 = 0 \tag{2}$$

Therefore, $F_{\text{in}} = F_{12}$ (and we will use this finding in writing the next batch of equilibrium equations).

$$\sum F_y(\uparrow +) = -2F_{\text{in}} + F_{Ay} + F_{By} = 0 \tag{3}$$

$$\sum F_z = F_{Az} + F_{Bz} = 0 \tag{4}$$

$$\sum M_{z@B}(\curvearrowleft +) = -2F_{\text{in}} a + F_{Ay}(a + a) = 0$$

From this set of equations, we find that:

$$F_{Ay} = F_{By} = F_{\text{in}} \quad \text{and} \quad F_{Ax} = F_{Az} = F_{Bz} = 0$$

We find the forces at C and D based on the free-body diagram for Shaft 2 by writing:

$$\sum M_{x@D}(\circlearrowleft +) = F_{\text{out}} r_1 - F_{21} r_1 = 0 \qquad (5)$$

Therefore, $F_{\text{out}} = F_{21}$. We also know from part (a), that $F_{21} = F_{12}$, therefore, $F_{\text{out}} = F_{\text{in}}$.

$$\sum F_x(\rightarrow +) = 0 \quad \Rightarrow \quad \boxed{F_{Cx} = 0} \qquad (1)$$

$$\sum F_y(\uparrow +) = 2F_{\text{in}} + F_{Cy} + F_{Dy} = 0 \qquad (2)$$

$$\sum F_z = F_{Cz} + F_{Dz} = 0 \qquad (3)$$

$$\sum M_{z@D}(\circlearrowleft +) = 2F_{\text{int}}a + F_{Cy}(a + a) = 0$$

From this set of equations, we find that:

$$\boxed{F_{Cy} = F_{Dy} = -F_{\text{in}} \quad \text{and} \quad F_{Cx} = F_{Cz} = F_{Dz} = 0}$$

Check The general law for two (frictionless) gears is that the moment multiplies by the ratio of the gear diameters and the speed is decreased by the same ratio. This is what we found in our analysis!

EXAMPLE 9.2.4

Reconsider the single-stage gear train shown in **Figure 1** and described in Example 9.2.3 when the coefficient of kinetic friction associated with the journal bearings is 0.25. If $r_1 = 25$ mm, $r_2 = 35$ mm, and the diameter of the shafts is 5 mm, determine the mechanical advantage of the gear train. Compare its efficiency to the case when the bearings are frictionless.

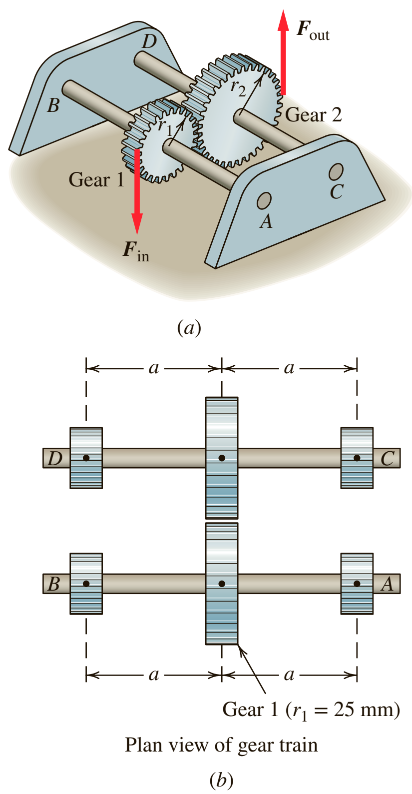


Figure 1 (*a*) A single-stage gear train (*b*) plan view of gear train.

Goal Determine the mechanical advantage of the gear train in **Figure 1**.

Given The same situation as in Example 9.2.3, except that kinetic friction of $\mu_k = 0.25$ acts on the 5-mm radii shafts in the journal bearings.

Assume Components are massless. Assume points B and D act as thrust bearings.

Draw **Figure 2** shows the free-body diagrams of each shaft and gear.

Formulate Equations and Solve:

The following forces are zero by inspection: F_{Az}, F_{Bx}, F_{Bz}, F_{Cz}, F_{Dx}, F_{Dz}.

From the free-body diagram of Shaft 1, symmetry, and Equation (7.18) that describes journal bearing friction:

$$\boldsymbol{M}_{\text{in}} = (25\text{ mm})(F_{\text{in}})\,\boldsymbol{i}$$

$$\boldsymbol{M}_{Ax} = -F_{A\,\text{friction}}\, r_1\,\boldsymbol{i} = -(0.25)(5\text{ mm})(F_{Ay})\,\boldsymbol{i}$$

$$\boldsymbol{M}_{Bx} = -F_{B\,\text{friction}}\, r_2\,\boldsymbol{i} = -(0.25)(5\text{ mm})(F_{By})\,\boldsymbol{i}$$

$$\sum F_{y(\uparrow+)} = F_{Ay} + F_{By} - F_{\text{in}} - F_{12} = 0 \Rightarrow F_{Ay} = F_{By} = \frac{(F_{\text{in}} + F_{12})}{2}$$

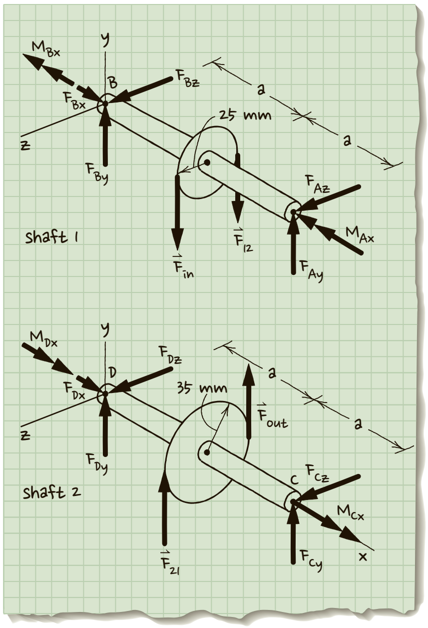


Figure 2 Free-body diagrams of the two shafts, including the effect of bearing friction.

$$\sum M_x(\curvearrowleft +) = (25\text{ mm})(F_{\text{in}}) - (25\text{ mm})(F_{12}) - (0.25)(5\text{ mm})(F_{Ay}) - (0.25)(5\text{ mm})(F_{By}) = 0$$

$$\Rightarrow F_{\text{in}} = \frac{26.25}{23.75}F_{12}$$

From the free-body diagram of Shaft 2 and symmetry:

$$\boldsymbol{M}_{\text{out}} = (35\text{ mm})(F_{\text{out}})\,\boldsymbol{i}$$

$$\boldsymbol{M}_{Cx} = (0.25)(5\text{ mm})(F_{Cy})\,\boldsymbol{i}$$

$$\boldsymbol{M}_{Dx} = (0.25)(5\text{ mm})(F_{Dy})\,\boldsymbol{i}$$

$$\sum F_{y(\uparrow +)} = 0 = F_{Cy} + F_{Dy} + F_{\text{out}} + F_{21} \rightarrow F_{Cy} = F_{Dy} = \frac{-(F_{\text{out}} + F_{21})}{2}$$

$$\sum M_x(\curvearrowleft +) = (35\text{ mm})(F_{\text{out}}) - (35\text{ mm})(F_{21}) + (0.25)(5\text{ mm})(F_{Cy}) + (0.25)(5\text{ mm})(F_{Dy}) = 0$$

$$\Rightarrow F_{\text{out}} = \frac{36.25}{33.75}F_{21}$$

From Newton's third law, we know that $F_{12} = F_{21}$.

$$\frac{M_{\text{out}}}{M_{\text{in}}} = \frac{35\text{ mm }F_{\text{out}}}{25\text{ mm }F_{\text{in}}} = \frac{(35\text{ mm})\dfrac{36.25}{33.75}F_{21}}{(25\text{ mm})\dfrac{26.25}{23.75}F_{12}} = 1.36$$

The mechanical advantage of the gear train is 1.36

The efficiency of the gear train with bearing friction, relative to one without bearing friction is defined as:

$$\text{Gear Train Efficiency} = 100 \times [M_{\text{out}}/M_{\text{in}}]_{w/\text{friction}}/[M_{\text{out}}/M_{\text{in}}]_{wo/\text{friction}}$$

Substituting in values found for the specific case of $r_1 = 25$ mm, $r_2 = 35$ mm, we find

$$\text{Gear Train Efficiency} = 100 \times (1.36/1.40) = 97.1 \text{ percent}$$

We interpret this to mean the gear train with friction is 97.1 percent as effective in converting the input moment into output moment; alternately we could say that there is a 2.9 percent "loss" in this conversion because of the presence of friction. What might this "lost" moment be converted into?

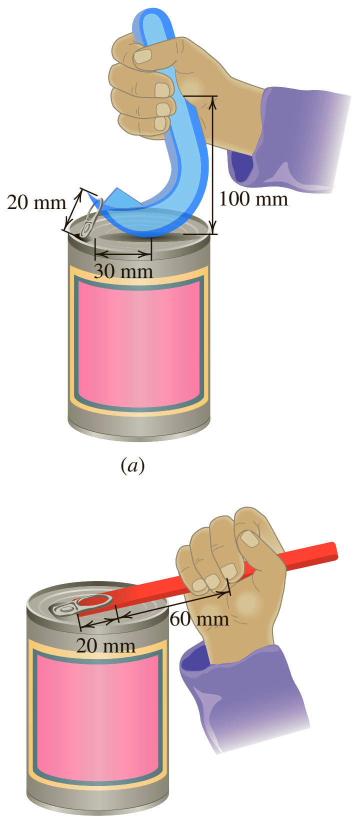
Check The friction on each shaft reduced the ratio by approximately the same amount. Check the moment balance for each shaft using alternate moment centers to ensure equilibrium.

EXERCISES 9.2

9.2.1. **[*]** Consider the two designs of tab-pullers shown in (*a*) and (*b*).

a. For each design, estimate its mechanical advantage. Approximate dimensions are provided to help with your estimates. Assume the angle between the top of the can and the tab is between 45 and 60 degrees. How do these numbers compare with using no device to open the tab? Make sure to include free-body diagrams in developing your answers.

b. Which device (if any) would you prefer to use and why?

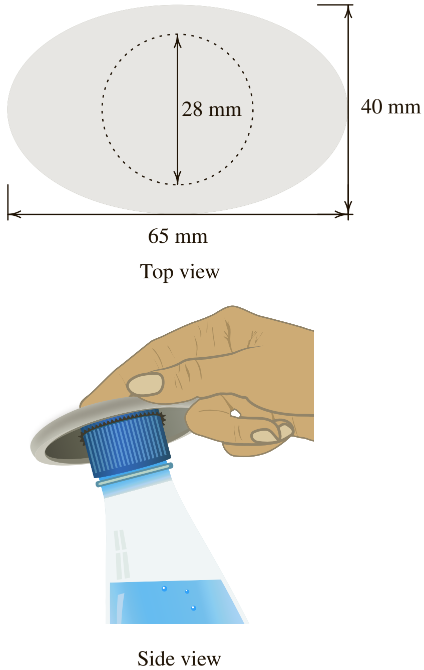


(*a*)

(*b*)

EX 9.2.1

9.2.2. **[*]** Consider the design of a twist-cap opener that fits over a bottle cap as shown. A typical bottle cap has a 28 mm diameter. What is your estimate of the mechanical advantage of this device? How does this compare to not using any device to open a cap? Make sure to include free-body diagrams in developing your answer.

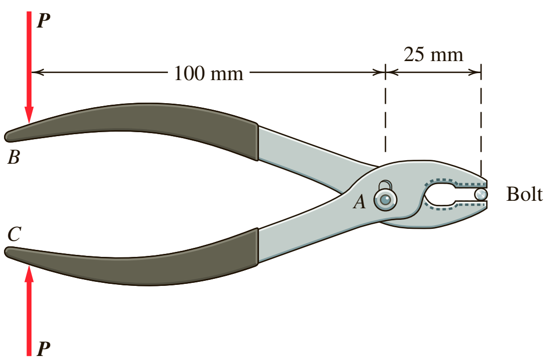


EX 9.2.2

9.2.3. **[*]** A pair of pliers grips the bolt as shown.

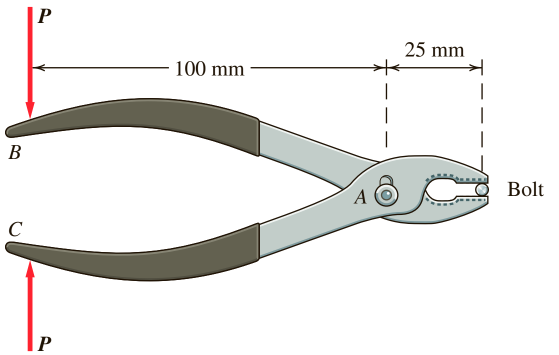
a. For the input force ***P***, determine the force exerted on the bolt.

b. Determine the mechanical advantage of this machine.



EX 9.2.3

9.2.4. **[*]** A pair of pliers grips the bolt as shown. Determine the shear force acting on the pin at *A*.

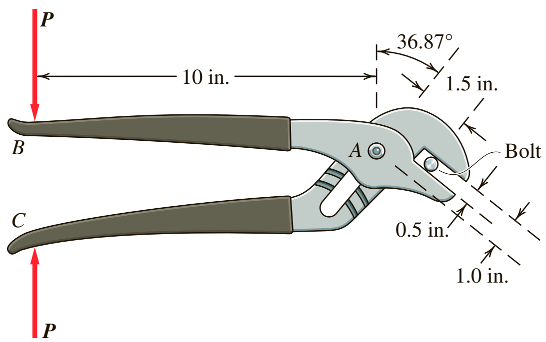


EX 9.2.4

9.2.5. [*] A pair of pliers grips the bolt as shown.

a. For the input force ***P***, determine the force exerted on the bolt.

b. Determine the mechanical advantage of this machine.



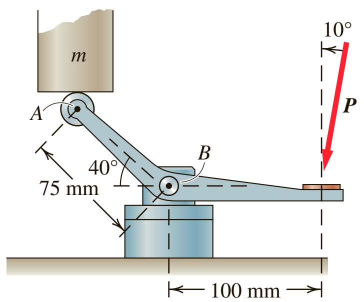
EX 9.2.5

9.2.6. [*] The foot-operated pedal raises a shaft of mass m = 100 kg that opens a trash can lid. For the position shown, determine

a. the magnitude of the necessary force ***P***

b. the mechanical advantage provided by the machine

c. the shear force acting on the pin at B.

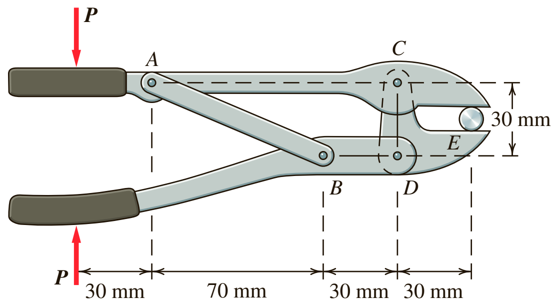


EX 9.2.6

9.2.7. [**] A pair of pliers grips the bolt as shown.

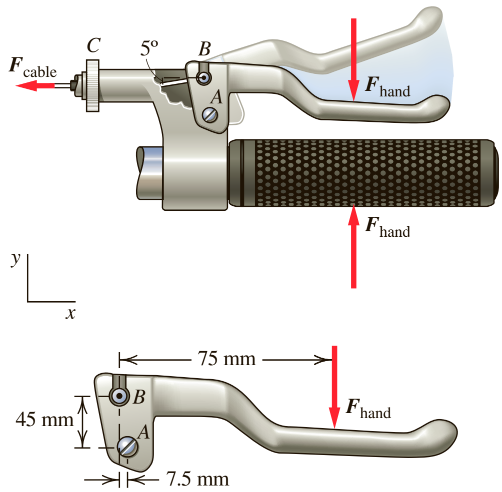
a. For the input force ***P***, determine the force exerted on the bolt.

b. Determine the mechanical advantage of this machine.



EX 9.2.7

9.2.8. [**] When a bicyclist applies a 60-N force to the bicycle brake hand lever, the lever compresses, which pulls on the cable. Determine the cable force and the mechanical advantage of the lever-cable system.

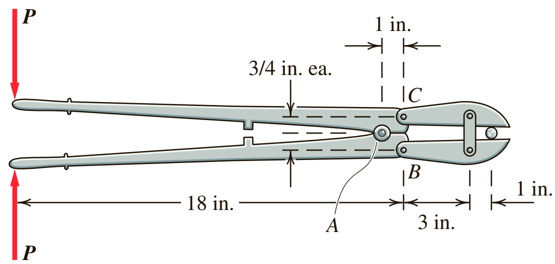


EX 9.2.8

9.2.9. [**] A pair of bolt cutters grips the bolt as shown.

a. For the input force ***P***, determine the force exerted on the bolt.

b. Determine the mechanical advantage of this machine.

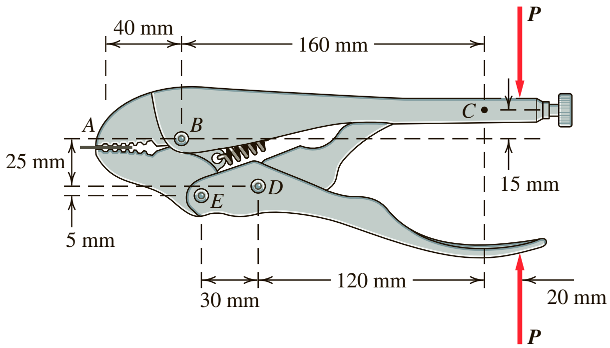


EX 9.2.9

9.2.10. [**] A pair of vise grips is used to grab a metal tab.

a. For the input force $\boldsymbol{P}$, determine the force exerted on the tab.

b. Determine the mechanical advantage of this machine.

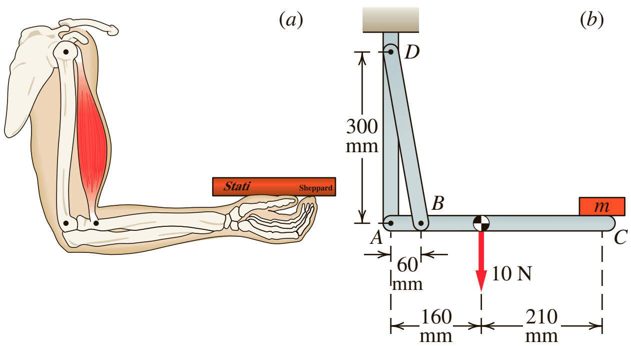


EX 9.2.10

9.2.11. [**] The bones and biceps muscle that make up a human arm are depicted in **Figure *a***. The arm is supporting a 2-kg mass in the horizontal position shown. The arm can be modeled as a frame, as in **Figure *b***, with the biceps muscle being represented as a two-force member (BD). If the forearm weighs 10 N, determine

a. the magnitude of the tension in the biceps muscle

b. the loads acting on the elbow joint at A

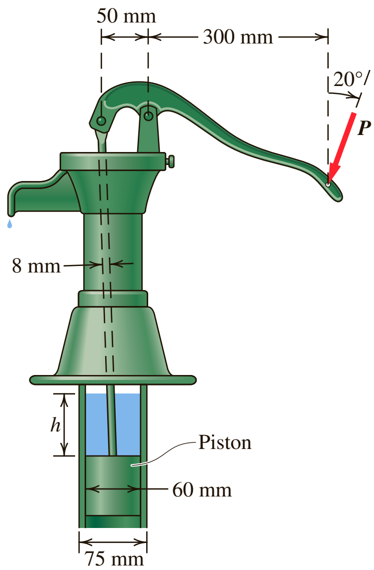


EX 9.2.11

9.2.12. [**] The hand-operated water pump is used to raise a column of water out of the ground. In the position shown, the operator applies a force $\boldsymbol{P}$ of 160 N.

a. Determine the mechanical advantage of this machine.

b. Determine the height h of the column of fresh water that can be raised.

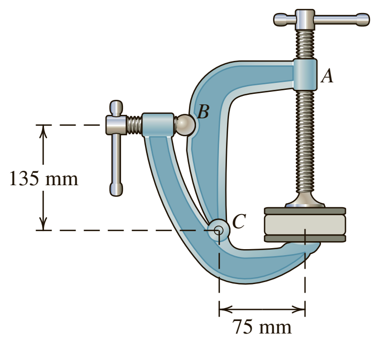


EX 9.2.12

9.2.13. [**] The dual grip C clamp shown uses two screws to supply the total clamping force. First the pieces are clamped with force P_{clamp} by tightening the screw at A. Then the screw at B is tightened against arm ABC to provide additional clamping force (ΔP_{clamp}). Determine

a. the force that acts on the arm ABC at B as a function of ΔP_{clamp}

b. the shear force acting on the pin at C due to ΔP_{clamp}



EX 9.2.13

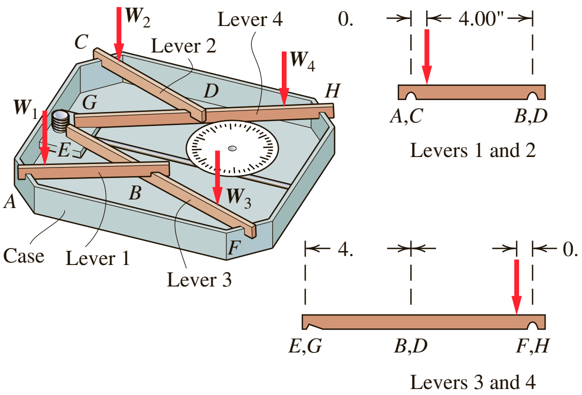
9.2.14. [***] The inside of an inexpensive bathroom scale consists of a series of levers. The platform (not shown in figure) distributes the total weight of the person being weighed as point loads W_1, W_2, W_3, and W_4.

a. Determine the vertical forces acting upward on the ends of levers 3 and 4 at E and G, respectively.

b. Determine the vertical forces acting upward on the levers at A, C, F, and H.

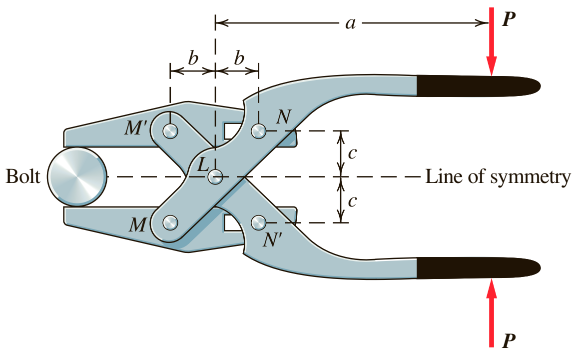
c. Based on **a** or **b**, determine what percentage of the total weight W of the person actuates the scale mechanism that levers 3 and 4 rest on at E and G.

d. Does the reading of the scale depend on where the person being weighed stands on the platform? (You should be able to use your answers from above to reason out an answer.)



EX 9.2.14

9.2.15. [***] For the grippers shown, find the magnitude of the gripping force F_{grip} as a function of the magnitude of the input force P and the geometric parameters given. Assume that the pins at N and N' slide freely in the slots. Express your answer in terms of mechanical advantage.



EX 9.2.15

9.2.16. [***] For the single-stage gear train shown, each gear is mounted on a 5-mm radius shaft that is supported by frictionless journal bearings. The gear train is rotating at a constant rate. Refer to Example 9.2.3.

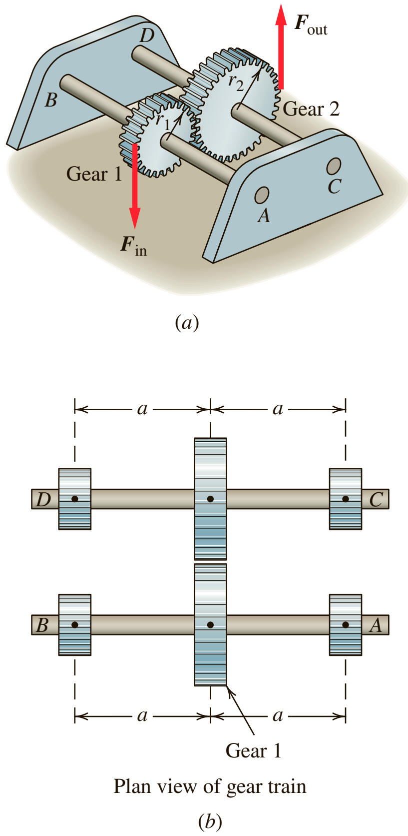
a. Confirm that the moment couples created by forces acting at the journal bearings balance M_{out} and M_{input}.

b. Choose a set of gears from Table 9.2.16 that would result in a mechanical advantage (the ratio of M_{out} and M_{input}) equal to 1.25.

c. At a gear ratio of 1.25, is the output shaft moving faster, slower, or at the same speed as the input shaft?

Table 9.2.16

G_1 radius (mm)	G_2 radius (mm)
20	25
25	30
25	35
30	35
35	25
40	20



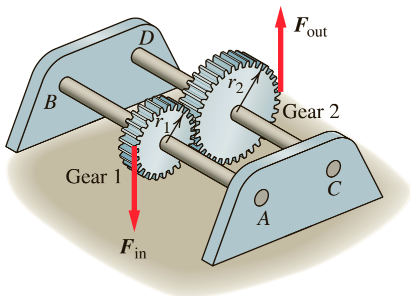
EX 9.2.16

9.2.17. [***] For the single-stage gear train shown, each gear is mounted on a 5-mm radius shaft that is supported by frictionless journal bearings. The gear train is rotating at a constant rate. Choose a set of gears from Table 9.2.17

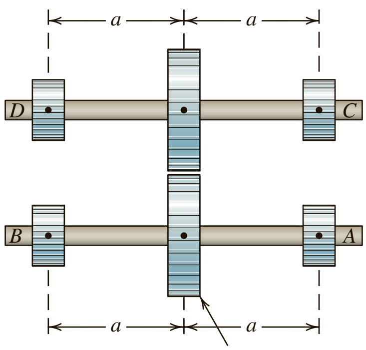
that will make the mechanical advantage of the gear train equal to 1.40. Also, at the gear ratio of 1.40 how many times faster or slower is the output shaft rotating than the input shaft?

Table 9.2.17

G_1 radius (mm)	G_2 radius (mm)
20	25
25	30
25	35
30	35
35	25
40	20



(a)



Plan view of gear train

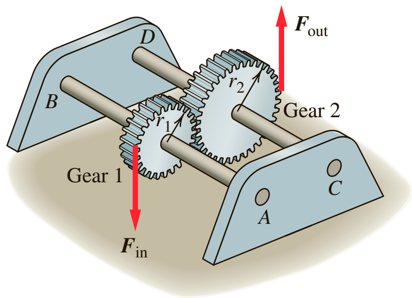
(b)

EX 9.2.17

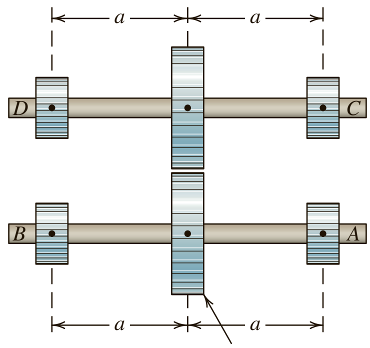
9.2.18. [***] For the single-stage gear train shown, each gear is mounted on a 5-mm radius shaft that is supported by frictionless journal bearings. The gear train is rotating at a constant rate. Choose a set of gears from **Table 9.2.18** that will make the mechanical advantage of the gear train equal to 0.50. Also, at the gear ratio of 0.50 how many times faster or slower is the output shaft rotating than the input shaft?

Table 9.2.18

G_1 radius (mm)	G_2 radius (mm)
20	25
25	30
25	35
30	35
35	25
40	20



(a)



Plan view of gear train

(b)

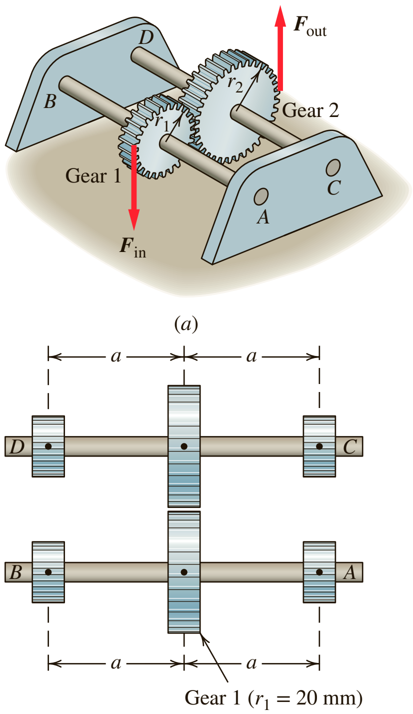
EX 9.2.18

9.2.19. [***] For the single-stage gear train shown, each gear is mounted on a 5-mm radius shaft that is supported by journal bearings with a coefficient of kinetic friction of 0.25. The gear train is rotating at a constant rate.

a. Determine the ratio of output M_{out} to input M_{input} in terms of the diameters of the gears and shafts

b. With $r_1 = 20$ mm and $r_2 = 25$ mm, calculate the mechanical advantage of this gear train, where mechanical advantage is defined as M_{out} to input M_{input}.

c. Calculate the efficiency is this gear train with friction, relative to the same gear train without friction.



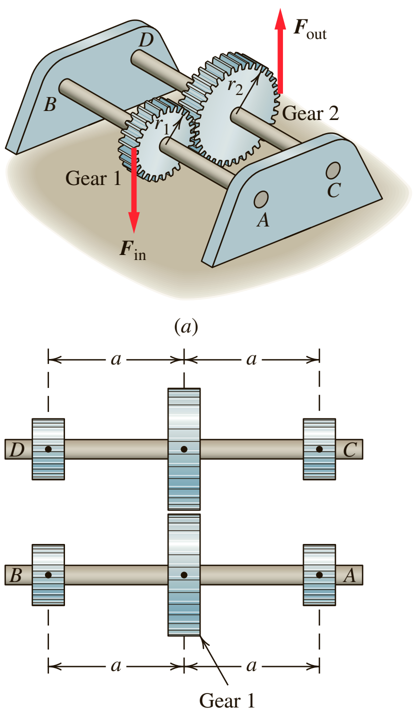
(a)

Plan view of gear train

(b)

EX 9.2.19

9.2.20. **[***]** For the single-stage gear train shown, each gear is mounted on a 5-mm radius shaft that is supported by journal bearings. Compare the bearing forces at A, B, C, and D for the gear train shown with and without bearing friction.



(a)

Plan view of gear train

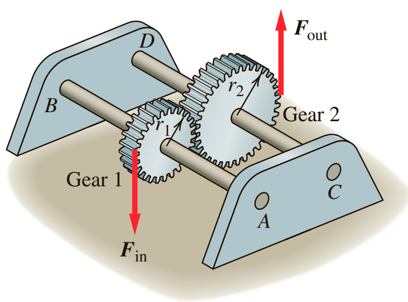
(b)

EX 9.2.20

9.2.21. **[***]** For the single-stage gear train shown, each gear is mounted on a 5-mm radius shaft that is supported by journal bearings with a coefficient of friction of 0.25. The gear train is rotating at a constant rate. Calculate the efficiency of each of the gear sets listed in **Table 9.2.21** with bearing friction. Present your answer in the form of a table.

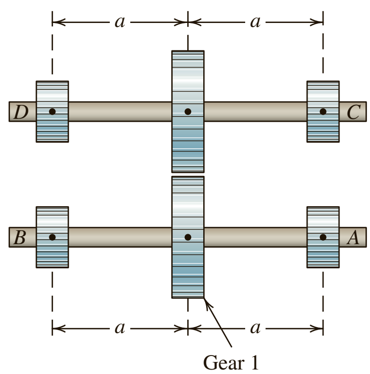
Table 9.2.21

G_1 radius (mm)	G_2 radius (mm)
20	25
25	30
25	35
30	35
35	25
40	20



(a)

EX 9.2.21(a)



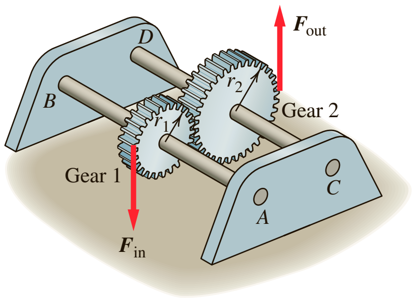
Plan view of gear train

(b)

EX 9.2.21(b)

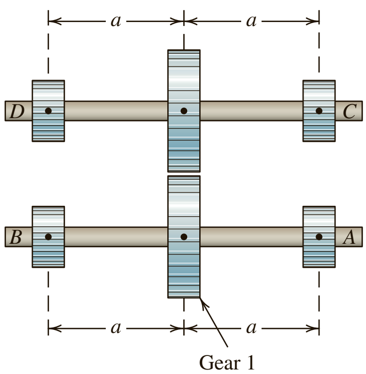
9.2.22. **[***]** For the single-stage gear train shown, each gear is mounted on a 5-mm radius shaft that is supported by journal bearings with friction. The gear train is rotating at a constant rate. With $r_1 = 20$ mm and $r_2 = 25$ mm,

calculate the maximum allowable coefficient of friction that results in an efficiency of at least 98 percent.



(*a*)

EX 9.2.22(a)

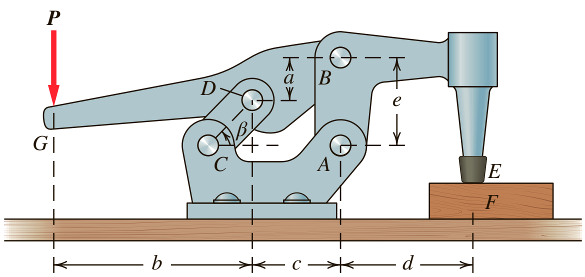


Plan view of gear train

(*b*)

EX 9.2.22(b)

9.2.23. [***] Consider the toggle clamp shown. For the dimensions in **Table 9.2.23** determine four possible toggle clamp geometries that result in a vertical clamping force at E that is 10 times greater than input force $\boldsymbol{P}$.



EX 9.2.23

Table 9.2.23

Dimensions of toggle
$c = 1.5$ in.
$d = 4$ in.
$e = 4$ in.
β is such that $\cos\beta = 4/5$, $\sin\beta = 3/5$
Due to size restrictions:
1 in. $\leq a \leq$ 2 in.
4 in. $\leq b \leq$ 7 in.

9.2.24. [***] For the bicycle shown, derive an expression that relates the magnitude of $\boldsymbol{F}_{\text{foot}}$ to the magnitude of $\boldsymbol{F}_{\text{friction}}$, where $\boldsymbol{F}_{\text{foot}}$ is the force of a foot pressing down on the pedal and $\boldsymbol{F}_{\text{friction}}$ is the friction (or traction) force pushing forward on the rear wheel. Your final solution should be expressed in terms of $N_{\text{chain ring}}$, $N_{\text{rear cog}}$, L_{crank}, and $R_{\text{rear wheel}}$ (where these variables are defined the figure).

Bicycle specifications:

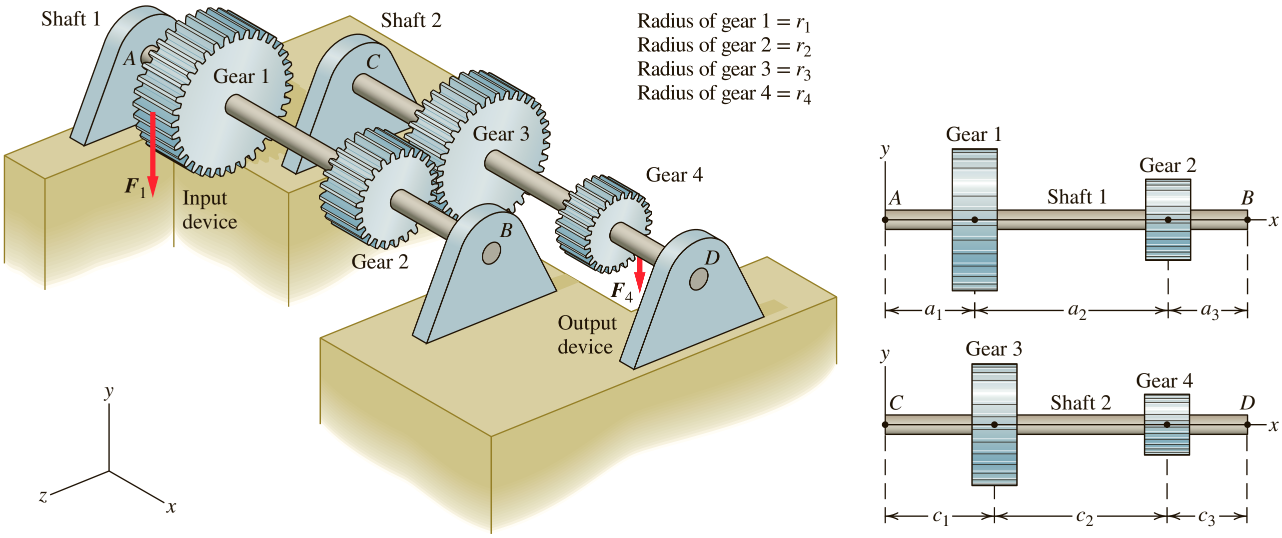
L_{crank} (length of crank) = 17.8 cm
$R_{\text{rear wheel}}$ (radius of rear wheel) = 34.3 cm
$R_{\text{chain ring}}$ (radius of chain ring)
$R_{\text{rear cog}}$ (radius of rear cog)
$N_{\text{chain ring}}$ (number of teeth on chain ring) = 50 teeth
$N_{\text{rear cog}}$ (number of teeth on rear cog) = 14 teeth
C_D (coefficient of drag) = 0.9
A_{frontal} (frontal area of cyclist + bicycle) = 0.50 m^2
W_{total} (weight of cyclist + bicycle) = 860 N

EX 9.2.24

9.2.25. **[***]** For the gear train shown the input force $\boldsymbol{F}_1$ causes Shaft 1 to rotate counterclockwise. Gear 2 meshes with Gear 3 and causes Shaft 2 to rotate clockwise and exert the output force $\boldsymbol{F}_4$ on an output device (not shown). Assume frictionless bearings.

a. Determine the forces applied to the shafts at journal bearing A, B, C, and D in terms of the input force $\boldsymbol{F}_1$.

b. For the shafts rotating at constant speed, determine the mechanical advantage in terms of the radii of various gears.

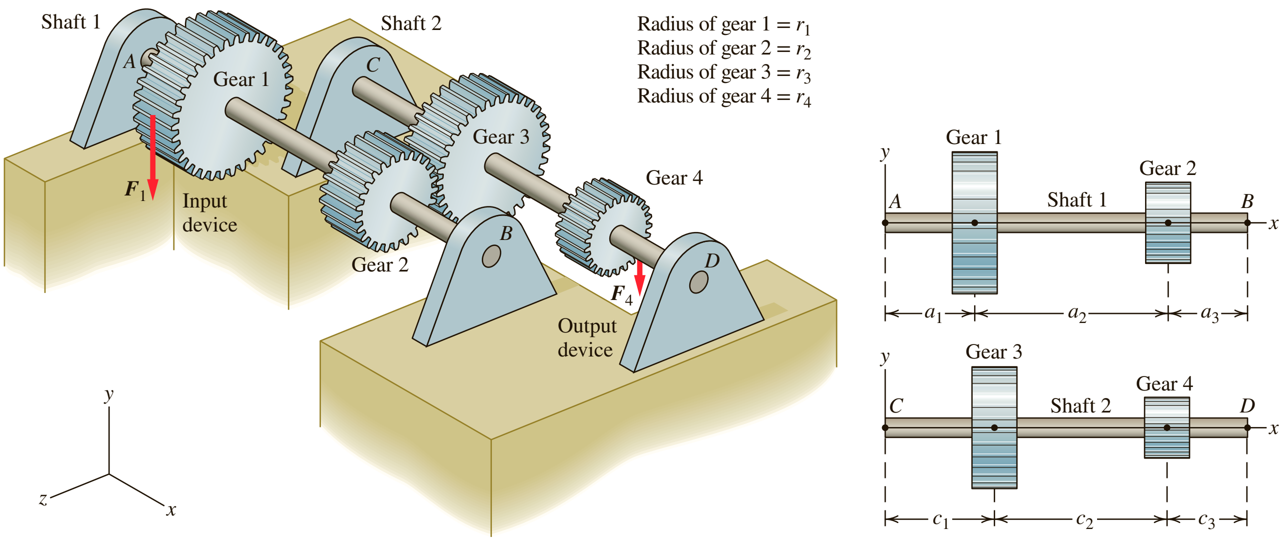


EX 9.2.25

9.2.26. **[***]** For the gear train shown the input force $\boldsymbol{F}_1$ causes Shaft 1 to rotate counterclockwise. Gear 2 meshes with Gear 3 and causes Shaft 2 to rotate clockwise and exert the output force $\boldsymbol{F}_4$ on an output device (not shown).

a. Calculate the mechanical advantage of this gear train in terms of the radii of various gears, and include the effect of friction in the journal bearings. The coefficient of kinetic friction is μ_k.

b. What is the efficiency of this gear train?



EX 9.2.26

9.3 DETERMINACY AND STABILITY IN FRAMES

Learning Objective: Determine whether a frame is statically determinate or indeterminate, or unstable.

In Chapter 8 we noted the general guideline for a nonplanar system to be statically determinate is to have no more than six unknown loads acting on it. Similarly, a planar system with no more than three unknowns is a good candidate for being statically determinate. But this guideline is not hard-and-fast; note that Examples 9.1.3 and 9.1.4 violate it as there are four unknown loads acting on the planar frame as a whole. Because the frame in these two examples is internally unstable (meaning that the frame's members are not rigidly connected to one another) an extra boundary support is needed to hold the frame fixed. We were able to look at parts of the system and generate additional linearly independent equations with which to find the unknown loads.

Like the trusses studied in Chapter 8 and structural systems more generally (Section 5.7), frames can be classified as statically determinate, statically indeterminate, or unstable.

- A frame is statically determinate if the equilibrium equations are sufficient to solve for all unknown member and support loads.
- A frame is statically indeterminate if it has *more* support loads and member loads than there are linearly independent equilibrium equations. In this case the deformation of frame members must be included in the equilibrium analysis; doing so generates additional equations for finding the unknown loads. Statically indeterminate frames are also called frames with redundancy, as they have more supports and/or members than are needed to be stable.
- An unstable frame has *fewer* support and member loads than there are equilibrium equations. Either the frame has too few members, causing it to be internally unstable or too few supports and thus is underconstrained. In either case, the frame is not in equilibrium and will move or deform excessively.

Determining that a frame is statically determinate is a good idea before attempting to complete an equilibrium analysis. If the frame is statically indeterminate, concepts from both static equilibrium *and* mechanics of materials will need to be applied. If it is unstable, the analysis cannot be done using static equilibrium conditions.

We can methodically determine whether a frame is determinate, indeterminate or unstable, as outlined in Table 9.2. Like the procedure that was outlined in **Tables 8.1** and **8.2** for determining the status of a truss, we compare the number of unique loads with the number of linearly independent equilibrium equations. Unlike the procedure for trusses (where all the members were two-force members), the procedure for frames must consider multiforce, two-force, and pinned members so the

Table 9.2 Analyzing the Determinacy and Stability of a Frame

Step		
1.	Mentally break the frame into its component parts. Include multiforce and two-force members, and pin connections if an external force acts on the pin.	
2.	Select one of the parts. Draw its free-body diagram. Show all the loads acting on it from supports, other members, and external loads.	Based on the free-body diagram, count the number of unique loads acting on the part; call this number $LOAD_{\text{unknowns}}$. Unique loads are ones that have not already been counted on another free-body diagram. Count the number of equilibrium equations associated with the member. This will be 6 for a nonplanar member, 3 for a planar member, 2 for a member that is a pin, and 1 for a member that is a two-force member. Call this number EQN.
3.	Repeat Step 2 for all other members.	
4.	Add up the number of $LOAD_{\text{unknowns}}$ and EQN: $\sum_{i=1}^{\text{number of parts}} LOAD_{\text{unknowns}} = [\text{Unique Loads}]$ $\sum_{i=1}^{\text{number of parts}} EQN = [\text{Linearly Independent Equations}]$	IF: [Unique Loads] = [Linearly Independent Equations], the frame meets the necessary condition to be labeled as statically determinate. Frames that are determinate obey this equality; however, not all frames that obey this equality are determinate. As such, the equality is a *necessary but not sufficient* condition. [Unique Loads] > [Linearly Independent Equations]; the frame meets the condition to be labeled as statically indeterminate. [Unique Loads] < [Linearly Independent Equations]; the frame meets the necessary (but not sufficient) condition to be labeled as underconstrained or internally unstable.

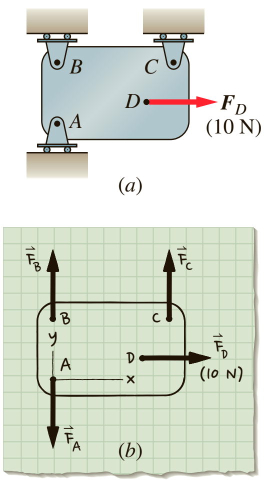
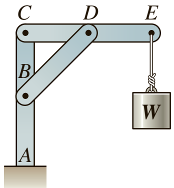


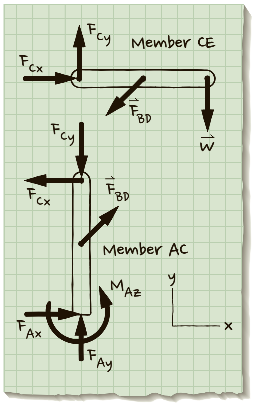
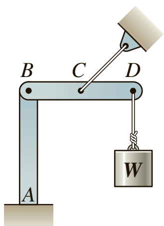
Figure 9.3.1 Frame meets the necessary but not the sufficient conditions to be statically determinate.

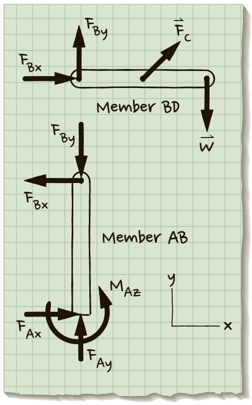
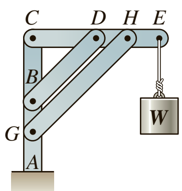
process is a little more complicated. **Table 9.3** provides some example applications of the process.

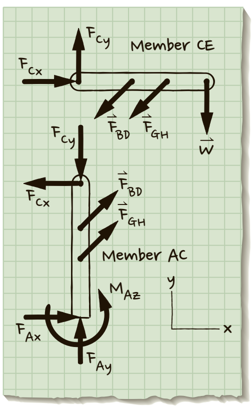
If the idea of conditions for determinacy being "necessary by not sufficient" seems confusing, consider member ABC (and its associated free-body diagram) shown in **Figure 9.3.1**. For this member, [Unique Loads] = 3 (F_A, F_B, and F_C), and [Linearly Independent Equations] = 3. Therefore, [Unique Loads] = [Linearly Independent Equations]. But, an inspection of its free-body diagram shows that member ABC will slide to the right due to the horizontal load $\boldsymbol{F_D}$ acting on it; it is in fact unstable. Even though this member meets the necessary condition for static determinacy, the supports are not sufficient for it to be stable (and determinate).

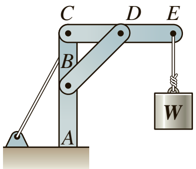
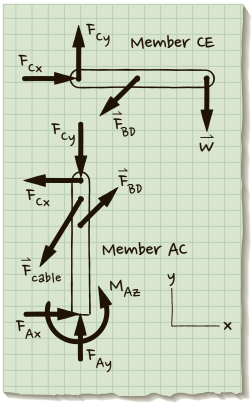
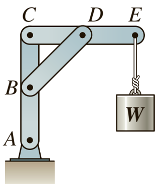
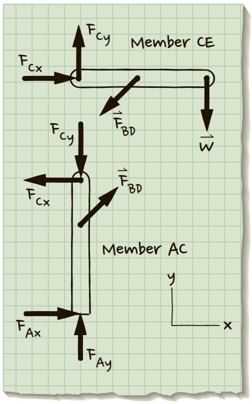
Table 9.3 Examples of Carrying Out the Determinacy and Stability Check

<table>
<tr><th>Case</th><th>Unique Loads</th><th>Linearly Independent Equations</th><th>Status:</th></tr>
<tr><td>Case A



</td><td>6</td><td>6</td><td>6 = 6, so the system meets the necessary condition for static determinacy</td></tr>
<tr><td>Case B



</td><td>6</td><td>6</td><td>6 = 6, so the system meets the necessary condition for static determinacy</td></tr>
<tr><td>Case C



</td><td>7</td><td>6</td><td>7 > 6, so the system meets the necessary condition for static indeterminacy. Notice that either member BD or member GH could be removed, and the frame would still be stable (and it would be determinate).</td></tr>
</table>

Case		Unique Loads	Linearly Independent Equations	Status:
Case D 		7	6	7 > 6, so the system meets the necessary condition for static indeterminacy. Notice that the cable could be removed, and the frame would still be stable (and it would be determinate).
Case E 		5	6	5 < 6, so the frame is underconstrained and will rotate about the pin at *A*.

Check out the following example of an application of this material.

- **Example 9.3.1 Determining Status of a Frame**

EXAMPLE 9.3.1

For the frame configurations in **Figure 1(a)** and **Figure 2** determine which of these labels apply to each frame: internally unstable, statically determinate, statically indeterminate, or underconstrained.

Frame 1 (Figure 1*a*): The frame is internally unstable because the force ***P*** causes members to rotate, as shown by the ghosted image. A frame in

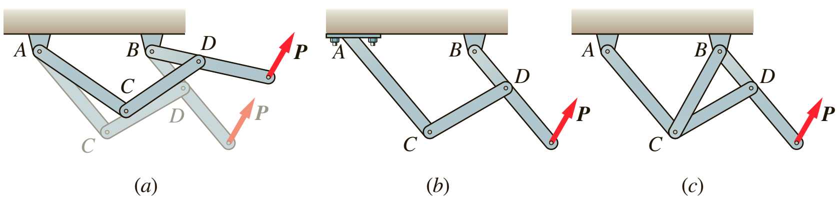


Figure 1 Frame 1 (*a*) is unstable. By changing support conditions (*b*) or adding a member (*c*), it can be made stable.

which members intentionally move is called a **mechanism**. An unstable frame with insufficient boundary conditions to prevent movement is also *underconstrained*. More members and/or boundary conditions are needed for this frame to be in equilibrium. For example, by changing the pin connection at A to a fixed support, as in **Figure 1*b***, we now have a stable frame that is statically determinate. Alternatively, by adding a diagonal member, as in **Figure 1*c***, we have created a stable frame that is statically determinate.

Frame 2 (Figure 2): The three members are sufficient to form a rigid frame, and removal of a member would make the frame internally unstable. The system has nine unknown loads (F_{Ax}, F_{Ay}, F_{Bx}, F_{By}, F_{Cx}, F_{Cy}, F_{Dx}, F_{Dy}, F_{Ey}) and nine linearly independent equilibrium equations can be written, making it *statically determinate*.

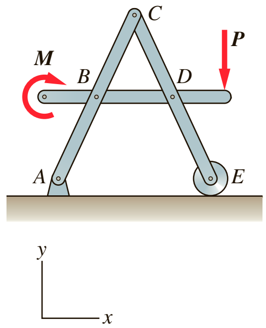
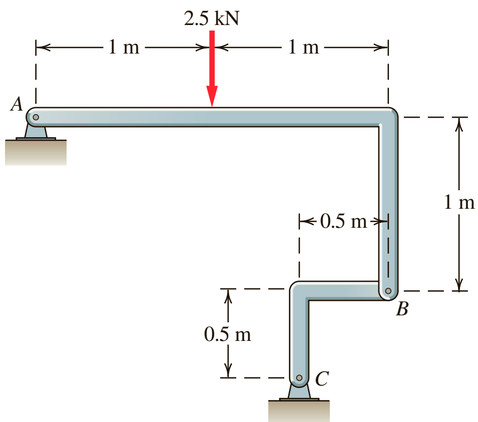


Figure 2 Frame 2 is stable and statically determinate.

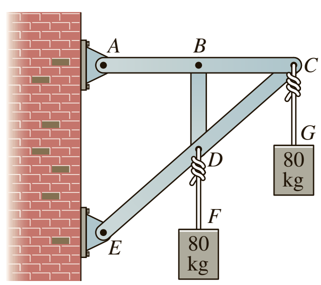
EXERCISES 9.3

9.3.1. [*] For the frame shown, determine which of these labels apply: statically determinate, statically indeterminate, or unstable.



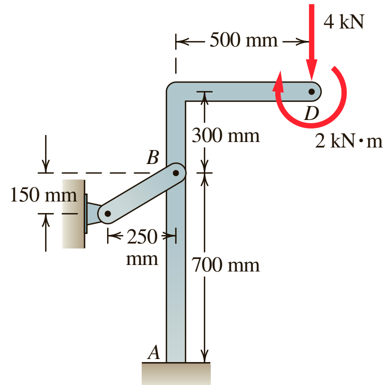
EX 9.3.1

9.3.2. [*] For the frame shown, determine which of these labels apply: statically determinate, statically indeterminate, or unstable.



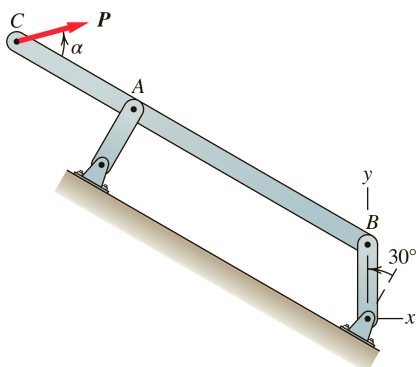
EX 9.3.2

9.3.3. [*] For the frame shown, determine which of these labels apply: statically determinate, statically indeterminate, or unstable.



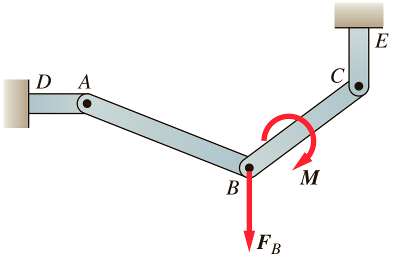
EX 9.3.3

9.3.4. [*] Explain why the frame shown is unstable. Suggest three different design changes that would make the frame stable.



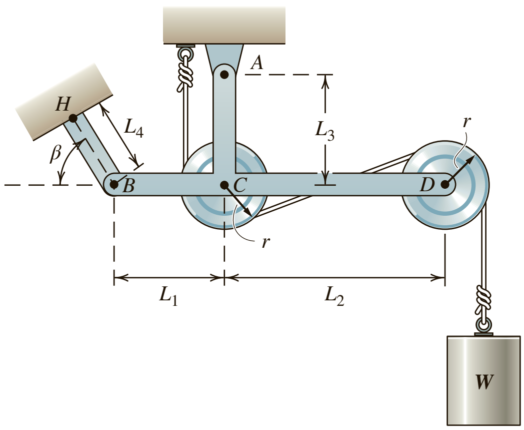
EX 9.3.4

9.3.5. [*] Determine which of the labels describe the frame shown: statically determinate, statically indeterminate, or unstable.



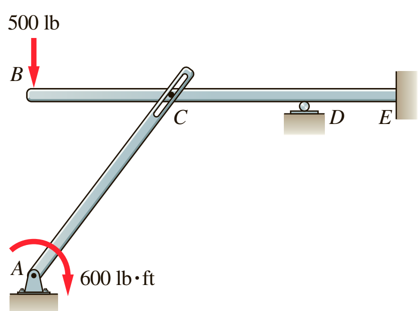
EX 9.3.5

9.3.6. [*] Determine which of the labels describe the frame shown: statically determinate, statically indeterminate, or unstable. Assume that the pulleys are frictionless.



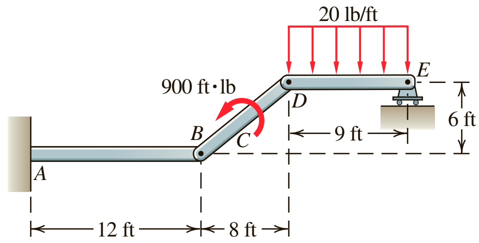
EX 9.3.6

9.3.7. [*] A frame is constructed from two members connected by a frictionless slot at C. Determine which of the labels describe the frame: statically determinate, statically indeterminate, or unstable.



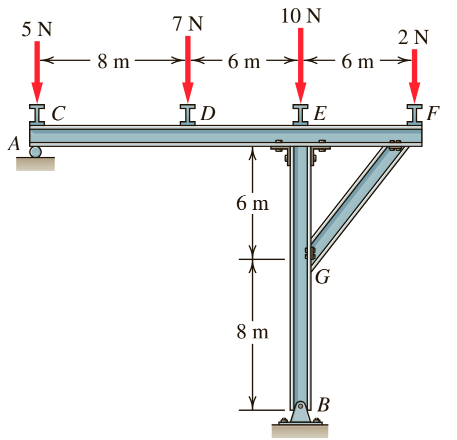
EX 9.3.7

9.3.8. [*] Determine which of the labels describe the frame shown: statically determinate, statically indeterminate, or unstable.



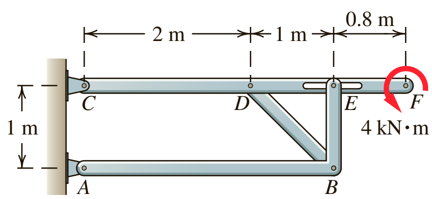
EX 9.3.8

9.3.9. [*] Determine which of the labels describe the frame shown: statically determinate, statically indeterminate, or unstable. Treat the connections at E, F, and G as pinned joints.



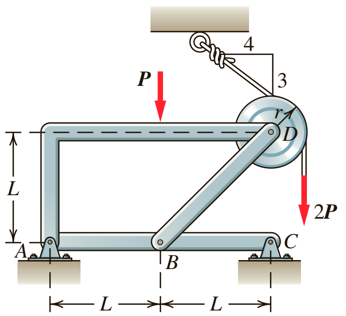
EX 9.3.9

9.3.10. [*] Determine which of the labels describe the frame shown: statically determinate, statically indeterminate, or unstable. Assume the slot at E is frictionless.



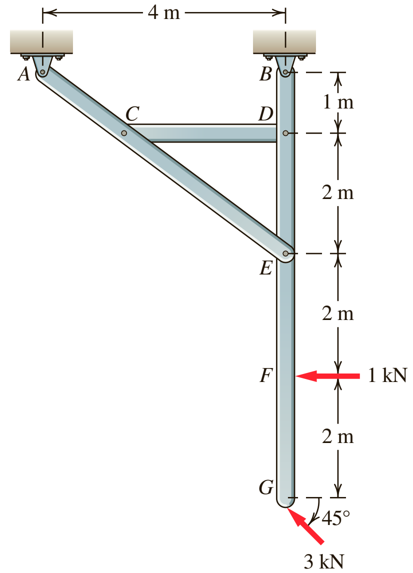
EX 9.3.10

9.3.11. [*] A load of $2P$ hangs from a frictionless pulley attached to the frame at D. Determine which of the labels describe the frame shown: statically determinate, statically indeterminate, or unstable.



EX 9.3.11

9.3.12. [*] Determine which of the labels describe the frame shown: statically determinate, statically indeterminate, or unstable.



EX 9.3.12

9.4 JUST THE FACTS

Defining and Analyzing Frames

A **frame** is a system designed to support loads, both forces and moments. It may be made up of a few members or many, but at least one member must be a **multiforce member**, which is any member that is not a two-force member. In addition, frames are classified as being, statically determinate, statically indeterminate, underconstrained or internally unstable and as **planar** or **nonplanar**.

Determinacy and Stability in Frames

The analysis of frames to find the loads acting on their boundaries and internal to the frame (between members) consists of making assumptions, drawing free-body diagrams, setting up and solving equilibrium equations, and summarizing results. This is the same analysis procedure we have developed in the preceding chapters, and it works for frames that are statically determinate. For a frame that is statically indeterminate (meaning there are more members and/or supports than there are equations of equilibrium), we must use concepts from mechanics of materials in addition to equilibrium conditions.

Defining and Analyzing Machines

A **machine** is a system designed to change the direction and/or magnitude of loads or motion. In considering machines, we often think in terms of the load *into* the system, the load *out of* the system, and the ratio of output to input. The basic analysis of a machine in equilibrium is identical to that of a frame in terms of analyzing separate members by drawing free-body diagrams, setting up the equilibrium equations, and then using these equations to find the unknown loads. Often the analysis of a machine also involves the calculation of the ratio of output load to

input load (referred to as the **mechanical advantage**) or output motion to input motion.

