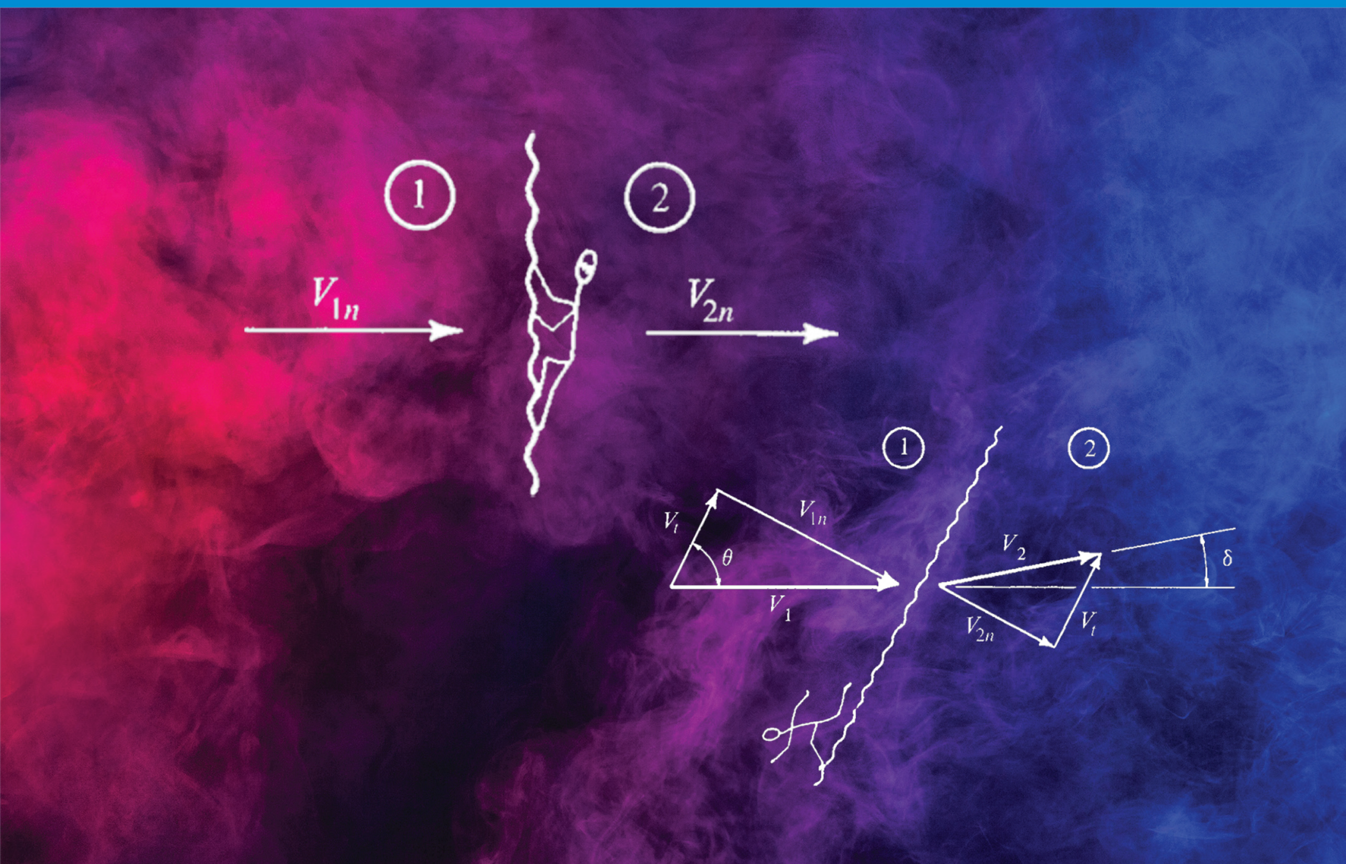
ROBERT D. ZUCKER | OSCAR BIBLARZ

FUNDAMENTALS OF GAS DYNAMICS

THIRD EDITION



with website

WILEY

FUNDAMENTALS OF GAS DYNAMICS

FUNDAMENTALS OF GAS DYNAMICS

Third Edition

DR. ROBERT D. ZUCKER†
DR. OSCAR BIBLARZ
Department of Mechanical and Aerospace Engineering
Naval Postgraduate School
Monterey, California, USA

WILEY

This edition first published 2020

Edition History
John Wiley & Sons Ltd (2e, 2002)

Registered Office
John Wiley & Sons, Inc., 111 River Street, Hoboken, NJ 07030, USA

Editorial Office
111 River Street, Hoboken, NJ 07030, USA

For details of our global editorial offices, customer services, and more information about Wiley products visit us at www.wiley.com.

Wiley also publishes its books in a variety of electronic formats and by print-on-demand. Some content that appears in standard print versions of this book may not be available in other formats.

Library of Congress Cataloging-in-Publication Data

Names: Zucker, Robert D., author. | Biblarz, Oscar, author.
Title: Fundamentals of gas dynamics / Robert D. Zucker, Oscar Biblarz.
Description: Third edition. | Hoboken : Wiley, 2020. | Includes index.
Identifiers: LCCN 2019024498 (print) | LCCN 2019024499 (ebook) | ISBN 9781119481706 (hardback) | ISBN 9781119481638 (adobe pdf) | ISBN 9781119481690 (epub)
Subjects: LCSH: Gas dynamics.
Classification: LCC QC168 .Z79 2020 (print) | LCC QC168 (ebook) | DDC 533/.2–dc23
LC record available at https://lccn.loc.gov/2019024498
LC ebook record available at https://lccn.loc.gov/2019024499

Cover design: Wiley
Cover image: Generated by book authors, © Stefano Buttafoco/Shutterstock

Set in 12/14pt Times by SPi Global, Pondicherry, India

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

Contents

Preface to Third Edition

A deliberate attempt has been made to retain the user-friendly and applications-oriented approach of previous editions. Beyond the added problems and examples, the following new applications are included: (1) the Shock Tube, (2) the Aerospike Nozzle, and (3) High-Energy Gas Lasers. Dimensional analysis and transonic flow concepts are presented but only succinctly discussed. We restrict all applications to below the *hypersonic range* to be consistent with our constant γ formulations, which also imply no molecular dissociation or chemical reactions for flows outside of the combustor region. Finally, with a few notable exceptions, all flows are taken to be steady and one-dimensional.

For this edition, all appendixes from the second edition have been retained. We have kept Appendixes G through J because oftentimes or at some locations, suitable electronic devices may be unavailable even though many students now are conversant and have access to programmable electronic calculators (hand-held, laptop, or table-top); also, in classroom or problem-session settings, it would be most unusual to have everyone with the same type of calculator. Furthermore, unlike graphs, tables show sufficient significant figures for meaningful answers in certain types of problems and tables may (with some experience) help to give the "big picture." All sketches and figures in the text have also been retained because they (and *T–s* diagrams in particular) are excellent learning tools.

We furthermore keep physical descriptions in both the EE and SI systems of units because the former is still used widely in the United States—most instructors are comfortable with them, and many users in the industry and laboratories have developed an intuitive sense for the magnitudes of the parameters and/or for the characteristic sizes of existing equipment. Moreover, the Gravitational System of units has wedged itself into the SI System with the kilogram-force (kgf) unit used

in Europe for gas–pressure measurements (its relevant g_c is given in Section 1.2). In Chapter 1, the previous Section 1.3 has been replaced with a new one that gives information on relevant fundamentals behind all systems of units and reasons why dimensionless quantities are so pervasive in gas dynamics—while real problems must deal with one or another system of units, nearly all graphing and tabulations in this book relate dimensionless quantities that are shown to be more general and more useful. Many examples in the book will show final answers given in both set of units because enough readers seem to have a preference for one or the other. Answers to Problems in the back of the book remain in either SI or EE. Appendixes K and L are in EE but are also available in SI (Keenan, J. H., Chao, J., and Kaye, J., *Gas Tables International Version*, 2nd Ed., Wiley, New York, 1983) or are posted on appropriate websites.

For this third edition, I would like to acknowledge the many helpful suggestions of Professor Garth Hobson. Mr. A. I. Biblarz developed our *Gasdynamics Calculator* as a direct companion to this book (available at https://www.oscarbiblarz.com/gascalculator), which can run on any electronic device that has a web browser; for perfect gases this calculator reproduces Appendixes G through J and includes software to calculate Oblique Shocks without the aid of the charts in Appendix D, having the Normal Shock formulations built-in. With the *Gasdynamics Calculator*, the user avoids any need for interpolation or extrapolation, and further manipulations needed to arrive at the desired final answers are easily done with any standard calculators. In addition to being a labor-saving aid, this calculator outputs some functions not tabulated in the appendixes and enables the user to conveniently tackle many problems that require more realistic values of γ (without defaulting to 1.40).

This third edition has been prepared by Oscar Biblarz. The book's organization and educational objectives introduced in previous editions have been retained—Bob Zucker, who passed away in 2011, strongly believed that rendering the logic of an engineering subject such as gas dynamics with relevant aspects of *educational technology* would appeal to beginning students as well as others interested in the field, and time has proven him right.

Monterey, CA, USA OSCAR BIBLARZ

Preface to Second Edition

This book is written for students in engineering and the physical sciences who want to learn the fundamentals of gas dynamics. It aims at the upper undergraduate level and thus requires a minimum of prerequisites. The writing style is informal and incorporates ideas in educational technology such as behavioral objectives, meaningful summaries, and check tests. Such features make this book well suited for self-study as well as for conventional course presentation. Sufficient material is included for a typical one-quarter or one-semester course, depending on the student's background.

Our approach in this book is to develop all basic relations on a rigorous basis with equations that are valid for the most general case of the unsteady, three-dimensional flow of an arbitrary fluid. These relations are then simplified to represent meaningful engineering problems for one- and two-dimensional steady flows. All basic internal and external flows are covered with practical applications which are interwoven throughout the text. Attention is focused on the assumptions made at every step of the analysis; emphasis is placed on the usefulness of T–s diagrams and the significance of any relevant loss terms.

Examples and problems are provided in both the English Engineering and SI systems of units. Homework problems range from the routine to the complex, with all charts and tables necessary for their solution included in the Appendixes.

The goals for the user should be not only to master the fundamental concepts but also to develop good problem-solving skills. After completing this book the student should be capable of pursuing the many references that are available on more advanced topics.

Professor Oscar Biblarz joined Robert D. Zucker as coauthor in the second edition. We have both taught gas dynamics from the first edition of this book for

many years. We both shared in the preparation of the new manuscript and in the proofreading. This second edition has been expanded to include (1) some material on conical shocks, (2) several sections showing how computer calculations can be helpful, and (3) an entire chapter on real gases, including simple methods to handle these problems. These topics have made the book more complete while retaining its original purpose and style.

We have gratefully acknowledged the help of Professors Raymond P. Shreeve and Garth V. Hobson of the Turbopropulsion Laboratory at the Naval Postgraduate School, particularly in the propulsion area. We also have mentioned that our many students throughout the years have provided the inspiration and motivation for preparing this material. In particular, for the first edition, we acknowledged Ernest Lewis, Allen Roessig, and Joseph Strada for their contributions beyond the classroom. We would also like to thank the Lockheed-Martin Aeronautics Company, General Electric Aircraft Engines, Pratt & Whitney Aircraft, the Boeing Company, and the National Physical Laboratory in the United Kingdom for providing photographs that illustrate various parts of the book. John Wiley & Sons should be recognized for understanding that the deliberate informal style of this book makes it a more effective teaching tool.

Professor Zucker owed a great deal to Newman Hall and Ascher Shapiro, whose books provided his first introduction to the area of compressible flow. Also, he thanked his wife, Polly, for sharing this endeavor with him for a second time.

Pebble Beach, CA ROBERT D. ZUCKER

Monterey, CA OSCAR BIBLARZ

To the Student

You do not need much background to enter the fascinating world of gas dynamics. However, it will be assumed that you have been exposed to college-level courses in calculus and thermodynamics. Specifically, you are expected to know the following:

1. Simple differentiation and integration, logarithms, and series expansions
2. The meaning of a partial derivative
3. Vectors and the significance of a dot product
4. How to draw and interpret free-body diagrams
5. How to resolve a force or other vector into its components
6. Newton's second law of motion and the units related to force and mass
7. About properties of fluids, particularly perfect gases
8. The zeroth, first, and second laws of thermodynamics

The first six prerequisites are very specific; the last two cover quite a bit of territory. In fact, a background in thermodynamics is so important to the study of gas dynamics that a review of the necessary concepts for control mass analysis is contained in Chapter 1. If you have recently completed a course in thermodynamics, you may skip most of Section 1.4, but you should *read the questions* at the end of the chapter. If you can answer these, press on! If any difficulties arise, refer back to the material in the chapter. Many of these equations will be used throughout the rest of the book. You may even gain more confidence by working some of the review problems in Chapter 1. In this third edition, Section 1.3 is new; it covers the often-unstated background for

why we prefer to use nondimensional quantities in gas dynamics and requires reading.

In Chapters 2 and 3 we convert the fundamental laws into a form needed for control volume analysis. If you have had a good course in fluid mechanics, much of this material should be familiar to you. A section on constant-density fluids is included to show the general applicability in that field and to tie in with any previous work that you have done in this area. If you have not studied fluid mechanics, do not worry. All the material that you need to know in this field is included. Because several special concepts are developed that are not treated in many thermodynamics and fluid mechanics courses, *read these chapters even if you have the relevant background*. They introduce the notation used, form the backbone of gas dynamics, and are referred to frequently in later chapters.

In Chapter 4, you are introduced to the characteristics of compressible fluids. Then in the following chapters, various basic flow phenomena are analyzed one by one: varying area, normal and oblique shocks, supersonic expansions and compressions, duct friction, and heat transfer. A wide variety of practical engineering problems can be solved using these concepts, and many of these problems are covered throughout the text. Examples of these include the off-design operation of supersonic nozzles, supersonic wind tunnels, blast waves and shock tubes, supersonic airfoils, some methods of flow measurement, and the choking from either friction or thermal effects. You will find that supersonic flows bring about special problems in that they do not seem to follow your intuition. In Chapter 11, you will be exposed to what goes on at the molecular level and a discussion of a laser based purely on gas dynamics. You will see how molecular structure at high temperatures affects real gas behavior and learn some relatively simple techniques to handle these situations.

Aircraft propulsion systems (with their air inlets, afterburners, and exit nozzles) represent a noteworthy application of nearly all the basic gas dynamics flow situations. Thus, in Chapter 12, we describe and analyze common air-breathing propulsion systems, including turbojets, turbofans, and turboprops. Other propulsion systems, such as rockets, ramjets, and pulsejets, are also covered.

A number of chapters contain material that shows how to use computer software in certain calculations. The aim is to indicate how particular software might be applied as a means of getting answers by using the same equations that could be worked manually or through the tables. The computer utility MAPLE is our choice, but if you have not studied it, do not worry. All the gas dynamics is presented in the sections preceding such applications so that all computer sections may be omitted. For students who have access to the "Word Wide Web," a Gasdynamics Calculator has been developed that can reproduce Appendices D, G, H, I, and J for all applicable values of γ. Access to a Web browser will allow students to view complementary figures and photographs that are presented in this book.

This book has been especially written for you, the student. We hope that its informal style will put you at ease and motivate you to keep reading on. Student comments on the previous editions indicate that this objective has been accomplished. Once you have read the first chapter, the remaining chapters follow a similar format. The following suggestions may help you optimize your study time. When you start each chapter, read the Introduction, as this will give you the general idea of what the chapter is all about. The next section contains a set of learning objectives (beginning with Chapter 2). These tell exactly what you should be able to do after completing the chapter successfully. Some objectives are marked *optional*, as they are only for the most serious students. Merely scan the objectives, as they would not mean much at first. However, *they will indicate important things to look for*. As you read the material, you may occasionally be asked to do something—complete a derivation, fill in a chart, draw a diagram, etc. Make an honest attempt to follow these instructions before proceeding further. You will not be asked to do something that you have not the background to do, and your active participation will help solidify important concepts and provide feedback on your progress.

As you complete each section, look back to see if any of the enumerated objectives have been covered. If so, make sure that you can do them. Write out the answers; these will help you in later studies. You may wish to make your own summary of important points in each chapter, and then see how well it agrees with the summary provided. After having worked a representative group of problems, you are ready to check your knowledge by taking the test at the end of the chapter. This should always be treated as a closed-book affair, with the exception of tables and charts in the appendices. If you have any difficulties with the check test, you should go back and restudy the appropriate sections. Do not proceed to the next chapter without completing the previous one satisfactorily.

Not all chapters are of the same length, most of them being a little long to tackle all at once. You might find it easier to break them into "bite-sized" pieces according to the Correlation Table that follows. Work on some problems on the first group of objectives and sections *before* proceeding. You should spend time on *each* study session working through the material. Learning can be fun; however, knowledge does not come free. We hope that this book will make the task of exploring gas dynamics more doable and more enjoyable.

Correlation Table for Sections, Objectives, and Problems

Chapter	Sections	Optional Section	Objectives	Optional Objectives	Problems	Optional Problem
1	1–3				Q: 1–8	Q: 1–9 P: 1–6
	4				Q: 10–34	
	4				P: 1–5	
2	1–5		1–3, 5	4	1–3, 5–6	4
	6		7, 9	6, 8	7–15	
3	1–7		1–9		1–14	
	8		10–12		15–22	
4	1–6		1–10		1–17	
5	1–6		1–7		1–8	
	7–10	11	8–12		9–23	24
6	1–5		1–7		1–6	
	6–8	9	8–10		7–19	
7	1–3		1–2		1–5	
	4–8		3–9	5	6–17	
	9–10	11	10–12		18–21	22
8	1–5		1–6	5	1–6	
	6–9	10	7–9		7–19	
9	1–6		1–7	2, 5	1–12	
	7–8, 10	9, 11	8–9, 11	10	13–22	23
10	1–6		1–7	2, 6	1–8	
	7–9	10	8–9, 11	10	9–21	22
11[a]	1–5		1–7		1–10	
	6–8		8	9	11–16	
12[a]	1–3		1–4		1–5	
	4–7		5–11	8, 9, 11	6–15	
	8–9		12–15	14	16–25	

[a] With very few exceptions, for more realistic answers, Problems in Chapters 11 and 12 can be worked or reworked using the *Gasdynamics Calculator* that accompanies this third edition without any need for Appendices D or G through J. To ascertain appropriate values of γ, use Appendices A and B, Appendix L, or Figure 11.2 (or related references) for the gases and temperatures specified.

About the Companion Website

This book is accompanied by a companion website:
www.wiley.com/go/zucker/gas

The website includes:

- Solution Manuals

Chapter 1

Definitions and Fundamental Principles

1.1 INTRODUCTION

It is assumed that before entering the world of gas dynamics you have had a reasonable background in mathematics (through calculus) together with a course in elementary thermodynamics. An exposure to basic fluid mechanics would be helpful but is not absolutely essential. The concepts used in fluid mechanics are relatively straightforward and can be developed as we need them. On the other hand, some of the concepts of thermodynamics are more abstract, and we must assume that you already understand the fundamental laws of thermodynamics as they apply to stationary systems. The extension of these laws to flow systems is so vital that we cover these systems in depth in Chapters 2 and 3.

This chapter is not intended to be a formal review of the courses noted above; rather, it should be viewed as a collection of the basic concepts and facts that will be used later. It should be understood that a great deal of background is omitted in this review and no attempt is made to prove each statement. Thus, if you have been away from this material for any length of time, you may find it necessary occasionally to refer to your notes or other textbooks to supplement this review. At the very least, the remainder of this chapter may be considered an assumed common ground of knowledge from which we shall venture forth.

At the end of this chapter a number of questions are presented for you to answer. No attempt should be made to continue further until you feel that you can answer a majority of these questions satisfactorily.

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

1.2 UNITS AND NOTATION

Dimension: a qualitative definition of a physical entity (such as time, length, force)
Unit: the appropriate magnitude of a dimension (such as seconds, feet, newtons)

In the United States much work in the area of thermo-gas dynamics (particularly in propulsion) continues to be done in the English Engineering (EE) system of units. However, most of the world is operating in the metric or International System (SI) of units. Thus, we shall use and review both systems, beginning with Table 1.1.

Variables

The equation

$$y = f(x) \tag{1.1}$$

indicates that a functional relation exists between the variables x and y. Further, it denotes that

x is the *independent variable,* whose value can be given any place within an appropriate range.
y is the *dependent variable,* whose value is fixed once x has been selected.

In most cases it is possible to interchange the dependent and independent variables and write

$$x = f(y) \tag{1.2}$$

Table 1.1 Systems of Units

	Basic Unit Used	
Dimension	English Engineering (EE)[a]	International System (SI)
Time	second (sec)	second (s)
Length	foot (ft)	meter (m)
Force	pound force (lbf)	newton (N)
Mass	pound mass (lbm)	kilogram (kg)
Temperature	Fahrenheit (°F)	Celsius (°C)
Absolute Temperature	Rankine (°R)	kelvin (K)

[a] *Caution:* Never say *pound*, as this is ambiguous. It is either a *pound force* or a *pound mass*. Only for mass at the Earth's surface is it unambiguous, because here a pound mass weighs a pound force.

Frequently, a variable will depend on more than one other variable. One might write

$$P = f(x, y, z) \tag{1.3}$$

indicating that the value of the dependent variable P is fixed once the values of the independent variables x, y, and z are selected.

Maximum and Minimum

If a plot is made of the functional relation $y = f(x)$, *maximum* and/or *minimum* points may be exhibited. At these points $dy/dx = 0$. If the point is a maximum, d^2y/dx^2 will be negative, whereas if it is a minimum, d^2y/dx^2 will be positive.

Force and Mass

In all systems of units, mechanical force and mass are related through *Newton's second law of motion*, which states that

$$\sum \mathbf{F} \propto \frac{d\left(\overrightarrow{\text{momentum}}\right)}{dt} \tag{1.4}$$

The proportionality factor is expressed as $1/g_c$, and thus

$$\sum \mathbf{F} = \frac{1}{g_c} \frac{d\left(\overrightarrow{\text{momentum}}\right)}{dt} \tag{1.5}$$

For an inertial mass that does not change with time, this becomes

$$\sum \mathbf{F} = \frac{m\mathbf{a}}{g_c} \tag{1.6}$$

where $\sum \mathbf{F}$ is the vector force summation acting on the inertial mass m and $\mathbf{a}$ is the vector acceleration of the mass.

In the EE system, we use the following definition:

A 1-pound force will give a 1-pound mass an acceleration of 32.174 ft/sec^2.

With this definition, we have

$$1 \text{ lbf} = \frac{1 \text{ lbm} \cdot 32.174 \text{ ft/sec}^2}{g_c}$$

and thus

$$g_c = 32.174 \frac{\text{lbm-ft}}{\text{lbf-sec}^2} \tag{1.7a}$$

Note that g_c is *not* the standard gravity (check the units). It is a proportionality factor whose value depends on the units being used. In further discussions, we shall take the numerical value of g_c to be 32.2 when using the EE system.

In other engineering fields of endeavor, such as statics and dynamics, the British Gravitational system (also known as the US customary system) is used. This is very similar to the EE system except that the unit of mass is the slug.

In this system of units, we follow the definition:

A 1-pound force will give a 1-slug mass an acceleration of 1 ft/sec^2.

Using this definition, we have

$$1 \text{ lbf} = \frac{1 \text{ slug} \cdot 1 \text{ ft/sec}^2}{g_c} \tag{1.7b}$$

and thus

$$g_c = 1 \frac{\text{slug-ft}}{\text{lbf-sec}^2}$$

Since g_c has the numerical value of unity, most authors drop this factor from the equations in the British Gravitational system. Consistent with the thermodynamics approach, we shall not use this system here. Comparison of the Engineering and Gravitational systems shows that 1 slug $\equiv$ 32.174 lbm.

In the SI system we use the following definition:

A 1-N force will give a 1-kg mass an acceleration of 1 m/sec^2.

Now equation (1.6) becomes

$$1 \text{ N} = \frac{1 \text{ kg} \cdot 1 \text{ m/s}^2}{g_c}$$

and thus

$$g_c = 1 \frac{\text{kg} \cdot \text{m}}{\text{N} \cdot \text{s}^2} \tag{1.7c}$$

Since g_c has the numerical value of unity (and uses the dynamical unit of mass, i.e., the kilogram) most authors omit this factor from equations in the SI system. However, we shall leave the symbol g_c in the equations so that you may use any system of units with less likelihood of making errors. Similar to the lbm/lbf notation, a gravitational-metric unit of force, the kilogram-force (kgf), is used in some countries outside the United States ($1\ \text{kgf} \equiv 9.807\ \text{N}$ so that here $g_c = 9.807\ \text{kg} \cdot \text{m/kgf} \cdot \text{s}^2$).

Density and Specific Volume

Density is the mass per unit volume and is given the symbol ρ. It has units of lbm/ft^3, kg/m^3, or slug/ft^3.

Specific volume is the volume per unit mass and is given the symbol $\boldsymbol{v}$. It has units of ft^3/lbm, m^3/kg, or ft^3/slug. Thus

$$\rho = \frac{1}{\boldsymbol{v}} \tag{1.8}$$

Specific weight is the weight (due to the gravity force) per unit volume and is customarily given the symbol γ. If we take a unit volume under the influence of gravity, its weight will be γ. Thus, from equation (1.6) we have

$$\gamma = \rho \frac{g}{g_c} \ \text{lbf/ft}^3 \quad \text{or} \quad \text{N/m}^3 \tag{1.9}$$

Note that mass, density, and specific volume *do not* depend on the value of the local gravity. Weight and specific weight *do* depend on gravity through the gravitational mass. We shall not refer to specific weight in this book; it is mentioned here only to distinguish it from density. Thus the symbol γ will be used for another purpose [see equation (1.39)].

Pressure

Pressure is the normal force per unit area and is given the symbol p. It has units of lbf/ft^2 (psf) or N/m^2. Several other units are commonly used, such as the pound per square inch (psi; lbf/in^2), the megapascal (MPa; 1×10^6 N/m^2), the bar (1×10^5 N/m^2), and the atmosphere (14.69 psi or 0.1013 MPa). Also in use are the kPa (1×10^3 N/m^2) and the kgf/m^2 (9.807 N/m^2). Often, we add an "a" to emphasize absolute pressures (psfa, psia).

Absolute pressure is measured with respect to a perfect vacuum.

Gage pressure is measured with respect to the surrounding (ambient) pressure:

$$p_{\text{abs}} = p_{\text{amb}} + p_{\text{gage}} \tag{1.10}$$

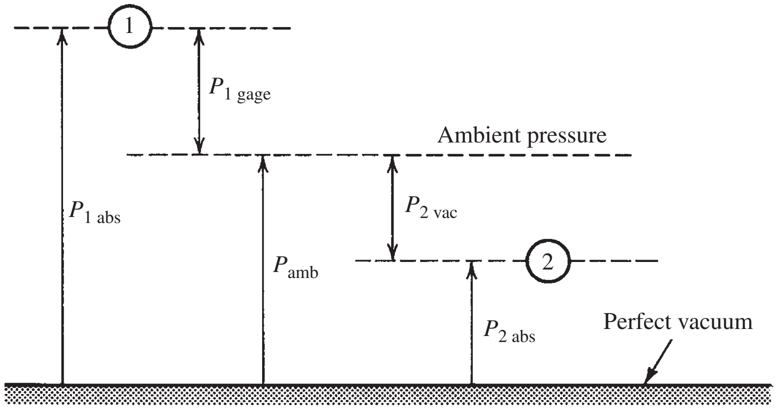


Figure 1.1 Absolute and gage pressures.

When the gage pressure is negative (i.e., the absolute pressure is below ambient) it is usually called a (positive) vacuum reading:

$$p_{abs} = p_{amb} - p_{vac} \tag{1.11}$$

Two pressure readings are shown in Figure 1.1. Case 1 shows the use of equation (1.10) and case 2 illustrates equation (1.11). It should be noted that the surrounding (ambient) pressure does not necessarily have to correspond to standard atmospheric pressure. However, when *no* other information is available, one has to assume that the surroundings are at 14.69 psi or 0.1013 MPa. Most often, equations require the use of absolute pressure, and for ambient pressure, we often use a numerical value of 14.7 when using the EE system and 0.1 MPa (1 bar) when using the SI system.

Temperature

Degrees Fahrenheit (or Celsius) can safely be used only when *differences* in temperature are involved. However, most equations require the use of absolute temperature in Rankine (or kelvin).

$$^\circ\mathrm{R} = {}^\circ\mathrm{F} + 459.67 \tag{1.12a}$$

$$\mathrm{K} = {}^\circ\mathrm{C} + 273.15 \tag{1.12b}$$

The values 460 and 273 will be often used in our calculations.

Viscosity

We shall be dealing with *common fluids,* which are defined as follows:

A common fluid is any substance that will continuously deform when subjected to a shear stress.

Thus the amount of deformation is of no significance (as it is with a solid), but rather, the *rate of deformation* is characteristic of each individual fluid and is indicated by the *viscosity:*

$$\text{Viscosity} \equiv \frac{\text{Shear stress}}{\text{Rate of angular deformation}} \tag{1.13}$$

Viscosity, sometimes called *absolute viscosity,* is given the symbol μ and has the unit lbf-sec/ft^2 or N·s/m^2.

For most common fluids, because viscosity is a function of the fluid, it varies with the fluid's state. Temperature has by far the greatest effect on viscosity, so most charts and tables display only this variable. Pressure has a slight effect on the viscosity of gases but a negligible effect on liquids.

A number of engineering computations use a combination of (absolute) viscosity and density. This *kinematic viscosity* is defined as

$$\nu \equiv \frac{\mu g_c}{\rho} \tag{1.14}$$

Kinematic viscosity has the unit ft^2/sec or m^2/s. We shall see more regarding viscosity in Chapter 9 when we deal with flow losses caused by duct friction.

Equation of State

In most of this book we consider all liquids as having constant density and all gases as following the perfect gas equation of state. Thus, for liquids we have the relation

$$\rho = \text{constant} \tag{1.15}$$

The perfect gas equation of state is derived from kinetic theory and neglects molecular volume and intermolecular forces. Thus it is accurate under conditions of relatively low density, which correspond to relatively low pressures and/or high temperatures. The form of the *perfect gas equation* normally used in gas dynamics is

$$p = \rho RT \tag{1.16}$$

where

$p \equiv$ absolute pressure	lbf/ft^2	or	N/m^2
$\rho \equiv$ density	lbm/ft^3	or	kg/m^3
$T \equiv$ absolute temperature	°R	or	K
$R \equiv$ *individual* gas constant	ft-lbf/lbm-°R	or	N·m/kg·K

The *individual* gas constant is found in the EE system by dividing 1545 by the molecular mass (MM) of the gas chemical constituents. In the SI system, R is found by dividing 8314 by the molecular mass. More exact numbers are given in Appendixes A and B. (The molecular mass has the same numerical value in lbm/lbmol or kg/kgmol.)

Example 1.1
The (effective) molecular mass of air is 28.97, what is its R?

$$R = \frac{1545}{28.97} = 53.3 \text{ ft-lbf/lbm-°R} \quad \text{or} \quad R = \frac{8314}{28.97} = 287 \text{ N·m/kg·K}$$

Example 1.2
Compute the density of air at 50 psia and 100°F.

$$\rho = \frac{p}{RT} = \frac{(50)(144)}{(53.3)(460+100)} = 0.241 \text{ lbm/ft}^3 \quad \left(\text{or } 3.86 \text{ kg/m}^3\right)$$

Properties of selected gases are given in Appendixes A and B. In many examples in this book we use EE units. However, there are numerous examples and problems in SI units. Some helpful conversion factors are also given in Appendixes A and B. You should become familiar with solving problems in both systems of units.

In Chapter 11 we discuss nonperfect or real gases and show how these may be handled. The simplifications that the perfect gas equation of state brings about are not only extremely useful but also accurate for ordinary gases because in most gas dynamics applications, low temperatures exist with low pressures and high temperatures with high pressures. In Chapter 11 we shall see that deviations from perfect-gas behavior become particularly important at high temperatures or at high pressures.

1.3 WHY WE USE NONDIMENSIONAL QUANTITIES

In this section we invite you to thumb through the rest of the book as we highlight details of what makes our gas dynamics "equation-packaging" different from many other engineering disciplines. We do this by introducing some properties of nondimensional quantities.

In Section 1.2 we introduced the subject of units and dimensions but as we look ahead (particularly in the Appendixes) all charts and Appendixes G to J are given in nondimensional quantities (recall that angles are already dimensionless—what is the difference between degrees and radians?). One reason for this is that, when properly done, dimensionless groups are more general and can therefore be more useful than their dimensional counterparts. Take for example the Mach number, equation (4.11), which is defined as the ratio of the gas velocity (V) divided by the speed of sound (a) of the medium [equation (4.5) or (4.10)]; the units drop out when a consistent set is used and we have devised a very useful variable at any given flow location, one that governs most of what we study in compressible flows. The Mach number combines three variables into one—in Chapter 4, you will be able to calculate that helium moving at 562 m/s and 800°C has the same Mach number as air at 100 m/s and 20°C (would the resulting Mach number change if we had calculated with equivalent numbers in the EE system of units?). You will further learn in Chapter 5 how and why the Mach number can continuously increase in nozzles and decrease in diffusers (see Figures 5.2 and 5.3).

As stated in Section 1.2, we use either the SI (metric) or the EE systems of units. We consider g_c as a proportionality factor between mechanical and other systems of units, that is, for those particularly involving the lbm and the kgf—another such proportionality factor J will be needed in equation (1.17) to convert from caloric to mechanical units, see equation (1.40). Measurements are necessary to physically describe any object or process, and every equation we use must be dimensionally homogeneous; but it is also true that physical laws need to be independent of any set of measurement units employed so nondimensional representations can be more intrinsic and often allow simplifications based on order of magnitude. As we absorb the problem's constants and other nonchanging parameters into its variables, dimensionless equations become more compact (can you see why?) than their original forms and more universal (no system of units), that is, they result in equations with reduced numbers of parameters, tailor-made for the problem under scrutiny. Such usefulness also applies to tabular and graphical displays.

Dimensionless Entities

Nondimensional quantities include both dimensionless parameters and dimensionless variables. Gas dynamics always involves three kinds of these quantities: universal constants like Pi (3.14159) or the Napierian logarithmic base (2.71828), dimensionless numbers like the Mach number (M) or Reynolds (Re) number, and the third kind being regular variables, for example, normalized one-dimensional isentropic variables like the area ratio ($A/A*$) or the temperature ratio (T/T_t). In addition to having no units, many dimensionless variables are deliberately normalized to be of *order one* (so that their values are close to unity over the range of interest) by appropriate scaling. With such carefully defined dimensionless entities we can more

easily distinguish physical domains and/or focus on flow regimes of importance. *Dimensional analysis* is the name of the discipline that deals with such concepts; this discipline is also used in the generalization of limited experimental results to other conditions and often permits the most compact graphical representations.

The dimensionless *numbers* most commonly utilized in gas dynamics are the Mach number (M) introduced in equation (4.11), the ratio of specific heats (γ) introduced in equation (1.39), the Reynolds number (Re) introduced in equation (9.48), the friction factor f introduced in equations (3.60) and (9.47), and the compressibility factor Z introduced in Chapter 11, together with some often-listed others like the Prandtl number (Pr) and the Knudsen number (Kn). Below, we briefly discuss this list together with some relevant details:

1. Mach number is the most important—$0 < M \leq 0.3$ is incompressible flow because the density remains practically constant, $0.3 < M < 1.0$ is subsonic compressible, $M \approx 1.0$ is transonic, $1.0 < M \leq 5.0$ is supersonic, and $M > 5.0$ is hypersonic because gas properties begin to change in polyatomic gases (including dissociation) from heating that arises from wall friction and from strong shocks.
2. γ is the next most important—it reflects "activated molecular structures" and varies with gas temperature in gases like air; its range is limited to $1.0 < \gamma \leq 1.67$ or 5/3 (see Appendixes A and B, and Chapter 11). In spite of this limited range, its variation is important because γ most often appears as an exponent.
3. Re conveys effects of flow viscosity such as friction, drag, and flow separation—for internal flows, the viscous force is proportional to an inside "wetted area," see equation (3.63). For external and developing internal flows, we use the symbol Re_L, where L is length, and use Re_D, where D is diameter, in fully developed tube flows. Turbulent duct flows where $\mathrm{Re}_\mathrm{D} > 3 \times 10^3$ are more *one-dimensional* than laminar where $\mathrm{Re}_D < 3 \times 10^3$, see Sections 2.3, 9.5, and Figures 2.1 and 2.2.
4. f—friction factor for viscous flow inside tubes, see equation (3.60) and Chapter 9.
5. Z—compressibility factor reflects deviations from the perfect gas law (introduced in Chapter 11), $Z \approx 1.0$ for many gases of interest.
6. Pr—(*optional*) the Prandtl number is the ratio of momentum to thermal diffusivities (in boundary layer development), $\mathrm{Pr} \leq 1.0$ for all perfect gases.
7. Kn—(*optional*) Knudsen number $\approx M/\mathrm{Re}^{1/2}$; it is the average distance gas particles travel before colliding with each other (or their mean-free-path) divided by a meaningful characteristic distance of the flow field such as an enclosure length.

In our gas dynamics only certain normalized *variables* are made to be of *order one;* the pressure, density, and temperature ratios in isentropic flows (Appendix

G) vary between zero and one because they are referenced to their stagnation condition (or highest value). The area ratios are greater or equal to one because they are referenced to the throat or minimum area (A^*) where $M = 1.0$ as shown in Figure 5.14. For normal shocks (Appendix H) properties are always normalized across the shock. And for both Fanno (Appendix I) and Rayleigh (Appendix J) flows, all ratios are referenced to two separate *-values where $M = 1.0$, at their respective entropy inflection points where the entropy change function S_{max}/R (or $(s^* - s)/R$) tends to zero; note that the *-values are different for Fanno and Rayleigh flow conditions (refer to Chapters 9 and 10 for details). All equations used to generate the values in Appendixes G through J need to be of the form *variable-ratio* $= f(M, \gamma)$. All charts in the appendixes are consistently nondimensional—the friction factor is shown versus Reynolds number (Appendix C); oblique-shock charts and conical-flow shock charts (Appendixes D and E) show relevant angles, Mach numbers, and static pressure ratios; and the Compressibility Factor chart (Appendix F) displays only dimensionless factors. All this may seem unnecessarily complicated now but rest assured that each reference value has been deliberately picked for maximum usefulness.

Dimensional Analysis (Optional)

Any system of units we use must contain all necessary physical dimensions and the discipline of dimensional analysis manipulates appropriate subsets of these. For example, looking at the momentum flux in equation (1.4), the physical quantities that represent momentum are the product of the inertial mass and its velocity. Velocity is a vector (having both magnitude and direction) and is made up of the time rate of change of the length in the flow direction, which in one-dimensional flow is often the Cartesian coordinate x, see Section 2.3 and Figure 8.13; the magnitude of the velocity needs to be based on an identifiable frame of reference. Ordinary space has three independent lengths, so the other two must occupy an area perpendicular to the flow direction and, in most general cases, we would deal with three *lengths* (L_x, L_y, L_z) together with the inertial *mass* (m) and the *time* (t). For thermal problems, we need to add the absolute *temperature* (T) because while heat is also a form of energy as shown by the *first law* [equation (1.17)] and may have units of Joules or ft-lbf (or Btus or calories), it always carries an entropy content whereas pure work has no entropy; thus, heat transfer is always associated with a temperature difference, T, an independent parameter [see *second law*, equation (1.28)]. In equation (1.4) at least three but up to six (three Ls, m, t, and T) dimensional parameters are necessary for nondimensional manipulations (in the case of symmetry, as with frictional flow inside circular ducts, we use only two lengths, x and r, see Section 9.9). Thus, dimensional analysis is more than just "making the units cancel out." We shall implicitly take inertial mass, gravitational mass, and molecular mass to be one and the same.

Beyond the gas dynamics presented in this book, applications of dimensional analysis have encompassed experimental techniques like *similitude* (geometrical similarity in testing) and *modeling* (the use of smaller specimens for testing in laboratory wind tunnels), as well as *approximation theory* (the development of approximate analytical solutions to complex sets of equations). For more details on dimensional analysis, consult References 1 and 2 as well as References 17, 19, and 20 under Gas Dynamics.

1.4 THERMODYNAMIC CONCEPTS FOR CONTROL MASS ANALYSIS

We apologize for the length of this section, but a good understanding of thermodynamic principles is essential to a study of gas dynamics.

General Definitions

Microscopic approach: deals with individual molecules, and with their motion and behavior, on a statistical basis. It depends on our understanding of the structure and behavior of molecular matter at the atomic level. This approach involves quantum mechanics. In this book we only utilize the microscopic approach in Section 11.3

Macroscopic approach: deals directly with the average behavior of molecules through observable and measurable properties (temperature, pressure, etc.). This classical approach involves no special assumptions regarding the molecular structure of matter and is based on the concept of a "continuum postulate" defined through the Knudsen number as $\text{Kn} < 0.01$. The macroscopic approach is used in this book throughout.

Control mass: a fixed quantity of mass that is being analyzed. It is separated from its surroundings by a boundary. A control mass is also referred to as a *closed system*. Although matter does not cross the boundary, energy may enter or leave the system.

Control volume: a region of space that is being analyzed. The boundary separating it from its surroundings is called the *control surface*. Matter as well as energy may cross the control surface, and thus a control volume is also referred to as an *open system*. We mostly use the control volume analysis and introduce it in Chapters 2 and 3.

Properties: characteristics that describe the state of a system, any quantity that has a definite value for each distinct state of a system (e.g., pressure, temperature, color, entropy). They have dimensions or similar attributes that relate to their numerical value.

Intensive property: depends only on the state of a system and is independent of its mass (e.g., temperature, pressure, specific volume).

Extensive property: depends on the mass of a system (e.g., internal energy, volume).

Types of properties:

1. *Observable*: readily or commonly measured (pressure, temperature, velocity, mass, etc.)
2. *Mathematical*: defined from combinations of other properties (density, specific heats, enthalpy, etc.)
3. *Derived*: arrived at as the result of analysis
   a. *Internal energy* (from the first law of thermodynamics)
   b. *Entropy* (from the second law of thermodynamics)

State change: comes about as the result of a change in any property.

Path or process: represents a series of consecutive states that define a unique path from one state to another. Some special processes often used:

Adiabatic	$\rightarrow$	no heat transfer
Isothermal	$\rightarrow$	$T = \text{constant}$
Isobaric	$\rightarrow$	$p = \text{constant}$
Isentropic	$\rightarrow$	$s = \text{constant}$

Cycle: a sequence of processes in which the system is returned to the original state.

Point functions: another way of saying *properties*, since they depend only on the state of the system and are independent of the history or process by which the state was attained.

Path functions: quantities that are *not* functions of the state of the system but rather depend on the path taken to move from one state to another during a given time interval. *Heat* and *work* are path functions that only manifest themselves as rates. They can only be observed crossing the system's boundaries *during* a process.

Laws of Classical Thermodynamics

0^2	Relation among properties
0	Thermal equilibrium
1	Conservation of energy
2	Degradation of energy (irreversibilities)

The 0^2 *law* (sometimes called the 00 *law*) is seldom listed as a formal law of thermodynamics; however, one should realize that without such a statement our entire

thermodynamic structure would collapse. This law states that we may assume the existence of a relation among the properties, that is, an *equation of state*. Such a relation or equation might be extremely complicated or even undefined, but as long as we know that such a relation exists, we can continue our studies. A relation or equation of state can also be given in the form of tabular or graphical information.

For a single component or pure substance only three *independent* properties are required to fix the state of the system. Care must be taken in the selection of these properties, for example, temperature and pressure become interdependent when the substance exists in more than one phase (as in a liquid together with its vapor). When dealing with a unit mass, only two independent properties are required to fix the state. Thus on a per unit mass basis one can express any property (P) in terms of two other known independent properties with a relation such as

$$P = f(x, y)$$

Now, if two systems are separated by a nonadiabatic wall (one that permits heat transfer), the state of each system will change until a new equilibrium state is reached for the combined system. The two systems are then said to be in *thermal equilibrium* with each other and will then have one property in common which we call the *temperature*.

The *zeroth* or 0 *law* states that two systems in thermal equilibrium with a third system are in thermal equilibrium with each other (and thus have the same temperature). Among other things, this concept allows the use of thermometers and their standardization.

First Law of Thermodynamics

The *first law* deals with conservation of energy, and it can be expressed in many equivalent ways. Heat and work are two different types of *energy in transit* and below we write this law before and after all transients die out (as a result both heat and work must be treated as path functions). In Chapter 2 we work with this law on a rate basis.

Heat is transferred from one system to another when such effect occurs solely as a result of a temperature difference between the two systems. Heat is always transferred from the system at the higher temperature to the one at the lower temperature.

Work is transferred from a system if the total external effect can be reduced to the raising of a mass in a gravity field. For a closed system that executes a complete *cycle*,

$$\sum Q = \sum W \tag{1.17}$$

where

$$Q = \text{heat transferred } \textit{into} \text{ the system is positive}$$

$$W = \text{work transferred } \textit{from} \text{ the system is positive}$$

Other sign conventions are sometimes used, but we shall adopt those above for this book.

For a closed system that executes a single *process*,

$$Q = W + \Delta E \tag{1.18}$$

where E represents the total energy of the system at the end states. On a unit mass basis, equation (1.18) is written as

$$q = w + \Delta e \tag{1.19}$$

The total energy may be broken down into (at least) three types:

$$e \equiv u + \frac{V^2}{2g_c} + \frac{g}{g_c}z \tag{1.20}$$

where

u = the intrinsic internal energy manifested by the thermal motion of the molecules within the system

$\frac{V^2}{2g_c}$ = the kinetic energy represented by the movement of the system as a whole

$\frac{g}{g_c}z$ = the potential energy caused by the position of the system in a field of gravity

It is sometimes necessary to include other types of energy (such as dissociation energy), but those mentioned above are the only ones that we are concerned with in this book.

For an infinitesimal process, we write equation (1.19) as

$$\delta q = \delta w + de \tag{1.21}$$

Note that since heat and work are *path functions* (i.e., they are a function of how the system gets from one state point to another during their respective transients), infinitesimal amounts of these quantities are not exact differentials and thus are

written as δq and δw. The infinitesimal change in internal energy is an exact differential since the internal energy is a point function or property. For a stationary system, equation (1.21) becomes

$$\delta q = \delta w + du \tag{1.22}$$

Note that we neglected potential energies for infinitesimal changes. Now, the reversible work done by pressure forces during a change of volume for a stationary system is

$$\delta w = p dv \tag{1.23}$$

Combination of the terms u and pv enters into many equations (particularly for open systems) and it is convenient to define the property *enthalpy* (h):

$$h \equiv u + pv \tag{1.24}$$

Enthalpy is a property since it is defined in terms of other properties. It is frequently used in differential form:

$$dh = du + d(pv) = du + pdv + vdp \tag{1.25}$$

Other examples of defined properties are the specific heats at constant pressure (c_p) and constant volume (c_v):

$$c_p \equiv \left(\frac{\partial h}{\partial T}\right)_p \tag{1.26}$$

$$c_v \equiv \left(\frac{\partial u}{\partial T}\right)_v \tag{1.27}$$

Second Law of Thermodynamics

The *second law* has been expressed in many equivalent forms. Perhaps the most common is the statement by Kelvin and Planck stating that it is impossible for an engine operating in a *cycle* to produce *net* work output when exchanging heat with only one temperature source. Although by itself this may not appear to be a profound statement, it leads the way to several corollaries and eventually to the establishment of a most important property (the entropy).

The *second law* also recognizes a degradation of energy quality by irreversible effects such as internal fluid friction, lack of pressure equilibrium between a system and its surroundings, and so on. All real processes have some degree of irreversibility present. In some cases, these effects are very small and we can envision an

ideal limiting condition that has none of these effects and thus is reversible. A *reversible process* is one in which *both* the system and its surroundings can be restored to their original states without any degradation of energy into heat.

By prudent application of the second law, it can be shown that the integral of $\delta Q/T$ for a reversible process is independent of the path. Thus this integral must represent the change of a *property,* which is called *entropy* (S):

$$\Delta S \equiv \int \frac{\delta Q_R}{T} \tag{1.28}$$

where the subscript R indicates that it must be applied to a process that can be reversed. An alternative expression on a unit mass basis for a differential process is

$$ds \equiv \frac{\delta q_R}{T} \tag{1.29}$$

In Chapter 9 we show how friction always increases the system's entropy and in Chapter 10 we show how heat transfer (unlike work) always contributes to the system's entropy; this contribution can add or subtract entropy amounts depending on the direction of the temperature gradient between the system and surroundings. Although you have no doubt used entropy for many calculations, plots, and so on, you probably do not have a good feeling for this property. In Chapter 3 we divide entropy changes into two parts (from internal and external contributions), and by using it in this fashion for the remainder of this book, we hope that you will gain a better understanding of this elusive "creature."

Property Relations

Some extremely important relations come from combinations of the first and second laws. Consider the first law for a stationary system that executes an infinitesimal process:

$$\delta q = \delta w + du \tag{1.22}$$

Considering each of these to be a reversible process,

$$\delta w = pdv \quad (1.23) \quad \text{and} \quad \delta q = Tds \qquad \text{(from 1.29)}$$

Substitution of these relations into the first law yields

$$Tds = du + pdv \tag{1.30}$$

Differentiating the enthalpy, we obtained

$$dh = du + pdv + vdp \tag{1.25}$$

Combining equations (1.30) and (1.25) produces

$$Tds = dh - vdp \tag{1.31}$$

Although the assumption of a reversible process was made to derive equations (1.30) and (1.31), the results are equations that *contain only properties and thus are valid relations to use between any end states,* whether reached reversibly or not. These are two important equations that are used throughout the book:

$$T\,ds = du + p\,dv \tag{1.30}$$

$$T\,ds = dh - v\,dp \tag{1.31}$$

If you are uncomfortable with the foregoing technique (one of making special assumptions to derive a relation which is then generalized to be always valid since it involves only properties), perhaps the following comments might be helpful. First let's write the first law in an alternative form (as some authors do):

$$\delta q - \delta w = du \tag{1.22a}$$

Since the internal energy is a property, changes in u depend only on the end states of a process. Let's now substitute an irreversible process *between the same end points* as our reversible process. Then du must remain the same for both the reversible and irreversible cases, with the following result:

$$(\delta q - \delta w)_{\text{rev}} = du = (\delta q - \delta w)_{\text{irrev}}$$

For example, the extra work that would be involved in an irreversible compression process must be compensated by *exactly the same amount of heat released* (an equivalent argument applies to an expansion). In this fashion, irreversible effects will appear to be "washed out" in equations (1.30) and (1.31) and we cannot tell from them whether a particular process is reversible or irreversible.

Perfect Gases

Recall that for a unit mass of a single component substance, any one property can be expressed as a function of at most *two* other independent properties. However, for substances that follow the perfect gas equation of state,

$$p = \rho RT \tag{1.16}$$

it can be shown (e.g., see p. 173 of Ref. 4) that *the internal energy and the enthalpy are functions of temperature only*. These are extremely important results, as they permit us to make many useful simplifications for perfect gases.

Consider the specific heat at constant volume:

$$c_v \equiv \left(\frac{\partial u}{\partial T}\right)_v \tag{1.27}$$

If $u = f(T)$ only, it does not matter whether the volume is held constant when computing c_v; thus the partial derivative becomes an ordinary derivative. Thus

$$c_v = \frac{du}{dt} \tag{1.32}$$

or

$$du = c_v dT \tag{1.33}$$

Similarly, for the specific heat at constant pressure, we can write for a perfect gas

$$dh = c_p dT \tag{1.34}$$

It is important to realize that equations (1.33) and (1.34) are applicable to *any and all* processes (as long as the gas behaves as a perfect gas). If the specific heats remain reasonably constant (normally good over limited temperature ranges), one can easily integrate equations (1.33) and (1.34):

$$\Delta u = c_v \Delta T \tag{1.35}$$

$$\Delta h = c_p \Delta T \tag{1.36}$$

In gas dynamics, one simplifies calculations by introducing an arbitrary base for internal energy. We let $u = 0$ when $T = 0$ (absolute). Then from the definition of enthalpy, h also equals zero when $T = 0$. Equations (1.35) and (1.36) can now be rewritten as

$$u = c_v T \tag{1.37}$$

$$h = c_p T \tag{1.38}$$

Typical values of the specific heats for air at normal temperature and pressure are $c_p = 0.240$ and $c_v = 0.171$ Btu/lbm-°R. Learn these numbers (or their SI equivalents)! You will use them often.

Other frequently used relations in connection with perfect gases are

$$\gamma \equiv \frac{c_p}{c_v} \tag{1.39}$$

$$c_p - c_v = \frac{R}{J} \tag{1.40}$$

Notice that the units conversion factor

$$J = 778 \text{ ft-lbf/Btu} \quad \text{or} \quad 4186 \text{ J/kcal} \tag{1.41}$$

has been introduced in equation (1.40) since the specific heats are normally given in units of Btu/lbm-°R or kcal/kg-K. The factor J will be omitted in future equations, and it will be left for you to consider when it is required. It is hoped that by this procedure, you will develop needed habits for checking units in all your work. What units are used for specific heat and R in the SI system? (See the table on gas properties in Appendix B.) Would the SI system of units require a J numerical factor in equation (1.40)?

Entropy Changes

The change in entropy between any two states can be obtained by integrating equation (1.29) along any reversible path or any combination of reversible paths connecting the points, with the following results for perfect gases:

$$\Delta s_{1-2} = c_p \ln \frac{v_2}{v_1} + c_v \ln \frac{p_2}{p_1} \tag{1.42}$$

$$\Delta s_{1-2} = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} \tag{1.43}$$

$$\Delta s_{1-2} = c_v \ln \frac{T_2}{T_1} + R \ln \frac{v_2}{v_1} \tag{1.44}$$

Remember, absolute values of pressures and temperatures must be used in these equations; volumes may be either total or specific, but both volumes must be of the same type. Pay close attention to the units on c_p, c_v, and R because they must be compatible.

Process Diagrams

Many processes dealing with gases can be represented by a so-called *polytropic process,* that is, one that follows the relation

$$pv^n = \text{const} = C_1 \tag{1.45}$$

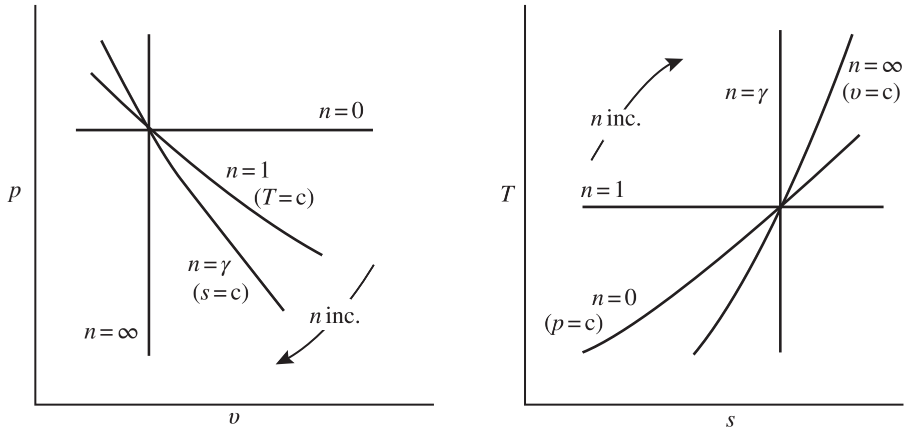


Figure 1.2 General polytropic process plots for perfect gases.

where n is the polytropic exponent, which can be any positive number. If the fluid is a perfect gas, the equation of state can be introduced into equation (1.45) to yield

$$Tv^{n-1} = \text{const} = C_2 \tag{1.46}$$

$$Tp^{(1-n)/n} = \text{const} = C_3 \tag{1.47}$$

Keep in mind that C_1, C_2, and C_3 in the equations above are different constants. It is interesting to note that certain values of n represent particular processes mentioned earlier:

$$n = 0 \rightarrow p = \text{const}$$

$$n = 1 \rightarrow T = \text{const}$$

$$n = \gamma \rightarrow s = \text{const}$$

$$n = \infty \rightarrow v = \text{const}$$

These plot in the p–v and T–s diagrams as shown in Figure 1.2; *learn these diagrams*! You should also be able to figure out how temperature and entropy vary in the p–v diagram and how pressure and volume vary in the T–s diagram. (Try drawing several T = const lines in the p–v plane. Which one represents the highest temperature?)

Consult References 3, 5, and 7 for additional information on Thermodynamic Concepts.

REVIEW QUESTIONS

Many of the questions that follow are based on concepts that you have covered in earlier physics and thermodynamic courses. State your answers as clearly and concisely as possible using any source that you wish (although almost all the material has been covered in

this chapter). It is best if you not proceed to Chapter 2 until you fully understand the correct answers to these questions and can write them down without reference to your notes.

1.1. State Newton's second law as you would apply it to a control mass.

1.2. Define a 1-pound force in terms of the acceleration it will give to a 1-pound mass. Give a similar definition for a newton in the SI system.

1.3. Explain the significance of g_c in Newton's second law. What are the magnitude *and units* of g_c in the English Engineering system? In the SI system?

1.4. What is the relation between degrees Fahrenheit and degrees Rankine? And the relation between degrees Celsius and Kelvin?

1.5. What is the relationship between density and specific volume?

1.6. Explain the difference between absolute and gage pressures.

1.7. What is the distinguishing characteristic of a fluid (as compared to a solid)? How is this related to viscosity?

1.8. In any given physical situation, why can there be a difference between the number of units and the number of dimensions? [Number of units being less or equal to number of dimensions.]

1.9. (Optional) Why is the ratio of the velocity at any point downstream of the throat of a supersonic nozzle to the velocity at the throat (where it equals the speed of sound), though dimensionless, *not* a Mach number?

1.10. Describe the difference between the microscopic and macroscopic approach in the analysis of fluid behavior.

1.11. Describe the control volume approach to problem analysis and contrast it to the control mass approach. What kinds of systems are these also called?

1.12. Describe a property and give at least three examples.

1.13. Properties may be categorized as either intensive or extensive. Define what is meant by each, and list examples of each type of property.

1.14. When dealing with a unit mass of a single component substance, how many independent properties are required to fix the state?

1.15. Why do we need an equation of state? Write down one with which you are familiar.

1.16. Define point functions and path functions. Give examples of each.

1.17. What is a thermodynamic process? What is a thermodynamic cycle?

1.18. How does the zeroth law of thermodynamics relate to temperature?

1.19. State the first law of thermodynamics for a closed system that is executing a single process.

1.20. What are the sign conventions used in this book for heat and work?

1.21. State any form of the second law of thermodynamics you are familiar with.

1.22. Define a reversible process for a thermodynamic system. Is any real process ever completely reversible?

1.23. What are some phenomena that cause processes to be irreversible?

1.24. Under what conditions is an isentropic process not a reversible adiabatic process?

1.25. Give the equations that define enthalpy and entropy.

1.26. Give differential expressions that relate entropy to
(a) internal energy
(b) enthalpy.

1.27. Define (in the form of partial derivatives) the specific heats c_v and c_p. Are these expressions valid for materials in any state?

1.28. State the perfect gas equation of state. Give a consistent set of units for each term in the equation.

1.29. For a perfect gas, the specific internal energy is a function of which state variables? How about the specific enthalpy?

1.30. Give expressions for Δu and Δh that are valid for perfect gases. Do these hold for any process?

1.31. For perfect gases, at what temperature do we arbitrarily assign $u = 0$ and $h = 0$?

1.32. State any one expression for the entropy change between two arbitrary points which is valid for a perfect gas.

1.33. If a perfect gas undergoes an isentropic process, what equation relates the pressure to the volume? Temperature to the volume? Temperature to the pressure?

1.34. Consider the general polytropic process (pv^n = const) for a perfect gas. In the p–v and T–s diagrams shown in Figure RQ1.34, label each process line with the correct value of n and identify which fluid property is being held constant.

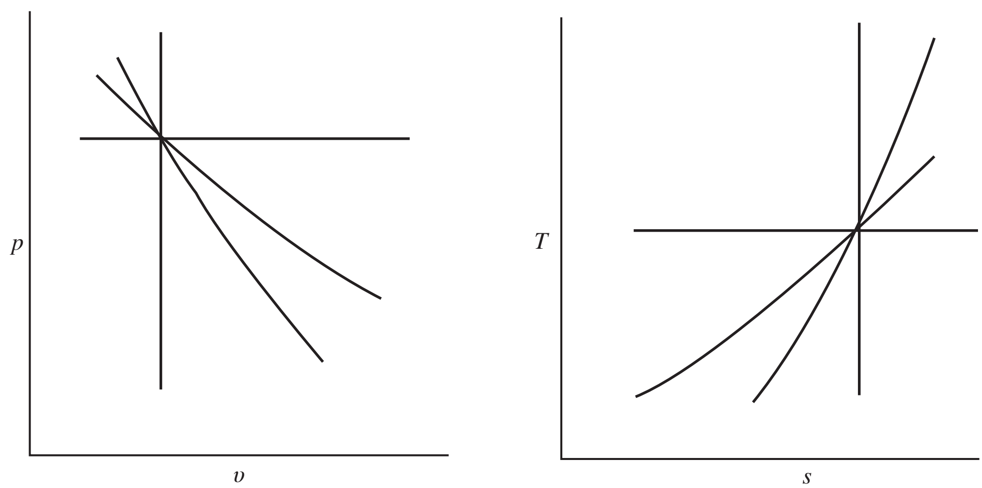


Figure RQ1.34

REVIEW PROBLEMS

If you have been away from thermodynamics for a long time, it might be useful to work the following problems.

1.1. How well does the relation $c_p = c_v + R$ represent the table of gas properties in Appendix A? Use entries for hydrogen.

1.2. A perfect gas having specific heats $c_v = 0.403$ Btu/lbm-°R and $c_p = 0.532$ Btu/lbm-°R undergoes a reversible polytropic process in which the polytropic exponent $n = 1.4$. Giving clear reasons, answer the following:

(a) Will there be any heat transfer in the process?

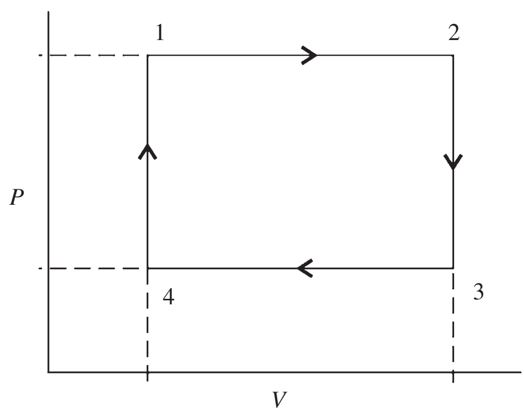
(b) Which would this process be *nearest,* a horizontal or a vertical line on a p–v or a T–s diagram? (Alternatively, state between which constant property lines the process lies.)

1.3. Nitrogen gas is reversibly compressed from 70°F and 14.7 psia to one-fourth of its original volume by (1) a $T = \text{const}$ process or (2) a $p = \text{const}$ process followed by a $v = \text{const}$ process to the same end point as (1).

(a) Which compression involves the least amount of work? Show clearly on a p–v diagram.

(b) Calculate the heat and work interaction for the isothermal compression.

1.4. For the reversible cycle shown in Figure RP1.4, compute the cyclic integrals $[\oint d(\cdot)]$ of dE, δQ, dH, δW, and dS.



$p_1 = p_2 = 1.0 \times 10^6$ Pa

$p_3 = p_4 = 0.4 \times 10^6$ Pa

$V_1 = V_4 = 0.6\,\text{m}^3$

$V_2 = V_3 = 1.0\,\text{m}^3$

Figure RP1.4

1.5. A perfect gas (methane) undergoes a reversible, polytropic process in which the polytropic exponent is 1.4.

(a) Using the first law, arrive at an expression for the heat transfer per unit mass solely as a function of the temperature difference ΔT. This should be some numerical value (use SI units).

(b) Would this heat transfer be equal to either the enthalpy change or the internal energy change for the same ΔT?

1.6. (Optional) The pressure in a perfect gas is proportional to the product of the density and the absolute temperature. By dimensional reasoning, find the units that the constant R should have in equation (1.16) if the pressure is measured in kgf/m^2, the temperature in degrees K, and the density in kg/m^3.

Chapter 2

Control Volume Analysis—Part I

2.1 INTRODUCTION

In the study of gas dynamics we are interested in examining gases that are *flowing*. The analysis of flow problems is based on the same fundamental principles that you have used in earlier courses in thermodynamics or fluid dynamics:

1. Conservation of mass
2. Conservation of energy
3. Newton's second law of motion

When applying these principles to the solution of specific problems, you must also know something about the properties of the fluid.

In Chapter 1 the concepts listed above were reviewed in a form applicable to a *control mass*. However, it is extremely difficult to approach flow problems from the control mass point of view. Thus it will first be necessary to develop some fundamental expressions that can be used to analyze *control volumes*. A technique is developed to transform our basic laws for a control mass into integral equations that are applicable to finite control volumes. Simplifications will then be made for special cases such as steady one-dimensional flow. We also analyze differential control volumes in order to arrive at some valuable differential relations. In this chapter we tackle mass and energy principles, and in Chapter 3, we discuss momentum concepts.

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

2.2 OBJECTIVES

After completing this chapter successfully, you should be able to:

1. State the basic concepts from which a study of gas dynamics proceeds.
2. Explain one-, two-, and three-dimensional flows.
3. Define steady flow.
4. (*Optional*) Compute the flow rate and average velocity from a multidimensional velocity profile.
5. Write the equation used to relate the material derivative of any extensive property to properties inside, and crossing the boundaries of, a control volume. Interpret in words the meaning of each term in the equation.
6. (*Optional*) Starting with the basic concepts or equations that are valid for a control mass, obtain the integral forms of the continuity and energy equations for a control volume.
7. Simplify the integral forms of the continuity and energy equations for a control volume for conditions of steady one-dimensional flow.
8. (*Optional*) Apply the simplified forms of the continuity and energy equations to differential control volumes.
9. Demonstrate the ability to apply continuity and energy concepts in an analysis of control volumes.

2.3 FLOW DIMENSIONALITY AND AVERAGE VELOCITY

As we observe fluid moving around, the various properties can be expressed as functions of location and time. Thus, in an ordinary rectangular Cartesian or *x*-*y*-*z* coordinate system, we could say in general that

$$V = f(x, y, z, t) \tag{2.1}$$

or

$$p = g(x, y, z, t) \tag{2.2}$$

Since it is necessary to specify three spatial coordinates and time, such dependence is called *three-dimensional unsteady flow*.

Two-dimensional unsteady flow would be represented by

$$V = f(x, y, t) \tag{2.3}$$

and *one-dimensional unsteady flow* by

$$V = f(x, t) \tag{2.4}$$

The assumption of one-dimensional flow is a simplification normally applied to flow systems, and the single coordinate is usually taken in the direction of flow. This is not necessarily *unidirectional flow*, as the direction of the flow duct might change. Another way of looking at one-dimensional flow is to say that at any given section (x-coordinate) all fluid properties are constant across the perpendicular cross section. Keep in mind that the properties can still change from section to section (as x changes).

The fundamental concepts reviewed in Chapter 1 were expressed in terms of a given mass of material (i.e., the control mass approach). When using the control mass approach we observe some property of the mass, such as enthalpy or internal energy. The (time) rate at which this property changes is called a *material derivative* (sometimes called a *total* or *substantial derivative*). It is written by various authors as $D(\cdot)/Dt$ or $d(\cdot)/dt$. Note that it is computed *as we follow the material around,* and thus it involves two contributions as explained below.

First, the property may change because the mass has moved to a new position (e.g., at the same instant of time the temperature in Tucson is different from that in Anchorage). This contribution to the material derivative is sometimes called the *convective derivative*.

Second, the property may change with time at any given position (e.g., even in Monterey the temperature varies from morning to night). This latter contribution is called the *local* or *partial derivative* with respect to time and is written $\partial(\cdot)/\partial t$. As an example, for a typical three-dimensional unsteady flow the material derivative of the pressure would be represented as

$$\frac{dp}{dt} = \underbrace{\frac{\partial p}{\partial x}\frac{dx}{dt} + \frac{\partial p}{\partial y}\frac{dy}{dt} + \frac{\partial p}{\partial z}\frac{dz}{dt}}_{\text{Convective derivative}} + \underbrace{\frac{\partial p}{\partial t}}_{\text{Local time derivative}} \tag{2.5}$$

If the fluid properties at every point are *independent* of time, we call this *steady flow*. Thus in steady flow the *partial derivative* of any property with respect to time is zero:

$$\frac{\partial(\cdot)}{\partial t} = 0 \text{ for steady flow} \tag{2.6}$$

Notice that this does not prevent properties from being different at different locations. Thus the material derivative may be nonzero even for the case of steady flow, due to the contribution of the convective portion.

Next we examine the problem of computing mass flow rates when the flow is not one-dimensional. Consider the flow of a real fluid in a circular duct. At low Reynolds numbers, where viscous forces predominate, the fluid tends to flow in layers without any energy exchange between adjacent layers. This is termed

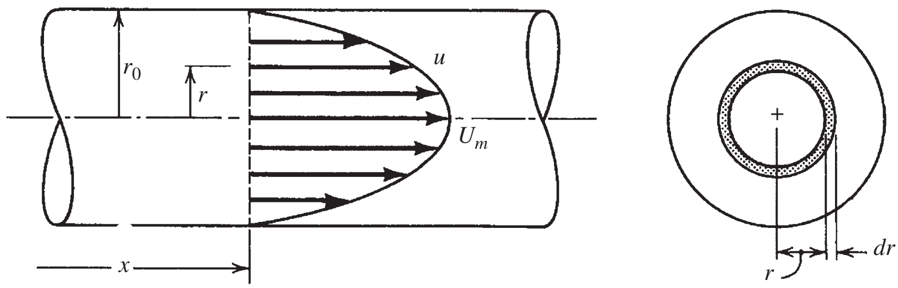


Figure 2.1 Velocity profile for laminar flow.

laminar flow, and we could readily establish (see p. 185 of Ref. 9) that the velocity profile for this case would be a paraboloid of revolution, a cross section of which is shown in Figure 2.1.

At any given cross section the velocity may be expressed as

$$u = U_m\left[1-\left(\frac{r}{r_0}\right)^2\right] \tag{2.7}$$

To compute the mass flow rate, we integrate:

$$\dot{m} = \text{mass flow rate} = \int_A \rho u dA \tag{2.8}$$

where

$$dA = 2\pi r dr \tag{2.9}$$

Assuming ρ to be a constant, carry out the indicated integration and *show* that

$$\dot{m} = \rho\left(\pi r_0^2\right)\frac{U_m}{2} = \rho A\frac{U_m}{2} \tag{2.10}$$

Note that for this multidimensional flow problem, when the flow rate is expressed as

$$\dot{m} = \rho A V \tag{2.11}$$

the velocity V is a *mass-average* velocity, which for this case equals $U_m/2$. Since the density was held constant during integration, V is more properly called an *area-averaged velocity*. But because there is generally little change in density *across* any given section, V is a reasonable mass average velocity. When expressing kinetic energy [see equation (1.20)] for laminar flows with this average V, a "correction factor" of 2.0 multiplies the square of the average velocity for fully developed laminar tube-flows (see Problem 2.4.).

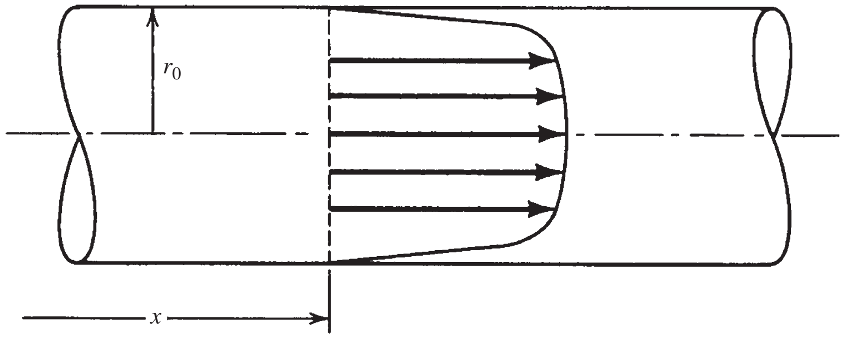


Figure 2.2 Velocity profile for turbulent flow.

As we move to higher Reynolds numbers, large inertia forces cause irregular velocity fluctuations in all directions, which in turn cause mixing between adjacent layers. The resulting energy transfer causes the fluid particles near the center to slow down while those particles next to the wall speed up. This produces the relatively flat velocity profile shown in Figure 2.2, which is typical of *turbulent flow*. Notice that for this type of flow, all particles at a given section have very nearly the same velocity, which very closely approximates a one-dimensional flow picture. Since most tube flows of engineering interest are well into this turbulent regime (after relatively short entrance regions), we can see why the assumption of one-dimensional flow is reasonably accurate. From Chapter 9 and data in Appendix C we find that flows are turbulent when $\mathrm{Re} > 3 \times 10^3$; this means that for atmospheric air at 60°F the product $VD > 8$ ft-in./sec (or 0.071 m^2/s), so for a 1-in.-diameter pipe when $V > 8$ ft/sec (or 2.8 m/s) the flow is in the turbulent region.

Streamlines and Streamtubes

As we progress through this book, we will occasionally mention the following:

Streamline: a line that is everywhere tangent to the velocity vectors of those fluid particles that are on the line

Streamtube: a flow passage that is formed by adjacent streamlines

By virtue of these definitions, no fluid particles ever cross a streamline. Hence an identifiable amount of fluid flows through a streamtube much as it does through a physical pipe.

2.4 TRANSFORMATION OF A MATERIAL DERIVATIVE TO A CONTROL VOLUME APPROACH

In most gas dynamics problems it will be more convenient to examine a fixed region in space, or a *control volume*. The fundamental equations were listed in Chapter 1 for the analysis of a control mass. We now ask ourselves what form

these equations take when applied to a control volume. In each case the troublesome term is the material derivative of an extensive property.

It will be simplest to show first how the material derivative of *any extensive property* transforms from a control mass to a control volume approach. The result will be a valuable general relation that can be used for many particular situations. Let

$N \equiv$ the total amount of any extensive property in a given mass

$\eta \equiv$ the amount of N per unit mass

Thus

$$N = \int \eta dm = \iiint \rho \eta d\tilde{v} = \int_v \rho \eta d\tilde{v} \tag{2.12}$$

where

$dm \equiv$ incremental element of mass
$d\tilde{v} \equiv$ incremental volume element

Note that for simplicity we are indicating the triple volume integral as $\int_v$.

Now let us consider what happens to the material derivative dN/dt. Recall that a material derivative is the (time) rate of change of a property computed as the mass moves around. Figure 2.3 shows an arbitrary mass at time t and the same mass at time $t + \Delta t$. Remember that this system is at all times composed of the same mass particles. If Δt is small enough, there will be an overlap of the two regions as shown in Figure 2.4, with the common region identified as region 2. At time t, the given mass particles occupy regions 1 and 2. At time $t + \Delta t$ the same mass particles occupy regions 2 and 3. We shall call the original confines of the mass (regions 1 and 2) the *control volume*.

We construct our material derivative from the mathematical definition

$$\frac{dN}{dt} \equiv \lim_{\Delta t \to 0} \left[\frac{(\text{final value of } N)_{t+\Delta t} - (\text{initial value of } N)_t}{\Delta t} \right] \tag{2.13}$$

where the final value of N is that N of regions 2 and 3 computed at time $t + \Delta t$, and the initial value of N is the N of regions 1 and 2 computed at time t.

A more specific expression is:

$$\frac{dN}{dt} = \lim_{\Delta t \to 0} \left[\frac{(N_2 + N_3)_{t+\Delta t} - (N_1 + N_2)_t}{\Delta t} \right] \tag{2.14}$$

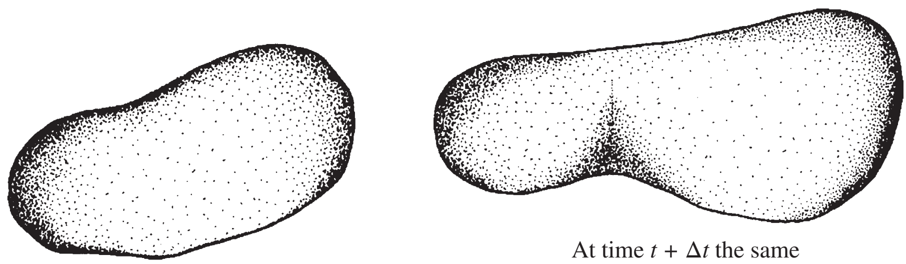


Figure 2.3 Identification of control mass.

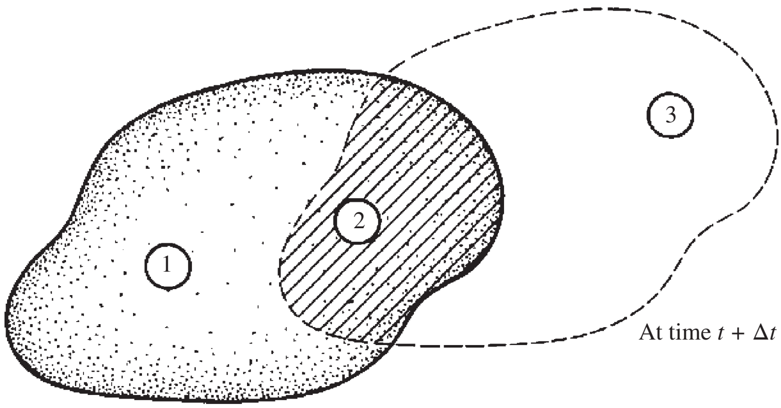


Figure 2.4 Control mass for small Δt.

First, consider the term

$$\lim_{\Delta t \to 0} \frac{N_3(t + \Delta t)}{\Delta t}$$

The numerator represents the amount of N in region 3 at time $t + \Delta t$, and by definition, *region 3 is formed by the fluid moving out of the control volume*. Let $\hat{n}$ be a unit normal, positive when pointing *outward* from the control volume. Also let dA be an increment of the surface area that separates regions 2 and 3, as shown in Figure 2.5.

$$\mathbf{V} \cdot \hat{n} = \text{component of } \mathbf{V} \perp \text{ to } dA$$

$$(\mathbf{V} \cdot \hat{n})dA = \text{incremental volumetric flow rate}$$

$$\rho(\mathbf{V} \cdot \hat{n})dA = \text{incremental mass flow rate}$$

$$\rho(\mathbf{V} \cdot \hat{n})dA\Delta t = \text{amount of mass that crossed } dA \text{ in time } \Delta t$$

$$\eta\rho(\mathbf{V} \cdot \hat{n})dA\Delta t = \text{amount of } N \text{ that crossed } dA \text{ in time } \Delta t$$

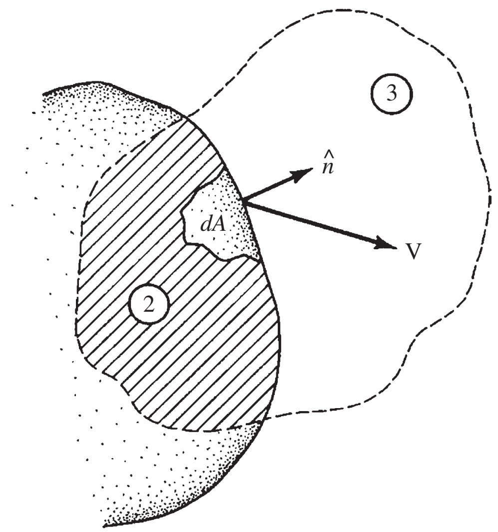


Figure 2.5 Flow out of control volume.

Thus

$$\int_{S_{\text{out}}} \eta\rho(\mathbf{V}\cdot\hat{n})dA\Delta t \approx \text{total amount of } N \text{ in region 3} \tag{2.15}$$

where $\int_{S_{\text{out}}}$ is a double integral over the surface where fluid *leaves* the control volume. The term in question becomes

$$\lim_{\Delta t\to 0}\frac{N_3(t+\Delta t)}{\Delta t} = \int_{S_{\text{out}}} \eta\rho(\mathbf{V}\cdot\hat{n})dA \tag{2.16}$$

This integral is called a *flux* or *rate* of N flow *out* of the control volume.

Since the Δt cancels, one might question the limit process. Actually, the integral expression in equation (2.15) is only approximately correct. This is because all the properties in this integral are going to be evaluated at the surface S at time t. Thus equation (2.15) is only approximate as written but *becomes exact in the limit* as Δt approaches zero.

Now let us consider the term

$$\lim_{\Delta t\to 0}\frac{N_1(t)}{\Delta t}$$

How has region 1 been formed? It has been formed by the original mass particles moving on (during time Δt), and *other fluid has moved into the control volume*. Thus we evaluate N_1 by the following procedure. Let $\hat{n}'$ be a unit normal, positive when pointing *inward* to the control volume, as shown in Figure 2.6.

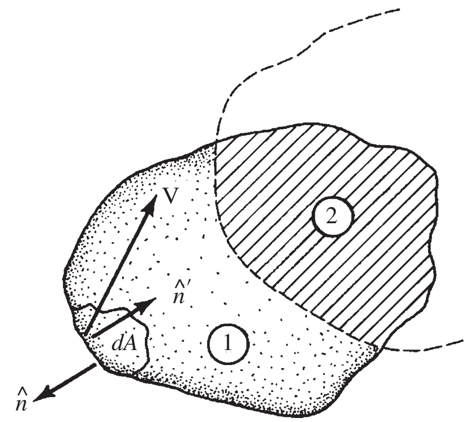


Figure 2.6 Flow into control volume.

Complete the following in words:

$$\mathbf{V}\cdot\hat{n}' =$$

$$(\mathbf{V}\cdot\hat{n}')dA =$$

$$\rho(\mathbf{V}\cdot\hat{n}')dA =$$

$$\rho(\mathbf{V}\cdot\hat{n}')dA\Delta t =$$

$$\eta\rho(\mathbf{V}\cdot\hat{n}')dA\Delta t =$$

It should be clear that

$$\int_{S_{\text{in}}} \eta\rho(\mathbf{V}\cdot\hat{n}')dA\Delta t \approx \text{total amount of } N \text{ in region 1} \tag{2.17}$$

and

$$\lim_{\Delta t\to 0} \frac{N_1(t)}{\Delta t} = \int_{S_{in}} \eta\rho(\mathbf{V}\cdot\hat{n}')dA \tag{2.18}$$

where $\int_{S_{\text{in}}}$ is a double integral over the surface where fluid enters the control volume. This term represents the N flux *into* the control volume.

Now look at the first and last terms of equation (2.14):

$$\lim_{\Delta t\to 0}\left[\frac{N_2(t+\Delta t)-N_2(t)}{\Delta t}\right] \quad \text{which by definition is} \quad \frac{\partial N_2}{\partial t}$$

Note that the partial derivative notation is used since the region of integration is fixed and time is the only independent parameter allowed to vary. Also note that as Δt approaches zero, region 2 approaches the original confines of the mass, which we have called the control volume. Thus

$$\lim_{\Delta t \to 0}\left[\frac{N_2(t+\Delta t)-N_2(t)}{\Delta t}\right]=\frac{\partial N_{\mathrm{cv}}}{\partial t}=\frac{\partial}{\partial t}\int_{\mathrm{cv}}\rho\eta\, d\tilde{v} \tag{2.19}$$

where cv stands for the control volume.

We now substitute into equation (2.14) all the terms that we have developed in equations (2.16), (2.18), and (2.19):

$$\frac{dN}{dt}=\frac{\partial}{\partial t}\int_{\mathrm{cv}}\rho\eta\, d\tilde{v}+\int_{S_{\mathrm{out}}}\eta\rho(\mathbf{V}\cdot\hat{n})dA-\int_{S_{\mathrm{in}}}\eta\rho(\mathbf{V}\cdot\hat{n}')dA \tag{2.20}$$

Noting that $\hat{n}=-\hat{n}'$, we can combine the last two terms into

$$\begin{aligned}&\int_{S_{\mathrm{out}}}\eta\rho(\mathbf{V}\cdot\hat{n})dA-\int_{S_{\mathrm{in}}}\eta\rho(\mathbf{V}\cdot\hat{n}')dA\\&=\int_{S_{\mathrm{out}}}\eta\rho(\mathbf{V}\cdot\hat{n})dA+\int_{S_{\mathrm{in}}}\eta\rho(\mathbf{V}\cdot\hat{n})dA=\int_{\mathrm{cs}}\eta\rho(\mathbf{V}\cdot\hat{n})dA\end{aligned} \tag{2.21}$$

where cs represents the entire control surface surrounding the control volume.

This term represents the *net* rate at which N passes *out* of the control volume (i.e., flow rate out minus flow rate in). The final transformation equation becomes

$$\boxed{\left(\frac{dN}{dt}\right)_{\text{material derivative}}=\underbrace{\frac{\partial}{\partial t}\int_{\mathrm{cv}}\eta\rho\, d\tilde{v}}_{\text{Triple integral}}+\underbrace{\int_{\mathrm{cs}}\eta\rho(\mathrm{V}\cdot\hat{n})\, dA}_{\text{Double integral}}} \tag{2.22}$$

This relation, known as *Reynolds's transport theorem,* can be interpreted in words as:

> The rate of chang e of N for a given mass as it is moving around is equal to the rate of change of N inside the control volume *plus* the *net* efflux (flow out minus flow in) of N from the control volume.

It is essential to note that we have not placed any restriction on N other than that it must be a mass-dependent (extensive) property. Thus N may be a scalar *or* a vector quantity. Examples of the application of this powerful transformation equation are provided in the next two sections and in Chapter 3.

2.5 CONSERVATION OF MASS

If we exclude from consideration the possibility of nuclear reactions, we can account separately for the conservation of mass and energy. Thus if we observe a given quantity of mass as it moves around, we can say by definition that the mass will remain fixed. Another way of stating this is that the material derivative of the mass is zero:

$$\boxed{\frac{d(\text{mass})}{dt} = 0} \tag{2.23}$$

This is the *continuity equation for a control mass*. What corresponding expression can we write for a control volume? To find out, we must transform the material derivative according to the relation developed in Section 2.4.

If N represents the total mass, η is the mass per unit mass, or one. Substitution into equation (2.22) yields

$$\frac{d(\text{mass})}{dt} = \frac{\partial}{\partial t}\int_{\text{cv}} \rho d\tilde{v} + \int_{\text{cs}} \rho(\mathbf{V}\cdot\hat{n})dA \tag{2.24}$$

But we know by equation (2.23) that this must be zero; thus the transformed equation is

$$\boxed{0 = \frac{\partial}{\partial t}\int_{\text{cv}} \rho d\tilde{v} + \int_{\text{cs}} \rho(\mathbf{V}\cdot\hat{n})dA} \tag{2.25}$$

This is the *continuity equation for a control volume*. State in words what each term represents. For steady flow, any partial derivative with respect to time is zero and the equation becomes

$$0 = \int_{\text{cs}} \rho(\mathbf{V}\cdot\hat{n})dA \tag{2.26}$$

Let us now evaluate the remaining integral for the case of one-dimensional flow. Figure 2.7 shows fluid crossing a portion of the control surface. Recall that for one-dimensional flow any fluid property will be constant over an entire cross section. Thus, both the density and the velocity can be brought out from under the integral sign. If the surface is always chosen perpendicular to V, the integral is very simple to evaluate:

$$\int \rho(\mathbf{V}\cdot\hat{n})dA = \rho\mathbf{V}\cdot\hat{n}\int dA = \rho VA$$

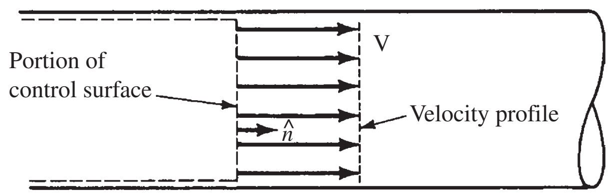


Figure 2.7 One-dimensional velocity profile.

This integral must be evaluated over the entire control surface, which yields

$$\int_{\text{cs}} \rho(\mathbf{V} \cdot \hat{n})dA = \sum \rho VA \tag{2.27}$$

This summation is taken over all sections where fluid crosses the control surface and is positive where fluid leaves the control volume (since $\mathbf{V} \cdot \hat{n}$ is positive here) and negative where fluid enters the control volume. For steady, one-dimensional flow, the continuity equation for a control volume becomes

$$\sum \rho AV = 0 \tag{2.28}$$

If there is only one section where fluid enters and one section where fluid leaves the control volume, this becomes

$$(\rho AV)_{\text{out}} - (\rho AV)_{\text{in}} = 0$$

or

$$(\rho AV)_{\text{out}} = (\rho AV)_{\text{in}} \tag{2.29}$$

We usually write this result as

$$\boxed{\dot{m} = \rho AV = \text{constant}} \tag{2.30}$$

Implicit in this expression is the fact that V is the component of velocity perpendicular to the area A. If the density ρ is in lbm per cubic foot, the area A is in square feet, and the velocity V is in feet per second, what are the units of the mass flow rate $\dot{m}$? What are the corresponding units for each term in SI?

Note that *as a result of steady flow* the mass flow rate into a control volume is equal to the mass flow rate out of the control volume. *The converse of this is not necessarily true*; that is, just because it is known that the flow rates into and out of a control volume are the same, this does not ensure that the flow is steady.

Example 2.1
Air flows steadily through a 1-in.-diameter section with a velocity of 1096 ft/sec. The temperature is 40°F and the pressure is 50 psia. The flow passage expands to 2 in. in diameter, and at this section, the pressure and temperature have dropped to 2.82 psia and −240°F, respectively. What is the average velocity at this section?

Knowing that

$$p = \rho RT \quad \text{and} \quad A = \frac{\pi D^2}{4}$$

for steady, one-dimensional flow, we obtain

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

$$\left[\frac{p_1}{RT_1}\right]\left[\frac{\pi D_1^2}{4}\right] V_1 = \left[\frac{p_2}{RT_2}\right]\left[\frac{\pi D_2^2}{4}\right] V_2$$

$$V_2 = V_1 \frac{D_1^2 p_1 T_2}{D_2^2 p_2 T_1} = (1096)\left(\frac{1}{2}\right)^2\left(\frac{50}{2.82}\right)\left(\frac{220}{500}\right)$$

$$V_2 = 2138\,\text{ft/sec} \quad (\text{or } 651.7\,\text{m/s})$$

An alternative form of the continuity equation can be obtained by differentiating equation (2.30). For steady one-dimensional flow this means that

$$d(\rho AV) = AVd\rho + \rho VdA + \rho AdV = 0 \tag{2.31}$$

Dividing by ρAV yields

$$\boxed{\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0} \tag{2.32}$$

This expression may also be obtained by *logarithmic differentiation* of equation (2.30).

This differential form of the continuity equation is useful in interpreting the changes that must occur as fluid flows through a duct, channel, or streamtube. This indicates that if mass is to be conserved, the changes in density, velocity, and cross-sectional area must compensate for one another. For example, if the area is constant ($dA = 0$), any increase in velocity must be accompanied by a corresponding decrease in density. We shall also use this form of the continuity equation in several future derivations.

2.6 CONSERVATION OF ENERGY

The first law of thermodynamics is a statement of conservation of energy. For a system composed of a given quantity of mass that undergoes a process, we can say that

$$\boxed{Q = W + \Delta E} \tag{1.18}$$

where

Q = the net heat transferred into the system
W = the net work done by the system
ΔE = the change in total energy of the system

This can also be written on a *rate basis* to yield an expression that is valid at any instant of time:

$$\frac{\delta Q}{dt} = \frac{\delta W}{dt} + \frac{dE}{dt} \tag{2.33}$$

We must carefully examine each term in this equation to clearly understand its significance. The terms $\delta Q/dt$ and $\delta W/dt$ represent instantaneous rates of heat and work transfer between the system and its surroundings. They are rates of transfer across the boundaries of the system. These terms are *not* material derivatives. (Recall that heat and work are not properties of a system.) On the other hand, energy is a property of the system and dE/dt is a material derivative.

We now ask what form the energy equation takes when applied to a control volume. To answer this, we must first transform the material derivative into equation (2.33) according to the relation developed in Section 2.4. If we let N be E, the total energy of the system, then η represents e, the energy per unit mass:

$$e = u + \frac{V^2}{2g_c} + \frac{g}{g_c}z \tag{1.20}$$

Substitution into equation (2.22) yields

$$\frac{dE}{dt} = \frac{\partial}{\partial t}\int_{\text{cv}} e\rho\, d\tilde{v} + \int_{\text{cs}} e\rho(\mathbf{V}\cdot\hat{n})\,dA \tag{2.34}$$

and the transformed equation that is applicable to a control volume becomes

$$\boxed{\frac{\delta Q}{dt} = \frac{\delta W}{dt} + \frac{\partial}{\partial t}\int_{\text{cv}} e\rho\, d\tilde{v} + \int_{\text{cs}} e\rho(\mathbf{V}\cdot\hat{n})\,dA} \tag{2.35}$$

In this case, $\delta Q/dt$ and $\delta W/dt$ represent instantaneous rates of heat and work transfer across the surface that surrounds the control volume. *State* in words what the other terms represent. [See the discussion following equation (2.22).]

For one-dimensional flow the last integral in equation (2.35) is simple to evaluate, as e, ρ, and V are constant over any given cross section. Assuming that the velocity V is perpendicular to the surface A, we have

$$\int_{\text{cs}} e\rho(\mathbf{V}\cdot\hat{n})\,dA = \sum e\rho V \int dA = \sum e\rho VA = \sum \dot{m}e \tag{2.36}$$

The summation is taken over all sections where fluid crosses the control surface and is positive where fluid leaves the control volume and negative where fluid enters the control volume.

In using equation (2.35) we must be careful to include all forms of work, whether done by pressure forces (from normal stresses) or shear forces (from tangential stresses). Figure 2.8 shows a simple control volume. Note that the control surface is chosen carefully so that there is no fluid motion or other action at the boundary except

a. where fluid enters and leaves the system, or
b. where a mechanical device such as a shaft crosses the boundaries of the system.

This prudent choice of the system boundary simplifies the calculation of the work quantities. For example, recall that for a real fluid there is no motion at the walls (e.g., see Figures 2.1 and 2.2). Thus the pressure and shear forces along the sidewalls perform no work since they do not move through any distance.

The rate at which work is transmitted out of the system by the mechanical device is called $\delta W_s/dt$ and is accomplished by shear stresses between the device and the fluid. (Think of the subscript *s* for shear stresses or *shaft work*.) The other work quantities considered are where fluid enters and leaves the system. Here the pressure forces do work to push fluid into or out of the control volume. The shaded area at the inlet in Figure 2.8 represents the fluid that enters the control volume during time *dt*. The work done here is

$$\delta W' = \mathbf{F} \cdot d\mathbf{x} = pA dx = pAV dt \tag{2.37}$$

The rate of doing work is

$$\frac{\delta W'}{dt} = pAV \tag{2.38}$$

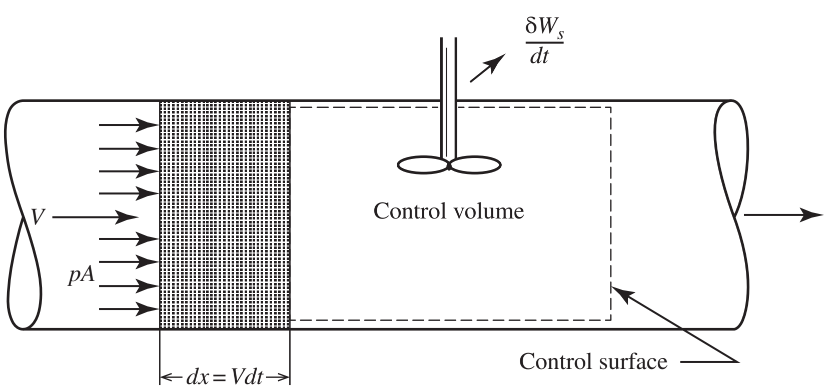


Figure 2.8 Identification of work quantities.

This is called *flow work* or *displacement work*. It can be expressed in a more meaningful form by introducing

$$\dot{m} = \rho A V \tag{2.11}$$

Thus the rate of doing flow work is ($1/\rho = v$)

$$pAV = p\frac{\dot{m}}{\rho} = \dot{m}pv \tag{2.39}$$

In general, this represents work done *by* the system (positive) to force fluid *out* of the control volume and represents work done *on* the system (negative) to force fluid *into* the control volume. Thus the total work

$$\frac{\delta W}{dt} = \frac{\delta W_s}{dt} + \sum \dot{m}pv$$

We may now rewrite our energy equation in a more useful form, which is applicable to one-dimensional flow. Notice how the flow work has been included in the last term:

$$\frac{\delta Q}{dt} = \frac{\delta W_s}{dt} + \frac{\partial}{\partial t}\int_{\text{cv}} e\rho d\tilde{v} + \sum \dot{m}(e + pv) \tag{2.40}$$

If we consider steady flow, the term involving the partial derivative with respect to time is zero. Thus for steady one-dimensional flow the energy equation for a control volume becomes

$$\boxed{\frac{\delta Q}{dt} = \frac{\delta W_s}{dt} + \sum \dot{m}(e + pv)} \tag{2.41}$$

If there is only one section where fluid leaves and one section where fluid enters the control volume, we have (from continuity)

$$\dot{m}_{\text{in}} = \dot{m}_{\text{out}} = \dot{m} \tag{2.42}$$

We may now divide equation (2.41) by $\dot{m}$:

$$\frac{1}{\dot{m}}\frac{\delta Q}{dt} = \frac{1}{\dot{m}}\frac{\delta W_s}{dt} + (e + pv)_{\text{out}} - (e + pv)_{\text{in}} \tag{2.43}$$

We now define

$$q \equiv \frac{1}{\dot{m}} \frac{\delta Q}{dt} \tag{2.44}$$

$$w_s \equiv \frac{1}{\dot{m}} \frac{\delta W_s}{dt} \tag{2.45}$$

where q and w_s represent rate quantities of heat and shaft work crossing the control surface per unit mass of fluid flowing. What are the units of q and w_s?

Our equation has become

$$q = w_s + (e + pv)_{\text{out}} - (e + pv)_{\text{in}} \tag{2.46}$$

This can be applied directly to the finite control volume shown in Figure 2.9, with the result

$$q = w_s + (e_2 + p_2 v_2) - (e_1 + p_1 v_1) \tag{2.47}$$

Detailed substitution for e [from equation (1.20)] yields

$$u_1 + p_1 v_1 + \frac{V_1^2}{2g_c} + \frac{g}{g_c} z_1 + q = u_2 + p_2 v_2 + \frac{V_2^2}{2g_c} + \frac{g}{g_c} z_2 + w_s \tag{2.48}$$

If we introduce the definition of enthalpy

$$h \equiv u + pv \tag{1.24}$$

the equation becomes more compact

$$\boxed{h_1 + \frac{V_1^2}{2g_c} + \frac{g}{g_c} z_1 + q = h_2 + \frac{V_2^2}{2g_c} + \frac{g}{g_c} z_2 + w_s} \tag{2.49}$$

This is the form of the energy equation that may be used to solve many problems. Can you list the assumptions that have been made to develop equation (2.49)?

Note that in Figure 2.9 for simplicity we have not drawn a dashed line completely surrounding the fluid inside the control volume. Rather, we have only identified sections where the fluid enters or leaves the control volume. This practice will generally be followed throughout the remainder of this book and should not cause any confusion. But remember, the analysis will always be made *for the fluid* inside the control volume.

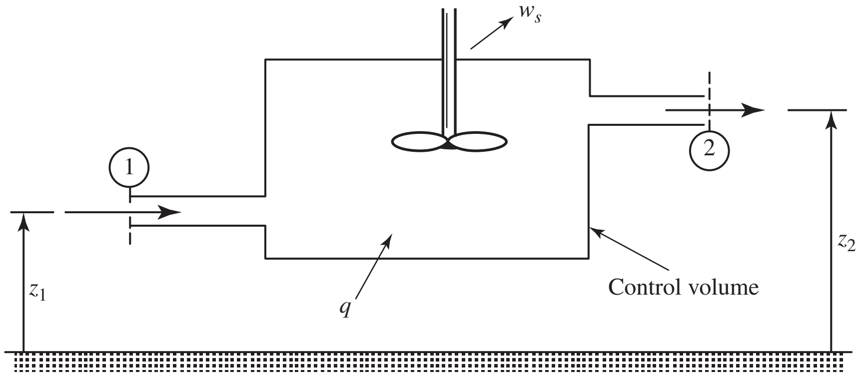


Figure 2.9 Finite control volume for energy analysis.

Example 2.2
Steam enters an ejector (Figure E2.2) at the rate of 0.1 lbm/sec with an enthalpy of 1300 Btu/lbm and negligible velocity. Water enters at the rate of 1.0 lbm/sec with an enthalpy of 40 Btu/lbm and negligible velocity. The mixture leaves the ejector with an enthalpy of 150 Btu/lbm and a velocity of 90 ft/sec. All potentials may be neglected. Determine the magnitude and direction of the heat transfer.

$$\dot{m}_1 = 0.1\,\text{lbm/sec} \quad V_1 \approx 0 \qquad h_1 = 1300\,\text{Btu/lbm}$$
$$\dot{m}_2 = 1.0\,\text{lbm/sec} \quad V_2 \approx 0 \qquad h_2 = 40\,\text{Btu/lbm}$$
$$V_3 = 90\,\text{ft/sec} \quad h_3 = 150\,\text{Btu/lbm}$$

Note the importance of making a sketch. It is necessary to establish the control volume and indicate clearly where fluid and energy cross the boundaries of the system. Identify these locations by number and list the given information with units. Make *logical* assumptions. Get in the habit of following this procedure for every problem.

Continuity:

$$\dot{m}_3 = \dot{m}_1 + \dot{m}_2 = 0.1 + 1.0 = 1.1\,\text{lbm/sec}$$

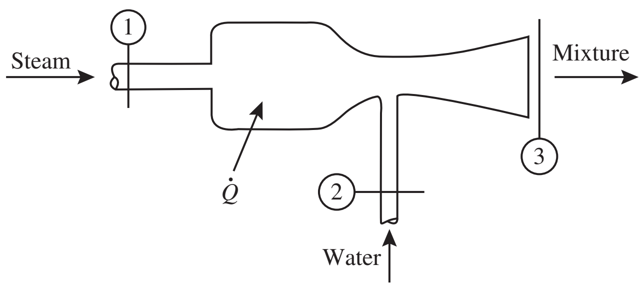


Figure E2.2

Energy:

$$\dot{m}_1\left(h_1+\frac{V_1^2}{2g_c}+\frac{g}{g_c}z_1\right)+\dot{m}_2\left(h_2+\frac{V_2^2}{2g_c}+\frac{g}{g_c}z_2\right)+\dot{Q}=\dot{m}_3\left(h_3+\frac{V_3^2}{2g_c}+\frac{g}{g_c}z_3\right)+\dot{W}_s$$

$$\dot{m}_1h_1+\dot{m}_2h_2+\dot{Q}=\dot{m}_3\left(h_3+\frac{V_3^2}{2g_c}\right)$$

$$(0.1)(1300)+(1.0)(40)+\dot{Q}=(1.1)\left[150+\frac{90^2}{(2)(32.2)(778)}\right]$$

$$130+40+\dot{Q}=(1.1)(150+0.162)=165.2$$

$$\dot{Q}=165.2-130-40=-4.8\,\text{Btu/sec}\quad(\text{or }-5.064\,\text{kW})$$

The minus sign indicates that heat is lost from the ejector.

Example 2.3
A horizontal duct of constant area contains CO_2 flowing isothermally (Figure E2.3). At a section where the pressure is 14 bar absolute, the average velocity is known to be 50 m/s. Farther downstream the pressure has dropped to 7 bar abs. Find the heat transfer. Assume one-dimensional flow.

$$\begin{array}{lll} p_1=14\times10^5\,\text{N/m}^2 & p_2=7\times10^5\,\text{N/m}^2 & w_{s(1-2)}=0\\ V_1=50\,\text{m/s} & V_2=? & q_{1-2}=?\\ z_1=z_2(\text{horizontal}) & A_1=A_2(\text{given}) & \end{array}$$

Energy:

$$h_1+\frac{V_1^2}{2g_c}+\frac{g}{g_c}z_1+q=h_2+\frac{V_2^2}{2g_c}+\frac{g}{g_c}z_2+w_s$$

Since perfect gas and isothermal, $\Delta h = c_p\,\Delta T = 0$ by equation (1.36), and thus

$$q_{1-2}=\frac{V_2^2-V_1^2}{2g_c}$$

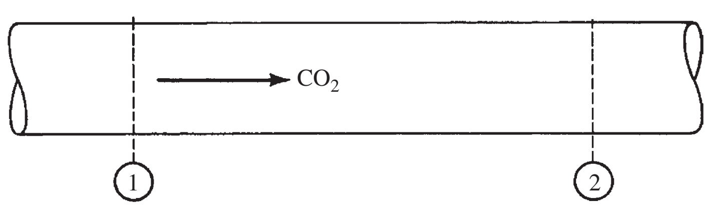


Figure E2.3

State:

$$\frac{p_1}{\rho_1 T_1} = \frac{p_2}{\rho_2 T_2} \rightarrow \frac{p_1}{p_2} = \frac{\rho_1}{\rho_2}$$

Continuity:

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

Show that

$$\frac{V_2}{V_1} = \frac{\rho_1}{\rho_2} = \frac{p_1}{p_2}$$

and thus

$$V_2 = \frac{p_1}{p_2} V_1 = \left(\frac{14 \times 10^5}{7 \times 10^5}\right)(50) = 100\,\text{m/s}$$

Returning to the energy equation, we have

$$q_{1-2} = \frac{V_2^2 - V_1^2}{2g_c} = \frac{100^2 - 50^2}{(2)(1)} = 3.750\,\text{kJ/kg} \quad (\text{or } 1.612\,\text{Btu/lbm})$$

[Note that if the flow had been *laminar* we would need to multiply each V^2 term above by 2.0, see Problem 2.4.]