When machines contain shafts that are held by dry or only partially lubricated journal bearings, the analysis may need to account for the presence of **journal bearing friction**. For a shaft of radius r, the friction creates a moment about a moment center at the center of the shaft of

$$M_o = r \cdot \mu F \tag{7.18}$$

in the direction *opposite* the rotation of the shaft. The force F in (7.18) is the radial force being carried by the bearing. For a shaft that is stationary relative to a journal bearing, the coefficient of static friction (μ_s) should be used in (7.18). If the shaft is rotating relative to the journal bearing, the coefficient of kinetic friction (μ_k) should be used in (7.18).

The analysis of machines that contain wheels may need to account for the presence of **rolling resistance**. Rolling resistance is caused by deformation of the rolling wheel and/or the surface on which it rolls and results in an additional force P being required to keep the wheel rolling at a constant speed. The additional force is

$$P = (a/r)L \tag{7.20}$$

where L is the downward force acting on the rolling wheel. The ratio of (a/r) is referred to as the **coefficient of rolling friction** (f_r).

When realities such as bearing friction or rolling resistance are present in a machine's operation, it is less than 100 percent efficient. **Mechanical efficiency** is the ratio of the reduction in a machine's output-to-input capability because of these realities, relative to its output-to-input capability without these realities.

SYSTEM ANALYSIS (SA) EXERCISES

SA9.1 A Self-Erecting Basketball Goal for the Reynolds Coliseum

There are two sets of portable basketball goals used in Reynolds Coliseum. One type has to be erected by hand (**Figure SA9.1.1**) and the other self-erects using a hydraulic cylinder.

The basketball team found out that your class is studying the design of mechanical systems and asks for your help. They are tired of having to erect the large goal by hand and wonder whether you could develop a better solution. **Figure SA9.1.2** shows the goal that they would like to have modified so it would erect itself with the assistance of a single hydraulic cylinder.

(a) Draw to scale the geometrical envelope for the entire deployment process of the goal shown in **Figure SA9.1.2**. Even better, use popsicle sticks and brads (serving as pins) to build a scale model.

(b) Now comes the critical decision—which structural element should be replaced with a hydraulic cylinder?

A note about how hydraulic cylinders work: the motion distance for erecting the goal is at most 80% of the length of the cylinder (see **Figure SA9.1.3*b***). Thus, the more stroke length you require, the longer the retracted cylinder has to be. (*Be aware that you need to have enough space to store the cylinder in the base of the goal.*)

(c) Finally, you need to make sure that the cylinder generates enough force. One way to do this is to find the position of your mechanism that requires the most force. You also need to assume the loads that impact your "machine." Assume that the backboard structure weighs 0.40 kN, the metal cantilever beam 0.55 kN, and the two vertical supports 0.15 kN each. At this time it should be possible to calculate the required *diameter of the hydraulic cylinder/piston* assuming that the hydraulic pump will provide a maximum of 2000 psi or 14 mPa. Don't forget to include the 100 psi back-pressure that is needed to keep the piston from a sudden jerk. Is the diameter you calculate the maximum or minimum that is required? (Include the reason behind your answer.)

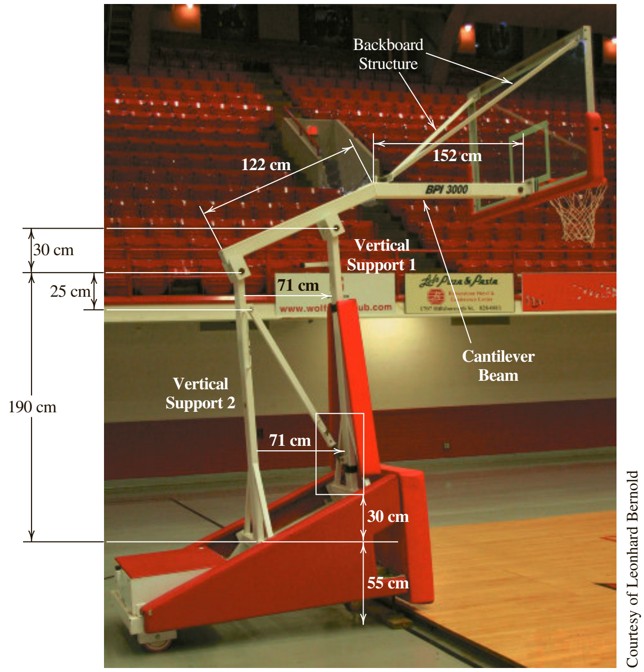


Figure SA9.1.2 Cumbersome basketball goal.

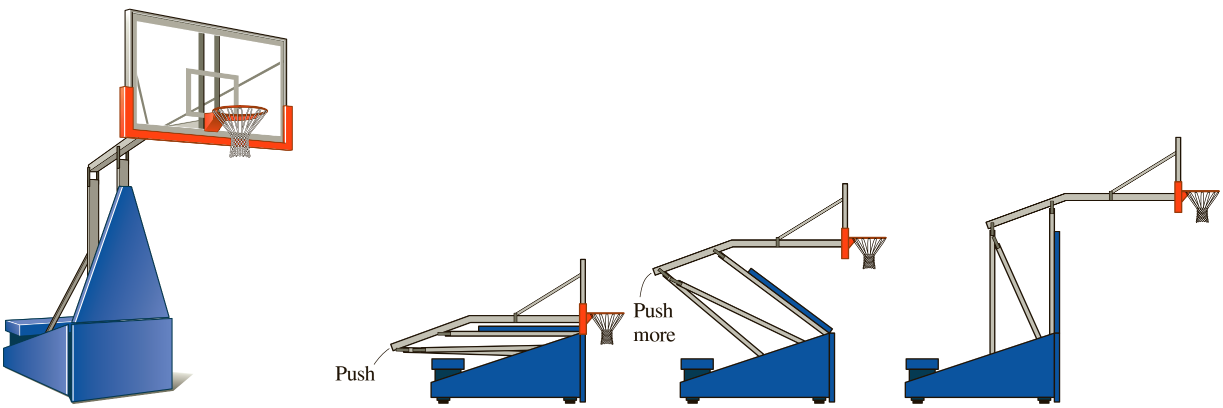


Figure SA9.1.1 Raising a basketball goal from a portable platform by hand.

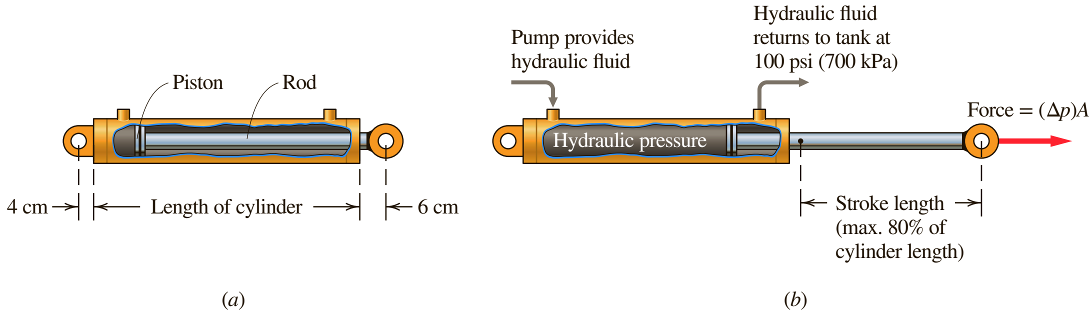


Figure SA9.1.3 Basics of force from a hydraulic cylinder: (*a*) totally retracted cylinder; (*b*) pressure difference creating a force at the end of rod.

SA9.2 Analysis of Bicycle Performance[3]

In this problem you will be considering shifting on a 3-speed bicycle. More specifically, you will consider what shifting speeds would be good and why there are multiple gears on a bicycle. To this end, consider the bicycle depicted in **Figure SA9.2.1**, with supporting date in **Table SA9.2.1**

(a) Derive an expression that relates the magnitude of $\boldsymbol{F}_{\text{friction}}$ to the velocity of the bicycle for the case where there is rolling resistance. Clearly state your source of data on the coefficient of rolling friction, along with all of your assumptions. Plot this expression. (This plot should look similar to the plot of $\boldsymbol{F}_{\text{friction}}$ versus velocity in **Figure SA9.2.2**, except that your plot includes rolling resistance.)

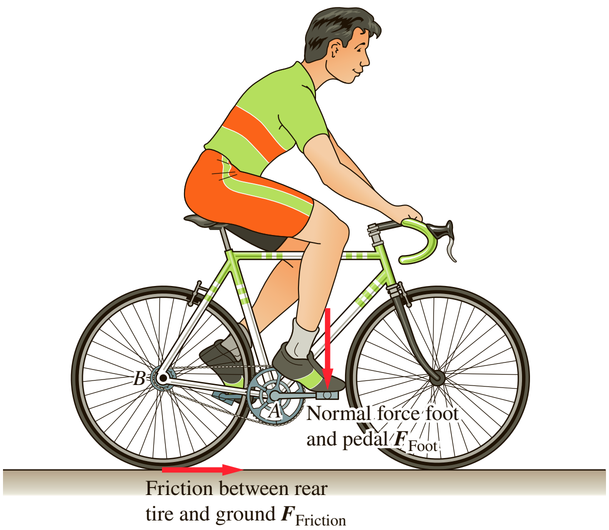


Figure SA9.2.1 Basics of bicycle under consideration.

Table SA9.2.1

Bicycle specifications
L_{crank} (length of crank) = 17.8 cm
$R_{\text{rear wheel}}$ (radius of rear wheel) = 34.3 cm
$N_{\text{rear cog}}$ (number of teeth on rear cog) = 14 teeth, 20 teeth, 28 teeth
$N_{\text{chain ring}}$ (number of teeth on chain ring) = 50 teeth
C_D (coefficient of drag) = 0.9
A_{Frontal} (frontal area of cyclist + bicycle) = 0.05 m^2
W_{Total} (weight of bicycle)

(b) For the bicycle in first gear, derive an expression that relates the magnitude of $\boldsymbol{F}_{\text{foot}}$ to the velocity of the bicycle. Add to the plot from (a).

(c) For the bicycle in second gear, derive an expression that relates the magnitude of $\boldsymbol{F}_{\text{foot}}$ to the velocity of the bicycle. Add to the plot from (a).

(d) For the bicycle in third gear, derive an expression that relates the magnitude of $\boldsymbol{F}_{\text{foot}}$ to the velocity of the bicycle. Add to the plot from (a). After completing part (d), you should have a plot that looks something like **Figure SA9.2.2**.

(e) Assuming that a cyclist in reasonable shape pedals at a rate of 80–110 rpm (revolutions per minute), at what bicycle velocities would you recommend that the cyclist shift from first to second, and from second to third gear? Mark these shifting velocities on the plot that you created above; this is your recommended shifting pattern. In addition, check that the foot forces

[3]Suggested material: Chapter 2 of *Bicycling Science*, 3rd edition, by D. G. Wilson (MIT Press, April 1, 2004).

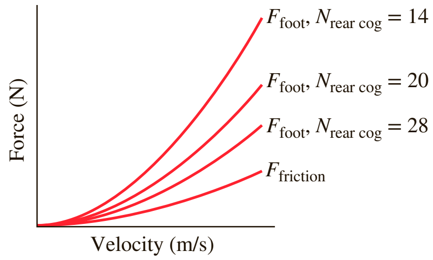


Figure SA9.2.2 Input force (F_{foot}) vs. velocity, for various rear cogs.

required by the cyclist for your recommended shifting pattern are reasonable.

(f) Using the plot you created as a "teaching prompt" (along with any figures that may be useful), write a description of why there are multiple gears on a bicycle. Your targeted audience for this description is a middle-school student.

SA9.3 Review of Appendix D, The Bicycle

In Appendix D a static analysis of bicycle performance was presented. More specifically, the question, "How fast can Merrill sprint toward the finish line?" was asked and was addressed by answering three interrelated sub-questions:

- What is the maximum force that Merrill can apply to the pedal?
- How is this force related to the friction force between the rear tire and the ground?
- How does the friction force relate to the drag force on the bicycle?

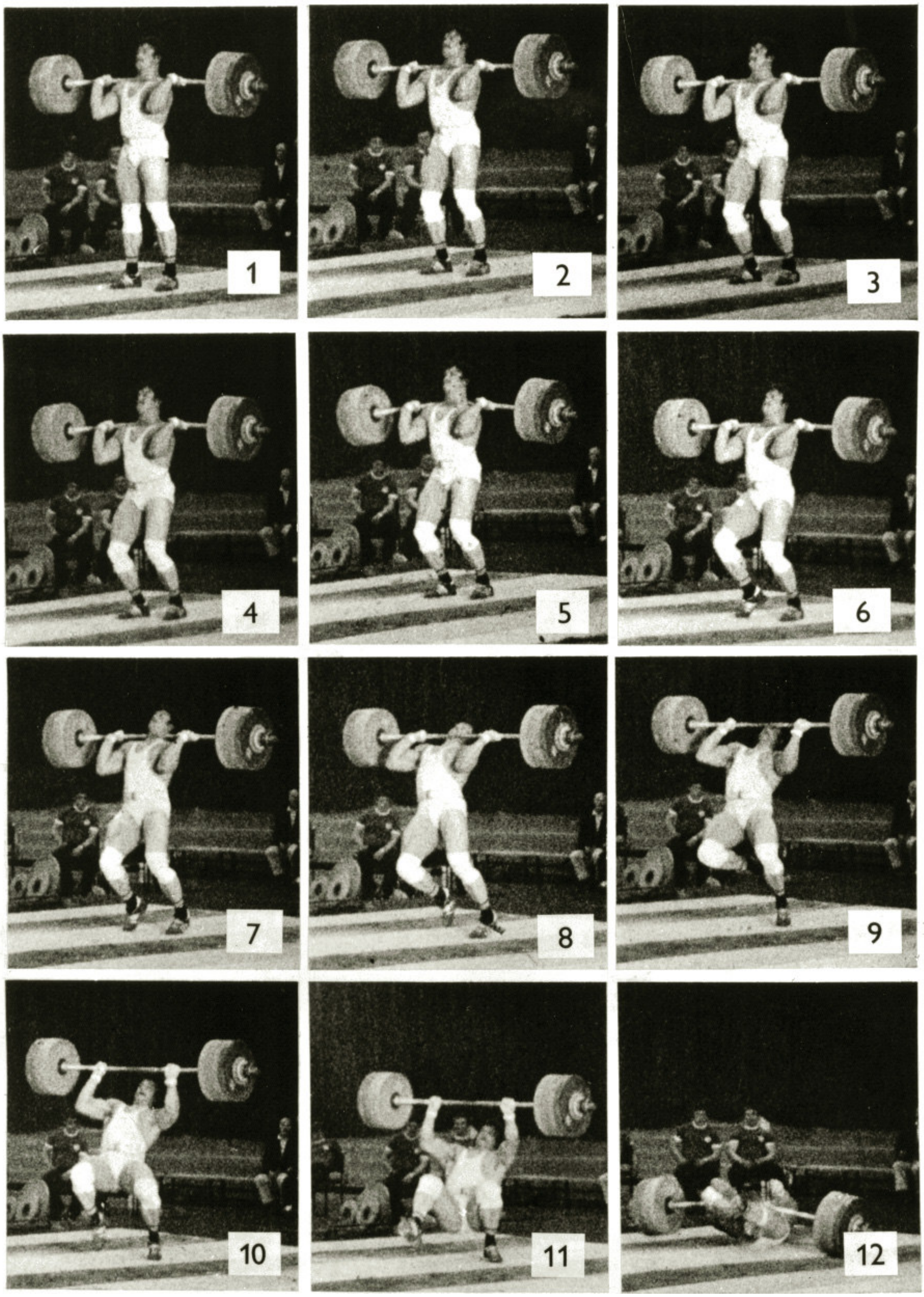
(a) Review the analysis presented in Appendix D. For the bicycle considered (with specifications given in **Figure D.2.1**), calculate the rate at which the cyclist would need to be pedaling when traveling at the maximum speed. Express your answer in revolutions per minute (rpm).

(b) If your answer in (a) is greater than 110 rpm (which it should be), it is not likely that Merrill will be able to achieve the maximum speed. This is because the ratio of $N_{chain\ ring}/N_{rear\ cog} = 50/14$ results in Merrill having to pedal too fast. (In general, bicyclists prefer to pedal at 80–110 rpm.) Specify whether the ratio of $N_{chain\ ring}/N_{rear\ cog}$ would need to increase or decrease in order to reduce Merrill's pedaling speed when he is traveling at the maximum speed. Provide an explanation of your answer.

(c) Based on calculations, recommend a ratio of $N_{chain\ ring}/N_{rear\ cog}$ that is more appropriate than 50/14.

SA9.4 A Heavy Load

The weightlifter shown in **Figure SA9.4.1** was in the process of starting the second phase of a clean-and-jerk maneuver when his patellar tendon ruptured. The goal of this problem is to determine the forces exerted on the patellar tendon and develop some intuition for the forces that we subject our bodies to. In the interest of completeness, it should be noted that the patellar tendon is not actually a tendon. It connects bone (patella) to bone (tibia) and is, more properly speaking, a ligament. The strength of the patellar tendon may depend on a variety of factors including its size, training history, rate of loading (tendons, ligaments, and most other soft tissues stiffen and strengthen when they are loaded quickly), orientation (which is why an athlete's form is crucial), and the presence or absence of injury. Strength estimates for patellar tendons from non-athletes, loaded relatively slowly, top out at approximately 10,000 N (per leg).



Courtesy of Karel Jelen, Ftvs UK-Prague

Figure SA9.4.1 Initiation and progression of a patellar tendon rupture by weightlifter in Olympic competition.

(a) Let's begin by assuming that the weightlifter has a mass m_w = 100 kg and is lifting a mass of 175 kg. Calculate the upward force, G_Y, that the ground exerts on the athlete's feet.

(b) Calculate the average force in each patellar tendon as a function of the angle θ (as shown in **Figure SA9.4.2*a***). At what position is the force the greatest?

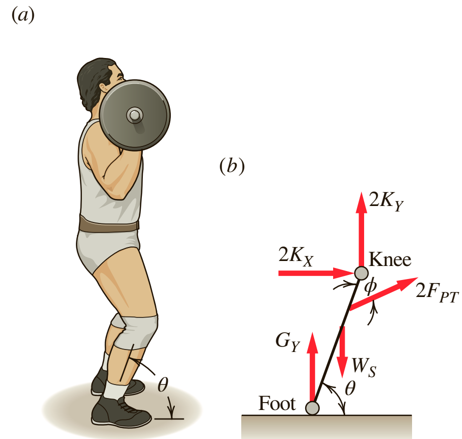


Figure SA9.4.2 (*a*) A schematic of a weightlifter just before he attempts to press the weight above his head. (*b*) A schematic of a weightlifter's lower legs just before he attempts to press the weight above his head. Note that we have included the reactions at both knees and both patellar tendons. Realistically, this problem requires a dynamic analysis, but we can learn a lot by starting with a static representation.

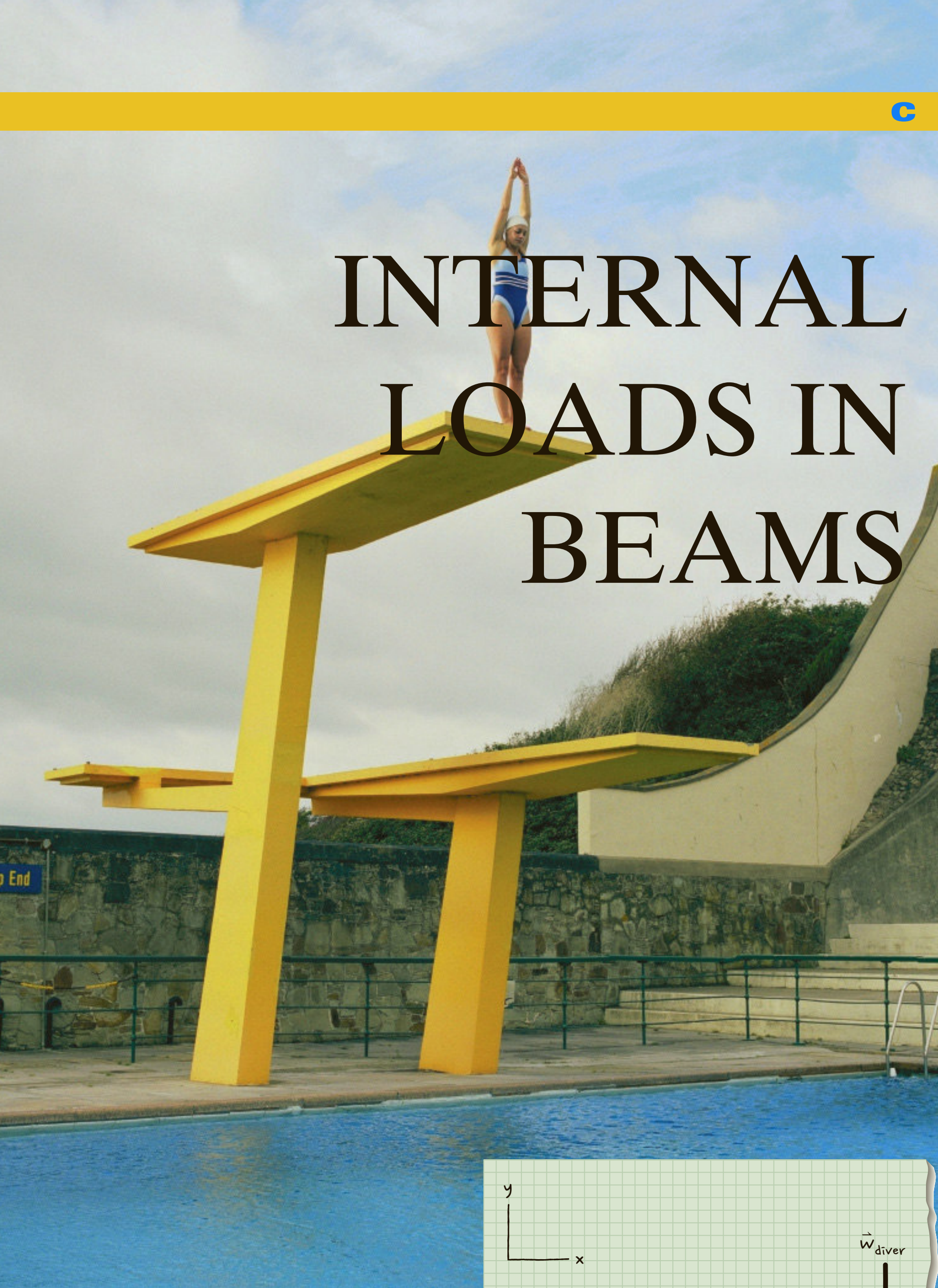
What happens if we change the angle θ (as shown in **Figure SA9.4.2*b***) at which the patellar tendon connects to the tibia by ±10%?

A few things you need to know and assume in order to get these calculations underway:

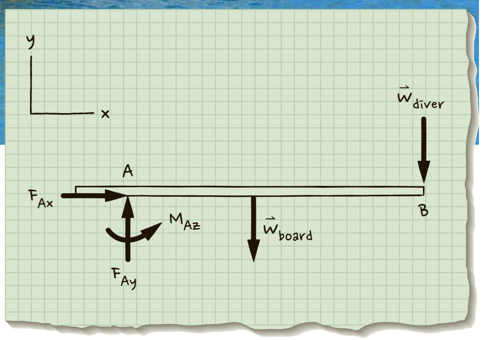
When combined, the lower legs and feet account for 13% of the weightlifter's total weight. We'll assume that this weight, W_s, acts in the middle of the lower leg (sometimes called the shank) as shown in the simplified model in **Figure SA9.4.2*b***. In addition, we'll assume that the shank has a total length, L_s, = 40 cm and that the patellar tendon attaches approximately 10 cm below the knee joint's center. To start the calculation, fix the angle between the patellar tendon's insertion point and the long axis of the tibia at $\theta = 15°$ and assume that it remains constant throughout the motion. In addition, allow the tibia angle θ, as shown in **Figure SA9.4.2**, to go from 45° to 90° at the completion of the lift.

(c) Using the 10,000 N patellar tendon number presented above, determine at what angle θ the weightlifter's patellar tendon fails? If we assume that, as a result of his training, his patellar tendon is much stronger than the average person's—perhaps on the order of 15,000 N—what is the factor of safety for this particular activity?

INTERNAL LOADS IN BEAMS



JFB/Getty Images, Inc.



If you look at the diving board shown you see that the diver's weight is applied at the end of the horizontal member *AB*—the diver is literally "out on a limb" (if we call the horizontal member the limb). This member is more commonly referred to as a beam. Beams are a fundamental component of structures and ubiquitous in the built environment.

Engineers are concerned with selecting beam sizes and materials that will ensure adequate structural performance. Part of this selection process involves calculating the loads internal to a beam. In this chapter we lay out how to systematically calculate the loads acting externally and internally on beams.

LEARNING OBJECTIVES

Upon completion of this chapter, you will be able to:

- **Recognize a beam and specific beam configurations.** (10.1)
- **Determine internal loads in a beam, as part of a comprehensive equilibrium analysis of the beam.** (10.2)
- **Present the internal loads determined from beam equilibrium analysis as a series of shear, bending moment, and axial force diagrams.** (10.3)
- **Relate the beam internal loads to one another.** (10.4)

10.1 DEFINING BEAMS AND RECOGNIZING BEAM CONFIGURATIONS

Learning Objective: Recognize a beam and specific beam configurations.

A member is called a **beam** if loads are applied perpendicular to its long axis. These loads, which may consist of forces and/or moments, are referred to as **lateral loads**. Because the loads are perpendicular to the long axis of the beam, they cause the beam to bend, as illustrated in Figure 10.1.1.

A beam is a particular type of multiforce member. The frame in Figure 10.1.2 contains two beams (CG and AE). The two-force member, BG, is not a beam because there are no loads acting perpendicular to its long axis.

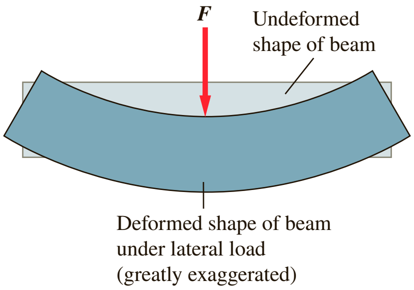


Figure 10.1.1 Loads applied to a beam cause it to bend.

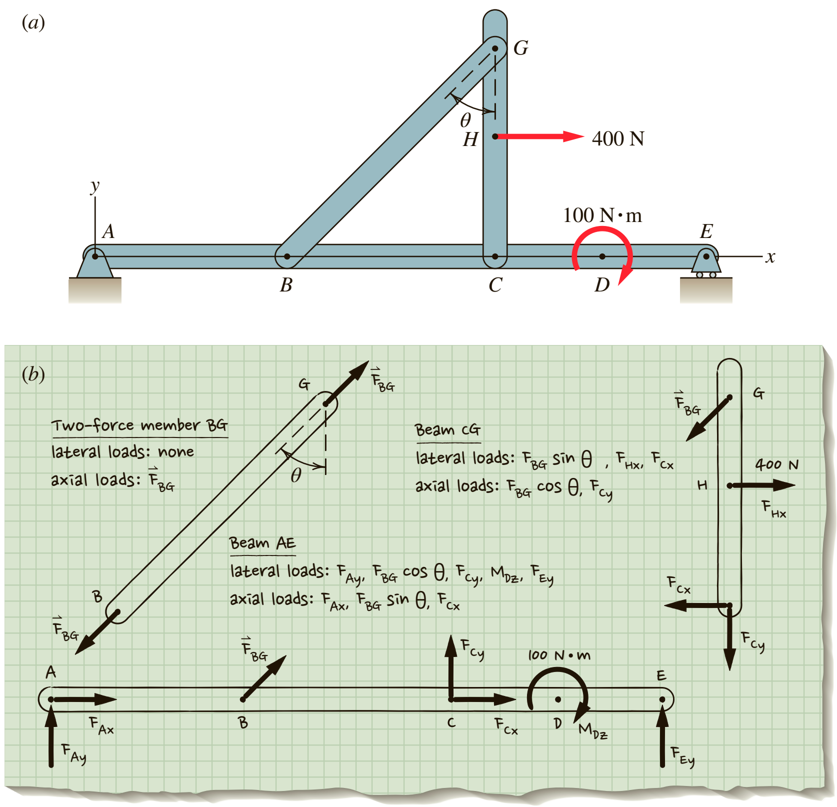


Figure 10.1.2 (a) A frame consisting of beams AE, and CG, and a two-force member BG; (b) free-body diagrams of the members.

Examples of systems that incorporate beams are trees (trunk and branches), the human skeleton (tibia, femur), building and bridge frames, automotive suspensions, skis, and swing sets (the metal tubes or wood members). In designing and evaluating beams, engineers must consider the beam's length and cross-sectional shape and the loads it must carry, as well as its material, weight, connection to the rest of the structure, cost, and availability of prefabricated beams. Beams are so common in our world that specialized analysis procedures have been developed to calculate the loads internal to a beam in order to assess the beam's capacity. In this section we present these procedures. This presentation involves some new vocabulary, a slight adaptation of several analysis steps, and some insights into beam behavior gained by revisiting the free-body diagram.

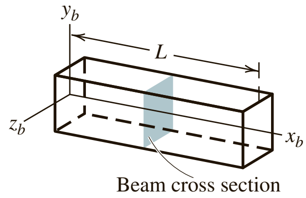


Figure 10.1.3 Beam coordinate system.

Beam Configurations

Geometric features that are important in describing a beam are its length and its cross-sectional area and shape. We set up a **beam coordinate system** with an x_b axis along the length (**Figure 10.1.3**)—we refer to this axis as the **longitudinal axis** or long axis. The other two axes (y_b and z_b) lie in the cross section of the beam, as shown. If the long axis of the beam is aligned with the x axis of the overall coordinate system, we can omit the b subscript in describing bending moments, shear forces, and axial forces; otherwise the b subscript helps in differentiating the beam's orientation from the global coordinate system.

Some beam configurations are given names based on how they are connected to the rest of the world. For example:

- A **cantilever beam** is fixed to the rest of the world at one end and has no supports at the other end (we call this the free end). A cantilever beam is generally represented as in **Figure 10.1.4*a***, where end (B) is fixed and end (A) is free.
- A **simply supported beam** is pinned to the rest of the world at one end and attached via a roller or rocker at the other end. **Figure 10.1.4*b*** shows a common representation of a simply supported beam, where end (C) is pinned and end (D) is supported by a roller.
- A **fixed-fixed beam** is fixed to the rest of the world at each end and is generally represented as in **Figure 10.1.4*c***.

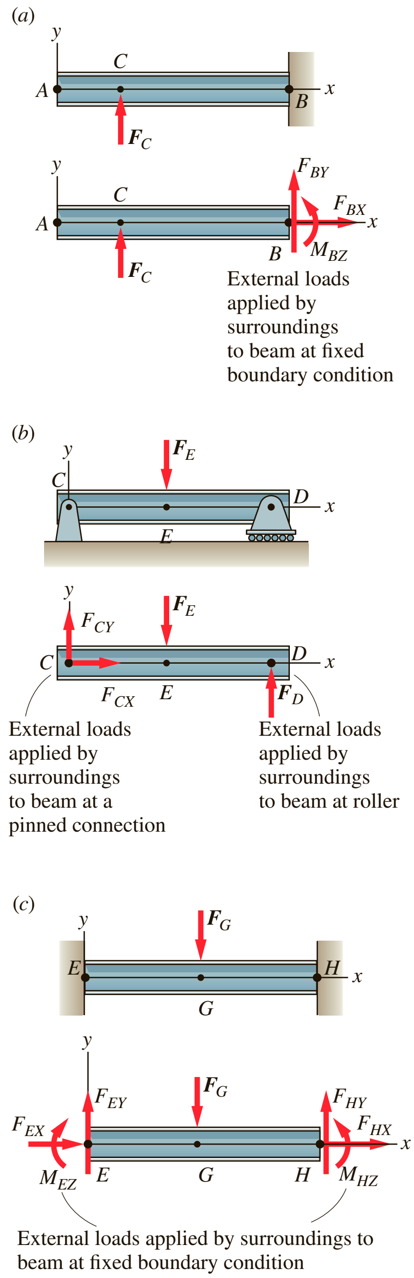


Figure 10.1.4 (a) Cantilever beam; (b) simply supported beam; (c) fixed-fixed beam.

There are many beam configurations, and **Figure 10.1.5** illustrates a few of the more common examples. The combination beam shown is a simply supported merged with a cantilever beam, but there are many examples of combination beams. The configurations in (a) are statically determinate, meaning that the conditions of equilibrium are sufficient for determining the loads acting on the beam at the supports. The configurations in (b) are statically indeterminate, and the conditions of equilibrium are not sufficient for determining the loads acting on the beam; additional relationships from mechanics of materials are needed. (Statically indeterminate systems were discussed in sections 5.7, 8.5, and 9.3).

If all of the external forces acting on a beam (single point forces and distributed forces) lie in a single plane and all external moments are about an axis perpendicular to this plane, the beam is a **planar beam**

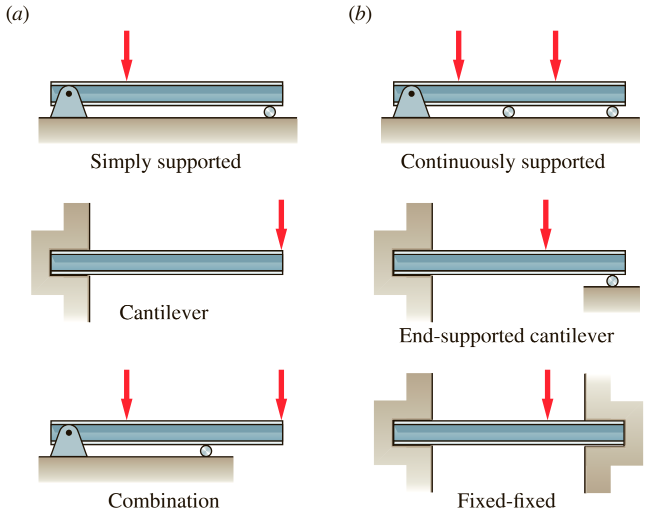


Figure 10.1.5 Various beam configurations: (*a*) statically determinate beams; (*b*) statically indeterminate beams.

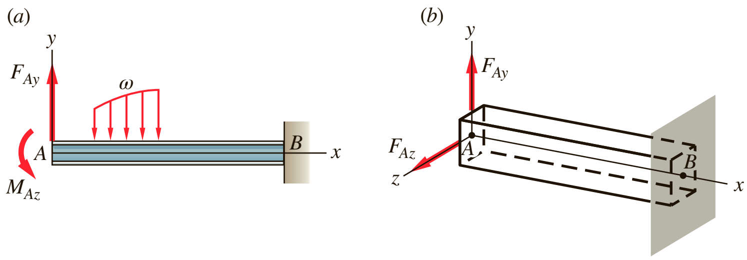


Figure 10.1.6 Fixed boundary condition at *B*: (*a*) planar beam; (*b*) nonplanar beam.

(also called a **two-dimensional beam**); otherwise it is a **nonplanar beam** (also called a **three-dimensional beam**). **Figure 10.1.6*a*** illustrates a planar cantilever beam, and **Figure 10.1.6*b*** illustrates a nonplanar cantilever beam.

Generally the first step in finding loads internal to a beam is to find the loads at its supports. When a beam is part of a larger system such as the frame in **Figure 10.1.2**, analysis of the beam may also involve finding the loads acting on the larger system.

Check out the following examples of applications of this material.

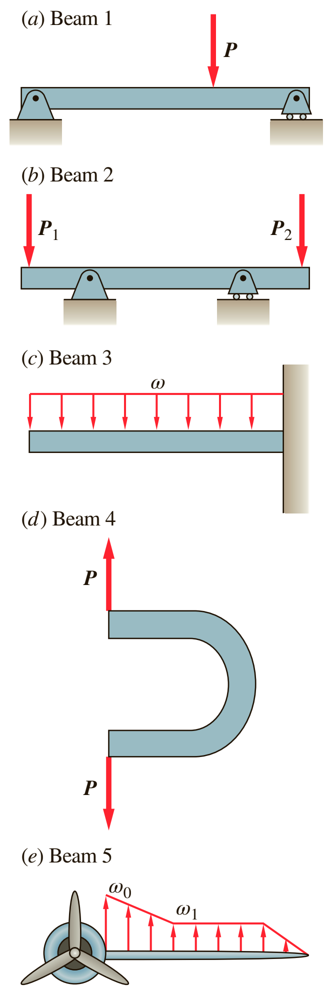
- **Example 10.1.1 Beam Identification**
- **Example 10.1.2 Determine Loads Acting on a Beam**

EXAMPLE 10.1.1

Describe the beam type and the loading for each beam presented.

Beam and Loading	Description
Beam 1 (Figure 1*a*)	Simply supported beam with a single concentrated force.
Beam 2 (**Figure 1*b***)	Combination beam with two concentrated loads, one at each end. This beam can also be described as a simply supported beam with overhangs.
Beam 3 (**Figure 1*c***)	Cantilever beam supporting a uniformly distributed load along its entire length.
Beam 4 (**Figure 1*d***)	Force-loaded curved beam. This beam is actually a two-force member. The straight two-force members that we studied in our analysis of trusses have axial forces as internal loads. A bent or curved two-force member also has internal bending moments and shear loads. Examples of curved beams include clamps, hooks, and bows.
Beam 5 (**Figure 1*e***)	Tapered airplane wing with a varying distributed load.
Beam 6 (**Figure 1*f***)	Cantilever beam of varying cross section with a concentrated load at the narrow end.
Beam 7 (**Figure 1*g***)	Round solid beam that acts as a shaft for the pulley system shown. The loading consists of concentrated loads at the locations of the pulleys.
Beam 8 (**Figure 1*h***)	L-shaped beam with a concentrated load, a triangular distributed load, and a uniformly distributed load.
Beam 9 (**Figure 1*i***)	Round, hollow, thin-walled cantilever beam with a concentrated load at the end.

Note: All the beams shown are statically determinate, and all the external reactions can be calculated using the planar equilibrium equations.



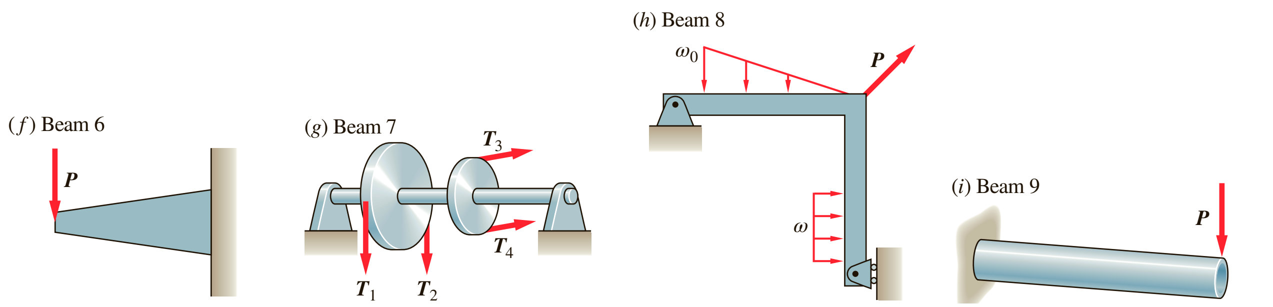


Figure 1 Beam configurations.

EXAMPLE 10.1.2

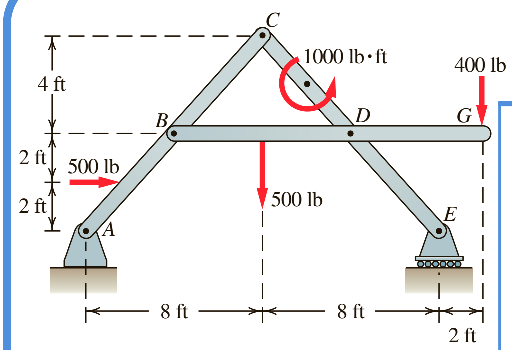


Figure 1 All three members of this frame behave as beams.

Beam *BDG* is part of the frame shown in **Figure 1** and is supported by pins at *B* and *D*. Determine the loads acting on *BDG* at pins *B* and *D*.

Goal Find the loads acting on beam *BDG* at pins *B* and *D*.

Given Information about the geometry and loading of the frame.

Assume The pins at *B* and *D* are frictionless, we can treat the system as planar, and that the weights of the members are negligible.

Draw When we isolate *BDG* (**Figure 2**) and analyze it for the loads at *B* and *D* we find that we have four unknown forces (F_{Bx}, F_{By}, F_{Dx}, F_{Dy}) and only three equations of planar equilibrium. Thus we need to analyze members that are connected to *BD* so we can reduce the number of unknowns for member *BDG*. The first step is to find the loads at supports *A*. To accomplish this we draw a free-body diagram (**Figure 3**) of the entire structure. We then disassemble the frame and use the free-body diagram of *ABC* to determine unknown loads at *B* (**Figure 4**). Alternatively we could find the load at support *E*, which would require us to analyze member *CDE* to find loads at *D*.

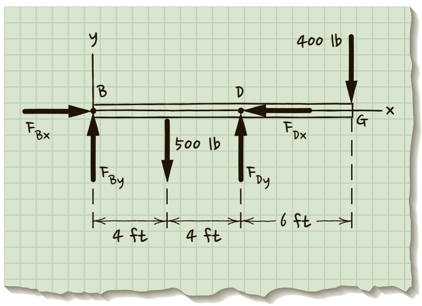


Figure 2 Free-body diagram of beam *BDG*.

Formulate Equations and Solve Based on the free-body diagram in **Figure 3**, we set up planar equilibrium equations (5.5) to find the loads at support *A*.

$$\sum M_{z@E}(\curvearrowleft +) = 0$$

$$-F_{Ay}(16\text{ ft}) - 500\text{ lb}(2\text{ ft}) + 500\text{ lb}(8\text{ ft}) - 400\text{ lb}(2\text{ ft}) + 1000\text{ lb}\cdot\text{ft} = 0$$

$$F_{Ay} = 200\text{ lb}$$

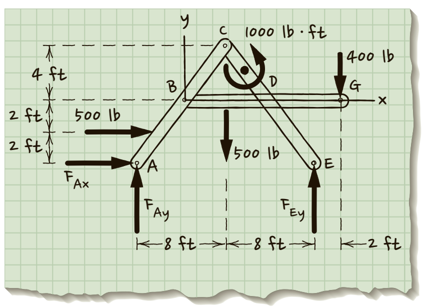


Figure 3 Free-body diagram of entire frame.

$$\sum F_x(\rightarrow +) = F_{Ax} + 500\text{ lb} = 0$$

$$F_{Ax} = -500\text{ lb}$$

We next analyze beam *BDG* to solve for the vertical loads at *B* and *D* (**Figure 2**). Based on (5.5C), with the moment center at *B*:

$$\sum M_{z@B}(\curvearrowleft +) = -500\text{ lb}(4\text{ ft}) + F_{Dy}(8\text{ ft}) - 400\text{ lb}(14\text{ ft}) = 0$$

$$F_{Dy} = 950\text{ lb} \qquad (1)$$

Based on (5.5B):

$$\sum F_y(\uparrow +) = F_{By} + F_{Dy} - 500\text{ lb} - 400\text{ lb} = 0$$

Substituting from (1) for F_{Dy},

$$F_{By} + 950\text{ lb} - 900\text{ lb} = 0$$

$$F_{By} = -50\text{ lb}$$

The minus sign indicates that F_{By} is acting downward on BDG and therefore upward on ABC.

Based on (5.5A):

$$\sum F_x(\rightarrow +) = F_{Bx} - F_{Dx} = 0$$

$$F_{Bx} = F_{Dx} \qquad (2)$$

We do not have another equation of planar equilibrium that we can apply to BDG to solve for F_{Bx} and F_{Dx}. However, since F_{Bx} also acts on ABC, we can analyze member ABC to solve for F_{Bx} (**Figure 4**).

We choose C as the moment center because then there is one unknown in the moment equilibrium equation.

$$\sum M_{z@C}(\curvearrowleft +) = 0$$

$$-200\text{ lb}(8\text{ ft}) - 500\text{ lb}(8\text{ ft}) + 500\text{ lb}(6\text{ ft}) - 50\text{ lb}(4\text{ ft}) - F_{Bx}(4\text{ ft}) = 0$$

$$\boxed{F_{Bx} = -700\text{ lb}} \qquad (3)$$

Finally, substitute (3) into (2) and solve for F_{Dx}:

$$-700\text{ lb} - F_{Dx} = 0$$

$$\boxed{F_{Dx} = -700\text{ lb}}$$

Alternately, we could represent the forces acting on the beam BDG as:

$$\mathbf{F}_B = -700\text{ lb}\,\boldsymbol{i} - 50\text{ lb}\,\boldsymbol{j}$$

$$\mathbf{F}_D = 700\text{ lb}\,\boldsymbol{i} + 950\text{ lb}\,\boldsymbol{j}$$

Comment F_{By} is pulling down on beam BDG to help counteract rotation about pin D that is caused by the 400 lb force at G. (The 500 lb force is also participating in counteracting that rotation).

Check To check our solution we can reconsider the free-body diagram of BDG in **Figure 2** and sum the moments about any point other than B. We do not want to sum the moments around B because we used B as the moment center in calculating the results. We want our check to consist of equations that are independent of those used in the analysis.

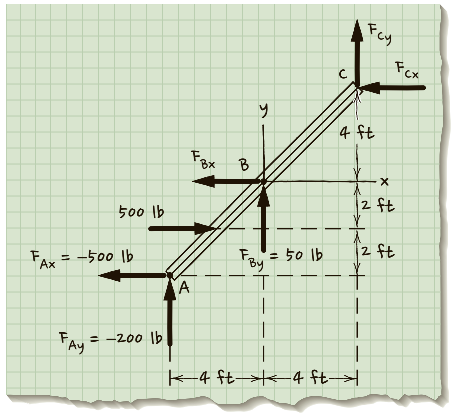
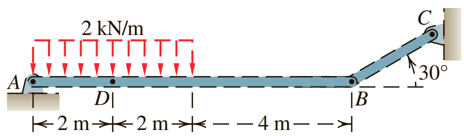


Figure 4 Free-body diagram of beam ABC.

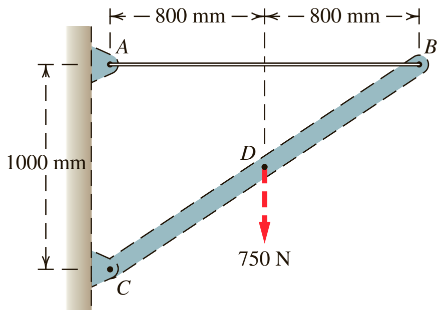
EXERCISES 10.1

10.1.1. [*] Identify the members in the structure shown that act as beams and whether they are planar or nonplanar beams.



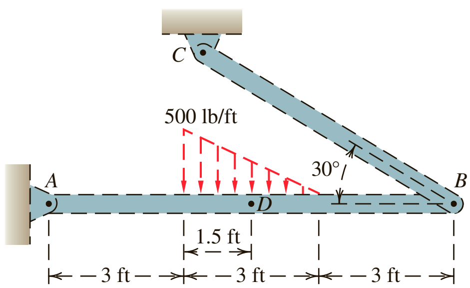
EX 10.1.1

10.1.2. [*] Identify the members in the structure shown that act as beams and whether they are planar or nonplanar beams.



EX 10.1.2

10.1.3. [*] Identify the members in the frame shown that act as beams and whether they are planar or nonplanar beams.

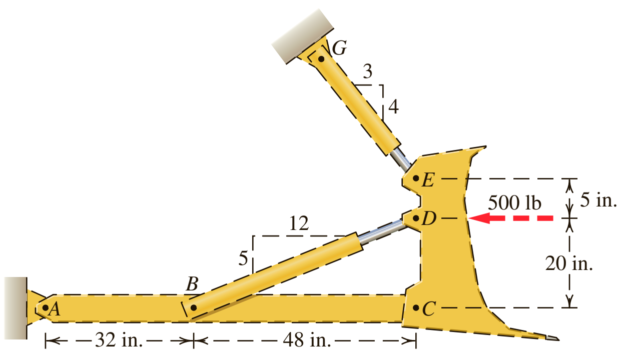


EX 10.1.3

10.1.4. [*] The bulldozer assembly, loaded with a resultant horizontal force of 500 lb, consists of push-arm *ABC*, hydraulic arms *BD* and *EG*, and blade *CDE*. All connections (*A*, *B*, *C*, *E*, *D*, and *G*) are frictionless pins. Consider members *ABC*, *BD*, and *EG*.

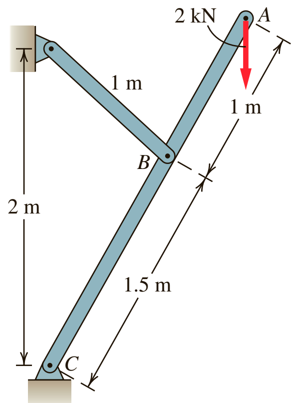
a. If we ignore the weight of the members, which members act as beams?

b. If we include the weight of the members, which members act as beams?



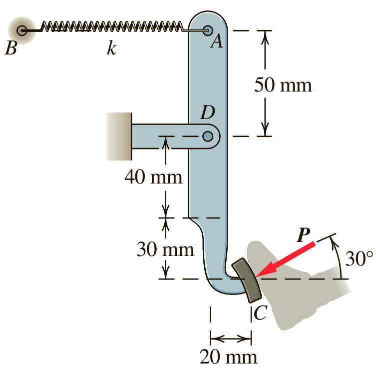
EX 10.1.4

10.1.5. [*] Determine the loads acting on beam *ABC* at *B* and *C*.



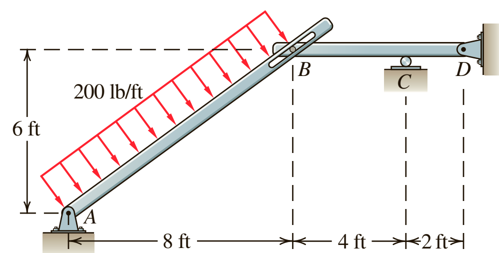
EX 10.1.5

10.1.6. [*] The spring attached to the brake pedal has a spring constant of 15 N/mm. In the configuration shown the spring is stretched 30 mm from its unstretched position. Determine the loads acting on beam *ADC* at *A*, *D*, and *C*.



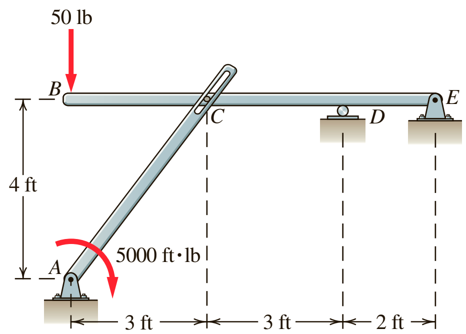
EX 10.1.6

10.1.7. [*] Determine the loads acting on beam *BCD* at *B*, *C*, and *D*.



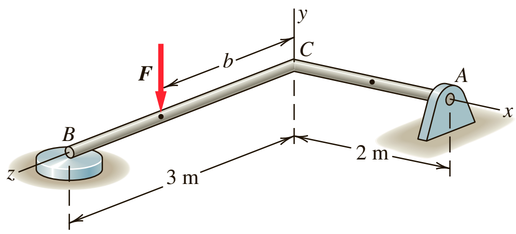
EX 10.1.7

10.1.8. [*] For the frame shown determine the loads acting on beam *BCDE* at points *C*, *D*, and *E*.



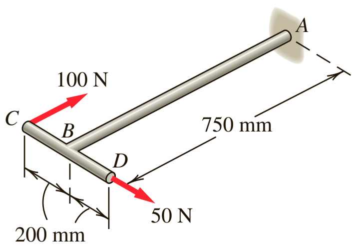
EX 10.1.8

10.1.9. [**] For the nonplanar structure shown identify whether beams *AC* and *BC* are planar or nonplanar beams.



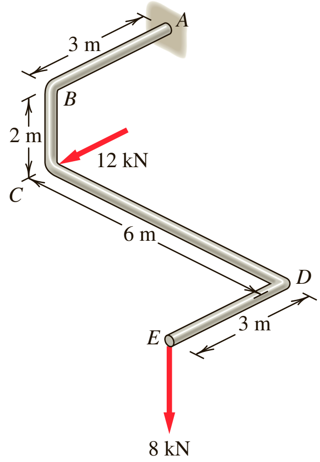
EX 10.1.9

10.1.10. [**] For the nonplanar structure shown identify whether members *AB*, *BC*, and *BD* behave as beams and whether they are planar or nonplanar beams.



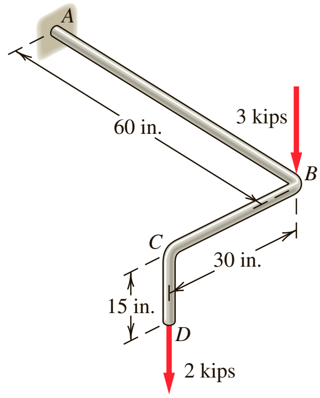
EX 10.1.10

10.1.11. [**] For the nonplanar structure shown identify whether members *AB*, *BC*, *CD*, and *DE* behave as beams and whether they are planar or nonplanar beams.



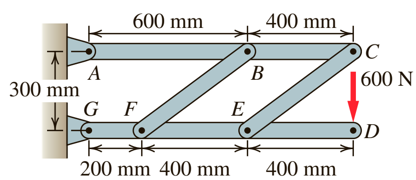
EX 10.1.11

10.1.12. [**] For the nonplanar structure shown identify whether members *AB*, *BC*, and *CD* behave as beams and whether they are planar or nonplanar beams.



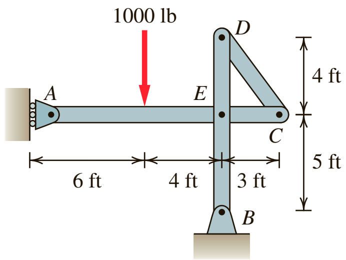
EX 10.1.12

10.1.13. [**] Beam *ABC* forms part of the frame shown. Determine the loads acting on beam *ABC* at *A*, *B*, and *C*.



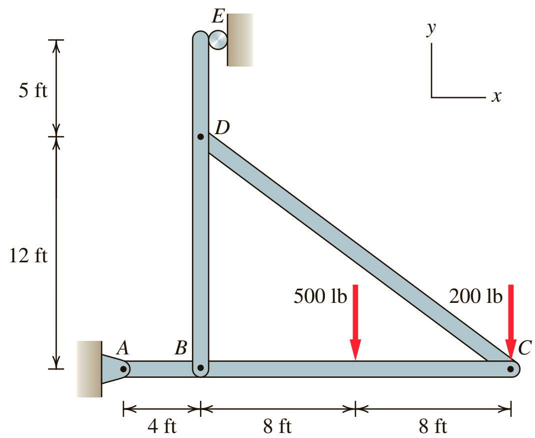
EX 10.1.13

10.1.14. [**] For beam *AEC* that forms part of the frame shown determine the loads acting at *A*, *E*, and *C*.



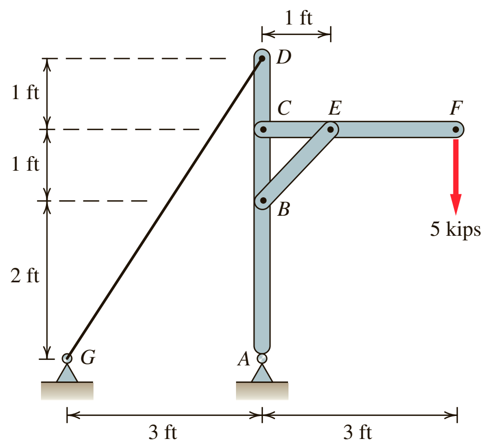
EX 10.1.14

10.1.15. [**] The 200-lb load acts on member *ABC* of the frame. Determine the resultant loads in the *x* and *y* directions acting on beam *ABC* at *A*, *B*, and *C*.



EX 10.1.15

10.1.16. [**] Frame *ABCDEF* is held vertical by cable *DG*. Determine the loads on beam *ABCD* acting at *A*, *B*, *C*, and *D*.



EX 10.1.16

10.2 BEAM INTERNAL LOADS

Learning Objective: Determine internal loads in a beam, as part of a comprehensive equilibrium analysis of the beam.

Consider the planar cantilever beam in **Figure 10.2.1*a***; a distributed force ω acts along the top surface of the beam. At point *P* along the long axis (x) indicated in the figure, we "cut" the beam to investigate the internal loads. At the location of the cut there are internal load pairs (according to Newton's third law) consisting of forces and moments, as illustrated in **Figure 10.2.1*b***.

If we now isolate the portion of the beam between *A* and *P*, the loads at the cut at *P* consist of a moment and forces that constitute half of internal load pairs (**Figure 10.2.1*c***). The moment, which is internal to a beam, is called a **bending moment**, the force in the plane of the cross section of the beam is called a **shear force**, and the force perpendicular to the cross section is called an **axial force**. Remember that these loads are internal to the beam when it is considered as a whole.

Bending moment acts about the z axis, and we denote it as M_{bz} (the *b* in the subscript stands for bending). Internal loads have their own widely accepted sign convention that is different from the sign convention we use for analyzing equilibrium. M_{bz} is defined to be positive when the beam is bent such that its top surface tends to concavity and its bottom surface

tends to convexity (**Figure 10.2.2*a***). The x axis runs through the centroid of the beam cross section, so that the beam's top surface is a plane defined by positive y values and its bottom surface is a plane defined by negative y values. This means that a **positive M_{bz}** leads to compression in the $+y$ surface and tension in the $-y$ surface. Conversely, a **negative M_{bz}** leads to tension in the $+y$ surface and compression in the bottom $-y$ surface (**Figure 10.2.2*b***).

Shear force acts in the y direction, and we denote it as V_y. It lies in the plane of the beam's cross section. Shear force tends to cause adjacent cross sections to slide parallel to each other in the y direction so that the beam takes on a skewed shape. The shear force is defined to be positive if it acts in the negative y direction on the right-hand side of the beam portion. **Figure 10.2.1*c*** shows a positive shear acting on the cut face for the portion of the beam between A and P. Because Newton's third law requires an equal and opposite force on the other side of the cut at P, we see that the positive shear acts upward on the left-hand side of portion PB in **Figure 10.2.1*d***.

Axial force acts in the x direction, and we denote it as N_x. It is perpendicular to the beam's cross section and is positive if it acts to pull (creating tension) on the cross section, as shown in **Figure 10.2.1*c***.

This discussion of bending moment, shear force, and axial force has been in terms of the portion of the beam from A to P, with positive as defined in **Figure 10.2.1*c***. It is important to understand that positive is defined as shown in **Figure 10.2.1*d*** for the portion of the beam from P to B. Here we are looking at the left-hand face of the beam portion. Notice that in both **Figures 10.2.1*c*** and **10.2.1*d*** the bending moment acts so as to create compression in the $+y$ surface and tension in the $-y$ surface. **Figure 10.2.3** summarizes the sign convention for internal forces in planar beams.

With nonplanar beams we are concerned with finding the internal loads M_{bz}, V_y, and N_x, but in addition there may be another bending moment (M_{by}), another shear force (V_z), and a moment about the x axis (M_{bx}, which is commonly called torque), as illustrated in **Figure 10.2.4**.

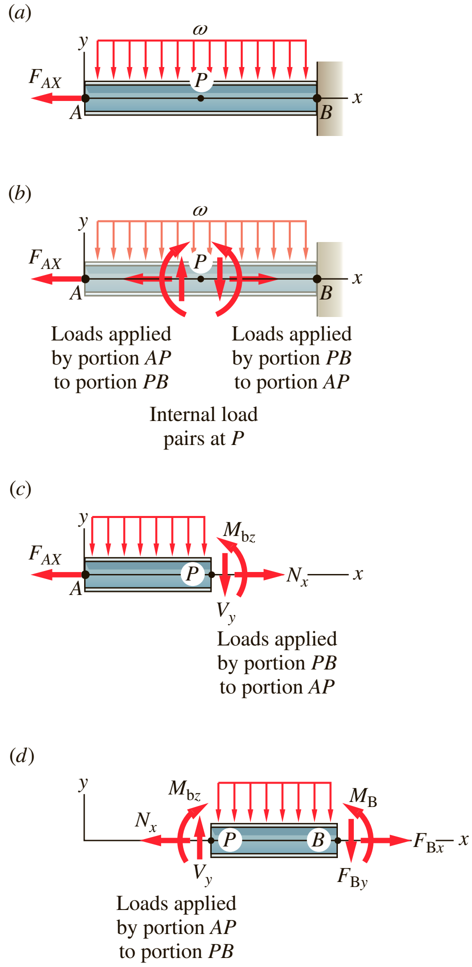


Figure 10.2.1 Loads at point P within a beam: (a) cantilever beam; (b) internal load pairs at point P; (c) free-body diagram of left-hand portion of beam; (d) free-body diagram of right-hand portion of beam.