Example 2.4
Air at 2200°R enters a turbine at the rate of 1.5 lbm/sec (Figure E2.4). The air expands through a pressure ratio of 15 and leaves at 1090°R. Velocities entering and leaving are negligible and there is no heat transfer. Calculate the horsepower (hp) output of the turbine.

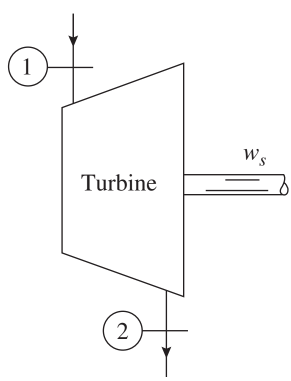


Figure E2.4

$$T_1 = 2200^\circ\text{R} \quad T_2 = 1090^\circ\text{R} \quad \dot{m} = 1.5\,\text{lbm/sec}$$
$$V_1 \approx 0 \qquad V_2 \approx 0 \qquad q = 0$$

Energy:

$$h_1 + \frac{V_1^2}{2g_c} + \frac{g}{g_c}z_1 + q = h_2 + \frac{V_2^2}{2g_c} + \frac{g}{g_c}z_2 + w_s$$

$$w_s = h_1 - h_2 = c_p(T_1 - T_2)$$
$$= (0.24)(2200 - 1090) = 266\,\text{Btu/lbm} \quad (\text{or } 619\,\text{kJ/kg})$$
$$\text{hp} = \dot{m}w_s\left(\frac{778}{550}\right) = (1.5)(266)\left(\frac{778}{550}\right) = 564\,\text{hp}$$

Differential Form of Energy Equation

One can also apply the energy equation to a differential control volume, as shown in Figure 2.10. We assume steady, one-dimensional flow. The properties of the fluid entering the control volume are designated as ρ, u, p, V, and so on. Fluid leaves the control volume with properties that have changed slightly as indicated by $\rho + d\rho$, $u + du$, and so on. Application of equation (2.46) to this differential control volume will produce

$$\delta q = \delta w_s + \left[(p + dp)(v + dv) + (u + du) + \frac{(V + dV)^2}{2g_c} + \frac{g}{g_c}(z + dz)\right] - \left(pv + u + \frac{V^2}{2g_c} + \frac{g}{g_c}z\right) \tag{2.50}$$

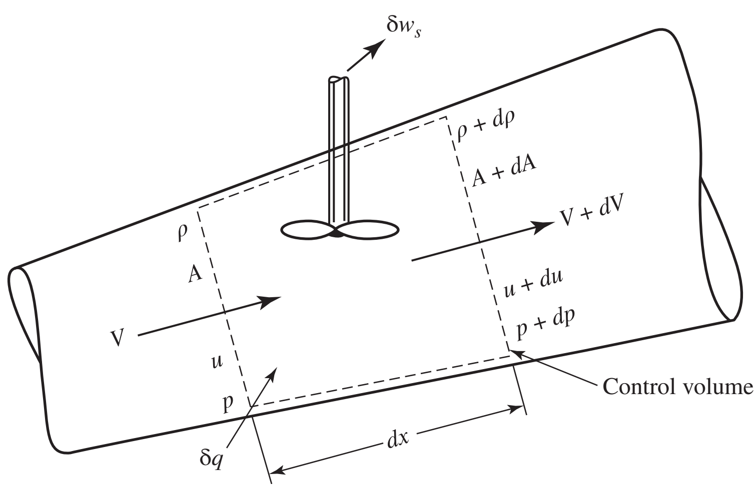


Figure 2.10 Energy analysis on infinitesimal control volume.

Expand equation (2.50), cancel like terms, and show that

$$\delta q = \delta w_s + p\,dv + v\,dp + dp\overset{\text{HOT}}{\cancel{dv}} + du + \frac{2V\,dV + (\overset{\text{HOT}}{\cancel{dV}})^2}{2g_c} + \frac{g}{g_c}\,dz \qquad (2.51)$$

As dx is allowed to approach zero, we can neglect any higher-order terms (HOT). Noting that

$$2VdV = dV^2$$

and

$$pdv + vdp = d(pv)$$

we obtain

$$\delta q = \delta w_s + d(pv) + du + \frac{dV^2}{2g_c} + \frac{g}{g_c}dz \qquad (2.52)$$

and since

$$dh = du + d(pv)$$

we have

$$\boxed{\delta q = \delta w_s + dh + \frac{dV^2}{2g_c} + \frac{g}{g_c}dz} \qquad (2.53)$$

This can be integrated directly to produce equation (2.49) for a finite control volume, but the differential form is frequently of considerable value by itself. The technique of analyzing a differential control volume is also an important one that we shall use many times.

2.7 SUMMARY

In the study of gas dynamics, as in any branch of fluid dynamics, most analyses are made on a control volume. We have shown how the material derivative of any mass-dependent property can be transformed into an equivalent expression for use with control volumes. We then applied this relation [equation (2.22)] to show how the basic laws regarding conservation of mass and energy can be converted from a control mass analysis into a form suitable for control volume analysis. Most of the work in this course will be done assuming steady one-dimensional flow; thus each general equation has been simplified for these conditions.

Care should be taken to approach each problem in a consistent and organized fashion. For a typical problem, the following steps should be taken:

1. Sketch the flow system and identify the control volume.
2. Label sections where fluid enters and leaves the control volume.
3. Note where energy (Q and W_s) crosses the control surface.
4. Record all known quantities with their units.
5. Make any *logical* assumptions regarding unknown information.
6. Solve for the unknowns by a systematic application of the basic equations.

The *basic concepts* that we have used so far are few in number:

State: a simple density relation such as $p = \rho RT$ or $\rho = \text{constant}$
Continuity: derived from conservation of mass
Energy: derived from conservation of energy

Some of the most frequently used equations that were developed in this unit are summarized below. Some are restricted to steady one-dimensional flow; others involve additional assumptions. You should know under what conditions each may be used.

1. *Mass flow rate past a section*

$$\dot{m} = \int_A \rho u dA \tag{2.8}$$

$u = \text{velocity perpendicular to } dA$

2. *Transformation of material derivative to control volume analysis*

$$\frac{dN}{dt} = \frac{\partial}{\partial t}\int_{\text{cv}} \eta\rho d\tilde{v} + \int_{\text{cs}} \eta\rho(\mathbf{V}\cdot\hat{n})dA \tag{2.22}$$

If one-dimensional,

$$\int_{\text{cs}} \eta\rho(\mathbf{V}\cdot\hat{n})dA = \sum \dot{m}\eta \tag{2.54}$$

If steady,

$$\frac{\partial(\cdot)}{\partial t} = 0 \tag{2.6}$$

3. *Mass conservation—continuity equation* $\begin{cases} N = \text{mass} \\ \eta = 1 \end{cases}$

$$\frac{\partial}{\partial t}\int_{\text{cv}} \rho d\tilde{v} + \int_{\text{cs}} \rho(\mathbf{V}\cdot\hat{n})dA = 0 \tag{2.25}$$

For steady one-dimensional flow,

$$\dot{m} = \rho AV = \text{const} \tag{2.30}$$

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0 \tag{2.32}$$

4. *Energy conservation—energy equation* $\begin{cases} N = E \\ \eta = e = u + V^2/2g_c + (g/g_c)z \end{cases}$

$$\frac{\delta Q}{dt} = \frac{\delta W}{dt} + \frac{\partial}{\partial t}\int_{\text{cv}} e\rho d\tilde{v} + \int_{\text{cs}} e\rho(\mathbf{V}\cdot\hat{n})dA \tag{2.35}$$

Introduce: $w = \text{shaft work } (w_s) + \text{flow work } (pv)$

For steady one-dimensional flow we get,

$$h_1 + \frac{V_1^2}{2g_c} + \frac{g}{g_c}z_1 + q = h_2 + \frac{V_2^2}{2g_c} + \frac{g}{g_c}z_2 + w_s \tag{2.49}$$

$$\delta q = \delta w_s + dh + \frac{dV^2}{2g_c} + \frac{g}{g_c}dz \tag{2.53}$$

PROBLEMS

Problem statements may occasionally give some irrelevant information; on the other hand, sometimes logical assumptions have to be made before a solution can be carried out. For example, unless specific information is given on potential differences, it is logical to assume that these are negligible; if no machine is present, it is reasonable to assume that $w_s = 0$, and so on. However, think carefully before arbitrarily eliminating terms from any equation—you may be eliminating a vital element from the problem. Check to see

whether there is some way to compute the desired quantity (such as calculating the enthalpy of a gas from its temperature). Properties of selected gases are provided in Appendixes A and B.

2.1. There is three-dimensional flow of an incompressible fluid in a duct of radius R. The velocity distribution at any section is hemispherical, with the maximum velocity U_m at the center and zero velocity at the wall. Show that the mass average velocity is $\frac{2}{3}U_m$.

2.2. A constant-density fluid flows between two flat parallel plates that are separated by a distance δ (Figure P2.2). Sketch the velocity distribution and compute the mass average velocity based on the velocity u given by:

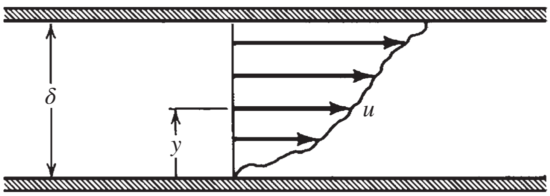


Figure P2.2

(a) $u = k_1 y$.
(b) $u = k_2 y^2$.
(c) $u = k_3(\delta y - y^2)$.
In each case, express your answer in terms of the maximum velocity U_m.

2.3. An incompressible fluid is flowing in a rectangular duct whose dimensions are 2 units in the Y-direction and 1 unit in the Z-direction. The velocity in the X-direction is given by $u = 3y^2 + 5z$. Compute the mass average velocity.

2.4. Laminar flow in circular ducts is *not* one-dimensional, but we may still use the equivalent mass-average velocity $V = U_m/2$ from equations (2.10) and (2.11) in our one-dimensional formulations. This velocity came from solving $\int u dA \equiv (\pi r_0^2)V$. Now, in the energy equation (2.36) we encounter the integral $\int \rho e(\mathbf{V}\cdot\hat{n})dA$ over the surface in Figure 2.1, where e has the kinetic energy content given in equation (1.20); this is equivalent to calculating $\int u^3 dA = 2.0(\pi r_0^2)V^3$ where V relates to U_m as indicated above. Verify that for laminar flow the resulting kinetic energy terms in equation (2.49) need to be multiplied by the factor 2.0 as mentioned above and in the text. Assume that the density remains constant along each cross section (Figure P2.4).

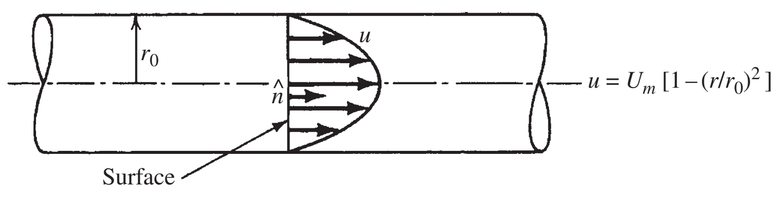


Figure P2.4

2.5. In a 10-in.-diameter duct the mass average velocity of water is 14 ft/sec.
(a) What is the average velocity if the diameter changes to 6 in.?
(b) Express the average velocity in terms of an arbitrary diameter.

2.6. Nitrogen flows in a constant-area duct. Conditions at section 1 are as follows: p_1 = 200 psia, T_1 = 90°F, and V_1 = 10 ft/sec. At section 2, we find that p_2 = 45 psia and T_2 = 90°F. Determine the velocity at section 2.

2.7. Steam enters a turbine with an enthalpy of 1600 Btu/lbm and a velocity of 100 ft/sec at a flow rate of 80,000 lbm/hr. The steam leaves the turbine with an enthalpy of 995 Btu/lbm and a velocity of 150 ft/sec. Compute the power output of the turbine, assuming it to be 100% efficient. Neglect any heat transfer and potential energy changes.

2.8. A flow of 2.0 lbm/sec of air is compressed from 14.7 psia and 60°F to 200 psia and 150°F. Cooling water circulates around the cylinders at the rate of 25 lbm/min. The water enters at 45°F and leaves at 130°F. (The specific heat of water is 1.0 Btu/lbm-°F.) Calculate the power required to compress the air, assuming negligible velocities at inlet and outlet.

2.9. Hydrogen expands isentropically from 15 bar absolute and 340 K to 3 bar absolute in a steady flow process without heat transfer.
(a) Compute the final velocity if the initial velocity is negligible.
(b) Compute the flow rate if the final duct size is 10 cm in diameter.

2.10. At a section where the diameter is 4 in., methane flows with a velocity of 50 ft/sec and a pressure of 85 psia. At a downstream section, where the diameter has increased to 6 in., the pressure is 45 psia. Assuming the flow to be isothermal, compute the heat transfer between the two locations.

2.11. Carbon dioxide flows in a horizontal duct at 7 bar abs. and 300 K with a velocity of 10 m/s. At a downstream location the pressure is 3.5 bar abs. and the temperature is 280 K. If 1.4×10^4 J/kg of heat is lost by the fluid between these locations:
(a) Determine the velocity at the second location.
(b) Compute the ratio of initial to final areas.

2.12. Hydrogen flows through a horizontal insulated duct. At section 1, the enthalpy is 2400 Btu/lbm, the density is 0.5 lbm/ft^3, and the velocity is 500 ft/sec. At a downstream section, h_2 = 2240 Btu/lbm and ρ_2 = 0.1 lbm/ft^3. No shaft work is done. Determine (a) the velocity at section 2 and (b) the ratio of areas.

2.13. Nitrogen, traveling at 12 m/s with a pressure of 14 bar abs. at a temperature of 800 K, enters a device with an area of 0.05 m^2. No work or heat transfer takes place. The temperature at the exit, where the area is 0.15 m^2, has dropped to 590 K. What are (a) the velocity and (b) pressure at the outlet section?

2.14. Cold water with an enthalpy of 8 Btu/lbm enters a heater at the rate of 5 lbm/sec with a velocity of 10 ft/sec and at a potential of 10 ft with respect to the other connections shown in Figure P2.14. Steam enters at the rate of 1 lbm/sec with a

velocity of 50 ft/sec and an enthalpy of 1350 Btu/lbm. These two streams mix in the heater, and hot water emerges with an enthalpy of 168 Btu/lbm and a velocity of 12 ft/sec.

(a) Determine the heat lost from the apparatus.

(b) What percentage error is involved if both kinetic and potential energy changes are neglected?

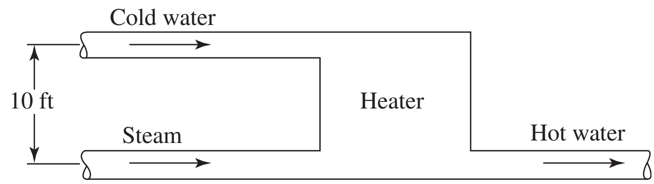


Figure P2.14

2.15. The control volume shown in Figure P2.15 has steady, incompressible flow, and all properties are uniform at the inlet and outlet. For $u_1 = 1.256$ MJ/kg and $u_2 = 1.340$ MJ/kg and $\rho = 10$ kg/m^3, calculate the work if there is a heat output of 0.35 MJ/kg.

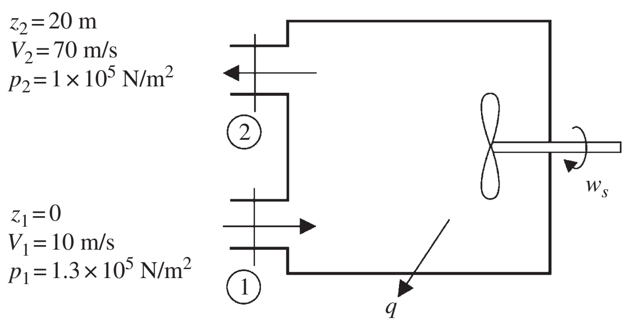


Figure P2.15

CHECK TEST

You should be able to complete this test without reference to material in the chapter.

2.1. Name the basic concepts (or equations) from which the study of gas dynamics proceeds.

2.2. Define *steady flow*. Explain what is meant by *one-dimensional flow*.

2.3. An incompressible fluid flows in a duct of radius r_0. At a particular location, the velocity distribution is $u = U_m[1 - (r/r_0)^2]$ and the distribution of an extensive property is $\beta = B_m[1 - (r/r_0)]$. Evaluate the integral $\int \rho\beta(\mathbf{V}\cdot\hat{n})\, dA$ at this location.

2.4. Write the equation used to relate the material derivative of any mass-dependent property to the properties inside, and crossing the boundaries of, a control volume. State in words what the integrals actually represent.

2.5. Simplify the integral $\int_{cs} \rho\beta(\mathbf{V}\cdot\hat{n})\, dA$ for the control volume shown in Figure CT2.5 if the flow is steady and one-dimensional. (*Careful:* β and ρ are constant across the areas shown but may vary from section to section.)

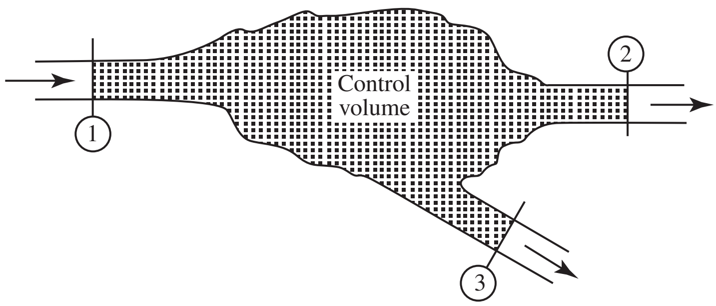


Figure CT2.5

2.6. Write the simplest form of the energy equation that you would use to analyze the control volume shown in Figure CT2.6. You may assume steady one-dimensional flow.

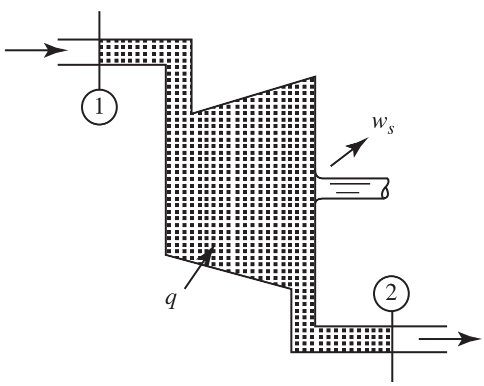


Figure CT2.6

2.7. Work Problem **2.13**.

Chapter 3

Control Volume Analysis—Part II

3.1 INTRODUCTION

We begin this chapter with a discussion of entropy, which happens to be one of the most useful thermodynamic properties in the study of gas dynamics. Entropy changes will be divided into two categories, to facilitate a better understanding of this important property. Next, we introduce the concept of a *stagnation* process. This leads to the *stagnation state* as a reference condition, which will be used throughout our remaining discussions. These ideas permit rewriting our energy equations in alternative forms from which interesting observations can be made.

We then investigate some of the consequences of a constant-density fluid. This leads to special relations that can be used not only for liquids but under certain conditions are excellent approximations for gases. At the close of the chapter we complete our basic set of equations by transforming Newton's second law for use in the analysis of control volumes. This is done for both finite and differential volume elements.

3.2 OBJECTIVES

After completing this chapter successfully, you should be able to:

1. Explain how entropy changes can be divided into two categories. Define and interpret each part.

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

2. Define an isentropic process and explain the relationships among reversible, adiabatic, and isentropic processes.
3. (*Optional*) Show that by introducing the concept of entropy and the definition of enthalpy, the path function heat (δQ) may be removed from the energy equation to yield an expression called the *pressure–energy equation*:

$$\frac{dp}{\rho}+\frac{dV^2}{2g_c}+\frac{g}{g_c}dz+Tds_i+\delta w_s=0$$

4. (*Optional*) Simplify the pressure-energy equation to obtain Bernoulli's equation. Specify all assumptions or restrictions that apply to Bernoulli's equation.
5. Explain the *stagnation state concept* and the difference between *static and stagnation properties*.
6. Define stagnation enthalpy by an equation that is valid for *any* fluid.
7. Draw an *h–s* diagram representing a flow system and indicate *static and stagnation points* for an arbitrary section.
8. (*Optional*) Introduce the stagnation concept into the energy equation and derive the *stagnation pressure–energy* equation:

$$\frac{dp_t}{\rho_t}+ds_e(T_t-T)+T_t ds_i+\delta w_s=0$$

9. Demonstrate the ability to apply continuity and energy concepts to the solution of typical flow problems with constant-density fluids.
10. (*Optional*) Obtain the integral form of the momentum equation for a control volume, given the basic concept or equation that is valid for a control mass.
11. Simplify the integral form of the momentum equation for a control volume for the conditions of steady one-dimensional flow.
12. Demonstrate the ability to apply momentum concepts in the analysis of control volumes.

3.3 COMMENTS ON ENTROPY

In Section 1.4 entropy changes were defined in the usual manner in terms of reversible processes:

$$\Delta S\equiv\int\frac{\delta Q_R}{T} \tag{1.28}$$

The term δQ_R is related to a fictitious reversible process (a rare happening indeed), and consequently, it may not represent the total entropy change in the process under consideration. It would seem more appropriate to work with the *actual* heat transfer for the real process. To accomplish this it is necessary to divide the entropy changes of any system into two categories. We shall follow the notation of Hall (Ref. 15). Let

$$dS \equiv dS_e + dS_i \tag{3.1}$$

The term dS_e represents that portion of entropy change caused by the actual heat transfer between the system and its (external) surroundings. It can be evaluated readily from

$$dS_e = \frac{\delta Q}{T} \tag{3.2}$$

One should note that dS_e can be either positive or negative, depending on the direction of heat transfer. If heat is removed from a system, δQ is negative and thus dS_e will be negative. Obviously, $dS_e = 0$ for any and all adiabatic processes.

The term dS_i represents that portion of entropy change caused by *irreversible* effects. Moreover, dS_i effects are strictly internal in nature, such as temperature and pressure gradients within the system or friction along the internal boundaries of the system. Note that this term depends on the process path and from observations we know that *all irreversibilities generate entropy* (i.e., cause the entropy of the system to increase). Thus, we could say that $dS_i \geq 0$. Obviously, $dS_i = 0$ only for a fully reversible process.

Recall that an isentropic process is the one with constant entropy. This is also represented by writing $dS = 0$. The equation

$$dS = dS_e + dS_i \tag{3.1}$$

confirms the well-known fact that a reversible-adiabatic process is also isentropic. It also clearly shows that the converse is not necessarily true: an isentropic process does not have to be reversible and adiabatic. If isentropic, we merely know that

$$dS = 0 = dS_e + dS_i \tag{3.3}$$

If an isentropic process is known to contain irreversibilities, what can be said about the direction of heat transfer? Note that dS_e and dS_i are unusual mathematical quantities and perhaps require a symbol other than the common one used for an exact differential. But in this book, we continue with the notation of equation (3.1) because it is the most commonly used.

Another familiar relation can be developed by taking the cyclic integral of equation (3.1):

$$\oint dS = \oint dS_e + \oint dS_i \tag{3.4}$$

Since a cyclic integral must be taken around a closed path and entropy (S) is a property,

$$\oint dS = 0 \tag{3.5}$$

We know that irreversible effects always generate entropy, so

$$\oint dS_i \geq 0 \tag{3.6}$$

with the equal sign holding only for a reversible cycle.

Thus

$$0 = \oint dS_e + (\geq 0) \tag{3.7}$$

and since

$$dS_e = \frac{\delta Q}{T} \tag{3.2}$$

then

$$\oint \frac{\delta Q}{T} \leq 0 \tag{3.8}$$

which is known as *the inequality of Clausius*.

The expressions above can be written for a unit mass, in which case we have

$$\boxed{\begin{aligned} ds &= ds_e + ds_i \qquad (3.9) \\ ds_e &= \frac{\delta q}{T} \qquad (3.10) \end{aligned}}$$

3.4 PRESSURE-ENERGY EQUATION

We are now ready to develop a very useful equation. Starting with the thermodynamic property relation

$$Tds = dh - vdp \tag{1.31}$$

we introduce $ds = ds_e + ds_i$ and $v = 1/\rho$, to obtain

$$Tds_e + Tds_i = dh - \frac{dp}{\rho}$$

or

$$dh = Tds_e + Tds_i + \frac{dp}{\rho} \tag{3.11}$$

Recalling the energy equation from Section 2.6,

$$\delta q = \delta w_s + dh + \frac{dV^2}{2g_c} + \frac{g}{g_c}dz \tag{2.53}$$

we now substitute for dh from (3.11) and obtain

$$\cancel{\delta} q = \delta w_s + \left(T \cancel{d} s_e + Tds_i + \frac{dp}{\rho} \right) + \frac{dV^2}{2g_c} + \frac{g}{g_c}dz \tag{3.12}$$

Recognize [from equation (3.10)] that $\delta q = Tds_e$ and we obtain a form of the energy equation which is often called the *pressure–energy equation*:

$$\boxed{\frac{dp}{\rho} + \frac{dV^2}{2g_c} + \frac{g}{g_c}dz + \delta w_s + T\,ds_i = 0} \tag{3.13}$$

Notice that even though the heat term (δq) does *not* appear in this equation, it is still applicable to cases that involve heat transfer.

Equation (3.13) can readily be simplified for special cases. For instance, if no shaft work crosses the boundary ($\delta w_s = 0$) and if there are no losses ($ds_i = 0$), then

$$\frac{dp}{\rho} + \frac{dV^2}{2g_c} + \frac{g}{g_c}dz = 0 \tag{3.14}$$

This is called *Euler's equation* and can be integrated when we know the functional relationship that exists between the pressure and density.

Example 3.1
Integrate Euler's equation for the case of isothermal flow of a perfect gas.

$$\int_1^2 \frac{dp}{\rho} + \int_1^2 \frac{dV^2}{2g_c} + \int_1^2 \frac{g}{g_c} dz = 0$$

For isothermal flow, $pv = const$ *or* p/ρ = c. *Thus,*

$$\int_1^2 \frac{dp}{\rho} = c\int_1^2 \frac{dp}{p} = c\ln\frac{p_2}{p_1} = \frac{p}{\rho}\ln\frac{p_2}{p_1} = RT\ln\frac{p_2}{p_1}$$

and

$$RT\ln\frac{p_2}{p_1} + \frac{V_2^2 - V_1^2}{2g_c} + \frac{g}{g_c}(z_2 - z_1) = 0$$

The important special case of incompressible fluids is considered in Section 3.7.

3.5 THE STAGNATION CONCEPT

When we speak of the thermodynamic state of a flowing medium and mention its properties (e.g., temperature, pressure), there may be some question as to what these properties actually represent or how they can be measured. Imagine that you have been miniaturized and put aboard a small submarine that is drifting along *with the fluid*. (An alternative might be to "saddle-up" a small fluid particle and take a ride.) If you had a thermometer and pressure gauge with you, they would indicate the temperature and pressure corresponding to the *static* state of the fluid, although the word *static* is usually omitted. Thus *the static properties are those that would be measured if you moved with the fluid*.

It is convenient to introduce the concept of a *stagnation state*. This is a reference state defined as that thermodynamic state which would exist if the fluid were brought to zero velocity and zero potential. To yield a consistent reference state, we must qualify how this *stagnation process* should be accomplished. The stagnation state must be reached

(1) without any energy exchange ($Q = W = 0$) and
(2) without losses.

By virtue of (1), $ds_e = 0$; and from (2), $ds_i = 0$. Thus *the stagnation process is isentropic*!

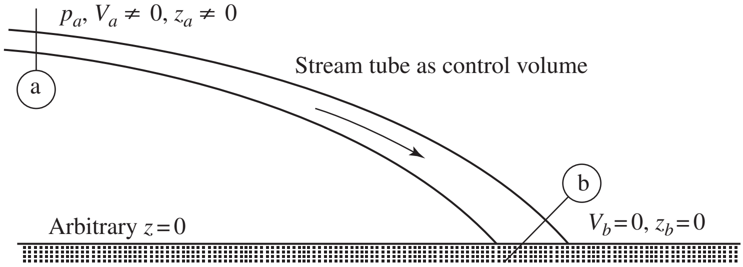


Figure 3.1 Stagnation process.

We can imagine the following example of actually carrying out the stagnation process. Consider a fluid that is flowing and has the static properties shown as (a) in Figure 3.1. At location (b) the fluid has been brought to zero velocity and zero potential under the foregoing restrictions. If we apply the energy equation to the control volume indicated for steady one-dimensional flow, we have

$$h_a + \frac{V_a^2}{2g_c} + \frac{g}{g_c}z_a + \cancel{q} = h_b + \cancel{\frac{V_b^2}{2g_c}} + \cancel{\frac{g}{g_c}z_b} + \cancel{w_s} \tag{2.49}$$

which simplifies to

$$h_a + \frac{V_a^2}{2g_c} + \frac{g}{g_c}z_a = h_b \tag{3.15}$$

But condition (b) represents the *stagnation state* corresponding to the *static state* (a). Thus we call h_b the *stagnation* or *total enthalpy* corresponding to state (a) and designate it as h_{ta}. Thus

$$h_{ta} = h_a + \frac{V_a^2}{2g_c} + \frac{g}{g_c}z_a \tag{3.16}$$

Or for any state, we have in general,

$$\boxed{h_t = h + \frac{V^2}{2g_c} + \frac{g}{g_c}z} \tag{3.17}$$

This is an important relation that is *always* valid. *Learn it*! When dealing with gases, potential changes can usually be neglected, and we write

$$h_t = h + \frac{V^2}{2g_c} \tag{3.18}$$

Example 3.2
Nitrogen at 500°R is flowing at 1800 ft/sec. What are the static and stagnation enthalpies?

$$h = c_p T = (0.248)(500) = 124 \text{ Btu/lbm}$$

$$\frac{V^2}{2g_c} = \frac{(1800)^2}{(2)(32.2)(778)} = 64.7 \text{ Btu/lbm}$$

$$h_t = h + \frac{V^2}{2g_c} = 124 + 64.7 = 188.7 \text{ Btu/lbm} \quad (\text{or } 438.9 \text{ kJ/kg})$$

Introduction of the stagnation (or total) enthalpy makes it possible to write equations in a more compact form. For example, the one-dimensional steady-flow energy equation

$$h_1 + \frac{V_1^2}{2g_c} + \frac{g}{g_c} z_1 + q = h_2 + \frac{V_2^2}{2g_c} + \frac{g}{g_c} z_2 + w_s \tag{2.49}$$

becomes

$$\boxed{h_{t1} + q = h_{t2} + w_s} \tag{3.19}$$

Furthermore,

$$\delta q = \delta w_s + dh + \frac{dV^2}{2g_c} + \frac{g}{g_c} dz \tag{2.53}$$

becomes

$$\boxed{\delta q = \delta w_s + dh_t} \tag{3.20}$$

Equation (3.19) [or (3.20)] shows that in any adiabatic, no-work, steady, one-dimensional flow system, the stagnation enthalpy remains constant, *irrespective of the losses*. What else can be said here if the fluid is a perfect gas?

You should realize here that the stagnation state is a *reference* state that may or may not actually exist in the flow system. Also, in general cases, each point or location in a flow system may have a different stagnation state, as shown in Figure 3.2. Remember that although the hypothetical process from 1 to 1_t must be reversible and adiabatic (as well as the process from 2 to 2_t), this in *no* way restricts the actual process that may exist in the flow system between 1 and 2.

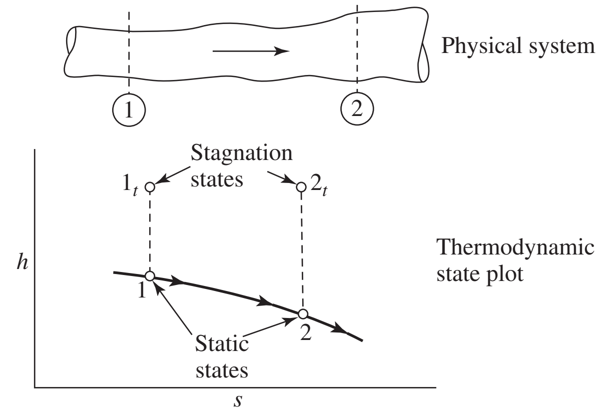


Figure 3.2 *h–s* diagram showing static and stagnation states.

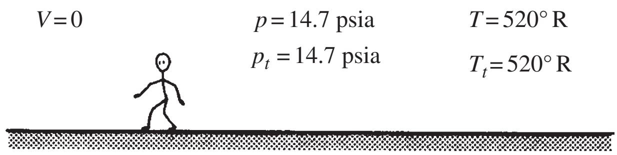


Figure 3.3 Earth as a frame of reference.

Also, one must realize that when the frame of reference for the control volume is changed, stagnation conditions change, although the static conditions remain the same. (Recall that static properties are defined as those that would be measured if the measuring devices moved with the fluid.) Consider still air with Earth as a reference frame (see Figure 3.3). In this case, since the velocity is zero (with respect to the frame of reference), the static and stagnation conditions are the same.

Now let's change the frame of reference by flying through this same air on a missile at 600 ft/sec (see Figure 3.4). As we look forward it appears that the air is coming at us at 600 ft/sec. The *static* pressure and temperature of the air remain *unchanged* at 14.7 psia and 520°R, respectively. However, in this case, the air has a velocity (with respect to the moving frame of reference) and thus the stagnation conditions are different from the static conditions. You should always remember that the stagnation reference state is completely dependent on the frame of reference used to specify velocities. (Changing the arbitrary $z = 0$ reference would also affect the stagnation conditions, but we shall not become involved with this situation.) You will soon learn how to compute stagnation properties other than enthalpy. Incidentally, is there any place in Figure 3.4 where the stagnation conditions *actually* exist? Is the fluid brought to rest at any missile location?

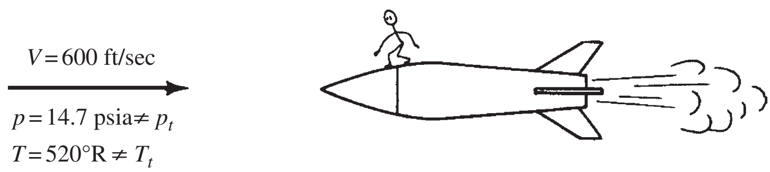


Figure 3.4 Missile as a frame of reference.

3.6 STAGNATION PRESSURE-ENERGY EQUATION

Consider the two section locations on the physical system shown in Figure 3.2. If we let the distance between these locations approach zero, we are dealing with an infinitesimal control volume with the thermodynamic states differentially separated, as shown in Figure 3.5. Also shown are the corresponding stagnation states for these two locations.

We may write the following property relation between points 1 and 2:

$$Tds = dh - vdp \tag{1.31}$$

Note that even though the stagnation states do not actually exist, they represent legitimate thermodynamic states, and thus any valid property relation or equation may be applied to these points. Thus we may also apply equation (1.31) between states 1_t and 2_t:

$$T_t ds_t = dh_t - v_t dp_t \tag{3.21}$$

However, because the stagnation state is reached reversibly and adiabatically

$$ds_t = ds \tag{3.22}$$

And from equation (3.9)

$$ds = ds_e + ds_i \tag{3.9}$$

Thus we may write

$$T_t(ds_e + ds_i) = dh_t - v_t dp_t \tag{3.23}$$

Recall the energy equation written in the form

$$\delta q = \delta w_s + dh_t \tag{3.20}$$

By substituting dh_t from equation (3.23) into (3.20), we obtain

$$\delta q = \delta w_s + T_t(ds_e + ds_i) + v_t dp_t \tag{3.24}$$

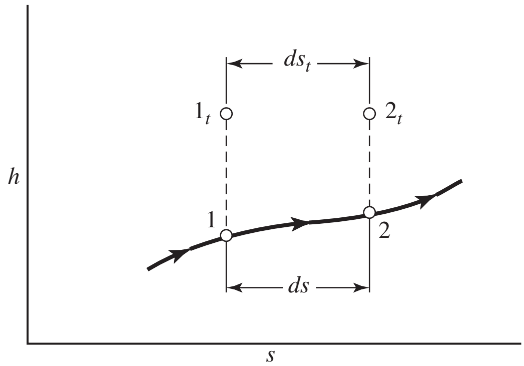


Figure 3.5 Infinitesimally separated static states with associated stagnation states.

Now also recall that

$$\delta q = T\,ds_e \tag{3.10}$$

Substitute equation (3.10) into (3.24) and note that $v_t = 1/\rho_t$ [from equation (1.8)] and you should obtain the following equation, called the *stagnation pressure–energy equation*:

$$\boxed{\frac{dp_t}{\rho_t} + ds_e(T_t - T) + T_t ds_i + \delta w_s = 0} \tag{3.25}$$

Consider what happens under the following assumptions:

a. There is no shaft work $\rightarrow \delta w_s = 0$
b. There is no heat transfer $\rightarrow ds_e = 0$
c. There are no losses $\rightarrow ds_i = 0$

Under these conditions, equation (3.25) becomes

$$\frac{dp_t}{\rho_t} = 0 \tag{3.26}$$

and since ρ_t cannot be infinite,

$$dp_t = 0$$

or

$$p_t = \text{constant} \tag{3.27}$$

Note that, in general, the total pressure will *not* remain constant; only under a very special set of circumstances will equation (3.27) hold true. What are these circumstances?

Many flow systems are adiabatic and contain no shaft work. For these systems,

$$\frac{dp_t}{\rho_t} + T_t ds_i = 0 \tag{3.28}$$

so that any losses are clearly reflected by a change in stagnation pressure. Will the stagnation pressure always increase or decrease if there are losses in this system? This important point will be discussed many times as we examine various flow systems in the remainder of the book.

3.7 CONSEQUENCES OF CONSTANT DENSITY

The density of a liquid is nearly constant and in Chapter 4 we shall see that under certain circumstances, gases change their density very little. Thus it will be useful to examine the form that some of our equations take for the special case of constant density.

Energy Relations

We start with the pressure–energy equation:

$$\frac{dp}{\rho} + \frac{dV^2}{2g_c} + \frac{g}{g_c}dz + \delta w_s + T ds_i = 0 \tag{3.13}$$

If $\rho = \text{const}$, we can easily integrate (3.13) between points 1 and 2 of a flow system:

$$\frac{p_2 - p_1}{\rho} + \frac{V_2^2 - V_1^2}{2g_c} + \frac{g}{g_c}(z_2 - z_1) + w_s + \int_1^2 T ds_i = 0$$

or

$$\frac{p_1}{\rho} + \frac{V_1^2}{2g_c} + \frac{g}{g_c}z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2g_c} + \frac{g}{g_c}z_2 + \int_1^2 T ds_i + w_s \tag{3.29}$$

Compare (3.29) to another form of the energy equation (namely, 2.48) and *show* that

$$\int_1^2 T ds_i = u_2 - u_1 - q \tag{3.30}$$

Does this result seem reasonable? To determine this, let us examine two extreme cases for the flow of a constant-density fluid. For the first case, assume that the system is perfectly insulated. Since the integral of Tds_i is a positive quantity, equation (3.30) shows that the losses (i.e., irreversible effects) will cause an increase in internal energy, which means there will be a temperature increase. Next, consider an isothermal system. For this case, how will losses manifest themselves?

For the flow of a constant-density fluid, "losses" must appear in some combination of the two forms described earlier. In either case, mechanical energy has been degraded into a less useful form—thermal energy. Thus, when dealing with constant-density fluids, we normally use a single loss term and generally refer to it as a *head loss* or *friction loss*, using the symbol h_ℓ or h_f in place of $\int Tds_i$. If you have studied fluid mechanics, you have undoubtedly used equation (3.29) in the form

$$\boxed{\frac{p_1}{\rho}+\frac{V_1^2}{2g_c}+\frac{g}{g_c}z_1=\frac{p_2}{\rho}+\frac{V_2^2}{2g_c}+\frac{g}{g_c}z_2+h_\ell+w_s} \tag{3.31}$$

How many restrictions and/or assumptions are embodied in equation (3.31)?

Example 3.3

A turbine extracts 300 ft-lbf/lbm of water flowing from a reservoir (Figure E3.3). Frictional losses amount to $8V_p^2/2g_c$, where V_p is the velocity in a 2-ft-diameter pipe. Compute the power output of the turbine if it is 100% efficient and the available potential is 350 ft.

$$p_1 = p_{\text{atm}} \qquad p_2 = p_{\text{atm}} \qquad w_s = 300\,\text{ft-lbf/lbm}$$

$$V_1 \approx 0 \qquad V_2 \approx 0 \qquad h_\ell = 8\frac{V_p^2}{2g_c}$$

$$z_1 = 350\,\text{ft} \qquad z_2 = 0$$

Note how the sections are chosen to make application of the energy equation easier.

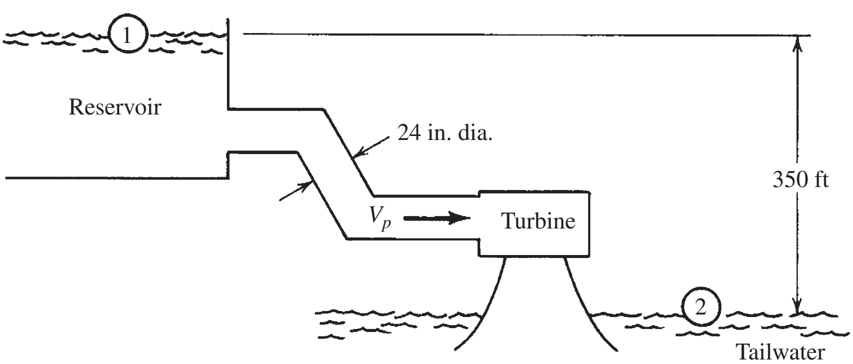


Figure E3.3

Energy:

$$\cancel{\frac{p_1}{\rho}} + \cancel{\frac{V_1^2}{2g_c}} + \frac{g}{g_c}z_1 = \cancel{\frac{p_2}{\rho}} + \cancel{\frac{V_2^2}{2g_c}} + \cancel{\frac{g}{g_c}z_2} + h_\ell + w_s$$

$$\left(\frac{32.2}{32.2}\right)(350) = \frac{8\,V_p^2}{2g_c} + 300$$

$$V_p^2 = \frac{2g_c(350-300)}{8} = 402.5$$

$$V_p = 20.1\ \text{ft/sec}$$

Flow rate:

$$\dot{m} = \rho AV = 62.4(\pi)20.1 = 3940\ \text{lbm/sec}$$

Power:

$$\text{hp} = \frac{\dot{m}w_s}{550} = \frac{(3940)(300)}{550} = 2150\ \text{hp} \quad (\text{or } 1.603\ \text{MW})$$

We can further restrict constant density flows to one in which no shaft work and no losses occur. In this case, equation (3.31) simplifies to

$$\frac{p_1}{\rho} + \frac{V_1^2}{2g_c} + \frac{g}{g_c}z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2g_c} + \frac{g}{g_c}z_2$$

or

$$\boxed{\frac{p}{\rho} + \frac{V^2}{2g_c} + \frac{g}{g_c}z = \text{const}} \tag{3.32}$$

This is called *Bernoulli's equation* and could also have been obtained directly by integrating Euler's equation (3.14) for a constant-density fluid. How many assumptions have been made to arrive at Bernoulli's equation?

Example 3.4

Water flows in a 6-in.-diameter duct with a velocity of 15 ft/sec. Within a short distance the duct converges to 3 in. in diameter. Find the pressure change if there are no losses between these two sections.

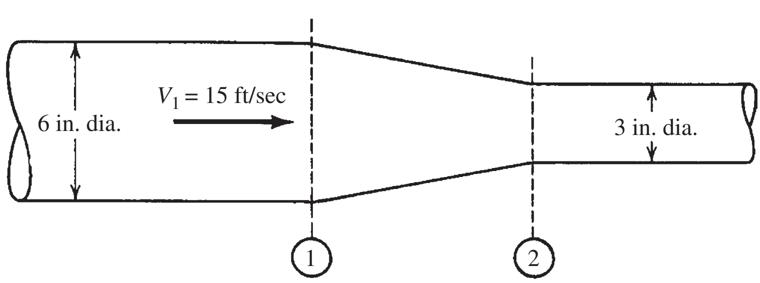


Figure E3.4

Bernoulli:

$$\frac{p_1}{\rho} + \frac{V_1^2}{2g_c} + \cancel{\frac{g}{g_c}}z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2g_c} + \cancel{\frac{g}{g_c}}z_2$$

$$p_1 - p_2 = \frac{\rho}{2g_c}\left(V_2^2 - V_1^2\right)$$

Continuity:

$$\cancel{\rho_1}A_1V_1 = \cancel{\rho_2}A_2V_2$$

$$V_2 = V_1\frac{A_1}{A_2} = V_1\left(\frac{D_1}{D_2}\right)^2 = (15)\left(\frac{6}{3}\right)^2 = 60\,\text{ft/sec}$$

Thus,

$$p_1 - p_2 = \frac{62.4}{(2)(32.2)}\left(60^2 - 15^2\right) = 3270\,\text{lbf/ft}^2 = 22.7\,\text{lbf/in}^2 \quad (\text{or } 0.1565\,\text{MPa})$$

Stagnation Relations

We start by considering the property relation:

$$Tds = du + p\,\cancel{dv} \tag{1.30}$$

Because for ρ = const, $dv = 0$. Then

$$Tds = du \tag{3.33}$$

Note that for any process if $ds = 0$, then $du = 0$. We also have, by definition,

$$c_v = \left(\frac{\partial u}{\partial T}\right)_v \tag{1.27}$$

But for a constant-density fluid, *every* process is one for which v = const. Thus for these fluids, we can drop the partial notation and write equation (1.27) as

$$c_v = \frac{du}{dT} \quad \text{or} \quad du = c_v dT \tag{3.34}$$

Note that here, for any process for which $du = 0$, $dT = 0$.

We now consider the stagnation process, which by virtue of its definition is isentropic, or $ds = 0$. From (3.33) we see that the internal energy does not change during the stagnation process.

$$u = u_t \quad \text{for} \quad \rho = \text{const} \tag{3.35}$$

From (3.34) it must then be that the temperature also does not change during the stagnation process.

$$T = T_t \quad \text{for} \quad \rho = \text{const} \tag{3.36}$$

Summarizing the above, we have shown that *for a constant-density fluid* the stagnation process is not only the one with constant entropy but also the one with constant temperature and internal energy. Let us continue and discover some other interesting relations.

From

$$h = u + pv \tag{1.24}$$

we have

$$dh = du + vdp + p\,\cancel{d}v \tag{3.37}$$

Let us integrate equation (3.37) between the static and stagnation states:

$$h_t - h = \cancel{(u_t - u)} + v(p_t - p) \tag{3.38}$$

But we know that

$$h_t = h + \frac{V^2}{2g_c} + \frac{g}{g_c}z \tag{3.17}$$

Combining these last two equations yields,

$$\left(\cancel{h} + \frac{V^2}{2g_c} + \frac{g}{g_c}z\right) - \cancel{h} = v(p_t - p)$$

which now becomes

$$\boxed{p_t = p + \frac{\rho V^2}{2g_c} + \rho\frac{g}{g_c}z} \qquad \rho = \text{const} \tag{3.39}$$

This equation may also be familiar to those of you who have studied fluid mechanics. It is imperative to note here that this relation between static and stagnation pressures *is valid **only** for a constant-density fluid.* In Section 4.5 we develop the corresponding relation for perfect gases.

Example 3.5
Water is flowing horizontally at a velocity of 20 m/s and has a pressure of 4 bar abs. What is the total pressure?

$$p_t = p + \frac{\rho V^2}{2g_c} + \rho \cancel{\frac{g}{g_c}} z$$

$$p_t = 4 \times 10^5 + \frac{(10^3)(20)^2}{(2)(1)} = 4 \times 10^5 + 2 \times 10^5$$

$$p_t = 6 \times 10^5 \,\mathrm{N/m^2}\,\text{abs.} \quad \text{or} \quad 0.6\,\mathrm{MPa} \quad (\text{or } 87.0\,\mathrm{psi})$$

In many problems you will be confronted by flow exiting a pipe or duct. To solve this type of problem, you must know the pressure at the duct exit. Incompressible flow must always adjust itself so that *the pressure at the duct exit exactly matches that of the surrounding ambient pressure* (which may or may not be the atmospheric pressure). In Section 5.7 you will find that in gases this is *true only for subsonic conditions;* but since the sonic speed in liquids can be shown to be much higher than the flow velocity, you will normally be dealing with "subsonic flow" in the cases involving liquids.

3.8 MOMENTUM EQUATION

If we observe the motion of a given quantity of mass, Newton's second law tells us that its linear momentum will be changed in direct proportion to the applied forces. This is expressed by the following equation:

$$\boxed{\sum \mathbf{F} = \frac{1}{g_c} \frac{d\left(\overrightarrow{\text{momentum}}\right)}{dt}} \tag{1.5}$$

We could write similar expressions relating torque and angular momentum, but we shall confine our discussion to linear momentum. Note that equation (1.5) is a vector relation and must be treated as such, or we must carefully work with components of the equation. In nearly all fluid flow problems, unbalanced forces exist and thus the momentum of the system being analyzed does not remain constant. Thus we shall properly avoid listing it as a conservation law.

Again, the question is: What corresponding expression can we write for a control volume? We note that the term on the right side of equation (1.5) is a material derivative and must be transformed according to the relation developed in

Section 2.4. If we let **N** be the linear momentum of the system, $\boldsymbol{\eta}$ then represents the momentum per unit mass, which is **V**. Substitution into equation (2.22) yields

$$\frac{d\left(\overrightarrow{\text{momentum}}\right)}{dt} = \frac{\partial}{\partial t}\int_{\text{cv}} \mathbf{V}\rho\, d\tilde{v} + \int_{\text{cs}} \mathbf{V}\rho(\mathbf{V}\cdot\hat{n})\, dA \tag{3.40}$$

and the transformed equation which is applicable to a control volume is

$$\boxed{\sum \mathbf{F} = \frac{1}{g_c}\frac{\partial}{\partial t}\int_{\text{cv}} \mathbf{V}\rho\, d\tilde{v} + \frac{1}{g_c}\int_{\text{cs}} \mathbf{V}\rho(\mathbf{V}\cdot\hat{n})\, dA} \tag{3.41}$$

This equation is usually called the *momentum* or *momentum flux equation*. The $\Sigma\mathbf{F}$ represents the summation of all forces *on the fluid within the control volume*. What do the other terms represent? [See the discussion following equation (2.22).]

In the solution of actual problems, one normally works with the components of the momentum equation. In fact, frequently, only one component is required for the solution of a problem. The x-component of this equation would appear as

$$\sum F_x = \frac{1}{g_c}\frac{\partial}{\partial t}\int_{\text{cv}} V_x\rho\, d\tilde{v} + \frac{1}{g_c}\int_{\text{cs}} V_x\rho(\mathbf{V}\cdot\hat{n})\, dA \tag{3.42}$$

Note carefully how the last term is written.

In the event that one-dimensional flow exists, the last integral in equation (3.41) is easy to evaluate, as ρ and **V** are constant over any given cross section. If we choose the surface A perpendicular to the velocity, then

$$\int_{\text{cs}} \mathbf{V}\rho(\mathbf{V}\cdot\hat{n})\, dA = \sum \mathbf{V}\rho V \int dA = \sum \mathbf{V}\rho V A = \sum \dot{m}\mathbf{V} \tag{3.43}$$

The summation is taken over all sections where fluid crosses the control surface and is positive where fluid leaves the control volume and negative where fluid enters the control volume. (Recall how $\hat{n}$ was chosen.)

If we now consider *steady flow*, the term involving the partial derivative with respect to time is zero. Thus for steady one-dimensional flow, the momentum equation for a control volume becomes

$$\sum \mathbf{F} = \frac{1}{g_c}\sum \dot{m}\mathbf{V} \tag{3.44}$$

If there is only one section where fluid enters and one section where fluid leaves the control volume, we know (from continuity) that

$$\dot{m}_{\text{in}} = \dot{m}_{\text{out}} = \dot{m} \tag{2.42}$$

and the momentum equation becomes

$$\boxed{\sum \mathbf{F} = \frac{\dot{m}}{g_c}(\mathbf{V}_{\text{out}} - \mathbf{V}_{\text{in}})} \tag{3.45}$$

This is the steady form of the momentum equation for a finite control volume. [In fully developed laminar tube flows (see Figure 2.1), we must include a correction factor when using mass-average velocities in equation (3.45); that is, use (4/3)V (see Problem 3.23).]

What assumptions have been fed into equation (3.45)? In using this relation, one must be sure to:

1. Identify the control volume.
2. Include all forces acting *on the fluid inside* the control volume.
3. Be extremely careful with the signs of all quantities.

Example 3.6

There is a steady one-dimensional flow of air through a 12-in.-diameter horizontal duct (Figure E3.6). At a section where the velocity is 460 ft/sec, the pressure is 50 psia and the temperature is 550°R. At a downstream section the velocity is 880 ft/sec and the pressure is 23.9 psia. Determine the total wall shearing force between these sections. Assume the flow to be turbulent and hence one-dimensional.

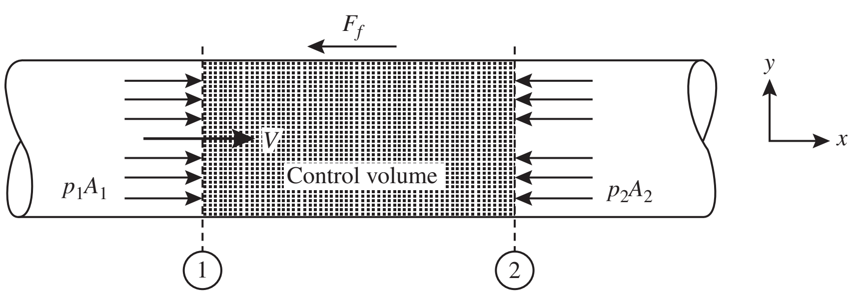


Figure E3.6

$$V_1 = 460\,\text{ft/sec} \quad V_2 = 880\,\text{ft/sec}$$
$$p_1 = 50\,\text{psia} \quad p_2 = 23.9\,\text{psia}$$
$$T_1 = 550°\text{R}$$

We establish a coordinate system and indicate the forces on the control volume. Let F_f represent the frictional force of the duct on the gas. We write the x-component of equation (3.45):

$$F_x = \frac{\dot{m}}{g_c}(V_{\text{out}_x} - V_{\text{in}_x})$$

$$p_1A_1 - p_2A_2 - F_f = \frac{\dot{m}}{g_c}(V_2 - V_1) = \frac{\rho_1 A_1 V_1}{g_c}(V_2 - V_1)$$

Note that any force in the negative direction must include a minus sign. We divide by $A = A_1 = A_2$:

$$p_1 - p_2 - \frac{F_f}{A} = \frac{\rho_1 V_1}{g_c}(V_2 - V_1)$$

$$\rho_1 = \frac{p_1}{RT_1} = \frac{(50)(144)}{(53.3)(550)} = 0.246\,\text{lbm/ft}^3$$

$$(50 - 23.9)(144) - \frac{F_f}{A} = \frac{(0.246)(460)}{32.2}(880 - 460)$$

$$3758 - \frac{F_f}{A} = 1476$$

$$F_f = (3758 - 1476)\pi(0.5)^2 = 1792\,\text{lbf} \quad (\text{or } 7.971\,\text{kN})$$

Example 3.7

Water flowing at the rate of 0.05 m^3/s has a velocity of 40 m/s. The jet strikes a vane and is deflected 120° (Figure E3.7). Friction along the vane is negligible and the entire system is exposed to the atmosphere. Potential changes can also be neglected. Determine the force necessary to hold the vane stationary.

$$p_1 = p_2 = p_{\text{atmos}} \qquad h_\ell = 0$$

$$z_1 = z_2 \qquad w_s = 0$$

Energy:

$$\cancel{\frac{p_1}{\rho}} + \frac{V_1^2}{2g_c} + \cancel{\frac{g}{g_c}z_1} = \cancel{\frac{p_2}{\rho}} + \frac{V_2^2}{2g_c} + \cancel{\frac{g}{g_c}z_2} + \cancel{h_\ell} + \cancel{w_s}$$

Thus

$$V_1 = V_2$$

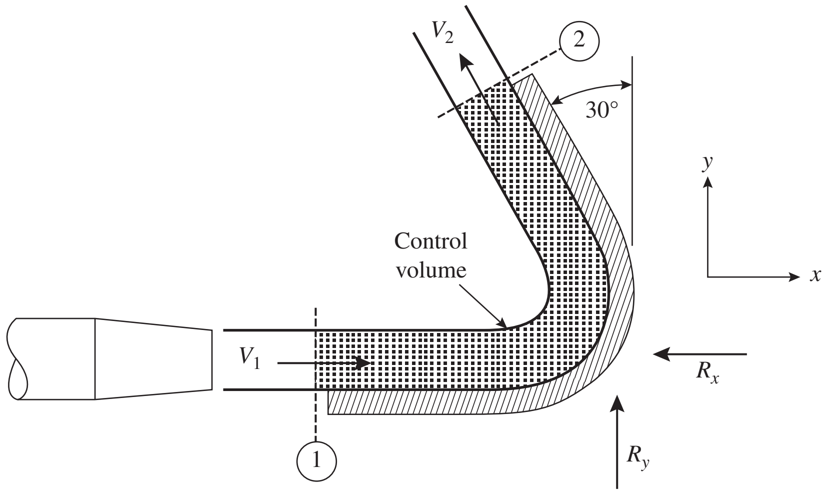


Figure E3.7

We indicate the force components of the vane on the fluid as R_x and R_y and put them on the diagram in assumed directions. (If we have guessed wrong, our answer will turn out to be negative.) For the x-component:

$$\sum F_x = \frac{\dot{m}}{g_c}(V_{2x} - V_{1x})$$

$$-R_x = \frac{\dot{m}}{g_c}[(-V_2 \sin 30) - V_1] = \frac{\dot{m}V_1}{g_c}(-\sin 30 - 1)$$

$$-R_x = \frac{(10^3)(0.05)(40)}{1}(-0.5 - 1)$$

$$R_x = 3.0\,\text{kN} \quad (\text{or}\, 674.4\,\text{lbf})$$

For the y-component:

$$\sum F_y = \frac{\dot{m}}{g_c}(V_{2y} - V_{1y})$$

$$R_y = \frac{\dot{m}}{g_c}[(V_2 \cos 30) - 0]$$

$$R_y = \frac{(10^3)(0.05)(40)}{1}(0.866)$$

$$R_y = 1.732\,\text{kN} \quad (\text{or } 389.4\,\text{lbf})$$

Note that the assumed directions for R_x and R_y were correct since the answers came out positive.

Differential Form of Momentum Equation

As a further indication of the meticulous care that must be exercised when using the momentum equation, we apply it to the differential control volume shown in Figure 3.6. Under conditions of steady, one-dimensional flow, the properties of the fluid entering the control volume are designated as ρ, V, p, and so on. Fluid leaves the control volume with slightly different properties, as indicated by $\rho + d\rho$, $V + dV$, and so on. The x-coordinate is chosen as positive in the direction of flow, and the positive z-direction is opposite to gravity. (Note that the x and z axes are not necessarily orthogonal here.)

Now that the control volume has been identified, we note all forces that act on it. The forces can be divided into two types:

1. *Surface forces*. These act on the control surface and from there indirectly on the fluid. These are either from normal or tangential stress components.
2. *Body forces*. These act directly on the fluid within the control volume. Examples of these are gravity and electromagnetic forces. We shall limit our discussion to gravity forces.

Thus we have

$F_1 \equiv$ Upstream pressure force

$F_2 \equiv$ Downstream pressure force

$F_3 \equiv$ Wall pressure force

$F_4 \equiv$ Wall friction force

$F_5 \equiv$ Gravity force

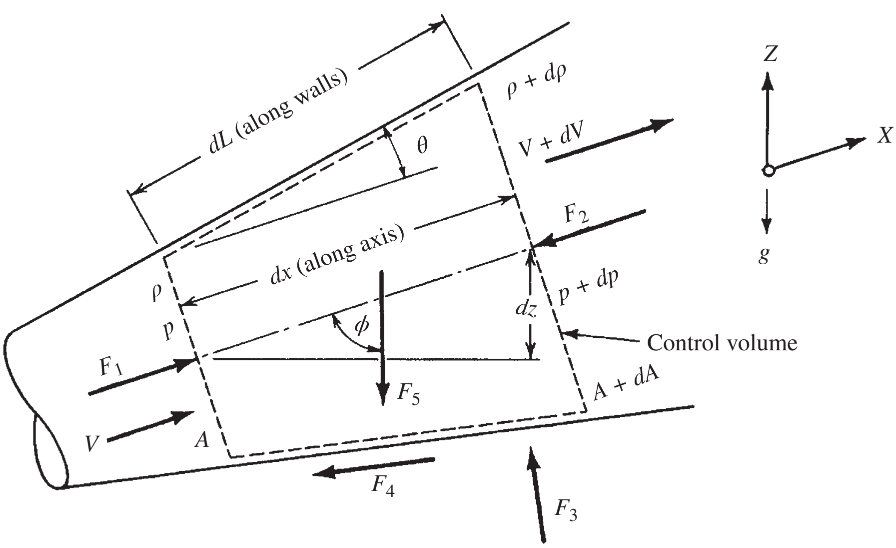


Figure 3.6 Momentum analysis on infinitesimal control volume.

It should be mentioned that wall forces F_3 and F_4 are usually lumped together into a single force called the *enclosure force* because it is extremely difficult to account for them separately in most finite control volumes. Fortunately, it is the total enclosure force that is of significance in the solution of these problems. However, in dealing with a differential control volume, it will be more instructive to separate each portion of the enclosure force as we have indicated.

We write the x-component of the momentum equation for steady one-dimensional flow:

$$\sum F_x = \frac{\dot{m}}{g_c}\left(V_{\text{out}_x} - V_{\text{in}_x}\right) \tag{3.46}$$

Now we proceed to evaluate the x-component of each force, taking care to indicate whether it is in the positive or negative direction.

$$F_{1x} = F_1 = (\text{pressure})(\text{area})$$

$$F_{1x} = pA \tag{3.47}$$

$$F_{2x} = -F_2 = -(\text{pressure})(\text{area})$$

$$F_{2x} = -(p + dp)(A + dA) = -(pA + p\,dA + A\,dp + \overset{\text{HOT}}{\cancel{dp\,dA}}) \tag{3.48}$$

Neglecting the higher-order term, this becomes

$$F_{2x} = -(pA + p\,dA + A\,dp) \tag{3.49}$$

The wall pressure force can be obtained with a mean pressure value:

$$F_{3x} = F_3 \sin\theta = [(\text{mean pressure})(\text{wall area})]\sin\theta$$

but dA = (wall area) sin θ; and thus

$$F_{3x} = \left(p + \frac{dp}{2}\right) dA \tag{3.50}$$

The same result could be obtained using principles of basic fluid mechanics, which show that a component of the pressure force can be computed by considering the pressure distribution over the projected area. Expanding and neglecting the higher order term, we have

$$F_{3x} = p\,dA \tag{3.51}$$

To compute the wall friction force, we define

$$\begin{aligned}
\tau_w &\equiv \text{the mean shear stress along the wall} \\
P &\equiv \text{the mean wetted perimeter} \\
F_{4x} &= -F_4\cos\theta = -[(\text{mean shear stress})(\text{wall area})]\cos\theta \\
F_{4x} &= \tau_w(PdL)\cos\theta
\end{aligned} \tag{3.52}$$

but $dx = dL\cos\theta$, and thus

$$F_{4x} = -\tau_w P\,dx \tag{3.53}$$

For the body force we have

$$\begin{aligned}
F_{5x} &= -F_5\cos\phi = -\left[(\text{volume})(\text{mean density})\frac{g}{g_c}\right]\cos\phi \\
F_{5x} &= -\left[\left(A+\frac{dA}{2}\right)dx\right]\left(\rho+\frac{d\rho}{2}\right)\frac{g}{g_c}\cos\phi
\end{aligned} \tag{3.54}$$

But $dx\cos\phi = dz$, and thus

$$F_{5x} = -\left(A+\frac{dA}{2}\right)\left(\rho+\frac{d\rho}{2}\right)\frac{g}{g_c}dz \tag{3.55}$$

Expand this and eliminate all the higher-order terms to show that

$$F_{5x} = -A\rho\frac{g}{g_c}dz \tag{3.56}$$

Summarizing the above, we have

$$\begin{aligned}
\sum F_x &= F_{1x}+F_{2x}+F_{3x}+F_{4x}+F_{5x} \\
\sum F_x &= \cancel{pA} - \left(\cancel{pA} + p\,\cancel{dA} + A\,dp\right) + p\,\cancel{dA} - \tau_w P\,dx - A\rho\frac{g}{g_c}dz \\
\sum F_x &= -A\,dp - \tau_w P\,dx - A\rho\frac{g}{g_x}dz
\end{aligned} \tag{3.57}$$

We now turn our attention to the right side of equation (3.46). Looking at Figure 3.6, we see that this is

$$\frac{\dot{m}}{g_c}(V_{\mathrm{out}_x} - V_{\mathrm{in}_x}) = \frac{\dot{m}}{g_c}[(V + dV) - V] = \frac{\dot{m}}{g_c} dV \tag{3.58}$$

Combining equations (3.57) and (3.58) yields the x-component of the momentum equation applied to a differential control volume:

$$\sum F_x = \frac{\dot{m}}{g_c}(V_{\mathrm{out}_x} - V_{\mathrm{in}_x}) \tag{3.46}$$

$$-A\,dp - \tau_w P\,dx - A\rho\frac{g}{g_c}dz = \frac{\dot{m}}{g_c}dV = \frac{\rho A V dV}{g_c} \tag{3.59}$$

Equation (3.59) can be put into a more useful form by introducing the concepts of the *friction factor* and *equivalent diameter*.

The *friction factor* (f) relates the average shear stress at the wall (τ_w) to the dynamic pressure in the following manner (more details given in Fanno flow, Section 9.9):

$$f \equiv \frac{4\tau_w}{\rho V^2/2g_c} \tag{3.60}$$

This is the *Darcy–Weisbach friction factor* and is the one we use in this book. Care should be taken when reading literature in this area since some authors use the *Fanning friction factor,* which is only one-fourth as large, due to omission of the factor of 4 in the definition.

Frequently, fluid flows through a noncircular cross section such as a rectangular duct. To handle these problems, an *equivalent diameter* has been devised, which is defined as

$$D_e \equiv \frac{4A}{P} \tag{3.61}$$

where

$A \equiv$ the cross-sectional area
$P \equiv$ the perimeter of the enclosure wetted by the fluid

Minor losses are neglected here. Note that if equation (3.61) is applied to a circular duct completely filled with fluid, the equivalent diameter is the same as the actual diameter.

Use the definitions given for the friction factor and the equivalent diameter and *show* that equation (3.59) can be rearranged to

$$\frac{dp}{\rho} + f\frac{V^2}{2g_c}\frac{dx}{D_e} + \frac{g}{g_c}dz + \frac{VdV}{g_c} = 0 \tag{3.62}$$

This is a very useful form of the momentum equation (written in the direction of flow) for steady one-dimensional flows through a differential control volume. The last term can be written in a more familiar form as

$$\boxed{\frac{dp}{\rho} + f\frac{V^2}{2g_c}\frac{dx}{D_e} + \frac{g}{g_c}dz + \frac{dV^2}{2g_c} = 0} \tag{3.63}$$

We shall use this equation in Chapter 9 when we discuss flow through ducts with friction.

It might be instructive at this time to compare equation (3.63) with equation (3.13). Recall that (3.13) was derived from energy considerations, whereas (3.63) was developed from momentum concepts. A comparison of this nature reinforces our division of the entropy, equation (3.1), for it shows that friction is a source of internal irreversibility

$$Tds_i = f\frac{V^2}{2g_c}\frac{dx}{D_e} \tag{3.64}$$

3.9 SUMMARY

We have taken a new look at entropy changes by separating them into two parts, one caused by heat transfer and the other caused by irreversible effects. We then introduced the concept of a stagnation reference state. These two ideas permitted the energy equation to be written in alternative forms called *pressure–energy equations*. Several interesting conclusions were drawn from these equations under appropriate assumptions.

Newton's second law was transformed into a form suitable for control volume analysis. Extreme care needs to be taken when the momentum equation is used. The following steps should be noted *in addition* to those listed in the summary for Chapter 2:

1. Establish a coordinate system for the control volume.
2. Indicate all forces acting *on the fluid* inside the control volume.
3. Be especially careful with the signs of vector quantities such as **F** and **V**.

Some of the most frequently used equations developed in this chapter are summarized below. Most are restricted to steady one-dimensional flow; others involve additional assumptions. You should determine under what conditions each may be used.

1. *Entropy division*

$$ds = ds_e + ds_i = \frac{\delta q}{T} + ds_i \qquad (3.9), (3.10)$$

   ds_e is positive or negative (depending on the direction of δq) and ds_i is always positive (denoting internal irreversibilities).

2. *Pressure–energy equation*

$$\frac{dp}{\rho} + \frac{dV^2}{2g_c} + \frac{g}{g_c}dz + \delta w_s + T ds_i = 0 \qquad (3.13)$$

3. *Stagnation concept (depends on reference frame)*

$$h_t = h + \frac{V^2}{2g_c} + \frac{g}{g_c}z \quad \text{(neglect } z \text{ for gases)}$$
$$s_t = s \qquad (3.17)$$

4. *Energy equation*

$$h_{t1} + q = h_{t2} + w_s \qquad (3.19)$$

$$\delta q = \delta w_s + dh_t \qquad (3.20)$$

$$\text{If } q = w_s = 0,\ h_t = \text{const.}$$

5. *Stagnation pressure–energy equation*

$$\frac{dp_t}{\rho_t} + ds_e(T_t - T) + T_t ds_i + \delta w_s = 0 \qquad (3.25)$$

$$\text{If } q = w_s = 0, \text{ and losses} = 0, \text{ then } p_t = \text{const.}$$

6. *Constant-density fluids*

$$\frac{p_1}{\rho} + \frac{V_1^2}{2g_c} + \frac{g}{g_c}z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2g_c} + \frac{g}{g_c}z_2 + h_\ell + w_s \qquad (3.31)$$

$$u = u_t \quad \text{and} \quad T = T_t \tag{3.35), (3.36}$$

$$p_t = p + \frac{\rho V^2}{2g_c} + \rho \frac{g}{g_c} z \tag{3.39}$$

7. *Second law of motion—momentum equation* $\begin{cases} \mathbf{N} = \overrightarrow{momentum} \\ \boldsymbol{\eta} = \mathbf{V} \end{cases}$

$$\sum \mathbf{F} = \frac{\partial}{\partial t} \int_{\text{cv}} \frac{\rho \mathbf{V}}{g_c} d\tilde{v} + \int_{\text{cs}} \frac{\rho \mathbf{V}}{g_c} (\mathbf{V} \cdot \hat{n}) dA \tag{3.41}$$

For steady, one-dimensional flow:

$$\sum \mathbf{F} = \frac{\dot{m}}{g_c} (\mathbf{V}_{\text{out}} - \mathbf{V}_{\text{in}}) \tag{3.45}$$

$$\frac{dp}{\rho} + f \frac{V^2}{2g_c} \frac{dx}{D_e} + \frac{g}{g_c} dz + \frac{dV^2}{2g_c} = 0 \tag{3.63}$$

Consult References 10–12 for additional information on Fluid Mechanics.

PROBLEMS

For those problems involving water, you may use $\rho = 62.4$ lbm/ft^3 or 1000 kg/m^3, and the specific heat of water equals 1 Btu/lbm-°R or 4.187 kJ/kg-K.

3.1. Compare the pressure–energy equation (3.13) for the case of no external work with the differential form of the momentum equation (3.63). Does the result seem reasonable?

3.2. Consider steady one-dimensional flow of a perfect gas in a horizontal insulated frictionless duct. Start with the pressure-energy equation and show that

$$\frac{V^2}{2g_c} + \frac{\gamma}{(\gamma - 1)} \frac{p}{\rho} = \text{const}$$

3.3. It is proposed to determine the flow rate through a pipeline from pressure measurements at two points of different cross-sectional areas. No energy transfers are involved ($q = w_s = 0$) and potential differences are negligible. Show that for the steady one-dimensional, frictionless flow of an incompressible fluid, the flow rate can be represented by

$$\dot{m} = A_1 A_2 \left[\frac{2\rho g_c (p_1 - p_2)}{A_1^2 - A_2^2} \right]^{1/2}$$

3.4. Pressure taps in a low-speed wind tunnel reveal the difference between stagnation and static pressure to be 0.5 psi. Calculate the test section air velocity under the assumption that the air density remains constant at 0.0765 lbm/ft^3.

3.5. Water flows through a duct of varying area. The difference in stagnation pressures between two sections is 4.5×10^5 N/m^2.

(a) If the water remains at a constant temperature, how much heat will be transferred in this length of duct?

(b) If the system is perfectly insulated against heat transfer, compute the temperature change of water as it flows through the duct.

3.6. The following information is known about the steady flow of methane through a horizontal insulated duct:

$$\begin{aligned}
&\text{Entering stagnation enthalpy} = 634\,\text{Btu/lbm}\\
&\text{Leaving static enthalpy} \qquad = 532\,\text{Btu/lbm}\\
&\text{Leaving static temperature} \;\; = 540^\circ\text{F}\\
&\text{Leaving static pressure} \qquad = 50\ \text{psia}
\end{aligned}$$

(a) Determine the outlet velocity.

(b) What is the stagnation temperature at the outlet?

(c) Determine the stagnation pressure at the outlet.

3.7. Under what conditions would it be possible to have an adiabatic flow process with a real fluid (with friction) and have the stagnation pressures at inlet and outlet to the system be the same? (*Hint:* Look at the stagnation pressure–energy equation.)

3.8. Simplify the stagnation pressure–energy equation (3.25) for the case of an incompressible fluid. Integrate the result and compare your answer to any other energy equation that you might use for an incompressible fluid [say, equation (3.29)].

3.9. An incompressible fluid (ρ = 55 lbm/ft^3) leaves the pipe shown in Figure P3.9 with a velocity of 15 ft/sec.

(a) Calculate the flow losses.

(b) Assume that all losses occur in the constant-area pipe and find the pressure at the entrance to the pipe.

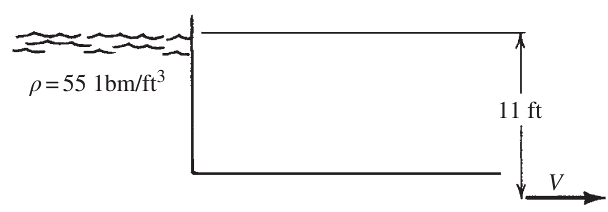


Figure P3.9

3.10. For the flow depicted in Figure P3.10, what Δz value is required to produce a jet velocity (V_j) of 30 m/s if the flow losses are $h_\ell = 15V_p^2/2g_c$?

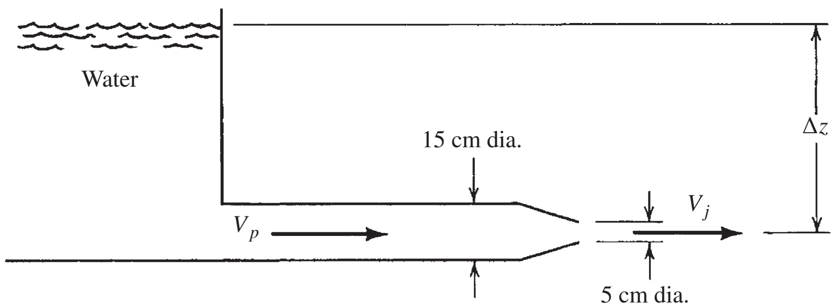


Figure P3.10

3.11. Water flows in a 2-ft-diameter duct under the following conditions: $p_1 = 55$ psia and $V_1 = 20$ ft/sec. At another section 12 ft below the first the diameter is 1 ft and the pressure $p_2 = 40$ psia.

(a) Compute the frictional losses between these two sections.

(b) Determine the direction of flow.

3.12. For Figure P3.12, find the pipe diameter required to produce a flow rate of 50 kg/s if the flow losses are $h_\ell = 6V^2/2g_c$.

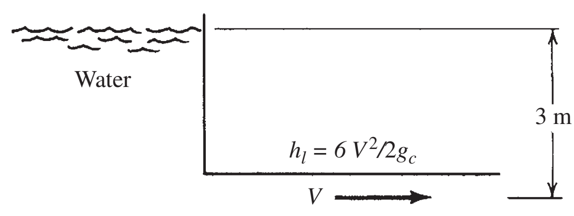


Figure P3.12

3.13. A pump at the surface of a lake expels a vertical jet of water (the water falls back into the lake).

(a) Discuss briefly (but clearly) all possible sources of irreversibilities in this situation.

(b) Now neglecting all losses that you discussed in part (a), what is the maximum height that the water may reach for $w_s = 35$ ft-lbf/lbm?

3.14. Which of the two pumping arrangements shown in Figure P3.14 is more desirable (i.e., less demanding of pump work)? You may neglect the minor loss at the elbow in arrangement (A).

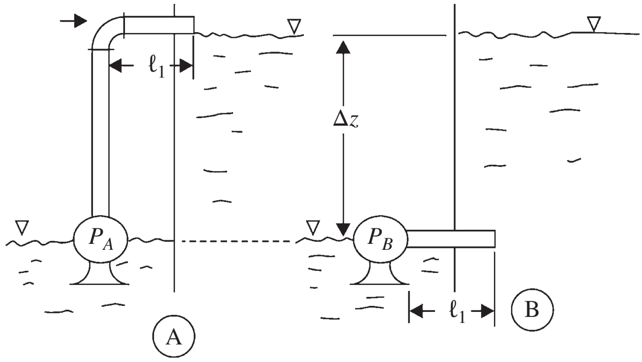


Figure P3.14

3.15. For a given mass, we can relate the moment of the applied force to the angular momentum by the following:

$$\sum \mathbf{M} = \frac{1}{g_c} \frac{d\left(\overrightarrow{\text{angular momentum}}\right)}{dt}$$

(a) What is the angular momentum per unit mass?

(b) What form does the equation above take for the analysis of a control volume?

3.16. An incompressible fluid flows through a 10 in. diameter horizontal constant-area pipe. At one section the pressure is 150 psia and 1000 ft downstream the pressure has dropped to 100 psia.

(a) Find the total frictional force exerted on the fluid by the pipe.

(b) Compute the average wall shear stress.

3.17. Methane gas flows through a horizontal constant-area pipe of 15 cm diameter. At section 1, $p_1 = 6$ bar abs., $T_1 = 66\,°\text{C}$, and $V_1 = 30$ m/s. At section 2, $T_2 = 38\,°\text{C}$ and $V_2 = 110$ m/s.

(a) Determine the pressure at section 2.

(b) Find the total wall frictional force.

(c) What is the heat transfer?

3.18. Seawater ($\rho = 64\ \text{lbm/ft}^3$) flows through the reducer shown in Figure P3.18 with $p_1 = 50$ psig. The flow losses between the two sections amount to $h_\ell = 5.0$ ft-1bf/lbm.

(a) Find V_2 and p_2.

(b) Determine the force exerted by the reducer on the seawater between sections 1 and 2.

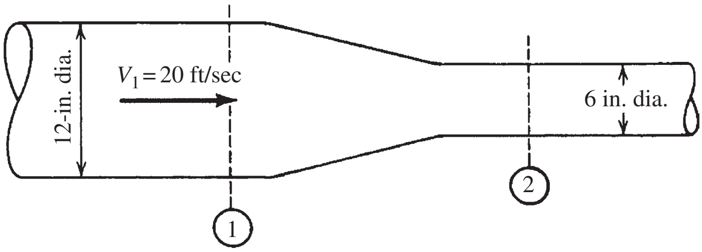


Figure P3.18

3.19. **(a)** Neglect all losses and compute the exit velocity from the tank shown in Figure P3.19.

(b) If the opening is 4 in. in diameter, determine the mass flow rate.

(c) Compute the force tending to push the tank along the floor.

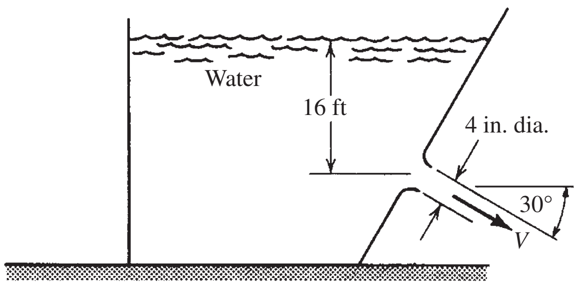


Figure P3.19

3.20. A jet of water with a velocity of 5 m/s has an area of 0.05 m^2. It strikes a 1 m thick concrete block at a point 2 m above the ground (Figure P3.20). After hitting the block, the water drops straight to the ground. What minimum weight must the block have in order not to tip over?

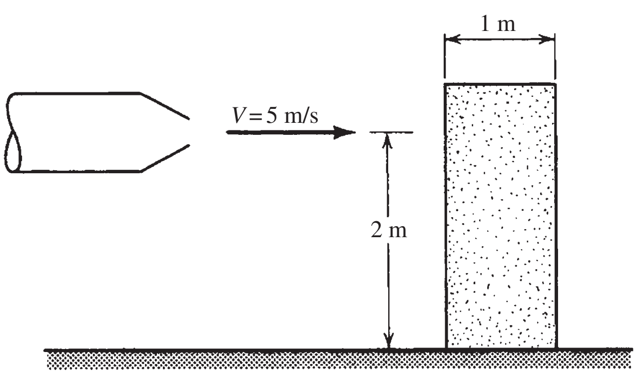


Figure P3.20

3.21. It is proposed to brake a racing car by opening an air scoop to deflect the air as shown in Figure P3.21. You may assume that the density of the air remains approximately constant at the inlet conditions of 14.7 psia and 60°F. Assume that there is no spillage—that all the air enters the inlet in the direction shown and the conditions specified. You may also assume that there is no change in the drag of the car when the air scoop is opened. What inlet area is needed to provide a braking force of 2000 1bf when traveling at 300 mph?

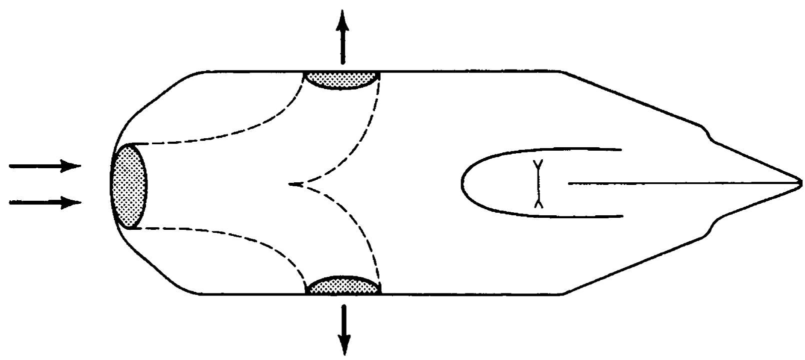


Figure P3.21

3.22. A fluid jet strikes a vane and is deflected through angle θ (Figure P3.22). For a given jet (fluid, area, and velocity are fixed), what deflection angle will cause the greatest x-component of force between the fluid and vane? You may assume an incompressible fluid and no friction along the vane. Set up the general problem and then differentiate to find the maximum.

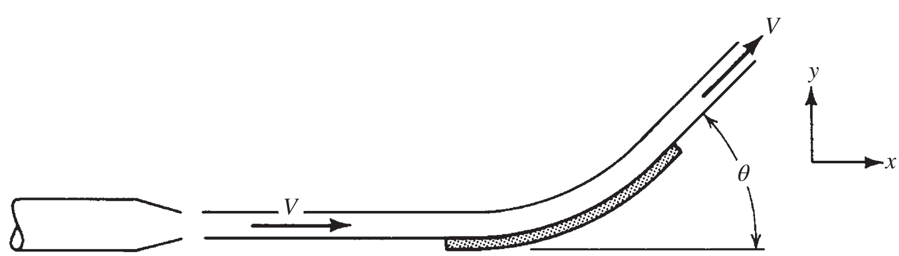


Figure P3.22

3.23. Laminar flow in circular ducts is not one-dimensional but we may use the equivalent mass-average velocity $V = U_m/2$ from equations (2.10) and (2.11) in our one-dimensional formulations. This velocity results from $\int u dA \equiv \left(\pi r_0^2\right) V$. Now, in the momentum equation (3.41) we encounter the integral $\int \rho \mathbf{V}(\mathbf{V}\cdot\hat{n})dA$ over the surface in Figure 2.1, here $\mathbf{V}$ comes from Newton's Second Law; this is equivalent

to calculating $\int u^2 dA = (4/3)(\pi r_0^2)V^2$ where V depends on U_m as indicated above. Verify that for laminar flow the resulting momentum terms in equation (3.45) need to be multiplied by the above factor 4/3 as mentioned in the text. Assume the density ρ remains constant along the cross section in Figure 2.1.

CHECK TEST

You should be able to complete this test with little or no reference to material in the chapter.

3.1. Entropy changes can be divided into two categories. Define these categories in words and where possible with equations. Comment on the sign of each part.

3.2. Given the differential form of the energy equation, derive the pressure–energy equation.

3.3. **(a)** Define the stagnation process. Be careful to state all conditions.

(b) Give a general equation for stagnation enthalpy that is valid for all substances.

(c) Under what conditions can you use the following equation?

$$\frac{p_t}{\rho} = \frac{p}{\rho} + \frac{V^2}{2g_c} + \frac{g}{g_c}z$$

3.4. One can use either person A (who is standing still) or person B (who is running) as a frame of reference (Figure CT3.4). Check the statement below that is correct.

(a) The stagnation pressure is the same for A and B.

(b) The static pressure is the same for A and B.

(c) Neither statement (a) nor (b) is correct.

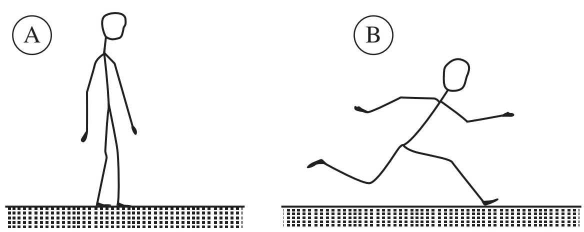


Figure CT3.4

3.5. Consider the case of steady one-dimensional flow with one stream in and one stream out of the control volume.

(a) Under what conditions can we say that the stagnation enthalpy remains constant? (Can p_t vary under these conditions?)

(b) If the conditions of part (a) are known to exist, what additional assumption is required before we can say that the stagnation pressure remains constant?

3.6. Under certain circumstances, the momentum equation is sometimes written in the following form when used to analyze a control volume:

$$\sum \mathbf{F} = \frac{\dot{m}}{g_c}(\mathbf{V}_r - \mathbf{V}_s)$$

(a) Which of the sections (r or s) represents the location where fluid enters the control volume?

(b) What circumstances must exist before you can use the equation in this form?

3.7. Work Problem 3.18.

Chapter 4

Introduction to Compressible Flow

4.1 INTRODUCTION

In the earlier chapters we developed the fundamental relations that are needed for our analysis of fluid flow. We have seen the special form that some of these take for the case of constant-density fluids. Our *main* interest from now on is incompressible fluids or gases. We shall soon learn that it is not uncommon to encounter gases that are traveling faster than the speed of sound. Furthermore, when in this situation, their behavior is quite different than when traveling slower than the speed of sound. Thus we begin by developing an expression for sonic speed through an arbitrary medium. This relation is then simplified for the case of perfect gases. We then examine subsonic and supersonic flows to gain some insight as to why their behavior is so different.

The Mach number is introduced as a key parameter for compressible flows, and we find that for a perfect gas it is very simple and useful to express our basic equations and many supplementary relations in terms of this new parameter. The chapter closes with a discussion of the significance of *h–s* and *T–s* diagrams and their importance in visualizing flow problems.

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

4.2 OBJECTIVES

After completing this chapter successfully, you should be able to:

1. Explain how sound propagates through any medium (solid, liquid, or gas).
2. Define *sonic speed*. State the basic differences between a *shock wave* and a *sound wave*.
3. (*Optional*) Starting with the continuity and momentum equations for steady, one-dimensional flow, utilize a control volume analysis to derive the general expression for the velocity of an infinitesimal pressure disturbance in an arbitrary medium.
4. State the relations for:
   a. Speed of sound in an arbitrary medium
   b. Speed of sound in a perfect gas
   c. Mach number
5. Discuss the propagation of signal waves from a moving body in a fluid by explaining *zone of action, zone of silence, Mach cone,* and *Mach angle*. Compare subsonic and supersonic flows using these concepts.
6. Write an equation for the stagnation enthalpy (h_t) of a perfect gas in terms of enthalpy (h), Mach number (M), and ratio of specific heats (γ).
7. Write an equation for the stagnation temperature (T_t) of a perfect gas in terms of temperature (T), Mach number (M), and ratio of specific heats (γ).
8. Write an equation for the stagnation pressure (p_t) of a perfect gas in terms of pressure (p), Mach number (M), and ratio of specific heats (γ).
9. (*Optional*) Demonstrate manipulative skills by developing simple relations in terms of Mach number for a perfect gas, such as

$$p_t = p\left(1 + \frac{\gamma - 1}{2}M^2\right)^{\gamma/(\gamma-1)}$$

10. Demonstrate the ability to utilize the concepts above in typical flow problems.

4.3 SONIC SPEED AND MACH NUMBER

We now examine the means by which disturbances pass through an elastic medium. A disturbance at a given point creates a region of compressed molecules that is passed along to its neighboring molecules and in so doing creates a *traveling wave*. Waves come in various *strengths,* which are measured by the

amplitude of the disturbance. The speed at which this disturbance is propagated through the medium is called the *wave speed*. This speed not only depends on the type of medium and its thermodynamic state but is also a function of the strength of the wave. The *stronger* the wave is, the *faster* it moves.

When we are dealing with waves of *large amplitude,* which involve relatively large changes in pressure and density, we call them *shock waves*. These will be studied in detail in Chapter 6. If, on the other hand, we are observing waves of *very small amplitude*, their speed is characteristic *only* of the medium and its state. These waves are of vital importance to our study since sound waves fall into this category. Furthermore, the presence of an object in a medium can only be felt by the object's sending out or reflecting infinitesimal pressure waves which propagate at the characteristic *sonic speed*.

Let us hypothesize how we might form an infinitesimal pressure wave and then apply the fundamental concepts to determine the wave speed. Consider a long constant-area tube filled with fluid and having a piston at one end, as shown in Figure 4.1. The fluid is initially at rest. At a certain instant the piston is given an incremental velocity dV to the left. The fluid particles immediately next to the piston are compressed a very small amount as they acquire the velocity of the piston.

As the piston (and these compressed particles) continue to move, the next group of fluid particles is compressed and the *wave front* is observed to propagate through the fluid at the characteristic *sonic speed* of magnitude a. All particles between the wave front and its piston are moving with velocity dV to the left and have been compressed from ρ to $\rho + d\rho$ and have increased their pressure from p to $p + dp$.

We next recognize that this is a difficult situation to analyze. Why? Because it is *unsteady* flow! [As you observe any given point in the tube, the properties change with time (e.g., pressure changes from p to $p + dp$ as the wave front passes).] This

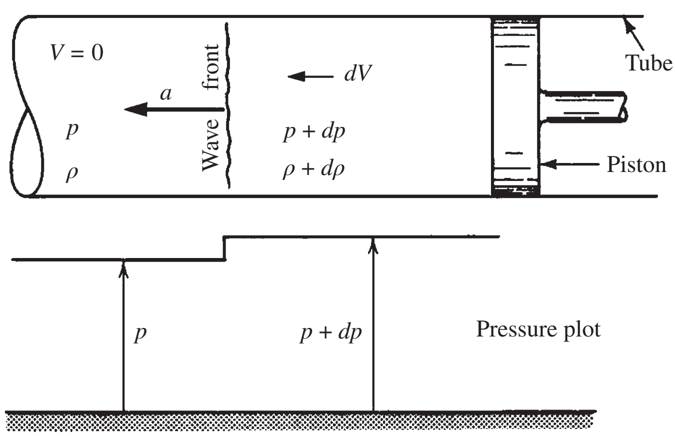


Figure 4.1 Initiation of infinitesimal pressure pulse.

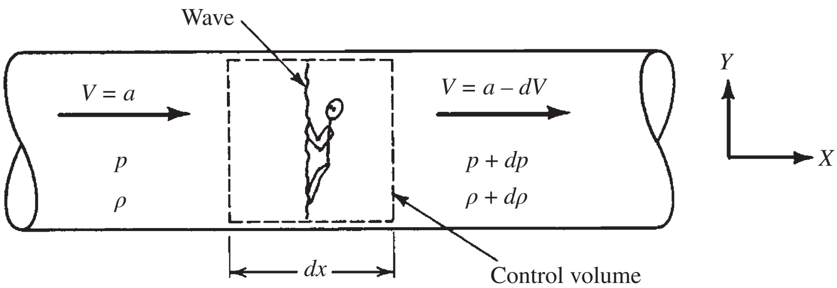


Figure 4.2 Steady-flow picture corresponding to Figure 4.1.

difficulty can easily be solved by superimposing on the entire flow field a constant velocity to the right of magnitude a. *This procedure changes the frame of reference to the wave front as it now appears as a stationary wave*. An alternative way of achieving this result is to jump on the wave front. Figure 4.2 shows the problem that we now have. *Note that changing the reference frame in this manner does not in any way alter the actual (static) thermodynamic properties of the fluid, although it will affect the stagnation conditions*. Since the wave front is extremely thin, we can use a control volume of infinitesimal thickness.

Continuity

For steady one-dimensional flow, we have

$$\dot{m} = \rho A V = \text{const} \tag{2.30}$$

But $A = \text{const}$; thus

$$\rho V = \text{const} \tag{4.1}$$

Application of this to our problem yields

$$\rho a = (\rho + d\rho)(a - dV)$$

Expanding gives us

$$\rho a = \rho a - \rho dV + a\,d\rho - \overset{\text{HOT}}{\cancel{d\rho\, dV}}$$

Cancelling ρa, neglecting the higher-order term and solving for dV, we have

$$dV = \frac{a\,d\rho}{\rho} \tag{4.2}$$

Momentum

Since the control volume has infinitesimal thickness, we can neglect any shear stresses along the walls. We shall write the x-component of the momentum equation, taking forces and velocity as positive if to the right. For steady one-dimensional flow, we may write

$$\sum F_x = \frac{\dot{m}}{g_c}(V_{\text{out}_x} - V_{\text{in}_x}) \tag{3.46}$$

$$\cancel{p}A - \left(\cancel{p} + dp\right)A = \frac{\rho A a}{g_c}\left[\left(\cancel{a} - dV\right) - \cancel{a}\right]$$

$$A dp = \frac{\rho A a}{g_c} dV$$

Canceling the area and solving for dV, we have

$$dV = \frac{g_c dp}{\rho a} \tag{4.3}$$

Equations (4.2) and (4.3) may now be combined to eliminate dV, with the result

$$a^2 = g_c \frac{dp}{d\rho} \tag{4.4}$$

However, the derivative $dp/d\rho$ is not unique. It depends entirely on the nature of the process. Thus it should really be written as a *partial* derivative with the appropriate subscript. But what subscript? What kind of process are we dealing with?

Remember, we are analyzing an infinitesimal disturbance. For this case, we may assume both negligible losses and no heat transfer as the wave passes through the fluid. Thus the process is both reversible and adiabatic, which means that it is isentropic. (Why?) After we have studied shock waves, we shall prove that very weak shock waves (i.e., very small disturbances) approach an isentropic process in the limit. Therefore, equation (4.4) should more properly be written as

$$\boxed{a^2 = g_c \left(\frac{\partial p}{\partial \rho}\right)_s} \tag{4.5}$$

This can be expressed in an "alternative standard form" by introducing the *bulk* or *volume modulus of elasticity* E_v. This is a relation between volume or density changes that occurs as a result of pressure fluctuations and is defined as

$$E_v \equiv -\boldsymbol{v}\left(\frac{\partial p}{\partial \boldsymbol{v}}\right)_s \equiv \rho\left(\frac{\partial p}{\partial \rho}\right)_s \tag{4.6}$$

Thus

$$\boxed{a^2 = g_c\left(\frac{E_v}{\rho}\right)} \tag{4.7}$$

Equations (4.5) and (4.7) are equivalent general relations for sonic speed through *any* medium. The bulk modulus is normally used in connection with liquids and solids for which it exceeds gaseous values. Table 4.1 gives some typical values of this modulus, the exact value depending on the temperature and pressure of the medium. For solids it also depends on the type of loading. The reciprocal of the bulk modulus is called the *compressibility*. What is the sonic speed in a truly *in*compressible fluid? [*Hint:* What is the value of $(\partial p/\partial \rho)_s$?]

Equation (4.5) is normally used for gases and this can be greatly simplified for the case of a gas that obeys the perfect gas law. For an isentropic process, we know that

$$p\boldsymbol{v}^\gamma = \text{const} \quad \text{or} \quad p = \rho^\gamma \text{const} \tag{4.8}$$

Thus

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \gamma\rho^{\gamma-1}\text{const}$$

But from (4.8), the constant = p/ρ^γ. Therefore,

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \gamma\rho^{\gamma-1}\frac{p}{\rho^\gamma} = \gamma\frac{p}{\rho} = \gamma RT$$

Table 4.1 Bulk Modulus Values for Common Non-Gaseous Media

Medium	Bulk Modulus (psi)
Oil	185,000–270,000
Water	300,000–400,000
Mercury	approx. 4,000,000
Steel	approx. 30,000,000

and from (4.5)

$$\boxed{a^2 = \gamma g_c RT} \tag{4.9}$$

or

$$\boxed{a = \sqrt{\gamma g_c RT}} \tag{4.10}$$

Notice that for perfect gases, the sonic speed is a function of the individual gas and temperature *only*.

Example 4.1
Compute the sonic speed in air at 70°F.

$$a^2 = \gamma g_c RT = (1.4)(32.2)(53.3)(460 + 70)$$

$$a = 1128\,\text{ft/sec} \quad (\text{or } 343.8\,\text{m/s})$$

Example 4.2
Sonic speed through carbon dioxide is 275 m/s. What is the temperature in Kelvin and Rankine?

$$a^2 = \gamma g_c RT$$

$$(275)^2 = (1.29)(1)(189)(T)$$

$$T = 310.2\,\text{K} \quad \text{and} \quad 558.4\text{°R}$$

Always keep in mind that in general sonic speed is a property of the fluid and varies with the thermodynamic state of the fluid at each location. *Only* for gases that can be treated as perfect is the sonic speed a function of temperature alone.

Mach Number

We define the *Mach number* as

$$\boxed{M \equiv \frac{V}{a}} \tag{4.11}$$

where

$V \equiv$ the velocity of the medium
$a \equiv$ sonic peed through the medium

It is important to realize that both V and a are computed *locally* for conditions that actually exist at any given point. If the velocity at one point in a flow system is twice that at another point, we *cannot* say that the Mach number has doubled. We must seek further information on the sonic speed, which has possibly also changed. (What property would we be of interested if the fluid were a perfect gas?)

As stated in Chapter 1, if the velocity is less than the local speed of sound, M is less than 1.0 and the flow is called *subsonic*. If the velocity is greater than the local speed of sound, M is greater than 1.0 and the flow is called *supersonic*. In Section 1.3 we gave information necessary for calculating a Mach number (which turns out to be $M = 0.291$ irrespective of the units used initially). We shall soon make you realize that the Mach number is the most important parameter in the analysis of compressible flows.

4.4 WAVE PROPAGATION

Let us examine a point disturbance that is at rest in a fluid. *Infinitesimal* pressure pulses are continually being emitted and thus they travel through the medium at *sonic* speeds in the form of spherical wave fronts. To simplify matters we shall keep track of only those pulses that are emitted every second. At the end of 3 seconds, the picture will appear as shown in Figure 4.3. Note that the wave fronts are concentric.

Now consider a similar problem in which the disturbance is no longer stationary. Assume that it is moving at a speed less than sonic speed, say $a/2$. Figure 4.4 shows such a situation at the end of 3 seconds. Note that the wave fronts are no longer concentric. Furthermore, the wave that was emitted at $t = 0$ is always in front of the disturbance itself. *Therefore, any person, object, or fluid particle located upstream will able to feel the wave fronts pass by and know that the disturbance is coming.*

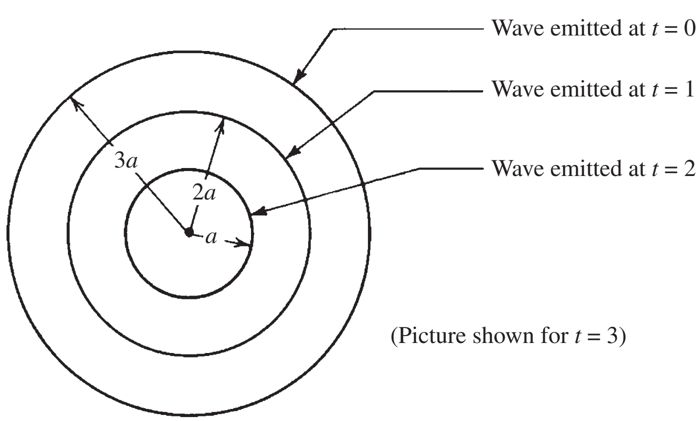


Figure 4.3 Wave fronts from a stationary disturbance.

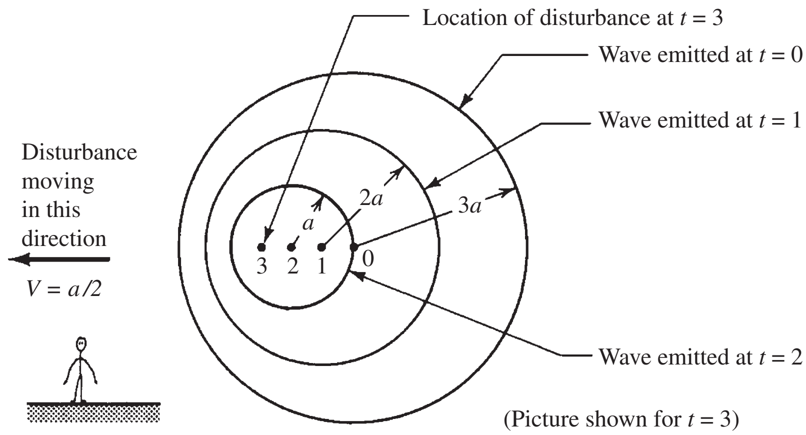


Figure 4.4 Wave fronts from subsonic disturbance.

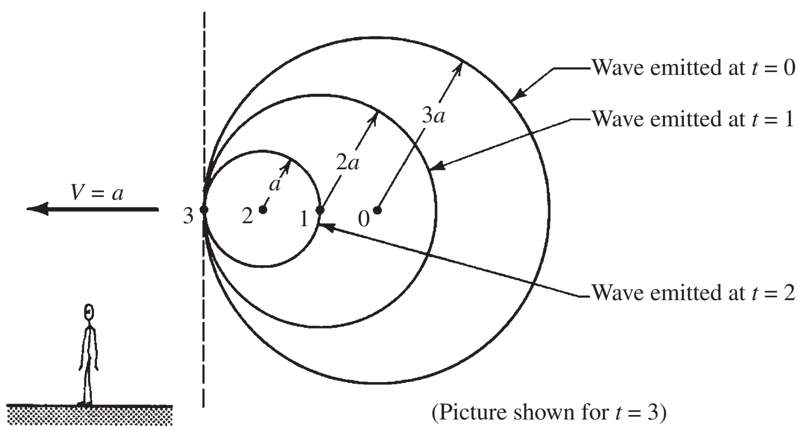


Figure 4.5 Wave fronts from sonic disturbance (a transonic flow condition).