Procedure for Finding Internal Loads in Beams

Let's say that we are interested in finding the internal loads (bending moment, shear force, and axial force) at a specified location P in a beam in

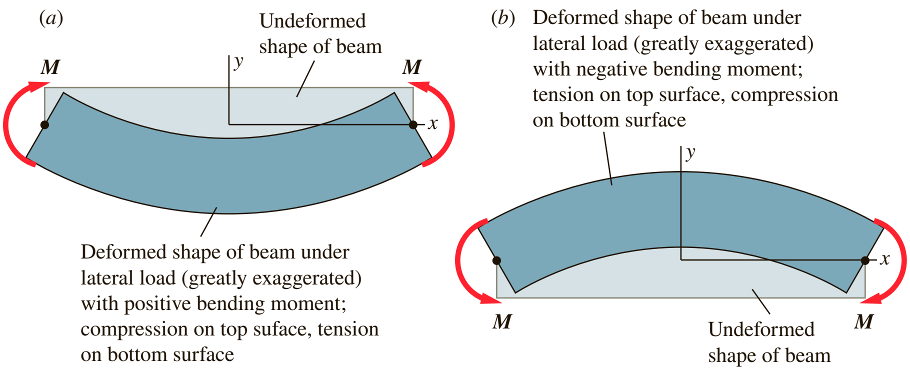


Figure 10.2.2 (a) Bending produced by positive bending moment M_b; (b) bending produced by negative bending moment M_b.

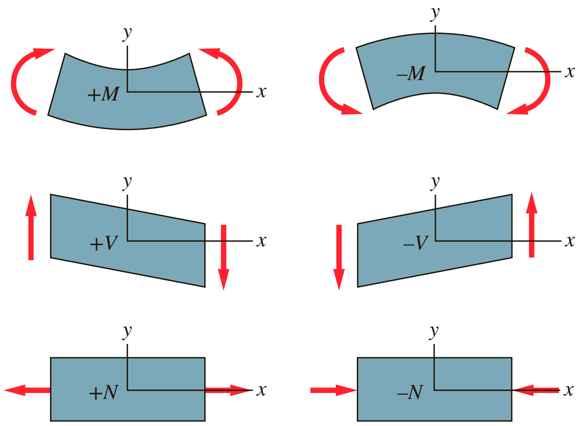


Figure 10.2.3 Summary of sign convention for internal forces.

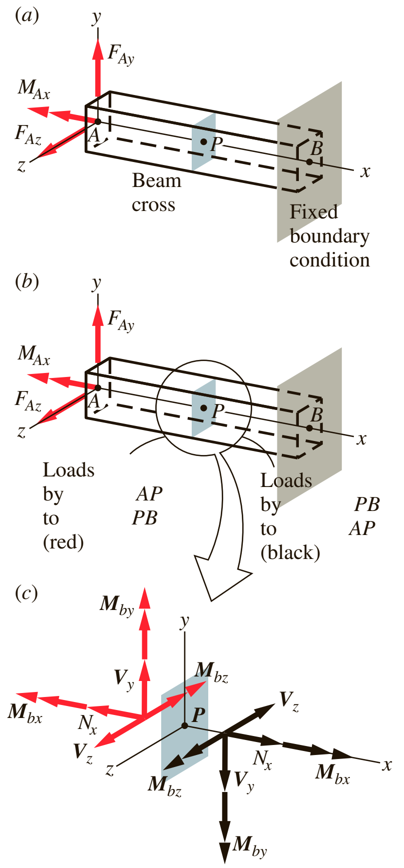


Figure 10.2.4 Loads at point P within a beam: (a) cantilever nonplanar beam with applied loads at A; (b) internal load pairs at point P; (c) detail of internal load pairs at P.

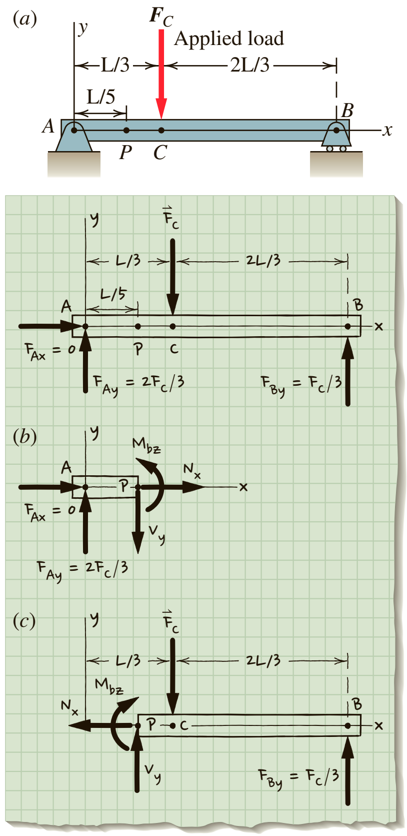


Figure 10.2.5 (a) A simply supported beam and its free-body diagram; (b) free-body diagram of left-hand portion; (c) free-body diagram of right-hand portion.

equilibrium (**Figure 10.2.5*a***). The analysis procedure consists of isolating a portion of the beam that includes a boundary cut through the beam at the specified position P, drawing a free-body diagram of the portion, then writing and solving equilibrium equations for the portion. For the beam

in **Figure 10.2.5*a*** we could isolate the left-hand portion *AP*, draw the associated free-body diagram (with positive bending moment and shear as shown in **Figure 10.2.5*b***), and solve the equilibrium equations for M_{bz}, V_y, and N_x. As an alternative, we could work with the right-hand portion and draw its free-body diagram (with positive bending moment and shear as shown in **Figure 10.2.5*c***). Both approaches result in the same values of internal loads. It's a good idea to analyze the portion that requires the fewest calculations to find the internal loads; just be sure to correctly draw positive bending moments and shear forces.

IMPORTANT NOTE! As we use the procedure for determining beam internal loads, we will switch back and forth between the sign convention for internal loads (for example, tension positive, compression negative) and the equilibrium sign convention (for example, all forces acting in the positive *x* direction are positive). Do not confuse these two different sign conventions. The first describes the direction of the internal forces based on how they deform the beam; the second describes the direction of the forces and moments relative to coordinate axes defined for the system and is used for writing and solving equilibrium equations.

Check out the following examples of applications of this material.
- **Example 10.2.1 Internal Loads in a Planar Simply Supported Beam**
- **Example 10.2.2 Internal Loads in a Planar Cantilever Beam**
- **Example 10.2.3 Internal Loads in a Nonplanar Beam**

EXAMPLE 10.2.1

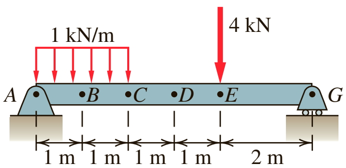


Figure 1 All loads act in the *xy* plane so this simply supported beam is treated as a planar beam.

A simply supported beam is loaded as shown in **Figure 1**. Determine the axial force, shear force, and bending moment at cross sections at *B* and *D*.

Goal Find the axial force, shear force, and bending moment at two locations (*B* and *D*).

Given Information about the dimensions and loading of the beam. Because all loads act in the *xy* plane, this is treated as a planar beam.

Assume The weight of the beam is negligible.

Draw First we need to find the loads at supports *A* and *G*. To accomplish this we draw a free-body diagram (**Figure 2**) of the entire beam. Later we will cut the beam at *B* and *D* and draw free-body diagrams of portions of the beam to find the internal loads at *B* and *D*.

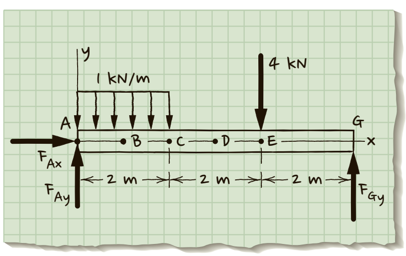


Figure 2 Free-body diagram of entire beam.

Formulate Equations and Solve First we use the equations for planar equilibrium to find the loads at supports *A* and *G* ($F_{Ax} = 0$, $F_{Ay} = 3$ kN, and $F_{Gy} = 3$ kN). (Calculations not shown.)

Internal Loads at B: We make a cut at *B* and draw a free-body diagram of the left portion of the beam (**Figure 3**). When drawing the diagram, we assume that the unknown internal loads are positive (using the internal load sign convention). We then apply the planar equilibrium equations (using the equilibrium sign convention):

$$\sum F_x(\rightarrow +) = 0 \quad \Rightarrow \quad N_x = 0$$

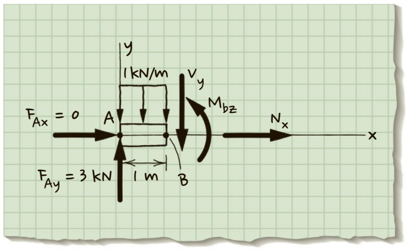


Figure 3 Free-body diagram of portion AB with M_{bz}, V_y, and N_x assumed positive.

We sum moments about the cut at B, allowing us to eliminate V_y from the equilibrium equation so that we have only one unknown. As we apply (5.5C), with the moment center at B, each position vector is measured from the cut to the load. For the distributed load, we measure the position vector from the cut to the centroid of the load.

$$\sum M_{z@B}(\curvearrowleft +) = -3\,\text{kN}(1\,\text{m}) + 1\frac{\text{kN}}{\text{m}}(1\,\text{m})(0.5\,\text{m}) + M_{bz} = 0$$

$$\boxed{M_{bz} = 2.5\,\text{kN}\cdot\text{m}}$$

Finally we solve for V_y:

$$\sum F_y(\uparrow +) = -V_y + 3\,\text{kN} - 1\frac{\text{kN}}{\text{m}}(1\,\text{m}) = 0$$

$$\boxed{V_y = 2\,\text{kN}}$$

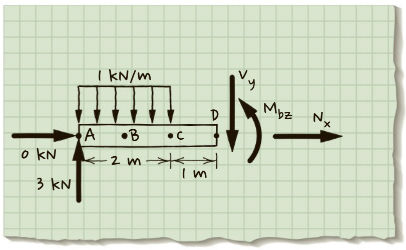


Figure 4 Free-body diagram of portion AD with M_{bz}, V_y, and N_x assumed positive.

Internal Loads at D: Using the free-body diagram in **Figure 4** as a reference, we calculate the internal forces at D using the same approach we used for cross section B.

$$\sum F_x(\rightarrow +) = 0 \quad \Rightarrow \quad \boxed{N_x = 0}$$

We determine M_D by summing moments with the moment center at D:

$$\sum M_{z@D}(\curvearrowleft +) = -3\,\text{kN}(3\,\text{m}) + 1\frac{\text{kN}}{\text{m}}(2\,\text{m})(2\,\text{m}) + M_{bz} = 0$$

$$\boxed{M_{bz} = 5\,\text{kN}\cdot\text{m}}$$

Finally we calculate V_y:

$$\sum F_y(\uparrow +) = -V_y + 3\,\text{kN} - 1\frac{\text{kN}}{\text{m}}(2\,\text{m}) = 0$$

$$\boxed{V_y = 1\,\text{kN}}$$

Comment: While it might be tempting to simplify the beam loading and calculations by replacing the distributed load by a 2-kN point load at B, this will lead to incorrect results. Prove this to yourself by drawing a free-body diagram of beam portion AB using the 2-kN replacement load and comparing it with Figure 3. Will you get the same M_{bz}?

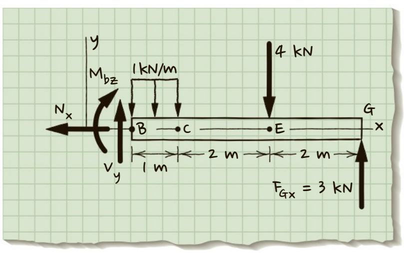


Figure 5 Free-body diagram of portion BG.

Check An alternative method of solving for the internal loads, and also a check on our solution, is to analyze the right portion of the beam (**Figure 5**).

$$\sum F_x(\rightarrow +) = 0 \quad \Rightarrow \quad N_x = 0$$

Based on **Figure 5** with the moment center at B:

$$\sum M_{z@B}(\curvearrowleft +) = -1\frac{\text{kN}}{\text{m}}(1\,\text{m})(0.5\,\text{m}) - 4\,\text{kN}(3\,\text{m}) + 3\,\text{kN}(5\,\text{m}) - M_{bz} = 0$$

$$M_{bz} = 2.5\,\text{kN}\cdot\text{m}$$

Summing vertical forces we get:

$$\sum F_y(\uparrow +) = V_y - 1\,\frac{\text{kN}}{\text{m}}(1\,\text{m}) - 4\,\text{kN} + 3\,\text{kN} = 0$$

$$V_y = 2\,\text{kN}$$

These are the same results we obtained earlier.

We can use a similar approach to check our results at D by analyzing the portion of the beam to the right of the cut at D.

EXAMPLE 10.2.2

The cantilever beam in **Figure 1** is loaded with a point force at the free end and a concentrated moment at a distance of $L/2$ from the free end. Determine the axial force, shear force, and bending moment at cross sections at B and D.

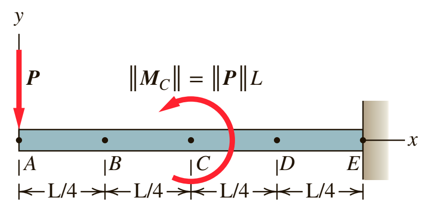


Figure 1 The force P is in the xy plane and the applied moment is about the z axis, so we treat this as a planar beam.

Goal Find the axial force, shear force, and bending moment at cross sections at B and D.

Given Information about the dimensions and loading of the beam. Because all of the applied loads cause the beam to bend in the xy plane, we can treat the system as planar.

Assume The weight of the beam is negligible.

Draw If we use a free-body diagram of the portion of the beam between the free end and the cross section of interest, we can determine the internal loads without calculating the loads at support E; therefore we do not need to start with a free-body diagram of the entire beam. Instead we cut the beam at B and analyze the portion of the beam to the left of the cut. We do the same for the cut at D. The free-body diagrams of the left portions of the beam for each cut are shown in **Figure 2** and **Figure 3**. In each diagram the unknown internal forces are assumed to be positive according to the internal load sign convention.

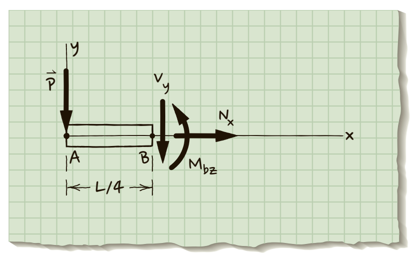


Figure 2 Free-body diagram of portion AB with M_{bz}, V_y, and N_x assumed positive.

Formulate Equations and Solve Using the free-body diagram of the left portion of the beam (**Figure 2**) as a reference we first sum moments about the cut at B:

$$\sum M_{z@B}(\curvearrowleft +) = P\left(\frac{L}{4}\right) + M_{bz} = 0$$

$$\boxed{M_{bz} = -\frac{PL}{4}}$$

The minus sign indicates that the bending moment is in the opposite direction from what we assumed. Since we assumed a positive bending moment, the bending moment is actually negative. This means that the beam is curving downward with the bottom surface of the beam in compression and the top in tension.

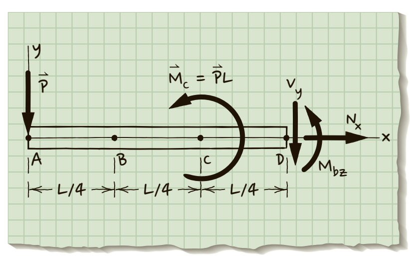


Figure 3 Free-body diagram of portion AD.

We use the other two planar equilibrium equations to solve for N_x and V_y:

$$\sum F_x(\rightarrow +) = 0 \quad \Rightarrow \quad \boxed{N_x = 0}$$

$$\sum F_y(\uparrow +) = -P - V_y = 0 \quad \Rightarrow \quad \boxed{V_y = -P}$$

The minus sign means that the shear force is acting in the opposite direction from that which was assumed and therefore is acting upward.

Before going on to calculate the internal loads at D, we reflect on the answers we obtained for the loads at B. **Figure 4** shows free-body diagrams of the left and right portions of the beam when it is cut at B and includes our calculated values from above. At the cut we see the equal and opposite force pairs and moment pairs that we expect because of Newton's third law. Looking at the portion of the beam to the left of B, we notice that the applied load at A and the shear force at B form a counterclockwise couple with a magnitude of $PL/4$. The clockwise internal bending moment of magnitude $PL/4$ at B maintains equilibrium for this portion of the beam.

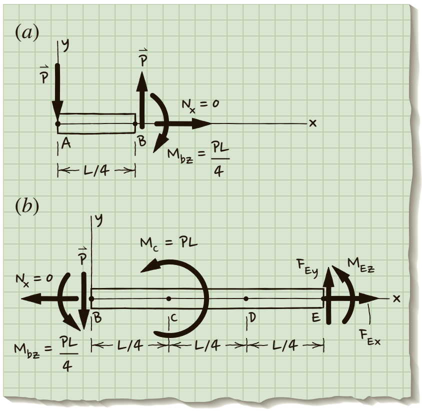


Figure 4 Calculated internal loads at B (a) acting on portion AB, and (b) acting on portion BE.

Now we find the internal loads at the cross section at D. Using **Figure 3** as a reference, we calculate the internal forces at D using the same approach we used for the cross section at B.

$$\sum M_{z@D}(\curvearrowleft +) = 0 = P\left(\frac{3L}{4}\right) + M_c + M_{bz}$$

Substituting for M_C (**Figure 1** indicates that $M_C = PL$), we get

$$P\left(\frac{3L}{4}\right) + PL + M_{bz} = 0 \quad \Rightarrow \quad \boxed{M_{bz} = -\frac{7PL}{4}}$$

Based on vertical and horizontal equilibrium:

$$\sum F_x(\rightarrow +) = 0 \quad \Rightarrow \quad \boxed{N_x = 0}$$

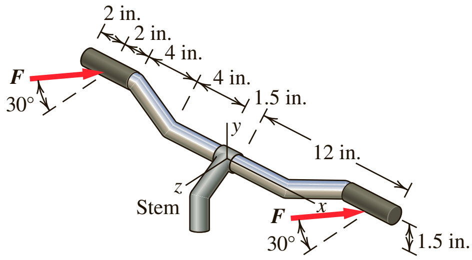
$$\sum F_y(\uparrow +) = -P - V_y = 0 \quad \Rightarrow \quad \boxed{V_y = -P}$$

There are no axial loads applied to the beam, and therefore the internal axial force is zero throughout the beam.

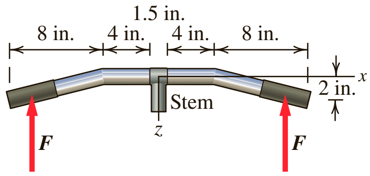
Check To check the results, we can analyze the portions of the beam to the right of the cuts at B and D. This requires us to first calculate the loads at support E.

EXAMPLE 10.2.3

A rider is pushing on each arm of the bicycle handlebars in **Figure 1** with a force of 50.0 lb. Each force is applied at 30° to the horizontal and at 2 in. from the end of the arm. Find the internal loads at the intersection of the handlebars and the stem.



(*a*) Isometric view



(*b*) Top view

Figure 1 The applied loads have components in the *xy* and *xz* planes, so this must be analyzed as a nonplanar beam.

Goal Find the internal forces and bending moments in the x, y, and z directions at the intersection of the handlebars and the stem.

Given Information about the dimensions and loading of the handlebars.

Assume The weight of the handlebars is negligible (no other information is given to estimate weight).

Draw Each arm of the handlebars is a cantilever beam supported by the stem, and loaded by force components in the y and z directions. We cut the handlebars at the stem, isolate the left arm, and draw the internal loads (Figure 2). To clarify the loading for the analysis we break the 50-lb force into components in the y and z directions ($F_y = -50 \sin 30° = -25.0$ lb and $F_z = -50 \cos 30° = -43.3$ lb). Because the system and its loading are symmetric, we know that both arms of the handlebars have the same internal loads.

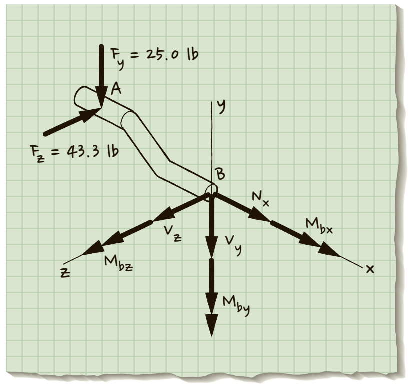


Figure 2 Free-body diagram of left handlebar.

Formulate Equations and Solve We apply the six nonplanar equilibrium equations to the free-body diagram in **Figure 2** to solve for the internal loads. Analyzing force equilibrium gives the following three results based on (5.3A), (5.3B), and (5.3C).

$$\sum F_x = 0 \quad \Rightarrow \quad \boxed{N_x = 0}$$

$$\sum F_y = 0 = -25.0\,\text{lb} - V_y \quad \Rightarrow \quad \boxed{V_y = -25.0\,\text{lb}}$$

$$\sum F_z = -43.3\,\text{lb} + V_z = 0 \quad \Rightarrow \quad \boxed{V_z = 43.3\,\text{lb}}$$

Because the handlebars are bent out of the *xy* plane the force F_y is offset from that plane by a distance d (Figure 3), creating a moment about the x axis. Similarly F_z is offset from the *xz* plane by 1.5 in., creating a moment about the x axis. To determine the distance d, we use similar triangles. From **Figure 1*b*** we know that the end of the arm is offset 2 in. from the x axis. In addition, ***F*** is applied 2 in. from the end of the arm; therefore analysis of similar triangles gives

$$\frac{d}{6\text{ in.}} = \frac{2\text{ in.}}{8\text{ in.}} \quad \Rightarrow \quad d = \frac{(6\text{ in.})2\text{ in.}}{8\text{ in.}} = 1.5\text{ in.}$$

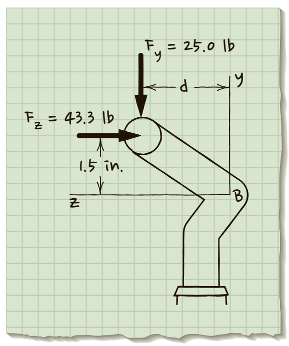


Figure 3 F_z and F_y generate moments about the x axis.

We now determine internal moments about the x, y, and z axes by analyzing moment equilibrium with the moment center at B in **Figure 2**. When writing our equations, we define positive moments as those that create a positive moment about an axis (the right hand rule is very helpful here for keeping signs straight).

$$\sum M_{x@B} = 0 = F_y d - F_z(1.5\,\text{in.}) + M_{bx}$$

$$(25.0\,\text{lb})(1.5\,\text{in.}) - (43.3\,\text{lb})(1.5\,\text{in.}) + M_{bx} = 0 \quad \Rightarrow \quad M_{bx} = 27.5\,\text{lb}\cdot\text{in.}$$

$$\sum M_{y@B} = 0 = -F_z(10\,\text{in.}) - M_{by}$$

$$-(43.3\,\text{lb})(10\,\text{in.}) - M_{by} = 0 \quad \Rightarrow \quad M_{by} = -433\,\text{lb}\cdot\text{in.}$$

$$\sum M_{z@B} = 0 = F_y(10\,\text{in.}) + M_{bz}$$

$$(25.0\,\text{lb})(10\,\text{in.}) + M_{bz} = 0 \quad \Rightarrow \quad M_{bz} = -250\,\text{lb}\cdot\text{in.}$$

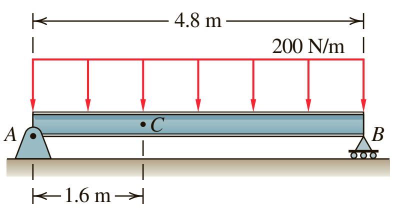
M_{bx} (often called a torque) causes twisting about the x axis. The moment M_{by} causes the handlebars to bend toward the front of the bicycle and M_{bz} causes them to bend downward. The negative signs in the results indicate that the internal loads are in the opposite direction from those shown in **Figure 2**.

Check To check our result, we can apply the calculated internal loads to the free-body diagram in **Figure 2** and perform an equilibrium analysis.

Question: Can you think of how using the cross product might have made the solution easier?

EXERCISES 10.2

10.2.1. [*] Determine the axial force, shear force, and bending moment acting on the cross section at point C of beam ACB.



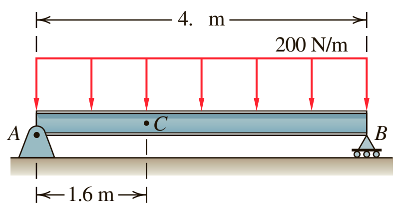
EX 10.2.1

10.2.2. [*] Replace the distributed load acting on beam ACB by a point load at the centroid of the distribution. For both the distributed load and the point load

a. draw a free-body diagram of beam portion AC

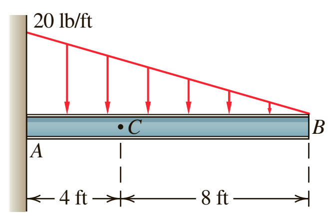
b. calculate V_y and M_{bz} acting on the cross section at point C

Do you get the same results? Discuss your conclusion.



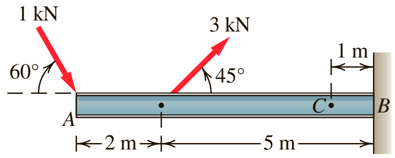
EX 10.2.2

10.2.3. [*] For the beam shown, determine the axial force, shear force, and bending moment acting on the cross section at point C.



EX 10.2.3

10.2.4. **[*]** Determine the axial force, shear force, and bending moment acting on the cross section at point C of the cantilever beam shown.



EX 10.2.4

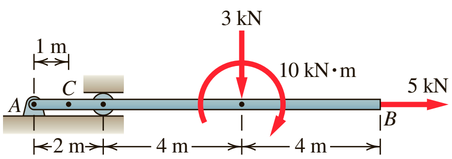
10.2.5. **[*]** An I-beam rests on roller supports at A and B. It is uniform with a mass of 100 kg and is 8 meters long. Determine

a. the loads that act on the beam at A and B if the beam is in equilibrium

b. the axial force, shear force, and bending moment acting on the cross section of the beam midway between A and B

EX 10.2.5

10.2.6. **[*]** Determine the axial force, shear force, and bending moment acting on the cross section of beam ACB at point C.



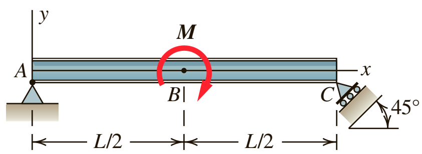
EX 10.2.6

10.2.7. **[*]** For beam ABC determine the axial force, shear force, and bending moment acting on a cross section:

a. at distance L/4 to the right of A.

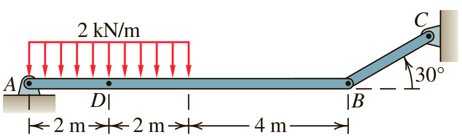
b. just to the left of B.

c. just to the right of B.



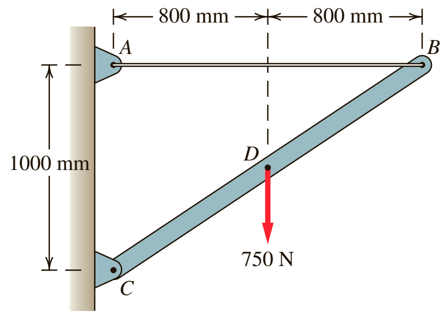
EX 10.2.7

10.2.8. **[*]** Determine the axial force, shear force, and bending moment acting on the cross section of beam ADB at point D. Note that member BC acts as a two-force member.



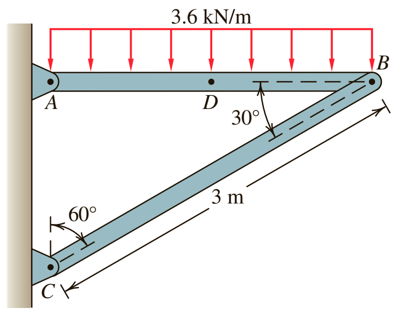
EX 10.2.8

10.2.9. **[*]** Determine the axial force, shear force, and bending moment acting on the cross section of beam BC just to the left of D. Remember to cut the beam perpendicular to its long axis.



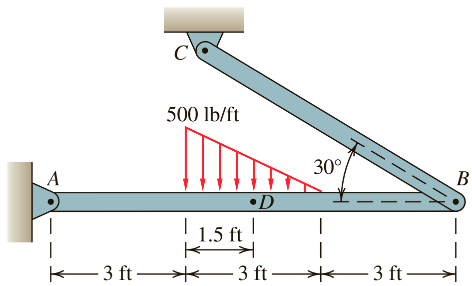
EX 10.2.9

10.2.10. **[*]** Determine the axial force, shear force, and bending moment acting on the cross section of beam ADB at point D, which is located midway between A and B.



EX 10.2.10

10.2.11. **[*]** Determine the axial force, shear force, and bending moment acting on the cross section of beam AB at D.



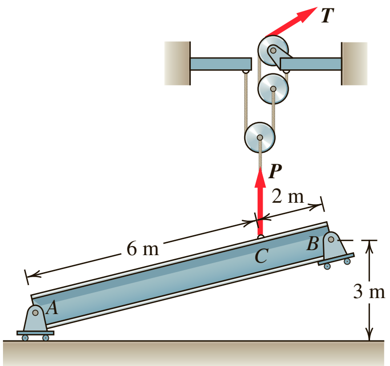
EX 10.2.11

10.2.12. **[*]** An overhead crane consisting of the three pulleys shown has been attached to the I-beam at C to move the beam. The beam is uniform with a mass of 500 kg. In the position shown, the beam is in equilibrium and end B is 3 m off the ground.

a. Determine the tension force $\boldsymbol{P}$ acting at C.

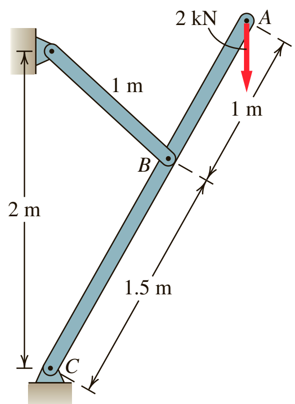
b. Determine the axial force, shear force, and bending moment acting on the cross section of the beam midway between A and B. Remember to cut the beam perpendicular to its long axis when you perform your analysis.

c. Estimate how many statics students would be required to apply the force $\boldsymbol{T}$ to hold the I-beam in the position shown. State your assumptions.



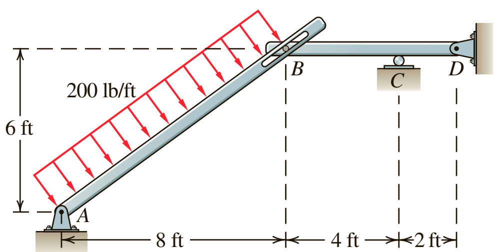
EX 10.2.12

10.2.13. [*] Determine the axial force, shear force, and bending moment acting on the cross section of beam ABC just below point B.



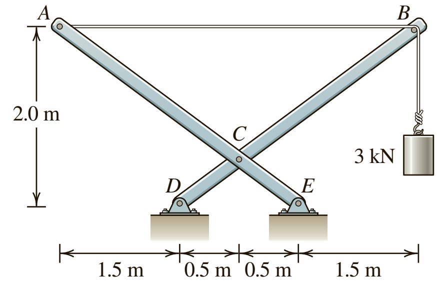
EX 10.2.13

10.2.14. [*] Determine the axial force, shear force, and bending moment acting on the cross section of beam BCD just to the left of point C.



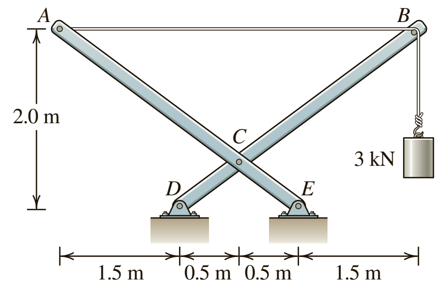
EX 10.2.14

10.2.15. [*] The frame shown supports a 3 kN load on a wire that is attached to A and runs over a frictionless pulley at B. Determine the axial force, shear force, and bending moment acting on the cross section of beam ACE midway between points C and E.



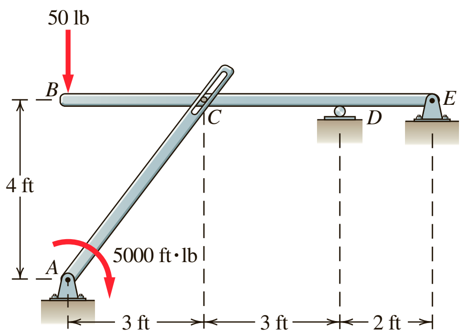
EX 10.2.15

10.2.16. [*] For the frame shown determine the axial force, shear force, and bending moment acting on the cross section of beam BCD just above point C and just below point C.



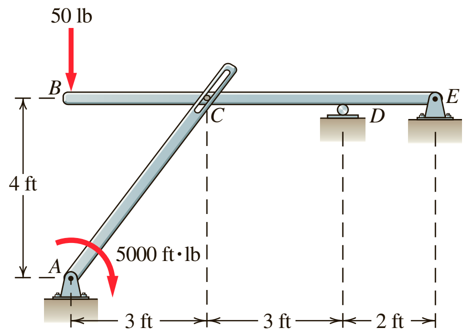
EX 10.2.16

10.2.17. [*] For the frame shown determine the axial force, shear force, and bending moment acting on the cross section of beam $BCDE$ midway between points D and E.



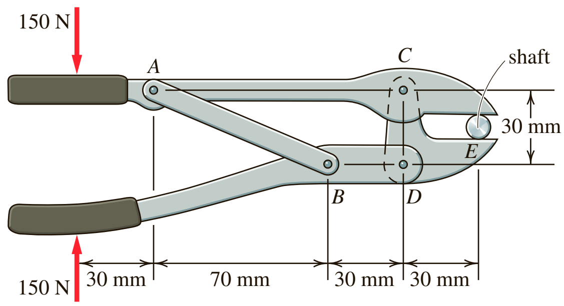
EX 10.2.17

10.2.18. [*] For the frame shown determine the axial force, shear force, and bending moment acting on the cross section of beam AC midway between points A and C.



EX 10.2.18

10.2.19. [*] Member *BD* of the pliers behaves as a beam. Determine the axial force, shear force, and bending moment acting on the cross section of member *BD* just to the left of the pin at *B* and just to the right of the pin at *B*.



EX 10.2.19

10.2.20. [*] A vendor has developed an unusual method to display 10 violins as shown. The PVC pipe tied to the truck frame behaves as a cantilever beam. Each violin weighs about 1 lb and they are hanging 1 ft apart with the first violin at 1 ft from the support. Ignore the weight of the PVC pipe.

a. Determine the locations and magnitudes of the maximum V_y and M in the beam.

b. The maximum bending capacity of the PVC occurs when the moment is 60 lb·ft. Can another violin be added to the beam? If so, where would you add it and why?

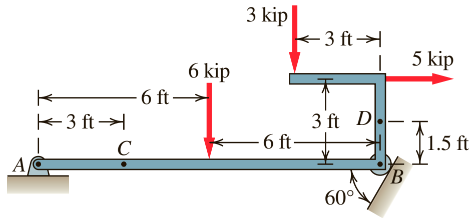
Thalia Anagnos

EX 10.2.20

10.2.21. [**] Determine the axial force, shear force, and bending moment acting on the cross section of the beam members at

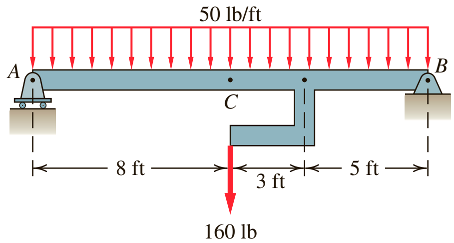
a. point *C*

b. point *D*



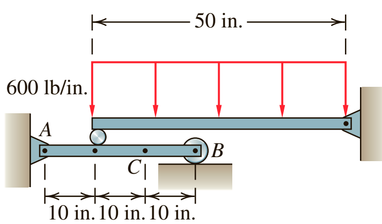
EX 10.2.21

10.2.22. [**] Determine the axial force, shear force, and bending moment acting on the cross section of the beam *AB* at point *C*.



EX 10.2.22

10.2.23. [**] Determine the axial force, shear force, and bending moment acting on the cross section of the beam *AB* at point *C*.

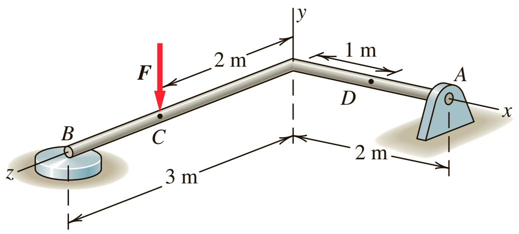


EX 10.2.23

10.2.24. [**] Determine the axial force and the magnitudes of the shear force, bending moment, and torsional moment acting on the cross section of the L-shaped beam

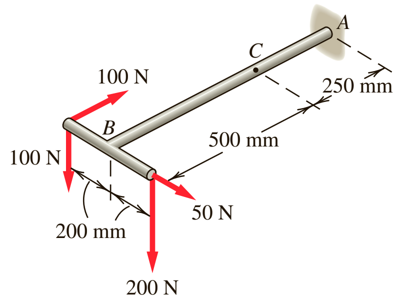
a. just to the right of *C*

b. at *D*



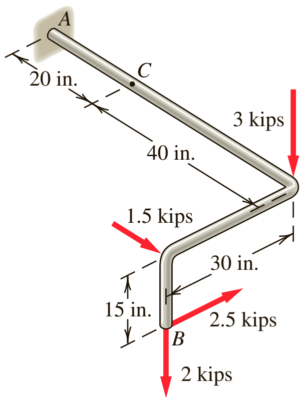
EX 10.2.24

10.2.25. [**] Determine the axial force and the magnitudes of the shear force, bending moment, and torsional moment acting on the cross section at location *C* of the T-beam shown.



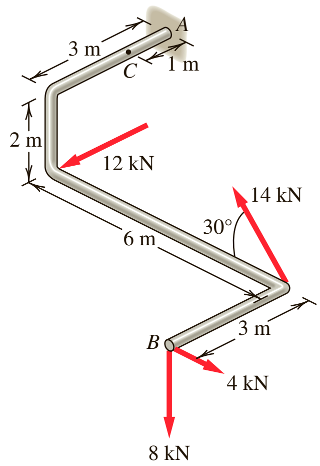
EX 10.2.25

10.2.26. [**] Determine the axial force and the magnitudes of the shear force, bending moment, and torsional moment acting on the cross section at location *C* of beam *ACB*.



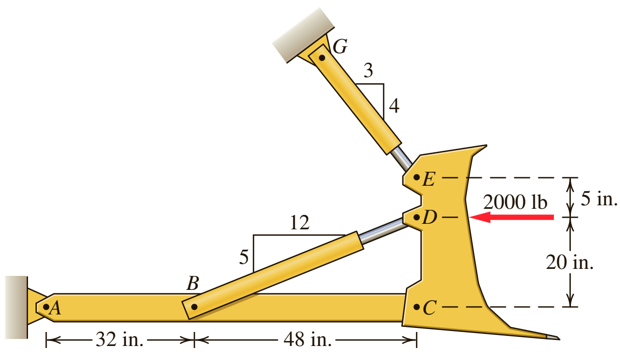
EX 10.2.26

10.2.27. [**] Determine the axial force and the magnitudes of the shear force, bending moment, and torsional moment acting on the cross section at location *C* of beam *ACB*.



EX 10.2.27

10.2.28. [**] The bulldozer shown pushes a dirt load with a resultant horizontal force of 2000 lb. The frame consists of push-arm *ABC*, hydraulic arms *BD* and *EG*, and blade *CDE*. All connections (*A*, *B*, *C*, *E*, *D*, and *G*) are frictionless pins. Ignore the weight of the members. Determine the axial force, shear force, and bending moment acting on the cross section of the beam *ABC* just to the right of point *B*.



EX 10.2.28

10.3 AXIAL FORCE, SHEAR FORCE, AND BENDING MOMENT DIAGRAMS

Learning Objective: Present the internal loads determined from beam equilibrium analysis as a series of shear, bending moment, and axial force diagrams.

In analyzing a beam you often will need to find the maximum bending moment and its location along the length of a beam because this is where the beam is likely to fail. We now illustrate how to do so for a cantilever

beam loaded by a uniformly distributed load along its top surface and an axial force at A (**Figure 10.3.1*a***).

Figure 10.3.1*a* shows a cantilever beam of length L extending from $x = 0$ (at A) to $x = L$ (at B). Consider a portion AP of the beam that extends from end A to some point P at $x < L$. Isolate this portion and draw its free-body diagram (**Figure 10.3.1*b***). The loads acting on the portion are the distributed force on the top and the point force F_{Ax} at $x = 0$. In addition, a bending moment, axial force, and shear force on the right are applied to portion AP by the rest of the beam (portion PB), which has been cut away. We will assume M_{bz}, V_y, and N_x positive, as drawn in **Figure 10.3.1*b***. (Notice that this same free-body diagram describes any portion of the beam that extends from the origin to some length $x < L$.)

We write the three equilibrium equations for a planar system based on this free-body diagram in order to create expressions for the bending moment, the shear force, and the axial force as functions of x:

$$\sum F_x\,(\rightarrow +) = 0 \quad \Rightarrow \quad -F_{Ax} + N_x = 0 \tag{10.1A}$$

$$\sum F_y(\uparrow +) = 0 \quad \Rightarrow \quad \underbrace{-\omega x}_{\text{force created by distributed load}} - V_y = 0 \tag{10.1B}$$

In order to write the moment equilibrium equation, we must select a moment center. Although the origin of the coordinate system shown in **Figure 10.3.1*a*** might seem like the obvious choice, we can simplify the math involved by choosing P, because then neither N_x nor V_y are included in the equation. We write the moment equilibrium equations about P as

$$\sum M_{z\,@\,P}(\curvearrowleft +) = 0 \Rightarrow \quad \underbrace{+\frac{x}{2}\,\omega x}_{\text{moment created by distributed load}} + M_{bz} = 0 \tag{10.1C}$$

Both terms in this equation are positive because both create a counterclockwise moment about a z axis through P.

Some rearranging of (10.1A), (10.1B), and (10.1C) allows us to write expressions for the shear and axial forces and bending moment as functions of x:

$N_x = F_{Ax}$	axial force	(10.2A)
$V_y = -\omega x$	shear force	(10.2B)
$M_{bz} = -\frac{1}{2}\omega x^2$	bending moment	(10.2C)

These expressions are valid for any x such that $0 < x < L$. For this loading scheme, the shear force and the bending moment are functions of x, but the axial force is independent of x, being the same everywhere along the length of the beam.

We plot these expressions in **Figures 10.3.1*c*, *d*, and *e*** to gain insight into how they change with x. These plots are referred to as the **axial force diagram**, **shear force diagram**, and **bending moment diagram**. The bending moment diagram is a convenient tool for identifying the

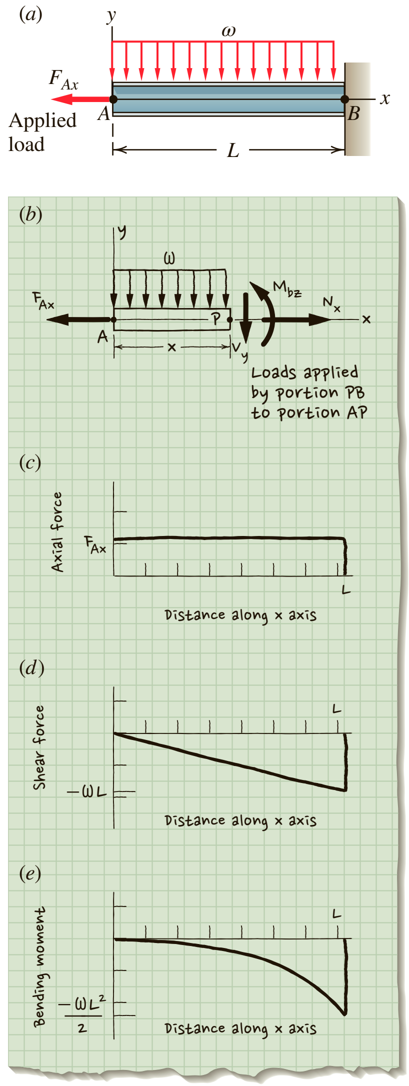


Figure 10.3.1 (*a*) Cantilever beam with distributed load causing negative bending moment; (*b*) free-body diagram of left-hand portion; (*c*) axial force diagram; (*d*) shear force diagram; (*e*) bending moment diagram.

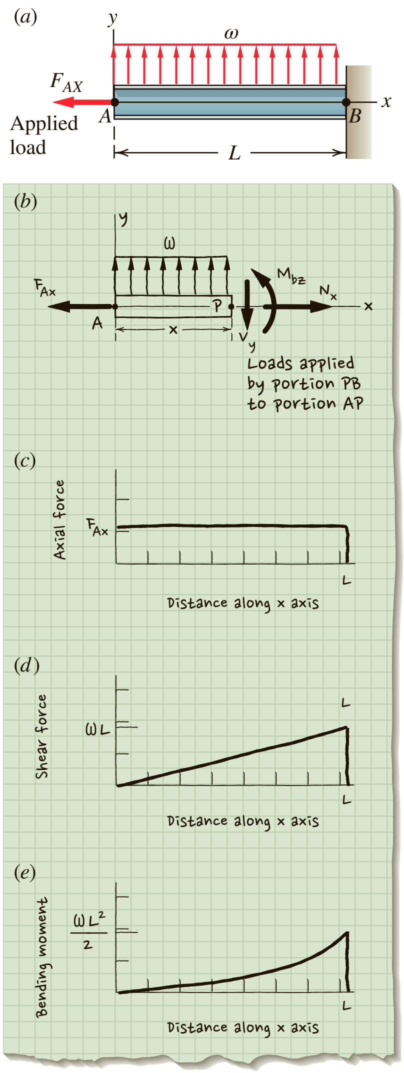


Figure 10.3.2 (*a*) Cantilever beam with distributed load causing positive bending moment; (*b*) free-body diagram of left-hand portion; (*c*) axial force diagram; (*d*) shear force diagram; (*e*) bending moment diagram.

maximum bending moment a beam must carry and at what point along the length of the beam it occurs. From **Figure 10.3.1*e*** the maximum moment and shear for the cantilever in our example occur at $x = L$; is this consistent with where you thought it would be? Is this where you would expect the beam to fail?

This cantilever beam example is repeated in **Figure 10.3.2** for the case in which the distributed load is acting upward. The solution is identical to that in **Figure 10.3.1**, except that the signs for the shear and bending moment are reversed.

The procedure we have illustrated in **Figure 10.3.1** for creating axial force, shear force, and bending moment diagrams works for any beam. In applying the procedure more generally, remember that it is necessary to

(a) create an additional free-body diagram wherever a new load is introduced along the beam length, and

(b) be consistent in defining positive and negative unknown bending moments and shear forces.

IMPORTANT NOTE! Do NOT replace a distributed load by an equivalent point load to determine the axial force, shear force, and moment diagrams. While it might seem that this replacement would simplify the calculations, it will lead to incorrect results. You can convince yourself of this by replacing the distributed load in Figure 10.3.1*a* with an equivalent point load at L/2 and drawing a free-body diagram.

Moment, Shear, and Axial Force Diagrams for Nonplanar Beams

The beams considered in **Figures 10.3.1** and **10.3.2** and Examples 10.3.1–10.3.4 are planar beams because the applied forces are all in a single plane and the moments are about an axis perpendicular to that plane. In contrast, the beam illustrated in **Figure 10.3.3** is a nonplanar beam. Point loads are applied in both the y and z directions at A causing the beam to bend in both directions. In addition M_{Ax} causes the beam to twist about the x axis.

At a point P along the length of the beam, up to six internal loads act (the actual number depends on the beam loading). These are the bending moment M_{bz} about the z axis and shear force V_y in the y direction that we identified for the planar beam, but also bending moment M_{by} about the y axis, moment M_{bx} (sometimes referred to as torque) about the x axis, and shear force V_z in the z direction. The procedure for determining internal loads for nonplanar beams is the same as for planar beam, but more involved because there are more unknowns. Bending moment diagrams for M_{bx}, M_{by}, and M_{bz}, shear diagrams for V_y and V_z, and an axial force diagram for N_x are required to get a complete picture of the loads internal to the beam.

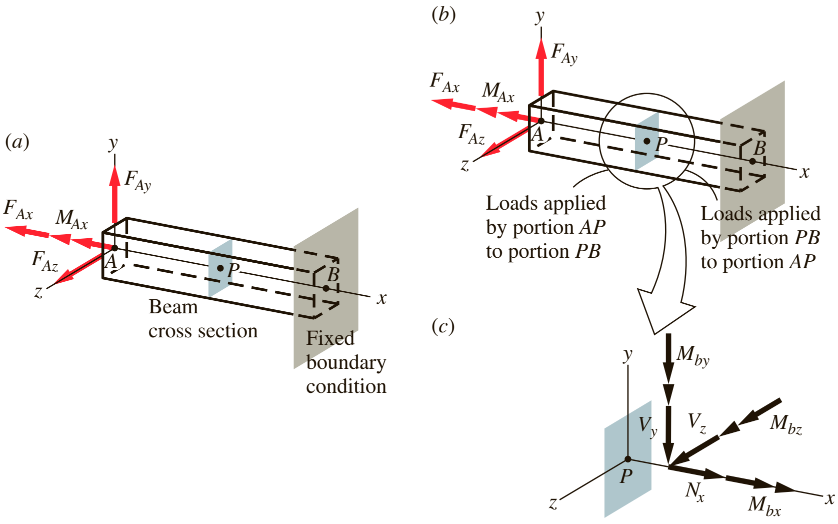


Figure 10.3.3 Loads internal to a nonplanar beam: (*a*) cantilever nonplanar beam with applied loads at *A*; (*b*) internal load pairs at point *P*; (*c*) detail of internal loads applied by portion *PB* to portion *AP* at *P*.

Check out the following examples of applications of this material.

- **Example 10.3.1 Shear, Moment, and Axial Force Diagram for a Simply Supported Beam**
- **Example 10.3.2 A Simple Beam with an Applied Moment**
- **Example 10.3.3 Beam with Distributed Load**
- **Example 10.3.4 Simply Supported Beam with an Overhang**

EXAMPLE 10.3.1

The simply supported beam in **Figure 1** is loaded with a point load at a distance $L/3$ to the right of support A. Determine the axial force, shear force, and bending moment diagrams for the beam.

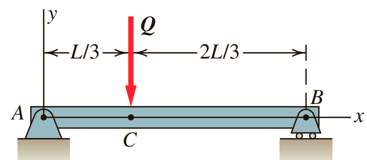


Figure 1 Simply supported beam with point load.

Goal For beam AB, find the axial force, shear force, and bending moment diagrams.

Given The dimensions, support conditions, and loading of the beam. The loads are all in one plane so we can treat beam as planar.

Assume The weight of the beam is negligible. We also assume A and B are ideal supports. This means the pin at A is frictionless and imparts no moment to the beam, and that the roller at B is frictionless and imparts no moment or axial force to the beam.

Draw By creating a free-body diagram of the entire beam and applying the conditions of equilibrium, we find the forces at supports A and B in terms of the applied force to be $F_{Ax} = 0$, $F_{Ay} = 2Q/3$, $F_{By} = Q/3$ (**Figure 2*a***).

We follow the same procedure we demonstrated for the cantilever beam and create a free-body diagram of a portion of the beam. However, because a point load is applied to beam AB, we need to create two

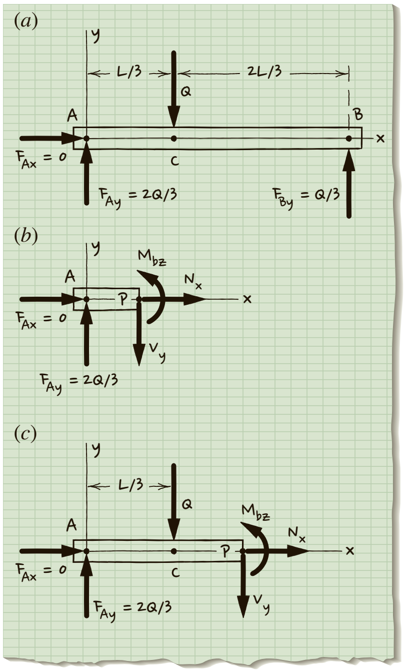


Figure 2 (*a*) Free-body diagram of entire beam; (*b*) free-body diagram with cut to the left of applied load; (*c*) free-body diagram with cut to the right of applied load.

free-body diagrams, one with P to the left of the load ($x < L/3$) (**Figure 2*b***) and the other with P to the right of the load at $x > L/3$ (**Figure 2*c***).

Formulate Equations and Solve Using the free-body diagram in **Figure 2*b*** ($x < L/3$), we write the planar equilibrium equations with the moment center at P:

$$\sum F_x(\rightarrow +) = 0 \quad \Rightarrow \quad F_{Ax} + N_x = 0$$

$$\sum F_y(\uparrow +) = 0 \quad \Rightarrow \quad \frac{2Q}{3} - V_y = 0$$

$$\sum M_{z@P}(\curvearrowleft +) = 0 \quad \Rightarrow -\frac{2Q}{3}x + M_{bz} = 0$$

These equations are valid for any x such that $0 \le x \le L/3$ and can be rearranged as

$$N_x = -F_{Ax} = 0 \qquad \text{axial force} \qquad (1A)$$

$$V_y = \frac{2Q}{3} \qquad \text{shear force} \qquad (1B)$$

$$M_{bz} = \frac{2Q}{3}x \qquad \text{bending moment} \qquad (1C)$$

Referring to the free-body diagram of the portion of the beam with the boundary at $x > L/3$ (**Figure 2*c***), we write a second set of planar equilibrium equations with moment center at P (where P is now at $L/3 \le x \le L$):

$$\sum F_x(\rightarrow +) = 0 \quad \Rightarrow \quad F_{Ax} + N_x = 0$$

$$\sum F_y(\uparrow +) = 0 \quad \Rightarrow \quad \frac{2Q}{3} - Q - V_y = 0$$

$$\sum M_{z@P}(\curvearrowleft +) = 0 \quad \Rightarrow \quad -\frac{2Q}{3}x + Q\left(x - \frac{L}{3}\right) + M_{bz} = 0$$

The moment equilibrium equation can be rearranged as

$$\frac{Q}{3}(x - L) + M_{bz} = 0$$

These equations are valid for any x such that $L/3 \le x \le L$ and can be rearranged as

$$N_x = -F_{Ax} = 0 \qquad \text{axial force} \qquad (2A)$$

$$V_y = -\frac{Q}{3} \qquad \text{shear force} \qquad (2B)$$

$$M_{bz} = \frac{Q}{3}(L - x) \qquad \text{bending moment} \qquad (2C)$$

We now create axial force, shear force, and bending moment diagrams for the simply supported beam. We use (1A)–(1C) to draw the diagrams for $0 \le x \le L/3$, and (2A)–(2C) for $L/3 \le x \le L$, as shown in **Figure 3*a*, *b*,** and ***c***.

We notice that at $x = L/3$, the bending moment equations for the two portions intersect at $M_{bz} = 2QL/9$ (this is the value of bending moment given by both (1C) and (2C) at $x = L/3$). At the same point, the shear force changes from positive to negative, represented by the step change on the shear force diagram.

From **Figure 3*c*** the largest magnitude of bending moment occurs at $x = L/3$; is this consistent with where you thought it would be? Is this where you would expect the beam to fail?

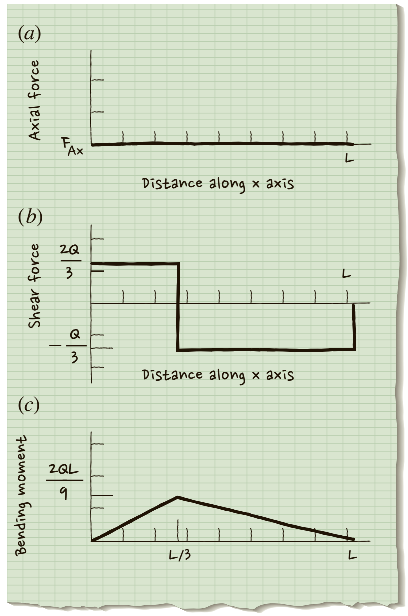


Figure 3 (*a*)–(*c*) axial force, shear force, and bending moment diagrams.

Check In the next section where we explore the mathematical relationship between V and M diagrams, we will discover that there are some consistencies between the two diagrams that help us to check if they are correct. In the meantime we can draw free-body diagrams and calculate values of N_x, V_y, and M_{bz} at several points and compare these results with the diagrams as a check.

Also, we can recognize that the shear diagram must be constant between applied point loads, and that the "jump" in the shear diagram at the location of the point load is equal in magnitude to the load. For this example V_y jumps from $2Q/3$ to $-Q/3$ at C. The total change is $-Q$, which is equal to the applied downward load.

EXAMPLE 10.3.2

A moment is applied at point B on the simply supported beam of length L in **Figure 1**. Determine the shear force and bending moment diagrams.

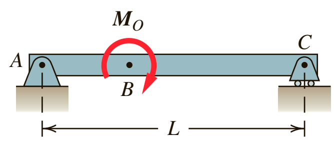


Figure 1 We model AC as a planar beam because loading lies in a single plane.

Goal Find the shear and bending moment diagrams for beam AC.

Given Information about the geometry and loading of the beam.

Assume The weight of the beam is negligible and supports are ideal.

Draw We draw a free-body diagram of the entire beam and solve for loads at the supports (**Figure 2*a***). Next, we create a coordinate system with its origin at A and make two different cuts in the beam: the first to the left of M_o, and the second to the right of M_o. We draw free-body diagrams of isolated portions AP when $x \le B$ (**Figure 2*b***), and when $x \ge B$ (**Figure 2*c***).

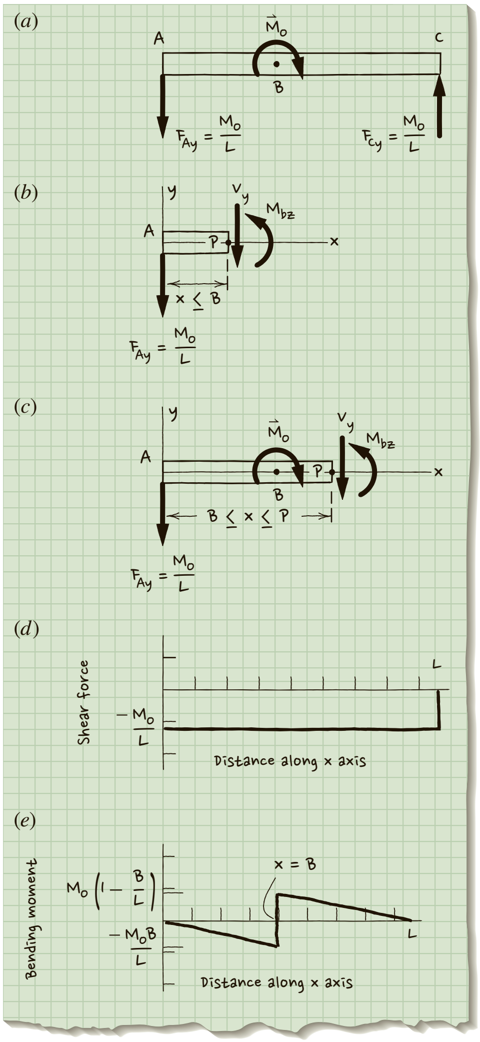


Figure 2 (*a*)–(*c*) Free-body diagrams for creating V and M diagrams; (*d*) shear diagram; (*e*) moment diagram.

Formulate Equations and Solve We analyze equilibrium of the beam portions and create equations that describe the shear and moment as a function of x.

For $0 \leq x \leq B$ (**Figure 2*b***):

$$\sum F_y(\uparrow +) = -\frac{M_o}{L} - V_y = 0$$

$$\boxed{V_y = -\frac{M_o}{L}} \qquad (1)$$

$$\sum M_{z@P}(\curvearrowleft +) = \frac{M_o}{L}x + M_{bz} = 0$$

$$\boxed{M_{bz} = -\frac{M_o}{L}x} \qquad (0 \leq x \leq B) \qquad (2)$$

For $B \leq x \leq L$ (**Figure 2*c***)

$$\sum F_y(\uparrow +) = -\frac{M_o}{L} - V_y = 0$$

$$\boxed{V_y = -\frac{M_o}{L}} \qquad (3)$$

$$\sum M_{z@P}(\curvearrowleft +) = \frac{M_o}{L}x + -M_o + M_{bz} = 0$$

$$\boxed{M_{bz} = M_o\left(1 - \frac{x}{L}\right)} \qquad (B \leq x \leq L) \qquad (4)$$

Using (1)–(4) we create the V and M diagrams (**Figures 2*d*** and **2*e***).