Next, let the disturbance move at exactly sonic speed. Figure 4.5 shows this case and you will note that all wave fronts coalesce on the left side and move along with the disturbance. After a long period of time this wave front would approximate a plane indicated by the dashed line. In this case, no region upstream is forewarned of the disturbance as the disturbance arrives at the same time as the wave front.

The only other case to consider is that of a disturbance moving at velocities greater than the speed of sound. Figure 4.6 shows a point disturbance moving at Mach number = 2.0 (twice sonic speed). The wave fronts have coalesced to form a cone with the disturbance at the apex. This is called a *Mach cone*. The region inside the cone is called the *zone of action* since it feels the presence of the waves. The outer region is called the *zone of silence,* as *this entire region*

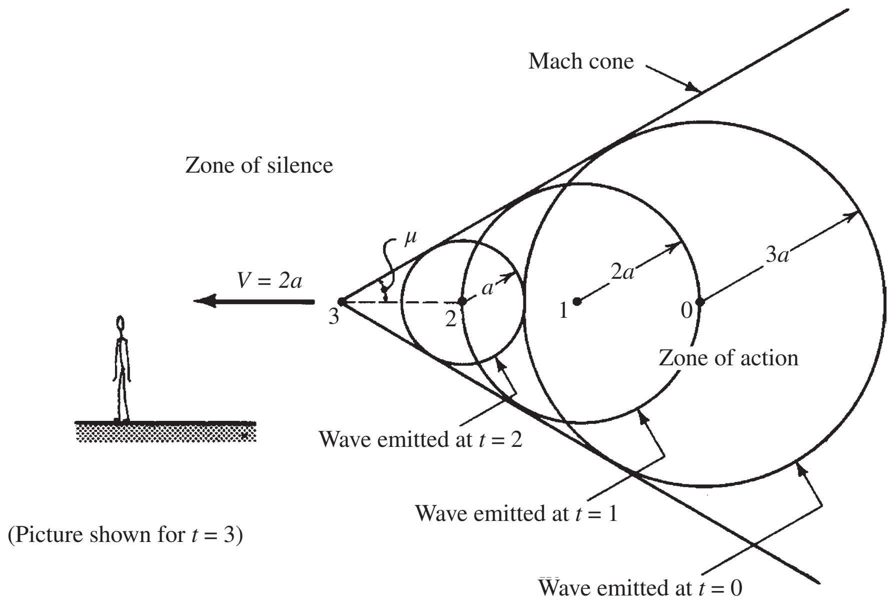


Figure 4.6 Wave fronts from supersonic disturbance.

is unaware of the disturbance. The surface of the Mach cone is sometimes referred to as a *Mach wave*; the half-angle at the apex is called the *Mach angle* and is given the symbol μ. It should be easy to see that

$$\boxed{\sin\mu = \frac{a}{V} = \frac{1}{M}} \tag{4.12}$$

In this section, we have discovered one of the most significant differences between subsonic and supersonic flow fields. In the subsonic case the fluid can "sense" the presence of an object and smoothly adjust its flow around the object. In supersonic flow, this is not possible, and thus flow adjustments occur rather abruptly in the form of shock or expansion waves. We study these in great detail in Chapters 6 through 8.

4.5 EQUATIONS FOR PERFECT GASES IN TERMS OF MACH NUMBER

In Section 4.4 we saw that supersonic and subsonic flows have totally different characteristics. This suggests that it would be desirable to use Mach number as a parameter in our basic equations. This can be done very easily for the flow of a perfect gas since in this case we have a simple equation of state *and* an explicit

expression for sonic speed. Development of some of the more important control volume relations follow.

Continuity

For steady one-dimensional flow with a single inlet and a single outlet, we have

$$\dot{m} = \rho A V = \text{const} \tag{2.30}$$

From the perfect gas equation of state,

$$\rho = \frac{p}{RT} \tag{1.16}$$

and from the definition of Mach number,

$$V = Ma \tag{4.11}$$

Also recall the expression for sonic speed in a perfect gas:

$$a = \sqrt{\gamma g_c RT} \tag{4.10}$$

Substitution of equations (1.16), (4.11), and (4.10) into (2.30) yields

$$\rho A V = \frac{p}{RT} AM \sqrt{\gamma g_c RT} = pAM \sqrt{\frac{\gamma g_c}{RT}}$$

Thus for steady one-dimensional flow of a perfect gas, the continuity equation becomes

$$\boxed{\dot{m} = pAM \sqrt{\frac{\gamma g_c}{RT}} = \text{const}} \tag{4.13}$$

Stagnation Relations

For gases we can neglect the potential term and write

$$h_t = h + \frac{V^2}{2g_c} \tag{3.18}$$

Knowing

$$V^2 = M^2 a^2 \qquad [\text{from}(4.11)]$$

and

$$a^2 = \gamma g_c RT \tag{4.9}$$

we have

$$h_t = h + \frac{M^2 \gamma g_c RT}{2g_c} = h + \frac{M^2 \gamma RT}{2} \tag{4.14}$$

From equations (1.39) and (1.40) we can write the specific heat at constant pressure in terms of γ and R. *Show* that

$$c_p = \frac{\gamma R}{\gamma - 1} \tag{4.15}$$

Combining (4.15) and (4.14), we have

$$h_t = h + M^2 \frac{\gamma - 1}{2} c_p T \tag{4.16}$$

But for a gas we can say that

$$h = c_p T \tag{1.38}$$

Thus

$$h_t = h + M^2 \frac{\gamma - 1}{2} h$$

or

$$\boxed{h_t = h\left(1 + \frac{\gamma - 1}{2} M^2\right)} \tag{4.17}$$

Using $h = c_p T$ and $h_t = c_p T_t$, this can further be written as

$$\boxed{T_t = T\left(1 + \frac{\gamma - 1}{2} M^2\right)} \tag{4.18}$$

Equations (4.17) and (4.18) are used frequently. *Memorize them!*

Now, the stagnation process is isentropic. Thus γ can be used as the exponent n in equation (1.47), and between any two points on the same isentropic, we have

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)} \tag{4.19}$$

Let point 1 refer to the static conditions, and point 2, the stagnation conditions. Then, combining (4.19) and (4.18) produces

$$\frac{p_t}{p} = \left(\frac{T_t}{T}\right)^{\gamma/(\gamma-1)} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/(\gamma-1)} \tag{4.20}$$

or

$$\boxed{p_t = p\left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/(\gamma-1)}} \tag{4.21}$$

This expression for total pressure is important. *Learn it!*

Example 4.3

Air flows with a velocity of 800 ft/sec and has a pressure of 30 psia and temperature of 600°R. Determine the stagnation pressure.

$$a = (\gamma g_c RT)^{1/2} = [(1.4)(32.2)(53.3)(600)]^{1/2} = 1201\ \text{ft/sec}$$

$$M = \frac{V}{a} = \frac{800}{1201} = 0.666$$

$$p_t = p\left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/\gamma-1} = 30\left[1 + \left(\frac{1.4-1}{2}\right)(0.666)^2\right]^{1.4/(1.4-1)}$$

$$p_t = (30)(1+0.0887)^{3.5} = (30)(1.346) = 40.4\ \text{psia} \quad (\text{or } 0.279\ \text{MPa})$$

Example 4.4

Hydrogen has a static temperature of 25°C and a stagnation temperature of 250°C. What is the Mach number?

$$T_t = T\left(1 + \frac{\gamma-1}{2}M^2\right)$$

$$(250+273) = (25+273)\left(1 + \frac{1.41-1}{2}M^2\right)$$

$$523 = (298)(1 + 0.205M^2)$$

$$M^2 = 3.683 \quad \text{and} \quad M = 1.92$$

Stagnation Pressure–Energy Equation

For steady one-dimensional flow, we have

$$\frac{dp_t}{\rho_t} + ds_e(T_t - T) + T_t ds_i + \delta w_s = 0 \tag{3.25}$$

For a perfect gas,

$$p_t = \rho_t R T_t \tag{4.22}$$

Substitute for the stagnation density and *show* that equation (3.25) can be written as

$$\boxed{\frac{dp_t}{p_t} + \frac{ds_e}{R}\left(1 - \frac{T}{T_t}\right) + \frac{ds_i}{R} + \frac{\delta w_s}{RT_t} = 0} \tag{4.23}$$

A large number of problems of interest are adiabatic and involve no shaft work. In this case, ds_e and δw_s are zero:

$$\frac{dp_t}{p_t} + \frac{ds_i}{R} = 0 \tag{4.24}$$

This equation can be integrated between any two points in the flow system to give

$$\ln\frac{p_{t2}}{p_{t1}} + \frac{s_{i2} - s_{i1}}{R} = 0 \tag{4.25}$$

But since $ds_e = 0$, $ds_i = ds$, and we really do not need to continue writing the subscript i under the entropy. Thus

$$\ln\frac{p_{t2}}{p_{t1}} = -\frac{s_2 - s_1}{R} \tag{4.26}$$

Taking the antilog, this becomes

$$\frac{p_{t2}}{p_{t1}} = e^{-(s_2 - s_1)/R} \tag{4.27}$$

or

$$\boxed{\frac{p_{t2}}{p_{t1}} = e^{-\Delta s/R}} \tag{4.28}$$

Watch your units when you use this equation! Total pressures must be absolute, and $\Delta s/R$ must be dimensionless. For the case of adiabatic no-work flow, Δs will *always* be positive in the flow direction. (Why?). Thus p_{t2} will always be less than p_{t1}. *Only* for the limiting case of no losses will the stagnation pressure remain constant throughout the flow.

This confirms previous knowledge gained from the stagnation pressure–energy equation: that for the case of an adiabatic, no-work system, without flow losses p_t = const for *any* fluid. Thus stagnation pressure is seen to be a very important parameter which in many systems uniquely reflects flow losses. Be careful to note, however, that the specific relation in equation (4.28) is applicable only to perfect gases, and even then only under certain flow conditions. (What are these conditions?)

Summarizing the above: For steady one-dimensional flow, we have

$$\delta q = \delta w_s + dh_t \tag{3.20}$$

Note that equation (3.20) is valid even when flow losses are present:

$$\text{if } \delta q = \delta w_s = 0, \quad \text{then } \quad h_t = \text{constant}$$

If in addition to the above, no internal losses occur, that is,

$$\text{if } \delta q = \delta w_s = ds_i = 0, \quad \text{then } \quad p_t = \text{constant}$$

Example 4.5

Oxygen flows in a constant-area, horizontal, insulated duct. Conditions at section 1 are $p_1 = 50$ psia, $T_1 = 600°$R, and $V_1 = 2860$ ft/sec. At a downstream section, the temperature is $T_2 = 1048°$R.

a. Determine M_1 and T_{t1}.

b. Find V_2 and p_2.

c. What is the entropy change between the two sections?

$$a_1 = (\gamma g_c R T_1)^{1/2} = [(1.4)(32.2)(48.3)(600)]^{1/2} = 1143\,\text{ft/sec}$$

a.
$$M_1 = \frac{V_1}{a_1} = \frac{2860}{1143} = 2.50$$

$$T_{t1} = T_1\left(1 + \frac{\gamma - 1}{2}M_1^2\right) = (600)\left[1 + \frac{1.4 - 1}{2}(2.5)^2\right] = 1350°\text{R} \quad (\text{or } 749.3\,\text{K})$$

b. *Energy*

$$h_{t1} + \cancel{q} = h_{t2} + \cancel{w_s}$$

$$h_{t1} = h_{t2}$$

and since this is a perfect gas, $T_{t1} = T_{t2}$.

$$T_{t2} = T_2\left(1 + \frac{\gamma - 1}{2}M_2^2\right)$$

$$1350 = (1048)\left(1 + \frac{1.4 - 1}{2}M_2^2\right) \quad \text{and} \quad M_2 = 1.20$$

$$V_2 = M_2 a_2 = (1.20)[(1.4)(32.2)(48.3)(1048)]^{1/2} = 1813\,\text{ft/sec} \quad (\text{or } 552.6\,\text{m/s})$$

Continuity:

$$\dot{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

but

$$A_1 = A_2 \quad \text{and} \quad \rho = \frac{p}{RT}$$

Thus

$$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2}$$

$$p_2 = \frac{V_1}{V_2}\frac{T_2}{T_1}p_1 = \left(\frac{2860}{1813}\right)\left(\frac{1048}{600}\right)(50) = 137.8\,\text{psia} \quad (\text{or } 0.950\,\text{MPa})$$

c. To obtain the entropy change, we need p_{t1} and p_{t2}.

$$p_{t1} = p_1\left(1 + \frac{\gamma - 1}{2}M_1^2\right)^{\gamma/(\gamma-1)} = (50)\left[1 + \frac{1.4 - 1}{2}(2.5)^2\right]^{1.4/(14-1)} = 854\,\text{psia}$$

Similarly,

$$p_{t2} = 334\,\text{psia}$$

$$e^{-\Delta s/R} = \frac{p_{t2}}{p_{t1}} = \frac{334}{854} = 0.391$$

$$\frac{\Delta s}{R} = \ln\frac{1}{0.391} = 0.939$$

$$\Delta s = \frac{(0.939)(48.3)}{(778)} = 0.0583\,\text{Btu/lbm-}^\circ\text{R} \quad (\text{or } 244.1\,\text{J/kg})$$

4.6 *h–s* AND *T–s* DIAGRAMS

Every problem should be approached with a simple sketch of the physical system and also a thermodynamic state diagram. Since losses affect entropy changes (through ds_i), one generally uses either an *h–s* or *T–s* diagram. In the case of perfect gases, enthalpy is a function of temperature only and therefore the *T–s* and *h–s* diagrams are identical except for scale.

Consider a steady one-dimensional flow of a perfect gas. Let us assume no heat transfer and no external work. From the energy equation

$$h_{t1} + \cancel{q} = h_{t2} + \cancel{w_s} \tag{3.19}$$

the stagnation enthalpy remains constant, and since it is a perfect gas, the total temperature is also constant. This is represented by the solid horizontal line in Figure 4.7. Two particular sections in the system have been indicated by 1 and 2. The actual process that takes place between these points is indicated on the *T–s* diagram.

Notice that although the stagnation conditions do not actually exist in the system, they are also shown on the diagram for reference. The distance between the static and stagnation points is indicative of the velocity that exists at that location (since gravity has been neglected). It can also be clearly seen that if there is a Δs_{1-2}, then $p_{t2} < p_{t1}$ and the relationship between stagnation pressure and flow losses is again verified.

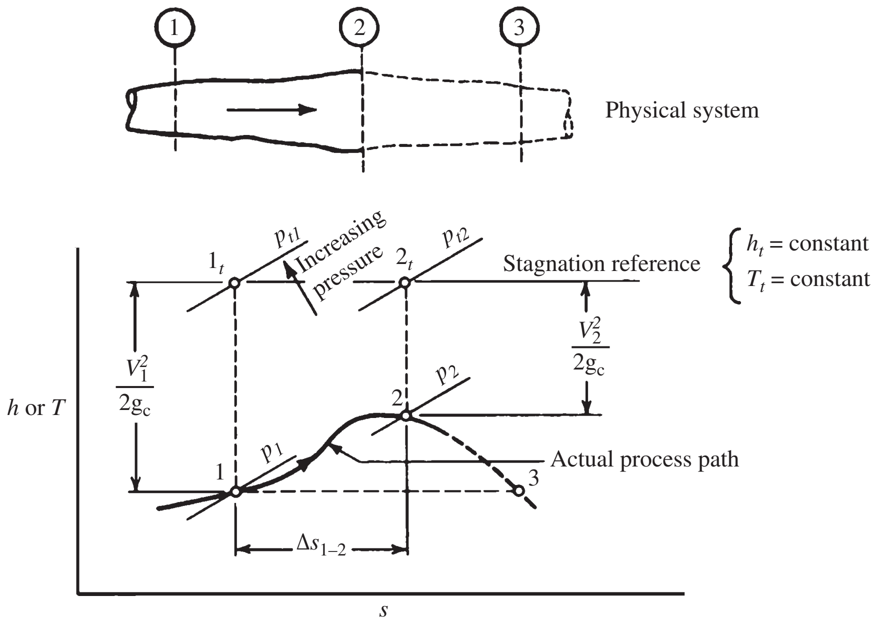


Figure 4.7 Stagnation reference states.

It is interesting to hypothesize a third section that just happens to be at the same enthalpy (and temperature) as the first. What else do these points have in common? The same velocity? Obviously! How about sonic speed? (Recall for gases that this is a function of temperature only.) This means that points 1 and 3 would also have the same Mach number (something that is not immediately obvious). One can now imagine that someplace on this diagram there is a horizontal line that represents the locus of points having a Mach number of unity. Between this line and the stagnation line lie all points in the subsonic regime. Below this line lie all points in the supersonic regime. These conclusions are based on certain assumptions. What are they?

4.7 SUMMARY

In general, pressure waves propagate at a speed that depends on the medium, its thermodynamic state, and on the strength of the wave. However, infinitesimal disturbances travel at a speed determined only by the medium and its state. Sound waves fall into this latter category. Our discussion of wave propagation and sonic speed brought out a basic difference between subsonic and supersonic flows. If subsonic, the flow can "sense" objects and flow smoothly around them. This is not possible in supersonic flow, and this topic will be discussed further after the appropriate background has been laid.

As you progress through the remainder of this book and analyze specific flow situations, it will become increasingly evident that gases behave quite differently in the supersonic regime than they do in our more familiar subsonic flow regime. Thus it will not be surprising to see how the Mach number becomes an important parameter. The significance of T–s diagrams as a key to problem visualization should not be overlooked.

Some of the most frequently used equations that were developed in this unit are summarized below. Most are restricted to the steady one-dimensional flow of any fluid, while others apply only to perfect gases. You should be able to determine under what conditions each may be used.

1. *Sonic speed* (propagation speed of infinitesimal pressure pulses)

$$a^2 = g_c\left(\frac{\partial p}{\partial \rho}\right)_s = g_c\frac{E_v}{\rho} \tag{4.5), (4.7}$$

$$M = \frac{V}{a} \quad (V \text{ and } a \text{ at the same location}) \tag{4.11}$$

$$\sin\mu = \frac{1}{M} \tag{4.12}$$

2. *Special relations for perfect gases*

$$a^2 = \gamma g_c RT \tag{4.9}$$

$$h_t = h\left(1 + \frac{\gamma - 1}{2}M^2\right) \tag{4.17}$$

$$T_t = T\left(1 + \frac{\gamma - 1}{2}M^2\right) \tag{4.18}$$

$$p_t = p\left(1 + \frac{\gamma - 1}{2}M^2\right)^{\gamma/(\gamma-1)} \tag{4.21}$$

$$\frac{dp_t}{p_t} + \frac{ds_e}{R}\left(1 - \frac{T}{T_t}\right) + \frac{ds_i}{R} + \frac{\delta w_s}{RT_t} = 0 \tag{4.23}$$

$$\frac{p_{t2}}{p_{t1}} = e^{-\Delta s/R} \quad \text{for} \quad Q = W = 0 \tag{4.28}$$

Consult References 13 and 14 for additional information on Gas Dynamics.

PROBLEMS

4.1. Compute and compare sonic speeds in air, water, and steel. Assume normal room temperature and pressure. For steel, use Table 4.1 and take $\rho = 0.284$ lbm/in^3. What do your results say about the relative magnitude of this speed in solids, liquids, and gases?

4.2. At what temperature and pressure would carbon monoxide, water vapor, and helium have the same speed of sound as standard air (288 K and 1 atm)?

4.3. Start with the relation for stagnation pressure that is valid for a perfect gas:

$$p_t = p\left(1 + \frac{\gamma - 1}{2}M^2\right)^{\gamma/(\gamma-1)}$$

Expand the right side in a binomial series and evaluate the result for small (but not zero) Mach numbers. Show that your answer can be written as

$$p_t = p + \frac{\rho V^2}{2g_c} + \text{HOT}$$

Remember, the higher-order terms are negligible only for very small Mach numbers. (See Problem 4.4.)

4.4. Measurement of airflow shows the static and stagnation pressures to be 30 and 32 psig, respectively. (Note that these are gage pressures.) Assume that $p_{amb} = 14.7$ psia and the temperature is 120°F.

(a) Find the flow velocity using equation (4.21).

(b) Now assume that the air flow is incompressible and calculate the velocity using equation (3.39).

(c) Repeat parts (a) and (b) for static and stagnation pressures of 30 and 80 psig, respectively.

(d) Can you reach any conclusions concerning when a gas may be treated as a constant-density fluid?

4.5. If $\gamma = 1.2$ and the fluid is a perfect gas, what Mach number will give a temperature ratio of $T/T_t = 0.909$? What will the ratio of p/p_t be for this flow?

4.6. Carbon dioxide with a temperature of 335 K and a pressure of 1.4×10^5 N/m^2 is flowing with a velocity of 200 m/s.

(a) Determine the sonic speed and Mach number.

(b) Determine the stagnation density.

4.7. The temperature of argon is 100°F, the pressure 42 psia, and the velocity 2264 ft/sec. Calculate the Mach number and stagnation pressure.

4.8. Helium flows in a duct with a temperature of 50°C, a pressure of 2.0 bar abs., and a total pressure of 5.3 bar abs. Determine the helium velocity in the duct.

4.9. An airplane flies 600 mph at an altitude of 16,500 ft, where the temperature is 0°F and the pressure is 1124 psfa. What temperature and pressure might you expect on the nose of the airplane?

4.10. Air flows at $M = 1.35$ and has a stagnation enthalpy of 4.5×10^5 J/kg. The stagnation pressure is 3.8×10^5 N/m^2. Determine the static conditions (pressure, temperature, and velocity).

4.11. A large chamber contains a perfect gas under conditions p_1, T_1, h_1, and so on. The gas is allowed to flow from the chamber (with $q = w_s = 0$). Show that because the initial energy is fixed the exit velocity cannot be greater than

$$V_{max} = a_1 \left(\frac{2}{\gamma - 1} \right)^{1/2}$$

If such velocity is the maximum, what is the corresponding Mach number?

4.12. Air flows steadily in an adiabatic duct where no shaft work is involved. At one section, the total pressure is 50 psia, and at another section, it is 67.3 psia. In which direction is the fluid flowing, and what is the entropy change between these two sections?

4.13. Methane gas flows in an adiabatic, no-work system with negligible change in potential. At one section $p_1 = 14$ bar abs., $T_1 = 500$ K, and $V_1 = 125$ m/s. At a downstream section $M_2 = 0.8$.

(a) Determine T_2 and V_2.

(b) Find p_2 assuming that there are no friction losses.

(c) What is the area ratio A_2/A_1?

4.14. Air flows through a constant-area, insulated passage. Entering conditions are $T_1 = 520°R$, $p_1 = 50$ psia, and $M_1 = 0.45$. At a point downstream, the Mach number is found to be unity.

(a) Solve for T_2 and p_2.

(b) What is the entropy change between these two sections?

(c) Determine the wall frictional force if the duct is 1 ft in diameter.

4.15. Carbon dioxide flows in a horizontal adiabatic, no-work system. Pressure and temperature at section 1 are 7 atm and 600 K. At a downstream section, $p_2 = 4$ atm., $T_2 = 550$ K, and the Mach number is $M_2 = 0.90$.

(a) Compute the velocity at the upstream location.

(b) What is the entropy change?

(c) Determine the area ratio A_2/A_1.

4.16. Oxygen with $T_{t1} = 1000°R$, $p_{t1} = 100$ psia, and $M_1 = 0.2$ enters a device with a cross-sectional area $A_1 = 1$ ft^2. There is no heat transfer, work transfer, or losses as the gas passes through the device and expands to 14.7 psia.

(a) Compute ρ_1, V_1, and $\dot{m}$.

(b) Compute M_2, T_2, V_2, ρ_2, and A_2.

(c) What force does the fluid exert on the device?

4.17. Consider steady, one-dimensional, constant-area, horizontal, isothermal flow of a perfect gas with no shaft work (Figure P4.17). The duct has a cross-sectional area A and perimeter P. Let τ_w be the shear stress at the wall.

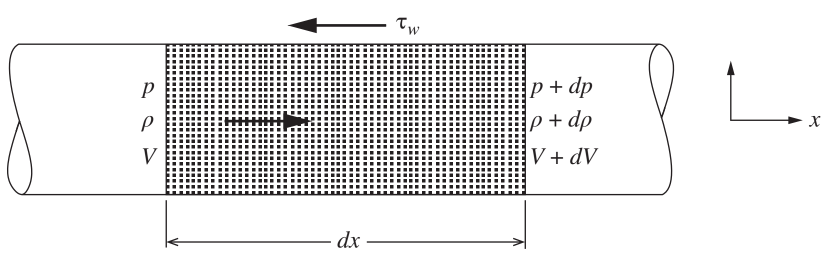


Figure P4.17

(a) Apply momentum concepts [equation (3.45)] and show that

$$-dp - f\frac{dx}{D_e}\frac{\rho V^2}{2g_c} = \frac{\rho V dV}{g_c}$$

(b) From the concept of continuity and the equation of state, show that

$$\frac{d\rho}{\rho} = \frac{dp}{p} = -\frac{dV}{V}$$

(c) Combine the results of parts (a) and (b) to show that

$$\frac{d\rho}{\rho} = \left[\frac{\gamma M^2}{2(\gamma M^2 - 1)}\right]\frac{f dx}{D_e}$$

CHECK TEST

You should be able to complete this test without reference to material in the chapter.

4.1. **(a)** Define Mach number and Mach angle.

(b) Give an expression that represents sonic speed in an arbitrary fluid.

(c) Give the relation used to compute sonic speed in a perfect gas.

4.2. Consider the steady, one-dimensional flow of a perfect gas with heat transfer. The T–s diagram (Figure CT4.2) shows both static and stagnation points at two locations in the system. It is known that $A = B$.

(a) Is heat transferred into or out of the system?

(b) Is $M_2 > M_1$, $M_2 = M_1$, or $M_2 < M_1$?

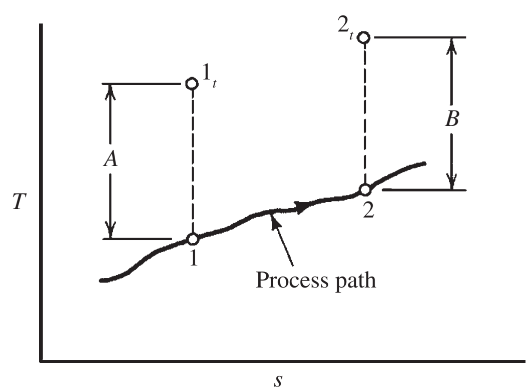


Figure CT4.2

4.3. State whether each of the following statements is true or false.

(a) Changing the frame of reference (or superposition of a velocity onto an existing flow) does not change the static enthalpy.

(b) Shock waves travel at sonic speed through a medium.

(c) In general, one can say that flow losses will show up as a decrease in stagnation enthalpy.

(d) The stagnation process is one of constant entropy.

(e) A Mach cone does not exist for subsonic flow.

4.4. Cite the conditions that are necessary for the stagnation temperature to remain constant in a flow system.

4.5. For steady flow of a perfect gas, the continuity equation can be written as

$$\dot{m} = f(p, M, T, \gamma, A, R, g_c) = \text{const}$$

Determine the precise function.

4.6. Work Problem 4.14.

Chapter 5

Varying-Area Adiabatic Flow

5.1 INTRODUCTION

Area changes, friction, and heat transfer are the most important factors that affect the properties in a flow system. Although some situations may involve the simultaneous effects of two or more of these factors, the majority of engineering problems are such that *only one of these factors becomes the dominant influence for any particular device*. Thus it is more than academic interest that leads to the separate study of each of the above-mentioned effects. In this manner it is possible to consider only the controlling factor and develop solutions that are within the realm of acceptable engineering accuracy.

In this chapter we study the general problem of varying-area flow under the assumptions of no heat transfer (adiabatic) and no shaft work. We first consider the flow of an arbitrary fluid without losses and determine how its properties are affected by area changes. The case of a perfect gas is then considered and simple working equations developed to aid in the solution of problems with and without flow losses. The latter case (isentropic flow) lends itself to the construction of tables that are used throughout the remainder of the book. The chapter closes with a brief discussion of the various ways in which nozzle and diffuser performance can be represented.

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

5.2 OBJECTIVES

After completing this chapter successfully, you should be able to:

1. (*Optional*) Simplify the basic equations for continuity and energy to relate differential changes in density, pressure, and velocity to the Mach number and a differential change in area for steady, one-dimensional flow through a varying-area passage with no losses.
2. Show graphically how pressure, density, velocity, and area vary in steady, one-dimensional, isentropic flow as the Mach number ranges from zero to supersonic values.
3. Compare the function of a nozzle to that of a diffuser. Sketch physical devices that perform as each for subsonic and supersonic flow.
4. (*Optional*) Derive the working equations for a perfect gas by relating property ratios between two points in adiabatic, no-work flow, as a function of the Mach number (M), ratio of specific heats (γ), and change in entropy (Δs).
5. Define the * reference condition and the properties associated with it (i.e., A^*, p^*, T^*, ρ^*, etc.). Identify what is the common feature of all * reference states.
6. Express the loss (Δs_i) (between two points in the flow) as a function of stagnation pressures (p_t) *or* reference areas (A^*). Identify under what conditions these relations are true.
7. State and interpret the relation between stagnation pressure (p_t) and reference area (A^*) for a process between two points in adiabatic no-work flow.
8. Explain how a converging nozzle performs with various receiver pressures. Do the same for the *isentropic* performance of a converging–diverging nozzle.
9. State what is meant by the *first* and *third critical* modes of supersonic nozzle operation. Given the area ratio of a converging–diverging nozzle, determine the operating pressure ratios that cause operation at the first and third critical points.
10. With the aid of an h–s diagram, give a suitable definition for both nozzle efficiency and diffuser performance.
11. Explain in your own words what is meant by a *choked* flow passage.
12. Demonstrate the ability to utilize the adiabatic and isentropic flow relations and the isentropic table to solve typical flow problems.

5.3 GENERAL FLUID WITH NO LOSSES

We first consider the general behavior of an arbitrary fluid. To isolate the effects of area change, we make the following assumptions:

Steady, one-dimensional flow	
Adiabatic	$\delta q = 0$ or $ds_e = 0$
No shaft work	$\delta w_s = 0$
Negligible potential	$dz = 0$
No losses	$ds_i = 0$

Our objective will be to obtain relations that indicate the variation of fluid properties with area changes *and* Mach number. In this manner we can more easily distinguish many important differences between subsonic and supersonic behavior. We start with the energy equation:

$$\delta q = \delta w_s + dh + \frac{dV^2}{2g_c} + \frac{g}{g_c} dz \tag{2.53}$$

But

$$\delta q = \delta w_s = 0$$

and

$$dz = 0$$

which leaves us with

$$0 = dh + \frac{dV^2}{2g_c} \tag{5.1}$$

or

$$dh = -\frac{VdV}{g_c} \tag{5.2}$$

We now introduce the property relation

$$Tds = dh - \frac{dp}{\rho} \tag{1.31}$$

Since our flow situation has been assumed to be adiabatic ($ds_e = 0$) and to contain no losses ($ds_i = 0$), it is also isentropic ($ds = 0$). Thus equation (1.31) becomes

$$dh = \frac{dp}{\rho} \tag{5.3}$$

We equate equations (5.2) and (5.3) to obtain

$$-\frac{VdV}{g_c} = \frac{dp}{\rho}$$

or

$$dV = -\frac{g_c dp}{\rho V} \tag{5.4}$$

We introduce this into equation (2.32) and the differential form of the continuity equation becomes

$$\frac{d\rho}{\rho} + \frac{dA}{A} - \frac{g_c dp}{\rho V^2} = 0 \tag{5.5}$$

Solve this for *dp/ρ* and *show* that

$$\frac{dp}{\rho} = \frac{V^2}{g_c}\left(\frac{d\rho}{\rho} + \frac{dA}{A}\right) \tag{5.6}$$

Recall the definition of sonic speed:

$$a^2 = g_c \left(\frac{\partial p}{\partial \rho}\right)_s \tag{4.5}$$

Since our flow *is* isentropic, we may drop the subscript and change the partial derivative to an ordinary derivative:

$$a^2 = g_c \frac{dp}{d\rho} \tag{5.7}$$

This permits equation (5.7) to be rearranged to

$$dp = \frac{a^2}{g_c} d\rho \tag{5.8}$$

Substituting this expression for *dp* into equation (5.6) yields

$$\frac{d\rho}{\rho} = \frac{V^2}{a^2}\left(\frac{d\rho}{\rho} + \frac{dA}{A}\right) \tag{5.9}$$

Introduce the definition of Mach number,

$$M^2 = \frac{V^2}{a^2} \tag{4.11}$$

and combine the terms in $d\rho/\rho$ to obtain the following relation between density and area changes:

$$\frac{d\rho}{\rho} = \left(\frac{M^2}{1-M^2}\right)\frac{dA}{A} \tag{5.10}$$

If we now substitute equation (5.10) into the differential form of the continuity equation (2.32), we can obtain a relation between velocity and area changes. *Show* that

$$\frac{dV}{V} = -\left(\frac{1}{1-M^2}\right)\frac{dA}{A} \tag{5.11}$$

Now equation (5.4) can be divided by *V* to yield

$$\frac{dV}{V} = -\frac{g_c dp}{\rho V^2} \tag{5.12}$$

If we equate (5.11) and (5.12), we can obtain a relation between pressure and area changes. *Show* that

$$dp = \frac{\rho V^2}{g_c}\left(\frac{1}{1-M^2}\right)\frac{dA}{A} \tag{5.13}$$

For convenience, we collect the three important relations that will be referred to in the analysis that follows:

$$\boxed{dp = \frac{\rho V^2}{g_c}\left(\frac{1}{1-M^2}\right)\frac{dA}{A}} \tag{5.13}$$

$$\boxed{\frac{d\rho}{\rho} = \left(\frac{M^2}{1-M^2}\right)\frac{dA}{A}} \tag{5.10}$$

$$\boxed{\frac{dV}{V} = -\left(\frac{1}{1-M^2}\right)\frac{dA}{A}} \tag{5.11}$$

Let us consider what is happening as fluid flows through a variable-area duct. For simplicity we shall *assume first that the pressure is always decreasing*. Thus dp is negative. From equation (5.13) you see that if $M < 1$, dA must be negative, indicating that the area is decreasing; whereas if $M > 1$, dA must be positive and the area is increasing.

Now continue to assume that the pressure is decreasing. Knowing the area variation you can now consider equation (5.10). Fill in the following blanks with the words *increasing* or *decreasing:* If $M < 1$ (and dA is _____), then $d\rho$ must be _____. If $M > 1$ (and dA is _____), then $d\rho$ must be _____.

Looking at equation (5.11) reveals that if $M < 1$ (and dA is _____) then, dV must be_____ meaning that velocity is _____, whereas if $M > 1$ (and dA is _____), then dV must be _____ and velocity is _____.

We summarize the above by saying that *as the pressure decreases*, the following variations occur:

		Subsonic ($M < 1$)	Supersonic ($M > 1$)
Area	A	Decreases	Increases
Density	ρ	Decreases	Decreases
Velocity	V	Increases	Increases

A similar chart could easily be made for the situation where pressure increases, but it is probably more convenient to express the above in an equivalent graphical form, as shown in Figure 5.1. The appropriate shape of these curves can easily be visualized if one combines equations (5.10) and (5.11) to eliminate the term dA/A with the following result:

$$\frac{d\rho}{\rho} = -M^2\frac{dV}{V} \tag{5.14}$$

From this equation we see that at low Mach numbers, density variations will be quite small, whereas at high Mach numbers the density changes *very* rapidly. (Eventually, as V becomes very large and ρ becomes very small, small density changes occur once again.) Putting it another way, the density remains nearly constant in the low subsonic regime ($d\rho \approx 0$), and only velocity changes compensate for area changes. [See the differential form of the continuity equation (2.32).]

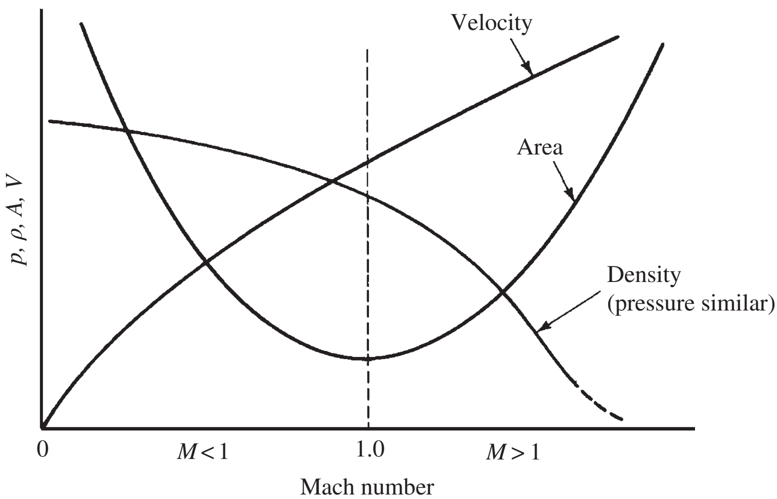


Figure 5.1 Property variation with area change.

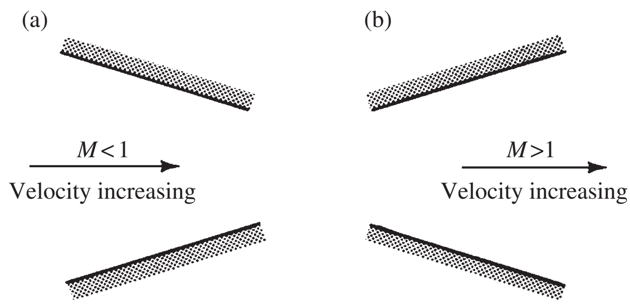


Figure 5.2 Nozzle configurations.

At a Mach number equal to unity, we reach a situation where density changes and velocity changes compensate for one another and thus no change in area is required ($dA = 0$). As we move on into the supersonic area, the density decreases so rapidly that the accompanying velocity change cannot accommodate the flow and thus the area must increase. We now recognize that this is another aspect of flow behavior which is exactly opposite in subsonic and supersonic flow. Consider next the operation of devices such as nozzles and diffusers.

A *nozzle* is a device that converts enthalpy (or pressure energy for the case of an incompressible fluid) into kinetic energy. From Figure 5.1 we see that an increase in velocity is accompanied by either an increase or decrease in area, depending on the Mach number. Figure 5.2 shows what these devices look like in the subsonic and supersonic flow regimes.

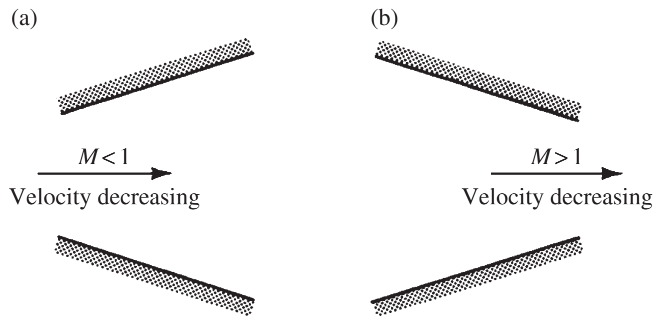


Figure 5.3 Diffuser configurations.

A *diffuser* is a device that converts kinetic energy into enthalpy (or pressure energy for the case of incompressible fluids). Figure 5.3 shows what these devices look like in the subsonic and supersonic regimes. Thus we see that the same piece of equipment can operate as either a nozzle or a diffuser, depending on the flow direction and regime.

Notice that a device is called a nozzle or a diffuser because of *what it does,* not what it looks like. Further consideration of Figures 5.1 and 5.2 leads to some interesting conclusions. If one attached only a converging section (see Figure 5.2*a*) to a high-pressure supply, one could never attain a flow greater than Mach 1, regardless of the pressure differential available. On the other hand, if we made a converging–diverging device (combination of Figure 5.2*a* and *b*), we see a means of accelerating the fluid into the supersonic regime, provided that the proper pressure differential exists. Specific examples of these cases are given later in the chapter.

Referring back to Figures 5.1 and 5.2, the transition from subsonic to supersonic flow at $M \approx 1.0$ is called the *transonic region*. Within this region, the *character of the flow* must change from what is mathematically termed "elliptic" to "hyperbolic," and within this transonic boundary, real flows become noticeably less one-dimensional. In other words, we must consider multidimensional descriptions. As mentioned earlier, in elliptic flows (which include the incompressible regime), the flow can anticipate barriers whereas supersonic flows cannot, because of wave fronts that must form shocks and that obey other hyperbolic flow constraints. Fortunately for *internal transonic flows,* such as those at the minimum area or *throat* of a supersonic nozzle, these deleterious phenomena are relatively limited and usually can be neglected. However, for *external flows* transonic regions tend to always be more complicated by flow separation and turbulence. This is one reason why in elementary gas-dynamic treatments we deal with *internal flows* almost exclusively—except for the purely supersonic regimes for which we may discuss certain airfoil configurations (see Section 8.7 on supersonic airfoils, e.g., flat plates, wedges, etc.) because their *boundary layers* have negligible effects.

5.4 PERFECT GASES WITH LOSSES

Now that we understand the general effects of area change in flow systems, we will develop some specific working equations for the case of a perfect gas. The term *working equations* will be used throughout this book to indicate relations between properties at arbitrary sections of a flow system written in terms of Mach numbers, an appropriate specific heat ratio, and a loss indicator such as Δs_i. An example of this for the system shown in Figure 5.4 is

$$\frac{p_2}{p_1} = f(M_1, M_2, \gamma, \Delta s_i) \tag{5.15}$$

We begin by feeding the following assumptions into our fundamental concepts of state, continuity, and energy:

Steady one-dimensional flow
Adiabatic
No shaft work
Perfect gas
Negligible potential changes

State
We have the perfect gas equation of state:

$$p = \rho RT \tag{1.16}$$

Continuity

$$\dot{m} = \rho AV = \text{const} \tag{2.30}$$

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \tag{5.16}$$

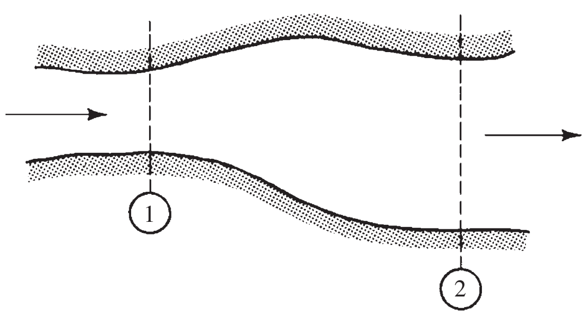


Figure 5.4 Varying-area flow system.

We first seek the area ratio

$$\frac{A_2}{A_1} = \frac{\rho_1 V_1}{\rho_2 V_2} \tag{5.17}$$

We substitute for the densities using the equation of state (1.16) and for velocities from the definition of Mach number (4.11):

$$\frac{A_2}{A_1} = \left(\frac{p_1}{RT_1}\right)\left(\frac{RT_2}{p_2}\right)\frac{M_1 a_1}{M_2 a_2} = \frac{p_1 T_2 M_1 a_1}{p_2 T_1 M_2 a_2} \tag{5.18}$$

Introduce the expression for the sonic speed of a perfect gas:

$$a = \sqrt{\gamma g_c RT} \tag{4.10}$$

and *show* that equation (5.18) becomes

$$\frac{A_2}{A_1} = \frac{p_1 M_1}{p_2 M_2}\left(\frac{T_2}{T_1}\right)^{1/2} \tag{5.19}$$

We must now find a means to express the pressure and temperature ratios in terms of M_1, M_2, γ, and Δs.

Energy

We start with

$$h_{t1} + q = h_{t2} + w_s \tag{3.19}$$

For an adiabatic, no-work process, this leads to

$$h_{t1} = h_{t2} \tag{5.20}$$

However, we can go further than this since we know that for a perfect gas, enthalpy is a function of temperature *only*. Thus

$$T_{t1} = T_{t2} \tag{5.21}$$

Recall from Chapter 4 that we developed a general relationship between static and stagnation temperatures for a perfect gas as

$$T_t = T\left(1 + \frac{\gamma - 1}{2}M^2\right) \tag{4.18}$$

Hence equation (5.21) can be written as

$$T_1\left(1+\frac{\gamma-1}{2}M_1^2\right)=T_2\left(1+\frac{\gamma-1}{2}M_2^2\right) \tag{5.22}$$

or

$$\frac{T_2}{T_1}=\frac{1+[(\gamma-1)/2]\,M_1^2}{1+[(\gamma-1)/2]\,M_2^2} \tag{5.23}$$

which is the ratio desired for insertion in equation (5.19). Note that no subscripts have been put on the specific heat ratio γ, which means we are assuming that $\gamma_1=\gamma_2$. This might be questioned since the specific heats c_p and c_v are known to vary somewhat with temperature. In Chapter 11 we explore real gas behavior and learn why these specific heats can vary and discover that their *ratio* (γ) does not exhibit significant changes except over large temperature ranges. Thus the assumption of constant γ generally leads to acceptable engineering accuracy except in hypersonic flows.

Recall from Chapter 4 that we also developed a general relationship between static and stagnation pressures for a perfect gas:

$$p_t=p\left(1+\frac{\gamma-1}{2}M^2\right)^{\gamma/(\gamma-1)} \tag{4.21}$$

Furthermore, the stagnation pressure–energy equation was easily integrated for the case of a perfect gas in adiabatic, no-work flow to yield

$$\frac{p_{t2}}{p_{t1}}=e^{-\Delta s/R} \tag{4.28}$$

If we introduce equation (4.21) into (4.28), we have

$$\frac{p_{t2}}{p_{t1}}=\frac{p_2}{p_1}\left(\frac{1+[(\gamma-1)/2]\,M_2^2}{1+[(\gamma-1)/2]\,M_1^2}\right)^{\gamma/(\gamma-1)}=e^{-\Delta s/R} \tag{5.24}$$

Rearrange this to obtain the ratio

$$\frac{p_1}{p_2}=\left(\frac{1+[(\gamma-1)/2]\,M_2^2}{1+[(\gamma-1)/2]\,M_1^2}\right)^{\gamma/(\gamma-1)}e^{+\Delta s/R} \tag{5.25}$$

We now have the desired information to accomplish the original objective. Direct substitution of equations (5.23) and (5.25) into (5.19) yields

$$\frac{A_2}{A_1} = \left[\left(\frac{1+[(\gamma-1)/2]\,M_2^2}{1+[(\gamma-1)/2]\,M_1^2}\right)^{\gamma/(\gamma-1)} e^{\Delta s/R}\right] \times \frac{M_1}{M_2}\left(\frac{1+[(\gamma-1)/2]\,M_1^2}{1+[(\gamma-1)/2]\,M_1^2}\right)^{1/2} \tag{5.26}$$

Show that this can be simplified to

$$\boxed{\frac{A_2}{A_1} = \frac{M_1}{M_2}\left(\frac{1+[(\gamma-1)/2]\,M_2^2}{1+[(\gamma-1)/2]\,M_1^2}\right)^{(\gamma+1)/2(\gamma-1)} e^{\Delta s/R}} \tag{5.27}$$

Note that to obtain this equation, we automatically utilized a number of other working equations, which for convenience we summarize below.

$$T_{t1} = T_{t2} \tag{5.21}$$

$$\frac{p_{t2}}{p_{t1}} = e^{-\Delta s/R} \tag{4.28}$$

$$\frac{T_2}{T_1} = \frac{1+[(\gamma-1)/2]\,M_1^2}{1+[(\gamma-1)/2]\,M_2^2} \tag{5.23}$$

$$\frac{p_2}{p_1} = \left(\frac{1+[(\gamma-1)/2]\,M_1^2}{1+[(\gamma-1)/2]\,M_2^2}\right)^{\gamma/(\gamma-1)} e^{-\Delta s/R} \tag{5.25}$$

From equations (1.16), (5.23), and (5.25) you should also be able to show that

$$\frac{\rho_2}{\rho_1} = \left(\frac{1+[(\gamma-1)/2]\,M_1^2}{1+[(\gamma-1)/2]\,M_2^2}\right)^{1/(\gamma-1)} e^{-\Delta s/R} \tag{5.28}$$

Example 5.1

Air flows in an adiabatic duct without friction. At one section the Mach number is 1.5, and farther downstream it has increased to 2.8. Find the area ratio.

For a frictionless, adiabatic system, $\Delta s = 0$. We substitute directly into equation (5.27):

$$\frac{A_2}{A_1} = \frac{1.5}{2.8}\left(\frac{1+[(1.4-1)/2](2.8)^2}{1+[(1.4-1)/2](1.5)^2}\right)^{(1.4+1)/2(1.4-1)} (1) = 2.98$$

This problem is very simple since both Mach numbers are known. The inverse problem (given A_1, A_2, and M_1, find M_2) is not so straightforward to calculate. We shall come back to this in Section 5.6 after we develop a new concept.

5.5 THE * REFERENCE CONCEPT

In Section 3.5 the concept of a stagnation reference state was introduced, which by the nature of its definition turned out to involve an isentropic process. Before going any further with the working equations developed in Section 5.4, it will be convenient to introduce another reference condition because, among other things, the stagnation state is not a feasible reference when dealing with area changes. (Why?) We denote this new reference state with a superscript * and define it as "that thermodynamic state which would exist if the fluid reached a Mach number of unity *by some particular process*." The italicized phrase is significant, for there are many processes by which we could reach Mach 1.0 from any given starting point, and they would each lead to a different thermodynamic state. As we analyze different flow phenomena we will be considering different types of processes, and thus we will be dealing with different * reference states (see Sections 9.5 and 10.5).

We consider here a * reference state reached under reversible-adiabatic conditions (i.e., by an isentropic process). Every point in the flow system has its own * reference state, just as it has its own stagnation reference state. As an illustration, consider a system that involves the flow of a perfect gas with no heat or work transfer. Figure 5.5 shows a *T–s* diagram indicating two points in such a flow system. Above each point is shown its stagnation reference state, and we now add the isentropic * reference state that is associated with each point. Not only is the stagnation line for the entire system a horizontal line, but in this system all * reference points will also lie on a horizontal line (see the discussion in Section 4.6). Is the flow subsonic or supersonic in the system depicted in Figure 5.5?

We now proceed to develop an extremely important relation. Keep in mind that * reference states may or may not exist in the system, but with appropriate area changes *they could exist,* and as such they represent legitimate section reference locations to be used with any of the equations that we developed earlier [such as equations (5.23), (5.25), (5.27), etc.]. Specifically, let us consider

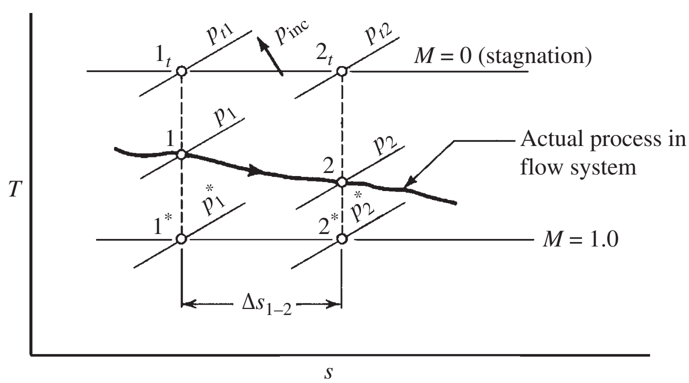


Figure 5.5 Isentropic * reference states.

$$\frac{A_2}{A_1} = \frac{M_1}{M_2}\left(\frac{1+[(\gamma-1)/2]\,M_2^2}{1+[(\gamma-1)/2]\,M_1^2}\right)^{(\gamma+1)/2(\gamma-1)} e^{\Delta s/R} \tag{5.27}$$

In this equation, points 1 and 2 represent *any* two points that could exist in a system (subject to the same assumptions that led to the development of the equation). We now apply equation (5.27) between points 1^* and 2^*. Thus

$$A_1 \Rightarrow A_1^* \quad M_1 \Rightarrow M_1^* \equiv 1$$
$$A_2 \Rightarrow A_2^* \quad M_2 \Rightarrow M_2^* \equiv 1$$

and we have:

$$\frac{A_2^*}{A_1^*} = \frac{1}{1}\left(\frac{1+[(\gamma-1)/2]1^2}{1+[(\gamma-1)/2]1^2}\right)^{(\gamma+1)/2(\gamma-1)} e^{\Delta s/R}$$

or

$$\boxed{\frac{A_2^*}{A_1^*} = e^{\Delta s/R}} \tag{5.29}$$

Before going further, it might be instructive to check this relation to see if it appears reasonable. First, take the case of no losses where $\Delta s = 0$. Then equation (5.29) says that $A_1^* = A_2^*$. Check Figure 5.5 for the case of $\Delta s_{1-2} = 0$. Under these conditions the diagram collapses into a single isentropic line on which 1_t is identical with 2_t and 1^* is the same point as 2^*. Under this condition, it should be obvious that A_1^* is the same as A_2^*.

Next, take the more general case where Δ_{S1-2} is nonzero. Assuming that these points exist in a flow system, they must pass the same amount of fluid, or

$$\dot{m} = \rho_1^* A_1^* V_1^* = \rho_2^* A_2^* V_2^* \tag{5.30}$$

Recall from Section 4.6 that since these state points are on the same horizontal line,

$$V_1^* = V_2^* \tag{5.31}$$

Similarly, we know that $T_1^* = T_2^*$, and from Figure 5.5 it is clear that $p_1^* > p_2^*$. Thus from the equation of state, we can easily determine that

$$\rho_2^* < \rho_1^* \tag{5.32}$$

Introduce equations (5.31) and (5.32) into (5.30) and *show* that for the case of $\Delta s_{1-2} > 0$,

$$A_2^* > A_1^* \tag{5.33}$$

which agrees with equation (5.29).

We have previously developed a relation between the stagnation pressures, which involves the same assumptions as equation (5.29):

$$\boxed{\frac{p_{t2}}{p_{t1}} = e^{-\Delta s/R}} \tag{4.28}$$

Check Figure 5.5 to convince yourself that this equation also appears to give reasonable answers for the special case of $\Delta s = 0$ and for the general case of $\Delta s > 0$.

We now multiply equation (5.29) by equation (4.28):

$$\frac{A_2^* p_{t2}}{A_1^* p_{t2}} = \left(e^{\Delta s/R}\right)\left(e^{-\Delta s/R}\right) = 1.0 \tag{5.34}$$

or

$$\boxed{p_{t1} A_1^* = p_{t2} A_2^*} \tag{5.35}$$

This is a most important relation that is frequently the key to problem solutions in adiabatic flows. Learn equation (5.35) and the conditions under which it applies.

5.6 ISENTROPIC TABLE

In Section 5.4 we considered the steady, one-dimensional flow of a perfect gas under the conditions of no heat and work transfer and negligible potential changes. Looking back over the working equations that were developed reveals that many of them do not include the loss term (Δs_i). In those where the loss term does appear, it takes the form of a simple multiplicative factor such as $e^{\Delta s/R}$. This leads to the natural use of the isentropic process as a standard for ideal performance with appropriate corrections made to account for losses when necessary. In a number of cases, we find that some actual processes are so efficient that they are very nearly isentropic and thus need no corrections.

If we simplify equation (5.27) for an isentropic process, it becomes

$$\frac{A_2}{A_1} = \frac{M_1}{M_2}\left(\frac{1 + [(\gamma-1)/2]\, M_2^2}{1 + [(\gamma-1)/2]\, M_1^2}\right)^{(\gamma+1)/2(\gamma-1)} \tag{5.36}$$

This is easy to solve for the area ratio if both Mach numbers are known (see Example 5.1), but let's consider a more typical problem. The physical situation is fixed (i.e., A_1 and A_2 are known). The fluid (and thus γ) is known, and the Mach number at one location (say, M_1) is known. Our problem is to solve for the Mach number (M_2) at the other location. Although this is not impossible, it is a lot of work without computer assistance.

We can simplify the solution by the introduction of the * reference state. Let point 2 be an arbitrary point in the flow system, and let its isentropic * point be point 1. Then

$$A_2 \Rightarrow A \quad M_2 \Rightarrow M \quad \text{(any value)}$$
$$A_1 \Rightarrow A^* \quad M_1 \Rightarrow 1$$

and equation (5.36) becomes

$$\frac{A}{A^*} = \frac{1}{M}\left(\frac{1+[(\gamma-1)/2]M^2}{(\gamma+1)/2}\right)^{(\gamma+1)/2(\gamma-1)} = f(M,\gamma) \tag{5.37}$$

We see that $A/A^* = f(M, \gamma)$, and we can easily construct a table giving values of A/A^* versus M for a particular γ. The problem posed earlier could then be solved as follows:

Given: γ, A_1, A_2, M_1, and isentropic flow.
Find: M_2.

We approach the solution by formulating the ratio $A_2/A_2{}^*$ in terms of known quantities.

$$\frac{A_2}{A_2^*} = \underbrace{\frac{A_2}{A_1}}_{\text{Given}}\ \underbrace{\frac{A_1}{A_1^*}}_{\substack{\text{A function of } M_1\text{; look}\\ \text{up in isentropic table}}}\ \underbrace{\frac{A_1^*}{A_2^*}}_{\substack{\text{Evaluated by equation (5.29) and}\\ \text{equals 1.0 if flow is isentropic}}} \tag{5.38}$$

Thus A_2/A_2^* can be calculated, and by entering the isentropic table with this value, M_2 can be determined. A *word of caution here!* The value of A_2/A_2^* will be found in *two* places in the table, as we are really solving equation (5.36), or the more general case equation (5.27), which is a quadratic for M_2. One value will be in the subsonic region and the other in the supersonic regime. You should have no difficulty determining which answer is correct when you consider the physical appearance of the system together with the concepts developed in Section 5.3.

Note that the general problem *with losses* can also be solved by the same technique as long as information is available concerning the loss. This could be given to us in the form of $A_1^*/A_2^*, p_{t2}/p_{t1}$, or possibly as Δs_{1-2}. All three of these represent equivalent ways of expressing the loss [through equations (4.28) and (5.29)].

We now realize that a key to the simplified problem solution is to have available a table of property ratios as a function of γ and only *one* Mach number. These are obtained by taking the equations developed in Section 5.4 and introducing a reference state, *either* the * reference condition (reached by an isentropic process) *or* the stagnation reference condition (reached by an isentropic process). We proceed with equation (5.23):

$$\frac{T_2}{T_1} = \frac{1 + [(\gamma-1)/2]\, M_1^2}{1 + [(\gamma-1)/2]\, M_2^2} \tag{5.23}$$

Let point 2 be any arbitrary point in the system and let its stagnation point be point 1. Then

$$\begin{aligned} &T_2 \Rightarrow T \quad M_2 \Rightarrow M \quad \text{(any value)} \\ &T_1 \Rightarrow T_t \quad M_1 \Rightarrow 0 \end{aligned}$$

and equation (5.23) becomes

$$\frac{T}{T_t} = \frac{1}{1 + [(\gamma-1)/2]M^2} = f(M,\gamma) \tag{5.39}$$

Equation (5.25) can be treated in a similar fashion. In this case we let 1 be the arbitrary point and its stagnation point is taken as 2. Then

$$\begin{aligned} &p_1 \Rightarrow p \quad M_1 \Rightarrow M \quad \text{(any value)} \\ &p_2 \Rightarrow p_t \quad M_2 \Rightarrow 0 \end{aligned}$$

and when we remember that the stagnation process is isentropic, equation (5.25) becomes

$$\frac{p}{p_t} = \left(\frac{1}{1 + [(\gamma-1)/2]M^2}\right)^{\gamma/(\gamma-1)} = f(M,\gamma) \tag{5.40}$$

Equations (5.39) and (5.40) are not surprising, as we have arrived at these previously by other methods [see equations (4.18) and (4.21)]. The tabulation of equation (5.40) may be used to solve problems in the same manner as the area ratio. For example, assume that we are

Given: γ, p_1, p_2, M_2, and Δs_{1-2} and asked to
Find: M_1.

To solve this problem, we seek the ratio p_1/p_{t1} in terms of known ratios:

$$\frac{p_1}{p_{t1}} = \underbrace{\frac{p_1}{p_2}}_{\text{Given}} \; \underbrace{\frac{p_2}{p_{t2}}}_{\substack{\text{A function of } M_2\text{; look}\\ \text{up in isentropic table}}} \; \underbrace{\frac{p_{t2}}{p_{t1}}}_{\substack{\text{Evaluated by equation (4.28)}\\ \text{as a function of } \Delta s_{1-2}}} \tag{5.41}$$

After calculating the value of p_1/p_{t1}, we enter the isentropic table and find M_1. Note that even though the flow from station 1 to 2 is *not* isentropic, the functions for p_1/p_{t1} and p_2/p_{t2} *are isentropic by definition*; thus the isentropic table can be used to solve such a problem. The connection *between* the two points is made through p_{t2}/p_{t1}, which involves the applicable entropy change.

We could continue to develop other isentropic relations as functions of the Mach number and γ. *Apply* the previous techniques to equation (5.28) and show that

$$\frac{\rho}{\rho_t} = \left(\frac{1}{1+[(\gamma-1)/2]M^2}\right)^{1/(\gamma-1)} \tag{5.42}$$

Another interesting relationship is the product of equations (5.40) and (5.37):

$$\frac{p}{p_t}\frac{A}{A^*} = f(M,\gamma) \tag{5.43}$$

Determine what unique function of M and γ is represented in equation (5.43). Since A/A^* and p/p_t are isentropic by definition, we should not be surprised that their product is listed in the isentropic table. But can these functions provide the connection *between* two locations in a flow system *with known losses*?

Recall that

$$\frac{p_{t2}}{p_{t1}} = e^{-\Delta s/R} \tag{4.28}$$

and

$$\frac{A_2^*}{A_1^*} = e^{\Delta s/R} \tag{5.29}$$

Thus, for adiabatic cases involving losses (Δs), changes in A^* are exactly compensated for by changes in p_t. This is true for all steady, one-dimensional flows of a perfect gas in an *adiabatic no-work* system. As already stated, we shall soon see that equation (5.43) provides the only direct means of solving certain types of problems.

Values of these isentropic flow parameters have been calculated from equations (5.37), (5.39), (5.40), and so on, and tabulated in Appendix G. To convince yourself that there is nothing magical about this table, you might want to check some of the numbers found in them opposite a particular Mach number. In fact, as stated in Problem 5.24, you could explore further with the *Gasdynamics Calculator* values of γ other than 1.4 (which is the only one included in Appendix G). In Section 5.10 we suggest alternatives to the use of the table. As you read the following examples, look up the numbers in the isentropic table to convince yourself that you know how to find them.

Example 5.2
You are now in a position to rework Example 5.1 with a minimum of calculation. Recall that $M_1 = 1.5$ and $M_2 = 2.8$.

$$\frac{A_2}{A_1} = \frac{A_2}{A_2^*}\frac{A_2^*}{A_1^*}\frac{A_1^*}{A_1} = (3.5001)(1)\left(\frac{1}{1.1762}\right) = 2.98$$

The following information (and Figure E5.3) are common to Examples 5.3 through 5.5. We are given the steady, one-dimensional flow of air ($\gamma = 1.4$), which can be treated as a perfect gas. Assume that $Q = W_s = 0$ and negligible potential changes. Moreover, take $A_1 = 2.0\ \text{ft}^2$ and $A_2 = 5.0\ \text{ft}^2$.

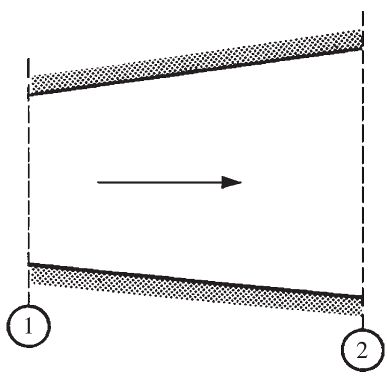


Figure E5.3

Example 5.3

Given that $M_1 = 1.0$ and $\Delta s_{1-2} = 0$. Find the possible values of M_2. To determine conditions at section 2 in Figure E5.3, we establish the ratio

$$\frac{A_2}{A_2^*} = \frac{A_2}{A_1}\frac{A_1}{A_1^*}\frac{A_1^*}{A_2^*} = \left(\frac{5}{2}\right)(1.000)\,(1) = 2.5$$

Equals 1.0 since isentropic

From isentropic table at $M = 1.0$

From given physical configuration

Look up $A/A^* = 2.5$ in the isentropic table and determine that M_2 can be either 0.24 or 2.44. We can't tell which Mach number exists without additional information.

Example 5.4

Given that $M_1 = 0.5$, $p_1 = 4$ bar, and $\Delta s_{1-2} = 0$, find M_2 and p_2.

$$\frac{A_2}{A_2^*} = \frac{A_2}{A_1}\frac{A_1}{A_1^*}\frac{A_1^*}{A_2^*} = \left(\frac{5}{2}\right)(1.3398)(1) = 3.35$$

Thus $M_2 \approx 0.175$. (Why isn't it 2.75?)

$$p_2 = \frac{p_2}{p_{t2}}\frac{p_{t2}}{p_{t1}}\frac{p_{t1}}{p_1}p_1 = (0.9788)(1)\left(\frac{1}{0.8430}\right)(4) = 4.64 \text{ bar}$$

Example 5.5

Given: $M_1 = 1.5$, $T_1 = 70°\text{F}$, and $\Delta s_{1-2} = 0$,

Find: M_2 and T_2.

(Calculate A_2/A_2^* and using Appendix G verify that $M_2 \approx 2.62$.)

Once M_2 is known, we can find T_2.

$$T_2 = \frac{T_2}{T_{t2}}\frac{T_{t2}}{T_{t1}}\frac{T_{t1}}{T_1}T_1 = (0.4214)(1)\left(\frac{1}{0.6897}\right)(530) = 324^\circ\text{R}$$

Why is $T_{t1} = T_{t2}$? (Write an energy equation between 1 and 2.)

Example 5.6

Oxygen flows into an insulated device with the following initial conditions: $p_1 = 20$ psia, $T_1 = 600°\text{R}$, and $V_1 = 2960$ ft/sec. After a short distance the area has converged from 6 to 2.5 ft^2 (Figure E5.6). You may assume steady, one-dimensional flow and a perfect gas. (See the table in Appendix A for gas properties.)

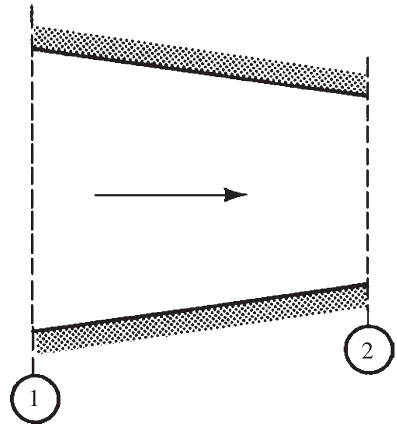


Figure E5.6

(a) Find M_1, p_{t1}, T_{t1}, and h_{t1}.
(b) If there are losses such that $\Delta s_{1-2} = 0.005$ Btu/1bm-°R, find M_2, p_2, and T_2.

(a) First, we determine conditions at station 1.

$$a_1 = (\gamma g_c \, RT_1)^{1/2} = [(1.4)(32.2)(48.3)(600)]^{1/2} = 1143 \, \text{ft/sec}$$

$$M_1 = \frac{V_1}{a_1} = \frac{2960}{1143} = 2.59$$

$$p_{t1} = \frac{p_{t1}}{p_1} p_1 = \left(\frac{1}{0.0509}\right)(20) = 393 \, \text{psia} \quad (\text{or } 2.71 \, \text{MPa})$$

$$T_{t1} = \frac{T_{t1}}{T_1} T_1 = \left(\frac{1}{0.4271}\right)(600) = 1405^{\circ}\text{R} \quad (\text{or } 781 \, \text{K})$$

$$h_{t1} = c_p T_{t1} = (0.218)\,(1405) = 306 \, \text{Btu/lbm} \quad (\text{or } 0.712 \, \text{MJ/kg})$$

(b) For a perfect gas with $q = w_s = 0$, $T_{t1} = T_{t2}$ (from an energy equation), and also from equation (529):

$$\frac{A_1^*}{A_2^*} = e^{-\Delta s/R} = e^{-(0.005)(778)/48.3} = 0.9226$$

Thus

$$\frac{A_2}{A_2^*} = \frac{A_2}{A_1}\frac{A_1}{A_1^*}\frac{A_1^*}{A_2^*} = \left(\frac{2.5}{6}\right)(2.8688)(0.9226) = 1.1028$$

From the isentropic table we find that $M_2 \approx$ _____. Why is the use of the isentropic table legitimate here when there are losses in the flow? Continue and compute p_2 and T_2.

$$p_2 = \qquad (p_2 \approx 117\,\text{psia} \;\text{ or }\; 0.806\,\text{MPa})$$

$$T_2 = \qquad \left(T_2 \approx 1019^{\circ}\text{R} \;\text{ or }\; 566\,\text{K}\right)$$

Could you find the velocity at section 2?

5.7 NOZZLE OPERATION

We will now start a discussion of nozzle operation and at the same time gain some experience in use of the isentropic table. Two types of nozzles are considered: a converging-only nozzle and a converging–diverging nozzle. We start by examining the physical situation shown in Figure 5.6. A source of air at 100 psia and 600°R is contained in a large tank where stagnation conditions prevail. Connected to the tank is a converging-only nozzle and it exhausts into an extremely large receiver where the pressure can be regulated. We can neglect frictional effects, as they can be very small in the short converging section.

If the receiver pressure is set at 100 psia (or 100% p_1), no flow results. Once the receiver pressure is lowered below 100 psia, air will flow from the supply tank. Since the supply tank has a large cross section relative to the nozzle outlet area, the velocities in the tank may be neglected. Thus $T_1 \approx T_{t1}$ and $p_1 \approx p_{t1}$. There is no shaft work, and we assume no heat transfer. We identify section 2 as the nozzle outlet.

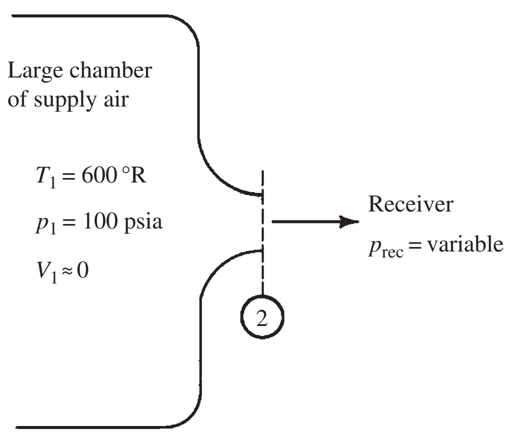


Figure 5.6 Converging-only nozzle.

Energy

$$
\begin{aligned}
h_{t1} + \cancel{q} &= h_{t2} + \cancel{w_s} \\
h_{t1} &= h_{t2}
\end{aligned}
\tag{3.19}
$$

and since we can treat this as a perfect gas,

$$T_{t1} = T_{t2}$$

It is important to recognize that here the receiver pressure is controlling the flow. The velocity will increase and the pressure will decrease as we progress through the nozzle until the pressure at the nozzle outlet equals that of the receiver. This will always be true *as long as* the nozzle outlet can "sense" the receiver pressure. Can you think of a situation where pressure pulses from the receiver could *not* be "felt" inside the nozzle? (Recall Section 4.4.)

Let us assume that

$$p_{\text{rec}} = 80.2\,\text{psia} \quad (\text{or } 80.2\%p_1)$$

Then

$$p_2 = p_{\text{rec}} = 80.2\,\text{psia}$$

and

$$\frac{p_2}{p_{t2}} = \frac{p_2}{p_{t1}}\frac{p_{t1}}{p_{t2}} = \left(\frac{80.2}{100}\right)(1) = 0.802$$

Note that $p_{t1} = p_{t2}$ by equation (4.28) since we are neglecting friction.

From the isentropic table corresponding to $p/p_t = 0.802$, we see that

$$M_2 = 0.57 \quad \text{and} \quad \frac{T_2}{T_{t2}} = 0.939$$

Thus

$$T_2 = \left(\frac{T_2}{T_{t2}}\right)T_{t2} = (0.939)(600) = 563^\circ\text{R}$$

$$a_2^2 = (1.4)(32.2)(53.3)(563)$$

$$a_2 = 1163\,\text{ft/sec}$$

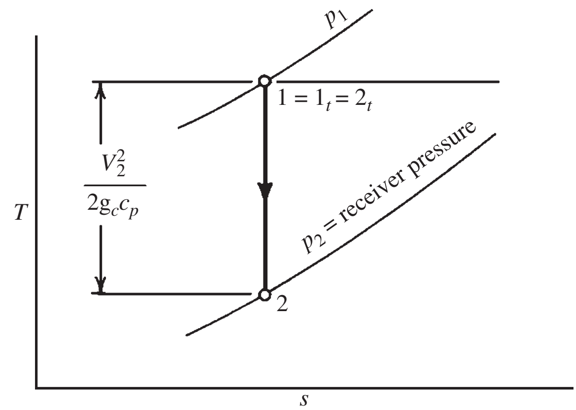


Figure 5.7 *T–s* diagram for converging-only nozzle.

and

$$V_2 = M_2 a_2 = (0.57)(1163) = 663\,\text{ft/sec}$$

Figure 5.7 shows this process on a *T–s* diagram as an isentropic expansion. If the pressure in the receiver were lowered further, the air would expand to this lower pressure and the Mach number and velocity would increase. Assume that the receiver pressure is now lowered to 52.83 psia (or 52.83% p_1). *Show* that

$$\frac{p_2}{p_{t2}} = 0.5283$$

and thus

$$M_2 = 1.00 \quad \text{with} \quad V_2 = 1096\,\text{ft/sec}$$

Notice that here the air velocity coming out of the nozzle is exactly sonic. If we now drop the receiver pressure below this *critical pressure* (52.83 psia), the nozzle has no way of adjusting to these conditions. Why not? Assume that the nozzle outlet pressure could continue to drop along with the receiver. This would mean that $p_2/p_{t2} < 0.5283$, which corresponds to a supersonic speed. We know that if the flow is to go supersonic, the area must reach a minimum and then increase (see Section 5.3). Thus for a *converging-only* nozzle, the flow is governed by the receiver pressure until sonic speed is reached at the nozzle outlet and *further reduction of the receiver pressure will have no effect on the flow conditions inside the nozzle*. Under these conditions, the nozzle is said to be *choked* and the nozzle outlet pressure remains at the *critical pressure*. Expansion to the receiver pressure has to take place *outside* the nozzle. Note that these values are only for $\gamma = 1.4$.

In reviewing this example, you should realize that there is nothing magical about a receiver pressure of 52.83 psia. The significant item is the *ratio* of the static to total pressure at the exit plane, which for the case of no losses is the *ratio* of the receiver pressure to the inlet pressure. With sonic speed at the exit, this *ratio* is 0.5283 for air.

The analysis above assumes that conditions within the supply tank remain constant. One should realize that the choked flow rate will change if, for example, the supply pressure or temperature is changed or the size of the throat (exit hole) is changed. It is instructive to take an alternative view of this situation. You are asked in Problem 5.9 to develop the following equation for isentropic flow:

$$\frac{\dot{m}}{A} = M\left(1 + \frac{\gamma - 1}{2}M^2\right)^{-(\gamma+1)/2(\gamma-1)} \left(\frac{\gamma g_c}{R}\right)^{1/2} \frac{p_t}{\sqrt{T_t}} \tag{5.44a}$$

Applying this equation to the outlet and considering choked flow, $M = 1.0$ and $A = A^*$, we then obtain

$$\left(\frac{\dot{m}}{A}\right)_{\max} = \frac{\dot{m}}{A^*} = \left[\frac{\gamma g_c}{R}\left(\frac{1}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}\right]^{1/2} \frac{p_t}{\sqrt{T_t}} \tag{5.44b}$$

For a given gas,

$$\frac{\dot{m}}{A^*} = \text{constant}\,\frac{p_t}{\sqrt{T_t}} \tag{5.44c}$$

We now look at four distinct possibilities:

1. For a fixed T_t, p_t, and $A^* \Rightarrow \dot{m}_{\max}$ constant.
2. For only p_t increasing $\Longrightarrow \dot{m}_{\max}$ increases.
3. For only T_t increasing $\Longrightarrow \dot{m}_{\max}$ decreases.
4. For only A^* increasing $\Longrightarrow \dot{m}_{\max}$ increases.

Figure 5.8 shows these possibilities in yet another way. A *dimensionless* form of equation (5.44*b*) that clearly shows its γ dependence may be written as,

$$\frac{\dot{m}\sqrt{RT_t/g_c}}{A^* p_t} = \left[\gamma\left(\frac{1}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}\right]^{1/2} = 0.0856 \quad \text{at} \quad \gamma = 1.40 \tag{5.44d}$$

where the right-hand side can only vary from 0.0143 at $\gamma = 1.2$ to 0.183 at $\gamma = 1.67$.

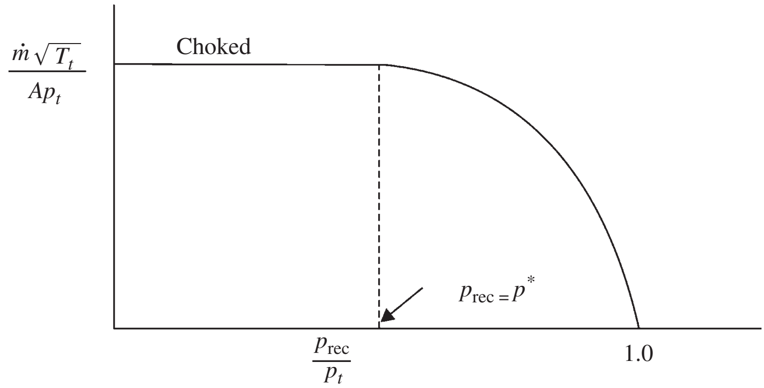


Figure 5.8 Operation of a converging-only nozzle at various back pressures.

Converging–Diverging Nozzle

Now let us examine the same chamber supply situation but with a converging–diverging nozzle (sometimes called a *DeLaval nozzle*), shown in Figures 5.9 and 5.10. We identify the *throat* (or section of minimum area) as 2 and the exit section as 3. The distinguishing physical characteristic of this type of nozzle is the *area ratio,* meaning the ratio of the exit area to the throat area. Assume this to be $A_3/A_2 = 2.494$. Keep in mind that the objective of using a converging–diverging nozzle is to obtain supersonic flow. Let us first examine the *design operating condition* for this nozzle. If the nozzle is to operate as desired, we know (see Section 5.3) that the flow will be subsonic from 1 to 2, sonic at 2, and supersonic from 2 to 3.

To quantify gas dynamic conditions that exist at the exit (under design operation), we seek the ratio A_3/A_3^*:

$$\frac{A_3}{A_3^*} = \frac{A_3}{A_2}\frac{A_2}{A_2^*}\frac{A_2^*}{A_3^*} = (2.494)(1)(1) = 2.494$$

Note that $A_2 = A_2^*$ since $M_2 = 1.0$, and $A_2^* = A_3^*$ by equation (5.29), as we are still assuming isentropic operation. We look for $A/A^* = 2.494$ in the *supersonic* section of the isentropic table and see that

$$M_3 = 2.44, \quad \frac{p_3}{p_{t3}} = 0.0643, \quad \text{and} \quad \frac{T_3}{T_{t3}} = 0.4565$$

Thus

$$p_3 = \frac{p_3}{p_{t3}}\frac{p_{t3}}{p_{t1}}p_{t1} = (0.0643)(1)(100) = 6.43\,\text{psia}$$

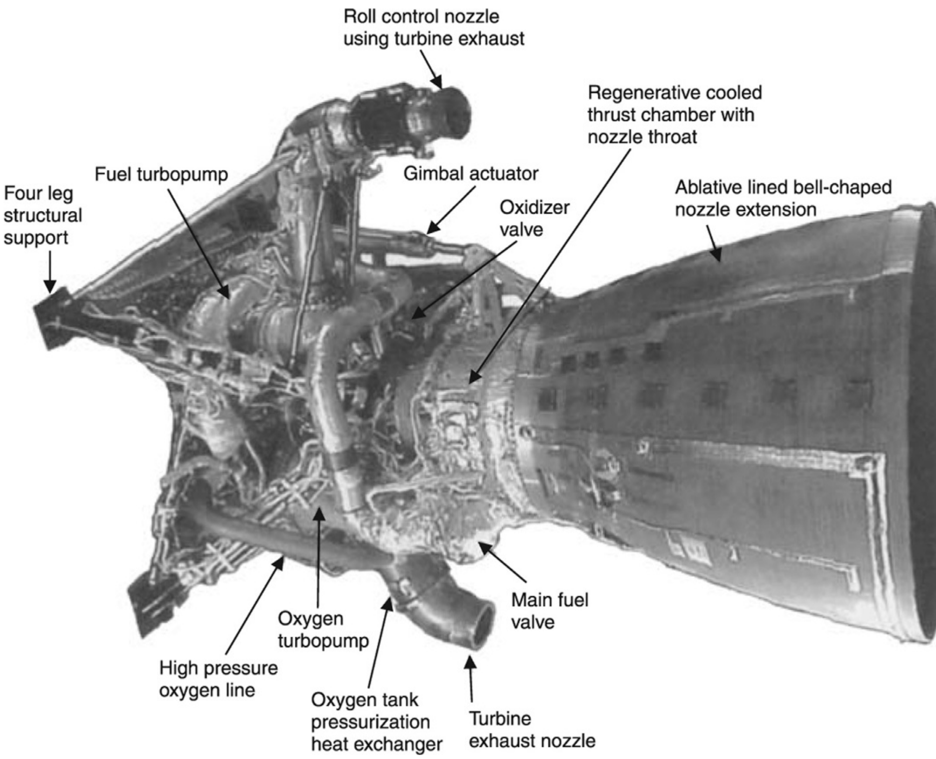


Figure 5.9 Typical converging–diverging rocket nozzle. (Courtesy of Aerojet Rocketdyne.)

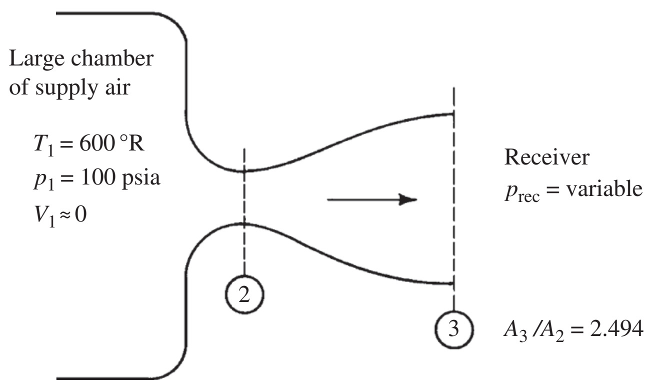


Figure 5.10 Converging–diverging nozzle schematic.

and to operate the nozzle at this *design condition* the receiver pressure *must be* at 6.43 psia. The pressure variation through the nozzle for this case is shown as curve "a" in Figure 5.11. This mode is sometimes referred to as *third critical*. From the temperature ratio T_3/T_{t3} we can easily compute T_3, a_3, and V_3 by the procedure shown previously.

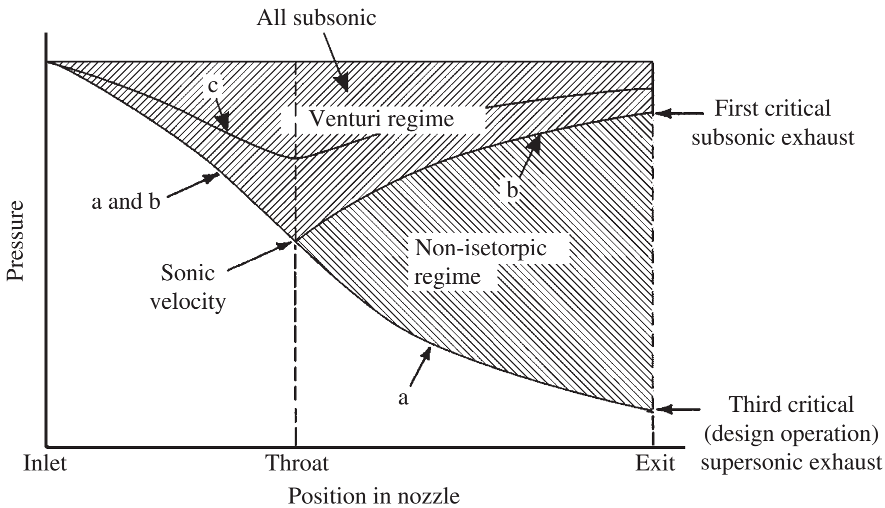


Figure 5.11 Pressure variations through converging–diverging nozzle.

One can also find $A/A^* = 2.494$ in the subsonic-section exhaust of the isentropic table. (Recall that these two answers come from the solution of a quadratic equation.) For this case

$$M_3 = 0.24, \quad \frac{p_3}{p_{t3}} = 0.9607, \quad \frac{T_3}{T_{t3}} = 0.9886$$

Thus

$$p_3 = \frac{p_3}{p_{t3}} \frac{p_{t3}}{p_{t1}} p_{t1} = (0.9607)(1)(100) = 96.07 \, \text{psia}$$

and to operate at this condition the receiver pressure *must be* at 96.07 psia. With this receiver pressure the flow is subsonic from 1 to 2, sonic at 2, and *subsonic* again from 2 to 3. The device is nowhere near its design condition and is really operating as a *venturi tube*, that is, the converging section is operating as a nozzle and the diverging section is operating as a diffuser. The pressure variation through the nozzle for this case is shown as curve "b" in Figure 5.11. This mode of operation is frequently called *first critical.*

Note that at both the first and third critical points, the flow variations are identical from the inlet to the throat. Once the receiver pressure has been lowered to 96.07 psia, Mach 1.0 exists in the throat and the flow is said to be *choked* as in Figure 5.8. *Further lowering of the receiver pressure will not change the mass flow rate* as given by equation (5.44*b*). Again, realize that it is not the pressure in the receiver by itself but rather the receiver pressure *relative* to the inlet pressure that determines the mode of operation.

Example 5.7
A converging–diverging nozzle with an area ratio of 3.0 exhausts into a receiver where the pressure is 1 bar. The nozzle is supplied by air at 22°C from a large chamber. At what pressure should the air in the chamber be for the nozzle to operate at its design condition (third critical point)? What will the outlet velocity be?

With reference to Figure 5.10, $A_3/A_2 = 3.0$:

$$\frac{A_3}{A_3{}^*} = \frac{A_3}{A_2}\frac{A_2}{A_2{}^*}\frac{A_2{}^*}{A_3{}^*} = (3.0)(1)(1) = 3.0$$

From the isentropic table:

$$M_3 = 2.64 \quad \frac{p_3}{p_{t3}} = 0.0471 \quad \frac{T_3}{T_{t3}} = 0.4177$$

$$p_1 = p_{t1} = \frac{p_{t1}}{p_{t3}}\frac{p_{t3}}{p_3}p_3 = (1)\left(\frac{1}{0.0471}\right)\left(1\times10^5\right) = 21.2\times10^5\,\mathrm{N/m^2}\ \ (\text{or}\,307.5\,\text{psia})$$

$$T_3 = \frac{T_3}{T_{t3}}\frac{T_{t3}}{T_{t1}}T_{t1} = (0.4177)(1)(22+273) = 123.2\ \mathrm{K}$$

$$V_3 = M_3 a_3 = (2.64)[(1.4)(1)(287)(123.2)]^{1/2} = 587\ \mathrm{m/s}\ \ (\text{or}\,1926\,\text{ft/sec})$$

We have discussed only two specific operating conditions, and one might ask what happens at other receiver pressures. We can state that the first and third critical points represent the only operating conditions that satisfy all the following criteria:

1. Mach 1.0 in the throat
2. Isentropic flow throughout the nozzle
3. Nozzle exit pressure equal to receiver pressure

With receiver pressures above the first critical, the nozzle operates as a venturi, and we never reach sonic speed in the throat. An example of this mode of operation is shown as curve "c" in Figure 5.11. The nozzle is no longer choked and the flow rate is less than the maximum. Conditions at the exit can be determined by the procedure shown previously for the converging-only nozzle. Then properties in the throat can be found if desired.

Operation between the first and third critical points is *not* isentropic. We shall learn later that under these conditions shocks will occur in either the diverging portion of the nozzle or after the exit. If the receiver pressure is below the third critical point, the nozzle operates *internally* as though it were at the design condition but expansion waves occur *outside* the nozzle. These operating modes will be discussed in detail after the appropriate background has been developed.

5.8 NOZZLE PERFORMANCE

We have seen that the isentropic operating conditions are relatively easy to determine. Friction losses can then be taken into account by one of several other methods. Direct information on the entropy change could be given, although this is usually not available. Sometimes equivalent information is provided in the form of the stagnation pressure ratio. Normally, however, nozzle performance is indicated by an *efficiency parameter,* which is defined as follows:

$$\boxed{\eta_n \equiv \frac{\text{Actual change in kinetic energy}}{\text{Ideal change in kinetic energy}}}$$

or

$$\eta_n \equiv \frac{\Delta \text{KE}_{\text{actual}}}{\Delta \text{KE}_{\text{ideal}}} \tag{5.45}$$

Since most nozzles involve negligible heat transfer (per unit mass of fluid flowing), we have from

$$h_{t1} + \cancel{q} = h_{t2} + \cancel{w_s} \tag{3.19}$$

$$h_{t1} = h_{t2} \tag{5.46}$$

Thus

$$h_1 + \frac{V_1^2}{2g_c} = h_2 + \frac{V_2^2}{2g_c} \tag{5.47a}$$

or

$$h_1 - h_2 = \frac{V_2^2 - V_1^2}{2g_c} \tag{5.47b}$$

Therefore, one normally sees the nozzle efficiency expressed as

$$\boxed{\eta_n \equiv \frac{\Delta h_{\text{actual}}}{\Delta h_{\text{ideal}}}} \tag{5.48}$$

With reference to Figure 5.12, this becomes

$$\eta_n = \frac{h_1 - h_2}{h_1 - h_{2s}} \tag{5.49}$$

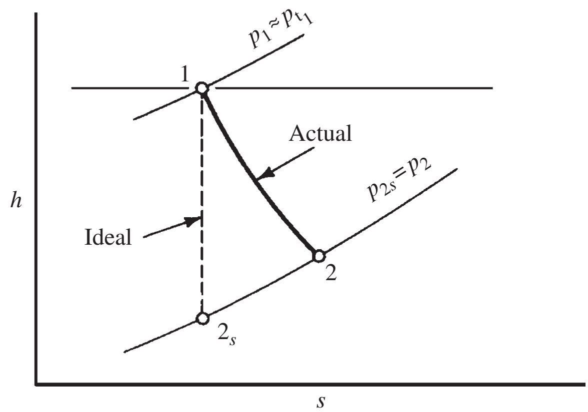


Figure 5.12 *h–s* diagram for a nozzle with losses.

Since nozzle outlet velocities are quite large (relative to the velocity at the inlet), one can normally neglect the inlet velocity with little error. This is the case shown in Figure 5.12. Also note that the *ideal* process is assumed to take place down at the actual available receiver pressure. This definition of nozzle efficiency and its application appear quite reasonable since nozzles are subjected to fixed (inlet and outlet) operating pressures and their purpose is to produce kinetic energy. The question might be how well and how quickly can η_n be used to determine the outlet state when losses are present.

Example 5.8

Air at 800°R and 80 psia feeds a converging-only nozzle having an efficiency of 96%. The receiver pressure is 50 psia. What is the actual nozzle outlet temperature?

Note that since $p_{rec}/p_{inlet} = 50/80 = 0.625 > 0.528$, the nozzle will not be choked, flow will be subsonic at the exit, and $p_2 = p_{rec}$ (see Figure 5.12).

$$\frac{p_{2s}}{p_{t2s}} = \frac{p_{2s}}{p_{t1}}\frac{p_{t1}}{p_{t2s}} = \left(\frac{50}{80}\right)(1) = 0.625$$

From table,

$$M_2 \approx 0.85 \quad \text{and} \quad \frac{T_{2s}}{T_{t2s}} = 0.8737$$

$$T_{2s} = \frac{T_{2s}}{T_{t2s}}\frac{T_{t2s}}{T_{t1}}T_{t1} = (0.8737)(1)(800) = 699^\circ \text{ R}$$

$$\eta_n = \frac{T_1 - T_2}{T_1 - T_{2s}} \qquad 0.96 = \frac{800 - T_2}{800 - 699}$$

$$T_2 = 703^\circ \text{ R} \ \ (\text{or } 390.5\,\text{K})$$

Can you find the actual outlet velocity?

Another method of expressing nozzle performance is with a *velocity coefficient,* which is defined as

$$C_v \equiv \frac{\text{Actual outlet velocity}}{\text{Ideal outlet velocity}} \tag{5.50}$$

Sometimes a *discharge coefficient* is used and is defined as

$$C_d \equiv \frac{\text{Actual mass flow rate}}{\text{Ideal mass flow rate}} \tag{5.51}$$

[In compressible flows C_d is somewhat larger than 1.0 due to non-one-dimensionalities.]

5.9 DIFFUSER PERFORMANCE

Although the common use of nozzle efficiency makes this parameter well understood by all engineers, there is no single parameter that is universally employed for diffusers. Nearly a dozen criteria have been suggested to indicate diffuser performance (see p. 392, Vol. 1 of Ref. 25). Two or three of these are the most popular, but unfortunately, even these are sometimes defined differently or called by different names. The following discussion refers to the *h–s* diagram shown in Figure 5.13.

Most of the propulsion industry uses the *total-pressure recovery factor* as a measure of diffuser performance. With reference to Figure 5.13, it is defined as

$$\boxed{\eta_r \equiv \frac{p_{t2}}{p_{t1}}} \tag{5.52}$$

This function is directly related to the area ratio A_1^*/A_2^* or the entropy change Δs_{1-2}, both of which we have previously shown to be equivalent loss indicators. As we shall see in Chapter 12, for propulsion applications this ratio is usually referred to the free-stream conditions rather than the diffuser inlet.

For a definition of diffuser efficiency analogous to that of a nozzle, we recall that the function of a diffuser is to convert kinetic energy into pressure energy; thus it is logical to compare the ideal and actual processes between the same two enthalpy levels that represent the same kinetic energy change. Therefore, a suitable definition of *diffuser efficiency* is

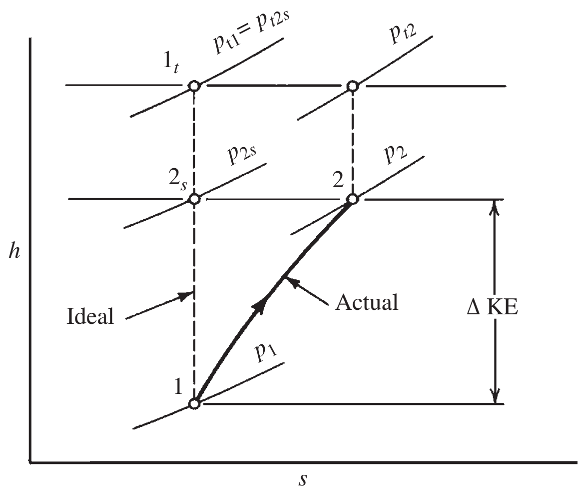


Figure 5.13 *h–s* diagram for a diffuser with losses.

$$\boxed{\eta_d \equiv \frac{\text{Actual pressure rise}}{\text{Ideal pressure rise}}} \tag{5.53}$$

or from Figure 5.13,

$$\eta_d \equiv \frac{p_2 - p_1}{p_{2s} - p_1} \tag{5.54}$$

You are again warned to be extremely cautious in accepting any performance figure for diffusers without also obtaining a precise definition of what is meant by the criterion.

Example 5.9

A steady flow of air at 650°R and 30 psia enters a diffuser with a Mach number of 0.8. The total-pressure recovery factor $\eta_r = 0.95$. Determine the static pressure and temperature at the exit if $M = 0.15$ at that section.

With reference to Figure 5.13,

$$p_2 = \frac{p_2}{p_{t2}} \frac{p_{t2}}{p_{t1}} \frac{p_{t1}}{p_1} p_1 = (0.9844)(0.95)\left(\frac{1}{0.6560}\right)(30) = 42.8\,\text{psia} \quad (\text{or } 0.295\,\text{MPa})$$

$$T_2 = \frac{T_2}{T_{t2}} \frac{T_{t2}}{T_{t1}} \frac{T_{t1}}{T_1} T_1 = (0.9955)(1)\left(\frac{1}{08865}\right)(650) = 730^{\circ}\text{R} \quad (\text{or } 405.5\,\text{K})$$

5.10 WHEN γ IS NOT EQUAL TO 1.4

In this section, as in the next few chapters, we present graphical information on one or more key parameter ratios as a function of the Mach number. This is done for several ratios of the specific heats (γ = 1.13, 1.4, and 1.67) to show overall trends. We will find that, within a certain range of Mach numbers, the tabulations in Appendix G for air at normal temperature and pressure, which represent the middle value of the variation ($\gamma = 1.4$), turn out to be satisfactory for other values of γ.

Figure 5.14 shows curves for p/p_t, T/T_t, and A/A^* in the interval $0.2 \leq M \leq 5$. Actually, compressible flow manifests itself in the range $M \geq 0.3$. Below this range, we can treat flows as constant density (see Section 3.7 and Problem 4.3). Moreover, we have deliberately chosen to remain below the hypersonic range, which is generally regarded to be the region $M \geq 5$, but the interval shown will be representative of many situations encountered in compressible flow. The curves in Figure 5.14 clearly show some significant trends:

a. As can be seen from Figure 5.14*a*, p/p_t is the least sensitive (of the three ratios plotted) to variations of γ. Below $M \approx 2.5$, the pressure ratio is well represented by the values tabulated in Appendix G for any γ.
b. Figure 5.14*b* shows that T/T_t is more sensitive than the pressure ratio to variations in γ. However, it shows relative insensitivity below $M \approx 0.8$ so that in this range the values tabulated in Appendix G could be used for any γ with little error.
c. The same can be said about A/A^*, as shown in Figure 5.14*c*, which turns out to be relatively insensitive to variations in γ below $M \approx 1.5$.

In summary, the tables in Appendix G may be used for estimates (within ±5%) for almost any value of γ between 1.13 and 1.67 within the Mach number ranges identified above. Strictly speaking, these curves are representative only for cases where γ variations are *negligible within the flow*. However, they offer hints as to what magnitude of changes are to be expected in other cases. Flows where γ *v*ariations are *not negligible within the flow* are treated in Chapter 11.

5.11 BEYOND THE TABLES

Tables in gas dynamics are extremely useful but they have limitations, such as:

1. They do not clearly show trends or the “big picture.”
2. There is almost always the need for interpolation.
3. They display only one or at most a few values of γ.
4. They do not necessarily have the required accuracy.

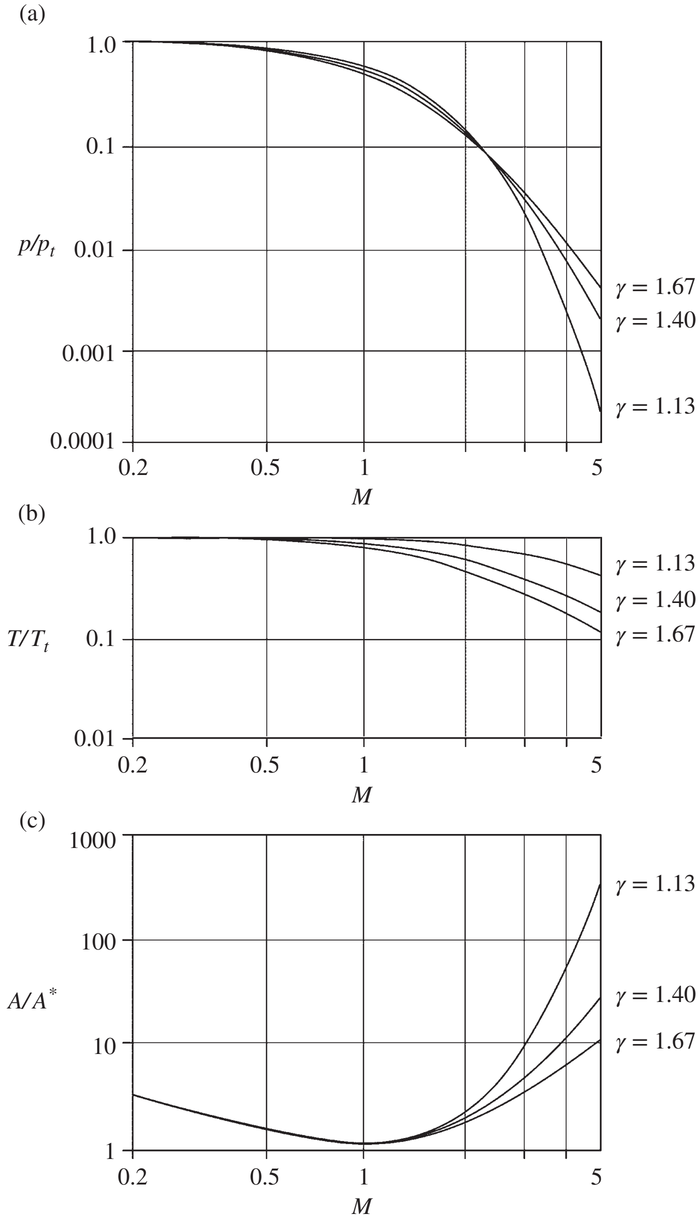


Figure 5.14 (a) Stagnation pressure ratio versus Mach number, (b) Stagnation temperature ratio versus Mach number, and (c) A/A^* area ratio versus Mach number for several values of γ.

Moreover, modern digital computers have made significant inroads for working problems, particularly when high-accuracy results and/or graphs outputs or sharing and reporting are required. Simply put, the computer can be programmed to do the hard (and the easy) numerical calculations. In previous editions of this book we deliberately avoided including any gas dynamics software (some of which is

commercially available) with the text material, preferring to present computer work as an adjunct to individual calculations. One reason has been that we want you to spend your time learning about the wonderful world of gas dynamics and not on how to manage the programming. Another reason is that both computers and packaged software evolve rather quickly, and in many cases the attention that must be paid just to use any particular software can soon be wasted. However, we now make available the *Gasdynamics Calculator* as a companion to this third edition, which will run on any modern electronic device that has a web browser. The notation in our calculator is the same as that in this book and it displays more functions than those shown in Appendixes G, H, I, and J. It also presents Oblique Shock numerical results that more than supplement the graphs in Appendix D.

(*Optional*) Once you have mastered the basics, however, we also feel that it is appropriate to discuss how certain answers might be obtained using the commercially available computer software. In this book we discuss how the computer utility MAPLE can be of help in solving problems in gas dynamics. MAPLE is a powerful computer environment for doing symbolic, numerical, and graphical work. It is the product of Waterloo Maple, Inc., and a recent version of MAPLE has a copyright of 2017. MAPLE is used routinely in many undergraduate engineering programs in the United States. Other software packages are also popular in engineering schools. One in particular is MATLAB, which can do things equivalent to those performed by MAPLE. MATLAB's real forte is in manipulating linear equations and in constructing tables. But we have chosen MAPLE because it can manipulate equations symbolically and because of its superior graphics. In our view, this makes MAPLE somewhat more attractive.

(*Optional*) We will present some simple examples to show how MAPLE can be used. The experienced programmer can go much beyond these exercises. This section is optional because we want you to concentrate mostly on the learning of gas dynamics and not spend extra time trying to demystify the computer approach. We focus on an example in Section 5.6, but the techniques must be understood to apply in general.

Example 5.10

In Example 5.6(a) calculations can be done from the formulas or by using the tables for p_{t1} and T_{t1}. In part (b), however, direct calculation of M_2 given A_2/A_2^* is more difficult because it involves equation (5.37), which cannot be solved explicitly for M.

$$\frac{A}{A^*} = \frac{1}{M}\left(\frac{1+[(\gamma-1)/2]M^2}{(\gamma+1)/2}\right)^{(\gamma+1)/2(\gamma-1)} = f(M,\gamma) \tag{5.37}$$

If we were given M_2, it would be simple to compute A_2/A_2^*.

But we are given A_2/A_2^* and we want to find M_2.

This is a problem where MAPLE can be useful because a built-in solver routine handles this type of problem easily.

First, we define some symbols: Let

$g \equiv \gamma$, a parameter (the ratio of the specific heats)
$X \equiv$ the independent variable (which in this case is M_2)
$Y \equiv$ the dependent variable (which in this case is A_2/A_2^*)

We need to introduce an index "m" to distinguish between subsonic and supersonic flow.

$$m \equiv \begin{cases} 1 & \text{for subsonic flow} \\ 10 & \text{for supersonic flow.} \end{cases}$$

Shown below is a copy from the exact MAPLE worksheet:

```
[>g: =1.4:Y:=1.1028:m: =10:
>fsolve(Y=(((1+(g-1)*(X^2)/2)/((g+1)/2)))^((g+1)/(2*(g-1)))/
    X, X, 1..m);
                              1.377333281          .
```

which is the desired answer.