Check We can check the value of the diagrams at several points, and also look at the diagram shapes. The shear force is constant between A and C, because no forces are applied between the supports. The bending moment diagram exhibits a discontinuity of magnitude M_o at the location of the applied moment.

EXAMPLE 10.3.3

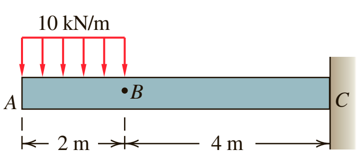


Figure 1 Cantilever beam with distributed load.

For the beam and loading shown in **Figure 1** draw the shear force and bending moment diagrams.

Goal Draw the shear force and bending moment diagrams for beam ABC.

Given Information about the dimensions and loading of the beam, which allow us to model AC as a planar beam.

Assume The weight of the beam is negligible and the fixed support at C is ideal.

Draw We require two free-body diagrams to develop the equations that describe the shear and moment diagrams for the beam. First we cut

the beam where the distributed load is applied (**Figure 2*a***). The free-body diagram for this isolated beam segment is valid for a cut anywhere between A and B—that is, for $0 \le x \le 2$ m. Then we cut between B and C, which is beyond the applied distributed load (**Figure 2*b***), and draw a free-body diagram that is valid for a cut such that $2\text{ m} \le x \le 6$ m.

Formulate Equations and Solve Analyzing equilibrium of the beam segments, we create equations that describe the shear and moment as a function of x.

For $0 \le x \le 2$ m (**Figure 2*a***):

In our equilibrium analysis, we must include only the portion of the distributed load that is acting on the segment of the beam we are considering. The length of the beam segment we have isolated to the left of the cut at P is x. Therefore the load is (10 kN/m)x, and:

$$\sum F_y(\uparrow +) = -10\frac{\text{kN}}{\text{m}}x - V_y = 0$$

$$\boxed{V_y = -10\frac{\text{kN}}{\text{m}}x} \qquad (0 \le x \le 2\text{ m}) \tag{1}$$

This equation describes a line from 0 kN at $x = 0$ m to –20 kN at $x = 2$ m.

We choose P as our moment center in the moment equilibrium equation. This means that we measure the moment arm from P to the centroid of the distributed load. For free-body diagram in **Figure 2*a*** that distance is $x/2$.

$$\sum M_{z@P}(\curvearrowleft +) = \left(10\frac{\text{kN}}{\text{m}}x\right)\frac{x}{2} + M_{bz} = 0$$

$$\boxed{M_{bz} = -\left(5\frac{\text{kN}}{\text{m}}\right)x^2} \qquad (0 \le x \le 2\text{ m}) \tag{2}$$

This equation describes a curve that starts from 0 kN·m at $x = 0$ m and bends parabolically down to –20 kN·m at $x = 2$ m.

For $2\text{ m} \le x \le 6$ m (**Figure 2*b***):

The entire distributed load is acting on the isolated portion of the beam shown in **Figure 2*b***. Therefore the load is (10 kN/m) (2 m) = 20 kN, and:

$$\sum F_y(\uparrow +) = -20\text{ kN} - V_y = 0$$

$$\boxed{V_y = -20\text{ kN}} \quad (2\text{ m} \le x \le 6\text{ m}) \tag{3}$$

For the moment equilibrium equation we again measure the moment arm from P to the centroid of the distributed load, which is 1 m to the right of A. Therefore the length of the moment arm is (x – 1 m).

$$\sum M_{z@P}(\curvearrowleft +) = 20\text{ kN}(x - 1\text{ m}) + M_{bz} = 0$$

$$\boxed{M_{bz} = 20\text{ kN}\cdot\text{m} - (20\text{ kN})x} \qquad (2\text{ m} \le x \le 6\text{ m}) \tag{4}$$

For this portion of the beam, the bending moment diagram is a straight line extending from –20 kN·m to –100 kN·m.

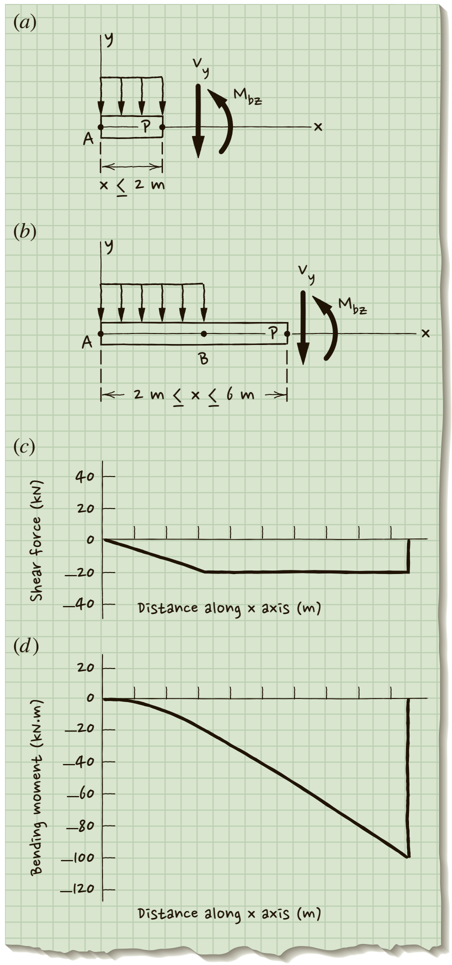


Figure 2 (*a*)–(*b*) free-body diagrams used to create V and M diagrams; (*c*) shear diagram; (*d*) moment diagram.

Using (1)–(4) we create the shear force and bending moment diagrams for the beam (**Figures 2*c*** and **2*d***).

Check In addition to checking the value of the diagrams at several points, we look at the shapes of the diagrams. The shear force is decreasing linearly between *A* and *B* due to the uniformly distributed load, and is constant between *B* and *C* where no loads are applied.

We illustrate another method for checking the shear and bending moment relationships in the next section.

EXAMPLE 10.3.4

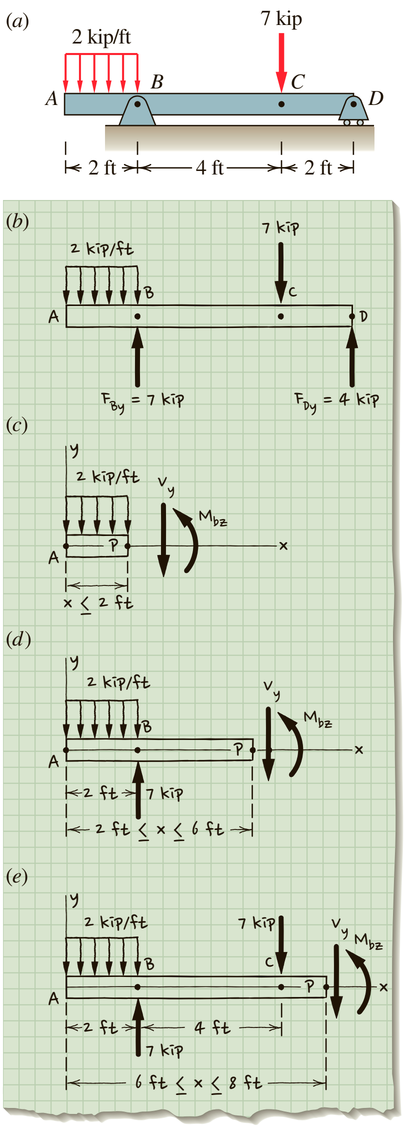


Figure 1 (*a*) Simply supported beam with overhang; (*b*) free-body diagram of entire beam; (*c*)–(*e*) free-body diagrams used to create *V* and *M* diagrams.

Given the beam and loading shown in **Figure 1*a***, draw the shear force and the bending moment diagrams. Locate the maximum bending moment for the beam.

Goal Draw the shear force and bending moment diagrams and locate the maximum bending moment.

Given Information about the dimensions and loading of the beam (indicating a planar beam model).

Assume The weight of the beam is negligible and the supports are ideal. This means the pin at *B* is frictionless and imparts no moment to the beam, and that the roller at *D* is frictionless and imparts no moment or axial force to the beam.

Draw We draw a free-body diagram of the beam and solve for the loads at *B* and *D* (**Figure 1*b***). We then create three free-body diagrams: one for a cut between *A* and *B*, one for a cut between *B* and *C*, and one for a cut between *C* and *D* (**Figures 1*c***, **1*d***, and **1*e***).

Formulate Equations and Solve We perform an equilibrium analysis of each of the three free-body diagrams to create equations that describe the shear and bending moment as a function of *x*.

For $0 \le x \le 2$ ft (**Figure 1*c***), equilibrium in the *y* direction is:

$$\sum F_y(\uparrow +) = -2\frac{\text{kip}}{\text{ft}}x - V_y = 0$$

$$\boxed{V_y = -2\frac{\text{kip}}{\text{ft}}x} \qquad (0 \le x \le 2\text{ ft}) \tag{1}$$

This equation describes a line from 0 kip at $x = 0$ ft to -4 kip at $x = 2$ ft. The moment equilibrium equation is:

$$\sum M_{z@P}(\curvearrowleft +) = \left(2\frac{\text{kip}}{\text{ft}}x\right)\frac{x}{2} + M_{bz} = 0$$

$$\boxed{M_{bz} = -\left(1\frac{\text{kip}}{\text{ft}}\right)x^2} \qquad (0 \le x \le 2\text{ ft}) \tag{2}$$

This equation describes a parabolic curve from 0 kip·ft at $x = 0$ ft to -4 kip·ft at $x = 2$ ft.

For 2 ft $\le x \le$ 6 ft (**Figure 1*d***), equilibrium in the y direction gives:

$$\sum F_y(\uparrow +) = -\left(2\,\frac{\text{kip}}{\text{ft}}\right)(2\text{ ft}) + 7\text{ kip} - V_y = 0$$

$$\boxed{V_y = 3\text{ kip}} \qquad (2 \le x \le 6\text{ ft}) \qquad (3)$$

Summing the moments about a z axis at the cut gives:

$$\sum M_{z@P}(\curvearrowleft +) = 0 = 2\,\frac{\text{kip}}{\text{ft}}(2\text{ ft})(x - 1\text{ ft}) - 7\text{ kip}(x - 2\text{ ft}) + M_{bz} = 0$$

With rearranging and simplifying this becomes

$$\boxed{M_{bz} = -10\text{ kip·ft} + (3\text{ kip})x} \qquad (2\text{ ft} \le x \le 6\text{ ft}) \qquad (4)$$

For this portion of the beam, the bending moment diagram is a straight line extending from -4 kip·ft at $x = 2$ ft to 8 kip·ft at $x = 6$ ft.

For 6 ft $\le x \le$ 8 ft (**Figure 1*e***), we get:

$$\sum F_y(\uparrow +) = -\left(2\,\frac{\text{kip}}{\text{ft}}\right)(2\text{ ft}) + 7\text{ kip} - 7\text{ kip} - V_y = 0$$

$$\boxed{V_y = -4\text{ kip}} \qquad (6\text{ ft} \le x \le 8\text{ ft}) \qquad (5)$$

and:

$$\sum M_{z@P} = 0(\curvearrowleft +)$$

$$2\,\frac{\text{kip}}{\text{ft}}(2\text{ ft})(x - 1\text{ ft}) - 7\text{ kip}(x - 2\text{ ft}) + 7\text{ kip}(x - 6\text{ ft}) + M_{bz} = 0$$

With rearranging and simplifying this becomes

$$\boxed{M_{bz} = 32\text{ kip·ft} - (4\text{ kip})x} \quad (6\text{ ft} \le x \le 8\text{ ft}) \qquad (6)$$

The bending moment diagram is a straight line extending from 8 kip·ft at $x = 6$ ft to 0 kip·ft at $x = 8$ ft.

Using (1)–(6) we create the shear force and bending moment diagrams for the beam (**Figures 2*b*** and **2*c***). We see from the bending moment diagram that the maximum moment is 8 kip·ft and it occurs at point C.

Check We can calculate V and M at several points and compare the values with the diagrams. We also can look at the shapes of the diagrams. The shear force is decreasing linearly between A and B due to the uniformly distributed load, and is constant between B and C, and between C and D where no loads are applied. We'll learn more about the shape of the moment diagram in the next section.

We will check the results for this example in the next section using the soon-to-be-developed relationships between ω, V_y, and M_{bz}.

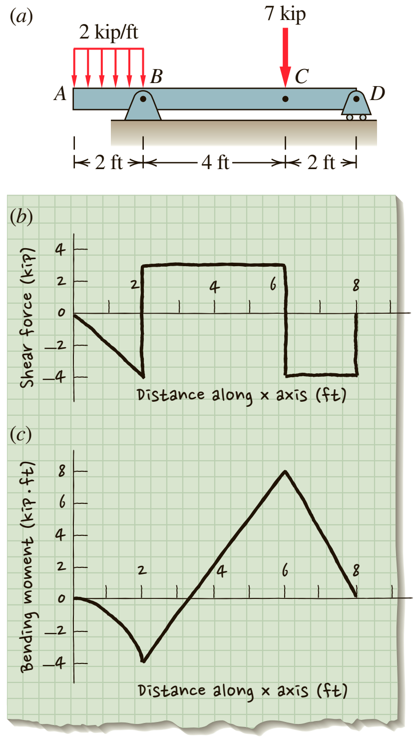


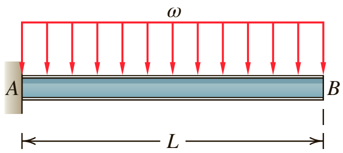
Figure 2 (*a*) Simply supported beam with overhang; (*b*) shear diagram; (*c*) moment diagram.

EXERCISES 10.3

10.3.1. [*] Consider the cantilever beam AB.

a. Draw the shear and bending moment diagrams.

b. Determine the bending moment with the largest magnitude and its location.



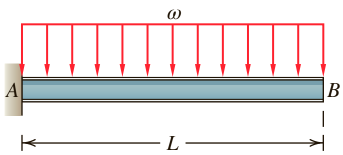
EX 10.3.1

10.3.2. [*] Replace the distributed load acting on beam AB by a point load at the centroid of the distribution. For both the distributed load and the point load

a. draw the free-body diagrams needed to determine the shear and moment diagrams.

b. determine the V_y and M_{bz} diagrams.

Do you get the same results? Discuss your conclusion.

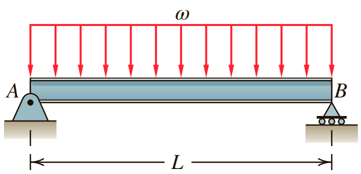


EX 10.3.2

10.3.3. [*] Consider the simply supported beam shown.

a. Draw the shear and bending moment diagrams.

b. Determine the bending moment with the largest magnitude and its location.



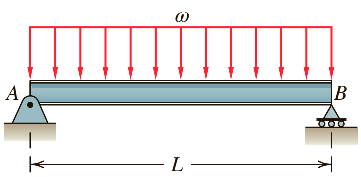
EX 10.3.3

10.3.4. [*] Replace the distributed load acting on beam AB by a point load at the centroid of the distribution. For both the distributed load and the point load

a. draw the free-body diagrams needed to determine the shear and moment diagrams.

b. determine the V_y and M_{bz} diagrams.

Do you get the same results? Discuss your conclusion.

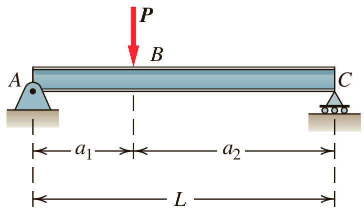


EX 10.3.4

10.3.5. [*] Consider the beam shown.

a. Draw the shear and bending moment diagrams.

b. Determine the bending moment with the largest magnitude and its location.

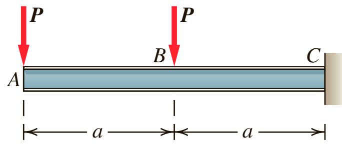


EX 10.3.5

10.3.6. [*] Consider beam ABC.

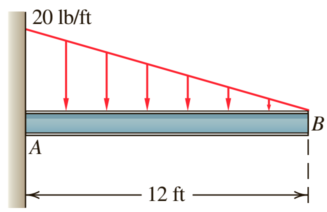
a. Draw the shear and bending moment diagrams.

b. Determine the bending moment with the largest magnitude and its location.



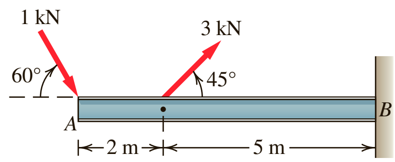
EX 10.3.6

10.3.7. [*] Determine the shear force and bending moment diagrams for the beam shown.



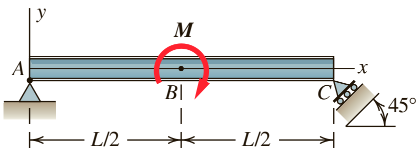
EX 10.3.7

10.3.8. [*] Determine the axial force, shear force, and bending moment diagrams for the beam shown.



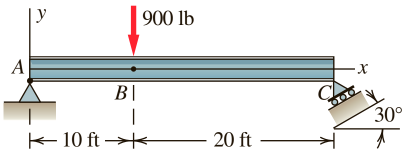
EX 10.3.8

10.3.9. [*] Determine the axial force, shear force, and bending moment diagrams for the beam shown.



EX 10.3.9

10.3.10. [*] Determine the axial force, shear force, and bending moment diagrams for the beam shown.

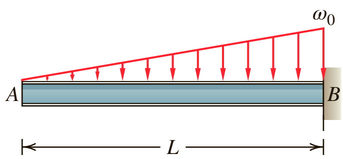


EX 10.3.10

10.3.11. [*] Consider the beam and loading shown.

a. Draw the shear and bending moment diagrams.

b. Determine the bending moment with the largest magnitude and its location.

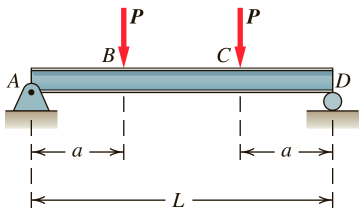


EX 10.3.11

10.3.12. [*] Consider the beam and loading shown.

a. Draw the shear and bending moment diagrams.

b. Determine the bending moment with the largest magnitude and its location.

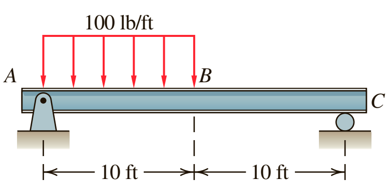


EX 10.3.12

10.3.13. [*] Consider the simply supported beam ABC.

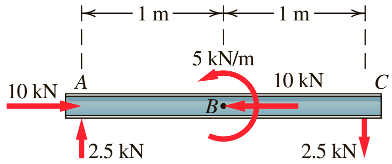
a. Draw the shear and bending moment diagrams.

b. Determine the bending moment with the largest magnitude and its location.



EX 10.3.13

10.3.14. [*] For the beam with the loading as shown, draw the axial force, shear force, and bending moment diagrams.

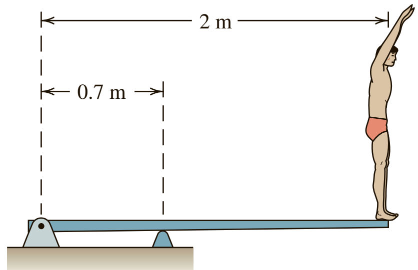


EX 10.3.14

10.3.15. [*] The diving board is supported by a pin at the left end and a smooth wedge at 0.7 m from the end. A 100-kg person stands on the free end.

a. Draw the shear and bending moment diagrams for the diving board.

b. Determine the bending moment with the largest magnitude and its location.

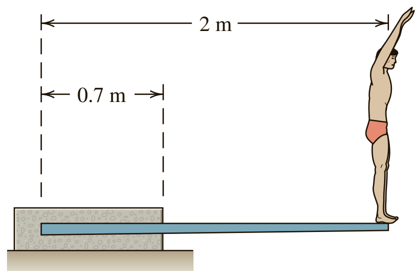


EX 10.3.15

10.3.16. [*] Consider the cantilevered diving board with a 100-kg person standing on the free end.

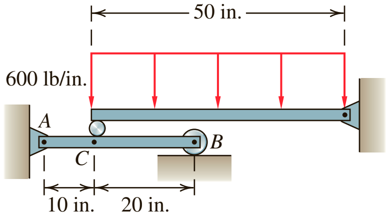
a. Draw the shear and bending moment diagrams for the diving board.

b. Determine the bending moment with the largest magnitude and its location.



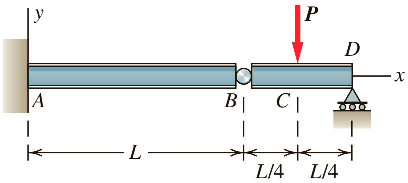
EX 10.3.16

10.3.17. [**] Beam AB supports a second beam loaded with a 600 lb/in. load. Determine the shear and moment diagrams for beam AB.



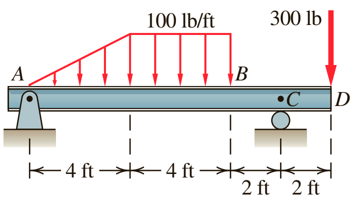
EX 10.3.17

10.3.18. [**] Beams AB and BCD are connected by a hinge at B. Determine the shear and moment diagrams for the beam system.



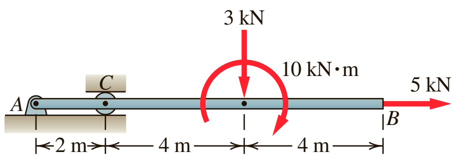
EX 10.3.18

10.3.19. [**] Consider the beam with an overhang.
a. Draw the shear and bending moment diagrams.
b. Determine the bending moment with the largest magnitude and its location.



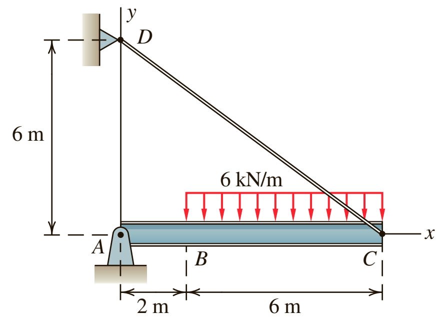
EX 10.3.19

10.3.20. [**] Determine the axial force, shear force, and bending moment diagrams for the beam shown.



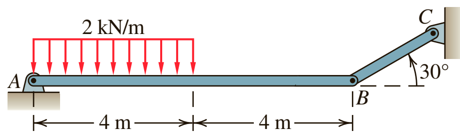
EX 10.3.20

10.3.21. [**] Beam ABC is supported by a pin at A and a cable that runs from C to D.
a. Draw the axial force, shear force, and bending moment diagrams.
b. Determine the bending moment with the largest magnitude and its location.



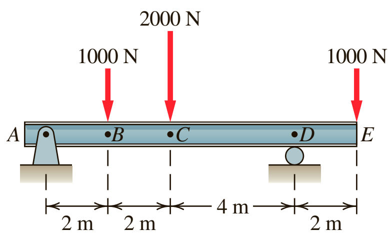
EX 10.3.21

10.3.22. [**] Determine the axial force, shear force, and bending moment diagrams for beam AB.



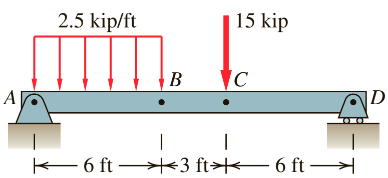
EX 10.3.22

10.3.23. [**] Consider the beam and loading shown.
a. Draw the shear and bending moment diagrams.
b. Determine the bending moment with the largest magnitude and its location.



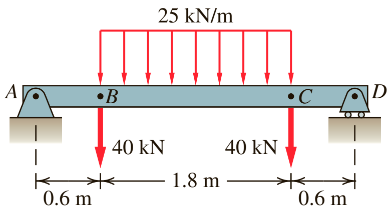
EX 10.3.23

10.3.24. [**] Consider the beam and loading shown.
a. Draw the shear and bending moment diagrams.
b. Determine the bending moment with the largest magnitude and its location.



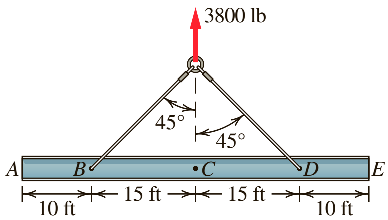
EX 10.3.24

10.3.25. [**] Consider the beam and loading shown.
a. Draw the shear and bending moment diagrams.
b. Determine the bending moment with the largest magnitude and its location.



EX 10.3.25

10.3.26. [**] A 50-ft steel beam that weighs 76 lb per foot is lifted by cables attached at *B* and *D*. Draw the axial force, shear force, and moment diagrams for the beam.

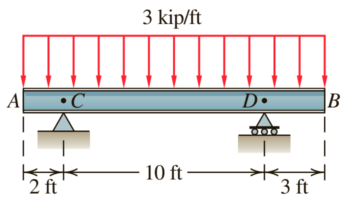


EX 10.3.26

10.3.27. [**] A beam and loading are shown.

a. Develop equations that describe *V* and *M* as a function of *x*.

b. Draw the shear and moment diagrams.

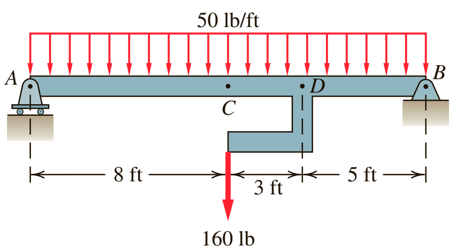


EX 10.3.27

10.3.28. [**] For beam *AB*:

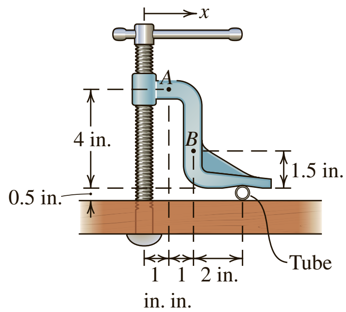
a. Write expressions for the bending moment as functions of *x*.

b. Draw the shear and moment diagrams.



EX 10.3.28

10.3.29. [**] A clamp is used to hold a 0.5-in. diameter tube on a bench top with a clamping force of 200 lb. Determine the axial force, shear force, and bending moment at points *A* and *B* on the clamp. Derive an expression for the bending moment along the top arm of the clamp as a function of *x*.

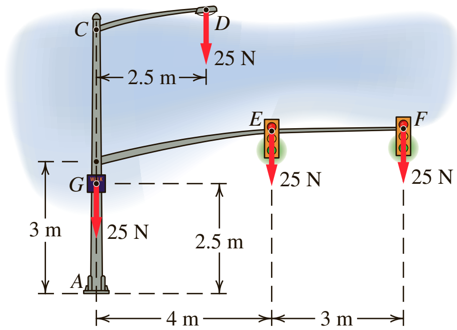


EX 10.3.29

10.3.30. [**] Consider the traffic signal with loading as shown.

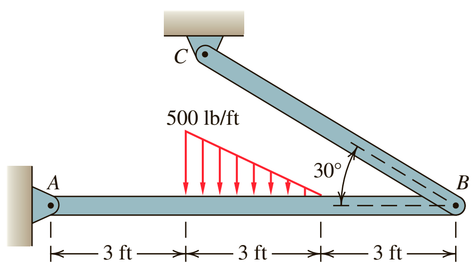
a. Draw the axial force, shear force, and bending moment diagrams for the beam *AC*.

b. For beam *AC* determine the bending moment with the largest magnitude and its location.



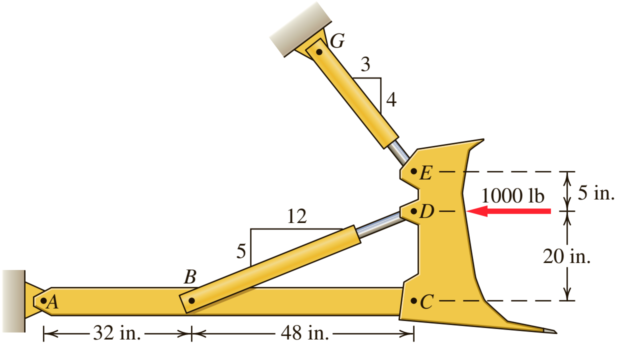
EX 10.3.30

10.3.31. [**] For beam *AB* shown determine the axial force, shear force, and moment diagrams.



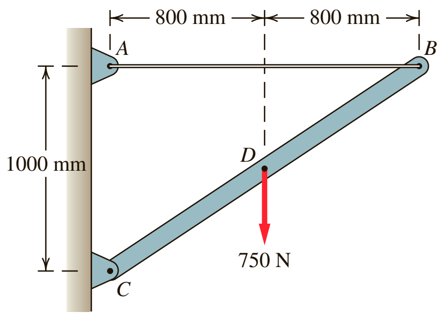
EX 10.3.31

10.3.32. [**] Beam *ABC* is part of the support mechanism for a front loader on a bulldozer. Determine the axial force, shear force, and moment diagrams.



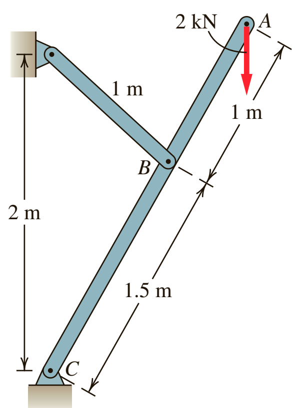
EX 10.3.32

10.3.33. [**] For beam BC determine the axial force, shear force, and moment diagrams. (Keep in mind that you want to cut the beam perpendicular to its long axis.)



EX 10.3.33

10.3.34. [**] For the beam ABC determine the axial force, shear force, and moment diagrams. (Keep in mind that you want to cut the beam perpendicular to its long axis.)

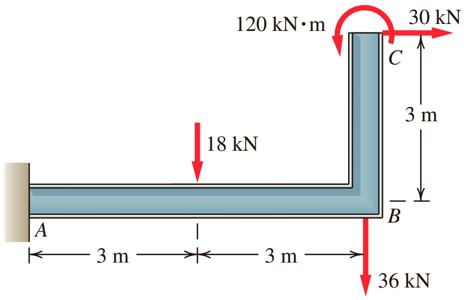


EX 10.3.34

10.3.35. [***] Consider the L-shaped beam with loading as shown.

a. Draw the axial force, shear force, and bending moment diagrams for the L-shaped beam. (*Hint:* Consider AB to be one beam and BC to be another beam.)

b. Determine the bending moment with the largest magnitude and its location.



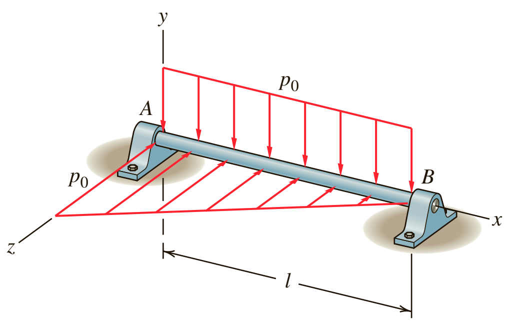
EX 10.3.35

10.3.36. [***] Bar AB is supported by a thrust bearing at A and a journal bearing at B, as shown. It is subjected to distributed loads in mutually perpendicular planes.

a. Write expressions for the bending moments M_{bz} and M_{by} as functions of x.

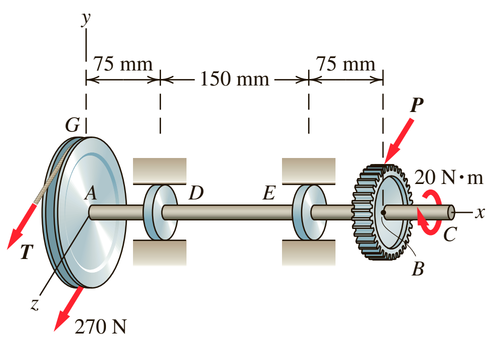
b. Combine the expressions for bending moment into a single expression for bending moment as a function of x. (Remember, moment is a vector quantity.)

c. Determine the loads acting on the bar at each of the bearings.



EX 10.3.36

10.3.37. [***] The shaft shown carries pulley A (diameter of 150 mm) and gear B (diameter of 100 mm). It is supported by frictionless bearings D and E. The gear B is subjected to the force $\boldsymbol{P} = -40\ \text{N}\ \boldsymbol{j} + 210\ \text{N}\boldsymbol{k}$. A moment (torque) of 20 N·m is applied at C to prevent rotation of the shaft.



EX 10.3.37

a. Determine the magnitude of the tension $\boldsymbol{T}$ in the belt at point G and the loads acting on the shaft at the bearing D and E. Belt force $\boldsymbol{T}$ and the 270 N force are in the z direction.

b. Write an expression for the bending moment M_{bz} as a function of x.

c. Write an expression for the bending moment M_{by} as a function of x.

d. Combine the expressions for bending moment into a single expression for bending moment as a function of x. (Remember, moment is a vector quantity.)

e. Write an expression for the twisting (torque) moment in the shaft as a function of x.

10.3.38. [***] Consider the freeway exit sign shown.

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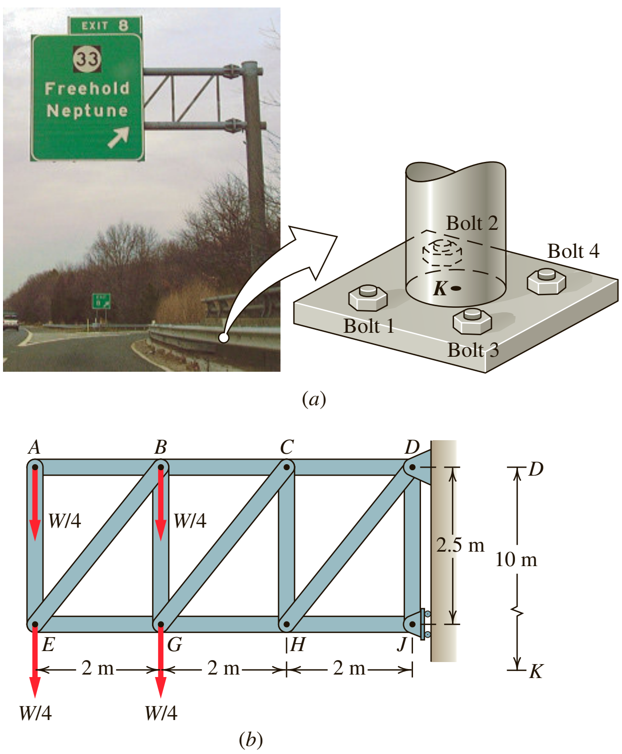


EX 10.3.38

a. Determine the loads that the horizontal boom (as shown in **E10.3.38*b***) applies to the vertical support KD if the system is in equilibrium.

b. Draw the shear, axial force, and bending moment diagrams for the beam KD.

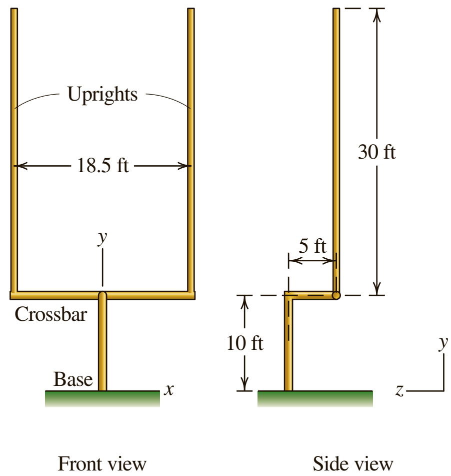
c. If the beam KD is in equilibrium, what loads act on it at its base? (You could use the diagrams in **b** to answer this question, or you could do additional calculations.)

d. Based on the loads you found in **c**, which of the four bolts do you expect to be in greatest tension (and why)?

10.3.39. [***] An NFL goal post consists of the base, crossbar, and uprights. The base rises 10 ft (the bottom 6 ft are usually padded) and extends forward 5 ft. The crossbar is 18.5 ft wide, and the uprights rise 30 ft above the crossbar. The base and crossbar are each 6 in. in diameter, and the uprights are 4 in. in diameter (assume they are solid).

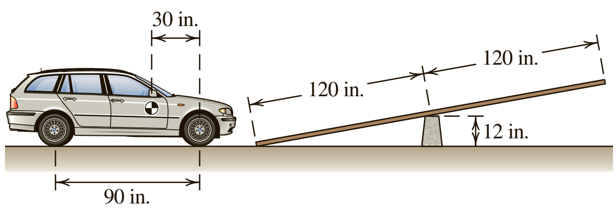
a. What are the loads acting at the bottom of the base if the goal post is in equilibrium? What is the maximum bending moment in the crossbar and where is it located?

b. Suppose that two exuberant fans have climbed onto the crossbar. One is positioned immediately adjacent to the left upright, and the other is halfway between the base and the left upright. Assume that each fan weighs $W = 150$ lbs. What are the loads acting at the bottom of the base if the goal post is in equilibrium? What is the maximum bending moment in the crossbar, and where is it located? Compare your answers with those from **a**.



EX 10.3.39

10.3.40. [***] To get a car weighing 2000 lb over a wall 12 in. high, two parallel boards are laid across the wall in front of the wheels as shown. The driver plans to drive the car onto the boards until they tip, allowing her to drive down the other side. If the bending moment in either board exceeds 35,000 lb·in, the board will break. Determine whether the boards are strong enough to allow the woman to drive the car slowly to the other side of the



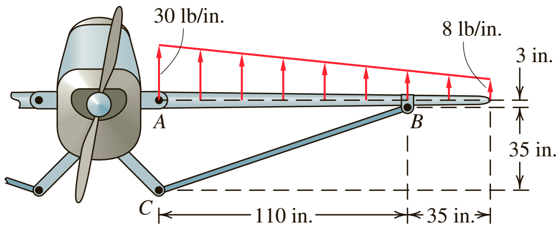
EX 10.3.40

wall. Neglect the inclination of the board when calculating V and M.

10.3.41. [***] Consider the airplane wing subjected to the lift load shown.

a. Draw the shear and bending moment diagrams. Be sure to include the load from strut BC.

b. Determine the bending moment with the largest magnitude and its location.



EX 10.3.41

10.4 BENDING MOMENT RELATED TO SHEAR FORCE AND NORMAL STRESS

Learning Objective: Relate the beam internal loads to one another.

The mathematical relationships between loads, shear forces, bending moments, and normal stresses give engineers powerful tools for analysis. In addition to determining important design values such as maximum moment or maximum normal stress, engineers also can use them for checking consistency between various analyses. In this section we develop these relationships and demonstrate their use in checking results and in determining design values for a system.

Bending Moment Related to Shear Force

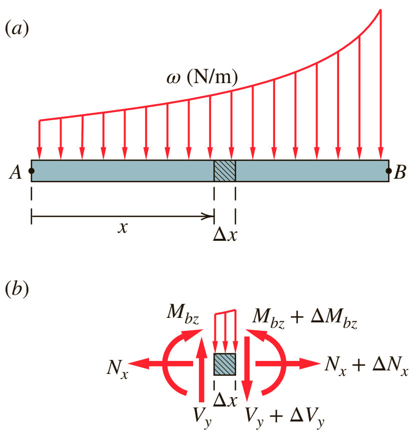


Figure 10.4.1 (*a*) Distributed load ω applied to beam AB; (*b*) free-body diagram of portion Δx of AB.

Figure 10.4.1*a* shows a beam that extends from A to B and is loaded by a distributed load ω. If we zoom in and consider moment equilibrium for a portion Δx of the beam (**Figure 10.4.1*b***) we can determine that the relationship between shear force and bending moment is

$$V_y = \frac{dM_{bz}}{dx} \tag{10.3}$$

This expression says that the shear force is equal to the slope of the bending moment curve. Equation (10.3) is a useful check on bending moment and shear force equations, as illustrated in Example 10.4.1.

Equilibrium in the vertical direction applied to the free-body diagram in **Figure 10.4.1*b*** results in a relationship between the distributed force ω and the shear force:

$$\omega = \frac{-dV_y}{dx} \tag{10.4A}$$

This expression says that the distributed force ω is the negative slope of the shear force curve. If we combine (10.3) and (10.4A) we get

$$-\omega = \frac{dV_y}{dx} = \frac{d^2 M_{bz}}{dx^2} \tag{10.4B}$$

Another way to examine these relationships is to look at the integrals. Integrating (10.3) we get an equation that says the change in the moment

diagram between A and B is the area under the shear diagram between A and B.

$$\int_A^B dM_{bz} = M_{bz@B} - M_{bz@A} = \int_A^B V_y\,dx \tag{10.5}$$

Similarly, we integrate (10.4A) to produce an equation in which the change in the shear diagram between A and B is the negative of the area under the distributed load curve between A and B.

$$\int_A^B dV_y = V_{y@B} - V_{y@A} = -\int_A^B \omega(x)\,dx \tag{10.6}$$

Equation (10.4B) can be used to find the bending moment if ω is a known and continuous function of x; two integrations are required. More generally, expressions (10.3)–(10.6) show that equilibrium imposes specific relationships between the shear force, bending moment, and distributed force acting along the length of a beam.

> **IMPORTANT NOTE!** Equations (10.3) through (10.6) require that you use the correct sign convention for ω, V, and M. The sign conventions for V and M are summarized in Figure 10.2.3, and ω is positive when downward.

Bending Moment Related to Stress

Apply two moments to a beam as in **Figure 10.4.2*a*** and the bottom of the beam stretches (because it is in tension) while the top contracts (because it is in compression). The tension force in the bottom of the beam and the compression force in the top form a couple (**Figure 10.4.2*b***). This couple is, in fact, what we have been calling "bending moment" (see **Figure 10.4.2*c***). Therefore we can legitimately replace the two antiparallel arrows representing the bending moment M_b in **Figure 10.4.2*b*** with the curved single arrow for M_b in **Figure 10.4.2*c***.

The tension and compression that make up the bending moment are not actually internal point forces as depicted in **Figure 10.4.2*b***, but rather are an internal "pressure" commonly referred to as **stress**. The tension force is created within the beam as a distribution of stresses that tend to pull the material, and the compression force is created within the beam as a distribution of stresses that then push the material. The complete distribution of stresses in the beam's cross section is shown in **Figure 10.4.2*d***. The stress in **Figure 10.4.2*d*** is called **normal stress** because it acts normal (perpendicular) to the cross section and its distribution is described by the relationship

$$\text{normal stress} = \sigma = \frac{-M_{bz}y}{I} \tag{10.7}$$

where M_{bz} is the bending moment at a particular location x along the length of the beam and y in the numerator is the y coordinate within the cross section where the stress is being calculated. The denominator is the

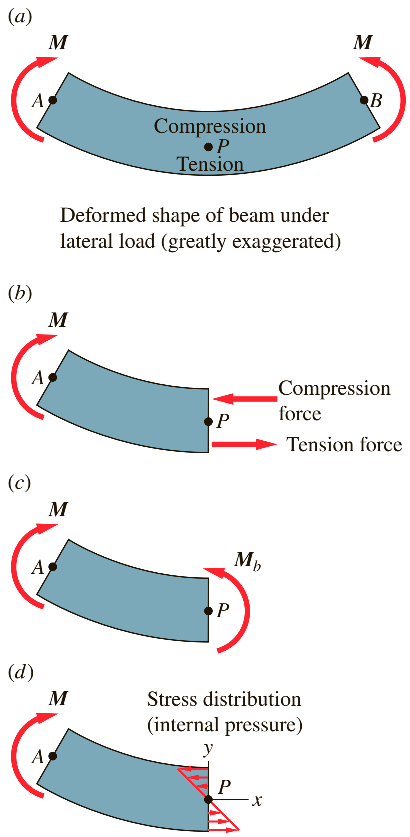


Figure 10.4.2 (*a*) Moment M applied to beam AB; (*b*) portion of beam AP showing compression force in top and tension force in bottom of cross section at P (these forces form a couple); (*c*) couple in (*b*) represented as the bending moment M_b; (*d*) bending moment created by distributed internal pressure (normal stress).

area moment of inertia, the scalar quantity that reflects the distribution of material in the beam cross section that we discussed in Section 6.4. Equation (10.7) is derived by considering equilibrium in a portion of a beam, and you can find its complete development in a mechanics of materials text. Stress is commonly expressed in N/mm^2 or MPa in the SI system, and in pounds/in^2 (psi) or kip/in^2 (ksi) in the English system. Notice that it has the same units as pressure.

Equation (10.7) shows that the normal stress is largest

- at the cross section of the beam where the bending moment M_{bz} is largest;
- at the location on that cross section with the largest value of y; (Because y is measured as shown in **Figure 10.4.2*d***, this means that stress is highest at the beam surface.)
- for equal bending moments, at the cross section with the smallest area moment of inertia.

Determining stress values in beams, and in systems more generally, is the focus of courses on mechanics of materials. Values of stress are compared with material capacity to determine whether the system is adequate (i.e., will not fail).

IMPORTANT NOTE! Equation (10.7) requires that the origin of the xy coordinate system be defined at the centroid of the cross section.

Check out the following examples of applications of this material.
- **Example 10.4.1 Using the Relationships between ω, V, and M**
- **Example 10.4.2 Calculating Beam Normal Stress**

EXAMPLE 10.4.1

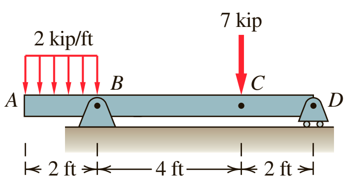


Figure 1 Simply supported beam with overhang.

Check the solution for Example 10.3.4 shown in **Figure 1** using (10.3) and (10.4).

$$V_y = \frac{dM_{bz}}{dx} \quad (10.3) \qquad \omega = -\frac{dV_y}{dx} \quad (10.4A) \qquad -\omega = \frac{d^2M_{bz}}{dx^2} \quad (10.4B)$$

Goal Examine the equations that describe the shear force and bending moment diagrams for Example 10.3.4 and verify that they satisfy equations (10.3) and (10.4).

Given Information about the dimensions and loading of the beam, and the equations for the V and M diagrams (See Example 10.3.4).

Assume The weight of the beam is negligible and the supports are ideal.

Formulate Equations and Solve We look at the equations for each segment.

Segment A–B **($0 \leq x \leq 2$ ft)** and free-body diagram in **Figure 2*a***:

$$M_{bz} = -\left(1\,\frac{\text{kip}}{\text{ft}}\right)x^2$$

Based on (10.3) we write:

$$V_y = \frac{dM_{bz}}{dx} = \frac{d}{dx}\left(-1\frac{\text{kip}}{\text{ft}}x^2\right) = -\left(2\frac{\text{kip}}{\text{ft}}\right)x$$

This agrees with (1) in Example 10.3.4 and with the shear diagram shown in Figure 3*a*.

Based on (10.4A) we write

$$\omega = -\frac{dV_y}{dx} = -\frac{d}{dx}\left(-2\frac{\text{kip}}{\text{ft}}x\right) = 2\frac{\text{kip}}{\text{ft}}$$

Yes, this agrees with the 2 kip/ft distributed load in the downward direction that is applied to the beam between A and B.

Another way to examine these relationships is to look at the integrals. Using (10.6), we get

$$V_{y@B} - V_{y@A} = -\int_A^B \omega(x)\,dx = -\int_0^{2\,\text{ft}} 2\frac{\text{kip}}{\text{ft}}dx$$

$$V_{y@B} = -2\frac{\text{kip}}{\text{ft}}x\Big|_0^{2\,\text{ft}} + \cancelto{0}{V_{y@A}} = -4\,\text{kip}$$

The shear is represented by a linearly decreasing function with a value of –4 kip at $x = 2$ ft as shown in Figure 3*a*.

We evaluate M_{bz} using the integral in (10.5),

$$M_{bz@B} - M_{bz@A} = \int_A^B V_y\,dx$$

$$M_{bz@B} = \int_0^{2\,\text{ft}} -2\frac{\text{kip}}{\text{ft}}x\,dx + \cancelto{0}{M_{bz@A}} = -1\frac{\text{kip}}{\text{ft}}x^2\Big|_0^{2\,\text{ft}} = -4\,\text{kip·ft}$$

The bending moment is represented by a quadratic equation with a value of –4 kip·ft at $x = 2$ ft as shown in Figure 3*b*.

Segment B–C **(2 ft ≤ *x* ≤ 6 ft)** and free-body diagram in **Figure 2*b***: Along this segment $\omega = 0$. Using (10.6) we get,

$$V_{y@C} - V_{y@B} = -\int_B^C 0\,dx = 0$$

indicating that the shear is a constant between B and C. In fact the shear is a constant 3 kip between B and C.

Now consider the moment by integrating the shear (10.5):

$$M_{bz@C} - M_{bz@B} = \int_B^C V_y\,dx = \int_{2\,\text{ft}}^{6\,\text{ft}} 3\,\text{kip}\,dx = (3\,\text{kip})x\Big|_{2\,\text{ft}}^{6\,\text{ft}} = 12\,\text{kip·ft}$$

This shows that M_{bz} is changing linearly between B and C, and the change is 12 kip·ft. It increases from –4 kip·ft at B to 8 kip·ft at C. This agrees with the previously derived moment diagram in Figure 3*b*.

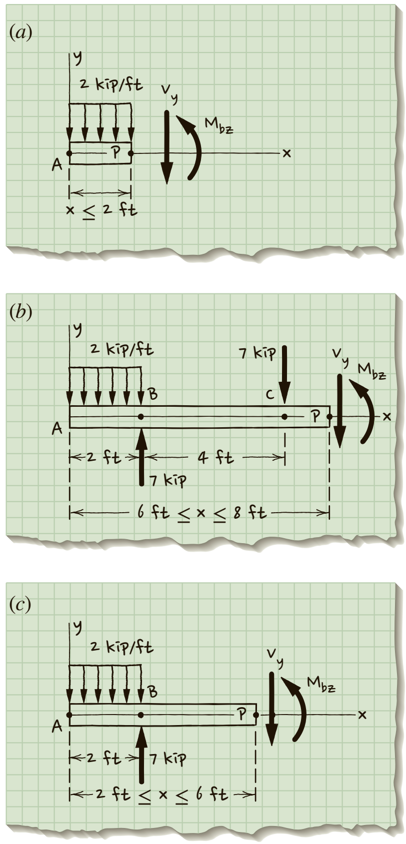


Figure 2 Free-body diagrams to create *V* and *M* diagrams.

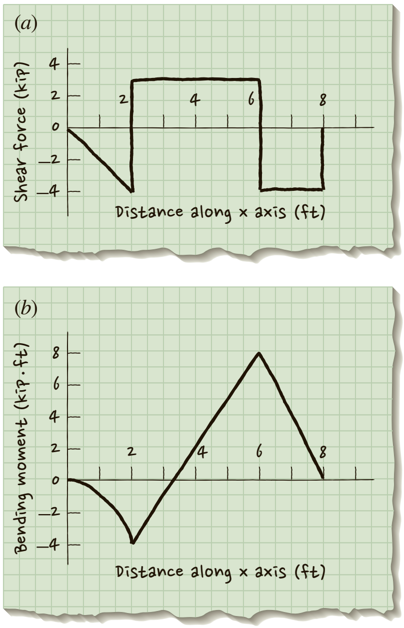


Figure 3 (*a*) shear diagram; (*b*) moment diagram developed in Example 10.3.4.

Segment C–D **(6 ft ≤ *x* ≤ 8 ft)** and free-body diagram in **Figure 2*c***: Along this segment $\omega = 0$. Integrating the distributed load (10.6) gives,

$$V_{y@D} - V_{y@C} = -\int_C^D 0\, dx = 0$$

indicating that the shear is a constant between C and D. The shear diagram indicates a constant –4 kip between C and D.

Finally we consider the moment between C and D using (10.5):

$$M_{bz@D} - M_{bz@C} = \int_C^D V_y\, dx = \int_{6\,\text{ft}}^{8\,\text{ft}} -4\,\text{kip}\, dx = (-4\,\text{kip})x\Big|_{6\,\text{ft}}^{8\,\text{ft}} = -8\,\text{kip·ft}$$

M_{bz} is changing linearly between C and D, from 8 kip·ft at C to 0 kip·ft at D.

Comment This type of analysis is very useful for checking the values, slopes, and shapes of the shear and bending moment diagrams. Often you don't need to carry out the integration explicitly, but instead check that the moment is linearly changing if the shear is constant, or the shear is linearly changing if the load is uniformly distributed, etc.

EXAMPLE 10.4.2

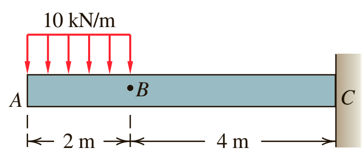


Figure 1 Cantilever beam analyzed in Example 10.3.3.

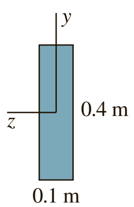


Figure 2 Cross-section of rectangular beam.

The cross section of the cantilever beam of Example 10.3.3 (**Figure 1**) is a rectangle 0.4 m high and 0.1 m wide (**Figure 2**). Determine the largest stress in the beam.

Goal Determine the maximum stress in beam ABC based on the moment diagram developed in Example 10.3.3 (**Figure 3**).

Given The beam cross section and the shear and bending moment diagrams developed in Example 10.3.3.

Assume Since no other information is given, assume the cross section of the beam is constant throughout its length.

Formulate Equations and Solve To calculate stress, we use (10.7)

$$\sigma = \frac{-M_{bz}y}{I}$$

We can examine the variation of stress throughout the beam. Since the cross section is assumed constant throughout the beam, the moment of inertia I and the distance from the centroid y in (10.7) are the same everywhere along the length of the beam. Thus the maximum stress will occur where $\|M_{bz}(x)\|$ is maximum. Examining the moment diagram in **Figure 3**, we see that the maximum moment of –100 kN·m occurs at C.

According to Appendix C, the area moment of inertia for a rectangular beam is $bh^3/12$, where b is the beam width and h is the height.

$$I = \frac{0.1\,\text{m}(0.4\,\text{m})^3}{12} = 0.533 \times 10^{-3}\,\text{m}^4$$

at the top of the beam

$$\sigma_{max} = -\frac{(-100\,\text{kN}\cdot\text{m})(0.2\,\text{m})}{0.533 \times 10^{-3}\,\text{m}^4}$$

$$= 37.5 \times 10^3 \frac{\text{N}}{\text{m}^2} = \boxed{37.5\,\text{MPa}}$$

at the bottom of the beam

$$\sigma_{max} = -\frac{(-100\,\text{kN}\cdot\text{m})(-0.2\,\text{m})}{0.533 \times 10^{-3}\,\text{m}^4}$$

$$= 37.5 \times 10^3 \frac{\text{N}}{\text{m}^2} = \boxed{-37.5\,\text{MPa}}$$

Answer $\sigma_{max} = 37.5$ MPa and occurs at $x = 6$ m. Because the beam is curving downward, the top of the beam experiences tensile stresses and the bottom of the beam experiences compressive stresses, as illustrated in Figure 4.

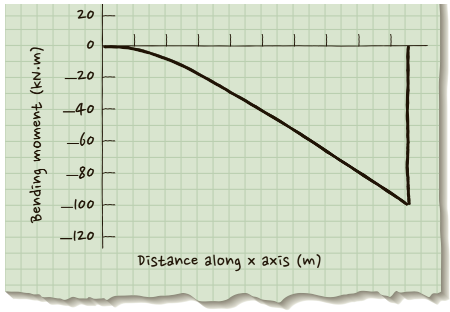


Figure 3 Moment diagram developed in Example 10.3.3.

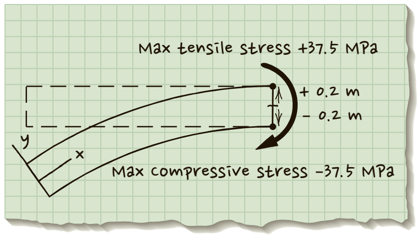


Figure 4 The beam bends downward due to the loading causing the top surface to be in tension and the bottom to be in compression.

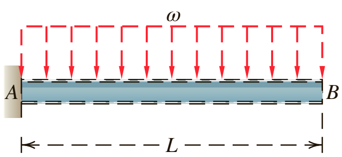
EXERCISES 10.4

10.4.1. [*] Consider the cantilever beam with a uniformly distributed load.

a. Use equilibrium to develop an equation to describe the moment in the beam as a function of x.

b. Differentiate the moment equation to determine the equation for the shear.

c. Develop an equation for the shear using equilibrium and check your answer for **b**.



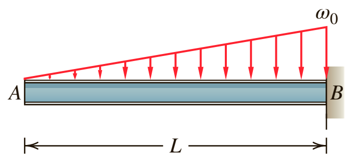
EX 10.4.1

10.4.2. [*] Consider the beam with a variable distributed load.

a. Use equilibrium to develop an equation to describe the moment in the beam as a function of x.

b. Differentiate the moment equation to determine the equation for the shear.

c. Develop an equation for the shear using equilibrium and check your answer for **b**.



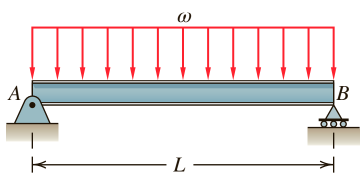
EX 10.4.2

10.4.3. [*] Consider the simply supported beam with a uniformly distributed load.

a. Use equilibrium to develop an equation to describe the shear in the beam as a function of x.

b. Integrate the shear equation to determine the equation for the bending moment.

c. Develop an equation for the bending moment using equilibrium and check your answer for **b**.



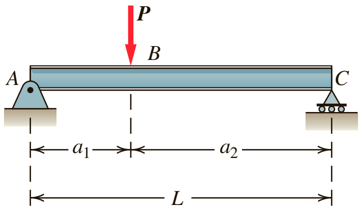
EX 10.4.3

10.4.4. [*] Consider beam ABC.

a. Use equilibrium to develop equations to describe the shear in the beam as a function of x.

b. Integrate the shear equations to determine the equations for the bending moment.

c. Develop equations for the bending moment using equilibrium and check your answer for **b**.



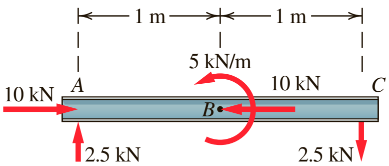
EX 10.4.4

10.4.5. [*] Consider beam ABC with the loading as shown.

a. Use equilibrium to develop equations to describe the moment in the beam as a function of x.

b. Differentiate the moment equations to determine the equations for the shear.

c. Develop equations for the shear force using equilibrium and check your answer for **b**.

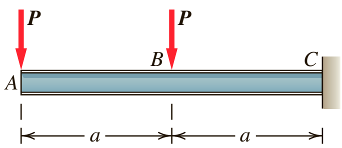


EX 10.4.5

10.4.6. [*] Consider beam ABC.

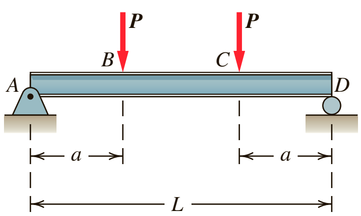
a. Draw the shear and bending moment diagrams for the beam.

b. Confirm that the shear and bending moments follow the relationships in (10.3) and (10.5).



EX 10.4.6

10.4.7. [*] Consider beam $ABCD$.



EX 10.4.7

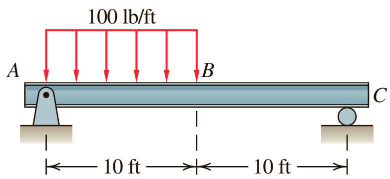
a. Draw the shear and bending moment diagrams for the beam.

b. Confirm that the shear and bending moments follow the relationships in (10.3) and (10.5).

10.4.8. [*] Consider beam ABC with loading as shown.

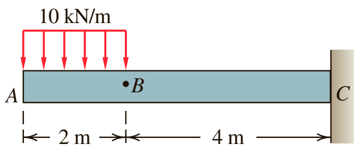
a. Draw the shear and bending moment diagrams for the beam.

b. Confirm that the shear and bending moments follow the relationships in (10.3) and (10.5).



EX 10.4.8

10.4.9. [*] The moment diagram for a cantilever beam loaded as shown was developed in Example 10.3.3. If the beam has a rectangular cross-section that is 0.4 m wide and 0.1 m high, what is the maximum stress in the beam? How does this compare with the results of Example 10.4.2?

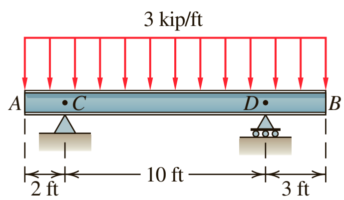


EX 10.4.9

10.4.10. [*] A beam and loading are shown.

a. Determine the maximum magnitude of the bending moment.

b. If the area moment of inertia I for the cross section shown in the figure is 124 in.^{4} and $y_{max} = \pm 5$ in., calculate the maximum stress due to bending.

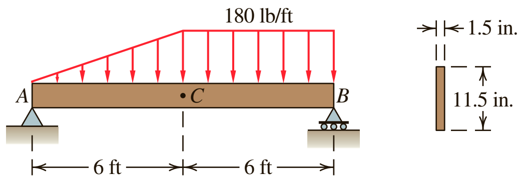


EX 10.4.10

10.4.11. [*] A beam and loading are shown.

a. Determine the maximum value (magnitude) of the bending moment.

b. If the area moment of inertia I for the cross section shown in the figure is 190 in.^{4} and $y_{max} = \pm 5.75$ in., calculate the maximum stress due to bending.

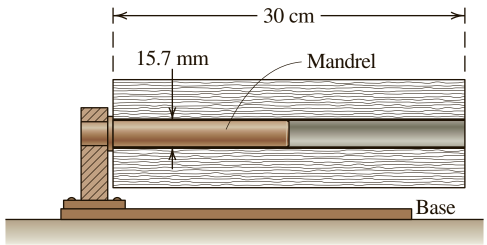


EX 10.4.11

10.4.12. [*] A paper roll storage rack is shown.

a. Determine the maximum value (magnitude) of the bending moment for the mandrel, which extends 50% into the roll. The paper roll weighs 2.0 N.

b. If the area moment of inertia I for the mandrel's cross section is 3.0×10^{-9} m^4, determine the maximum stress in the mandrel and specify its location.



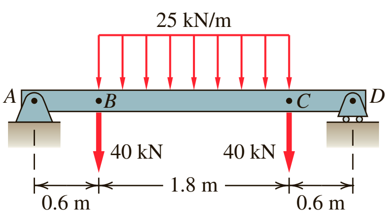
EX 10.4.12

10.4.13. [**] Consider beam $ABCD$.

a. Use equilibrium to develop equations to describe the moment in the beam as a function of x.

b. Differentiate the moment equations to determine the equations for the shear.

c. Develop equations for the shear using equilibrium and check your answer for **b**.



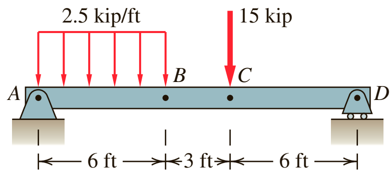
EX 10.4.13

10.4.14. [**] Consider the beam with loading as shown.

a. Use equilibrium to develop equations to describe the moment in the beam as a function of x.

b. Differentiate the moment equations to determine the equations for the shear.

c. Develop equations for the shear using equilibrium and check your answer for **b**.



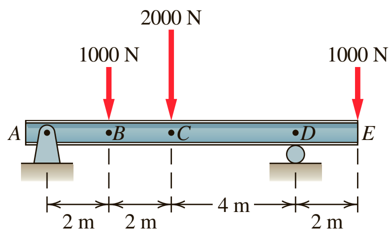
EX 10.4.14

10.4.15. [**] Consider beam AE.

a. Use equilibrium to develop equations to describe the shear in the beam as a function of x.

b. Integrate the shear equations to determine the equations for the bending moment.

c. Develop equations for the bending moment using equilibrium and check your answer for **b**.

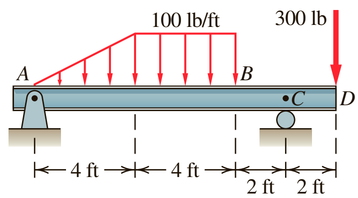


EX 10.4.15

10.4.16. [**] Consider beam $ABCD$.

a. Use equilibrium to develop equations to describe the shear in the beam as a function of x.

b. Differentiate the shear equations and compare the results with the applied loads.

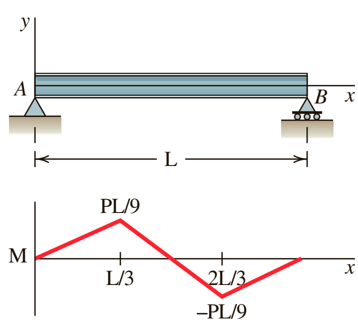


EX 10.4.16

10.4.17. [**] The loading on the simply supported beam AB generates the moment diagram shown.

a. Determine the shear diagram.

b. Determine the loading on the beam.



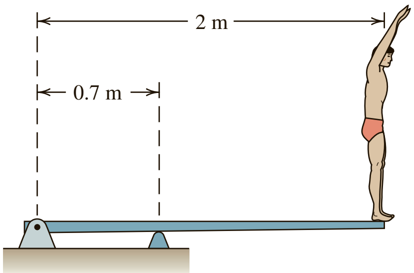
EX 10.4.17

10.4.18. [**] The overhung diving board is made from a solid plank 0.44 m wide and 0.04 m thick.

a. Determine the maximum stress in the diving board if a 630 N person is standing on the free end.

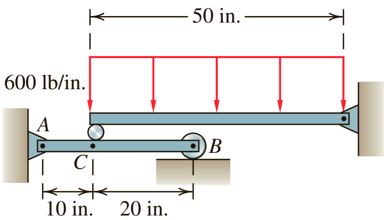
b. Assuming that the diving board can sustain a maximum stress of 40 MPa before failure, and that when a person jumps on the board it is equivalent to applying double his weight, determine the weight of the heaviest person who can jump on the board.

c. Does the result in **b** seem like a reasonable value so that the diving board is unlikely to fail under normal conditions?



EX 10.4.18

10.4.19. [**] Beam AB is fabricated from a 1.5-inch wide rectangular aluminum bar with a maximum allowable bending stress of 20 ksi. Determine the minimum height of beam AB so that the allowable stress is not exceeded.

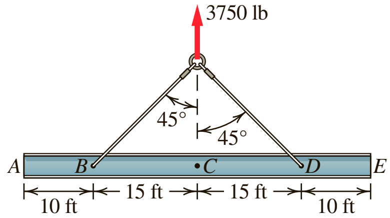


EX 10.4.19

10.4.20. [**] The steel I-beam shown weighs 75 lb per foot and is being lifted by cables attached at B and D. The beam is a standard S 20 × 75 section (h = 18 in. and $I = 1280$ in.^{4}).

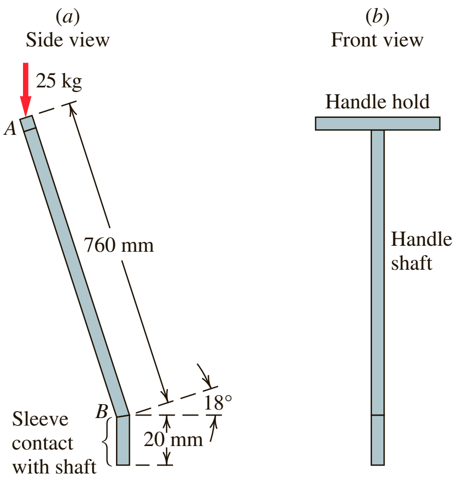
a. Determine the location and magnitude of the maximum stress in the beam.

b. If the rigging is changed so that the cables are attached to points A and E instead of B and D, determine the location and magnitude of the maximum stress in the beam. Compare your answer with part **a**.



EX 10.4.20

10.4.21. [***] One of the design loads for the handle of the MoneyMaker Plus water pump discussed in **SA 10.1.1** is caused by the user leaning on the handle. This 25-kg vertical load applied at A is shown in (a). Assuming the handle is fabricated from 20 × 20 × 2 (mm) square steel tubing, determine if the maximum normal stress (due to bending) in handle section AB is less than $\sigma_{allow} = 220$ MPa.



EX 10.4.21

10.5 JUST THE FACTS

Engineers are concerned with selecting beam sizes and materials that will ensure adequate structural performance. Part of this selection process involves calculating the loads internal to a beam. In this chapter we considered how to systematically calculate the loads acting externally and internally on beams and cables.

Defining Beams and Recognizing Beam Configurations

A member is called a **beam** if loads are applied perpendicular to its long axis. These loads, which may consist of forces and/or moments, are referred to as **lateral loads**. Geometric features that are important in describing a beam are its length and its cross-sectional area and shape. We set up a **beam coordinate system** with an x_b axis along the long axis. The other two axes (y_b and z_b) lie in the cross section of the beam with their origin at the cross-section centroid.

Some beam configurations are given names (**cantilever beam, simply supported beam, fixed-fixed beam**). These configurations are illustrated in **Figure 10.1.5**. A beam is classified as a **planar beam** if all of the applied external forces lie in a single plane and all external moments are about an axis perpendicular to this plane; otherwise it is a **nonplanar beam**.

Beam Internal Loads

Axial Force, Shear Force, and Bending Moment Diagrams

The loads internal to a beam consist of a **bending moment** (M_b), **shear force** (V), and **axial force** (N). Static analysis of beams consists of finding these internal loads at various locations along the long axis of the beam. These internal loads can be depicted in terms of **bending moment, shear force**, and **axial force diagrams**.

Bending Moment Related to Shear Force and Normal Stress

Bending moment and shear are related by

$$\boxed{V_y = \frac{dM_{bz}}{dx}} \tag{10.3}$$

Furthermore, a distributed force ω acting on the beam is related to the shear force by

$$\boxed{\omega = \frac{-dV_y}{dx}} \tag{10.4A}$$

The bending moment is also related to internal "pressure" in the beam, commonly referred to as **normal stress** by

$$\boxed{\text{normal stress} = \sigma = \frac{-M_{bz}y}{I}} \tag{10.7}$$

where M_{bz} is the bending moment at a particular location x along the length of the beam and y in the numerator is the y coordinate within the cross section where the stress is being calculated. The denominator is the **area moment of inertia**, a scalar quantity that reflects the distribution of material in the beam cross section that was developed in Section 6.4.

SYSTEM ANALYSIS (SA) EXERCISES

SA10.1 Handle Design for the Money-Maker Plus Water Pump[1]

The Story of KickStart: Headquartered in Nairobi, Kenya, KickStart, formerly ApproTEC (Appropriate Technologies for Enterprise Creation) is a nonprofit social enterprise started in 1991 by Stanford University graduate and 2016 Engineering Hero Martin Fisher. KickStart designs and mass-markets high-quality income-generating tools particularly suited for poor farmers in rural Africa. KickStart's best-selling pump was the Super-MoneyMaker, a 2 cylinder piston pump that was driven by a single operator stepping back and forth between two treadles. As of 2016, KickStart has sold over 300,000 MoneyMaker pumps that have empowered more than 1.1 Million people to lift themselves out of extreme poverty. Illustrations of the pumps presently offered appear at the end of this question.

The MoneyMaker Plus Water Pump: In 2001, KickStart designed the MoneyMaker Plus water pump, shown in Figure SA10.1.1, which stayed in production until 2008. The MoneyMaker Plus is similar to the Super-MoneyMaker pump, though with one cylinder rather than two, and was designed with the goal to reduce the cost 50 percent to make it more affordable for farmers. The cost of the entire pump was extremely constrained—excluding manufacturing, was 1600 Kenya Shillings (approx. \$20 USD). In operation, it is capable of pulling water from 6 m (from a water source such as a creek) and pushing it another 12 m (up a hill or to power mechanical sprinklers). The KickStart pump designers hoped to use a handle consisting of a steel square hollow section (SHS) shaft and sleeve; this handle design had been used successfully for the Super-MoneyMaker pump. The square handle shaft fits snugly in a square sleeve tube with a stopper at the end and is easily removable from the pump for easy transport.

The proposed geometry of the handle is shown in Figure SA10.1.2 and is based on successfully tested design prototypes. The handle is important because it helps users maintain their balance and shift their weight smoothly from one foot to the other to operate the pump. This pump is a bit trickier to operate than previous ones because the user tips her weight back and forth as if her feet were on a mini seesaw with the handle coming up from between her feet near the pivot (Figure SA10.1.3). The user tips back and forth a bit, so the handle is necessary to maintain balance, particularly with new users. Experienced users occasionally lean on the handle (not recommended as that wastes body weight that could be applied to pumping), but otherwise hold the handle lightly during operation.

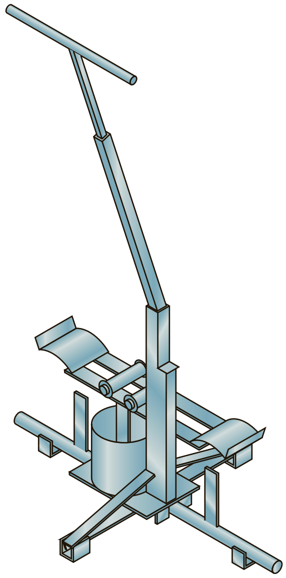


Figure SA10.1.1 MoneyMaker Plus water pump.

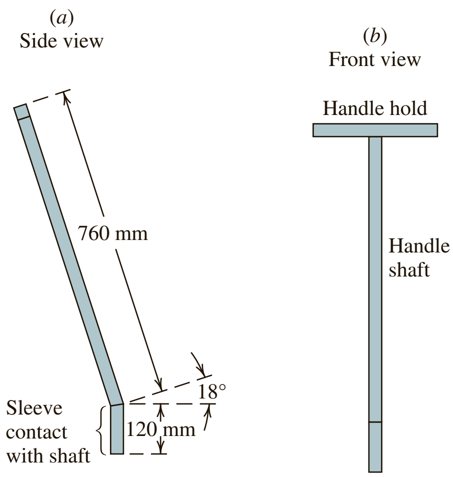


Figure SA10.1.2 Design of pump handle: (*a*) side view; (*b*) front view.

[1]This systems analysis problem was created by Dr. Krista Donaldson. At the time she authored this problem she was a Ph.D. student working on her thesis related to increasing manufacturing capacity in developing countries. More information about KickStart can be found at http://kickstart.org/

Figure SA10.1.3 Operation of pump.

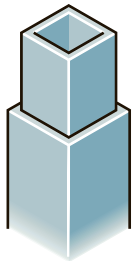


Figure SA10.1.4 Detail of shaft-sleeve connection.

Of the available tubing sizes for the handle, only two shaft–sleeve pairs (**Figure SA10.1.4**) are suitable for ergonomic and pump size constraints, and interface snugly (and somewhat consistently) in practice. The SHS pairs are:

- $20 \times 20 \times 2$ with a $25 \times 25 \times 2$
- $25 \times 25 \times 2$ (or $\times 3$) with a $30 \times 30 \times 2$

(units in mm). The smaller SHS pair is preferable as it is cheaper. Taking into account the low grade of steel to be used and a safety factor, the maximum allowable normal stress is $\sigma_{allow} = 220$ MPa.

Your Assignment: As an employee of KickStart, you have been asked to size the handle tubing to ensure that it will not fail. More specifically, you must determine whether the preferred $20 \times 20 \times 2/25 \times 25 \times 2$ shaft—sleeve combination is structurally appropriate for this pump. If it is not, you must determine whether any of the available combinations will work (and if not, it is back to the drawing board!).

Based on discussions with fellow KickStart engineers about potential pump uses, you define worst case scenarios for pump handle loads as listed in **Table SA10.1.1**.

Pick what you think is the worst of the worst case scenarios and check that the proposed shaft–sleeve pairs will not fail.

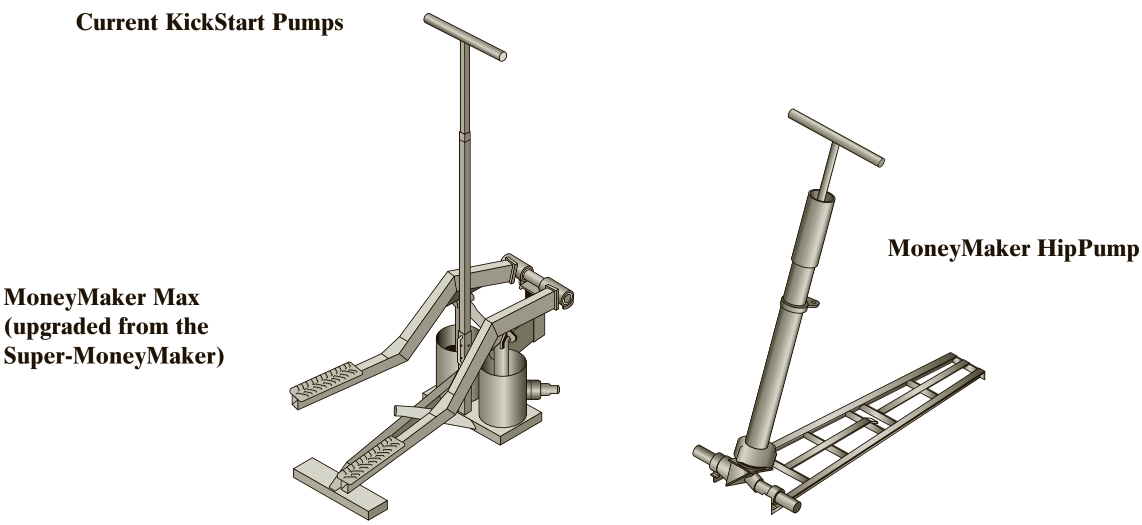
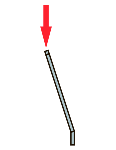
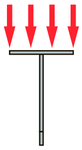
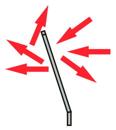
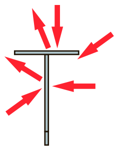
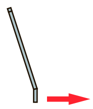
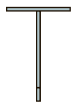
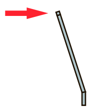
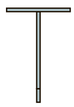


Table SA10.1.1 Handle Usage Patterns and Possible Worst Case Scenarios

Case	Side View	Front View	Description
1			User leans on handle pumping (evident during testing). This would result in a downward force of approximately 1/3 the body weight of the user.*
2			User shakes handle vigorously to determine how sturdy the pump is (observed behavior with customers). The force associated with this is likely to be less than 1/3 of the body weight of the user, since there is no body weight behind it.
3			The pump is cantilevered by its handle when picked up (perhaps dirt is in the sleeve and the handle gets stuck). The mass of the pump with water in the valve box and piston cylinder and mud caked to the bottom is at most 10 kg. In testing, the tester was observed to cantilever it over his shoulder like a hobo carries his stick because he was experimenting with how he might carry it easily while carrying something else with his other hand. In practice, this would likely be the case with a real farmer. The farmer, generally being a woman, would probably also have a baby on her front and a basket holding vegetables on her head.
4			The user loses her/his balance during operation and grasps the handle to keep from toppling. If falling, the user could direct all of her/his mass (75 kg) at the handle. But the pump was not designed to make sure that people don't fall off (this would be too expensive), so assuming the pump would not be fixed into the ground, it is estimated that about 50 kg of the user's mass will be directed horizontally at the top of the handle should it tip over.

*KickStart assumes user mass to be around 75 kg. This might seem low, but men in the targeted pump regions tend to be much smaller and thinner than Americans; the average woman, on the other hand, is often bigger than the average man, but is still likely to be less than 75 kg.

SA10.2 Form Follows Function[2]

Karl Culmann (1821–1881) was one of the foremost structural engineers of his day. One of his major breakthroughs was the development of graphical statics, a technique used to visualize forces and aid in structural design. This time-consuming analysis method was an integral part of engineering programs up until the introduction of computer modeling in the 1970s. Many truss bridges are built using this fundamental engineering principle in which the moment diagram gives an approximation of the superstructure's finished form.

(a) Draw the shear and bending moment diagrams for the two simply supported beams shown in Figure SA10.2.1. Compare the bending moment diagrams

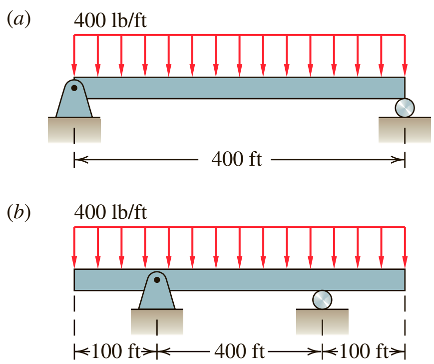
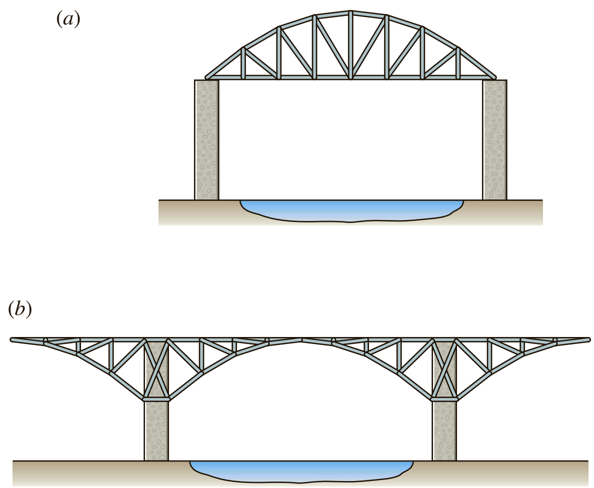


Figure SA10.2.1 Two simply supported beams.

[2] Phrase coined by the American architect Louis Sullivan in his article, "The Tall Office Building Artistically Considered," first published in 1896. To be found in I. Athey, ed. *Kindergarten Chats (revised 1918) and Other Writings.* New York 1947: 202–13. More on this phrase and its implications for design work can be found at http://www.geocities.com/Athens/2360/jm-eng.fffhai.html.



From D'Arcy Tompson's *On Growth and Form* (Dover, 1992)

Figure SA10.2.2 Two common bridge designs that incorporate Culmann's methodology.

with the companion truss bridges in **Figure SA10.2.2**. Does it matter whether the superstructure is on the top or the bottom of the bridge? What does switching the location of the superstructure from the top to the bottom do to the individual function of the load-bearing elements?

(b) The spine of quadruped animals is very similar to a cantilever bridge in both form and function. Draw the shear and moment diagrams of the simplified American bison spine in **Figure SA10.2.3**. Compare your diagrams to the skeletal structure of the animal shown in **Figure SA10.2.4**. What are the functions of the legs and the processes originating from the spine? Are there any relationships between the size of the processes and the moment diagram? What elements missing in the skeletal structure are needed to complete the "bridge"?

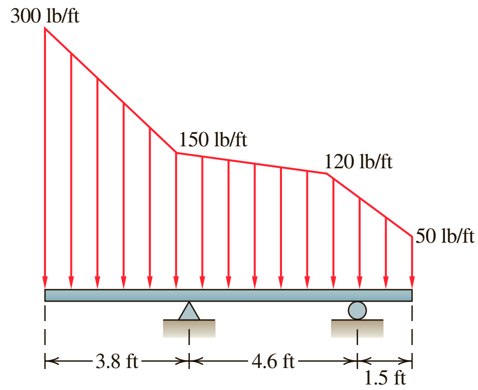
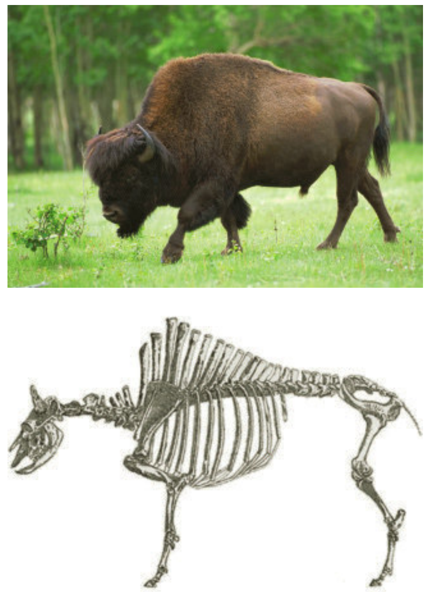


Figure SA10.2.3 A simplified model of an American bison's spine.



Superstock

Lawrence C. Todd, Colorado State University

Figure SA10.2.4 The skeletal structure of an American bison.

SA10.3 Hoover Dam

Hoover Dam is one of the greatest monuments to industrial strength and human audacity (**Figure SA10.3.1**). The dam itself rises approximately 726 feet above the floor of the Black Canyon between Nevada and Arizona. It is 1244 feet long, 660 feet thick at its base, 45 feet thick at its crest, and weighs 5,500,000 tons. The lake it created, Lake Mead, backs up 110 miles behind the dam and is nearly 500 feet deep. It contains enough water to submerge the state of Connecticut under 10 feet of water or to supply 5000 gallons of water to every person in the world. The concrete dam was cast as a single solid piece. Pouring started in June 1933 and continued 24 hours a day, seven days a week until May 1935.

Derrick James/Corbis Documentary/Getty Images

Education Images/UIG/Getty Images

Figure SA10.3.1 Images of Hoover Dam: (*a*) from above; (*b*) from its base (adapted from government publications).

The engineer responsible for this feat was Frank Crowe. He studied civil engineering at the University of Maine and, after a summer internship working on the drainage basin of the Yellowstone River, joined the U.S. government's Reclamation Service. Crowe was widely considered the best construction engineer in the United States and was eventually hired by the Six Companies to spearhead their efforts to obtain the Hoover Dam contract. He was the first to use cableways to transport people and supplies to the worksite (**Figure SA10.3.2**). In many instances the cables were required to support loads of several tons. In addition, because falling rocks were always a hazard during the construction of the dam, the first hard hats were developed by the workers using baseball caps and tar as raw materials. It is worth noting that after painstaking calculations and recalculations and four years of construction, nobody knew for sure that the dam would work! The only way to be certain was to seal the diversion tunnels and hope for the best.

Model the dam as a simple cantilever that rises up out of the bottom of Black Canyon. Given the weight of the dam and the depth of the water behind it, calculate the loads acting on the dam at its base.

Bureau of Reclamation/U.S. Dept. of the Interior

Figure SA10.3.2 Transporting equipment and materials at Hoover Dam (postcard from Hoover Dam).

SA10.4 How Much Load Does a Main Column Carry?

In Chapter 5 we learned how the 4.0-kN panels were lifted onto the top of the 48.0 m wide and 12.5–15.0 m high Reynolds Coliseum (see **Figure SA5.4.2**). As the panels were placed onto the flanges of the T-beams spanning the 6.0-m distance between the I-beam girders, the column section of each frame experienced increasing loads (**Figures SA10.4.1** and **SA10.4.2**). After the panels were firmly in place, a crew applied tar and several layers of roofing paper to create a water barrier weighing 50 N/m^2.

In order to assess the effects of the different loading conditions you are asked to calculate the effect of the construction sequence on the forces in the columns.

(a) What is the total load at point *D*, as indicated in **Figure SA10.4.3*a***, right after the field-spanning steel girders have been erected? (You can neglect the lightweight trusses.)

(b) Next, the T-beams spanning the space between the girders are installed. What are the loads at *A*, *B*, and *C* in **Figure SA10.4.2** after they are in place?

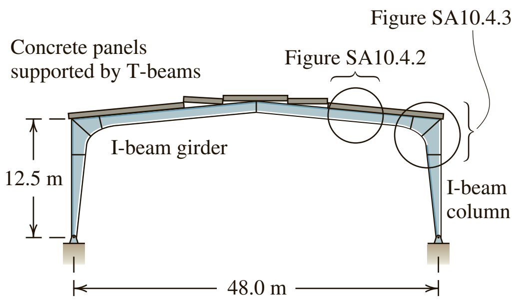


Figure SA10.4.1 Review of Reynolds Coliseum structure introduced in Chapter 5.

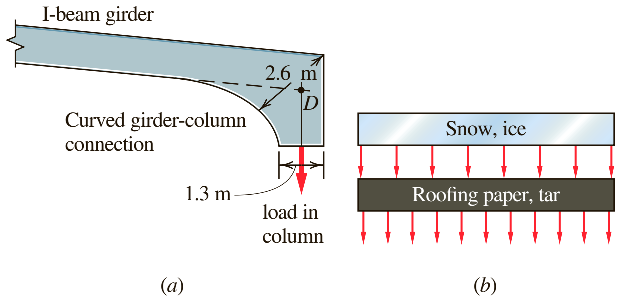


Figure SA10.4.3 (*a*) Main girder connects to column; (*b*) loads on roof.

(c) In Chapter 5 we learned how the panels were placed. Develop and sketch a placement sequence from beginning to end for 1949 or today (label rows parallel to the ridge of the roof, spans between adjacent girders, and panels according to their location within a row and a span).

(d) Based on the sequence you proposed in (c), plot the steady increase of the forces at *A*, *B*, and *C* in **Figure SA10.4.2** as panels in three rows and three spans are put in place.

(e) Pick a girder column and plot the load in point *D* as a function of completed spans and rows per your sequence designed in (c).

(f) If you assume that two crews work in parallel, how would you have to adjust your plots in (e)?

(g) Finally, calculate the loads at *A*, *B*, *C*, and *D* after the water barrier has been installed.

There are additional loading cases that we need to consider—namely, wind, snow, and ice. Since these loads vary with the location of the building (e.g., coastal region, mountains), one needs to find those values that are valid for the area where the building is to be built.

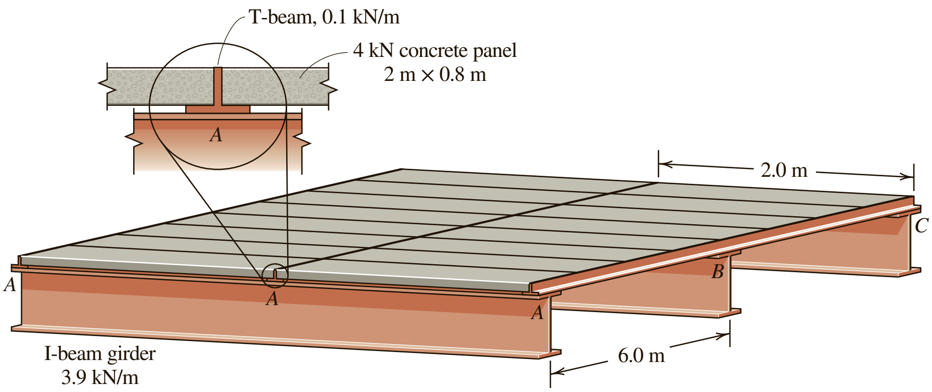


Figure SA10.4.2 Schematic of roof support with girder: T-beams and panels (to locate, see Figure SA10.4.1).

(h) Find the 100-year wind speed and depth of snow and ice accumulation for your area using data from the National Weather Service (i.e., maximum wind speeds, snow, and ice that can be measured once every 100 years). If you live in an area that receives little or no snow and ice, find the data for Houghton, Michigan.

(i) Convert the snow or ice that you found to a snow/ice pressure that would be applied to the roof. Before determining the additional load acting at D in **Figure SA10.4.3*a*** due to the snow and ice, draw the distributed load as shown in **Figure SA10.4.3*b***. (*Hint:* Powder snow = 50–100 kg/m^3, wet snow = 300–350 kg/m^3, glacier ice = 600–700 kg/m^3.)

(j) **Figure SA10.4.4** presents a given wind load of 0.6 kN/m^2 resulting from a wind of approximately 75 mph. (*Comment:* This number is different from the number for a girder because now we are considering a vertical wall that measures 15 meters. As you can find in any building design book or website, wind loads or pressures depend on a variety of different factors that go beyond statics.) Based on what you found in (h), adjust the wind load in **Figure SA10.4.4** up or down.

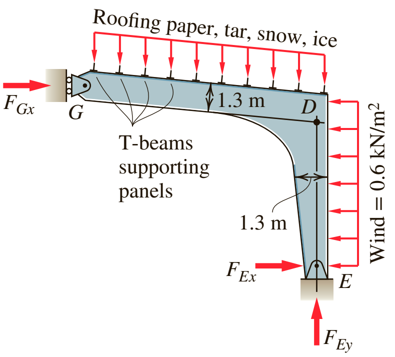


Figure SA10.4.4 Simplified model of symmetrical girder frame with total load calculation model.

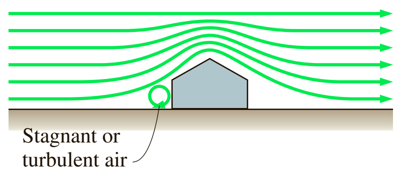


Figure SA10.4.5 Basics of wind speed and wind pressure on a building.

(k) How would you introduce the wind forces on the two sides of a sloped roof as shown in **Figure SA10.4.5**? Draw the approximate pressure distribution of a cross section for the totally sealed coliseum due to the wind pressure (assume that the leeward-side pressure is 1/3 of the windward side pressure).

(l) Let's assume that the girder frame shown in **Figure SA10.4.4** behaves as indicated (pin in E and roller in G). Develop the moment, shear, and axial force diagrams for the girder when the building is just completed (without external loads of wind, snow, and ice).

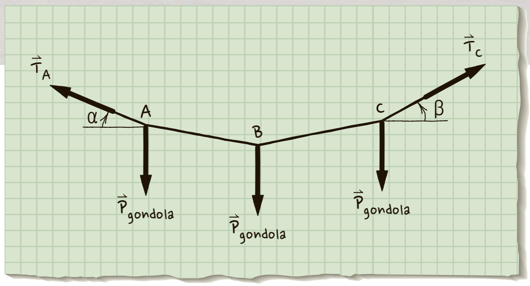
(m) How large and where are the *maximum* moment, shear, and axial forces with and without the external loads for the area that you picked?

(n) Assume that you are the building inspector and the town requires a safety factor of 1.2 (multiplication factor from theoretical to design values of moment, shear, and axial forces). Decide and defend minimum loads, moments, shear, and axial forces that you would expect the designer to use. (Expect the design engineer to argue against you since every additional amount of force that must be considered costs more money. Be sure that you have the proper arguments!)

CHAPTER 11

INTERNAL LOADS IN CABLES

Anthony Mayatt/Getty Images, Inc.



A cable is a tension-only member constructed from a large number of wound wires. It is an efficient element, meaning it can carry large loads compared to its weight. The forces acting on a cable may be distributed along its length or applied at a point. Examples of structures that incorporate cables are ski lifts, amusement park rides, circus tents, suspension bridges, railings on boats, and electric power lines. Like beams, cables are a fundamental structural element requiring engineering design and analysis.

During design, engineers are concerned with selecting cable sizes and materials to ensure that performance criteria are met. Part of this selection process involves calculating the loads internal to the cable. In this chapter we lay out how to systematically model a cable's deformed shape and calculate its internal loads when subjected to point loads or distributed loads.

LEARNING OBJECTIVES

Upon completion of this chapter, you will be able to:

- **Determine the deflected shape and carry out equilibrium analysis of a cable subjected to point loads.** (11.1)
- **Determine the deflected shape and carry out equilibrium analysis of a cable subjected to distributed loads.** (11.2)

11.1 CABLES WITH POINT LOADS

Learning Objective: Determine the deflected shape and carry out equilibrium analysis of a cable subjected to point loads.

Cables are an example of **tension-only members**, meaning they collapse when subjected to compression forces. Therefore, cables are unlike springs and two-force members in trusses that support both internal compression forces and tension forces. Instead, the internal forces in cables are only tension forces. When a cable is subjected to loads perpendicular to its long axis, it changes shape so that the internal axial forces are aligned to equilibrate the system. Thus cables are also unlike beams, which develop internal shear forces and bending moments. Other tension-only members are ropes and chains.

Engineers undertake analyses to predict the tension in a cable so that they can select a cable with a cross-sectional area that has enough capacity to transmit the tension without failing. They also need to be able to predict the shape of the cable so that they will know the geometry of the fully loaded structure.

We consider four possible cable loadings, as illustrated in **Figure 11.1.1**. For each of the four cases of a cable of length L suspended between anchors A and B in **Figure 11.1.1** we are interested in finding the tension force $\boldsymbol{T}$ anywhere along the length. We are also interested in finding the shape of the cable. In this section we discuss case I in which the cable is fully taut and the applied forces (at A and B) act along the cable. We also discuss case II, where concentrated forces act on the cable such as the traffic lights strung on a cable shown in **Figure 11.1.2**. The next section analyzes case III, where the cable is only loaded by its own weight, and case IV where the cable is carrying a distributed load and the cable's weight is small relative to this distributed load.

Thalia Anagnos

Figure 11.1.2 The traffic lights apply point loads to the cable from which they are hung.

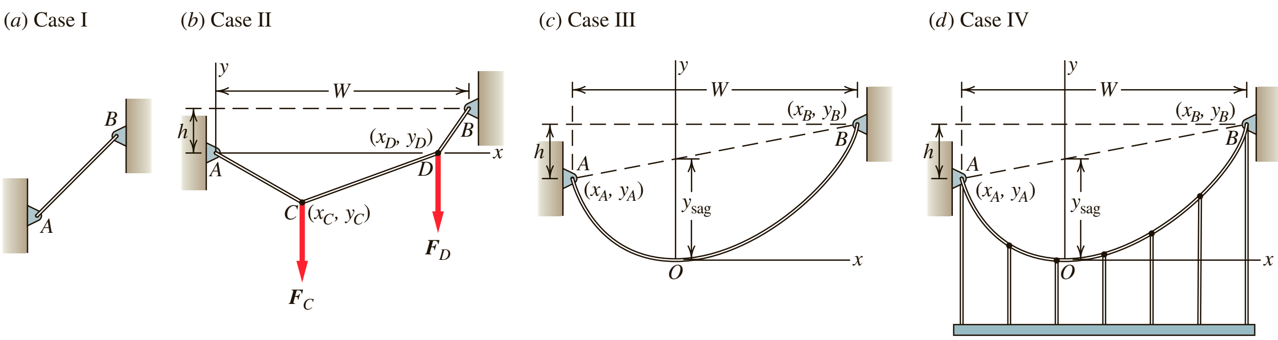


Figure 11.1.1 Examples of cable configurations and loadings.

Procedures for Finding Shape and Tension in Cables with Point Loads

Case I: Taut Cable. Consider the cable illustrated in **Figure 11.1.1*a***. With the cable pulled taut between the anchors its shape is simply a line between A and B, and the tension in the cable acts along this line. Thus the cable behaves like a two-force member. This case should look familiar as this is an analysis we have done in Section 5.4.2.

Case II: Concentrated Forces. When a cable is loaded with concentrated forces, its shape is defined by the locations of each point load. As shown in **Figure 11.1.1*b***, the cable takes the form of several taut line segments. Each segment is subjected to a constant tensile force aligned with the segment. It is important to remember in performing this analysis that we are assuming that the weight of the cable is negligible in comparison to the loads, and so we can ignore the cable weight.

Considering that we know the distances h and W, coordinates x_C and x_D, the forces F_C and F_D, and the total cable length, L, we can find the tension in each of the segments (T_{AC}, T_{CD}, T_{DB}), and y coordinates y_C and y_D, at C and D, respectively. We need five equations to find these five unknowns (T_{AC}, T_{CD}, T_{DB}, y_C, y_D). Four of these equations come from writing force equilibrium equations (ΣF_x, ΣF_y) based on free-body diagrams at both points C and D (**Figure 11.1.3**). The fifth equation relates the total cable length L to y_C, y_D, x_C, x_D, h, and W. These five equations can be solved for the five unknowns; unfortunately, this type of problem is hard to solve by hand. If the value of y_C or y_D is specified, the number of unknowns is reduced by one and it becomes easier to solve for the internal forces by hand.

As you perform cable analyses, keep in mind that if all of the applied loads are vertical the horizontal component of the tension in the cable will be constant throughout the length of the cable. In addition, the segment of the cable with the steepest slope will experience the largest tensile force.

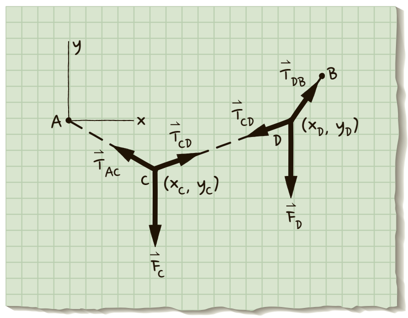


Figure 11.1.3 Free-body diagrams at points C and D.

Check out the following example of an application of this material.
- **Example 11.1.1 Flexible Cable with Concentrated Loads**

EXAMPLE 11.1.1

The cable shown in **Figure 1** supports a 500-lb load at C and a 250-lb load at D. Support A is 6 ft below B, and C is 3 ft below support A. Determine the vertical location of point D (y_D) and the maximum tension in the cable.

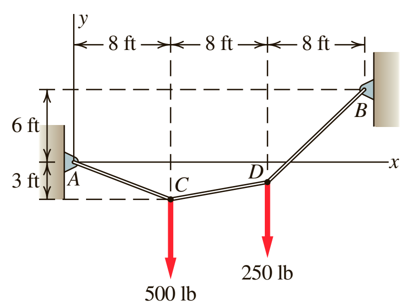


Figure 1 Cable supporting two point loads.

Goal Find the vertical coordinate of point D (y_D) and the maximum tension in the cable.

Given Information about the cable geometry and applied loads, and a coordinate system with origin at A.

Assume The weight of the cable is negligible compared to the loads and the cable is uniform.

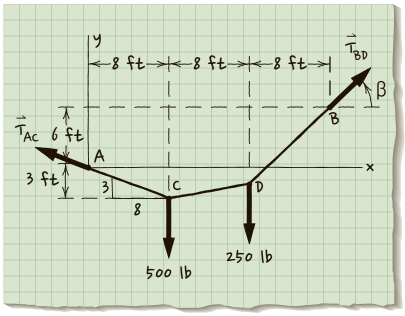


Figure 2 Free-body diagram of entire cable.

Draw Figure 2 is a free-body diagram of the entire system. The cable pulls on the supports at A and B and the supports pull back on the cable. This information allows us to draw the tension at A (T_{AC}) in a known direction (we know the slope because we know, based on **Figure 1**, that $x_C = 8$ ft and $y_C = -3$ ft). We also draw the tension at B (T_{BD}), but since we do not know y_D, we show it at an unknown angle β.

Formulate Equations and Solve Based on the free-body diagram of the entire system (**Figure 2**), we write the equilibrium equation:

$$\sum M_{z@A}(\curvearrowleft +) = 0$$

$$-500\text{ lb}(8\text{ ft}) - 250\text{ lb}(16\text{ ft}) + T_{BD}\sin\beta(24\text{ ft}) - T_{BD}\cos\beta\,(6\text{ ft}) = 0$$

$$-4000\text{ lb}\cdot\text{ft} + T_{BD}\sin\beta\,(12\text{ ft}) - T_{BD}\cos\beta(3\text{ ft}) = 0 \qquad (1)$$

Also, we can write:

$$\sum M_{z@C}(\curvearrowleft +) = -250\text{ lb}(8\text{ ft}) + T_{BD}\sin\beta(16\text{ ft}) - T_{BD}\cos\beta(9\text{ ft}) = 0$$

$$-2000\text{ lb}\cdot\text{ft} + T_{BD}\sin\beta(16\text{ ft}) - T_{BD}\cos\beta(9\text{ ft}) = 0 \qquad (2)$$

Solving (1) and (2) simultaneously gives

$$\beta = 36.9° \qquad \text{and} \qquad \boxed{T_{BD} = 834\text{ lb}}$$

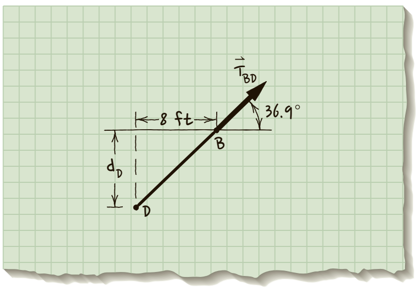


Figure 3 Orientation of T_{BD} at B.

We now use this value of β to find the vertical coordinate of D (y_D). Based on the geometry in **Figure 3**, we write

$$\tan\beta = \tan 36.9° = \frac{d_D}{8\text{ ft}}$$

Solving for d_D, we find

$$d_D = 6.00\text{ ft}$$

If point D is 6 feet below anchor B, it is at the same level as anchor A. This means that the y coordinate of point D relative to the xy coordinate system defined in **Figure 1** is

$$\boxed{y_D = 0.0\text{ ft}}$$

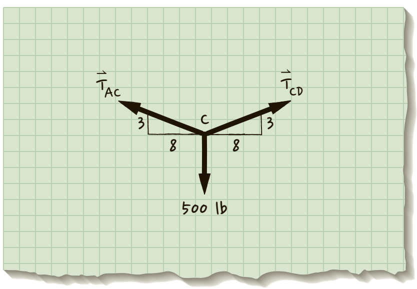


Figure 4 Free-body diagram of particle C.

To find the maximum tension in the cable, we could calculate the tension in the other cable segments. Based on equilibrium conditions applied to the free-body diagram in **Figure 4** we find that $T_{CD} = T_{AC} =$ 712 lb. Therefore the maximum tension in the cable is:

$$\boxed{T_{\max} = T_{BD} = 834\text{ lb}}$$

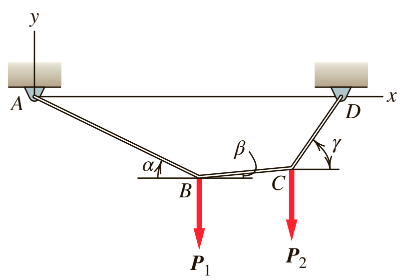
Alternatively, we could use the knowledge that when all of the loads are vertical the maximum tension occurs in the segment with the steepest slope relative to horizontal. The steepest segment is BD and therefore $T_{BD} = T_{\max}$.

Check We can check our results by using the free-body diagram in **Figure 2** and the calculated values of T_{BD} and T_{AC} to write the moment equilibrium equation about a moment center at D.

Note: Because no horizontal forces are applied to the cable, the horizontal force component is constant along the cable. In other words, we can show that $T_{CAx} = T_{CDx} = T_{BDx} = 667$ lb. This is also a good check of our results.

EXERCISES 11.1

11.1.1. [*] The geometry of the cable is such that $0° \leq \beta < \alpha < \gamma < 90°$. Use this cable system to show that the largest cable tension occurs in the segment with the steepest slope. (Hint: start with the fact that the horizontal component of the cable tension is a constant if all applied loads are vertical.)

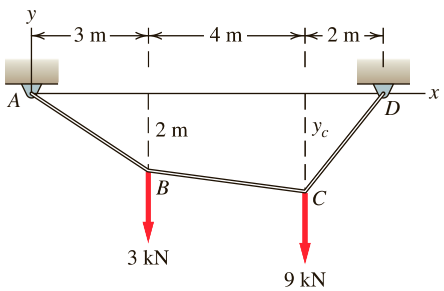


EX 11.1.1

11.1.2. [*] Cable $ABCD$ supports two loads. Determine

a. the vertical location of point C (y_C).

b. the maximum tension in the cable.

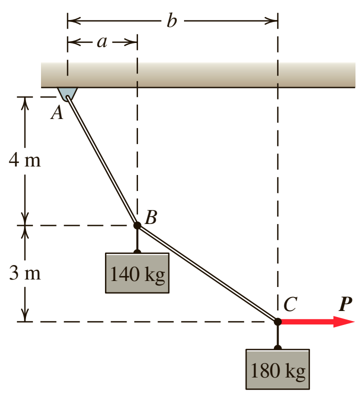


EX 11.1.2

11.1.3. [*] Cable ABC supports two masses as shown. Distance $b = 7$ m. Determine

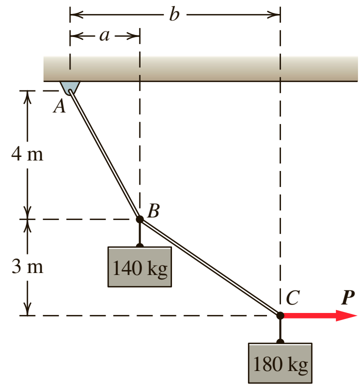
a. the required magnitude of the force $\boldsymbol{P}$

b. the corresponding distance a



EX 11.1.3

11.1.4. [*] Cable ABC supports two masses as shown. Determine the distances a and b when a horizontal force $\boldsymbol{P}$ of magnitude 1000 N is applied at C.

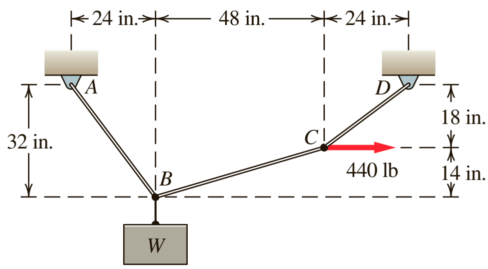


EX 11.1.4

11.1.5. [*] The cable $ABCD$ is maintained in the position shown by the 440-lb force applied at C and the block attached at B. Determine

a. the required weight W of the block

b. the tension in each portion of the cable

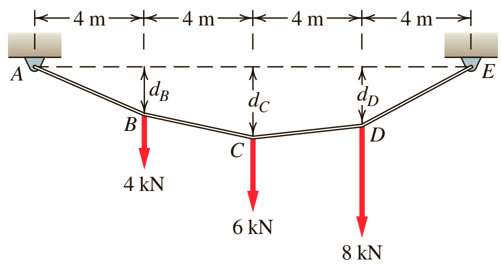


EX 11.1.5

11.1.6. [*] For the cable shown, $d_C = 7$ m. Determine

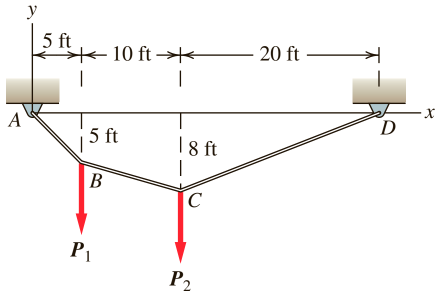
a. the vertical and horizontal components of the force acting on the cable at E

b. the maximum tension in the cable



EX 11.1.6

11.1.7. [**] The maximum tension in the cable shown is $T_{max} = 1000$ lb. Determine the magnitude of the loads P_1 and P_2.

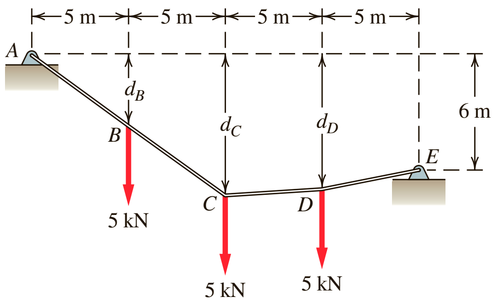


EX 11.1.7

11.1.8. [**] Three loads are suspended as shown from the cable. Distance $d_C = 7$ m. Determine

a. the vector components of the forces acting at A and E

b. the maximum tension in the cable

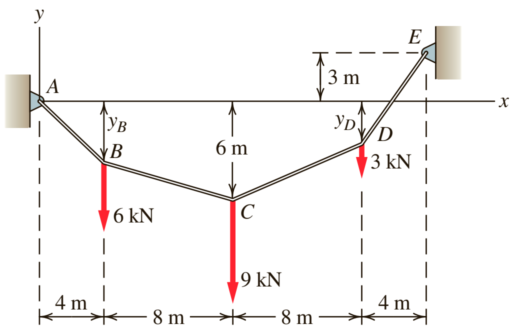


EX 11.1.8

11.1.9. [**] A cable carries three forces, as shown. Determine

a. the locations y_B and y_D of points B and D, respectively

b. the maximum cable tension and its location



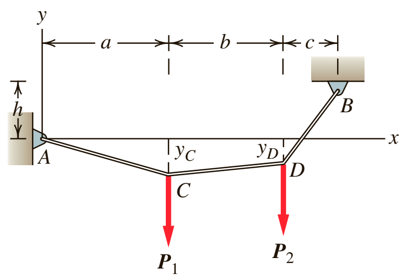
EX 11.1.9

11.1.10. [*, computer]** Develop a computer program or a spreadsheet to analyze the cable shown for any values of the known variables $P_1, P_2, a, b, c, h,$ *and* y_C. Your output should include y_D, T_{AC}, T_{CD}, T_{DB}, and the length of the cable.

a. Check your program by comparing it with Example 11.1.1.

b. Using the same cable, vary h from 0 to 6, and keep all other inputs constant. For what value of h does $\|T_{AC}\| = \|T_{DB}\|$? Can you explain why they must be equal for this value of h?

c. For what value of h is the length of the cable minimized?



EX 11.1.10

11.1.11. [*, computer]** Using the same computer program or spreadsheet you developed in **11.1.10** and the data from Example 11.1.1, vary a and b so that P_1 moves from $x = 0$ to $x = 16$ ft.

a. What happens when you move P_1 to $x = 0$? Explain why you get this result.

b. As you move P_1 from $x = 0$ to $x = 16$ ft, how does y_D vary? Where does $y_D = 0$?

c. If you wanted to limit the maximum tension in the cable to 625 lb, where would you place P_1?

11.2 CABLES WITH DISTRIBUTED LOADS

Learning Objective: Determine the deflected shape and carry out equilibrium analysis of a cable subjected to distributed loads.

All cables are subjected to distributed loads because they must support their own weight. However, if the point loads are very large compared to the cable weight, we can ignore the cable weight and still perform a reasonably accurate analysis, as we did in Case II. Similarly, if the uniformly

distributed load is very large compared to the weight of the cable, we can ignore the cable weight as we do in Case IV. In the design and analysis of some cables, such as electrical power lines, the cable weight is the dominant design load.

In both cases of a cable subjected to distributed loads (Cases III and IV), we make the assumptions that the cable is inextensible (cannot stretch) and that it is incapable of developing internal shear and bending forces. We also assume the cable shape is defined relative to its lowest point (**Figures 11.1.1*c*** and **11.1.1*d***). We define this lowest point as the origin, and supports A and B are at coordinates (x_A, y_A) and (x_B, y_B), respectively. The maximum vertical distance from the line connecting the two anchors to the cable is called the **sag**, y_{sag}. Other important dimensions shown in the figures are the horizontal distance between the supports, called the **span** (W), and the difference in height between A and B (h). The ratio of sag to span (y_{sag}/W) is often a consideration in designing cable systems.

Procedures for Finding Shape and Tension in Cables with Distributed Loads

Case III: Cable Weight Only. Consider the cable illustrated in **Figure 11.2.1**. We wish to find the cable's shape and tension when it is loaded by its own weight (which we denote as μ in force/length). The shape of the cable is found by considering a free-body diagram of an infinitesimally small portion of the cable (**Figure 11.2.2**). Application of equilibrium equations to this free-body diagram results in equations for the shape (curve) and slope of the cable:

$$y = \frac{T_o}{\mu}\left(\cosh\frac{\mu x}{T_O} - 1\right) \tag{11.1A}$$

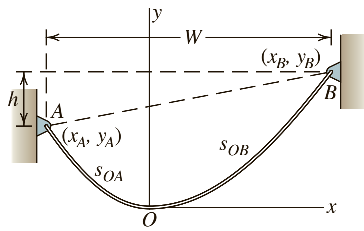


Figure 11.2.1 Cable of uniform weight μ (force/length).

where cosh is the hyperbolic cosine, and T_O is the constant horizontal force component everywhere in the cable. This curve is defined relative to the origin O noted in the figure. The particular shape described by (11.1A) is referred to as a **catenary curve**.[1] The slope of the curve is given as

$$\frac{dy}{dx} = \sinh\frac{\mu x}{T_O} = \frac{e^{\mu x/T_O} - e^{-\mu x/T_O}}{2} \tag{11.1B}$$

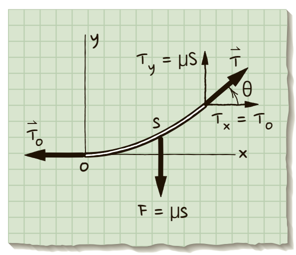


Figure 11.2.2 Free-body diagram of a portion of the cable of length s.

where sinh is the hyperbolic sine. The complete development of (11.1A) and (11.1B) is presented in Note 1 at the end of this chapter.

To use (11.1A) to plot the shape of a particular cable requires that we find the value of T_O and define an origin that is located at the lowest point in the cable (we define it relative to anchors A and B at x_A, y_A and x_B, y_B, respectively). We need five equations to find the five unknowns $(x_A, y_A, x_B, y_B, T_O)$.[2]

[1] The word *catenary* comes from the Latin word *catena*, which means chain.

[2] These five equations are numbered III.1 through III.5 to indicate that they are associated with Case III.

Two equations can be written based on the horizontal span (W) and vertical (h) spacing of the two anchors:

$$W = -x_A + x_B \quad \text{or} \quad x_A = x_B - W \tag{III.1}$$

and

$$h = |y_B - y_A| \tag{III.2}$$

Two additional equations result from writing (11.1A) for anchor points A and B. At A, $x = x_A$:

$$y_A = \frac{T_o}{\mu}\left(\cosh \frac{\mu x_A}{T_o} - 1\right) \tag{III.3}$$

and at B, $x = x_B$:

$$y_B = \frac{T_O}{\mu}\left(\cosh \frac{\mu x_B}{T_O} - 1\right) \tag{III.4}$$

Finally, based on the geometry depicted in **Figure 11.2.1**, we write the total cable length (L) as the sum of the cable lengths to the left (s_{OA}) and right (s_{OB}) of the origin. The geometry in **Figure 11.2.2** shows us that $dy/dx = \tan\theta = \mu s/T_O$, which, when combined with (11.1B), results in an expression for cable length:

$$s = \frac{T_O}{\mu}\left(\sinh \frac{\mu x}{T_O}\right)$$

Therefore,

$$L = -s_{OA} + s_{OB} = \frac{T_O}{\mu}\left(-\sinh \frac{\mu x_A}{T_O} + \sinh \frac{\mu x_B}{T_O}\right) \tag{III.5}$$

These five equations (III.1–III.5) can be used to find the five unknowns T_O, x_A, x_B, y_A, and y_B.

When anchors A and B are at different heights (i.e., $h \neq 0$), these five nonlinear equations must be solved to find the five unknowns. If the anchor points are at the same height, $-x_A = x_B = W/2$, and $y_A = y_B = y_{\text{sag}}$, equation (III.3) or (III.4) in combination with (III.5) can then be used to find T_O and y_{sag}, as illustrated in Example 11.2.1.

Catenary cable tension: Equilibrium conditions applied to the free-body diagram in **Figure 11.2.2** result in an expression for tension anywhere along the length of a cable loaded by its own weight:

$$T = T_O \cosh \frac{\mu x}{T_O} \underbrace{=}_{\substack{\text{using} \\ (11.1\text{A})}} T_O + \mu y \tag{11.2}$$

This expression describes the tension at any location (x, y) along the catenary cable—it tells us that the tension is maximum at the greatest distance from the origin (i.e., where x and y are maximum, which will be

at the anchor farthest from the origin). Details of the development of (11.2) are contained in Note 1 at the end of this chapter.

Case IV: Uniform Weight Hanging from Cable. Consider the cable illustrated in **Figure 11.2.3*a***. We wish to find the cable's shape and tension when a uniform weight hangs from it—this description closely approximates a suspension bridge with a roadway hanging from it. We denote the uniform weight as ω in force/horizontal length. The shape of the cable is found by considering a free-body diagram of an infinitesimally small portion of the cable (**Figure 11.2.3*b***). We can derive an equation for the curve describing the cable configuration:

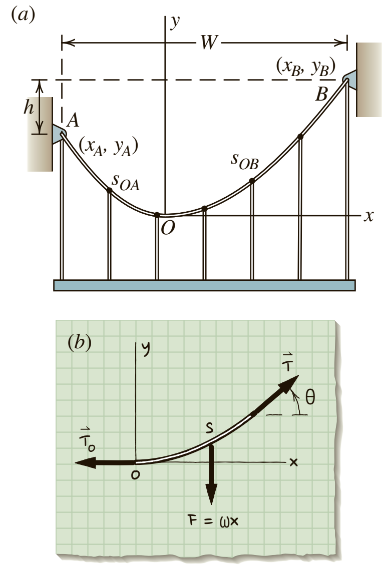


Figure 11.2.3 (*a*) Uniform load on cable ω; (*b*) free-body diagram of a portion of the cable of length *s*.

$$y = \frac{\omega x^2}{2T_O} \tag{11.3A}$$

where T_O is the constant horizontal force component everywhere in the cable. This curve is defined relative to the origin O noted in the figure, which is the lowest point of the cable. The particular shape described by (11.3A) is referred to as a **parabolic curve**. The slope of the curve is given as

$$\frac{dy}{dx} = \frac{\omega x}{T_O} \tag{11.3B}$$

The complete development of (11.3A) and (11.3B) is contained in Note 2 at the end of this chapter.

To plot the shape of a particular cable using (11.3A) requires that we find a value of T_O and that we define an origin that is located at the lowest point in the cable, which is defined relative to anchors A and B at (x_A, y_A) and (x_B, y_B). We need five equations to find the five unknowns $(x_A, y_A, x_B, y_B, T_O)$.[3]

Two equations can be written based on the horizontal span (W) and vertical (h) spacing of the two anchors:

$$W = -x_A + x_B \tag{IV.1}$$

and

$$h = |y_B - y_A| \tag{IV.2}$$

Two additional equations result from writing (11.3A) for anchor points A and B.