We discuss now details of the MAPLE solution. If you are familiar with these, skip to the next paragraph. We must assume that the numerical value outputted is X because that is what we asked for in the executable statement with "fsolve( )," which terminates in a semicolon. Statements terminated in a colon are also executed but no return is asked for.

Example 5.11

We continue with this problem, as this is a good opportunity to show how MAPLE can help you avoid interpolation. If you are on the same worksheet, MAPLE remembers the values of g, ***Y***, and X. We are now looking for the ratio of static to stagnation temperature, which is given the symbol ***Z*.** This ratio comes from equation (5.39):

$$\frac{T}{T_t} = \frac{1}{1+[(\gamma-1)/2]M^2} = f(M,\gamma) \tag{5.39}$$

Shown below are the precise inputs and program that you use in the computer.

```
[>X:=1.3773:
  >Z:=1/(1+(g-1)*(X^2)/2);
                              Z:=.7249575776
```

Now we can calculate the static temperature by the usual method.

$$T_2 = \frac{T_2}{T_{t2}}\frac{T_{t2}}{T_{t1}}T_{t1} = (0.725)(1)(1405) = 1019^\circ \text{R}$$

The static pressure (p_2) can be found by a similar procedure.

5.12 SUMMARY

We analyzed a general adiabatic but varying-area configuration and found that properties vary in a radically different manner depending on whether the flow is subsonic or supersonic. The case of a perfect gas enabled the development of simple working equations for flow analysis. We then introduced the concept of a * reference state. The combination of the * and the stagnation reference states led to the development of the isentropic table, which greatly aids in finding solution to problems. Deviations from isentropic flow may be handled by introducing appropriate factors or efficiency criteria.

A large number of useful equations were developed; however, most of these are of the type that need not be memorized. Equations (5.10), (5.11), and (5.13) were used for the general analysis of varying-area flow, and these are summarized in the middle of Section 5.3. The working equations that apply to a perfect gas are summarized at the end of Section 5.4 and are (4.28), (5.21), (5.23), (5.25), (5.27), and (5.28). Equations used as a basis for the isentropic table are numbered (5.37), (5.39), (5.40), (5.42), and (5.43) and are located in Section 5.6.

Those equations that are most frequently used are summarized below. You should be familiar with the conditions under which each may be used. Go back and review the equations listed in previous summaries, particularly those in Chapter 4.

1. *For steady one-dimensional flow of a perfect gas when* $Q = W_s = 0$

$$\frac{p_{t2}}{p_{t1}} = e^{-\Delta s/R} \tag{4.28}$$

$$\frac{A_2^*}{A_1^*} = e^{\Delta s/R} \tag{5.29}$$

$$p_{t1}A_1^* = p_{t2}A_2^* \tag{5.35}$$

2. *Nozzle performance.*
Nozzle efficiency (between same pressures):

$$\eta_n \equiv \frac{\Delta\ \text{KE}_{\text{actual}}}{\Delta\ \text{KE}_{\text{ideal}}} = \frac{h_1 - h_2}{h_1 - h_{2s}} \tag{5.45), (5.49}$$

3. *Diffuser performance.*
Total-pressure recovery factor:

$$\eta_r \equiv \frac{p_{t2}}{p_{t1}} \tag{5.52}$$

or diffuser efficiency (between the same enthalpies):

$$\eta_d \equiv \frac{\text{Actual pressure rise}}{\text{Ideal pressure rise}} = \frac{p_2 - p_1}{p_{2s} - p_1} \tag{5.53), (5.54}$$

PROBLEMS

5.1. The following information is common to both parts (a) and (b). Nitrogen flows through a diverging section with $A_1 = 1.5\ \text{ft}^2$ and $A_2 = 4.5\ \text{ft}^2$. You may assume steady, one-dimensional flow, $Q = W_s = 0$, negligible potential changes, and no losses.

(a) If $M_1 = 0.7$ and $p_1 = 70$ psia, find M_2 and p_2.

(b) If $M_1 = 1.7$ and $T_1 = 95°\text{F}$, find M_2 and T_2.

5.2. Air enters a converging section where $A_1 = 0.50\ \text{m}^2$. At a downstream section, $A_2 = 0.25\ \text{m}^2$, $M_2 = 1.0$, and $\Delta s_{1-2} = 0$. It is known that $p_2 > p_1$. Find the initial Mach number (M_1) and the temperature ratio (T_2/T_1).

5.3. Oxygen flows into an insulated device with initial conditions as follows: $p_1 = 30$ psia, $T_1 = 750°\text{R}$, and $V_1 = 639$ ft/sec. The area changes from $A_1 = 6\ \text{ft}^2$ to $A_2 = 5\ \text{ft}^2$.

(a) Compute M_1, p_{t1}, and T_{t1}.

(b) Is this device a nozzle or diffuser?

(c) Determine M_2, p_2, and T_2 if there are no losses.

5.4. Air flows with $T_1 = 250$ K, $p_1 = 3$ bar abs., $p_{t1} = 3.4$ bar abs., and the cross-sectional area $A_1 = 0.40\ \text{m}^2$. The flow is isentropic to a point where $A_2 = 0.30\ \text{m}^2$. Determine the temperature at section 2.

5.5. The following information is known about the steady flow of air through an adiabatic system:

At section 1, $T_1 = 556°R$, p_1, = 28.0 psia

At section 2, $T_2 = 70°F$, $T_{t2} = 109°F$, p_2, = 18 psia

(a) Find M_2, V_2, and p_{t2}.

(b) Determine M_1, V_1, and p_{t1}.

(c) Compute the area ratio A_2/A_1.

(d) Sketch a physical diagram of the system along with a *T–s* diagram.

5.6. Assuming the flow of a perfect gas in an adiabatic, no-work system, show that the sonic speed corresponding to the stagnation conditions (a_t) is related to the sonic speed where the Mach number is unity (a^*) by the following equation:

$$\frac{a^*}{a_t} = \left(\frac{2}{\gamma + 1}\right)^{1/2}$$

5.7. Carbon monoxide flows through an adiabatic system. $M_1 = 4.0$ and $p_{t1} = 45$ psia. At a point downstream, $M_2 = 1.8$ and $p_2 = 7.0$ psia.

(a) Are there losses in this system? If so, compute Δs.

(b) Determine the ratio of A_2/A_1.

5.8. Two venturi meters are installed in a 30-cm-diameter duct that is insulated (Figure P5.8). The conditions are such that sonic flow exists at each throat (i.e., $M_1 = M_4 = 1.0$). Although each venturi is isentropic, the connecting duct has friction and hence losses exist between sections 2 and 3. $p_1 = 3$ bar abs. and $p_4 = 2.5$ bar abs. If the diameter at section 1 is 15 cm and the fluid is air:

(a) Compute Δs for the connecting duct.

(b) Find the diameter at section 4.

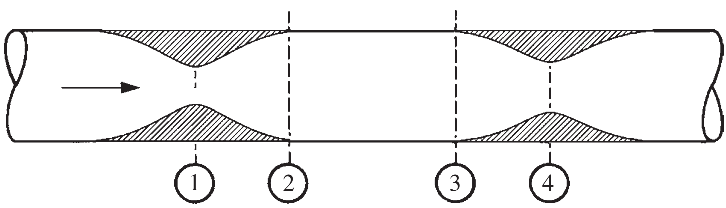


Figure P5.8

5.9. Starting with the mass flow rate form given as equation (2.30), derive the following relation [equation (5.44*a*)]:

$$\frac{\dot{m}}{A} = M\left(1 + [(\gamma-1)/2]M^2\right)^{-(\gamma+1)/2(\gamma-1)}\left(\frac{\gamma g_c}{R}\right)^{1/2}\frac{p_t}{\sqrt{T_t}}$$

5.10. A smooth 3-in.-diameter hole is punched into the side of a large chamber where oxygen is stored at 500°R and 150 psia. Assume frictionless flow.

(a) Compute the initial mass flow rate from the chamber if the surrounding pressure is 15.0 psia.

(b) What is the flow rate if the pressure of the surroundings is lowered to zero?

(c) What is the flow rate if the chamber pressure is raised to 300 psia?

5.11. Nitrogen is stored in a large chamber under conditions of 450 K and 1.5×10^5 N/m^2. The gas leaves the chamber through a convergent-only nozzle whose outlet area is 30 cm^2. The ambient room pressure is 1×10^5 N/m^2 and there are no losses.

(a) What is the velocity of the nitrogen at the nozzle exit?

(b) What is the mass flow rate?

(c) What is the maximum flow rate that could be obtained by lowering the ambient pressure?

5.12. A converging-only nozzle has an efficiency of 96%. Air enters with negligible velocity at a pressure of 150 psia and a temperature of 750°R. The receiver pressure is 100 psia. What are the actual outlet temperature, Mach number, and velocity?

5.13. A large chamber contains air at 80 psia and 600°R. The air enters a converging–diverging nozzle which has an area ratio (exit to throat) of 3.0.

(a) What pressure must exist in the receiver for the nozzle to operate at its first critical point?

(b) What should the receiver pressure be for third critical (design point) operation?

(c) If operating at its third critical point, what are the density and velocity of the air at the nozzle exit plane?

5.14. Air enters a convergent–divergent nozzle at 20 bar abs. and 40°C. At the end of the nozzle the pressure is 2.0 bar abs. Assume a frictionless adiabatic process. The throat area is 20 cm^2.

(a) What is the area at the nozzle exit?

(b) What is the mass flow rate in kg/s?

5.15. A converging–diverging nozzle is designed to operate with an exit Mach number of $M = 2.25$. It is fed by a large chamber of oxygen at 15.0 psia and 600°R and exhausts into the room at 14.7 psia. Assuming the losses to be negligible, compute the velocity in the nozzle throat.

5.16. A converging–diverging nozzle (Figure P5.16) discharges air into a receiver where the static pressure is 15 psia. A 1-ft^2 duct feeds the nozzle with air at 100 psia, 800°R, and a velocity such that the Mach number $M_1 = 0.3$. The exit area is such that the pressure at the nozzle exit exactly matches the receiver pressure. Assume steady, one-dimensional flow, perfect gas, and so on. The nozzle is adiabatic and there are no losses.

(a) Calculate the flow rate.

(b) Determine the throat area.

(c) Calculate the exit area.

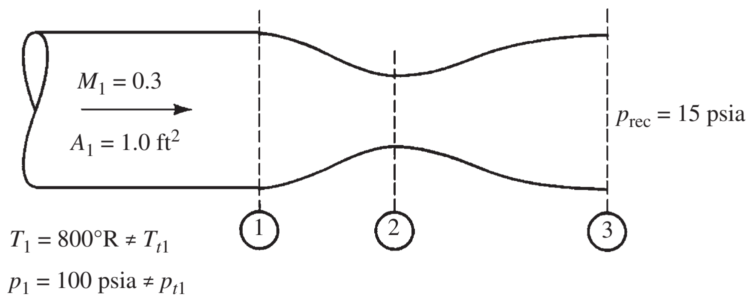


Figure P5.16

5.17. Ten kilograms per second of air is flowing in an adiabatic system. At one section, the pressure is $2.0 \times 10^5\ N/m^2$, the temperature is 650°C, and the area is 50 cm^2. At a downstream section $M_2 = 1.2$.

(a) Sketch the general shape of the system.

(b) Find A_2 if the flow is frictionless.

(c) Find A_2 if there is an entropy change between these two sections of 42 J/kg-K.

5.18. Carbon monoxide is expanded adiabatically from 100 psia, 540°F and negligible velocity through a converging–diverging nozzle to a pressure of 20 psia.

(a) What is the ideal exit Mach number?

(b) If the actual exit Mach number is found to be $M = 1.6$, what is the nozzle efficiency?

(c) What is the entropy change for the flow?

(d) Draw a T–s diagram showing the ideal and actual processes. Indicate pertinent temperatures, pressures, etc.

5.19. Air enters a converging–diverging nozzle with $T_1 = 22°C$, $p_1 = 10$ bar abs., and V_1 0. The exit Mach number is 2.0, the exit area is 0.25 m^2, and the nozzle efficiency is 0.95.

(a) What are the actual exit values of T, p, and p_t?

(b) What is the ideal exit Mach number?

(c) Assume that all the losses occur in the diverging portion of the nozzle and compute the throat area.

(d) What is the mass flow rate?

5.20. A diffuser receives air at 500°R, 18 psia, and a velocity of 750 ft/sec. The diffuser has an efficiency of 90% [as defined by equation (5.54)] and discharges the air with a velocity of 150 ft/sec.

(a) What is the pressure of the discharge air?

(b) What is the total-pressure recovery factor as given by equation (5.52)?

(c) Determine the area ratio of the diffuser.

5.21. (*Optional*) Consider the steady, one-dimensional flow of a perfect gas through a horizontal system with no shaft work. No frictional losses are involved, but area changes and *heat transfer* effects provide a flow at constant temperature.

(a) Start with the pressure-energy equation and develop

$$\frac{p_2}{p_1} = e^{(\gamma/2)\left(M_1^2 - M_2^2\right)}$$

$$\frac{p_{t2}}{p_{t1}} = e^{(\gamma/2)\left(M_1^2 - M_2^2\right)} \left(\frac{1 + [(\gamma-1)/2]\, M_2^2}{1 + [(\gamma-1)/2]\, M_1^2}\right)^{\gamma/(\gamma-1)}$$

(b) From the continuity equation show that

$$\frac{A_1}{A_2} = \frac{M_2}{M_1} e^{(\gamma/2)\left(M_1^2 - M_2^2\right)}$$

(c) By letting M_1 be any Mach number and $M_2 = 1.0$, write the expression for A/A^*. Show that the section of minimum area occurs at $M = 1/\sqrt{\gamma}$.

5.22. Consider the steady, one-dimensional flow of a perfect gas through a horizontal system with no heat transfer or shaft work. *Friction effects* are present, but area changes cause the flow to be at a constant Mach number.

(a) Recall the arguments of Section 4.6 and determine what other properties remain constant in this flow.

(b) Apply the concepts of continuity and momentum [equation (3.63)] to show that

$$D_2 - D_1 = \frac{fM^2\gamma}{4}(x_2 - x_1)$$

You may assume a circular duct and a constant friction factor.

5.23. Assume that a supersonic nozzle operating isentropically delivers air at an exit Mach number of 2.8. The entrance conditions are 180 psia, 1000°R, and near-zero Mach number.

(a) Find the area ratio A_3/A_2 and the mass flow rate per unit throat area.

(b) What are the receiver pressure and temperature?

(c) If the entire diverging portion of the nozzle were suddenly to detach, what would the Mach number and $\dot{m}/A$ be at the new outlet?

5.24. (*Optional*) Examine available tables of isentropic flow parameters for $\gamma \neq 1.4$ in the literature, like Reference 20, and compare them to results from our *Gasdynamics Calculator*. (Useful values γ = 1.2, 1.3, or 1.67.). Look at the following headings: M, p/p_t, T/T_t, ρ/ρ_t, and A/A^*. How do the actual numbers compare?

CHECK TEST

You should be able to complete this test without reference to the material in this chapter.

5.1. Define the * reference condition.

5.2. In adiabatic, no-work flow, losses can be expressed by three different parameters. List these parameters and show how they are related to one another.

5.3. In the T–s diagram (Figure CT5.3), point 1 represents a stagnation condition. Proceeding isentropically from 1, the flow reaches a Mach number of unity at 1^*. Point 2 represents another stagnation condition in the same flow system. Assuming that the fluid is a perfect gas, locate the corresponding isentropic 2^* and prove that T_2^* is either greater than, equal to, or less than T_1^*.

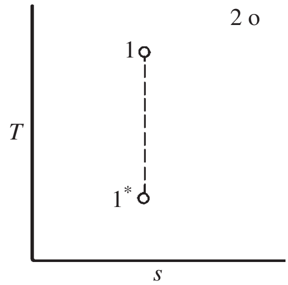


Figure CT5.3

5.4. A supersonic nozzle is fed by a large chamber and produces Mach 3.0 at the exit (Figure CT5.4). Sketch curves (to no particular scale) that show how properties vary through the nozzle as the Mach number increases from zero to 3.0.

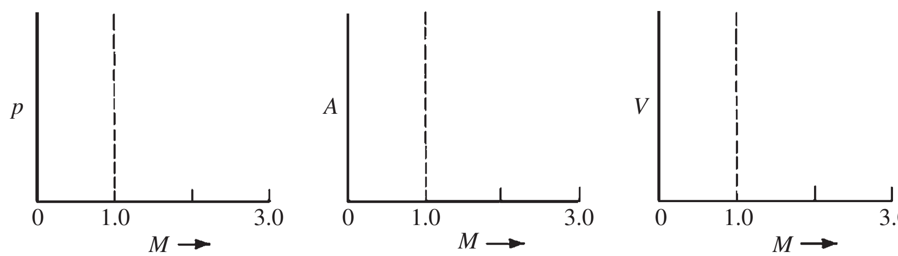


Figure CT5.4

5.5. Give a suitable definition for nozzle efficiency in terms of enthalpies. Sketch an h–s diagram to identify your state points.

5.6. Air flows steadily with no losses through a converging–diverging nozzle with an area ratio of 1.50. Conditions in the supply chamber are $\boldsymbol{T}$ = 500°R and $\boldsymbol{p}$ = 150 psia.

(a) To choke the flow, to what pressure must the receiver be lowered?

(b) If the nozzle is choked, determine the density and velocity at the throat.

(c) If the receiver is at the pressure determined in part (a) and the diverging portion of the nozzle is removed, what will the exit Mach number be?

5.7. For steady, one-dimensional flow of a perfect gas in an adiabatic, no-work system, derive the working relation between the temperatures at two locations:

$$\frac{T_2}{T_1} = f(M_1, M_2, \gamma)$$

5.8. Work Problem 5.20.

Chapter 6

Standing Normal Shocks

6.1 INTRODUCTION

Up to this point we have considered only continuous flows, flow systems in which state changes occur continuously and thus whose processes can easily be identified and plotted. Recall from Section 4.3 that *infinitesimal* pressure disturbances are called sound waves and these travel at a characteristic speed that is determined by the medium and its thermodynamic state. In Chapters 6 and 7 we turn our attention to some *finite* pressure disturbances which are frequently encountered. Although finite pressure disturbances incorporate large changes in fluid properties, their thickness is extremely small. Typical thicknesses are on the order of a few molecular mean free paths, and thus macroscopically they appear as *discontinuities* in the flow and they are called *shock waves*.

Due to the complex interactions involved, microscopic analysis of the changes within a shock wave is beyond the scope of this book. Thus we deal only with the properties that exist on each side of the discontinuity. We first consider a *standing normal shock,* a stationary wave front that is perpendicular to the direction of flow. We will discover why this phenomenon is found only when supersonic flows exist and that it is actually a form of a compression process. We apply the basic concepts of gas dynamics to analyze a shock wave in an arbitrary fluid and then develop working equations for perfect gases. This procedure leads naturally to the compilation of tabular information which greatly simplifies problem solution. The chapter closes with a discussion of shocks found in the diverging portion of supersonic nozzles.

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

6.2 OBJECTIVES

After completing this chapter successfully, you should be able to:

1. List the assumptions used to analyze a standing normal shock.
2. Given the continuity, energy, and momentum equations for steady one-dimensional flow, utilize control volume analysis to derive the relations between properties on each side of a standing normal shock for an arbitrary fluid.
3. (*Optional*) Starting with the basic shock equations for an arbitrary fluid, derive the working equations for a perfect gas relating property ratios on each side of a standing normal shock as a function of Mach number (M) and specific heat ratio (γ).
4. (*Optional*) Given the working equations for a perfect gas, show that a unique relationship must exist between the Mach numbers before and after a standing normal shock.
5. (*Optional*) Explain how a normal-shock table may be developed that gives property ratios across the shock in terms of only the Mach number before the shock.
6. Sketch a normal-shock *process* on a T–s diagram, indicating pertinent features, such as static and total pressures, static and total temperatures, and velocities. Show each of the preceding features before and after the shock.
7. Explain why *expansion shocks* cannot exist.
8. Describe the *second critical* mode of supersonic nozzle operation. Given the area ratio of a converging–diverging nozzle, determine the operating pressure ratio that causes operation at the second critical point.
9. Describe how a converging–diverging nozzle operates between first and second critical points.
10. Demonstrate the ability to solve typical standing normal-shock problems by use of tables and equations.

6.3 SHOCK ANALYSIS: GENERAL FLUID

Figure 6.1 shows a standing normal shock in a section of varying area. We first establish a control volume that includes the shock region and an infinitesimal amount of fluid on each side of the shock. In this manner we deal only with the changes that occur across the shock. It is important to recognize that since the shock wave is so thin (about 10^{-6} m), a control volume chosen in the manner

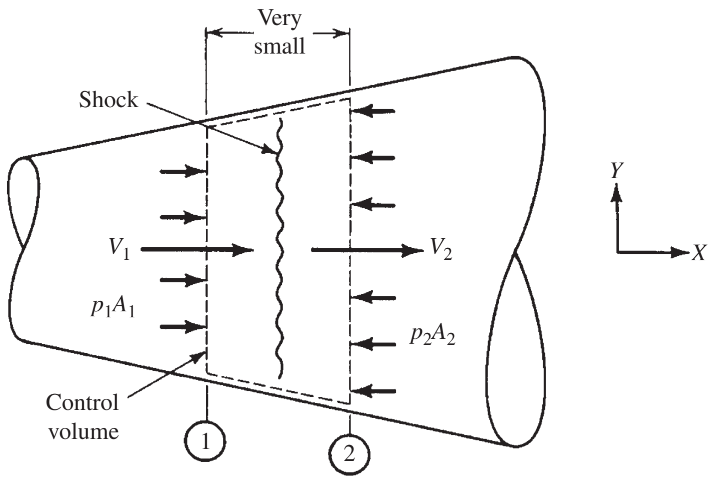


Figure 6.1 Control volume for shock analysis.

described above is extremely thin in the x-direction. This permits the following simplifications to be made without introducing error in the analysis:

1. The area on both sides of the shock may be considered to be the same.
2. There is negligible surface in contact with the wall, and thus frictional effects may be omitted.

We begin by applying the basic concepts of continuity, energy, and momentum under the following assumptions:

Steady one-dimensional flow	
Adiabatic	$\delta q = 0$ or $ds_e = 0$
No shaft work	$\delta w_s = 0$
Neglect potential	$dz = 0$
Constant area	$A_1 = A_2$
Neglect wall shear	

Continuity

$$\dot{m} = \rho A V \tag{2.30}$$

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \tag{6.1}$$

But since the area is constant,

$$\boxed{\rho_1 V_1 = \rho_2 V_2} \tag{6.2}$$

Energy

We start with

$$h_{t1} + q = h_{t2} + w_s \tag{3.19}$$

For adiabatic and no work, this becomes

$$h_{t1} = h_{t2} \tag{6.3}$$

or

$$\boxed{h_1 + \frac{V_1^2}{2g_c} = h_2 + \frac{V_2^2}{2g_c}} \tag{6.4}$$

Momentum

The x-component of the momentum equation for steady one-dimensional flow is

$$\sum F_x = \frac{\dot{m}}{g_c}(V_{\text{out}_x} - V_{\text{in}_x}) \tag{3.46}$$

which when applied to Figure 6.1 becomes

$$\sum F_x = \frac{\dot{m}}{g_c}(V_{2x} - V_{1x}) \tag{6.5}$$

From Figure 6.1 we can also see that the force summation is

$$\sum F_x = p_1 A_1 - p_2 A_2 = (p_1 - p_2) A \tag{6.6}$$

Thus the momentum equation in the direction of flow becomes

$$(p_1 - p_2)A = \frac{\dot{m}}{g_c}(V_2 - V_1) = \frac{\rho A V}{g_c}(V_2 - V_1) \tag{6.7}$$

With $\dot{m}$ written as ρAV, we can cancel the area from both sides. Now the ρV remaining can be written as either $\rho_1 V_1$ or $\rho_2 V_2$ [see equation (6.2)] and equation (6.7) becomes

$$p_1 - p_2 = \frac{\rho_2 V_2^2 - \rho_1 V_1^2}{g_c} \tag{6.8}$$

or

$$\boxed{p_1 + \frac{\rho_1 V_1^2}{g_c} = p_2 + \frac{\rho_2 V_2^2}{g_c}} \tag{6.9}$$

For the general case of an arbitrary fluid, we have arrived at *three* governing equations: (6.2), (6.4), and (6.9). A typical problem would be: Knowing the fluid and the conditions before the shock, predict the conditions that would exist after the shock. The unknown parameters are then *four* in number (ρ_2, p_2, h_2, V_2), which requires additional information for a problem solution. The missing information is supplied in the form of property relations for the fluid involved. For general fluids (not perfect gases), this leads to iterative-type solutions which modern digital computers handle quite adequately.

6.4 WORKING EQUATIONS FOR PERFECT GASES

In Section 6.3 we have seen that a typical normal-shock problem has four unknowns, which can be found through the use of the three governing equations (from continuity, energy, and momentum concepts) plus additional information on property relations. For the case of a perfect gas, this additional information is supplied in the form of the equation of state and the assumption of constant specific heats. We now proceed to develop working equations in terms of Mach numbers and the specific heat ratio.

Continuity

We start with the continuity equation developed in Section 6.3:

$$\rho_1 V_1 = \rho_2 V_2 \tag{6.2}$$

Substitute for the density from the perfect gas equation of state:

$$p = \rho RT \tag{1.16}$$

and for the velocity from equations (4.10) and (4.11):

$$V = Ma = M\sqrt{\gamma g_c RT} \tag{6.10}$$

Show that the continuity equation can now be written as

$$\frac{p_1 M_1}{\sqrt{T_1}} = \frac{p_2 M_2}{\sqrt{T_2}} \tag{6.11}$$

Energy

From Section 6.3 we have

$$h_{t1} = h_{t2} \tag{6.3}$$

But since we are now restricted to a perfect gas for which enthalpy is a function of temperature *only,* we can say that

$$T_{t1} = T_{t2} \tag{6.12}$$

Recall from Chapter 4 that for a perfect gas with constant specific heats,

$$T_t = T\left(1 + \frac{\gamma - 1}{2} M^2\right) \tag{4.18}$$

Hence the energy equation across a standing normal shock can be written as

$$T_1\left(1 + \frac{\gamma - 1}{2} M_1^2\right) = T_2\left(1 + \frac{\gamma - 1}{2} M_2^2\right) \tag{6.13}$$

Momentum

The momentum equation in the direction of flow was seen to be

$$p_1 + \frac{\rho_1 V_1^2}{g_c} = p_2 + \frac{\rho_2 V_2^2}{g_c} \tag{6.9}$$

Substitutions are made for the density from the equation of state (1.16) and for the velocity from equation (6.10):

$$p_1 + \left(\frac{p_1}{RT_1}\right)\left(\frac{M_1^2 \gamma g_c R T_1}{g_c}\right) = p_2 + \left(\frac{p_2}{RT_2}\right)\left(\frac{M_2^2 \gamma g_c R T_2}{g_c}\right) \tag{6.14}$$

and the momentum equation becomes

$$p_1\left(1+\gamma M_1^2\right) = p_2\left(1+\gamma M_2^2\right) \tag{6.15}$$

The governing equations for a standing normal shock have now been simplified for a perfect gas and for convenience are summarized below.

$$\frac{p_1 M_1}{\sqrt{T_1}} = \frac{p_2 M_2}{\sqrt{T_2}} \tag{6.11}$$

$$T_1\left(1+\frac{\gamma-1}{2}M_1^2\right) = T_2\left(1+\frac{\gamma-1}{2}M_2^2\right) \tag{6.13}$$

$$p_1\left(1+\gamma M_1^2\right) = p_2\left(1+\gamma M_2^2\right) \tag{6.15}$$

There are seven variables involved in these equations:

$$\gamma,\ p_1,\ M_1,\ T_1,\ p_2,\ M_2,\ T_2$$

Once the gas is identified, γ is known, and a given state preceding the shock fixes p_1, M_1, and T_1. Thus equations (6.11), (6.13), and (6.15) are sufficient to solve for the unknowns after the shock: p_2, M_2, and T_2.

Rather than struggle through the details of the solution for every shock problem that we encounter, let's solve it once and for all right now. We proceed to combine the equations above and derive an expression for M_2 in terms of the information given. First, we rewrite equation (6.11) as

$$\frac{p_1 M_1}{p_2 M_2} = \sqrt{\frac{T_1}{T_2}} \tag{6.16}$$

and equation (6.13) as

$$\sqrt{\frac{T_1}{T_2}} = \left(\frac{1 + [(\gamma - 1)/2]\, M_2^2}{1 + [(\gamma - 1)/2]\, M_1^2} \right)^{1/2} \tag{6.17}$$

and equation (6.15) as

$$\frac{p_1}{p_2} = \frac{1 + \gamma M_2^2}{1 + \gamma M_1^2} \tag{6.18}$$

We then substitute equations (6.17) and (6.18) into equation (6.16), which yields

$$\left(\frac{1 + \gamma M_2^2}{1 + \gamma M_1^2} \right) \frac{M_1}{M_2} = \left(\frac{1 + [(\gamma - 1)/2]\, M_2^2}{1 + [(\gamma - 1)/2]\, M_1^2} \right)^{1/2} \tag{6.19}$$

At this point notice that M_2 is a function of only M_1 and γ. A trivial solution of this is seen to be $M_1 = M_2$, which represents the degenerate case of no shock. To solve the nontrivial case, we square equation (6.19), cross-multiply, and arrange the result as a quadratic in M_2^2:

$$A\left(M_2^2\right)^2 + BM_2^2 + C = 0 \tag{6.20}$$

where A, B, and C are functions of M_1 and γ. Only if you have considerable motivation should you attempt to carry out the tedious algebra (or to utilize a computer utility, see Section 6.9) required to show that the solution of this quadratic is

$$\boxed{M_2^2 = \frac{M_1^2 + 2/(\gamma - 1)}{[2\gamma/(\gamma - 1)]\, M_1^2 - 1}} \tag{6.21}$$

For our typical shock problem the Mach number after the shock is computed with the aid of equation (6.21), and then T_2 and p_2 can easily be found from equations (6.13) and (6.15). To complete the picture, the total pressures p_{t1} and p_{t2} can be computed in the usual manner. It turns out that since M_1 is supersonic, M_2 will always be subsonic and a typical problem is shown on the T–s diagram in Figure 6.2.

The end points 1 and 2 (before and after the shock) are well-defined states, but the changes that occur within the shock do not follow equilibrium processes in the usual thermodynamic sense. For this reason the shock *process* is usually shown by a dashed or wiggly line. Note that when points 1 and 2 are located on the T–s

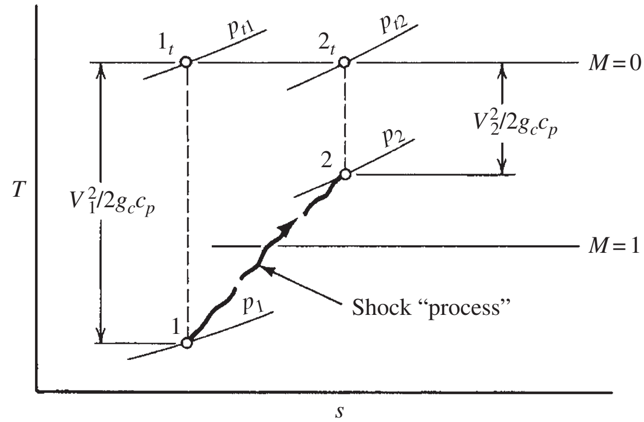


Figure 6.2 *T–s* diagram for typical normal shock.

diagram, it can immediately be seen that an entropy change (increase) is involved in the shock process. This is discussed in greater detail in the next section.

Example 6.1
Helium is flowing at a Mach number of 1.80 and enters a normal shock. Determine the pressure ratio across the shock.

We use equation (6.21) to find the Mach number after the shock and (6.15) to obtain the pressure ratio.

$$M_2^2 = \frac{M_1^2 + 2/(\gamma - 1)}{[2\gamma/(\gamma - 1)]\, M_1^2 - 1} = \frac{(1.8)^2 + 2/(1.67 - 1)}{[(2 \times 1.67)/(1.67 - 1)](1.8)^2 - 1} = 0.411$$

$$M_2 = 0.641$$

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} = \frac{1 + (1.67)(1.8)^2}{1 + (1.67)(0.411)} = 3.80$$

6.5 NORMAL-SHOCK TABLE

We have found that for any given fluid with a specific set of conditions entering a normal shock there is one and only one set of conditions that can result after the shock. An iterative solution results for fluids that cannot be treated as a perfect gas, whereas the case of the perfect gas produces explicit solutions. The latter case opens the door to further simplifications since equation (6.21) yields the exit Mach number M_2 for any given inlet Mach number M_1 and we can now eliminate M_2 from all previous equations.

For example, equation (6.13) can be solved for the temperature ratio

$$\frac{T_2}{T_1} = \frac{1 + [(\gamma - 1)/2]\, M_1^2}{1 + [(\gamma - 1)/2]\, M_2^2} \tag{6.22}$$

If we now eliminate M_2 by the use of equation (6.21), the result will be

$$\frac{T_2}{T_1} = \frac{\left\{1 + [(\gamma - 1)/2]\, M_1^2\right\}\left\{[2\gamma/(\gamma - 1)]\, M_1^2 - 1\right\}}{\left[(\gamma + 1)^2/2(\gamma - 1)\right] M_1^2} \tag{6.23}$$

Similarly, equation (6.15) can be solved for the pressure ratio

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \tag{6.24}$$

and elimination of M_2 through the use of equation (6.21) will produce

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma + 1} M_1^2 - \frac{\gamma - 1}{\gamma + 1} \tag{6.25}$$

If you are very persistent (and in need of algebraic exercise or want to do it with a computer), you might carry out the development of equations (6.23) and (6.25). Also, these can be combined to form the density ratio

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)\, M_1^2}{(\gamma - 1)\, M_1^2 + 2} \tag{6.26}$$

Other interesting ratios can be developed, each as a function of only M_1 and γ. For example, since

$$p_t = p\left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma - 1)} \tag{4.21}$$

we may write

$$\frac{p_{t2}}{p_{t1}} = \frac{p_2}{p_1}\left(\frac{1 + [(\gamma - 1)/2]\, M_2^2}{1 + [(\gamma - 1)/2]\, M_1^2}\right)^{\gamma/(\gamma - 1)} \tag{6.27}$$

The ratio p_2/p_1 can be eliminated by equation (6.25) with the following result:

$$\frac{p_{t2}}{p_{t1}} = \left(\frac{[(\gamma+1)/2]\,M_1^2}{1+[(\gamma-1)/2]\,M_1^2}\right)^{\gamma/(\gamma-1)} \left[\frac{2\gamma}{\gamma+1}M_1^2 - \frac{\gamma-1}{\gamma+1}\right]^{1/(1-\gamma)} \tag{6.28}$$

Equation (6.28) is extremely important since the stagnation pressure ratio is related to the entropy change through equation (4.28):

$$\frac{p_{t2}}{p_{t1}} = e^{-\Delta s/R} \tag{4.28}$$

In fact, we could combine equations (4.28) and (6.28) to obtain an explicit relation for Δs as a function of M_1 and γ.

Note that for a given fluid (γ known), the equations (6.23), (6.25), (6.26), and (6.28) express property ratios as a function of the entering Mach number only. This suggests that we could easily construct a table giving values of M_2, T_2/T_1, p_2/p_1, ρ_2/ρ_1, p_{t2}/p_{t1}, and so on, versus M_1 for a particular γ. Such a table of normal-shock parameters is given in Appendix H. This table greatly aids problem solution, as the following example shows.

Example 6.2

The fluid is air and it can be treated as a perfect gas. If the conditions before the shock are: $M_1 = 2.0$, $p_1 = 20$ psia, and $T_1 = 500°\text{R}$; determine the conditions after the shock and the entropy change across the shock.

First, we compute p_{t1} with the aid of the isentropic table.

$$p_{t1} = \frac{p_{t1}}{p_1}p_1 = \left(\frac{1}{0.1278}\right)(20) = 156.5 \text{ psia}$$

Now from the normal-shock table opposite $M_1 = 2.0$, we find

$$M_2 = 0.57735 \quad \frac{p_2}{p_1} = 4.5000 \quad \frac{T_2}{T_1} = 1.6875 \quad \frac{p_{t2}}{p_{t1}} = 0.72087$$

Thus

$$p_2 = \frac{p_2}{p_1}p_1 = (4.5)(20) = 90 \text{ psia} \quad (\text{or } 0.620\,\text{MPa})$$

$$T_2 = \frac{T_2}{T_1}T_1 = (1.6875)(500) = 844°\text{R} \quad (\text{or } 469\,\text{K})$$

$$p_{t2} = \frac{p_{t2}}{p_{t1}}p_{t1} = (0.72087)(156.5) = 112.8\,\text{psia} \quad (\text{or } 0.777\,\text{MPa})$$

Or p_{t2} can be computed with the aid of the isentropic table:

$$p_{t2} = \frac{p_{t2}}{p_2} p_2 = \left(\frac{1}{0.7978}\right)(90) = 112.8 \text{ psia}$$

To compute the entropy change, we use equation (4.28):

$$\frac{p_{t2}}{p_{t1}} = 0.72087 = e^{-\Delta s/R}$$

$$\frac{\Delta s}{R} = 0.3273$$

$$\Delta s = \frac{(0.3273)(53.3)}{778} = 0.0224 \text{ Btu/lbm-°R} \quad (\text{or } 93.79\,\text{J/kg-K})$$

It is interesting to note that as far as the governing equations are concerned, the problem in Example 6.2 could be completely reversed. The fundamental relations of continuity (6.11), energy (6.13), and momentum (6.15) would be satisfied completely if we changed the problem to $M_1 = 0.577$, $p_1 = 90$ psia, $T_1 = 844$°R, with the resulting $M_2 = 2.0$, $p_2 = 20$ psia, and $T_2 = 500$°R (which would represent an *expansion shock*). However, in the latter case the entropy change would be *negative,* which clearly violates the second law of thermodynamics for an adiabatic no-work system.

Example 6.2 and the accompanying discussion clearly show that every shock phenomenon is a one-way process (i.e., irreversible). It is always a compression shock, and for a normal shock the flow is always supersonic before the shock and subsonic after the shock. One can note from the shock tables that as M_1 increases, the pressure, temperature, and density ratios increase, indicating a stronger shock (or compression). One can also note that as M_1 increases, p_{t2}/p_{t1} decreases, which means that the entropy change increases. Thus *as the strength of the shock increases, the fluid losses also increase*.

Example 6.3

Air has a temperature and pressure of 300 K and 2 bar abs., respectively. It is flowing with a velocity of 868 m/s and enters a normal shock. Determine the density before and after the shock.

$$\rho_1 = \frac{p_1}{RT_1} = \frac{2 \times 10^5}{(287)(300)} = 2.32 \text{ kg/m}^3$$

$$a_1 = (\gamma g_c R T_1)^{1/2} = [(1.4)(1)(287)(300)]^{1/2} = 347 \text{ m/s}$$

$$M_1 = \frac{V_1}{a_1} = \frac{868}{347} = 2.50$$

From the shock table we obtain

$$\frac{\rho_2}{\rho_1} = \frac{p_2}{p_1}\frac{T_1}{T_2} = (7.125)\left(\frac{1}{2.1375}\right) = 3.333$$

$$\rho_2 = 3.3333\rho_1 = (3.3333)(2.32) = 7.73\,\text{kg/m}^3 \quad \left(\text{or } 0.483\,\text{lbm/ft}^3\right)$$

Example 6.4

Oxygen enters the converging section shown in Figure E6.4 and a normal shock occurs at the exit. The entering Mach number is 2.8 and the area ratio $A_1/A_2 = 1.7$. Compute the overall static temperature ratio T_3/T_1. Neglect all frictional losses.

$$\frac{A_2}{A_2^*} = \frac{A_2}{A_1}\frac{A_1}{A_1^*}\frac{A_1^*}{A_2^*} = \left(\frac{1}{1.7}\right)(3.5001)(1) = 2.06$$

Thus $M_2 \approx 2.23$, and from the shock table we get

$$M_3 = 0.5431 \quad \text{and} \quad \frac{T_3}{T_2} = 1.8835$$

$$\frac{T_3}{T_1} = \frac{T_3}{T_2}\frac{T_2}{T_{t2}}\frac{T_{t2}}{T_{t1}}\frac{T_{t1}}{T_1} = (1.8835)(0.5014)(1)\frac{1}{0.3894} = 2.43$$

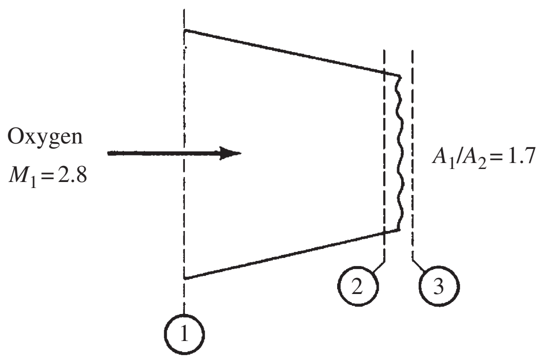


Figure E6.4

We can also develop here a relation for the velocity change across a standing normal shock for use in Chapter 7. Starting with the basic continuity equation

$$\rho_1 V_1 = \rho_2 V_2 \tag{6.2}$$

we introduce the density relation from (6.26):

$$\frac{V_2}{V_1} = \frac{\rho_1}{\rho_2} = \frac{(\gamma-1)\,M_1^2+2}{(\gamma+1)\,M_1^2} \tag{6.29}$$

and subtract 1 from each side:

$$\frac{V_2-V_1}{V_1} = \frac{(\gamma-1)\,M_1^2+2-(\gamma+1)\,M_1^2}{(\gamma+1)\,M_1^2} \tag{6.30}$$

$$\frac{V_2-V_1}{M_1 a_1} = \frac{2\left(1-M_1^2\right)}{(\gamma+1)\,M_1^2} \tag{6.31}$$

or

$$\boxed{\frac{V_1-V_2}{a_1} = \left(\frac{2}{(\gamma+1)}\right)\left(\frac{M_1^2-1}{M_1}\right)} \tag{6.32}$$

This is another parameter that is purely a function of M_1 and γ and thus has been added to our normal shock table as $\Delta V/a_1$. Its usefulness for solving certain types of problems will become apparent in Chapter 7.

6.6 SHOCKS IN NOZZLES

In Section 5.7 we discussed the isentropic operations of a converging-diverging nozzle. Remember that this type of nozzle is physically distinguished by a specific *area ratio,* the ratio of the exit area to the throat area. Furthermore, its flow conditions are determined by the *operating pressure ratio,* the ratio of the receiver pressure to the inlet stagnation pressure. We identified two significant critical pressure ratios. For any pressure ratio above the first critical point, the nozzle is not choked and has subsonic flow throughout (typical venturi operation). The first critical point represents flow that is subsonic in both the convergent and divergent sections but is choked with a Mach number of 1.0 at the throat. The third critical point represents operation at the design condition with subsonic flow in the converging section and supersonic flow in the entire diverging section. It is also choked with Mach 1.0 in the throat. The first and third critical points are the only operating points that have all of the following: (1) isentropic flow throughout, (2) a Mach number of 1.0 at the throat, and (3) exit pressure equal to receiver pressure.

Remember that with subsonic flow at the exit, the exit pressure *must* equal the receiver pressure. Imposing a pressure ratio slightly below that of the first critical point presents a difficulty in that there is no way that *isentropic* flow can meet the boundary condition of pressure equilibrium at the exit. However, there is nothing to prevent a *non-isentropic* flow adjustment from occurring within the nozzle. This internal adjustment takes the form of a standing normal shock, which we now know always involves an entropy change.

As the pressure ratio is lowered below the first critical point, a normal shock forms just downstream of the throat. The remainder of the *nozzle* is now acting as a diffuser since after the shock the flow is subsonic and the area is increasing. The shock will locate itself in a position such that the pressure changes that occur ahead of the shock, across the shock, and downstream of the shock will produce a pressure that *exactly matches the outlet pressure*. In other words, *the operating pressure ratio determines the nozzle location and the strength of the shock*. An example of this mode of operation is shown in Figure 6.3. As the pressure ratio is lowered further, the shock continues to move toward the exit. When the shock is located at the exit plane, this condition is referred to as the *second critical point*.

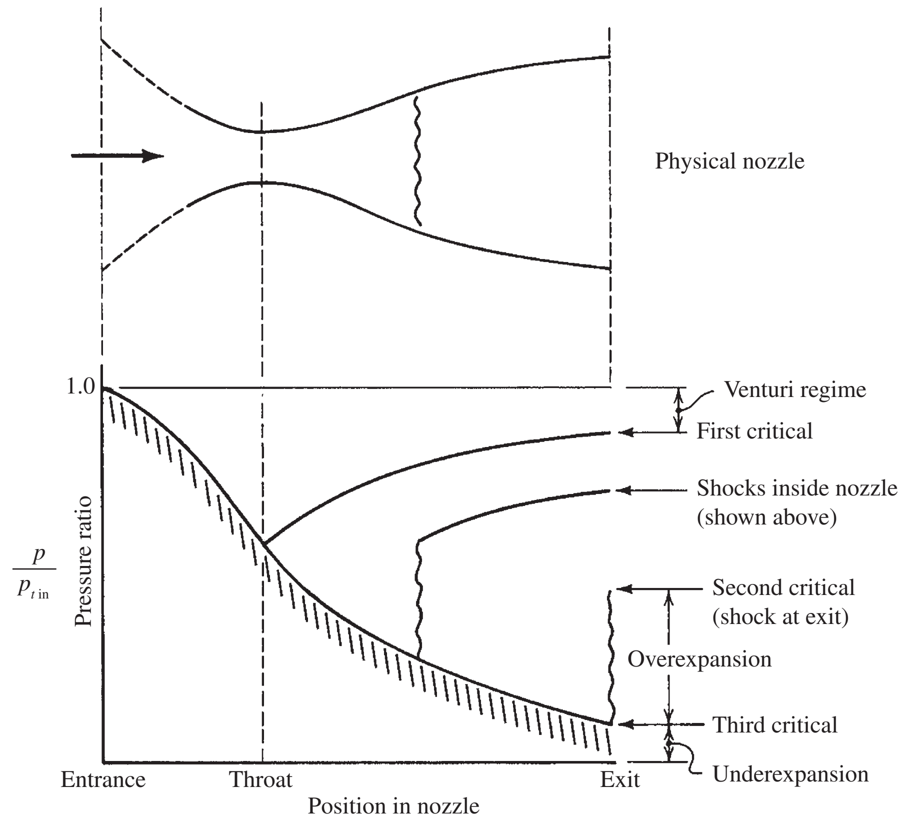


Figure 6.3 Operating modes for DeLaval nozzle.

We have been ignoring boundary layer effects that are always present due to fluid viscosity. These effects sometimes cause what are known as *lambda shocks*. It is important for you to understand that real flows are often somewhat more complicated than the idealizations that we are describing.

If the operating pressure ratio is between the second and third critical points, compression takes place *outside* the nozzle. This condition is called *overexpansion* (i.e., the flow has been expanded too far within the nozzle). If the receiver pressure is below the third critical point, expansion takes place *outside* the nozzle. This condition is called *underexpansion*. We investigate these flow conditions in Chapters 7 and 8 after some necessary background has been covered.

For the present we proceed to investigate the operational regime between the first and second critical points. Let us work with the same nozzle and inlet conditions that we used in Section 5.7. The nozzle has an area ratio of 2.494 and is fed by air at 100 psia and 600°R from a large tank. Thus the inlet conditions are essentially stagnation. For these fixed inlet conditions, we previously found that a receiver pressure of 96.07 psia (an *operating pressure ratio* of 0.9607) identifies the first critical point and a receiver pressure of 6.426 psia (an *operating pressure ratio* of 0.06426) exists at the third critical point.

What receiver pressure do we need to operate at the second critical point? Figure 6.4 shows such a condition and you should recognize that the entire nozzle up to the shock itself is operating at its design or third critical condition.

From the isentropic table at $A/A^* = 2.494$, we have

$$M_3 = 2.44 \quad \text{and} \quad \frac{p_3}{p_{t3}} = 0.06426$$

From the normal-shock table for $M_3 = 2.44$, we have

$$M_4 = 0.5189 \quad \text{and} \quad \frac{p_4}{p_3} = 6.7792$$

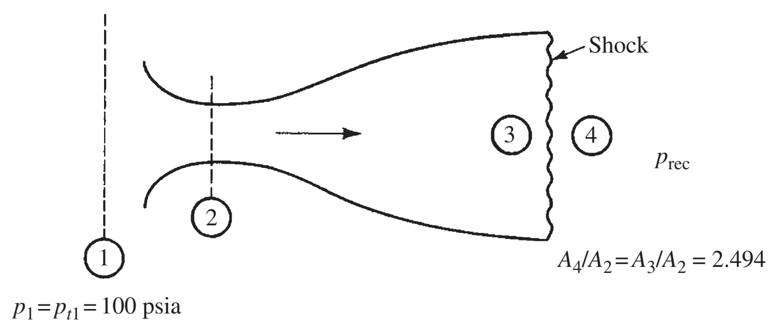


Figure 6.4 Operation at the second critical.

and the operating pressure ratio will be

$$\frac{p_{rec}}{p_{t1}} = \frac{p_4}{p_{t1}} = \frac{p_4}{p_3}\frac{p_3}{p_{t3}}\frac{p_{t3}}{p_{t1}} = (6.7792)(0.06426)(1) = 0.436$$

or for $p_1 = p_{t1} = 100$ psia,

$$p_4 = p_{rec} = 43.6 \text{ psia}$$

Thus for our converging-diverging nozzle with an area ratio of 2.494, any operating pressure ratio between 0.9607 and 0.436 will cause a normal shock to be located someplace inside the diverging portion of the nozzle.

Suppose now that we are given an operating pressure ratio of 0.60. The logical question to ask is: Where is the shock? This situation is shown in Figure 6.5. We must take advantage of the only two available pieces of information and from these construct a solution. We know that

$$\frac{A_5}{A_2} = 2.494 \quad \text{and} \quad \frac{p_5}{p_{t1}} = 0.60$$

We may also assume that all losses occur across the shock and we know that $M_2 = 1.0$. It might also be helpful to visualize the flow on a T–s diagram, and this is shown in Figure 6.6. Since there are no losses up to the shock, we know that

$$A_2 = A_1^*$$

Thus

$$\frac{A_5}{A_2}\frac{p_5}{p_{t1}} = \frac{A_5}{A_1^*}\frac{p_5}{p_{t1}} \tag{6.33}$$

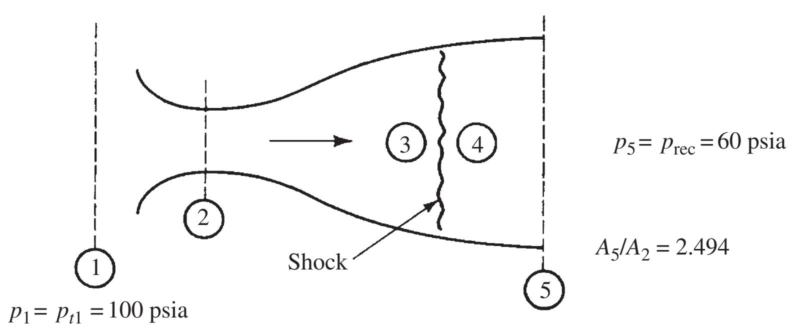


Figure 6.5 DeLaval nozzle with normal shock in diverging section.

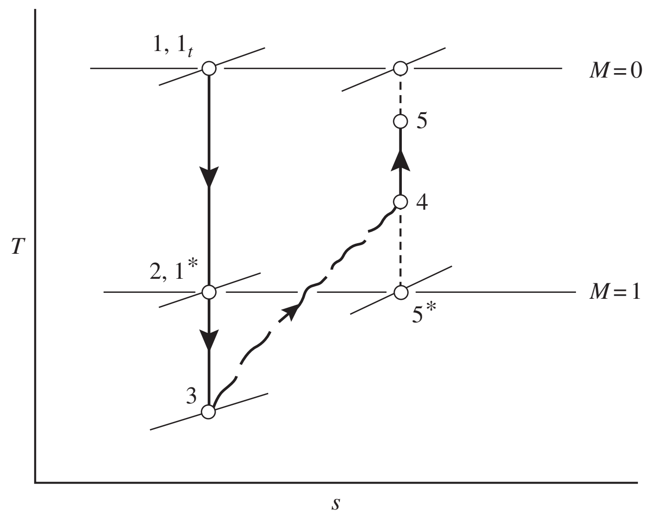


Figure 6.6 *T–s* diagram for DeLaval nozzle with normal shock. (For physical picture see Figure 6.5.)