At A, $x = x_A$:

$$y_A = \frac{\omega x_A^2}{2T_O} \tag{IV.3}$$

[3]We number these equations IV.1 through IV.5 to indicate that they are associated with Case IV.

and at B, $x = x_B$:

$$y_B = \frac{\omega x_B^2}{2T_O} \tag{IV.4}$$

Finally, based on the geometry depicted in **Figure 11.2.3*b***, we can write the total cable length (L) as the sum of the cable lengths to the left (s_{oA}) and right (s_{oB}) of the origin. The cable lengths are found based on integrating the differential relationship $ds = [(dx)^2 + (dy)^2]^{1/2}$, as detailed in Note 2. Therefore,

$$L = -s_{OA} + s_{OB} = -\left[x_A + \frac{\omega^2 x_A^3}{6T_O^2} - \frac{\omega^4 x_A^5}{40T_O^4} + \cdots\right] + \left[x_B + \frac{\omega^2 x_B^3}{6T_O^2} - \frac{\omega^4 x_B^5}{40T_O^4} + \cdots\right] \tag{IV.5A}$$

Equation IV.5A can be reformulated if we first rewrite (11.3A) as

$$T_O = \frac{\omega x_B^2}{2y_B} = \frac{\omega x_A^2}{2y_A}$$

Substituting this into equation IV.5A results in an alternate expression for the length of the cable, written in terms of the x and y coordinates of the supports:

$$L = -s_{OA} + s_{OB} = -x_A\left[1 + \frac{2}{3}\left(\frac{y_A}{x_A}\right)^2 - \frac{2}{5}\left(\frac{y_A}{x_A}\right)^4\right] + x_B\left[1 + \frac{2}{3}\left(\frac{y_B}{x_B}\right)^2 - \frac{2}{5}\left(\frac{y_B}{x_B}\right)^4\right] \tag{IV.5B}$$

These five equations (IV.1–IV.5) can be solved for the five unknowns T_O, x_A, x_B, y_A, and y_B.

When anchors A and B are at different heights (i.e., $h \neq 0$), these nonlinear five equations must be solved to find the five unknowns. If the anchor points are at the same height, $-x_A = x_B = W/2$, and $y_A = y_B = y_{sag}$, equation (IV.3) or (IV.4) and (IV.5) can then be used to find T_O and y_{sag}.

Parabolic cable tension: Equilibrium conditions applied to the free-body diagram in **Figure 11.2.3*b*** result in an expression for tension anywhere along the length of a cable loaded by uniform load:

$$T^2 = \omega^2 x^2 + T_O^2 \quad \text{or} \quad T = \sqrt{\omega^2 x^2 + T_O^2} \tag{11.4}$$

This expression describes the tension at any location (x, y) along the cable—it tells us that the tension is maximum at the greatest distance from the origin (i.e., where x and y are maximum, which will be at the anchor farthest from the origin).

Comparison of Cases III and IV. The shape and tension of a cable loaded by its own weight (catenary curve) may be approximated by solution for a cable loaded by a constant weight (parabolic) when the sag-to-span ratio (y_{sag}/W) is small. A small sag-to-span ratio means a fairly taut cable, and the uniform distribution of weight along the cable is not much different from the same load intensity distributed uniformly along the horizontal.

Check out the following examples of applications of this material.

- **Example 11.2.1 Catenary Curve with Supports at Same Height**
- **Example 11.2.2 Catenary with Supports at Different Elevations**
- **Example 11.2.3 Uniformly Loaded Cable with Supports at Same Height**
- **Example 11.2.4 Uniformly Loaded Cable with Supports at Unequal Heights**
- **Example 11.2.5 Catenary versus Parabolic**

EXAMPLE 11.2.1

An electrical power line weighs 1 lb/ft and is anchored to towers of the same height (Figure 1). The span between towers is 400 ft, and the maximum design sag is restricted to 80 ft. Determine the length of the cable and the maximum tension.

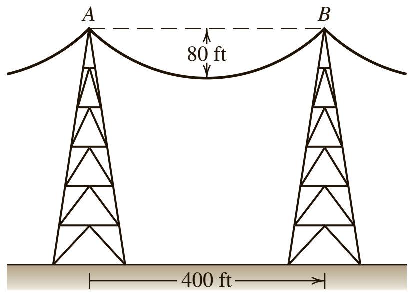


Figure 1 A power line under its own weight hangs in the shape of a catenary curve.

Goal Find the length of the cable and the maximum tension.

Given Information about the geometry and weight of the power line as well as the maximum allowable sag.

Assume The cable is inextensible and the weight of the cable is the only significant loading.

Draw Because of symmetry, the lowest point in the cable occurs halfway between the two towers. This is where we locate the origin of the coordinate system on the free-body diagram (Figure 2) of a segment of the cable.

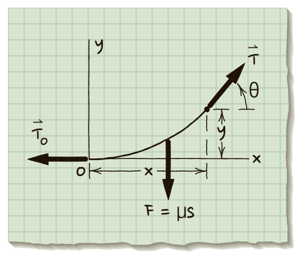


Figure 2 Free-body diagram of cable segment; the coordinate system origin is at the midpoint of the span.

Formulate Equations and Solve Since the only significant load on the cable is its own weight, the cable will take on a catenary shape, and the equations developed under Case III are relevant.

To determine the maximum tension in the cable we first determine the horizontal force in the cable, T_O, which we calculate using (11.1A):

$$y = \frac{T_O}{\mu}\left[\cosh\left(\frac{\mu x}{T_O}\right) - 1\right]$$

At the tower at B, $x_B = 200$ ft, $y_B = 80$ ft, and $\mu = 1$ lb/ft, therefore:

$$80\text{ ft} = \frac{T_O}{1\text{ lb/ft}}\left[\cosh\left(\frac{1\text{ lb/ft}(200\text{ ft})}{T_O}\right) - 1\right]$$

We solve this equation using a trial-and-error approach with the result $T_O = 262.3$ lb.

Using (11.2), we now solve for T_{max} at $x = 200$ ft, $y = 80$ ft:

$$T_{max} = T_O + \mu y = 262.3\text{ lb} + 1\text{ lb/ft}(80\text{ ft}) \Rightarrow \boxed{T_{max} = 342\text{ lb}}$$

This maximum tension occurs where the cable attaches to the towers and where its slope is the steepest.

We find the total cable length between the two towers by using (III.5), remembering that tower A is at (–200 ft, 80 ft) and tower B is at (200 ft, 80 ft):

$$L = -s_{OA} + s_{OB} = \frac{T_O}{\mu}\left[-\sinh\left(\frac{\mu x_A}{T_O}\right) + \sinh\left(\frac{\mu x_B}{T_O}\right)\right]$$

$$L = \frac{262.3\text{ lb}}{1\text{ lb/ft}}\left[-\sinh\left(\frac{1\text{ lb/ft}(-200\text{ ft})}{262.3\text{ lb}}\right) + \sinh\left(\frac{1\text{ lb/ft}(200\text{ ft})}{262.3\text{ lb}}\right)\right] \Rightarrow \boxed{L = 440\text{ ft}}$$

Check Aside from rechecking calculations, there are no alternative calculations that serve as a check for this problem.

EXAMPLE 11.2.2

Two pairs of civil engineering students use surveying tapes that weigh 0.2 N/m to measure a distance $W = 29.0$ m. Each pair pulls horizontally on the ends of the tape with a force of 50 N. One pair is instructed to have both ends of the tape at the same elevation (Pair 1; **Figure 1*a***). The other pair is instructed to have one end 0.1 m above the other end (Pair 2; **Figure 1*b***). What length will Pair 1 measure on their tape? What length will Pair 2 measure on their tape? How much effect does the 0.1 m in elevation difference have on the measurement?

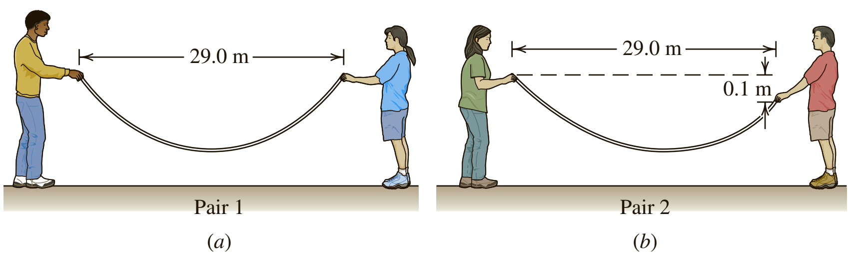


Figure 1 The surveying tape, supported only at the ends and loaded along its length only by its own weight, assumes the shape of a catenary.

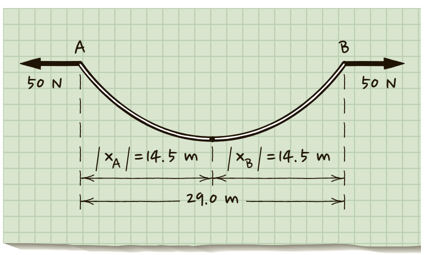


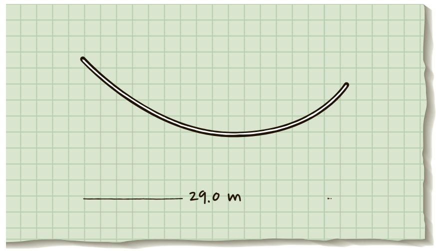
Figure 2 The origin is in the center when the supports are at the same elevation.

Goal Find the measurements taken by two pairs of engineering students, then consider how their measurements differ.

Given The weight of the tape, the horizontal force on the tape, and the locations of the supports.

Assume The tape is inextensible (i.e., it does not stretch), the error due to temperature variation can be ignored, and the error due to inexact markings on the tape can be ignored.

Draw We locate the origin for the two cases: the supports at the same elevation (**Figure 2**) and the supports at different elevations (**Figure 3**).


29.0 m

EXAMPLE 11.2.3

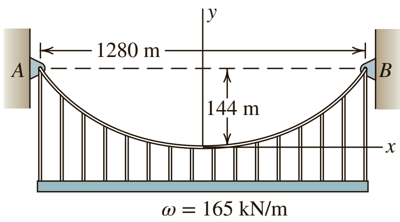


Figure 1 Model of a main cable on the center span of the Golden Gate Bridge.

The cable shown in **Figure 1** is a model of one of the main cables on the center span of the Golden Gate Bridge. Each main cable supports one half the roadbed resulting in a load of 165 kN/m of horizontal length. The towers are 1280 m apart, and the sag of the cable is 144 m. Determine the cable tension at midspan, the maximum tension, and the total length of the cable.

Goal Find the cable tension at midspan, the maximum tension, and the total length of the cable.

Given Information about the cable geometry and loading.

Assume The cable is inextensible and that the weight of the cable is negligible relative to the loading.

Formulate Equations and Solve Given the conditions of the problem, the equations developed for Case IV (uniformly distributed load) apply and the cable will conform to a parabolic shape. Based on the geometry in **Figure 1**, we know that

$$|x_A| = |x_B| = 640\text{ m},\quad y_A = y_B = 144\text{ m},\quad \text{and}\quad \omega = 165\text{ kN/m}$$

The cable tension at midspan is T_O since the cable is horizontal at midspan. Applying (11.3A) at B allows us to solve for T_O:

$$y = \frac{\omega x^2}{2T_O} = 144\text{ m} = \frac{\left(165\,\dfrac{\text{kN}}{\text{m}}\right)(640\text{ m})^2}{2\,T_O}$$

Rearranging gives $\Rightarrow$ $T_O = 234,667\text{ kN}$ $\Rightarrow$ $\boxed{T_O = 235\text{ MN}}$

Using (11.4), and noting that the maximum tension occurs at the supports, we find $T_{\max}$:

$$T_{\max} = T_{@\,x=640\text{ m}} = \sqrt{\left(165\,\frac{\text{kN}}{\text{m}}\right)^2(640\text{ m})^2 + (234,667\text{ kN})^2} = 257,332\text{ kN}$$

$$\boxed{T_{\max} = 257\text{ MN}}$$

Finally, to find the length of the cable L, substitute $x_A = -640$ m, $x_B = 640$ m, $T_O = 234{,}667$ kN, and $\omega = 165$ kN/m into equation (IV.5A):

$$L = -s_{OA} + s_{OB} = -\left[x_A + \frac{\omega^2 x_A^3}{6T_O^2} - \frac{\omega^4 x_A^5}{40\,T_O^4}\right] + \left[x_B + \frac{\omega^2 x_B^3}{6\,T_O^2} - \frac{\omega^4 x_B^5}{40\,T_O^4}\right] \Rightarrow \boxed{L = 1322\text{ m}}$$

Alternately, (IV.5B) could have been used to find L.

Check Aside from rechecking calculations, there are no alternative calculations that serve as a check for this problem.

EXAMPLE 11.2.4

A 150-ft foot bridge crossing a gorge is suspended from two cables. Due to the natural terrain, the right support is 12 ft higher than the left support. The weight of the bridge deck and handrails supported by the cables is 60 lb/ft (**Figure 1**) and the lowest point of the cables is 10 ft below the left support. Determine the maximum and minimum cable tensions, the angle between the cables and the horizontal at the right support, and the total length of each cable.

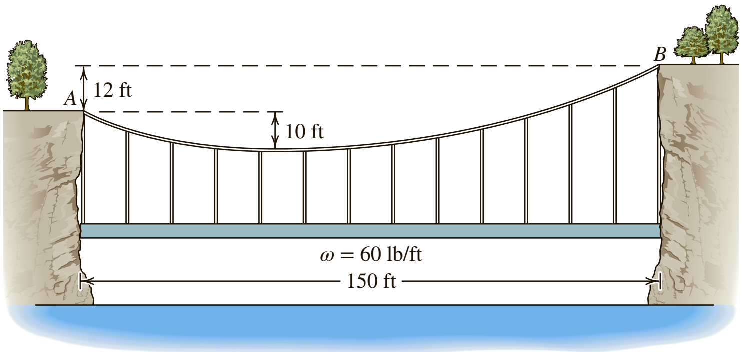


Figure 1 The suspension cables hang in a parabolic curve when the uniformly applied load (the deck) is large compared to the weight of the cables.

Goal For each cable find the maximum and minimum cable tensions, the angle between the cable and the horizontal at B, and the total length.

Given Information about the cable geometry and loading.

Assume The cable is inextensible and the cable weight is negligible relative to the loading.

Draw We define a coordinate system with its origin at the lowest point of the cable, the location of which is unknown (**Figure 2**). We define x_B as the x coordinate of support B; then the coordinate $x_A = x_B - 150$ ft.

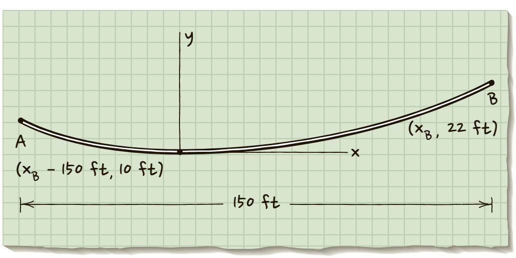


Figure 2 The location of the coordinate system origin is unknown.

Formulate Equations and Solve The load is shared equally by the two suspension cables; therefore $\omega = 30$ lb/ft for each cable. We locate the origin by applying (11.3A) at supports A and B:

$$\text{At } A: \qquad y_A = 10\text{ ft} = \frac{(30\text{ lb/ft})(x_B - 150\text{ ft})^2}{2T_o} \tag{1}$$

$$\text{At } B: \qquad y_B = 22\text{ ft} = \frac{(30\text{ lb/ft})(x_B)^2}{2T_o} \tag{2}$$

We eliminate T_O by dividing (1) by (2), which gives

$$\frac{10\text{ ft}}{22\text{ ft}} = \frac{(x_B - 150\text{ ft})^2}{x_B^2}$$

Rearranging and simplifying produces an equation that can be solved using the quadratic formula:

$$x_B^2 - 550x_B + 41{,}250 = 0$$

$$x_B = \frac{550 \pm \sqrt{(-550)^2 - 4(1)(41{,}250)}}{2(1)} = 275\text{ ft} \pm 185.4\text{ ft} \quad \Rightarrow \quad x_B = 460.4\text{ ft or } 89.6\text{ ft}$$

We discard $x_B = 460.4$ ft because it is not a realistic solution when the constraints of this problem are considered. Therefore $x_B = 89.6$ ft.

Substituting this value of x_B into (2), we solve for the minimum cable tension T_O

$$y_B = \frac{\omega\, x_B^2}{2\,T_O} = 22\text{ ft} = \frac{(30\text{ lb/ft})(89.6\text{ ft})^2}{2\,T_O} \quad \Rightarrow \quad \boxed{T_O = 5475\text{ lb}}$$

T_{max} occurs at support B, because the slope of the cable is steepest there. Using (11.4), we find T_{max}:

$$T_{max} = T_{@x=89.6\text{ ft}} = \sqrt{\left(30\,\frac{\text{lb}}{\text{ft}}\right)^2 (89.6\text{ ft})^2 + (5474\text{ lb})^2} \quad \Rightarrow \quad \boxed{T_{max} = 6098\text{ lb}}$$

The tensile force is tangent to the cable at any point along the cable. Therefore, the angle the cable force makes with the horizontal and the angle the cable makes with the horizontal are the same. We call this angle θ_B at support B. **Figure 3** shows the relationship between the cable force at B and the horizontal component of that force.

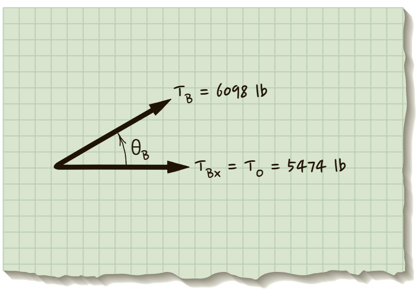


Figure 3 The horizontal component of the cable force at B is T_O.

$$T_{Bx} = T_B \cos\theta_B \quad \Rightarrow \quad \theta_B = \cos^{-1}\left(\frac{5474\text{ lb}}{6098\text{ lb}}\right) \quad \Rightarrow \quad \boxed{\theta_B = 26.2^\circ}$$

Finally, to find the length of the cable L, substitute $x_A = 89.6$ ft–150 ft = –60.4 ft, $x_B = 89.6$ ft, $T_O = 5474$ lb, and $\omega = 30$ lb/ft into (IV.5A):

$$L = -s_{OA} + s_{OB} = -\left[x_A + \frac{\omega^2 x_A^3}{6\,T_O^2} - \frac{\omega^4 x_A^5}{40\,T_O^4}\right] + \left[x_B + \frac{\omega^2 x_B^3}{6\,T_O^2} - \frac{\omega^4 x_B^5}{40\,T_O^4}\right]$$

$$\boxed{L = 154.6\text{ ft}}$$

Check Aside from rechecking calculations, there are no alternative calculations that serve as a check for this problem.

EXAMPLE 11.2.5

For the electrical power line depicted in **Figure 1**, the maximum design sag is restricted to 80 ft. Determine the length of the cable and the maximum tension assuming that the shape of the cable is parabolic. Compare your answers with those from Example 11.2.1 in which the cable was modeled as a catenary.

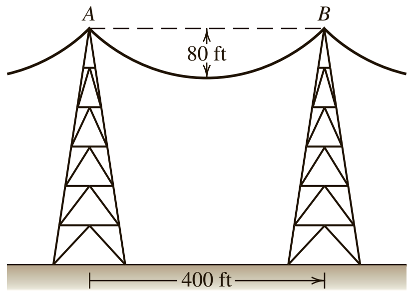


Figure 1 A power line under its own weight hangs in the shape of a catenary curve.

Goal Find L and T_{max} using the parabolic approach, then compare the result with Example 11.2.1.

Given Information about the geometry and weight of the power line as well as a specific design constraint.

Assume The cable is inextensible.

Draw **Figure 2** represents the location of the origin for either a catenary or a parabolic shape.

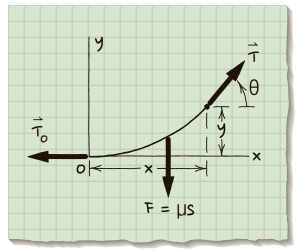


Figure 2 Free-body diagram of cable segment; the coordinate system origin is at the midpoint of the span.

Formulate Equations and Solve In the parabolic method approach, the weight of the cable is modeled as a uniform weight per *horizontal distance* (ω) hanging from the cable, which we calculate by dividing the weight of the cable by the distance between the towers.

$$\omega = \frac{F_{\text{cable}}}{W}$$

where

$$F_{\text{cable}} = \mu L = (1\ \text{lb/ft})\,L$$

Equation (IV.5B) is independent of ω, so we can find the length L of the cable using $x_A = -200$ ft, $y_A = 80$ ft, $x_B = 200$ ft, and $y_B = 80$ ft:

$$L = -x_A\left[1 + \frac{2}{3}\left(\frac{y_A}{x_A}\right)^2 - \frac{2}{5}\left(\frac{y_A}{x_A}\right)^4\right] + x_B\left[1 + \frac{2}{3}\left(\frac{y_B}{x_B}\right)^2 - \frac{2}{5}\left(\frac{y_B}{x_B}\right)^4\right]$$

$$= -(-200\ \text{ft})\left[1 + \frac{2}{3}\left(\frac{80\ \text{ft}}{-200\ \text{ft}}\right)^2 - \frac{2}{5}\left(\frac{80\ \text{ft}}{-200\ \text{ft}}\right)^4\right] + 200\ \text{ft}\left[1 + \frac{2}{3}\left(\frac{80\ \text{ft}}{200\ \text{ft}}\right)^2 - \frac{2}{5}\left(\frac{80\ \text{ft}}{200\ \text{ft}}\right)^4\right] \Rightarrow \boxed{L = 438.6\ \text{ft}}$$

Therefore

$$\omega = \frac{F_{\text{cable}}}{W} = \frac{(1\ \text{lb/ft})438.6\ \text{ft}}{400\ \text{ft}} = 1.096\ \text{lb/ft}$$

Equation (11.3A) describes the parabolic shape of the hanging cable, which is constrained by the 80-ft sag and the 400-ft distance between the towers. We rearrange (11.3A) to solve for T_O:

$$T_O = \frac{\omega x^2}{2y} \underbrace{=}_{x=200\ \text{ft},\ y=80\ \text{ft}} \frac{1.096\ \text{lb/ft}(200\ \text{ft})^2}{2(80\ \text{ft})} = 274.1\ \text{lb}$$

Using (11.4), we find T_{max}, which occurs at either of the towers:

$$T_{max} = \sqrt{\omega^2 x_A^2 + T_O^2} = \sqrt{(1.096\ \text{lb/ft})^2(200\ \text{ft})^2 + (274.1\ \text{lb})^2} \Rightarrow \boxed{T_{max} = 351.0\ \text{lb}}$$

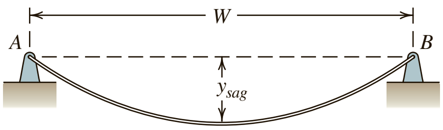
In Example 11.2.1 T_{max} = 342 lb and L = 440 ft. For this example (sag-to-span ratio y_{sag}/W = 80/400 = 0.2), the difference in the answers is not large. For larger sag-to-span ratios, the catenary approach will generate more accurate solutions than the parabolic approach for the self-weight loading case. As a general rule, the parabolic approach is only recommended for sag-to-span ratios of 0.1 or less.

EXERCISES 11.2

11.2.1. [*] A uniform cable of weight/length μ is suspended between two points A and B, as shown. Values of y_{sag} (the sag) and W (the distance between A and B) are known.

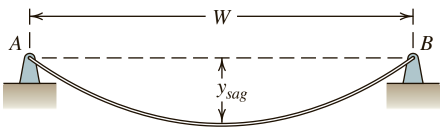
a. Describe, in words, your procedure for finding the minimum and maximum tension in the cable and the length of the cable.

b. Apply your procedure to find minimum and maximum tension and cable length for the case when μ = 10 lb/ft, y_{sag} = 100 ft, and W = 500 ft.



EX 11.2.1

11.2.2. [*] A uniform 0.5-lb/ft cable is suspended between two supports at equal elevation. The span (W) is 250 ft and the sag is 100 ft. Determine the minimum and maximum tensions in the cable and the length of the cable.



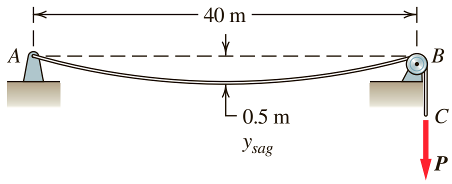
EX 11.2.2

11.2.3. [*] A cable supports a mass of 12 kg/m of its own length and is suspended between two points at the same level. It supports only its own weight. Determine the tension at midspan (T_O), the maximum tension, and the total cable length for the following conditions:

	Span	Sag
Case 1	300 m	60 m
Case 2	200 m	40 m

11.2.4. [*] A cable is attached to a support at A, passes over a small ideal pulley at B, and supports a force $\boldsymbol{P}$, as shown. The sag of the cable (y_{sag}) is 0.5 m, and the mass per unit length of the cable is 1.0 kg/m. Neglect the weight of the cable portion from B to C. Determine

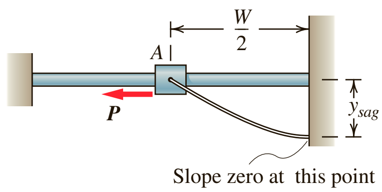
a. the magnitude of the load $\boldsymbol{P}$ to maintain equilibrium
b. the slope of the cable just to the left of B
c. the total length of the cable from A to B



EX 11.2.4

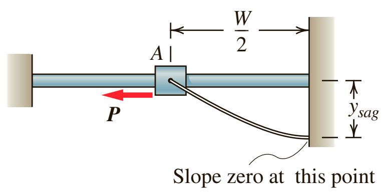
11.2.5. [*] A uniform 0.4-kg/m cable is attached to collar A, which may slide along the frictionless horizontal bar, as shown. If a horizontal force of 10 N is required to hold the collar in place when $W/2$ = 900 mm, determine

a. y_{sag}
b. the length of the cable



EX 11.2.5

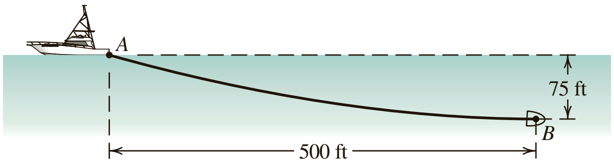
11.2.6. [*] For the cable shown, determine the horizontal force $\boldsymbol{P}$ that is required to hold the collar such that $W/2$ = 600 mm, and y_{sag} = 360 mm.



EX 11.2.6

11.2.7. [*] A marine research boat collecting water samples tows a cable and water sampling vessel behind, as shown. The cable weighs 2 lb/ft and is horizontal at B. Assume the buoyancy effects of the water on the cable

are negligible. Determine the horizontal tension T_O in the cable at the water sampling vessel.



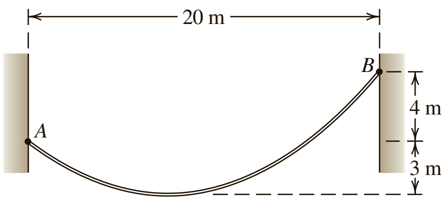
EX 11.2.7

11.2.8. [*] The cable shown hangs under its own weight ($\mu = 1$ N/m) and behaves as a catenary. Finding y_{sag} for this problem requires solving simultaneous equations numerically and is explored in **Exercise 11.2.24**. You can simplify the analysis and obtain an approximate solution by modeling this as a parabolic cable. Using the parabolic model:

a. determine x_A and x_B

b. convince yourself that x_A and x_B are independent of the value of μ when W, y_A, and y_B are specified

c. determine y_{sag} and L. Are these two quantities independent of μ in this problem?



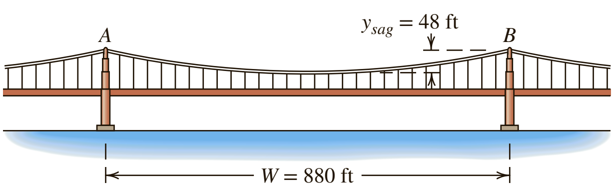
EX 11.2.8

11.2.9. [*] A uniformly distributed roadway is carried by the two main cables of a suspension bridge, as shown. Each cable supports ω = 1200 lb per horizontal foot, with span W = 880 ft and y_{sag} = 48 ft. Each cable is supported at the towers at A and B at the same elevation. Determine

a. the tension in the main cable at a support and the minimum tension in the main cable

b. the length of the main cable between A and B if it takes on a parabolic shape

c. the length of the main cable between A and B if it takes on a catenary shape (Compare the answer to **b.**)



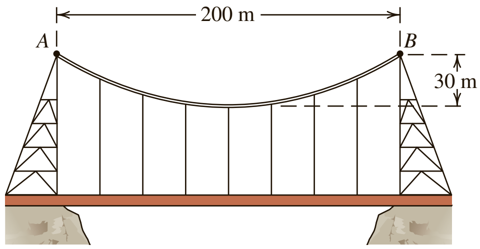
EX 11.2.9

11.2.10. [*] A water conduit that crosses a canyon is suspended from a cable, as shown. When full of flowing water, the conduit weighs 30 kN per meter of length.

a. Determine the maximum tension in the cable.

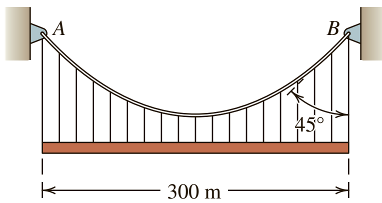
b. Find the length of the cable.

c. When water is not flowing through the conduit, the conduit weighs 15 kN per meter of length. How might the towers A and B be designed so as to maintain the same cable sag as in the fully loaded condition? Why would it be desirable to maintain constant cable sag?



EX 11.2.10

11.2.11. [*] Cable AB carries a uniform load of 500 N/m and has a span of 300 m, as shown. If the tangent to the cable at either end is at 45° from the vertical, determine y_{sag} and the maximum tension in the cable.



EX 11.2.11

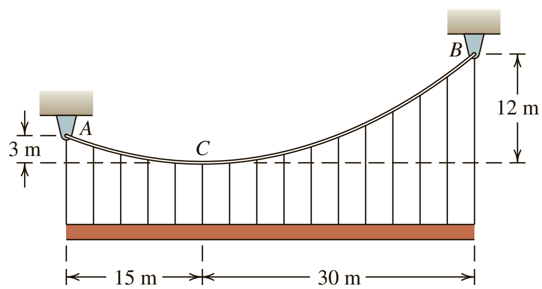
11.2.12. [*] A suspension bridge roadway is supported by a cable anchored at A and B, as shown. The roadway weighs 3 kN per meter of horizontal length. Determine

a. the minimum and maximum tension in the cable

b. the cable tensions at the supports A and B

c. the length of the cable

d. y_{sag}



EX 11.2.12

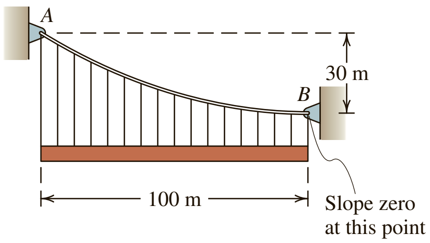
11.2.13. [*] A suspension bridge supports a 150-kN load that is uniformly distributed over the bridge span. The cable slope is zero at B. Determine

a. the minimum and maximum tension in the cable

b. the cable tensions at the supports A and B

c. the location on the cable where the tension is the average of the values determined in **a**

d. y_{sag}

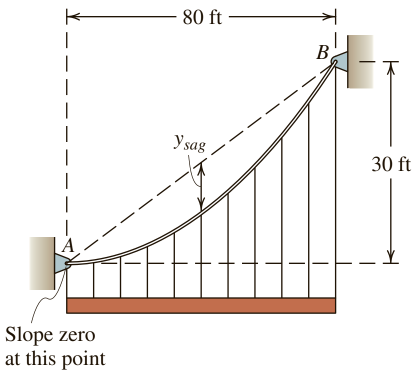


EX 11.2.13

11.2.14. [*] A cable supports a uniform load of 100 lb/ft and is anchored at A and B, as shown. At A, the cable is horizontal. Determine

a. the minimum and maximum tension in the cable

b. the length of the cable using both the parabolic and catenary methods (Compare the two solutions.)



EX 11.2.14

11.2.15. [*] When Joseph Strauss designed the Golden Gate Bridge, he specified the height of the towers, the length of the center span (W = 1280 m), and the sag of the main cable (y_{sag} = 143 m). If he had specified shorter towers so that the allowable sag was only 130 m, by what percentage would the horizontal component of the cable tension T_O have changed? (Assume ω = 165 kN/m.)

11.2.16. [**] A cable supporting only its own weight hangs from supports at the same height. Determine the sag-to-span ratio y_{sag}/W if the total weight equals the minimum tension in the cable.

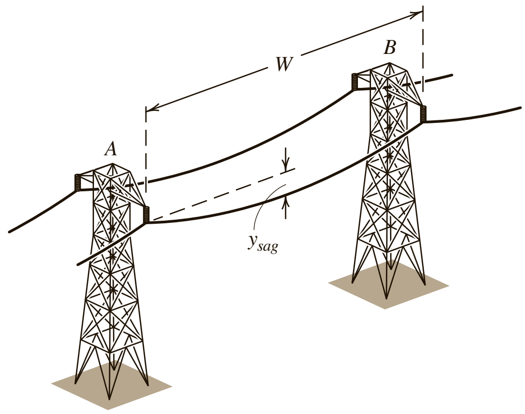
11.2.17. [**] A cable supporting only its own weight hangs from supports at the same height. Determine the sag-to-span ratio y_{sag}/W if the total weight equals the maximum tension in the cable.

11.2.18. [**] The mass per unit length of a cable supporting its own weight is 0.4 kg/m. Determine the two values of y_{sag} for which the maximum tension in the cable is 250 N if W = 70 m.

11.2.19. [**] Two electrical transmission cables, each having a weight per unit length of 2 lb/ft, are suspended between and pinned to a series of towers, as shown. If W = 400 ft and y_{sag} = 5 ft, determine

a. the horizontal and vertical components of the resultant force exerted by segment AB of one cable on tower A

b. the loads acting on the base of tower A



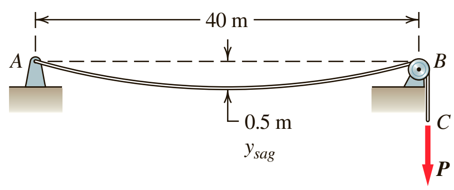
EX 11.2.19

11.2.20. [**] A cable is attached to a support at A, passes over a small ideal pulley at B, and supports a force $\boldsymbol{P}$, as shown. The sag of the cable (y_{sag}) varies from 0.25 to 5 m. The mass per unit length of the cable is 1.0 kg/m.

a. Create a plot of the required force $\boldsymbol{P}$ as a function of y_{sag}.

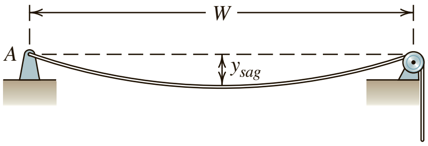
b. Create a plot of the cable length (between A and B) as a function of y_{sag}.

c. If the maximum cable tension is to be limited to 1000 N, specify the maximum force $\boldsymbol{P}$ that should be applied to the cable.



EX 11.2.20

11.2.21. [**] A uniform cord 750-cm long and weighing 0.1 N/cm passes over a frictionless pulley at B and is attached to a rigid support at A. For a span W = 250 cm, determine the smaller of the two values of y_{sag} for which the cord is in equilibrium.



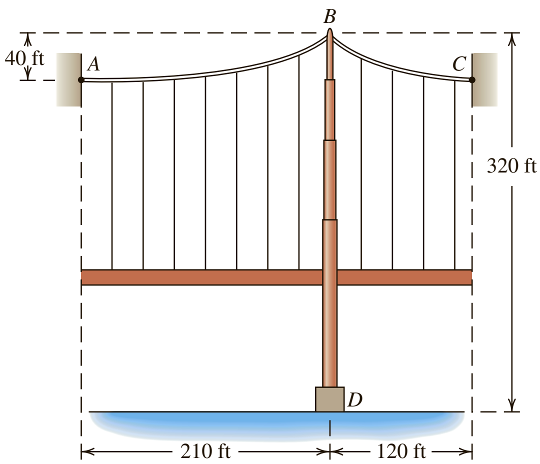
EX 11.2.21

11.2.22. [**] A suspension bridge roadway is supported by two cables AB and BC. One end of each cable is pinned to a tower at B. The other ends of the cables are anchored at the same elevation at A and C, respectively, where the cables have horizontal slopes. The roadway load is 200 lb per horizontal foot. Determine

a. the resultant of the forces exerted on tower B by the cables and the forces exerted on the supports at A and C

b. the lengths of cables AB and BC

c. the loads acting on the tower at its base D

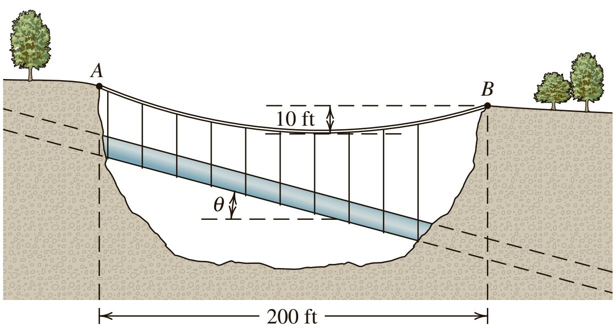


EX 11.2.22

11.2.23. [**] Cable AB carries a pipeline, whose weight per unit length is 220 lb/ft. If $\theta = 20°$ and B is located 10 ft below A, determine

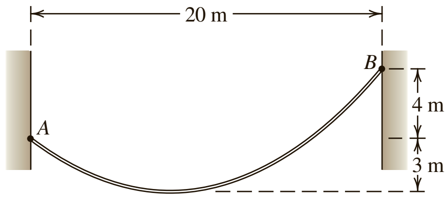
a. the maximum tension in the cable

b. the length of the cable



E11.2.23

11.2.24. [***] A cable has a unit weight of $\mu = 1$ N/m. Using a catenary approach, determine its total length and y_{sag}. You may want to start by modeling the cable as a parabolic cable to develop initial estimates of x_A, x_B, and T_O for the numerical solution (see **Exercise 11.2.8**).



EX 11.2.24

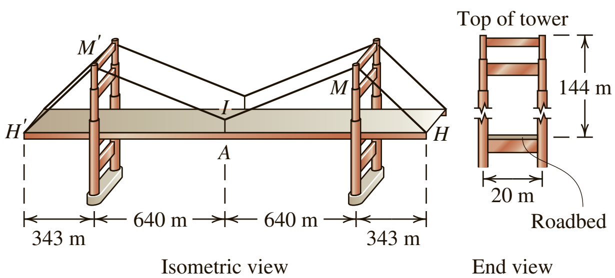
11.2.25. [***] A very simple model of the Golden Gate Bridge is used to perform a preliminary "back of the envelope" analysis to get a quick idea of the magnitudes of loads and internal forces. (This type of simple model could be useful in checking whether the output of a computer analysis looks reasonable or providing a first guess at member sizes for a design.)

The model consists of the bridge deck, a 10-m long suspender on each side at the middle of the bridge, the two towers, and four truss links to represent each main cable. Assuming that the bridge deck weighs 330 kN/m and that each main cable carries an equal share of the bridge load, calculate the force in members IA and IM. How do these values compare with

(a) the forces acting on the links in the simplified model in **Figure SA11.1.2**, or

(b) the theoretical force developed in Example 11.2.3?

Another approximate model of the Golden Gate Bridge is explored in SA11.1.



EX 11.2.25

11.3 JUST THE FACTS

Cables are constructed from a large number of wound wires. Engineers undertake analysis to calculate the cable tension so that they can design a cable whose material and cross-sectional area ensure sufficient capacity to transmit the tension without failing. Designers also need to be able to predict the shape of the cable so that they will know the geometry of the fully loaded structure. In this chapter we considered how to systematically model a cable's deformed shape and calculate its internal loads when subjected to point loads or distributed loads.

Cables with Point Loads

The forces acting on cables may be distributed along their length or may be point forces. The internal forces in cables are limited to tension forces. The deformed shape of the cable is directly related to the loads, in particular, the ratio of **sag** (maximum *vertical* distance from the line connecting the two anchors to the cable) to **span** (*horizontal* distance between the supports) is often a consideration in designing systems with cables.

In this chapter we considered four possible cable loadings. They are

- Case I—the cable is fully taut and the force acts along the cable.
- Case II—concentrated forces are applied perpendicular to the cable and the cable adjusts so that there are taut straight-line segments between the concentrated forces.
- Case III—the cable is loaded solely by its own weight. Its shape is a **catenary curve** and the tension varies along the length.
- Case IV—the cable is carrying a distributed load and the cable's weight is small relative to this distributed load. Its shape is a **parabolic curve** and the tension varies along the length.

Cables with Distributed Loads

The equation for a catenary curve, which describes the shape of a cable loaded by its own weight (Case III), is

$$y = \frac{T_o}{\mu}\left(\cosh\frac{\mu x}{T_O} - 1\right) \tag{11.1A}$$

where cosh is the hyperbolic cosine, μ is the weight per unit cable length, and T_O is the constant horizontal force component everywhere in the cable. This curve is defined relative to the origin O, which is located at the lowest point on the curve. The slope of the curve is given as

$$\frac{dy}{dx} = \sinh\frac{\mu x}{T_O} = \frac{e^{\mu x/T_O} - e^{-\mu x/T_O}}{2} \tag{11.1B}$$

where sinh is the hyperbolic sine.

The tension at any location (x, y) along the catenary cable is given by:

$$T = T_O \cosh\frac{\mu x}{T_O} = T_O + \mu y \tag{11.2}$$

This expression tells us that the tension is maximum at the greatest distance from the origin (i.e., where x and y are maximum, which will be at the anchor farthest from the origin), and minimum at the origin.

The shape of a cable when a uniform weight hangs from it (Case IV) is described by a parabola:

$$y = \frac{\omega x^2}{2T_O} \tag{11.3A}$$

where T_O is the constant horizontal force component everywhere in the cable and ω is the distributed load per horizontal length. The parabolic curve is defined relative to the origin O, which is the lowest point of the cable. The slope of the curve is given as

$$\frac{dy}{dx} = \frac{\omega x}{T_O} \tag{11.3B}$$

The tension at any location (x, y) along the length of a cable loaded by uniform load is expressed by:

$$T^2 = \omega^2 x^2 + T_O^2 \quad \text{or} \quad T = \sqrt{\omega^2 x^2 + T_O^2} \tag{11.4}$$

The tension is maximum at the greatest distance from the origin and minimum at the origin.

NOTES

Note 1: Development of Expressions that Describe Catenary Cables

Development of (11.1A) and (11.1B). Using **Figure 11.2.2**, we write the equations of equilibrium.

$$\sum F_x = T\cos\theta - T_O = 0 \qquad \Rightarrow \qquad T\cos\theta = T_O \tag{1}$$

$$\sum F_y = T\sin\theta - \mu s = 0 \qquad \Rightarrow \qquad T\sin\theta = \mu s \tag{2}$$

Dividing (2) by (1),

$$\tan\theta = \frac{\mu s}{T_O} \tag{3}$$

Recognizing that $\tan\theta = \dfrac{dy}{dx}$, we can rewrite (3) as

$$\frac{dy}{dx} = \frac{\mu s}{T_O} \tag{4}$$

Differentiating (4) with respect to x,

$$\frac{d^2y}{dx^2} = \frac{\mu}{T_O}\frac{ds}{dx} \tag{5}$$

We now write cable distance s as a function of x and y as

$$ds^2 = dx^2 + dy^2$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{6}$$

and substitute (6) into (5):

$$\frac{d^2y}{dx^2} = \frac{\mu}{T_O}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{7}$$

Calling $\frac{dy}{dx} = q$, we can rewrite (7) as

$$\frac{dq/dx}{\sqrt{1+q^2}} = \frac{\mu}{T_O} \tag{8}$$

Integration of (8) with the boundary condition ($x = 0, \frac{dy}{dx} = 0$) results in

$$q = \frac{dy}{dx} = \sinh\frac{\mu x}{T_O}$$

which can be rewritten in terms of exponentials as

$$\frac{dy}{dx} = \sinh\frac{\mu x}{T_O} = \frac{e^{\mu x/T_O} - e^{-\mu x/T_O}}{2} \tag{9}$$

Integration of (9) with the boundary condition ($x = 0$, $y = 0$) results in

$$y = \frac{T_O}{\mu}\left(\cosh\frac{\mu x}{T_O} - 1\right) \tag{10}$$

Equation (9) is the same as (11.1B), and (10) is the same as (11.1A).

Development of (11.2) for Catenary Cable Tension With the expression for slope in (9), we can rewrite (4) as

$$s = \frac{T_O}{\mu}\sinh\frac{\mu x}{T_O} \tag{11}$$

Squaring (1) and (2) and adding, we have

$$T^2 = T_0^2 + \mu^2 s^2 \tag{12}$$

Substituting (11) into (12) and rearranging we write an expression for the tension T:

$$\boxed{T = T_O \cosh \frac{\mu x}{T_O}} \tag{13A}$$

Equation (13A) can be written in an alternate form by substituting in for the hyperbolic cosine from (10):

$$\boxed{T = T_O + \mu y} \tag{13B}$$

Equations (13A) and (13B) are the same as (11.2).

Note 2: Development of Expressions that Describe Parabolic Cables

Development of (11.3A) and (11.3B). Using **Figure 11.2.3*b***, we write the equations of equilibrium.

$$\sum F_x = T\cos\theta - T_O = 0 \quad \Rightarrow \quad T\cos\theta = T_O \tag{1}$$

$$\sum F_y = T\sin\theta - \omega x = 0 \quad \Rightarrow \quad T\sin\theta = \omega x \tag{2}$$

Dividing (2) by (1),

$$\tan\theta = \frac{\omega x}{T_O} \tag{3}$$

Recognizing that $\tan\theta = \dfrac{dy}{dx}$, we can rewrite (3) as

$$\boxed{\frac{dy}{dx} = \frac{\omega x}{T_O}} \tag{4}$$

This is the same as (11.3B).

Integrating (4) with respect to x with the boundary condition ($x = 0$, $y = 0$) results in

$$\boxed{y = \frac{\omega x^2}{2T_O}} \tag{5}$$

which is the same as (11.3A).

Development of Expression for Cable Length (IV.5A). A segment of the cable ds can be written as

$$ds^2 = dx^2 + dy^2$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{6}$$

Now substitute (4) into (6):

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{\omega x}{T_O}\right)^2} \tag{7}$$

Rearranging and integration for the segment s_{OA} that runs from $x = 0$ to $x = x_A$, we have

$$\int_0^{s_{OA}} ds = \int_0^{x_A} \sqrt{1 + \left(\frac{\omega x}{T_O}\right)^2}\, dx \tag{8}$$

Although this expression can be integrated in closed form, for computational purposes it is more convenient to express the radical as a convergent series and then integrate term by term. For this purpose we use the binomial expression

$$\left(1 + \left(\frac{\omega x}{T_O}\right)^2\right)^n = 1 + n\left(\frac{\omega x}{T_O}\right)^2 + \frac{n(n-1)}{2!}\left(\frac{\omega x}{T_O}\right)^4 + \frac{n(n-1)(n-2)}{3!}\left(\frac{\omega x}{T_O}\right)^6 + \cdots \tag{9}$$

which converges for $\left(\frac{\omega x}{T_O}\right)^2 < 1$. Setting $n = \frac{1}{2}$ in (9) gives the expression

$$s_{OA} = \int_0^{x_A} \left(1 + \frac{\omega^2 x^2}{2T_O^2} - \frac{\omega^4 x^4}{8T_O^4} + \cdots\right) dx = \left[x_A + \frac{\omega^2 x_A^3}{6T_O^2} - \frac{\omega^4 x_A^5}{40T_O^4} + \cdots\right] \tag{10}$$

Notice that this is the same as the first term on the right-hand side of equation IV.5A. A similar expression can be created for s_{OB}.

SYSTEM ANALYSIS (SA) EXERCISES

SA11.1 Golden Gate Bridge Approximate Analysis 1

Undertaking analysis of structures often requires that engineers make assumptions about the structure's behavior. These approximations are commonly simplifications or idealizations about the behavior made because not everything is known about the structure and/or because of constraints on the resources available for the analysis. In the case study in Appendix E an analysis of the Golden Gate Bridge is undertaken in which the main cables are approximated as a series of links (**Figure SA 11.1.1**). The approximate analysis gives the following:

Member	**Force**
IJ and *IJ′*	237 MN (tension)
IA	26.4 MN (tension)
JK	240 MN (tension)
JB	26.4 MN (tension)

The following tasks are designed to further explore the approximation of the main cables as presented in Appendix E:

(a) Draw a free-body diagram of joint K and use the geometry as defined in **Table SA 11.1.1** to find the force in member KL.

(b) Draw a free-body diagram of joint L and use the geometry as defined in **Table SA 11.1.1** to find the force in member LM.

(c) By calculating the horizontal components of the forces acting on links IJ, JK, KL, and LM, show that the horizontal force in the main cable remains constant between any two points along its length.

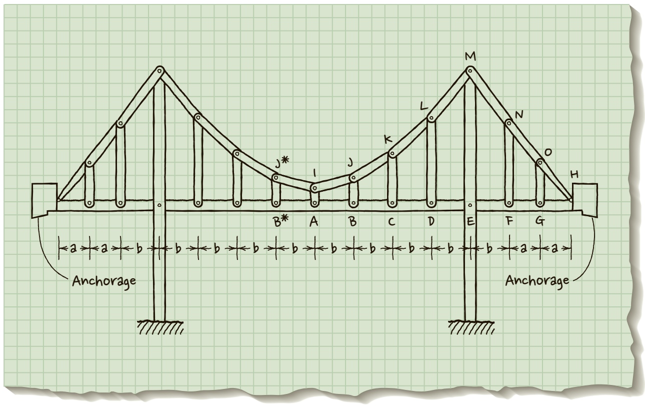


Figure SA11.1.1 Golden Gate cables modeled as two-force members.

Table SA11.1.1 Geometry of "Main Cable" in Our Approximate Model

Pin	Height above Bridge Deck (meters)	Distance from Center of Bridge (meters)
I	10.0	0
J and J*	18.9	160
K	45.8	320
L	90.4	480
M	153.0	640
N	71.4	800
O	32.8	891.5
H	0	983

(d) Compare the tension acting on links IJ and IJ' to T_O, and the tension acting on LM to T_{max} presented in **Figure SA 11.1.2**. Based on this comparison, do you think that the links serve as a reasonable approximation to the continuous cable? How might the approximation be improved?

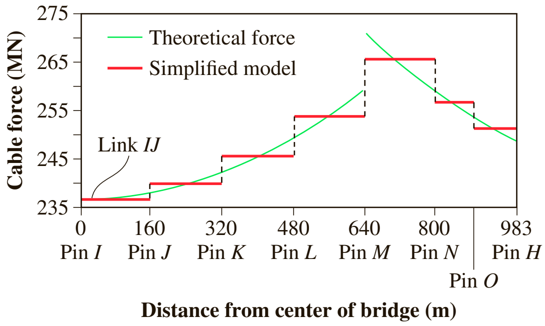


Figure SA11.1.2 Forces calculated from the simplified analytical model of the main cable compared to the actual forces exerted on the cable.

SA 11.2 Hoover Dam

Frank Crowe, the chief engineer of the Hoover Dam, was the first to use cableways to transport people and supplies to the worksite (**Figure SA11.2.1**). In many instances the cables were required to support loads of several tons.

(a) Consider a simple model of the cableways that Frank Crowe employed during the construction of Hoover Dam (**Figure SA11.2.2**). Model link BE as a particle and assume that $\alpha = 25°$, $\beta = 30°$, the mass of the car and its load is approximately 12,000 kg, and the tension in cable segment EG is negligible. Find the tension in the support cable ABC and the traction cable DE. What uncertainty in these forces is associated with α?

(b) To support the kind of loads found in (a), locked-coil track strand wire rope (shown in **Figure SA11.2.2*b***) was used. This particular wrapping style helps to distribute the contact loads exerted by the pulleys on the cable. **Table SA11.2.1** gives cable strengths (adapted from *Marks' Standard Handbook*, page 10-9). Choose a factor of safety and a grade of cable, then select an appropriate diameter for cables ABC and DE.

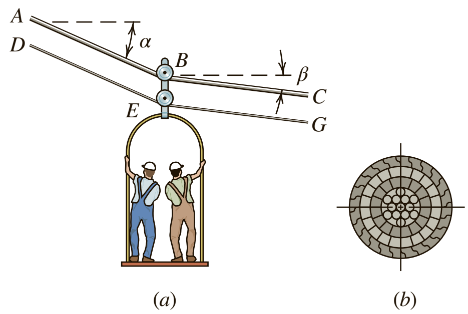


Figure SA11.2.2 (*a*) Cableways similar to those used to build Hoover Dam; (*b*) cross-section of the locked-coil track strand wire rope that was used to support the heavy loads (*Marks' Standard Handbook*).

Bureau of Reclamation/U.S. Dept. of the Interior

Figure SA 11.2.1 Transporting equipment and materials at Hoover Dam (postcard from Hoover Dam).

Table SA11.2.1 **Capacity Data for Cables**

Diameter		Special Grade		Standard Grade	
in.	**mm**	**Short ton**	**Tonne**	**Short Ton**	**Tonne**
0.750	19.1	31.5	28.6	25	22.7
0.875	22.2	41.5	37.6	32	29.0
1.00	25.4	52.5	47.6	42	38.1
1.125	28.6	66.0	59.9	54	49.0
1.250	31.8	81.0	73.5	65	59.0
1.375	34.9	100.0	90.7	78	70.8
1.500	38.1	120.5	109.3	93	84.4
1.625	41.3	140.0	127.0	108	98.0
1.750	44.5	165.0	150	125	113.4
1.875	47.6	187.5	170	138	125.2
2.00	50.8	215	195	158	143
2.250	57.2	280	254		
2.500	63.5	345	313		
2.750	69.9	420	381		
3.00	76.2	500	454		
3.250	82.6	580	526		
3.500	88.9	690	626		
3.750	95.3	785	712		
4.00	101.6	880	798		

SELECTED TOPICS IN MATHEMATICS

A.1 Algebra

1. Quadratic equation

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, b^2 \geq 4ac \text{ for real roots}$$

2. Logarithms

$$b^x = y, x = \log_b y$$

Natural logarithms

$$b = e = 2.718282$$

$$e^x = y, x = \log_e y = \ln y$$

$$\log(ab) = \log a + \log b$$

$$\log(a/b) = \log a - \log b$$

$$\log(1/n) = -\log n$$

$$\log a^n = n \log a$$

$$\log 1 = 0$$

$$\log_{10} x = 0.4343 \ln x$$

3. Determinants

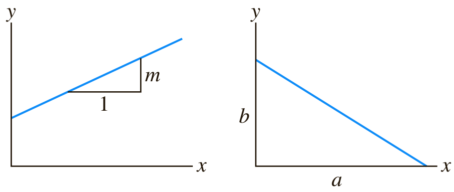
2nd order

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

3rd order

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{aligned} &+a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) = \\ &+a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 \\ &-a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2 \end{aligned}$$

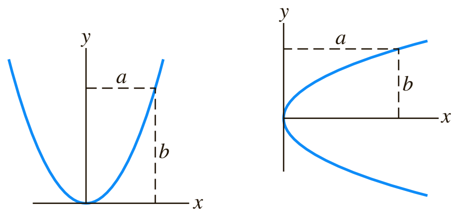
A.2 Analytic Geometry



$y = a + mx$ $\qquad$ $\dfrac{x}{a} + \dfrac{y}{b} = 1$

(*a*) $\qquad$ (*b*)

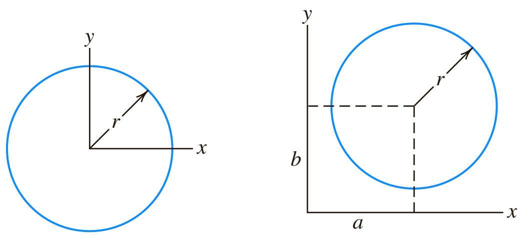
Figure A.2.1 Straight line.



$y = b\dfrac{x^2}{a^2}$ $\qquad$ $x = a\dfrac{y^2}{b^2}$

(*a*) $\qquad$ (*b*)

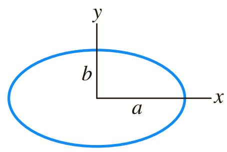
Figure A.2.3 Parabola.



$x^2 + y^2 = r^2$ $\qquad$ $(x - a)^2 + (y - b)^2 = r^2$

(*a*) $\qquad$ (*b*)

Figure A.2.2 Circle.



$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Figure A.2.4 Ellipse.

A.3 Trigonometry

1. Definitions

$\sin\theta = a/c \qquad \csc\theta = c/a$

$\cos\theta = b/c \qquad \sec\theta = c/b$

$\tan\theta = a/b \qquad \cot\theta = b/a$

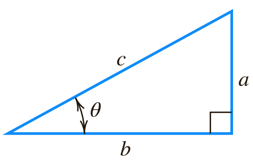
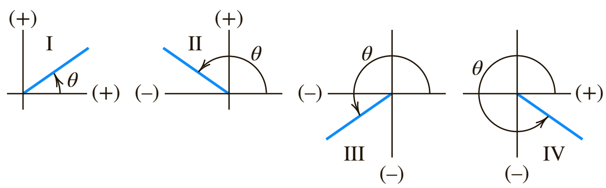


Figure A.3.1

2. Signs in the four quadrants



	I	II	III	IV
$\sin\theta$	+	+	–	–
$\cos\theta$	+	–	–	+
$\tan\theta$	+	–	+	–
$\csc\theta$	+	+	–	–
$\sec\theta$	+	–	–	+
$\cot\theta$	+	–	+	–

Figure A.3.2

3. Miscellaneous relations

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$
$$\sin\frac{\theta}{2} = \sqrt{\tfrac{1}{2}(1 - \cos\theta)}$$
$$\cos\frac{\theta}{2} = \sqrt{\tfrac{1}{2}(1 + \cos\theta)}$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin(a \pm b) = \sin a\cos b \pm \cos a\sin b$$
$$\cos(a \pm b) = \cos a\cos b \mp \sin a\sin b$$

4. Law of sines

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

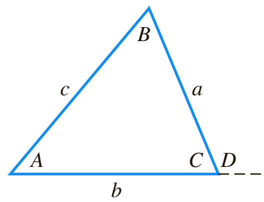


Figure A.3.3

5. Law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$c^2 = a^2 + b^2 + 2ab\cos D$$

A.4 The Cross Product (Vector Product)

The cross product or vector product of two vectors (in the case at hand, $\boldsymbol{r}$ and $\boldsymbol{F}$, in that order) is, by definition, a third vector that is perpendicular to the plane defined by the two vectors. Call this third vector $\boldsymbol{M}$ (for moment). We can write the cross product in terms of the $\boldsymbol{r}$ (the position vector) and $\boldsymbol{F}$ (the force vector) in terms of their respective components as

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = (r_x\boldsymbol{i} + r_y\boldsymbol{j} + r_z\boldsymbol{k}) \times (F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k}) \qquad (3.7\text{B})$$

By applying the distributive and associative laws of vector multiplication to (3.7B) and noting that

$$\begin{array}{lll} \boldsymbol{i}\times\boldsymbol{i} = 0 & \boldsymbol{j}\times\boldsymbol{i} = -\boldsymbol{k} & \boldsymbol{k}\times\boldsymbol{i} = \boldsymbol{j} \\ \boldsymbol{i}\times\boldsymbol{j} = \boldsymbol{k} & \boldsymbol{j}\times\boldsymbol{j} = 0 & \boldsymbol{k}\times\boldsymbol{j} = -\boldsymbol{i} \\ \boldsymbol{i}\times\boldsymbol{k} = -\boldsymbol{j} & \boldsymbol{j}\times\boldsymbol{k} = \boldsymbol{i} & \boldsymbol{k}\times\boldsymbol{k} = 0 \end{array} \qquad (\text{A}.4a)$$

we can simplify (3.7B) as

$$\boxed{\boldsymbol{M} = \boldsymbol{r}\times\boldsymbol{F} = \underbrace{(+r_yF_z - r_zF_y)}_{M_x}\boldsymbol{i} + \underbrace{(+r_zF_x - r_xF_z)}_{M_y}\boldsymbol{j} + \underbrace{(+r_xF_y - r_yF_x)}_{M_z}\boldsymbol{k}} \qquad (3.8)$$

The cross product of the position vector and the force vector, as represented in (3.8), can be written in matrix form as:

$$\boxed{\boldsymbol{M} = \boldsymbol{r}\times\boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}} \qquad (3.12)$$

The form in (3.12) specifies that the **determinant** of the matrix be taken. This determinant consisting of three rows and three columns may be evaluated by repeating the first and second columns and forming

products along each diagonal line. The sum of the products obtained along the colored lines is then subtracted from the sum of the products obtained along the black lines; this is identical to the expression on the righthand side of (3.8).

$$\begin{array}{ccccc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ r_x & r_y & r_z & r_x & r_y \\ F_x & F_y & F_z & F_x & F_y \end{array} \tag{A.4b}$$

PHYSICAL QUANTITIES

B.1 Physical Properties

	Density kg/m³	Specific weight lb/ft³		Density kg/m³	Specific weight lb/ft³
Air (20°C, 1 atm)	1.2062	0.08	Mercury	13570	847.13
Aluminum	2690	167.93	Nickel	8800	549.36
Bone	500	31.21	Nylon	1150	71.79
Brick	2000	124.85	Oil (av.)	900	56.18
Concrete (av.)	2400	149.82	Oxygen (0°C, 1 atm)	1.43	0.09
Copper	8910	556.22	Platinum	21500	1342.18
Cork	250	15.61	Polyethylene	1000	62.43
Diamond	3300	206.01	Silver	10500	655.48
Earth (dry, av.)	1280	79.91	Steel	7830	488.80
(wet, av.)	1760	109.87	Styrofoam	100	6.24
Glass	2590	161.69	Titanium	3080	192.28
Gold	19300	1204.84	Tungsten	19300	1204.84
Helium (0°C, 1 atm)	0.178	0.01	Uranium	18700	1167.38
Hydrogen (0°C, 1 atm)	0.09	0.01	Water (fresh)	1000	62.43
Ice	900	56.18	(seawater)	1030	64.30
Iron (cast)	7210	450.10	Wood (Balsa wood)	120	7.49
Lead	11370	709.79	(hard oak)	800	49.94
Mammals (most)	1000	62.43	(soft pine)	480	29.96

B.2 Solar System Constants

BODY	MEAN DISTANCE TO SUN km (mi)	PERIOD OR ORBIT solar days	MEAN DIAMETER km (mi)	MASS RELATIVE TO EARTH
Sun	—	—	1 392 000 (865 000)	333 000
Moon	384 398* (238 854)*	27.32	3 476 (2 160)	0.0123
Mercury	57.3×10^6 (35.6×10^6)	87.97	5 000 (3 100)	0.054
Venus	108×10^6 (67.2×10^6)	224.70	12 400 (7 700)	0.815
Earth	149.6×10^6 (92.96×10^6)	365.26	12 742† (7 918)†	1.000
Mars	227.9×10^6 (141.6×10^6)	686.98	6 788 (4 218)	0.107

*Mass of earth is 5.976×10^{24} kg (4.05×10^{23} lb · s^2/ft).

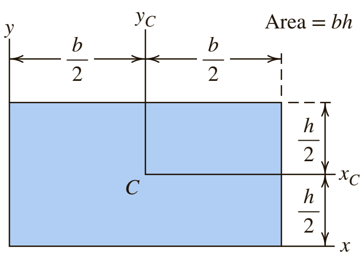
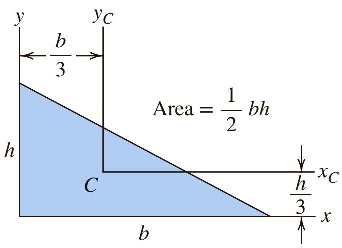
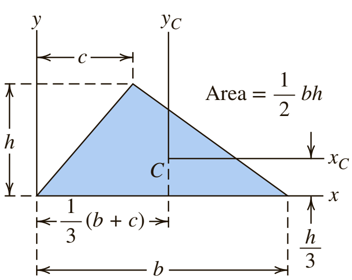
B.3 Conversion Factors from U.S. Customary Units

PHYSICAL QUANTITY	U.S. CUSTOMARY UNIT	= SI EQUIVALENT
	BASIC UNITS	
Length	1 foot (ft)	$= 3.048(10^{-1})$ meter (m)
	1 inch (in.)	$= 2.54(10^{-2})$ meter (m)
	1 mile (U.S. statute)	$= 1.6093(10^{3})$ meter (m)
Mass	1 slug (lb $\cdot$ s^3/ft)	$= 1.4594(10)$ kilogram (kg)
	1 pound mass (lbm)	$= 4.5359(10^{-1})$ kilogram (kg)
	DERIVED UNITS	
Acceleration	1 foot/second^2 (ft/s^2)	$= 3.048(10^{-1})$ meter/second^2 (m/s^2)
	1 inch/second^2 (in./s^2)	$= 2.54(10^{-2})$ meter/second^2 (m/s^2)
Area	1 foot^2 (ft^2)	$= 9.2903(10^{-2})$ meter^2 (m^2)
	1 inch^2 (in.^{2})	$= 6.4516(10^{-4})$ meter^2 (m^2)
Density	1 slug/foot^3 (lb $\cdot$ s^2/ft^4)	$= 5.1537(10^{2})$ kilogram/meter^3 (kg/m^3)
	1 pound mass/foot^3 (lbm/ft^3)	$= 1.6018(10)$ kilogram/meter^3 (kg/m^3)
Energy and Work	(1 joule = 1 meter-newton)	
	1 foot-pound (ft $\cdot$ lb)	$= 1.3558$ joules (J)
	1 kilowatt-hour (kW $\cdot$ hr)	$= 3.60(10^{6})$ joules (J)
	1 British thermal unit (Btu)	$= 1.0551(10^{3})$ joules (J)
Force	(1 newton = 1 kilogram-meter/second^2)	
	1 pound (lb)	$= 4.4482$ newtons (N)
	1 kip (1000 lb)	$= 4.4482(10^{3})$ newtons (N)
Power	(1 watt = 1 joule/second)	
	1 foot-pound/second (ft $\cdot$ lb/s)	$= 1.3558$ watt (W)
	1 horsepower (hp)	$= 7.4570(10^{2})$ watt (W)
Pressure and Stress	(1 pascal = 1 newton/meter^2)	
	1 pound/foot^2 (lb/ft^2)	$= 4.7880(10)$ pascal (Pa)
	1 pound/inch^2 (lb/in.^{2})	$= 6.8948(10^{3})$ pascal (Pa)
	1 atmosphere (standard, 14.7 lb/in.^{2})	$= 1.0133(10^{5})$ pascal (Pa)
Speed	1 foot/second (ft/s)	$= 3.048(10^{-1})$ meter/second (m/s)
	1 mile/hr	$= 4.4704(10^{-1})$ meter/second (m/s)
	1 mile/hr	$= 1.6093$ kilometer/hour (km/hr)
Volume	1 Foot^3 (ft^3)	$= 2.8317(10^{-2})$ meter^3 (m^3)
	1 inch^3 (in.^{3})	$= 1.6387(10^{-5})$ meter^3 (m^3)
	1 gallon (U.S. liquid)	$= 3.7854(10^{-3})$ meter^3 (m^3)

APPENDIX C

PROPERTIES OF AREAS AND VOLUMES

C.1 Areas, Centroids, and Area Moments of Inertia

Shape	Area Moment of Inertia
 **Figure C.1.1** Rectangle.	$I_{x_C} = \dfrac{bh^3}{12}$ $I_{y_C} = \dfrac{b^3h}{12}$ $I_x = \dfrac{bh^3}{3}$ $I_y = \dfrac{b^3h}{3}$
 **Figure C.1.2** Right triangle.	$I_{x_C} = \dfrac{bh^3}{36}$ $I_{y_C} = \dfrac{b^3h}{36}$ $I_x = \dfrac{bh^3}{12}$ $I_y = \dfrac{b^3h}{12}$
 **Figure C.1.3** Scalene triangle.	$I_{x_C} = \dfrac{bh^3}{36}$ $I_{y_C} = \dfrac{bh}{36}(b^2 + c^2 - bc)$ $I_x = \dfrac{bh^3}{12}$

Shape	Area Moment of Inertia

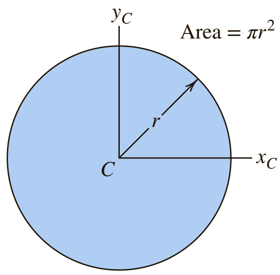


Figure C.1.4 Circle.

$$I_{x_C} = I_{y_C} = \frac{1}{4}\pi r^4$$

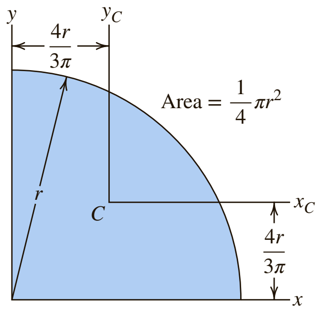


Figure C.1.5 Quarter circle.

$$I_{x_C} = I_{y_C} = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) r^4$$

$$I_x = I_y = \frac{\pi r^4}{16}$$

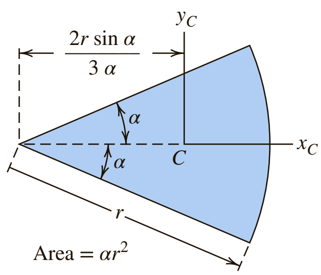


Figure C.1.6 Circular sector.

$$I_{x_C} = \frac{r^4}{4}\left(\alpha - \frac{\sin 2\alpha}{2}\right)$$

$$I_{y_C} = \frac{r^4}{4}\left(\alpha + \frac{\sin 2\alpha}{2}\right)$$

Shape	Area Moment of Inertia

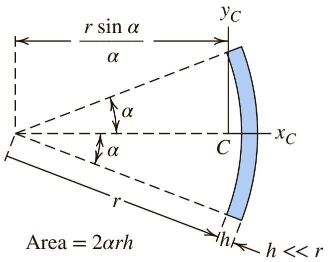


Figure C.1.7 Circular arc.

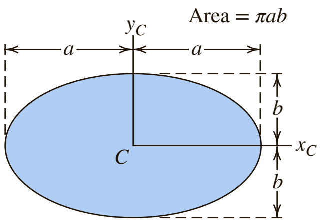


Figure C.1.8 Ellipse.

$$I_{x_C} = \frac{\pi}{4} ab^3$$

$$I_{y_C} = \frac{\pi}{4} a^3 b$$

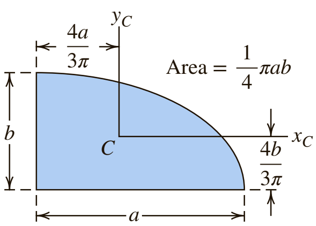


Figure C.1.9 Quarter ellipse.

$$I_{x_C} = \left(\frac{9\pi^2 - 64}{144\pi}\right) ab^3$$

$$I_{y_C} = \left(\frac{9\pi^2 - 64}{144\pi}\right) a^3 b$$

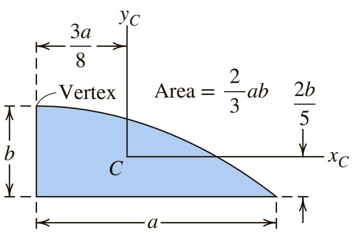
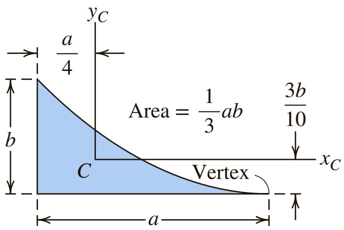


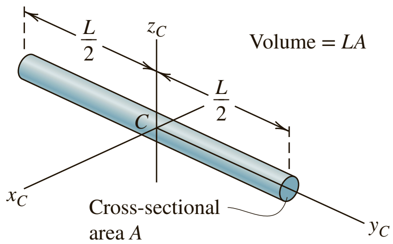
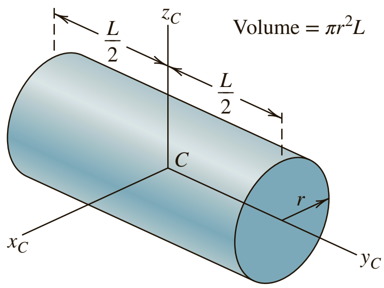
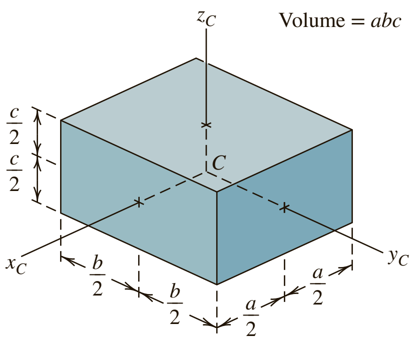
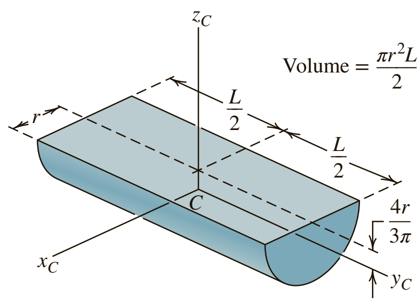
Figure C.1.10 Parabolic section.

$$I_{x_C} = \frac{8}{175} ab^3$$

$$I_{y_C} = \frac{19}{480} a^3 b$$

Shape	Area Moment of Inertia
 **Figure C.1.11** Parabolic spandrel.	$I_{x_C} = \frac{19}{1050}ab^3$ $I_{y_C} = \frac{1}{80}a^3b$

C.2 Volumes and Centroids

Shape	Shape
 **Figure C.2.1** Uniform slender rod.	 **Figure C.2.3** Cylinder.
 **Figure C.2.2** Rectangular parallelepiped.	 **Figure C.2.4** Semicylinder.

Shape

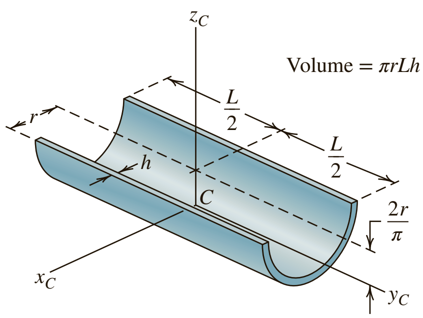


Figure C.2.5 Semicylindrical shell.

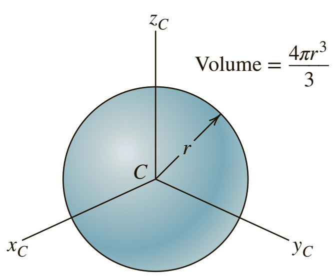


Figure C.2.6 Sphere.

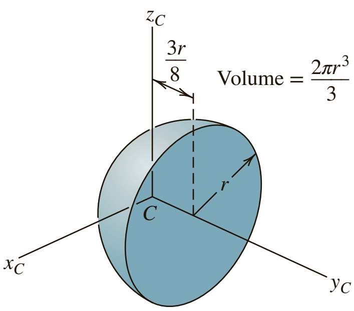


Figure C.2.7 Hemisphere.

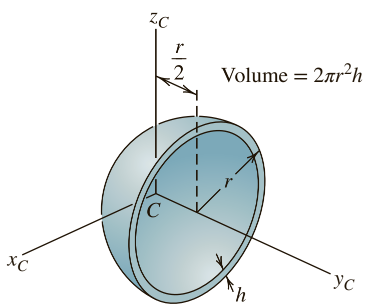


Figure C.2.8 Hemispherical shell.

Shape

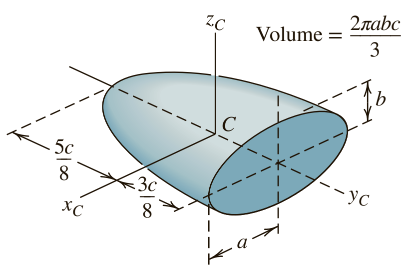


Figure C.2.9 Semiellipsoid.

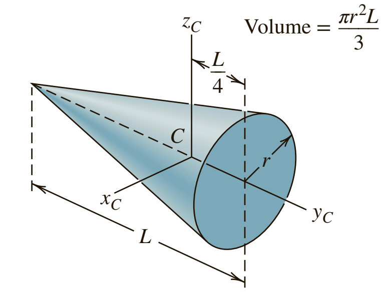


Figure C.2.10 Cone.

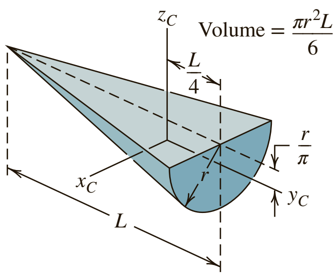


Figure C.2.11 Semicone.

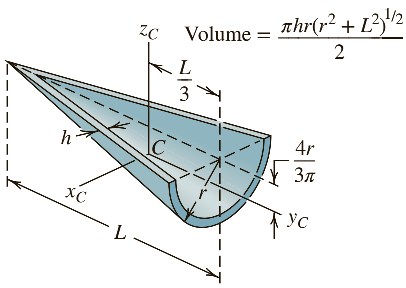


Figure C.2.12 Semiconical shell.

Shape	Shape

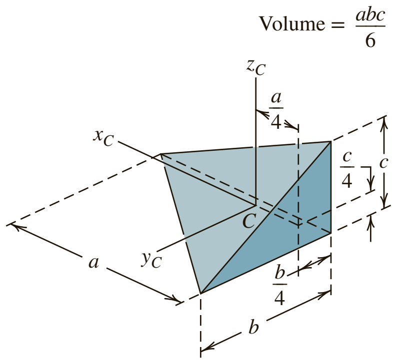


Figure C.2.13 Orthogonal tetrahedron.

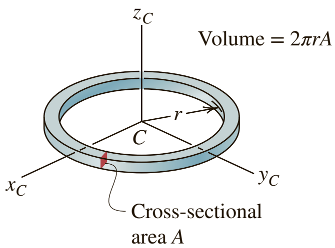


Figure C.2.17 Thin circular ring.

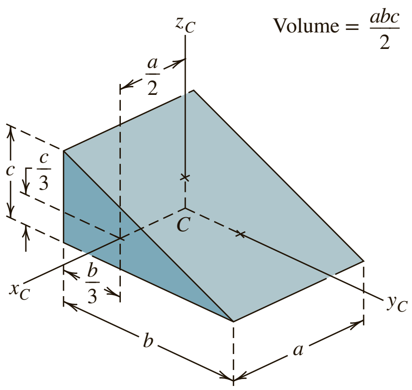


Figure C.2.14 Right triangular prism.

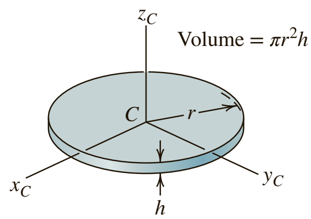


Figure C.2.18 Circular plate.

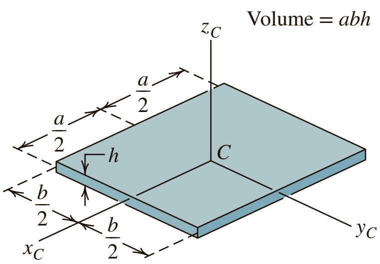


Figure C.2.15 Rectangular plate.

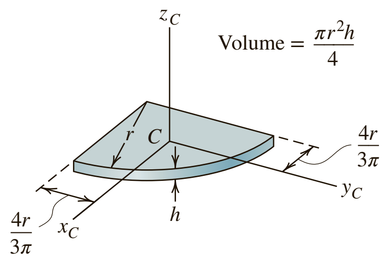


Figure C.2.19 Quarter circular plate.

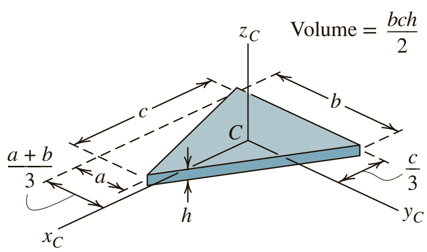


Figure C.2.16 Triangular plate.

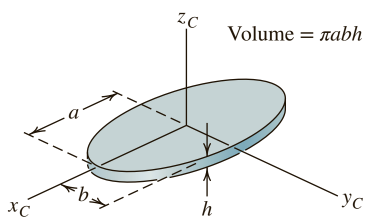


Figure C.2.20 Elliptical plate.

CASE STUDY: THE BICYCLE

In this case we illustrate how the basic concepts of statics help us understand how a bicycle moves. We've chosen the bicycle because most people have had firsthand experience with it and because its "machinery" is so visible. Engineers concerned with modifying parts of the bicycle base their work on the ideas presented in this chapter—an overview of how a bicycle works and a description of how to quantify the forces acting on one. These engineers might be motivated by a need to eliminate a part failure, to reduce manufacturing costs, or to improve performance.

All of the major statics concepts introduced in this case are covered in greater detail in the main chapters. The two main purposes of this case are to give you an overview of these concepts in the context of an engineered artifact you are likely to be familiar with and to illustrate the utility of these concepts in describing the artifact in a quantitative manner. In other words, we want to whet your appetite for the rest of the book!

Before reading further, we urge you to

1. draw a bicycle from memory, labeling all of the components you know, then
2. study a bicycle, taking particular note of the relationships between pedals, chain, and wheels.

Completing these two tasks is likely to make our discussion of how a bicycle works all the more real to you. You may even want to see the 1979 Academy Award-winning movie *Breaking Away* (directed by Peter Yates) to get into a bicycling frame of mind.

OBJECTIVES

On completion of this case, you will be able to:

- **Describe the types of forces that act on a moving bicycle**
- **Isolate a cyclist and bicycle from the rest of the world and identify the external forces acting on them**
- **Use Newton's first law to answer questions about the performance of a cyclist and bicycle**
- **Appreciate the usefulness of a procedure known as static analysis**

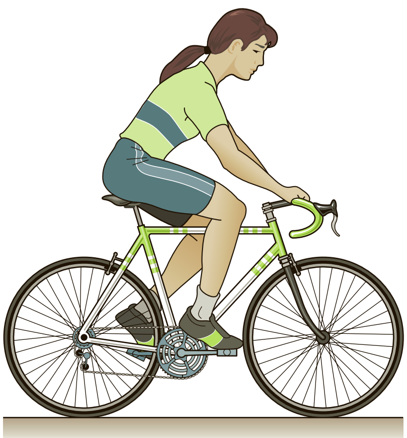


Figure D.1.1 A cyclist and bicycle.

D.1 THE FORCES OF BICYCLING

All of us experience forces—all the pushes, pulls, tugs, and shoves of everyday life. Some of these forces are welcome—for example, the updown-sideways jostles of a roller coaster ride—and some are unwelcome—like the thud and bump when two cars roll into each other. Forces operate can openers, automobiles, ski lifts, and airplanes and are what buildings, bridges, and ships must stand up to.

Forces are also what operate a bicycle. So what forces are involved? To answer this question, take a minute to list as many as you can of the forces acting in **Figure D.1.1** for a bicycle moving along level ground at a constant velocity.

Your list probably includes most (or all, if you're very good) of the following:

Weight of bicycle
Weight of cyclist
Normal force between foot and pedal
Normal force between cyclist and seat
Normal force between hands and handlebars
Normal force between front tire and ground
Normal force between rear tire and ground
Friction force between cyclist and seat
Friction force between hands and handlebars
Friction force between rear tire and ground[1]
Tension in brake cables
Tension in shifter cables
Tension in chain
Compression in seat tube
Drag force exerted by air on cyclist
Drag force exerted by air on bicycle

Note that this list is organized by categories of forces: **weight**, **normal force**, **friction force**, **tension**, **compression**, and **drag force**. A lot more is said about these categories in Chapter 2.

Which forces from the list are relevant to our analysis depends on which portion of the system we are interested in. We draw a boundary around the portion of interest and consider only those forces that

[1]If the bicycle was traveling up or down a slope or was accelerating, there would also be a friction force between the front tire and the ground.

Table D.1 Forces Involved in Cycling

All forces	A. External forces acting on cyclist	B. External forces acting on bicycle	C. External forces acting on cyclist–bicycle
Weight of bicycle		Weight of bicycle	Weight of bicycle
Weight of cyclist	Weight of cyclist		Weight of cyclist
Normal force between foot and pedal	Normal force between foot and pedal	Normal force between foot and pedal	
Normal force between cyclist and seat	Normal force between cyclist and seat	Normal force between cyclist and seat	
Normal force between hands and handlebars	Normal force between hands and handlebars	Normal force between hands and handlebars	
Normal force between front tire and ground		Normal force between front tire and ground	Normal force between front tire and ground
Normal force between rear tire and ground		Normal force between rear tire and ground	Normal force between rear tire and ground
Friction force between cyclist and seat	Friction force between cyclist and seat	Friction force between cyclist and seat	
Friction force between hands and handlebars	Friction force between hands and handlebars	Friction force between hands and handlebars	
Friction force between rear tire and ground		Friction force between rear tire and ground	Friction force between rear tire and ground
Tension in brake cables			
Tension in shifter cables			
Tension in chain			
Compression in seat tube			
Drag force exerted by air on cyclist	Drag force exerted by air on cyclist		Drag force exerted by air on cyclist
Drag force exerted by air on bicycle		Drag force exerted by air on bicycle	Drag force exerted by air on bicycle

act either on the boundary or across the boundary. We call these the **external forces**. In analyzing a static system, only external forces are relevant.

To illustrate the process of identifying external forces, let's suppose we are interested only in the cyclist. The external forces acting on her are listed in column A of **Table D.1** and shown in the free-body diagram of **Figure D.1.2**. All the forces not included in column A are not acting on the boundary surrounding the cyclist and therefore do not act on her. Because they are not acting on her, they are not shown in the free-body diagram.

Alternatively, say we are interested in the bicycle. The external forces are listed in column B of **Table D.1** and presented in a free-body diagram in **Figure D.1.3*a***. Two of the forces not included in column B are *beyond* the boundary surrounding the bicycle (weight of cyclist and drag force exerted by air on cyclist). The others not included are completely *within* the boundary and are therefore called **internal forces**. Internal

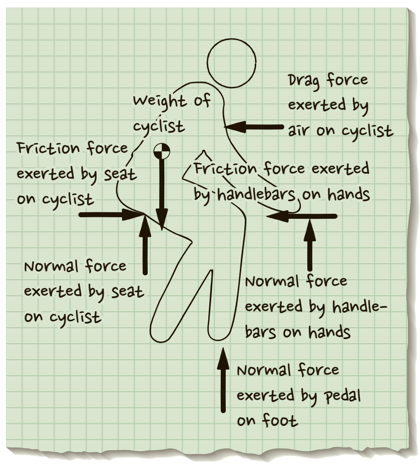


Figure D.1.2 Free-body diagram of cyclist.

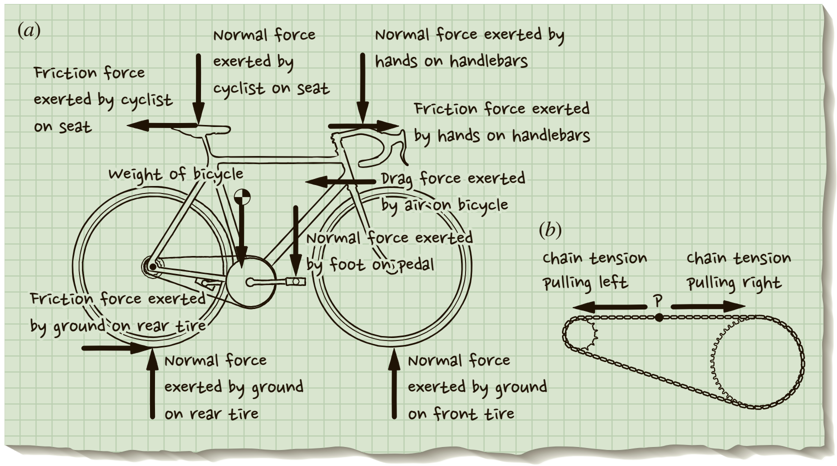


Figure D.1.3 (*a*) Free-body diagram for bicycle; (*b*) internal chain forces at *P*.

forces come in equal and opposite force pairs (as described by Newton's third law), and therefore cancel each another. For example, at point *P* in the chain in **Figure D.1.3*b***, the tension of the chain pulling to the right at *P* is canceled by the tension of the chain pulling to the left, as illustrated in the figure.

Finally, suppose we are interested in the combination of cyclist and bicycle. The external forces for this system are listed in column C of **Table D.1** and presented in a free-body diagram in **Figure D.1.4*a***. Any forces shown in **Figure D.1.2** or **Figure D.1.3** but not here are completely *within* the boundary and so are internal forces. For example, the force between the cyclist and the bicycle seat consists of a downward normal force exerted by the cyclist on the bicycle seat canceled by an equal and opposite upward normal force exerted by the seat on the cyclist (**Figure D.1.4*b***).

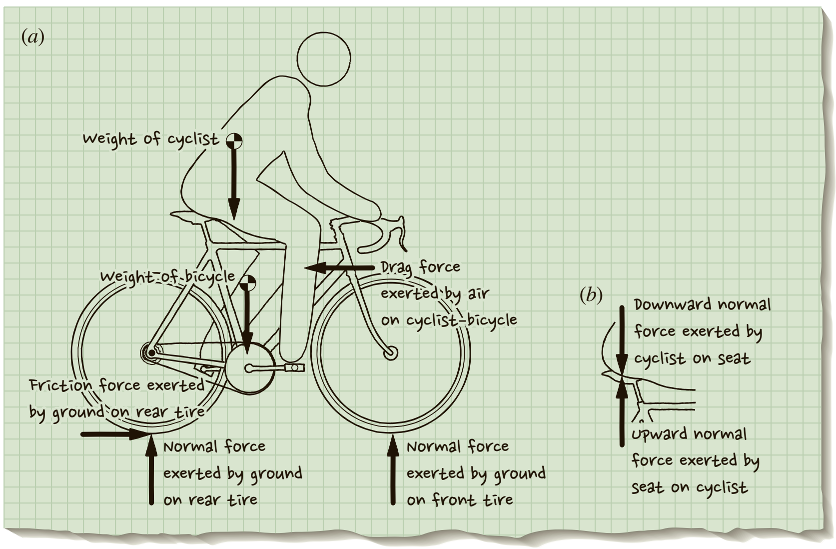


Figure D.1.4 (*a*) Free-body diagram for cyclist–bicycle combination; (*b*) internal seat–cyclist forces.

Summary

In this section the key ideas are:

1. The forces involved in bicycling are weight, normal, friction, tension, compression, and drag forces.
2. We represent an object and the forces acting on it in a free-body diagram. The forces in the diagram are those that act on or across a boundary surrounding the object; these are the external forces.
3. Forces that are within the boundary come in pairs, and we call them internal forces. Each force pair consists of equal and opposite forces that cancel each other. Internal forces are not shown in a free-body diagram.
4. Forces that are internal to one object may be external to parts of the object—for example, the normal force between foot and pedal is an external force when we consider either the cyclist alone or the bicycle alone, but is an internal force when we consider the cyclist–bicycle combination.

D.2 WHAT IS THE MAXIMUM SPEED?

Now that we have introduced the major forces involved in bicycling, we consider how forces affect bicycle performance. More specifically, suppose that an average engineering student named Merrill is riding the bicycle shown in **Figure D.2.1** in a race. *How fast could he sprint toward the finish line?*

One way to answer this question is to have Merrill perform the task and for us to measure his speed. Several trials would be necessary, and

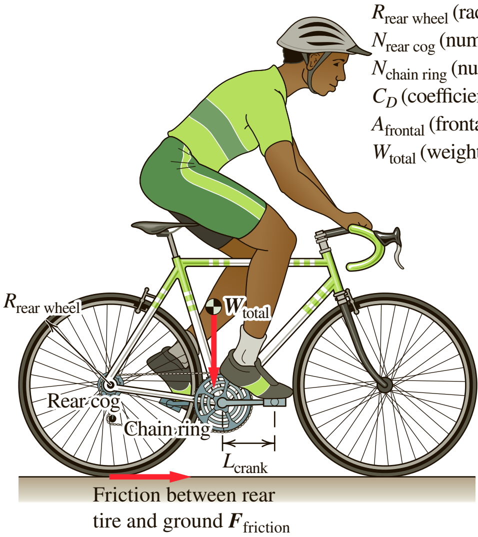


Figure D.2.1 Cyclist–bicycle combination with assumptions.

we would need to factor in how representative his performance was on that particular day. Another approach is for us to come up with an estimate based on the application of Newton's first law; in other words, we could perform static analysis. We now step through this latter approach, making the assumptions listed in **Figure D.2.1**.

Our analysis to find out how fast Merrill can sprint toward the finish line consists of addressing three interrelated subquestions:

1. What is the maximum force Merrill can apply to each pedal?
2. How is this force related to the friction force between the rear tire and the ground?
3. How does the friction force relate to the drag force on the bicycle?

In answering each subquestion, we will identify external forces, draw a free-body diagram, and apply Newton's first law.

1. What Is the Maximum Force Merrill Can Apply to Each Pedal? An initial estimate might be Merrill's weight based on him not sitting on the bicycle but rather standing on the pedals with all of his weight on one foot. In order to come up with a better estimate, consider the setup in **Figure D.2.2*a***. A student stands on a bathroom scale; if she is on earth, the scale should read her weight on earth. Now she pushes up on the lip of a cabinet next to her (**Figure D.2.2*b***)—does the reading on the scale go up, go down, or remain the same? To answer this question, consider the free-body diagram of the student, shown in **Figure D.2.2*c***. Since she pushes up on the lip, we show how the lip pushes down on her (therefore $\boldsymbol{F}_{\substack{\text{scale pushing}\\ \text{on student}}}$ is in the downward direction). Newton's first law tells us that because the student is not accelerating vertically, the sum of the forces acting in the vertical direction is zero. This means that

$$\begin{aligned} \boldsymbol{F}_{\substack{\text{scale pushing}\\ \text{on student}}} - \boldsymbol{W}_{\text{student}} - \boldsymbol{F}_{\substack{\text{cabinet pushing}\\ \text{on student}}} &= 0 \\ \boldsymbol{F}_{\substack{\text{scale pushing}\\ \text{on student}}} &= \boldsymbol{W}_{\text{student}} + \boldsymbol{F}_{\substack{\text{cabinet pushing}\\ \text{on student}}} \end{aligned} \tag{D.1}$$

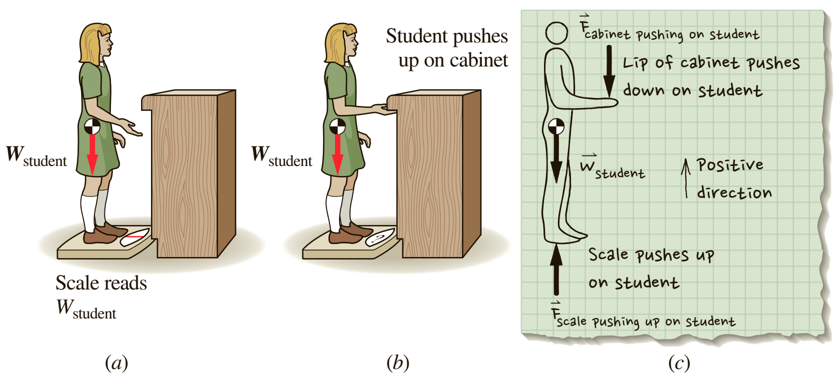


Figure D.2.2 (*a*) Student standing on bathroom scale; (*b*) student pushes up a cabinet; (*c*) free-body diagram of student.

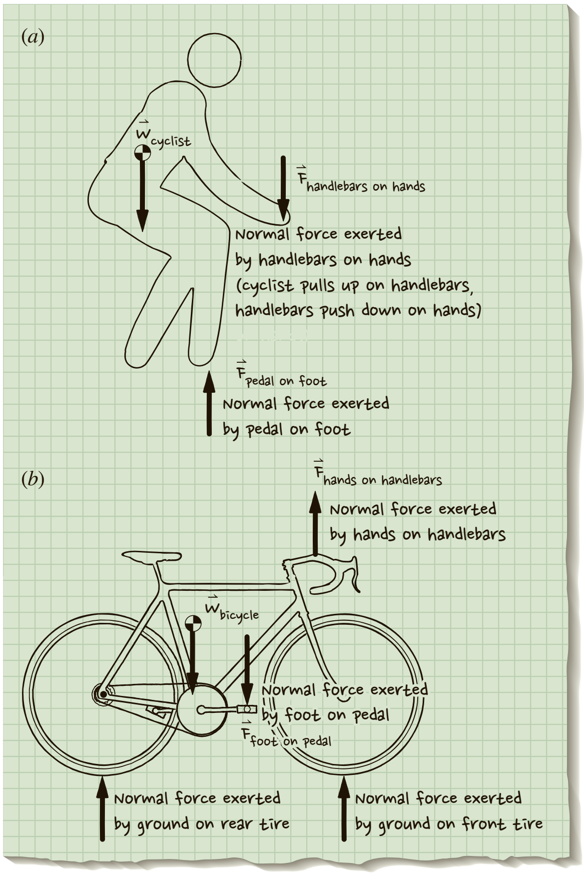


Figure D.2.3 (*a*) Free-body diagram of Merrill; (*b*) freebody diagram of Merrill's bicycle.

Therefore, the reading on the scale, which is equal to $\boldsymbol{F}_{\text{scale pushing on student}}$ increases as the student pushes up on the cabinet.

The connection between this bathroom scale example and Merrill's pedal force is that, if he is pulling up on the bicycle handlebars and not sitting on the bicycle seat, the magnitude of the force he applies to the pedal is greater than his weight. To see how this is so, look at Figure D.2.3*a*. As with the student standing on the scale, the sum of the forces acting on Merrill in the vertical direction must be zero because he's not accelerating in that direction. Thus

$$\boldsymbol{F}_{\text{pedal on foot}} = \boldsymbol{W}_{\text{cyclist}} + \boldsymbol{F}_{\text{handlebars on hands}}$$

Because $\boldsymbol{F}_{\text{foot on pedal}}$, the force Merrill exerts on the pedal, is the normal force opposing the force $\boldsymbol{F}_{\text{pedal on foot}}$, their magnitudes are the same, meaning that Merrill's force $\boldsymbol{F}_{\text{foot on pedal}}$ on the pedal is greater than his weight!

Answer to Question 1

We estimate that Merrill is able to apply a force to the pedal that is greater than his weight. What makes this additional force possible is the force he can exert on the handlebars, and a ballpark estimate of the magnitude of that force is how much weight he can lift. Since an average engineering student weighs 750 N and can repetitively lift 400 N,* we estimate that Merrill is able to apply at most 1150 N (= 750 N + 400 N) to the pedal with his foot. If Merrill does not realize that by pulling up on the handlebars he can increase the force he can apply to the pedal, our estimate of how much force he can apply to the pedal is simply his weight of 750 N. Therefore our complete estimate of the force Merrill is able to apply to the pedal is between 750 N and 1150 N (assuming that Merrill is in a semi-erect position, not sitting on the seat). In other words, $\|\boldsymbol{F}_{\text{foot on pedal}}\|$ in **Figure D.2.3*b*** is estimated to be between 750 and 1150 N. Its exact size will depend on how skilled Merrill is in combining the action of legs and arms in cycling.

*Based on data from 70 engineering students, fall 2001.

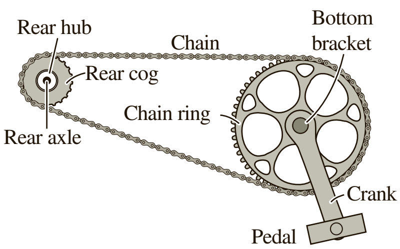


Figure D.2.4 Bicycle powertrain and its components.

2. How Is This Force Related to the Friction Force Between the Rear Tire and the Ground? To answer this question, we consider how a bicycle converts the downward push of a bicyclist's foot on the pedal to a forward shove that moves the bicycle (and cyclist) forward. The machinery that accomplishes this is called the powertrain. The important components of the powertrain for a one-speed bicycle, shown in **Figure D.2.4**, are pedals, cranks, bottom bracket, chain ring, chain, rear cog, rear hub, and rear axle.

To see how pushing down on either pedal is related to the horizontal forward push exerted on the rear wheel that makes the bicycle move forward, we trace the force from the foot through the various components of the powertrain. **Figure D.2.5*a*** shows $\boldsymbol{F}_{\text{foot on pedal}}$ applied where the cyclist's foot engages the pedal and $\boldsymbol{F}_{\text{friction, ground on tire}}$ applied at the rear wheel, which moves the bicycle to the right.

The pedal is connected to the rest of the bicycle by the crank, which offsets the pedal from the center of the bottom bracket. This offset results in the force $\boldsymbol{F}_{\text{foot on pedal}}$ causing the crank to rotate in the clockwise direction. This tendency to rotate is given a special name—**moment**—and is illustrated in **Figure D.2.5*b***.

As the crank rotates, it turns the chain ring. Teeth around the circumference of the chain ring fit into and pull the chain (**Figure D.2.5*c***). The chain, in turn, pulls on another toothed disk, called the rear cog (**Figure D.2.5*d***). Because the chain pulling force is offset from the center of rotation of the rear wheel, that force causes a clockwise moment about the rear hub. The rear cog is part of the rear hub, and so the whole rear wheel tends to turn in the clockwise direction. As this happens, the tire pushes backward on the ground (to the left in our example). In response, the ground pushes forward on the tire, as indicated by the force $\boldsymbol{F}_{\text{friction, ground on tire}}$ in **Figure D.2.5*d***. This is how the vertical input force $\boldsymbol{F}_{\text{foot on pedal}}$ gets converted into a horizontal push force $F_{\text{friction, ground on tire}}$.

By drawing free-body diagrams for the powertrain components and then evaluating the forces in these diagrams, we can convert the description of the powertrain given above into a mathematical relationship between $\|\boldsymbol{F}_{\text{foot on pedal}}\|$ and $\|\boldsymbol{F}_{\text{friction, ground on tire}}\|$ (you can explore

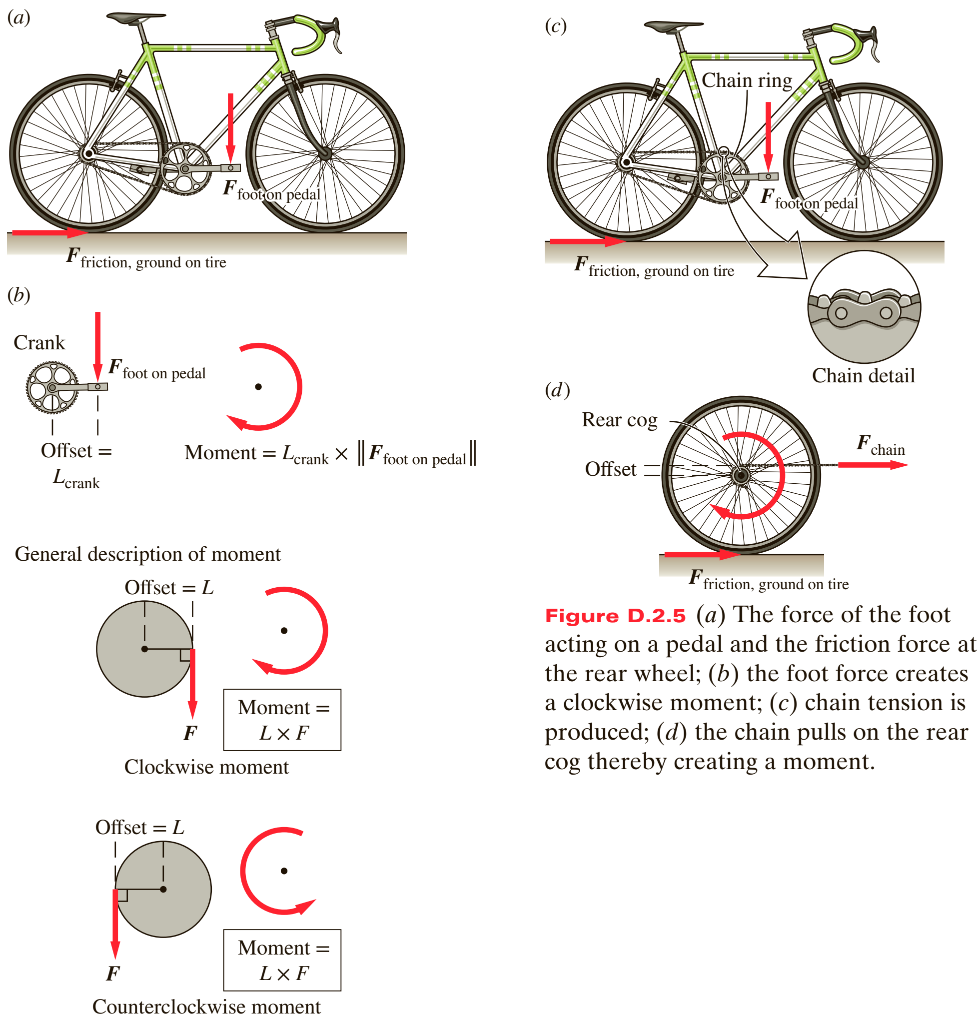


Figure D.2.5 (*a*) The force of the foot acting on a pedal and the friction force at the rear wheel; (*b*) the foot force creates a clockwise moment; (*c*) chain tension is produced; (*d*) the chain pulls on the rear cog thereby creating a moment.

this further in the exercises in section 9.2 on defining and analyzing machines.):

$$\left\|\boldsymbol{F}_{\text{friction, ground on tire}}\right\| = \left\|\boldsymbol{F}_{\text{foot on pedal}}\right\| \left(\frac{L_{\text{crank}}}{R_{\text{wheel}}}\right)\left(\frac{N_{\text{rear cog}}}{N_{\text{chain ring}}}\right) \tag{D.2}$$

where

$\left\|\boldsymbol{F}_{\text{friction, ground on tire}}\right\|$ is the magnitude of the friction force where the ground pushes on the rear tire

$\left\|\boldsymbol{F}_{\text{foot on pedal}}\right\|$ is the magnitude of the cyclist's foot pressing on the pedal

L_{crank} is the length of the crank

R_{wheel} is the radius of the rear wheel

$N_{\text{rear cog}}$ is the number of teeth around the circumference of the rear cog

$N_{\text{chain ring}}$ is the number of teeth around the circumference of the chain ring

Notice in (D.2) that $\left\|\boldsymbol{F}_{\text{foot on pedal}}\right\|$ is multiplied by a factor that is a product of sizes of bicycle components and therefore this factor is a function only of the geometry of the bicycle. Based on the information in **Figure D.2.1**, this factor is

$$\left(\frac{17.8\text{ cm}}{34.3\text{ cm}}\right)\left(\frac{14\text{ teeth}}{50\text{ teeth}}\right) = 0.145$$

for Merrill's bicycle; therefore, the magnitude of $\boldsymbol{F}_{\text{friction, ground on tire}}$ (the force that pushes the bicycle forward) is $0.145\left\|\boldsymbol{F}_{\text{foot on pedal}}\right\|$.

Answer to Question 2

A bicycle's powertrain, comprising pedals, cranks, toothed cogs, chain, and tire-wheel assembly, converts the force of a foot pushing down on a pedal. The conversion includes changing the direction from a downward force ($\boldsymbol{F}_{\text{foot on pedal}}$) to a forward force ($\boldsymbol{F}_{\text{friction, ground on tire}}$) and changing the magnitude of the force. In the case of Merrill and his bicycle, the magnitude of the force is reduced by a factor of 0.145: $\left\|\boldsymbol{F}_{\text{friction, ground on tire}}\right\| = 0.145\left\|F_{\text{foot on pedal}}\right\|$. We estimated in question 1 that $750\text{ N} < \left\|\boldsymbol{F}_{\text{foot on pedal}}\right\| < 1150\text{ N}$. This means that $109\text{ N} < \left\|\boldsymbol{F}_{\text{friction, ground on tire}}\right\| < 167\text{ N}$.

3. How Does the Friction Force Relate to the Drag Force on the Bicycle? Figure D.2.6 is a free-body diagram of the cyclist–bicycle combination. Newton's first law applied to this diagram tells us that because Merrill and his bicycle are moving at a constant velocity (an assumption we've made; see **Figure D.2.1**), there is no net force in the horizontal direction. This means that $\boldsymbol{F}_{\text{friction, ground on tire}}$,

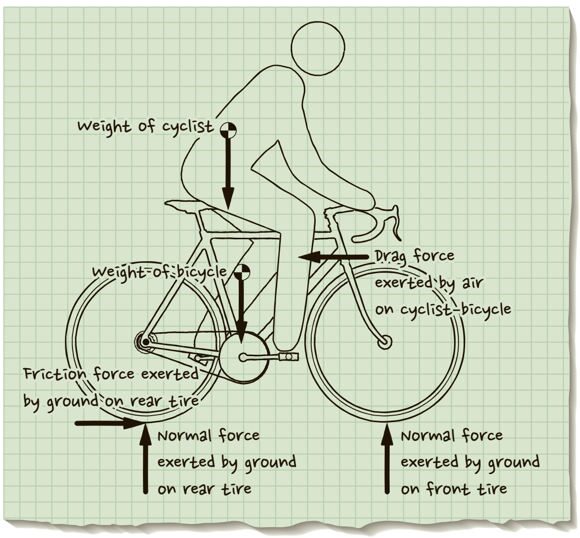


Figure D.2.6 Free-body diagram of Merrill and his bicycle.

which is pushing the bicycle forward, is canceled by the drag force $\boldsymbol{F}_{\text{drag}}$ on the bicycle:

$$\begin{aligned} \boldsymbol{F}_{\text{friction}} - \boldsymbol{F}_{\text{drag}} &= 0 \\ \boldsymbol{F}_{\text{friction}} &= \boldsymbol{F}_{\text{drag}} \end{aligned} \qquad \text{(D.3)}$$

Based on our answer to question 2, we estimate that this drag force is $109\ \text{N} < \|\boldsymbol{F}_{\text{drag}}\| < 167\ \text{N}$.

Our final step is to relate the drag force to the velocity of the air moving around the bicycle. For blunt objects moving at relatively low velocity near the ground (e.g., bicycles and cyclists), this relationship takes the form[2]

$$\|\boldsymbol{F}_{\text{drag}}\| = \left(\frac{C_d \rho A}{2}\right) V^2 \qquad \text{(D.4)}$$

where

C_d is the drag coefficient
ρ is the density of air (kg/m^3)
A is the frontal area of the cyclist–bicycle (m^2)
V is the velocity of air moving around the bicycle (m/s)

Answer to Question 3

Newton's first law was used to equate the horizontal forces acting on the bicycle–cyclist combination. The horizontal forces are friction force where the rear tire contacts the road and a drag force. In calculating the answer to question 2 we found that $109\ \text{N} < \|\boldsymbol{F}_{\text{friction, ground on tire}}\| < 167\ \text{N}$. Therefore, the drag force has this same range in magnitude. Finally, we modeled the drag force as a function of V^2, the air velocity squared in (D.4), to find the velocity of Merrill and his bicycle.

Our answer to the question of how fast Merrill could sprint toward the finish line is $20.1\ \text{m/s} < V < 24.9\ \text{m/s}$ (or $43\ \text{mph} < V < 54\ \text{mph}$). The answer is a range because Merrill's speed depends on how skilled he is in pulling up on the handlebars. In addition, the answer is dependent on the particular values of C_d, ρ, and A, and the assumptions listed in **Figure D.2.1**.

We should not just accept this answer, but should compare it to our expectations and experience. Do you think that 43 mph is a reasonable answer? What about 54 mph? Also, are there any additional factors that we should consider? We will come back to this point in the next section.

Figure D.2.7 shows a plot of (D.4) for Merrill's bicycle. With $109\ \text{N} < \|\boldsymbol{F}_{\text{drag}}\| < 167\ \text{N}$, we read off the plot that $20.1\ \text{m/s} < V < 24.9\ \text{m/s}$ (or $43\ \text{mph} < V < 54\ \text{mph}$). Since Merrill is bicycling on a windless day (one of our assumptions), the magnitude of the wind velocity is equal to the magnitude of Merrill's velocity.

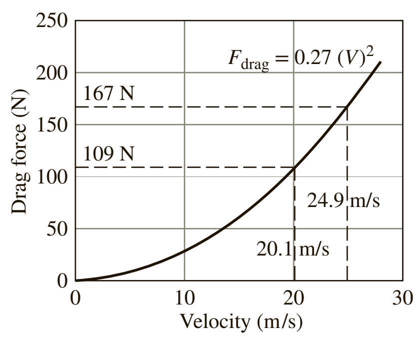


Figure D.2.7 Drag force versus velocity of air moving past bicycle.

D.3 ADDING MORE REALITY

Our discussion up to now has assumed a constant foot force always oriented at a right angle to the crank, as **Figures D.2.3*b*** and **D.2.5*a*** show. In real bicycling, the push on the pedal is not constant as the cyclist's

[2]See *Bicycling Science*, 2nd edition, R. Rowland Whitt and D. G. Wilson (Cambridge, Mass.: The MIT Press, 1990).

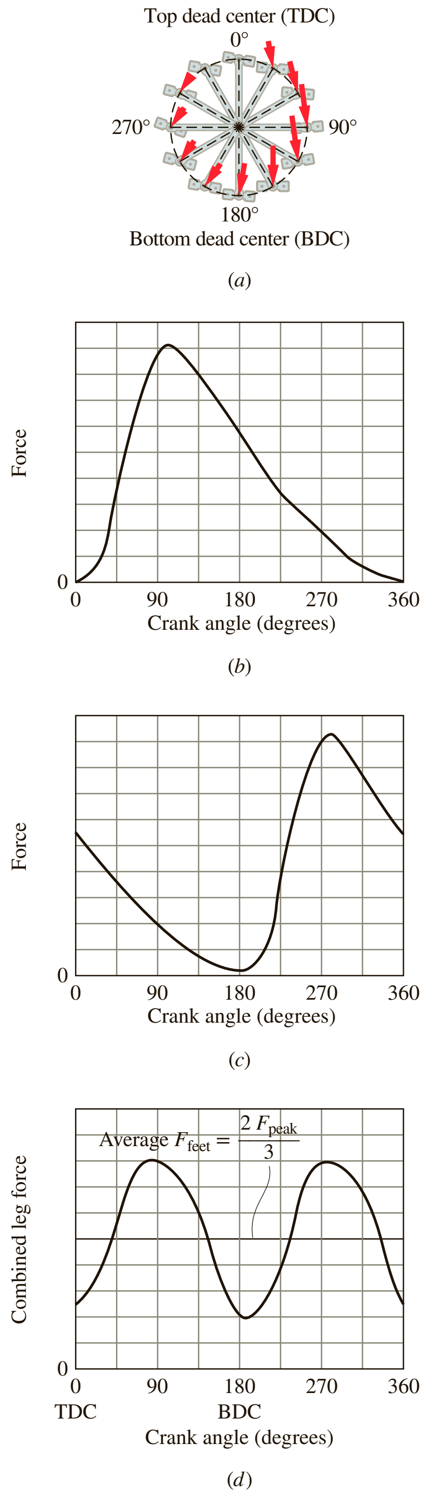


Figure D.3.1 (*a*) Force direction as right foot travels 360°; (*b*) right foot force; (*c*) left foot force; (*d*) total foot force (left and right) perpendicular to the crank.

foot moves 360°. **Figure D.3.1*a*** shows, for a cyclist's right foot, the force applied to the pedal as it travels around the 360° pedaling cycle, and **Figure D.3.1*b*** shows a graph of this force during the rotation. There are three things to note from these figures:

1. The force vector hardly ever has the vertical orientation we have assumed up to now in our discussion.
2. The applied force varies greatly around the 360° of travel, and we can examine this force by looking at the crank angle. This angle is defined as being 0° when the crank points straight up (with the pedal end of the crank being the "point" end) and increases as the pedal moves clockwise. At a crank angle of 90°, the applied force is maximum, while near top dead center (the top position), there is no applied force.
3. On the upstroke between 180° and 360°, the cyclist's input does not contribute to the input moment that propels the bicycle forward.

The pedaling force exerted by the cyclist's left foot is shown in **Figure D.3.1*c***. Superimposing the action of the right foot and the left foot, and factoring in that only the component of force perpendicular to the crank creates a moment about the bottom bracket, we get the effective input force profile shown in **Figure D.3.1*d***. Also shown in the figure is the average value of the input force, which is about of $\frac{2}{3}$ the peak force. This force level represents the average force applied over a 360° leg cycle.

We could now redo our calculations and include this average force applied over a 360° leg cycle to find Merrill's velocity; this is one of the exercises included at the end of the case.

In estimating Merrill's velocity from the data given in **Figure D.2.1**, we did not consider the rate at which his legs would need to be spinning in order to achieve a given velocity. The next step of our analysis should include a check to answer the questions:

- When traveling at 20.1 m/s, at what rate are his legs spinning?
- Could Merrill actually accomplish this rate for the given leg force?

This check is included as an exercise at the end of the case.

D.4 JUST THE FACTS

In this case we have examined the question: How fast could Merrill, racing on a bicycle, sprint toward the finish line? By looking at how forces are transferred from the pedal to the chain ring through the chain to the rear cog to the wheel to the ground, we were able to relate the downward push exerted by the cyclist on the pedal ($\boldsymbol{F}_{foot}$) to the forward push force exerted on the rear wheel ($\boldsymbol{F}_{friction,\ ground\ on\ tire}$). Then we related this push force to the drag force and velocity. This examination involved making assumptions, creating free-body diagrams, and applying Newton's first law. This process of assuming, drawing, and applying Newton's first law is the heart of static analysis.

SYSTEM ANALYSIS (SA) EXERCISES

SAD.1 Exploring a Bicycle

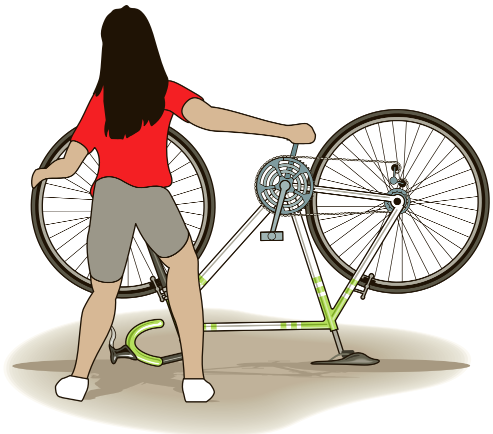
1. Through drawings engineers convey ideas about objects. The following drawing exercises are about gaining comfort in conveying information about a bicycle wheel.
   (a) Draw as many circles as you can on an $8.5 \times 11''$ sheet of paper in 15 seconds, then convert one of the circles into a bicycle wheel.
   (b) Draw as many rectangles as you can on an $8.5 \times 11''$ sheet of paper in 15 seconds, then convert one of the rectangles into a bicycle wheel.
   (c) Inspect a bicycle wheel. How representative of the structure of a bicycle wheel are your drawings in (a) and (b)? (If you had trouble with (b), imagine what a bicycle wheel must look like to a bird flying overhead.)
2. Examine the following on a bicycle:
   (a) *Chain and chain ring:* How many teeth are on the largest chain ring? What is its diameter? Create a sketch that shows the interaction of the chain and chain ring. State any assumptions that you make in taking the measurements.
   (b) *Chain and the smallest rear cog:* How many teeth are on the smallest rear cog? What is its diameter? Create a sketch that shows the interaction of the chain and rear cog. State any assumptions that you make in taking the measurements.
   (c) *Pedal and crank:* What is the distance from the center of the pedal pivot point to the center of the bottom bracket? Would you describe the connection between the pedal and crank as (I) fixed (the pedal and crank always keep the same orientation relative to one another), or (II) connected but not fixed relative to one another?
   (d) *Rear wheel assembly:* What is the diameter of the rear wheel? State any assumptions that you make in taking the measurements.
3. Consider the force profile presented in **Figure D.3.1*d***. When the right pedal is in the configurations listed below, how much moment is created about the bottom bracket and what is the direction (clockwise or counterclockwise) of the moment? F_{peak} in **Figure D.3.1*d*** is at 800 N. Show all work.
   (a) 90° configuration
   (b) 180° configuration
   (c) 270° configuration
4. Following the same reasoning that was used to generate the free-body diagrams associated with **Table D.1**, generate a list of the external forces acting on each of the following systems when a cyclist and bicycle are traveling at constant velocity. Also draw a free-body diagram of each system:
   (a) Front wheel assembly
   (b) Rear wheel assembly
   (c) Front fork
5. When Merrill is traveling at 20.1 m/s, at what rate are his legs spinning (in revolutions/minute)? Is this a sustainable pedaling rate?
6. (a) In Section D.3, we discuss an average pedaling force of $\frac{2}{3}$ the peak. If this is indeed the average pedaling force, calculate the maximum velocity you estimate that Merrill will be able to travel. Include all supporting calculations.
   (b) When traveling at the velocity you calculated in (a), at what rpm (revolutions per minute) will Merrill's legs be rotating? How does the rpm you just calculated compare with the preferred rpm of 90–110 rev/min? Include all supporting calculations.

SAD.2 Analysis of Bicycle Performance

This exercise is about describing the powertrain of a bicycle. Flip a bicycle upside down on a table, as shown in **Figure SAD.2.1*a***. While turning one of the pedals, use your other hand to shift through the front and rear gears (as defined in **Figure SAD.2.1*b***) in order to answer the following two questions:

1. Qualitative analysis

(a) In low gear (smallest chain ring, largest rear cog), is the rear wheel rotating faster or slower than in high gear (largest chain ring, smallest rear cog)?



(*a*)

Rear cogs and front chain rings when viewed from above in (*a*).

(*b*)

Figure SAD.2.1 Experimenting with gear ratios.

(b) In high gear (largest chain ring, smallest rear cog), is more or less friction force applied to the ground than in low gear (smallest chain ring, largest rear cog)?[3]

2. Quantitative analysis

(a) For your bicycle, record the following information:

Bicycle make: ________________________

Number of front chain rings: ________________________

Number of teeth on each front chain ring ($N_{\text{chain ring}}$):

____

Approximate diameter of each front chain ring:

____

Number of rear sprockets: ________________________

Number of teeth on each rear cog ($N_{\text{rear cog}}$):

____

Length of crank arm: ________________________

Diameter of rear tire: ________________________

Given here is some useful background material on bicycle gearing, gear ratios, and gear selection. Let:

L_{crank} be the length of the crank

$R_{\text{chainwheel}}$ be the radius of the chain ring (front sprocket)

$R_{\text{rear cog}}$ be the radius of the rear cog (rear sprocket)

R_{wheel} be the radius of the rear wheel ($= D/2$)

(A) We are able to determine that the rotational speed of the rear wheel assembly ($\omega_{\text{rear wheel}}$) is related to the rotational speed of the front sprocket ($\omega_{\text{chain ring}}$) by:

$$\omega_{\text{real wheel}} = \omega_{\text{chain ring}}(R_{\text{chain ring}}/R_{\text{rear cog}}) \tag{1a}$$

Furthermore, we can also write that

$$(R_{\text{chain ring}}/R_{\text{rear cog}}) = (N_{\text{chain ring}}/N_{\text{rear cog}}) \tag{1b}$$

which, when substituted into (1a), gives us:

$$\boxed{\omega_{\text{real wheel}}/\omega_{\text{chain ring}} = N_{\text{chain ring}}/N_{\text{rear cog}}} \tag{1c}$$

(B) The product ($N_{\text{chain ring}}D/N_{\text{rear cog}}$) is commonly called the **gear-inch number *G***. Therefore we can write (D.2) as

$$\|\boldsymbol{F}_{\text{friction, ground on tire}}\| = \|\boldsymbol{F}_{\text{foot on pedal}}\|\frac{2L_{\text{crank}}}{G} \tag{2a}$$

or

$$\boxed{\frac{\|\boldsymbol{F}_{\text{friction, ground on tire}}\|}{\|\boldsymbol{F}_{\text{foot on pedal}}\|} = \frac{2L_{\text{crank}}}{G}} \tag{2b}$$

Let's apply (1c) and (2b) for the author's 10-speed bicycle:

The chain ring has sprockets with 40 and 52 teeth.

The rear cogs have 14, 17, 20, 24, and 28 teeth.

The crank is 6 inches long, and the diameter of the rear tire is 27 inches.

In low gear ($G = 39$), $\|\boldsymbol{F}_{\text{friction, ground on tire}}\|$ is 31% of $\|\boldsymbol{F}_{\text{foot on pedal}}\|$ and the rear wheel spins 1.43 times for every revolution of her legs. In high gear ($G = 100$), $\|\boldsymbol{F}_{\text{friction, ground on tire}}\|$ is 12% of $\|\boldsymbol{F}_{\text{foot on pedal}}\|$ and the rear wheel spins 3.71 times for every revolution of her legs.

(b) Use the data recorded in (a) to complete a table similar to **Table SAD.2.1** for your bicycle. You may find it useful to create a spreadsheet. Include a printout of your table.

(c) In high gear, what is the value of G (the gear-inch number)? What is the ratio of $\|\boldsymbol{F}_{\text{friction, ground on tire}}\|$ to $\|\boldsymbol{F}_{\text{foot on pedal}}\|$? What is the ratio of $\omega_{\text{rear wheel}}$ to $\omega_{\text{chain ring}}$?

(d) In low gear, what is the value of G (the gear-inch number)? What is the ratio of $\|\boldsymbol{F}_{\text{friction, ground on tire}}\|$ to $\|\boldsymbol{F}_{\text{foot on pedal}}\|$? What is the ratio of $\omega_{\text{rear wheel}}$ to $\omega_{\text{chain ring}}$?

(e) Based on the data you presented in (c) and (d), complete the following statement for your bicycle: In low gear ($G =$ ________), $\|\boldsymbol{F}_{\text{friction, ground on tire}}\|$ is ________% of $\|\boldsymbol{F}_{\text{foot on pedal}}\|$ and the rear wheel spins ________ times for every revolution of my legs. In high gear ($G =$ ________), $\|\boldsymbol{F}_{\text{friction, ground on tire}}\|$ is ________% of $\|\boldsymbol{F}_{\text{foot on pedal}}\|$ and the rear wheel spins ________ times for every revolution of my legs.

(f) Write up a brief comparison of the author's 10-speed bicycle and the bicycle you just explored.

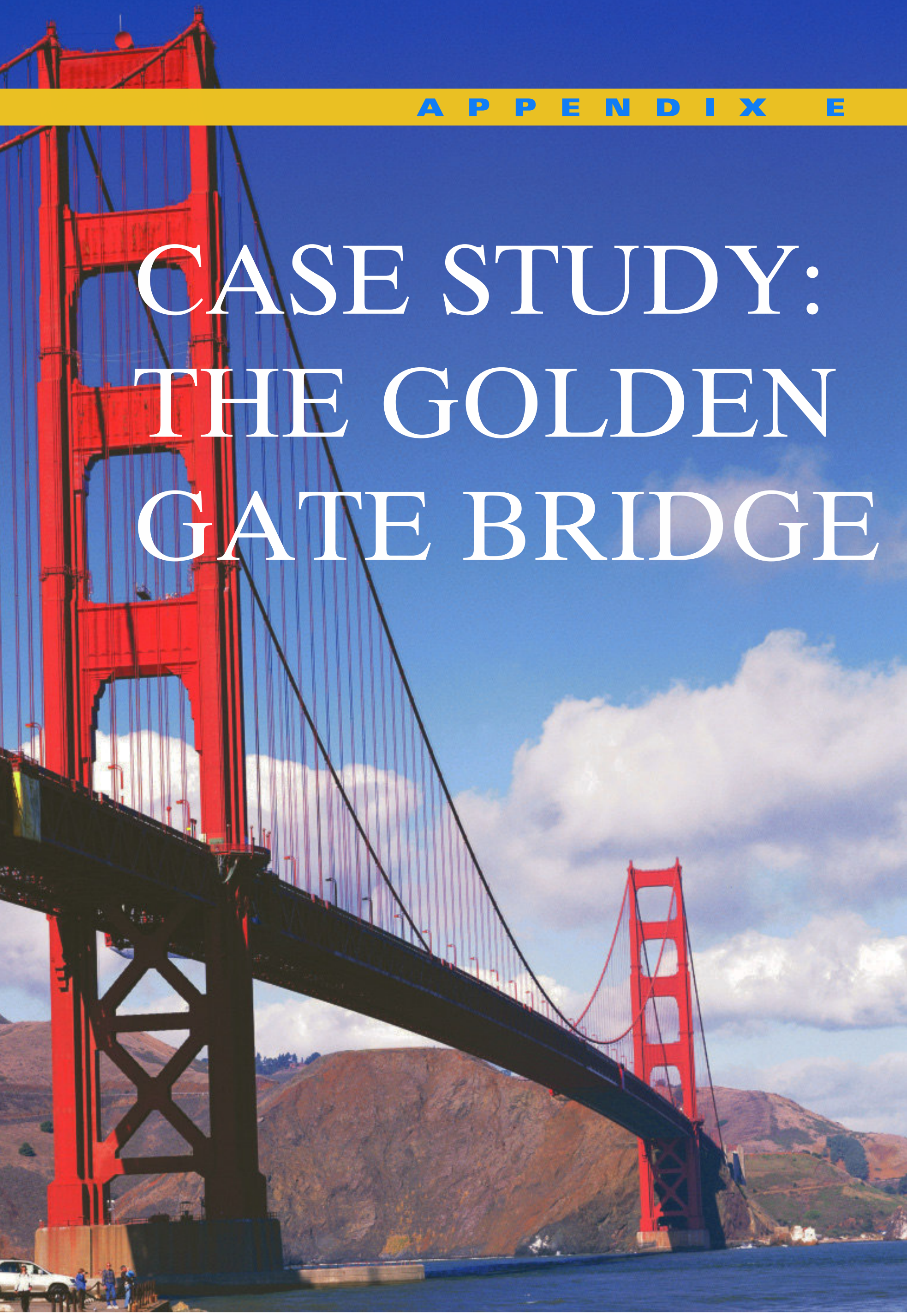
[3]While pedaling, put your bicycle in high gear. Place a 2 × 4 on top of the rear wheel in order to create resistance—note how difficult it is to increase pedaling to reach a rate of one revolution per second. Repeat in low gear.

Table SAD.2.1 Performance Data on Author's Bicycle

	$N_{\text{rear cog}}$	14	17	20	24	28
$N_{\text{chain ring}} = 40$	G = $(N_{\text{chain ring}} D / N_{\text{rear cog}})$	77	64	54	45	39
	$\omega_{\text{rear wheel}} / \omega_{\text{chain ring}}$ (1c)	2.86	2.35	2.00	1.67	1.43
	$\dfrac{\lVert \boldsymbol{F}_{\text{friction}} \rVert}{\lVert \boldsymbol{F}_{\text{foot on pedal}} \rVert}$ (2b)	0.16	0.19	0.22	0.27	0.31
$N_{\text{chain ring}} = 52$	G = $(N_{\text{chain ring}} D / N_{\text{rear cog}})$	100	83	70	59	50
	$\omega_{\text{rear wheel}} / \omega_{\text{chain ring}}$ (1c)	3.71	3.06	2.60	2.17	1.86
	$\dfrac{\lVert \boldsymbol{F}_{\text{friction}} \rVert}{\lVert \boldsymbol{F}_{\text{foot on pedal}} \rVert}$ (2b)	0.12	0.15	0.17	0.21	0.24

APPENDIX E

CASE STUDY: THE GOLDEN GATE BRIDGE



Eddie Hironka/The Image Bank/Getty Images

In this case, we use statics concepts to understand how a bridge functions and why the Golden Gate Bridge is shaped the way it is.[1] We have chosen the Golden Gate Bridge because it is one of the most recognized and beautiful structures in the world. Its graceful lines, Art Deco details, and spectacular views make it a popular stop for tourists from around the world. When it was completed in 1937, the 1280-meter suspension span was the longest in the world (now the 13th longest). Today more than 100,000 vehicles cross the bridge every day. It has been subjected to earthquakes, strong winds, and swift tides, and yet it continues to perform its function of linking the headlands on the two sides of the entrance to San Francisco Bay.[2] Engineers designing the bridge used statics concepts first to evaluate all the loads the bridge could potentially experience and then to design the members to resist these loads.

Before reading further, think about different types of bridges you have seen.

1. Sketch at least two of them and label the parts you know.
2. Identify the locations where the forces exerted on the bridge are transferred to the ground.

You may want to go to a bridge near you and study how it is built and, particularly, how it is attached to the ground. There are also many excellent pictures of bridges on the Internet. This task will provide good background for the discussion in this chapter.

[1]The bridge is named after the entrance to San Francisco Bay, which was named the Golden Gate by John Charles Fremont in the mid-1800s.

[2]The Golden Gate Bridge District maintains a website (http://www.goldengate.org/) if you are interested in more information about this bridge.

◆ OBJECTIVES

On completion of this case you will be able to:

- **Describe the types of forces that act on a bridge**
- **Isolate components of a suspension bridge and identify the external forces**
- **Use Newton's first law to answer questions about the transfer of loads from one bridge component to another**
- **Compare the analysis of an approximate model with a more "exact" model**

E.1 A WALK ACROSS THE BRIDGE

The main components of a suspension bridge that carry the loads from the bridge deck down to the ground are shown in **Figure E.1.1**. Cars, trucks, trains, and people travel on the bridge deck, which hangs from suspenders hung from main cables that are draped over towers and attached to anchorages at each end of the bridge. The towers are embedded deep in the ground and supported by massive concrete foundations.

Imagine that you are standing in the middle of the Golden Gate Bridge. How would your weight be transferred to the ground? To get a feel for how the bridge works, let's build a model. You'll need two large paper clips, a piece of string about 3 meters long, a pencil, and six heavy books. Set aside two of the books that are about the same size (you will use them in a moment for the towers). Tie one end of the string around two of the remaining four books, tie the other end around the other two books, and place the two piles about 2 meters apart with the string lying slack between them. These piles are the two anchorages. Stand the two books you set aside on end, one about 30 cm to the right of the left anchorage and the other about 30 cm to the left of the right anchorage, as in **Figure E.1.2*a***. Drape the string (the main cable) over the top of the towers, hook the two paper clips (the suspenders) onto the string, and slide the pencil (the bridge deck) into the paper clips. To represent yourself standing on the bridge, push down on the pencil.

Now modify the model by removing the anchorages and tying the string directly around the two towers (**Figure E.1.2*b***). What happens when you push down on the pencil this time? Does the system collapse? Yes, the towers fall over and the bridge collapses because the towers are being pulled toward each other (inward) by the force in the main cable. In the first bridge model, the cable tied to the anchorage provides

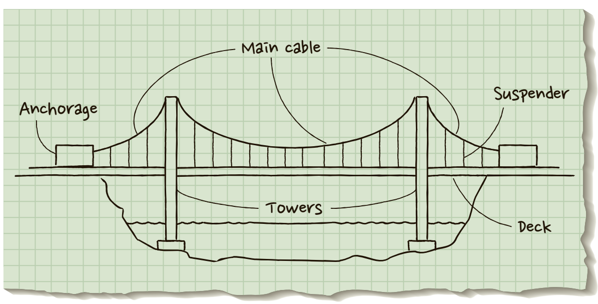


Figure E.1.1 The basic components of a suspension bridge.

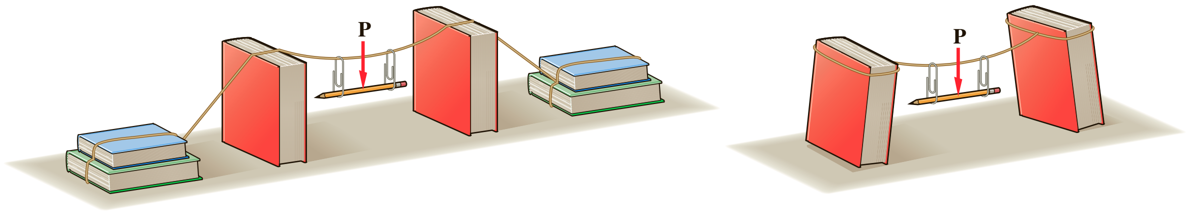


Figure E.1.2 A model of a suspension bridge made from string, pencil, paper clips, and books (*a*) with anchorages and (*b*) without anchorages.

an outward force on the towers to balance the inward force. This outward force is transferred to the table at the anchorages through friction. When the anchorages are removed and the main cable is tied directly to the towers, there is no outward force pulling on the towers to balance the inward force. Furthermore, unless we glue the upright books to the table, there is no way for the table to pull on the bottom of the towers to keep them from tipping. These two models demonstrate how all the bridge components work together to make a complete "load path" to transfer loads from bridge to ground.[3]

For a more systematic view of how the loads exerted on the bridge deck are related to the forces exerted by the ground on each tower base ($\boldsymbol{F}_{\text{ground, tower}}$) and on each anchorage ($\boldsymbol{F}_{\text{normal, anchorage}}$ and $\boldsymbol{F}_{\text{friction, anchorage}}$), shown in **Figure E.1.3**, we will trace the forces through all the bridge components. We will think in terms of the **free-body diagram** for each component of our model when we put a load on it.

1. Start with the deck. You push down on the deck (pencil), and it pulls down on the suspenders (paper clips). As a result, tension is developed in the suspenders as they pull up on the deck ($\boldsymbol{T}_{\text{suspender, deck}}$ in **Figure E.1.4*a***) and down on the main cable ($\boldsymbol{T}_{\text{suspender, cable}}$ in **Figure E.1.4*b***), just as in a game of tug-of-war the rope pulls on the two teams at its ends. As the suspenders pull down on the main cable (string), a tension force is created in the cable throughout its length. (The tension force is what took the slack out of the string in your model.)

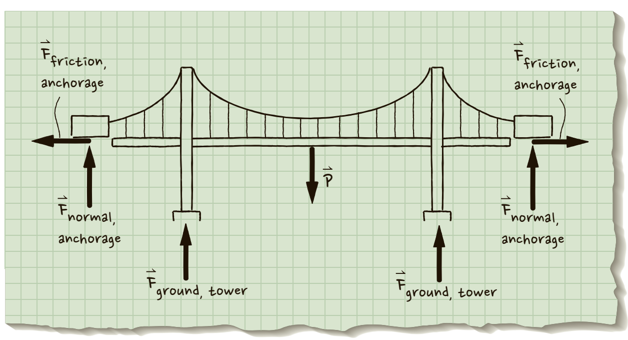


Figure E.1.3 Forces on the suspension bridge as a result of a load $\boldsymbol{P}$ exerted on the deck.

[3]Exercise modified from http://www.pbs.org/wgbh/nova/bridge/meetsusp.html.

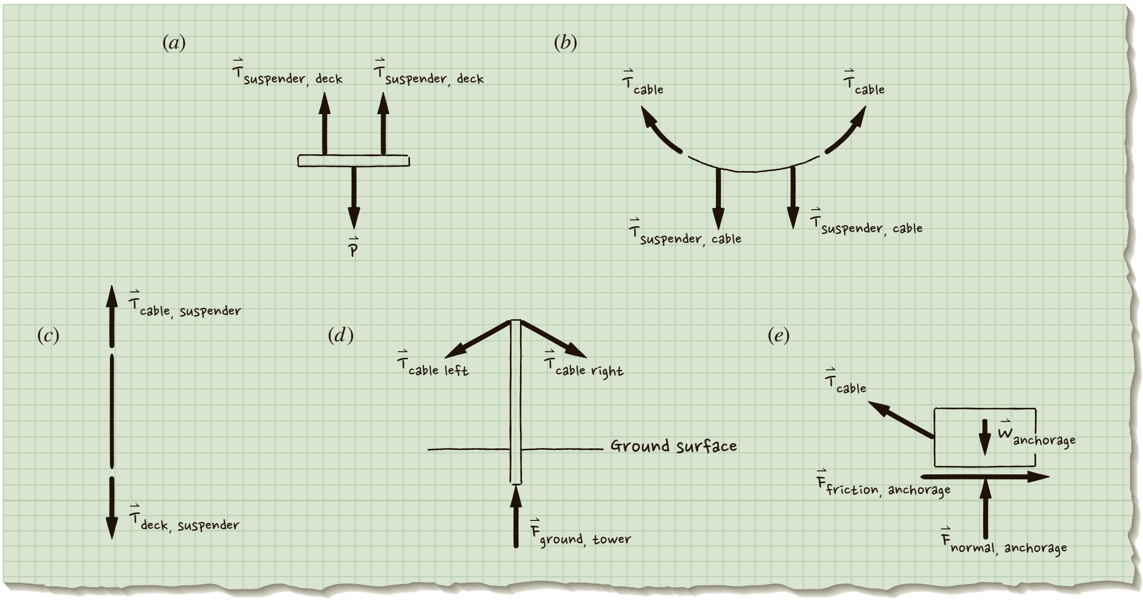


Figure E.1.4 Free-body diagrams for the bridge components: (*a*) The suspenders pull up on the bridge deck with a force $\boldsymbol{T}_{\text{suspender, deck}}$; (*b*) the suspenders pull down on the main cable, causing tension $\boldsymbol{T}_{\text{cable}}$ to develop in the cable; (*c*) the suspender is in tension with the main cable pulling up and the deck pulling down; (*d*) the main cable forces are counteracted by the force $\boldsymbol{F}_{\text{ground, tower}}$ exerted by the ground on the tower; (*e*) the main cable force $\boldsymbol{T}_{\text{cable}}$ is counteracted by the weight of the anchorage $\boldsymbol{W}_{\text{anchorage}}$ and the forces that the ground exerts on the anchorage $\boldsymbol{F}_{\text{normal, anchorage}}$ and $\boldsymbol{F}_{\text{friction, anchorage}}$.

Note that as the suspender pulls down on the main cable, the cable pulls up on the suspender with an equal and opposite force. **Figure E.1.4*c*** shows a free-body diagram of a suspender. The main cable pulls up on the suspender with a force $\boldsymbol{T}_{\text{cable, suspender}}$ and the deck pulls down with a force $\boldsymbol{T}_{\text{deck, suspender}}$. Since these are the only two forces acting on the suspender, Newton's first law requires that they must be equal. For simplicity, from now on we will call these tensile forces $\boldsymbol{T}_{\text{suspender}}$.

2. Now follow the main cable to the towers. Where the main cable passes over the top of a tower (upright book), the cable is sloping away from the tower on both sides. This orientation causes the cable to pull down on the tower (**Figure E.1.4*d***). To counteract this, the tower pushes up on the cable. The force of the cable pulling down on the towers is transferred to the ground through the towers and creates the force $\boldsymbol{F}_{\text{ground, tower}}$ shown in **Figures E.1.3** and **E.1.4*d***. Because the tower is being pushed on at the top and bottom, it is in compression.

3. Continue tracing the main cable to the right anchorage. Because the cable is embedded in the anchorage (stacked books), it pulls on the anchorage with a large force ($\boldsymbol{T}_{\text{cable}}$ in **Figure E.1.4*e***). Though $\boldsymbol{T}_{\text{cable}}$ is pulling upward, the anchorage is kept from lifting off the ground by its enormous weight ($\boldsymbol{W}_{\text{anchorage}}$). The ground exerts both a normal contact force ($\boldsymbol{F}_{\text{normal, anchorage}}$) and a friction force ($\boldsymbol{F}_{\text{friction, anchorage}}$) on the anchorage, and the anchorage is kept from sliding by this friction force. As we see in Chapter 7, the maximum size of the friction force is limited by

the weight of the anchorage. In order to develop an adequate friction force, the anchorage must be very heavy. (Experiment with this by using lightweight books for the anchorages in your model. Do lightweight anchorages slide across the table?) The estimate of the magnitude of the cable force exerted on the anchorage is used in designing the weight of the anchorage block.

Thinking about the role each structural component plays in transferring the load on the bridge deck clarifies why your model collapsed when you removed the anchorages. The bridge would collapse if the main cables were not securely anchored into the ground at each end. Even though the main cables of the Golden Gate Bridge are very slender (0.92 m diameter), they are able to transfer thousands of kilonewtons of tensile force to the ground. To prevent uplift and sliding, each anchorage contains more than 20,000 m^3 of concrete and weighs more than 530,000 kN.

We just traveled through the bridge's **load path**, which is the route of the loads as they are transferred from one structural member to another. In studying the load path of any structure, think of the structure as a series of interconnected pipes and imagine pouring water into one end and watching the water exit at the other end. In the case of a suspension bridge, you pour water into the deck, and it flows from deck to suspenders to cables to towers and anchorages and then to the ground, where it exits the "pipe."

Summary

In this section the key ideas are:

1. A simple physical model can be used to gain an understanding of a complex structure.
2. Forces acting on the bridge are transferred from one component to another and then to the ground. The load path is the route of the loads as they are transferred from one component to another.
3. The cables and suspenders on the bridge are in tension. The bridge towers are in compression. The anchorages are kept from sliding through friction forces.

E.2 HOW HEAVY SHOULD THE ANCHORAGES BE?

Now that we have laid out in a general manner the forces acting on the components of the Golden Gate Bridge and how those forces are transferred to the ground, we will answer the same question Joseph B. Strauss and his team of engineers had to answer when they designed the bridge—how heavy should the anchorages be?

As is common in engineering analysis, we will make several assumptions to create a simplified analytical model. This will allow us to develop some equations that provide reasonable estimates of the forces acting on the components. Later in the book, we will use more complex assumptions and equations to perform a more exact analysis. We can

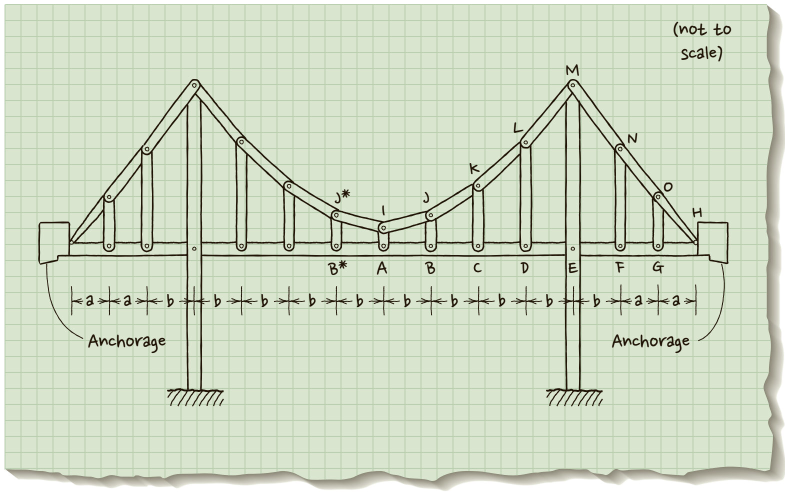


Figure E.2.1 Approximate representation of the Golden Gate Bridge using pinned links to model the main cables and suspenders.

then compare our approximate analysis with the more exact analysis to investigate how our simplifying assumptions have affected our results.

In order to answer the question about the weight of the anchorages, we must work our way through the load path and answer three interrelated subquestions:

1. How large is the force pulling on each of the suspenders?
2. What is the tension force in each main cable?
3. What forces on the anchorage would cause uplift or sliding?

What Assumptions Are We Making?

First, we replace each main cable by a series of links that mimic the geometry of the Golden Gate Bridge and are connected to one another with pins (**Figure E.2.1** and **Table E.1**). This will allow us to complete an approximate analysis of the main cable using simple applications of Newton's first and third laws. Second, we assume that the only loads

Table E.1 Geometry of "Main Cable" in Our Approximate Model

Pin	Height above Bridge Deck (meters)	Distance from Center of Bridge (meters)
I	10.0	0
J and J*	18.9	160
K	45.8	320
L	90.4	480
M	153.0	640
N	71.4	800
O	32.8	891.5
H	0	983

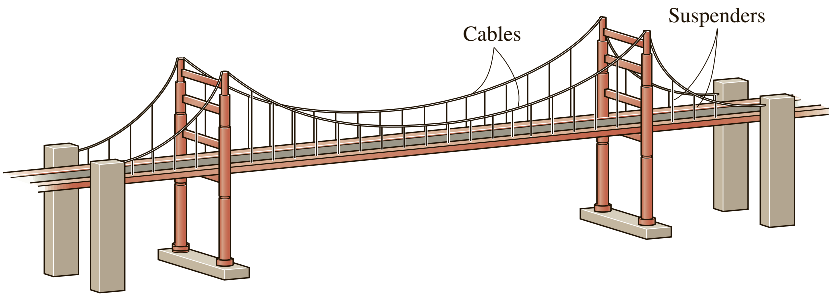


Figure E.2.2 A suspension bridge has two main cables, each taking one-half of the load.

acting on the bridge are **gravity loads** (vertical loads exerted by people, vehicles, and the weight of the bridge). This allows us to ignore such horizontal loads as winds or earthquakes. Third, we assume that the weight of the main cable is much less than the weight of the bridge deck and vehicles and therefore its weight can be ignored in our analysis. This assumption allows us to assume that the main cable is in the shape of a parabola. (For the Golden Gate Bridge, the combined weight of the deck and vehicles is more than seven times the weight of the main cables, so ignoring the weight of the cables is a reasonable assumption for a preliminary analysis.)

1. How Large Is the Force Pulling on Each of the Suspenders? When we look at the bridge from the orientation shown in **Figure E.2.2**, we are reminded that there are two main cables, each attached to the deck with suspenders. When the total load is distributed uniformly across the width of the bridge, each main cable supports one half the load.

To calculate the force exerted by the deck on one suspender, we start by calling the weight per unit length of deck $\boldsymbol{w}$. We assume that all the suspenders on each side are evenly spaced along the length of the deck and that the distance between any two suspenders on the same side of the bridge is b. We then slice a length of deck out of our model (**Figure E.2.3*a***), making our first cut halfway between two suspenders and our second cut a distance b away from the first cut. Thus the length of the deck slice is b, and there is one suspender attached on each side at the midpoint of length b (**Figure E.2.3*b***). Applying Newton's first law, we can say that because the bridge is not moving, the sum of the forces in the vertical direction will be zero. This means that the force exerted by the suspenders pulling up ($2\boldsymbol{T}_{\text{suspender}}$) must equal the weight of the deck slice pulling down ($\boldsymbol{W}_{\text{deck slice}}$):

$$\left\|2\boldsymbol{T}_{\text{suspender}}\right\| - \left\|\boldsymbol{W}_{\text{deck slice}}\right\| = 0$$

$$\left\|2\boldsymbol{T}_{\text{suspender}}\right\| - \|\boldsymbol{w}\|b = 0$$

$$\left\|\boldsymbol{T}_{\text{suspender}}\right\| = \frac{\|\boldsymbol{w}\|b}{2} \qquad \text{(E.1)}$$

We can now use this relationship to calculate the force on each suspender in our model.

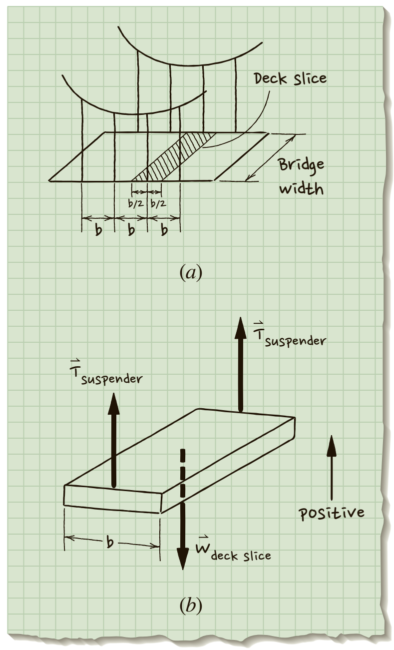


Figure E.2.3 (*a*) A slice of bridge deck of length b is used for analysis. (*b*) The weight of the slice of bridge deck, pulling down, is balanced by the force exerted by the suspenders, pulling up.

Answer to Question 1

For our simplified model, the suspenders are 160 m apart and the deck weighs 330 kN/m (Table E.2). The total weight of the slice we are analyzing is 52,800 kN = (330 kN/m)(160 m). Then the force of each suspender pulling up on the bridge deck is 26,400 kN = (330 kN/m)(160 m)/2. We write our final answer in meganewtons (see **Table 1.2**): $\|\boldsymbol{T}_{\text{suspender}}\| = 26.4$ MN.

Table E.2 **Properties of the Golden Gate Bridge***

Structural Property	Quantity
Length of main span (distance between towers)	1280 m
Length of one side span	343 m
Width of bridge	27 m
Height of each tower above road deck	152 m
Height of each tower	227 m
Maximum sag in main cable	144 m
Weight of cable per one horizontal meter	48.7 kN/m
Diameter of one main cable with wrapping	0.92 m
Number of wires in each cable	27,572
Hanger spacing	15.2 m
Weight per unit length of bridge deck	330 kN/m
Weight of one tower	196,000 kN
Weight of one anchorage	530,000 kN

*Data from www.goldengatebridge.org/research/factsGGBDesign.html and Abdel-Ghaffar and Scanlan (1985).

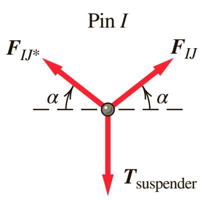


Figure E.2.4 Free-body diagram of pin I.

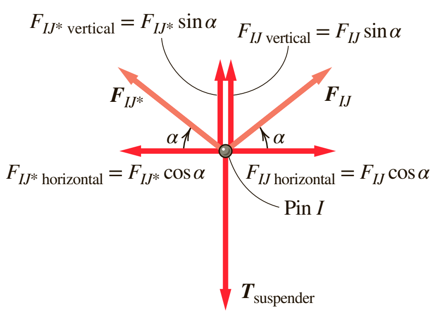


Figure E.2.5 Force balance: Equilibrium means that the forces pulling to the left equal the forces pulling to the right on pin I. Similarly, the forces pulling down equal the forces pulling up.

2. What Is the Tension Force in Each Main Cable? In our model, we are representing each of the main cables by a series of links attached by pins. Figure E.2.4 shows the forces acting on the pin at the bridge center, the pin labeled I in **Figure E.2.1**. We use two-letter subscripts to identify the link forces acting on any pin: the first letter indicates the pin we are currently evaluating, and the second letter indicates the other pin the link is attached to. For example, $\boldsymbol{F}_{IJ}$ symbolizes the force that link IJ exerts on pin I and $\boldsymbol{F}_{JI}$ symbolizes the force that link IJ exerts on pin J. A free-body diagram of link IJ would show that $\|\boldsymbol{F}_{IJ}\| = \|\boldsymbol{F}_{JI}\|$. In our analysis here of pin I, therefore, the two link forces are $\boldsymbol{F}_{IJ^*}$ and $\boldsymbol{F}_{IJ}$.

The suspender force $\boldsymbol{T}_{\text{suspender}}$ pulls down on the pin, and the forces $\boldsymbol{F}_{IJ}$ and $\boldsymbol{F}_{IJ^*}$ in the links IJ and IJ^* each pull away from the pin along the long axes of the links. Because the pin's state of motion is not changing, Newton's first law requires that the sum of the forces in the horizontal direction as well as the sum of the forces in the vertical direction be zero (Figure E.2.5). Looking first at the horizontal forces:

$$\left\|\boldsymbol{F}_{IJ\,\text{horizontal}}\right\| - \left\|\boldsymbol{F}_{IJ^*\,\text{horizontal}}\right\| = 0$$

$$\|\boldsymbol{F}_{IJ}\|\cos\alpha - \|\boldsymbol{F}_{IJ^*}\|\cos\alpha = 0$$

$$\|\boldsymbol{F}_{IJ^*}\| = \|\boldsymbol{F}_{IJ}\| \tag{E.2A}$$

Now looking at the vertical forces:

$$\left\|\boldsymbol{F}_{IJ^*\,\text{vertical}}\right\| + \left\|\boldsymbol{F}_{IJ\,\text{vertical}}\right\| - \left\|\boldsymbol{T}_{\text{suspender}}\right\| = 0$$

$$\left\|\boldsymbol{F}_{IJ^*}\right\|\sin\alpha + \left\|\boldsymbol{F}_{IJ}\right\|\sin\alpha - \left\|\boldsymbol{T}_{\text{suspender}}\right\| = 0 \qquad \text{(E.2B)}$$

Substituting from (E.2A) into (E.2B) and rearranging gives

$$\left\|\boldsymbol{F}_{IJ}\right\| = \frac{\left\|\boldsymbol{T}_{\text{suspender}}\right\|}{2\sin\alpha} \qquad \text{(E.3)}$$

The angle α can be determined from the geometry of the bridge shown in **Figure E.2.1** and in **Table E.1** data.[4] Figure E.2.6 shows a blow-up of a segment cut out of the center of the bridge. From the dimensions shown in the figure, $\cos\alpha = 160/160.25 = 0.998$ and $\sin\alpha = 8.9/160.25 = 0.0556$. From (E.3) and our known value $\|\boldsymbol{T}_{\text{suspender}}\| = 26.4$ MN, we see that $\|\boldsymbol{F}_{IJ}\| = \|\boldsymbol{F}_{IJ^*}\| = 237$ MN. The result of this calculation indicates that the force in the main cable is quite large—a force of 237 MN is equivalent to the weight of about 18,000 automobiles.

The next step is to draw a free-body diagram of the pin at J, as shown in Figure E.2.7. Once again, requiring the sum of the forces in the horizontal direction to be zero gives

$$\left\|\boldsymbol{F}_{JK\,\text{horizontal}}\right\| - \left\|\boldsymbol{F}_{JI\,\text{horizontal}}\right\| = 0$$

$$\left\|\boldsymbol{F}_{JK}\right\|\cos\beta - \left\|\boldsymbol{F}_{JI}\right\|\cos\alpha = 0$$

We determine β from **Table E.1**, which shows that the vertical distance from J to K is 45.8 m − 18.9 m = 26.9 m. This distance is the length of the side opposite β in the right triangle suggested in **Figure E.2.7**. **Table E.1** also shows that the side adjacent to β is 320 m − 160 m = 160 m. Using these values in the hypotenuse formula from footnote 4 gives 162.25 m for the hypotenuse length in the right triangle suggested in **Figure E.2.7**. Therefore $\cos\beta$ = adjacent side/hypotenuse = 160/162.25 = 0.986. This gives

$$\left\|\boldsymbol{F}_{JK}\right\| = \left\|\boldsymbol{F}_{JI}\right\|\frac{\cos\alpha}{\cos\beta} = (237\text{ MN})\frac{0.998}{0.986} = 240\text{ MN}$$

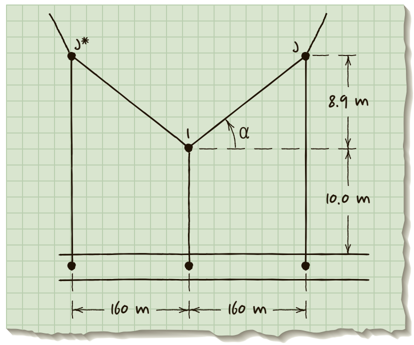


Figure E.2.6 Blow-up of a segment at the center of the bridge.

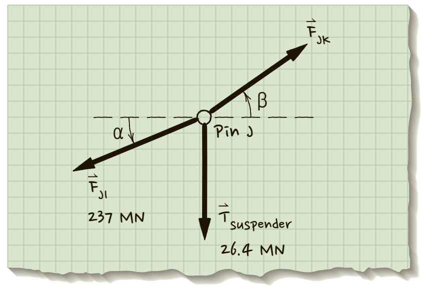
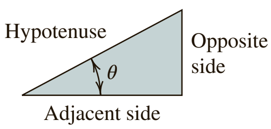


Figure E.2.7 Free-body diagram of pin J.

[4]For a right triangle, the lengths of the two sides and the hypotenuse are sufficient to determine angles and sines and cosines of those angles.



For the triangle shown here, "adjacent side" is the length of the side adjacent to the angle θ, "opposite side" is the length of the side opposite the angle θ, and "hypotenuse" is the length of the hypotenuse = (adjacent side^2 + opposite side^2)$^{1/2}$. Then $\sin\theta$ = opposite side/hypotenuse and $\cos\theta$ = adjacent side/hypotenuse.

To complete the analysis, we repeat the same type of calculation for each pin as we move across the bridge. The next pin is pin K. Note that as we move along the center span of the bridge from the center toward the towers, the force on each successive link increases, with the largest force being on link LM. On the side span, we find the force decreasing as we move from the tower toward the anchorage.

Answer to Question 2

Figure E.2.8 compares our simplified analytic model (red lines) with the results of a more exact analysis (green curves) as presented in Chapter 11. This figure shows that our simplified analysis provides a good estimate of the force exerted on the main cable. Near the center of the bridge, the results vary by less than 0.1%, and near either tower the variation is no more than 3%. Variations occur because the links, being a series of straight lines, cannot exactly duplicate the geometry of the parabolic cable. If we were to modify our approximate model by making the links shorter and the suspenders closer together, we could more closely approximate the actual bridge geometry and would converge to the parabolic cable solution in **Figure E.2.8**.

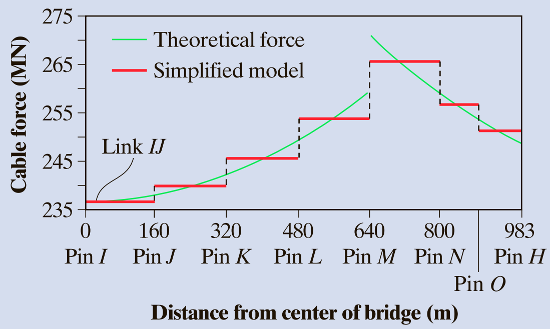


Figure E.2.8 Forces calculated from the simplified analytical model of the main cable are a good approximation of the actual forces exerted on the cable.

Why do you think there is a discontinuity in the theoretical curve in **Figure E.2.8** at the tower (pin M)? This discontinuity occurs because of the change in the orientation of the cable as it is draped over the top of the tower. The horizontal component (T_h) of the cable force remains constant throughout the length of the cable. The magnitude of the cable force at any location is $T = T_h/\cos\theta$, where θ is the angle between the cable and the horizontal. At the tower, the angle changes abruptly and consequently so does the cable force.

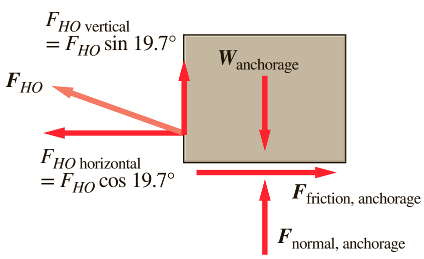


Figure E.2.9 Friction forces and the weight of the anchorage counteract the pull of the main cable.

3. What Forces on the Anchorage Would Cause Uplift or Sliding?

The main cable is pulling on the anchorage at the right end of the bridge with a very large force directed upward and to the left (**Figure E.1.4*e*** and Figure E.2.9). In our model this force is represented by $\boldsymbol{F}_{HO}$, the force of link HO pulling on the anchorage at H. The anchorage is kept from sliding to the left by the large friction force $\boldsymbol{F}_{\text{friction, anchorage}}$ developed between it and the ground. It is kept from lifting off the ground by its heavy weight. In designing the Golden Gate Bridge anchorages, engineers had to make each anchorage heavy enough to prevent either sliding or uplift.

What minimum weight of the anchorage will prevent it from lifting? Newton's first law tells us that if the anchorage is stationary, the sum of the forces in the vertical and horizontal directions must be zero. Looking at **Figure E.2.9**, this means that

$$\left\|\boldsymbol{F}_{HO\text{ vertical}}\right\| + \left\|\boldsymbol{F}_{\text{normal, anchorage}}\right\| - \left\|\boldsymbol{W}_{\text{anchorage}}\right\| = 0 \qquad \text{(E.4)}$$

$$-\left\|\boldsymbol{F}_{HO\text{ horizontal}}\right\| + \left\|\boldsymbol{F}_{\text{frictin, anchorage}}\right\| = 0 \qquad \text{(E.5)}$$

If the anchorage were to lift off the ground, the normal force shown in **Figure E.2.9** would be zero. We can calculate the weight of the anchorage when the normal force is zero by equating the forces in the vertical direction to zero and eliminating the normal force from (E.4):

$$\left\|\boldsymbol{F}_{HO\text{ vertical}}\right\| - \left\|\boldsymbol{W}_{\text{anchorage at uplift}}\right\| = 0 \tag{E.6}$$

$$\left\|\boldsymbol{W}_{\text{anchorage at uplift}}\right\| = \left\|\boldsymbol{F}_{HO\text{ vertical}}\right\|$$

$$\left\|\boldsymbol{W}_{\text{anchorage at uplift}}\right\| = (251.2\text{ MN})(\sin 19.7°) = 84.7\text{ MN}$$

If the anchorage weighs less than 84.7 MN, it will lift off the ground, and if it weighs more it will not.

The friction force required to prevent sliding can be determined from (E.5):

$$\left\|\boldsymbol{F}_{\text{friction, anchorage}}\right\| = \left\|\boldsymbol{F}_{HO\text{ horizontal}}\right\|$$

$$\left\|\boldsymbol{F}_{\text{friction, anchorage}}\right\| = (251.2\text{ MN})(\cos 19.7°) = 236.5\text{ MN} \tag{E.7}$$

The next question we want to ask is how heavy the anchorage must be to develop this large friction force. As presented in Chapter 7, the maximum friction force ($\boldsymbol{F}_{\text{friction max}}$) that can be produced between the ground and the anchorage depends on normal force between the anchorage and the ground and the roughness of contact between the two surfaces, reflected in the **coefficient of friction**, μ_{static}. For rough materials such as rock and concrete, μ_{static} could be in the range from 0.5 to 0.7. For our example we shall use $\mu_{\text{static}} = 0.6$. The relationship is expressed mathematically as

$$\boldsymbol{F}_{\text{friction max}} = \mu_{\text{static}}\boldsymbol{F}_{\text{normal, anchorage}} \tag{E.8}$$

In (E.7) we determined that the friction force needed to prevent sliding is 236.5 MN, which must be developed by the roughness between the anchorage and the ground as expressed by (E.8). Therefore

$$\left\|\boldsymbol{F}_{\text{friction, anchorage}}\right\| = \left\|\boldsymbol{F}_{\text{friction max}}\right\| = \mu_{\text{static}}\left\|\boldsymbol{F}_{\text{normal anchorage}}\right\| = 236.5\text{ MN}$$

$$\left\|\boldsymbol{F}_{\text{normal, anchorage}}\right\| = \frac{236.5\text{ MN}}{0.6} = 394\text{ MN}$$

Finally, the weight of anchorage required to produce a normal force of 394 MN is found from (E.4):

$$\begin{aligned}\left\|\boldsymbol{W}_{\text{anchorage required}}\right\| &= \left\|\boldsymbol{F}_{\text{normal, anchorage}}\right\| + \left\|\boldsymbol{F}_{HO\text{ vertical}}\right\| \\ &= 394\text{ MN} + 84.7\text{ MN} = 479\text{ MN}\end{aligned} \tag{E.9}$$

Answer to Question 3

This tells us that the anchorage must weigh more than 84.7 MN to prevent uplift and more than 479 MN to prevent sliding. In fact, on the Golden Gate Bridge each anchorage weighs about 530 MN, which satisfies both conditions.

E.3 ADDING MORE REALITY

Not all bridges are suspension bridges. Engineers use different design solutions after considering many issues, such as distance to be spanned, types of loads to be carried, strength of the rock available for the foundation, type of material to be used, aesthetics, and cost. Suspension bridges are typically used for spanning large distances, on the order of 600 to 2000 meters. For shorter distances, designers might use beam, arch, or truss bridges. Beam bridges, typically seen as freeway overpasses, are inexpensive to build and efficient for spanning distances of 75 meters or less. Arch bridges, developed by the ancient Romans, are useful for spanning distances from 100 to 400 meters. Truss bridges have the advantage of being lightweight and can be built up from a series of short members.

If you study the Golden Gate Bridge, you will see that it is made up of several types of bridges. For example, the south approach consists of a steel arch, five truss spans, and a series of steel beam bridges. Each bridge type has a different mechanism for transferring loads to the ground.

Up to this point we have assumed that the Golden Gate Bridge is not moving and that only gravity forces act on it. In fact, the bridge is moving all the time and is subjected to a number of dynamic loads, including earthquakes, wind loads, vehicle loads, and the action of strong tidal currents. The currents and the wind impart sideways loads on the towers, causing them to sway approximately 0.3 m from side to side. Vehicular traffic is another source of bridge movement, and as you stand on the bridge sidewalk, you can feel the vibrations of the deck as the cars and trucks drive by.

In extreme cases, wind loads can cause a bridge deck to oscillate and twist wildly, possibly leading to a collapse, as was the case in 1940 on the Tacoma Narrows Bridge. The deck acts like an airfoil as the wind passes by and causes the deck to lift and fall. As the deck goes up and down, changes in the geometry of the main cables cause the towers to sway shoreward and channelward as much as 0.5 m. Thermal expansion and contraction of the main cables also causes the deck to move. Design calculations indicate that at its center, the deck of the Golden Gate Bridge can deflect downward 3.3 m and upward 1.8 m as a result of temperature and other loading.

Because the Golden Gate Bridge is not far from the San Andreas and Hayward faults, it is periodically subjected to earthquakes. During an earthquake, the ground accelerates vertically and horizontally, causing **inertial forces** to act on the bridge. The inertia of a structure causes it to resist any sudden movement of its base, so that the upper parts of the structure deform relative to the base (**Figure E.3.1**). A unique feature of inertial forces caused by earthquakes is that they are proportional to the weight of the structure—the heavier the structure, the larger the forces. The motion of the bridge during an earthquake is very complex, consisting of horizontal and vertical vibrations as well as a twisting of the deck and towers. Calculation of bridge deflections and the resulting forces requires a **dynamic analysis**. Although a static analysis can provide preliminary estimates of the earthquake and wind

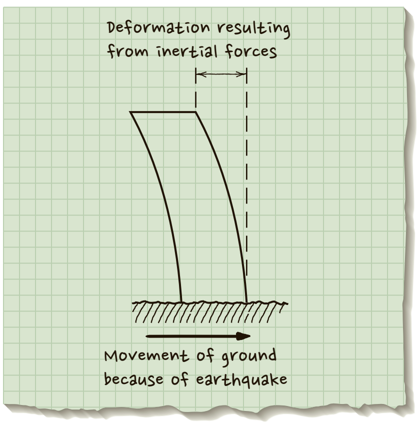


Figure E.3.1 Deformation of a structure subjected to earthquake inertial forces.

forces acting on each bridge component, a dynamic analysis will provide more accurate results.

E.4 JUST THE FACTS

In this case we examined the question of how heavy the anchorages should be for the Golden Gate Bridge. By analyzing the loads acting on a suspension bridge as they are transferred from the bridge deck to the suspenders and then through the main cables to the anchorages, we were able to find the forces of the cables pulling on the anchorages. We used a simplified analytical model to calculate an approximate solution to the forces in the bridge's main cable. We then compared our approximate analysis with a more exact solution. We examined how heavy the anchorages must be to prevent both uplift and sliding. The analysis involved making assumptions, creating free-body diagrams, and then applying Newton's first law.

E.5 REFERENCES

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Golden Gate Bridge, Highway and Transportation District: www.goldengate bridge.org/research/

SYSTEM ANALYSIS (SA) EXERCISES

SAE.1 Exploring a Suspension Bridge

1. Reconstruct the model of a suspension bridge in Section E.1 (**Figure E.1.2*a***). Push down on the pencil and feel how much force the system can resist before the anchorages start to slide.

   Now remove one of the books from each of the anchorages. Push on the pencil again. Can the system resist more or less force than before? How does the system fail?

   Try making other alterations to the model to examine the effect on the system capacity (i.e., the force it can support) and the failure mechanism. Examples of alterations you can implement include:

   (a) Adding a book to each of the anchorages so there are three books for each

   (b) Inserting a shiny (slippery) piece of paper between the table surface and the books that serve as anchorages

   (c) Inserting a rough piece of cloth or carpet between the table surface and the books that serve as anchorages

   (d) Shortening the string so that it is tight across the books that serve as the towers

   (e) Moving the towers very close to the anchorages or very close together

2. Assuming that the geometry of the Golden Gate Bridge remains unchanged, double the weight per unit length of the bridge deck to 660 kN/m and calculate the force $\boldsymbol{F}_{IJ}$ acting on member *IJ* (**Figures E.2.1** and **E.2.4**).

   (a) How much does $\boldsymbol{F}_{IJ}$ change?

   (b) How much will doubling the weight per unit length of the bridge deck change the force $\boldsymbol{F}_{HO}$ pulling on the anchorage? Explain your answer.

3. If $\boldsymbol{F}_{HO}$ is doubled (**Figure E.2.9**), by how much will the required weight of the anchorage increase? Explain your answer.

SAE.2 Exploring a Beam Bridge

Whereas suspension bridges are efficient for spanning distances of about 600 m to 2000 m, beam bridges are often used to span short distances. A common example of a beam bridge is a freeway overpass. To model a beam bridge, you need three books. Place two books on end about 20 cm apart to serve as the piers and lay the third book across the two to create the bridge deck as shown in **Figure SAE.2.1**.

Load the bridge in two ways:

1. Push straight down on top of the deck with your hand.
2. Push horizontally on the deck with your hand.

For each of these loading cases:

(a) Draw a free-body diagram for each component of the bridge to trace the load path as $\boldsymbol{F}_{\text{hand}}$ is transferred to the ground. State any assumptions you are making in drawing the diagrams.

(b) Explain why the bridge in (*b*) of the figure fell over and how you might alter the design so that it wouldn't.

(c) Now replace the book that is modeling the bridge deck with a piece of cardboard or thick paper. Push straight down on the deck with your hand. Describe the behavior of the deck.

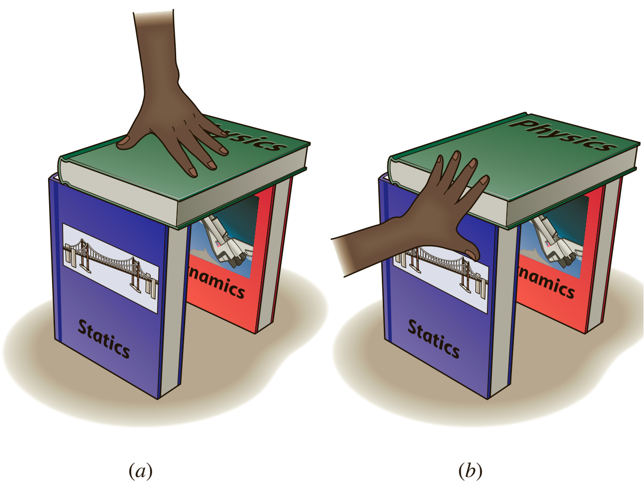


Figure SAE.2.1 Beam bridge modeled with books (*a*) loaded with a vertical force and (*b*) loaded with a horizontal force.

SAE.3 Exploring an Arch Bridge

An arch bridge is made up of a bridge deck supported by an arch that is connected at both ends to supports called abutments. The load on the bridge is transferred along the curve of the arch to the abutments. The arch bridge can span larger distances than a beam bridge (100 to 400 meters). To understand the load transfer in an arch bridge and the function of the abutments, you can build a model.

- You need a one-pint (or larger) container like the type used to package cottage cheese, sour cream, or delicatessen food.
- Cut the container in half along its diameter so that it makes two semicircular pieces.
- To complete the arch, cut off the bottom of the container and the stiffening ring (or lip) at the top (**Figure SAE.3.1*a***).

Load the arch by pressing down on the center as shown in **Figure SAE.3.1*b***.

(a) Is the arch in tension or compression? What happens to the ends of the arch?

To prevent the arch from collapsing, you must add abutments. To model your abutments, have a friend place her or his hands at the intersections of the arch and the ground as shown in **Figure SAE.3.1*c***. Again load the arch by pressing down on the center as shown in **Figure SAE.3.1*d***.

(b) How do the abutments affect how the ends of the arch move?

(c) Is the arch pushing or pulling on the abutments?

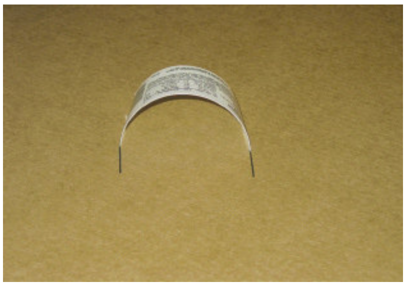
(d) What prevents the abutments from sliding?

(e) With the abutments in place, does the bridge provide more or less resistance to the push of your finger?

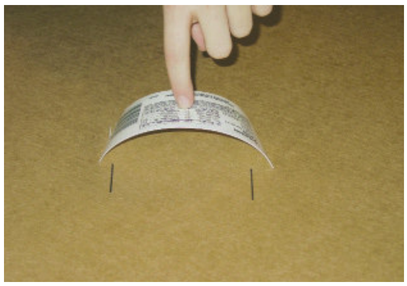
(f) Draw a free-body diagram showing all of the forces acting on one abutment.

(g) Review the analysis of the Golden Gate Bridge anchorage (**Figure E.2.9** and (E.4) to (E.9)) and explain how the weight of the arch bridge abutment is important in preventing it from sliding. To test your reasoning, model the abutments using something relatively lightweight such as CD-ROM cases. When you load the bridge, do the lightweight abutments move? Now press down on the lightweight abutments and load the bridge. Do the abutments move when the extra weight is on them?

(h) Are there any forces pulling up on the abutment? To determine the required weight of an arch bridge abutment, how would the analysis differ from the analysis of the suspension bridge anchorage?

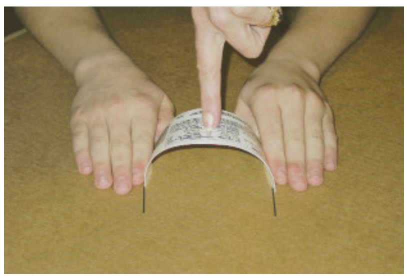


(*a*) Arch bridge made from cottage cheese container



(*b*) Loaded arch without abutments

(*c*) Abutments added to arch bridge



(*d*) Loaded arch with abutments

Jeffrey Koseff

Figure SAE.3.1

SAE.4 Exploring a Cable-Stayed Bridge

It is possible to confuse cable-stayed bridges with suspension bridges because both types use cables to hold up the bridge deck. However, the two bridge types have different mechanisms for transferring loads to the ground. In a suspension bridge, the main cables are draped over the tops of the towers and pull on the anchorages. Both the towers and the anchorages transfer the loads to the ground. In a cable-stayed bridge, the cables are attached directly to the towers and only the towers transfer the loads to the ground. As shown in **Figure SAE.4.1**, the cables, which are in tension, pull up on the deck and down on the tower. The tower, which is in compression, transfers a downward force to the ground.

To understand the load transfer in a cable-stayed bridge, we can build a model.[5] The cables can be attached to the towers in a number of patterns, but for our model we will use a fan pattern. In this pattern all of the cables are attached to the top of the tower, and then each cable is attached to a different point along the length of the bridge deck.

You need two pieces of string, one about 1.5 meters long and the other 2 meters long, and a partner to help you. Use your arms to model the bridge deck by holding both arms out horizontally to the side. You should be able to feel your muscles holding up your arms. Model the bridge tower with the trunk of your body and your head, and use the tower to support the cables that support the bridge deck. Have your partner tie the 1.5-m piece of string to each of your elbows, with the middle of the string lying on top of your head. The string acts as a stay-cable and holds your elbows up, but your hands and lower arms are hanging downward with little support. You should feel less stress on your muscles.

Have your partner tie the 2-m piece of string to each wrist, with the middle of the string lying on top of your head, making sure both strings are still taut. The bridge is now supported by two stay-cables, and your lower arms are also supported as shown in **Figure SAE.4.2**.

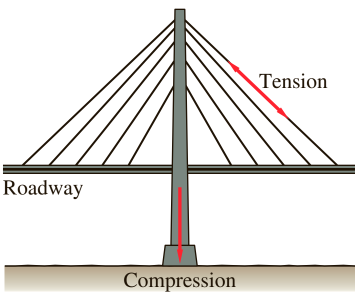


Figure SAE.4.1 Load path for cable-stayed bridge.

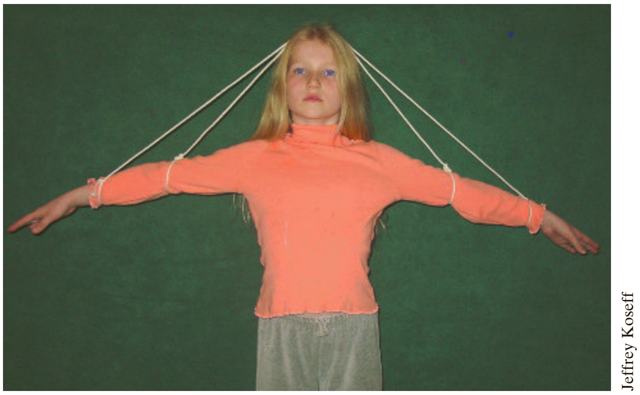


Figure SAE.4.2 Model of cable-stayed bridge.

Describe the load path of the cable-stayed bridge by answering these questions:

(a) What forces acting on the bridge deck are being transferred to the ground?

(b) Which components of the bridge are in tension?

(c) In which parts of your body do you feel a compression force? (**Figure SAE.4.1** does not show all of the compressive forces acting on the bridge.)

(d) How is the load on the top of your head transferred to the ground?

Assuming that the cables are attached to the deck at equal intervals and each carries an equal portion of the weight of bridge deck, the cables attached farther from the tower are subjected to larger tension forces than those attached closer to the tower. **Figure SAE.4.3** is an incomplete free-body diagram of a deck slice showing the force pulling on cable stay 1 and the weight of the slice.

(e) Using the variables shown in **Figure SAE.4.3**, convince yourself that the steeper cables, which are attached closer to the tower, are subjected to smaller tension forces. For this exercise, the cable number increases as you approach the tower.

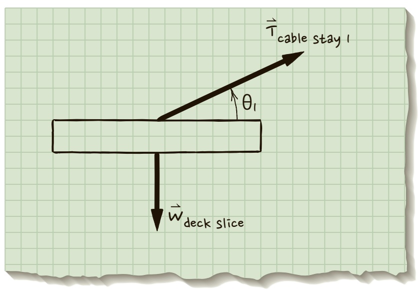


Figure SAE.4.3 Tension force on cable stay 1 and the weight of the bridge deck slice.

[5] Adapted from http://www.pbs.org/wgbh/nova/bridge/meetcable.html.

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