We also know from equation (5.35) that for the case of adiabatic no-work flow of a perfect gas,

$$A_1^* p_{t1} = A_5^* p_{t5} \tag{6.34}$$

Thus

$$\frac{A_5\ p_5}{A_1^*\ p_{t1}} = \frac{A_5\ p_5}{A_5^*\ p_{t5}}$$

In summary:

$$\underbrace{\frac{A_5}{A_2}\frac{p_5}{p_{t1}}}_{\text{known}} = \frac{A_5}{A_1^*}\frac{p_5}{p_{t1}} = \frac{A_5 p_5}{A_5^* p_{t5}} \tag{6.35}$$

$$(2.494)\ (0.6) \qquad = \qquad 1.4964$$

Note that we have manipulated the known information into an expression with all similar station subscripts. In Section 5.6 we showed with equation (5.43) that the ratio $pA/p_t A^*$ is a simple function of M and γ and thus is listed in the isentropic table. A check in the table shows that the exit Mach number is $M_5 \approx 0.38$.

To locate the shock, seek the ratio

$$\frac{p_{t5}}{p_{t1}} = \frac{p_{t5}}{p_5}\frac{p_5}{p_{t1}} = \left(\frac{1}{0.9052}\right)(0.6) = 0.664$$

Given

From isentropic table at $M = 0.38$

and since all the loss is assumed to take place across the shock, we have

$$p_{t5} = p_{t4} \quad \text{and} \quad p_{t1} = p_{t3}$$

Thus

$$\frac{p_{t4}}{p_{t3}} = \frac{p_{t5}}{p_{t1}} = 0.664$$

Knowing the total pressure ratio across the shock, we can determine from the normal shock table that $M_3 \approx 2.12$, and then from the isentropic table we note that this Mach number will occur at an area ratio of about $A_3/A_3^* = A_3/A_2 = 1.869$.

We see that if we are given a physical converging–diverging nozzle (area ratio is known) and an operating pressure ratio between the first and second critical points, it is a simple matter to determine the position and strength of the normal shock in the diverging section.

Example 6.5

A *converging–diverging* nozzle has an area ratio of 3.50. At off-design conditions, the exit Mach number is observed to be 0.3. What operating pressure ratio would cause this situation?

Using the section numbering system of Figure 6.5, for $M_3 = 0.3$, we have

$$\frac{p_5 A_5}{p_{t5} A_5^*} = 1.9119$$

$$\frac{p_5}{p_{t1}} = \frac{p_5 A_5}{p_{t5} A_5^*}\left(\frac{p_{t5} A_5^*}{p_{t1} A_1^*}\right)\frac{A_1^* A_2}{A_2 A_5} = (1.9119)(1)(1)\left(\frac{1}{3.50}\right) = 0.546$$

Could you now find the shock location and Mach number?

Example 6.6

Air enters a converging-diverging nozzle that has an overall area ratio of 1.76. A normal shock occurs at a section where the area is 1.19 times that of the throat. Neglect all friction losses and find the operating pressure ratio. Again, we use the numbering system shown in Figure 6.5.

From the isentropic table at $A_3/A_2 = 1.19$, $M_3 = 1.52$.
From the shock table, $M_4 = 0.6941$ and $p_{t4}/p_{t3} = 0.9233$. Then

$$\frac{A_5}{A_5^*} = \frac{A_5}{A_2}\frac{A_2}{A_4}\frac{A_4}{A_4^*}\frac{A_4^*}{A_5^*} = (1.76)\left(\frac{1}{1.19}\right)(1.0988)(1) = 1.625$$

Thus $M_5 \approx 0.389$.

$$\frac{p_5}{p_{t1}} = \frac{p_5}{p_{t5}}\frac{p_{t5}}{p_{t4}}\frac{p_{t4}}{p_{t3}}\frac{p_{t3}}{p_{t1}} = (0.9007)(1)(0.9233)(1) = 0.832$$

6.7 SUPERSONIC WIND TUNNEL OPERATION

To provide a test section with supersonic flow requires a converging–diverging nozzle. To operate economically, the nozzle–test-section combination must be followed by a diffusing section which also must be converging–diverging. This configuration presents some interesting problems in flow analysis. Starting up such a wind tunnel is another example of nozzle operation at pressure ratios above the second critical point. Figure 6.7 shows a typical tunnel in its most *unfavorable* operating condition, which occurs at startup. A brief analysis of the situation follows.

As the exhauster is started, this reduces the pressure and produces flow through the tunnel. At first the flow is subsonic throughout, but at increased power settings the exhauster reduces pressures still further and causes increased flow rates until the nozzle throat (section 2) becomes choked. At this point the nozzle is operating at its first critical condition. As power is increased further, a normal shock is formed just downstream of the throat, and if the tunnel pressure is decreased continuously, the shock will move down the diverging portion of the nozzle and pass rapidly through the test section and into the diffuser. Figure 6.8 shows this general running condition, which is called the *most favorable condition*.

We return to Figure 6.7, which shows the shock located in the test section. The variation of Mach number throughout the flow system is also shown for this case. This is called the *most unfavorable condition* because the shock occurs at the highest possible Mach number and thus the losses are greatest. We should also point out that the diffuser throat (section 5) must be sized for this condition. Let us see how this is done.

Recall the relation $p_t A* =$ constant. Thus

$$p_{t2}A_2^* = p_{t5}A_5^*$$

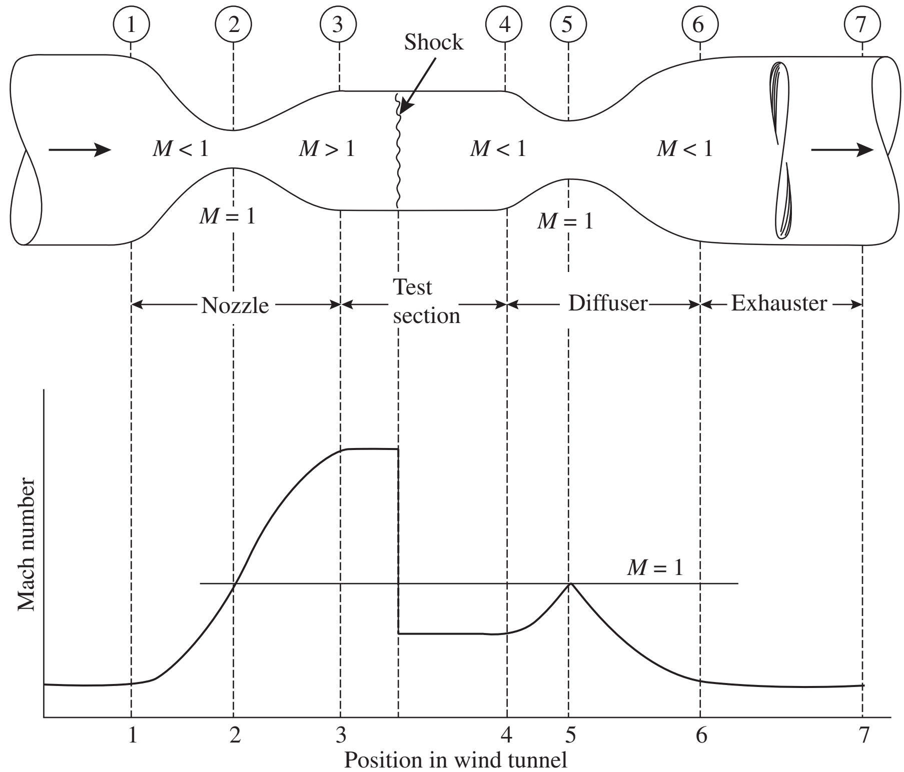


Figure 6.7 Supersonic tunnel at startup (with associated Mach number variation).

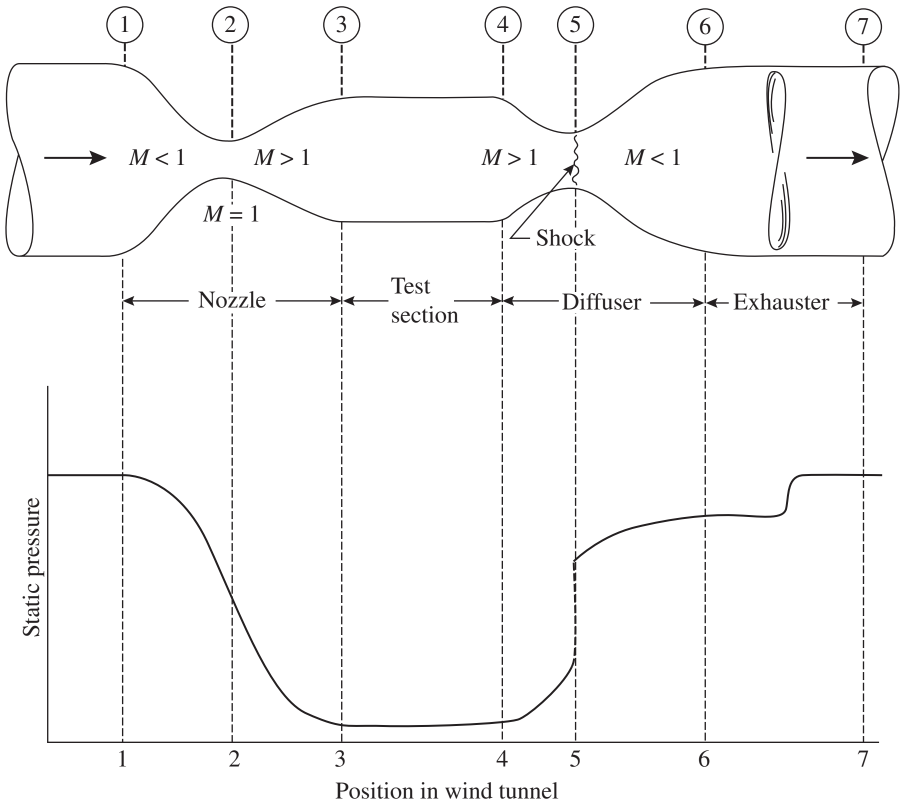


Figure 6.8 Supersonic tunnel in running condition (with associated pressure variation).

But since Mach 1 exists at both sections 2 and 5 (during startup),

$$A_2 = A_2^* \quad \text{and} \quad A_5 = A_5^*$$

Hence

$$p_{t2}A_2 = p_{t5}A_5 \tag{6.36}$$

Due to the shock losses (and other friction losses), we know that $p_{t5} < p_{t2}$, and therefore A_5 must be greater than A_2. Knowing the test-section-design Mach number fixes the shock strength in this unfavorable condition and A_5 is easily determined from equation (6.36). Keep in mind that this represents a *minimum* area for the diffuser throat. If it is made any smaller than this, the tunnel could never be started (i.e., we could never get the shock into and through the test section). In fact, if A_5 is made too small, the flow will choke first in this throat and never get a chance to reach sonic conditions in section 2.

Once the shock has passed into the diffuser throat, knowing that $A_5 > A_2$ we realize that the tunnel can never run with sonic speed at section 5. Thus, to operate as a diffuser, there must be a shock at this point, as shown in Figure 6.8. We have also shown the pressure variation through the tunnel for this running condition.

To keep the losses during running at a minimum, the shock in the diffuser should occur at the lowest possible Mach number, which means a small throat. However, we have seen that it is necessary to have a large diffuser throat in order to start the tunnel. A solution to this dilemma would be to construct a diffuser with a variable throat area. After startup, A_5 could be decreased, with a corresponding decrease in shock strength and operating power. However, the power required for any installation must always be computed on the basis of the unfavorable startup condition.

Although the supersonic wind tunnel is used primarily for aeronautically oriented work, its operation serves to solidify many of the important concepts of variable-area flow, normal shocks, and their associated flow losses. Equally important is the fact that it begins to focus our attention on some practical design applications.

6.8 WHEN γ IS NOT EQUAL TO 1.4

As indicated in Chapter 5, we discuss the effects that changes from $\gamma = 1.4$ bring about. Figures 6.9 and 6.10 show curves for T_2/T_1 and p_2/p_1 versus Mach number in the interval $1 \leq M \leq 5$ entering the shock. This is done for various ratios of the specific heats ($\gamma = 1.13$, 1.4, and 1.67).

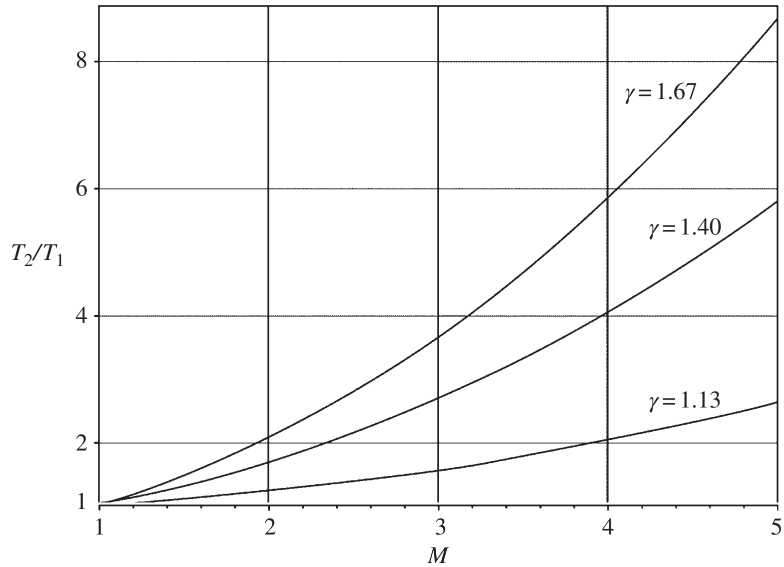


Figure 6.9 Temperature ratio across a normal shock versus Mach number for various values of γ.

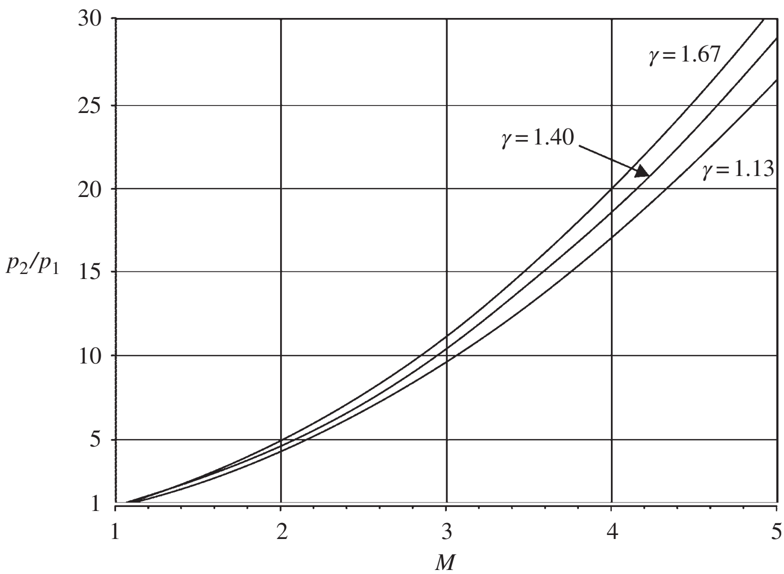


Figure 6.10 Pressure ratio across a normal shock versus Mach number for various values of γ.

1. Figure 6.9 depicts T_2/T_1 across a normal-shock wave. As can be seen in the figure, the temperature ratio is very sensitive to γ.
2. On the other hand, as shown in Figure 6.10, the pressure ratio across the normal shock is relatively less sensitive to γ. Below $M \approx 1.5$ the pressure ratio tabulated in Appendix H could be used with little error for any γ.

Strictly speaking, these curves are representative only for cases where γ variations are *negligible within the flow*. However, they offer hints as to what magnitude of changes are to be expected in other cases. Flows where γ-variations are *not negligible within the flow* are treated in Chapter 11.

Besides reproducing all the values tabulated in Appendix H, the *Gasdynamics Calculator* generates values for the relevant shock parameters in the range $1.0 < \gamma \leq 1.67$.

6.9 (OPTIONAL) BEYOND THE TABLES

As illustrated in Chapter 5, one can eliminate a lot of interpolation and get accurate answers for any ratio of the specific heats γ and/or any Mach number by using a computer utility such as MAPLE. For instance, we can easily calculate the left-hand side of equations (6.21), (6.23), (6.25), (6.26), and (6.28) to a high degree of precision given M_1 and γ (or calculate any one of the three variables given the other two).

Example 6.7
Let's go back to Example 6.3, where the density ratio across the shock is desired. We can compute this from equation (6.26):

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\,M_1^2}{(\gamma-1)\,M_1^2+2} \tag{6.26}$$

Let

$g \equiv \gamma$, a parameter (the ratio of specific heats)
$X \equiv$ the independent variable (which in this case is M_1)
$Y =$ the dependent variable (which in this case is ρ_2/ρ_1)

Listed below are the precise inputs and program that you use in the computer.

```
[ > g :=1.4:  X:= 2.5:
[ > Y:=((g+1)*X^2)/((g-1)*X^2+2);
                              Y:=3.333333333
```

which is the desired answer.

A rather unique capability of MAPLE is its ability to solve equations symbolically (in contrast to strictly numerically). This comes in handy when trying to reproduce proofs of somewhat complicated algebraic expressions.

Example 6.8
Suppose that we want to solve for M_2 in equation (6.19):

$$\left(\frac{1+\gamma M_2^2}{1+\gamma M_1^2}\right)\frac{M_1}{M_2}=\left(\frac{1+[(\gamma-1)/2]\,M_2^2}{1+[(\gamma-1)/2]\,M_1^2}\right)^{1/2} \tag{6.19}$$

Let

$g \equiv \gamma$, a parameter (the ratio of specific heats)
$X \equiv$ the independent variable (which in this case is M_1^2)
$Y \equiv$ the dependent variable (which in this case is M_2^2)

Listed below are the precise inputs and program that you use in the computer.

```
>solve((((1+g*Y)^2)/((1 + g*X)^2))*(X/Y) = (2 +
(g-1)*Y)/(2 + (g-1)*X), Y);
```

$$X,\ \frac{2+Xg-X}{-g+1+2Xg}.$$

which are the desired answers.

Above are the two roots of Y (or M_2^2), because we are solving a quadratic. With some manipulation, we can get the second or nontrivial root to look like equation (6.21). It is easier to check it by substituting in some numbers and comparing results with the normal-shock table.

The type of calculations shown above can be integrated into more sophisticated programs to handle most situations as has been done in the *Gasdynamics Calculator*.

6.10 SUMMARY

We examined stationary discontinuities of a type perpendicular to the flow. These are finite pressure disturbances and are called *standing normal shock waves*. If conditions are known ahead of a shock, a precise set of conditions must exist after the shock. Explicit solutions can be obtained for the case of a perfect gas and these lend themselves to tabulation for various specific heat ratios.

Shocks are found only in supersonic flows, and these flows are always subsonic after a normal shock. The shock wave is a type of compression process, although a rather inefficient one since relatively large losses is involved in the process. (What

has been lost?) Shocks provide a means of flow adjustment to meet imposed pressure boundary conditions in supersonic flows.

As in Chapter 5, most of the equations in this chapter need not be memorized. However, you should be completely familiar with the fundamental relations that apply to all fluids across a normal shock. These are equations (6.2), (6.4), and (6.9). Essentially, these say that the end points of a shock have three things in common:

1. The same mass flow per unit area
2. The same stagnation enthalpy
3. The same value of $p + \rho V^2/g_c$

The working equations that apply to perfect gases, equations (6.11), (6.13), and (6.15), are summarized in Section 6.4. In Section 6.5 we developed equation (6.32) and noted that it will be very useful in solving certain types of problems. You should also be familiar with the various ratios that have been tabulated in Appendix H. Just knowing what kind of information you have available is frequently very helpful in setting up a problem solution.

PROBLEMS

Unless otherwise indicated, you may assume that there is no friction in any of the following flow systems; thus the only losses are those generated by shocks.

6.1. A standing normal shock occurs in air that is flowing at a Mach number of 1.8.
- **(a)** What are the pressure, temperature, and density ratios across the shock?
- **(b)** Compute the entropy change for the air as it passes through the shock.
- **(c)** Repeat part (b) for flows at $M = 2.8$ and 3.8. What pattern do you see?

6.2. The difference between the total and static pressure before a shock is 75 psi. What is the maximum static pressure that can exist at this point ahead of the shock? The gas is oxygen. (*Hint:* Start by finding the static and total pressures ahead of the shock for the limiting case of $M = 1.0$.)

6.3. In an arbitrary perfect gas, the Mach number before a shock is infinite.
- **(a)** Determine a general expression for the Mach number after the shock. What is the value of this expression for $\gamma = 1.4$?
- **(b)** Determine general expressions for the ratios p_2/p_1, T_2/T_1, ρ_2/ρ_1, and p_{t2}/p_{t1}. Do these agree with the values shown in Appendix H for $\gamma = 1.4$?

6.4. It is known that sonic speed exists in each throat of the system shown in Figure P6.4. The entropy change for the air is 0.062 Btu/lbm-°R. Negligible friction exists in the duct. Determine the area ratios A_3/A_1 and A_2/A_1.

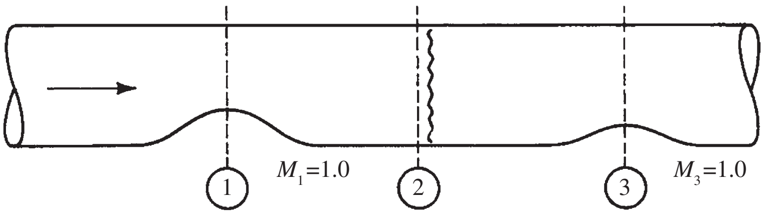


Figure P6.4

6.5. Air flows in the system shown in Figure P6.5. It is known that the Mach number after the shock is $M_3 = 0.52$. Considering p_1 and p_2, it is also known that one of these pressures is twice the other.

(a) Compute the Mach number at section 1.

(b) What is the area ratio A_1/A_2?

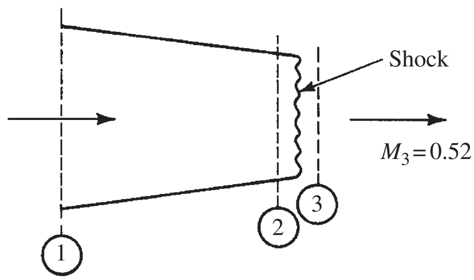


Figure P6.5

6.6. A shock stands at the inlet to the system shown in Figure P6.6. The free-stream Mach number is $M_1 = 2.90$, the fluid is nitrogen, $A_2 = 0.25\ \text{m}^2$, and $A_3 = 0.20\ \text{m}^2$. Find the outlet Mach number and the temperature ratio T_3/T_1.

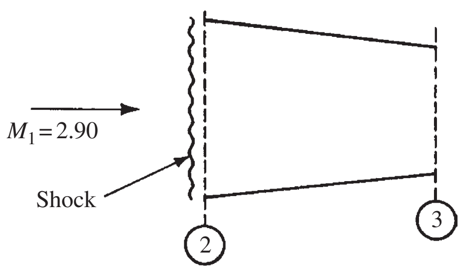


Figure P6.6

6.7. A converging–diverging nozzle is designed to produce a Mach number of 2.5 with air.

(a) What operating pressure ratio (p_{rec}/p_t inlet) will cause this nozzle to operate at the first, second, and third critical points?

(b) If the inlet stagnation pressure is 150 psia, what receiver pressures represent operation at these critical points?

(c) Suppose that the receiver pressure were fixed at 15 psia. What inlet pressures are necessary to cause operation at the critical points?

6.8. Air enters a convergent–divergent nozzle at 20×10^5 N/m^2 and 40°C. The receiver pressure is 2×10^5 N/m^2 and the nozzle throat area is 10 cm^2.

(a) What should the exit area be for the design conditions above (i.e., to operate at third critical?)

(b) With the nozzle area fixed at the value determined in part (a) and the inlet pressure held at 20×10^5 N/m^2, what receiver pressure would cause a shock to stand at the exit?

(c) What receiver pressure would place the shock at the throat?

6.9. In Figure P6.9, $M_1 = 3.0$ and $A_1 = 2.0$ ft^2. If the fluid is carbon monoxide and the shock occurs at an area of 1.8 ft^2, what is the minimum area possible for section 4?

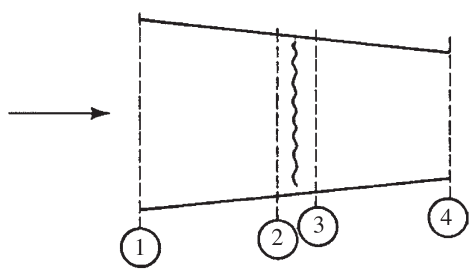


Figure P6.9

6.10. A converging–diverging nozzle has an area ratio of 7.8 but is not being operated at its design pressure ratio. Consequently, a normal shock is found in the diverging section at an area twice that of the throat. The fluid is oxygen.

(a) Find the Mach number at the exit and the operating pressure ratio.

(b) What is the entropy change through the nozzle if there is negligible friction?

6.11. The diverging section of a supersonic nozzle is formed from the frustum of a cone. When operating at its third critical point with nitrogen, the exit Mach number is 2.6. Compute the operating pressure ratio that will locate a normal shock as shown in Figure P6.11.

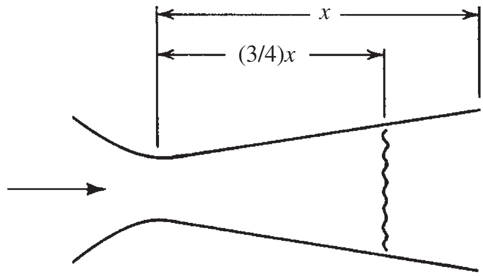


Figure P6.11

6.12. A converging–diverging nozzle receives air from a tank at 100 psia and 600°R. The pressure is 28.0 psia immediately preceding a plane shock that is located in the diverging section. The Mach number at the exit is 0.5 and the flow rate is 10 lbm/sec. Determine:

(a) The throat area.

(b) The area at which the shock is located.

(c) The outlet pressure required to operate the nozzle in the manner described above.

(d) The outlet area.

(e) The design Mach number.

6.13. Air enters a device with a Mach number of $M_1 = 2.0$ and leaves with $M_2 = 0.25$. The ratio of exit to inlet area is $A_2/A_1 = 3.0$.

(a) Find the static pressure ratio p_2/p_1.

(b) Determine the stagnation pressure ratio p_{t2}/p_{t1}.

6.14. Oxygen, with $p_t = 95.5$ psia, enters a diverging section of area 3.0 ft^2. At the outlet the area is 4.5 ft^2, the Mach number is 0.43, and the static pressure is 75.3 psia. Determine the possible values of Mach number that could exist at the inlet.

6.15. A converging–diverging nozzle has an area ratio of 3.0. The stagnation pressure at the inlet is 8.0 bar and the receiver pressure is 3.5 bar. Assume that $\gamma = 1.4$.

(a) Compute the critical operating pressure ratios for the nozzle and show that a shock is located within the diverging section.

(b) Compute the Mach number at the outlet.

(c) Compute the shock location (area) and the Mach number before the shock.

6.16. Nitrogen flows through a converging–diverging nozzle designed to operate at a Mach number of 3.0. If it is subjected to an operating pressure ratio of 0.5:

(a) Determine the Mach number at the exit.

(b) What is the entropy change in the nozzle?

(c) Compute the area ratio at the shock location.

(d) What value of the operating pressure ratio would be required to move the shock to the exit?

6.17. Consider a converging–diverging nozzle feeding air from a reservoir at p_1 and T_1. The exit area is $A_e = 4A_2$, where A_2 is the area at the throat. The back pressure p_{rec} is steadily reduced from an initial $p_{\text{rec}} = p_1$.

- **(a)** Determine the receiver pressures (in terms of p_1) that would cause this nozzle to operate at first, second, and third critical points.
- **(b)** Explain how the nozzle would be operating at the following back pressures: **(i)** $p_{\text{rec}} = p_1$; **(ii)** $p_{\text{rec}} = 0.990p_1$; **(iii)** $p_{\text{rec}} = 0.53p_1$; **(iv)** $p_{\text{rec}} = 0.03p_1$.

6.18. Draw a detailed T–s diagram corresponding to the *supersonic tunnel startup* condition (Figure 6.7). Identify the various stations (i.e., 1, 2, 3, etc.) in your diagram. You may assume no heat transfer and no frictional losses in the system.

6.19. Consider the wind tunnel shown in Figures 6.7 and 6.8. Atmospheric air enters the system with a pressure and temperature of 14.7 psia and 80°F, respectively, and has negligible velocity at section 1. The test section has a cross-sectional area of 1 ft^2 and operates at a Mach number of 2.5. You may assume that the diffuser reduces the velocity to approximately zero so that the final exhaust to the atmosphere has a negligible velocity. The system is fully insulated and there are negligible friction losses. Find:

- **(a)** The throat area of the nozzle.
- **(b)** The mass flow rate.
- **(c)** The minimum possible throat area of the diffuser.
- **(d)** The total pressure entering the exhauster at startup (Figure 6.7).
- **(e)** The total pressure entering the exhauster when running (Figure 6.8).
- **(f)** The hp value required for the exhauster (based on an isentropic compression).

6.20. Rework Problem **6.1** (a) and (b) for helium, water vapor, and carbon dioxide using the *Gasdynamics Calculator* and γ entries in Appendix A or B.

CHECK TEST

You should be able to complete this test without reference to material in the chapter.

6.1. Given the continuity, energy, and momentum equations in a form suitable for steady one-dimensional flow, analyze a standing normal shock in an arbitrary fluid. Then simplify your results for the case of a perfect gas.

6.2. Fill in the following blanks with *increases, decreases,* or *remains constant*. Across a standing normal shock, the

- **(a)** Temperature _____
- **(b)** Stagnation pressure _____
- **(c)** Velocity _____
- **(d)** Density _____

6.3. Consider a converging–diverging nozzle with an area ratio of 3.0 and assume operation with a perfect gas ($\gamma = 1.4$). Determine the operating pressure ratios that would cause operation at the first, second, and third critical points.

6.4. Sketch a *T–s* diagram for a standing normal shock in a perfect gas. Indicate static and total pressures, static and total temperatures, and velocities (both before and after the shock).

6.5. Nitrogen flows in an insulated variable-area system with friction. The area ratio is $A_2/A_1 = 2.0$ and the static pressure ratio is $p_2/p_1 = 0.20$. The Mach number at section 2 is $M_2 = 3.0$.

(a) What is the Mach number at section 1?

(b) Is the gas flowing from 1 to 2 or from 2 to 1?

6.6. A large chamber contains air at 100 psia and 600°R. A converging–diverging nozzle with an area ratio of 2.50 is connected to the chamber and the receiver pressure is 60 psia.

(a) Determine the outlet Mach number and velocity.

(b) Find the Δs value across the shock.

(c) Draw a *T–s* diagram for the flow through the nozzle.

Chapter 7

Moving and Oblique Shocks

7.1 INTRODUCTION

In Section 4.3 we superimposed a uniform velocity on a traveling sound wave so that we could obtain a standing wave and analyze it by the use of steady flow equations. We use precisely the same technique in this chapter to compare standing and moving normal shocks. Recall that velocity superposition does not affect the *static* thermodynamic state of a fluid but does change the *stagnation* conditions (see Section 3.5).

We then superimpose a velocity *tangential* to a standing normal shock and find that this results in the formation of an *oblique shock*, one in which the wave front is at an angle of other than 90° to the approaching flow. The case of an oblique shock in a perfect gas will then be analyzed in detail, and as you might suspect, these results lend themselves to the construction of tables and charts that greatly aid problem solving. We then discuss a number of situations where oblique shocks can be found, along with an investigation of the boundary conditions that control shock formation. This chapter includes material on conical shocks and their solution and also introduces the *shock tube*.

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

7.2 OBJECTIVES

After completing this chapter successfully, you should be able to:

1. Identify the properties that remain constant and the properties that change when a uniform velocity is superimposed on another flow field.
2. Describe how *moving* normal shocks can be analyzed with the relations developed for *standing* normal shocks.
3. Explain how an oblique shock can be described by the superposition on a normal shock of a tangential flow field.
4. Sketch an oblique shock and define the *shock angle* and *deflection angle*.
5. (*Optional*) Analyze an oblique shock in a perfect gas and develop the relation among shock angle, deflection angle, and entering Mach number.
6. Describe the general results of oblique-shock analysis in terms of a diagram such as shock angle versus inlet Mach number for various deflection angles.
7. Distinguish between weak and strong shocks. Know when each might result.
8. Describe the conditions that cause a detached shock to form.
9. State what operating conditions will cause an oblique shock to form at a supersonic nozzle exit.
10. Explain the main reason why (three-dimensional) conical shocks and (two-dimensional) wedge shocks differ quantitatively.
11. Describe the function and operation of a *shock tube*.
12. Demonstrate the ability to solve typical problems involving moving normal shocks or oblique shocks (planar or conical) by the use of the appropriate equations and tables or charts.

7.3 NORMAL VELOCITY SUPERPOSITION: MOVING NORMAL SHOCKS

Let us consider a plane shock wave that is moving into a stationary fluid such as shown in Figure 7.1. Such a wave could be found traveling down a *shock tube* or could have originated from a distant explosive device in open air. In the latter case the shock travels out from the explosion point in the form of a spherical wave front. However, eventually the radius of curvature becomes so large that it may be treated as a planar wave front with little error. A typical problem might be to determine the conditions that exist after the passage of the shock front, assuming that we know the original conditions and the speed of the shock wave.

In Figure 7.1 we are on the ground viewing a normal shock that is moving to the left at a constant velocity V_s into standard sea-level air. This is an *unsteady* picture, and we seek a means to make this fit the analysis made in Chapter 6. To do this we superimpose on the entire flow field a velocity of V_s to the right. An alternative

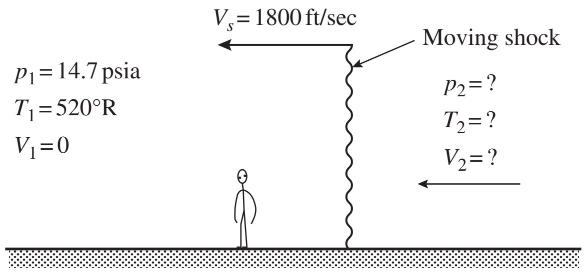


Figure 7.1 Moving normal shock with ground as reference.

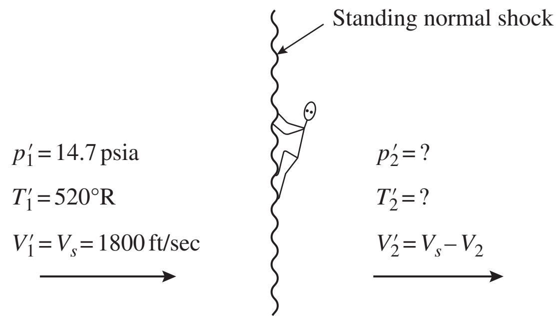


Figure 7.2 Moving shock transformed into stationary shock.

way of accomplishing the same effect is to get on the shock wave and go for a ride, as shown in Figure 7.2.

By either method, the result is to *change the frame of reference* to the shock wave, and thus it appears to be a standing normal shock.

Example 7.1
The shock is given as moving at 1800 ft/sec into air at 14.7 psia and 520°R. Solve the problem represented in Figure 7.2 by the methods developed in Chapter 6.

$$a_1' = \sqrt{\gamma g_c R T_1'} = \sqrt{(1.4)(32.2)(53.3)(520)} = 1118 \text{ ft/sec}$$

$$M_1' = \frac{V_1'}{a_1'} = \frac{1800}{1118} = 1.61$$

From the normal-shock table we find that

$$M_2' = 0.6655 \quad \frac{p_2'}{p_1'} = 2.8575 \quad \frac{T_2'}{T_1'} = 1.3949$$

Thus

$$p_2' = \frac{p_2'}{p_1'} p_1' = (2.8575)(14.7) = 42.0 \text{ psia} = p_2$$

$$T_2' = \frac{T_2'}{T_1'} T_1' = (1.3949)(520) = 725°\text{R} = T_2$$

$$a_2' = \sqrt{\gamma g_c R T_2'} = \sqrt{(1.4)(32.2)(53.3)(725)} = 1320 \text{ ft/sec} = a_2$$

$$V_2' = M_2' a_2' = (0.6655)(1320) = 878 \text{ ft/sec}$$

$$V_2 = V_s - V_2' = 1800 - 878 = 922 \text{ ft/sec} \quad (\text{or } 281 \text{ m/s})$$

Therefore, after the shock passes (referring now to Figure 7.1), the pressure and temperature will be 42 psia and 725°R, respectively, and the air will have acquired a ground velocity of 922 ft/sec to the left. It will be interesting to compute and compare the stagnation pressures in each case. Notice that they are completely different because of the change in reference that has taken place.

For Figure 7.1:

$$p_{t1} = p_1 = 14.7 \text{ psia}$$

$$M_2 = \frac{V_2}{a_2} = \frac{922}{1320} = 0.698$$

$$p_{t2} = \frac{p_{t2}}{p_2} p_2 = \left(\frac{1}{0.7222}\right)(42) = 58.2 \text{ psia} \quad (\text{or } 0.401 \text{ MPa})$$

For Figure 7.2:

$$p_{t1}' = \frac{p_{t1}'}{p_1} p_1' = \left(\frac{1}{0.2318}\right)(14.7) = 63.4 \text{ psia}$$

$$p_{t2}' = \frac{p_{t2}'}{p_2'} p_2' = \left(\frac{1}{0.7430}\right)(42) = 56.5 \text{ psia} \quad (\text{or } 0.389 \text{ MPa})$$

For the steady flow picture, $p_{t2}' < p_{t1}'$ as expected. However, note that this decrease in stagnation pressure does not occur for the unsteady case. You might compute the stagnation temperatures on each side of the shock for the unsteady and steady flow cases. Would you expect $T_{t2} = T_{t1}$? How about T_{t1}' and T_{t2}'?

Another type of moving shock is illustrated in Figure 7.3, where air is flowing through a duct under known conditions and a downstream valve is suddenly

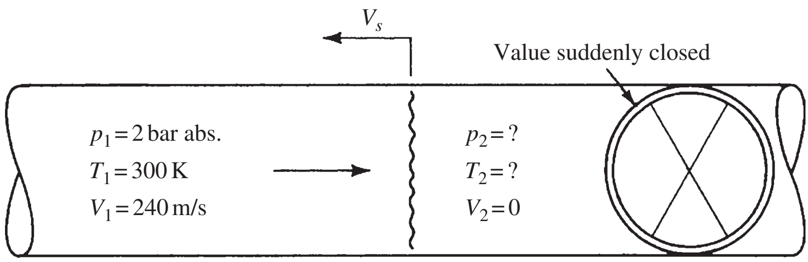


Figure 7.3 Moving normal shock in duct.

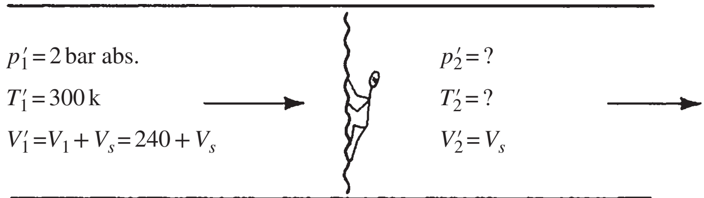


Figure 7.4 Moving shock transformed into stationary shock.

closed. The fluid is compressed as it is quickly brought to rest. This results in a shock wave propagating back through the duct as shown. In this case the problem is not only to determine the conditions that exist after passage of the shock but also to predict the speed of the shock wave.

This can also be viewed as the *reflection* of a shock wave, similar to what happens at the closed end of a *shock tube*. Our procedure is exactly the same as before. We hop on the shock wave and with this new frame of reference we have the standing normal-shock problem shown in Figure 7.4. (We have merely superimposed the constant velocity V_s on the entire flow field.) Solution to this problem, however, is not as straightforward as in Example 7.1 for the reason that the velocity of the shock wave is unknown. Since V_s is unknown, V_1' is unknown and M_1' cannot be calculated. We could approach this as a trial-and-error problem, but a direct solution is available to us. Recall the relation for the velocity difference across a normal shock that was developed in Chapter 6 [equation (6.32)]. Applied to Figure 7.4, this becomes

$$\frac{V_1' - V_2'}{a_1'} = \left(\frac{2}{\gamma + 1}\right)\left(\frac{M_1'^2 - 1}{M_1'}\right) \tag{7.1}$$

Example 7.2
Solve for V_s with the information given in Figure 7.4.

$$a_1' = \left(\gamma g_c R T_1'\right)^{1/2} = [(1.4)(1)(287)(300)]^{1/2} = 347\,\mathrm{m/s}$$

$$\frac{V_1' - V_2'}{a_1'} = \frac{240}{347} = 0.6916$$

Entering 0.6916 for $\Delta V/a_1$ into the normal-shock table (Appendix H), we see that

$$M_1' \approx 1.5, \quad M_2' = 0.7011, \quad T_2'/T_1' = 1.3202, \quad \text{and} \quad p_2'/p_1' = 2.4583.$$

$$p_2' = (2.4583)(2) = 4.92 \text{ bar abs.} = p_2$$

$$T_2' = (1.3202)(300) = 396 \text{ K} = T_2$$

$$a_2' = \sqrt{(1.4)(1)(287)(396)} = 399 \text{ m/s}$$

$$V_2' = M_2' a_2' = (0.7011)(399) = 280 \text{ m/s} \,(\text{or}\, 918.68\,\text{ft/sec}) = V_s$$

Do not forget that the *static* temperatures and pressures obtained in solutions to problems of this type are the desired answers to the original problem but that the velocities and Mach numbers for the standing-shock problem are *not* the same as those in the original moving-shock problem.

7.4 TANGENTIAL VELOCITY SUPERPOSITION: OBLIQUE SHOCKS

We now reconsider the standing normal shock as shown in Figure 7.5. To emphasize the fact that these velocities are normal to the shock front, we have labeled them V_{1n} and V_{2n}. Recall that the velocity is decreased as the fluid passes through a

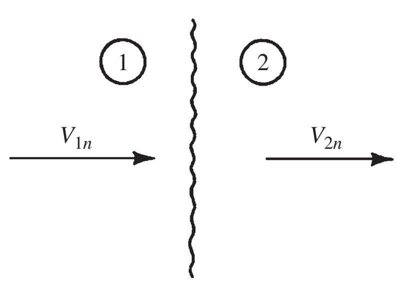


Figure 7.5 Standing normal shock.

shock wave, and thus $V_{1n} > V_{2n}$. Also remember that for this type of shock, V_{1n} must always be supersonic and V_{2n} is always subsonic.

Now let us superimpose on the entire flow field a velocity of magnitude V_t, which is *perpendicular* to V_{1n} and V_{2n}. This is equivalent to running *along* the shock front at a speed of V_t. The resulting picture is shown in Figure 7.6. As before, we realize that constant velocity superposition does not affect the *static* states of the fluid. What does change?

We would prefer to view this picture in a slightly different manner. If we concentrate on the total velocity (rather than its components) by making it horizontal, we see the flow as illustrated in Figure 7.7 and immediately notice several things:

1. The shock is no longer normal to the approaching flow vector; hence it is called an *oblique shock*.
2. The downstream flow direction has been deflected *away* from its original one.
3. V_1 must still be supersonic.
4. V_2 could also be supersonic (if V_t is large enough).

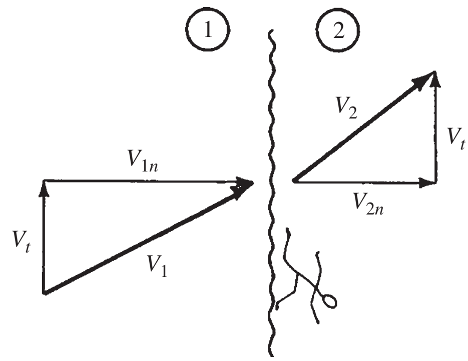


Figure 7.6 Standing normal shock plus tangential velocity.

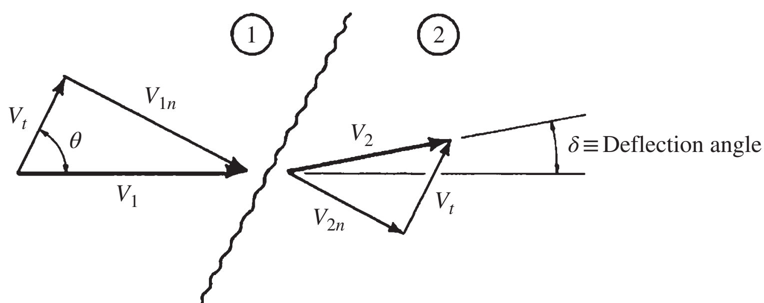


Figure 7.7 Oblique shock with θ- and δ-angle definitions.

We define the *shock angle* θ as the acute angle between the approaching flow (V_1) and the shock front. The *deflection angle* δ is the angle through which the flow velocity vector has been deflected upward from the horizontal approach.

Viewing the oblique shock in this way, as a combination of a normal shock and a tangential velocity, permits one to use the normal-shock equations and table to solve oblique-shock problems for perfect gases provided that proper care is taken.

$$V_{1n} = V_1 \sin\theta \tag{7.2}$$

Since sonic speed is a function of static temperature only,

$$a_{1n} = a_1 \tag{7.3}$$

Dividing (7.2) by (7.3), we have

$$\frac{V_{1n}}{a_{1n}} = \frac{V_1 \sin\theta}{a_1} \tag{7.4}$$

or

$$\boxed{M_{1n} = M_1 \sin\theta} \tag{7.5}$$

Thus, if we know the approaching Mach number (M_1) and the shock angle (θ), the normal-shock equations and table can be utilized by using the *normal Mach number* (M_{1n}). This procedure can be used to obtain *static* temperature and pressure changes across the shock because these are unaltered by the superposition of V_t on the original normal-shock picture.

Let us now investigate the range of possible shock angles that may exist for a given Mach number. We know that for a shock to exist,

$$M_{1n} \geq 1 \tag{7.6}$$

Thus

$$M_1 \sin\theta \geq 1 \tag{7.7}$$

and the minimum θ will occur when $M_1 \sin \theta = 1$, or

$$\theta_{\min} = \sin^{-1} \frac{1}{M_1} \tag{7.8}$$

Recall that this is the same expression that was developed for the Mach angle μ, equation (4.12). Hence *the Mach angle is the minimum possible shock angle*. Note that this only is a limiting condition and because no shock really exists since for this case, $M_{1n} = 1.0$. For this reason, these are called *Mach waves* or *Mach lines* rather than shock waves. The *maximum* value that θ can achieve is obviously 90°. This is another limiting condition and represents our now familiar normal shock.

Notice that as the shock angle θ decreases from 90° to the Mach angle μ, M_{1n} decreases from M_1 to 1.0. Since the strength of a shock is dependent on the *normal* Mach number, we have the means to produce a shock of *any strength* equal to or less than the normal shock. Do you see here any possible application of this information for the case of a converging–diverging nozzle with an operating pressure ratio someplace between the second and third critical points? We shall return to this thought in Section 7.8.

The following example is presented to provide a better understanding of the correlation between oblique and normal shocks.

Example 7.3

With the information shown in Figure E7.3*a*, we proceed to compute the conditions following the shock.

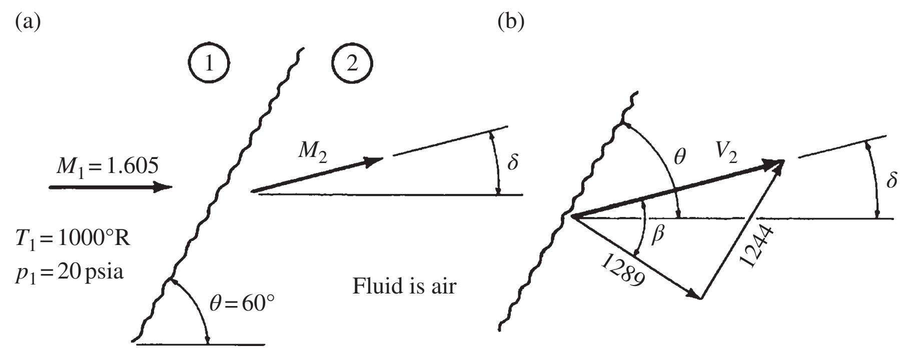


Figure E7.3

$$
\begin{aligned}
a_1 &= (\gamma g_c R T_1)^{1/2} && = [(1.4)(32.2)(53.3)(1000)]^{1/2} && = 1550 \text{ ft/sec} \\
V_1 &= M_1 a_1 && = (1.605)(1550) && = 2488 \text{ ft/sec} \\
M_{1n} &= M_1 \sin\theta && = 1.605 \sin 60° && = 1.39 \\
V_{1n} &= M_{1n} a_1 && = (1.39)(1550) && = 2155 \text{ ft/sec} \\
V_t &= V_1 \cos\theta && = 2488 \cos 60° && = 1244 \text{ ft/sec}
\end{aligned}
$$

Using information from the normal-shock table at $M_{1n} = 1.39$, we find that $M_{2n} = 0.7440$, $T_2/T_1 = 1.2483$, $p_2/p_1 = 2.0875$, and $p_{t2}/p_{t1} = 0.9607$. Remember that the static temperatures and pressures are the same whether we are talking about the normal shock or the oblique shock.

$$
\begin{aligned}
p_2 &= \frac{p_2}{p_1} p_1 && = (2.0875)(20) && = 41.7 \text{ psia} \\
T_2 &= \frac{T_2}{T_1} T_1 && = (1.2483)(1000) && = 1248°\text{R} \\
a_2 &= (\gamma g_c R T_2)^{1/2} && = [(1.4)(32.2)(53.3)(1248)]^{1/2} && = 1732 \text{ ft/sec} \\
V_{2n} &= M_{2n} a_2 && = (0.7440)(1732) && = 1289 \text{ ft/sec} \\
V_2 &= V_{1t} && = V_t && = 1244 \text{ ft/sec} \\
V_2 &= \left[(V_{2n})^2 + (V_{2t})^2\right]^{1/2} && = \left[(1289)^2 + (1244)^2\right]^{1/2} && = 1791 \text{ ft/sec} \\
M_2 &= \frac{V_2}{a_2} && = \frac{1791}{1732} && = 1.034
\end{aligned}
$$

Note that although the *normal component* is subsonic after the shock, the total velocity after the shock is supersonic in this case.

We now calculate the deflection angle (Figure E7.3*b*).

$$\tan\beta = \frac{1244}{1289} = 0.9651 \quad \beta = 44°$$

$$90 - \theta = \beta - \delta$$

Thus

$$\delta = \theta - 90 + \beta = 60 - 90 + 44 = 14°$$

Once δ is known, an alternative calculation for M_2 using equation (7.5) would be

$$\boxed{M_2 = \frac{M_{2n}}{\sin(\theta - \delta)}} \tag{7.5a}$$

$$M_2 = \frac{0.7440}{\sin(60 - 14)} = 1.034$$

Example 7.4

For the conditions in Example 7.3, compute the stagnation pressures and temperatures.

$$p_{t1} = \frac{p_{t1}}{p_1} p_1 = \left(\frac{1}{0.2335}\right)(20) = 85.7 \text{ psia} \quad (\text{or } 0.591\,\text{MPa})$$

$$p_{t2} = \frac{p_{t2}}{p_2} p_2 = \left(\frac{1}{0.5075}\right)(41.7) = 82.2 \text{ psia} \quad (\text{or } 0.567\,\text{MPa})$$

If we looked at the normal-shock problem and computed stagnation pressures on the basis of the *normal* Mach numbers, we would have

$$p_{t1n} = \left(\frac{p_{t1}}{p_1}\right)_n p_1 = \left(\frac{1}{0.3187}\right)(20) = 62.8 \text{ psia}$$

$$p_{t2n} = \left(\frac{p_{t2}}{p_2}\right)_n p_2 = \left(\frac{1}{0.6925}\right)(41.7) = 60.2 \text{ psia}$$

We now proceed to calculate the stagnation temperatures and show that for the actual oblique-shock problem, $T_t = 1515°\text{R}$, and for the normal-shock problem, $T_t = 1386°\text{R}$. All of these static and stagnation pressures and temperatures are shown in the *T–s* diagram of Figure E7.4. This clearly shows the effect of superimposing the tangential velocity on top of the normal-shock problem with the corresponding change in stagnation reference. It is interesting to note that the *ratio* of stagnation pressures is the same whether figured from the oblique-shock problem or the normal-shock problem.

$$\frac{p_{t2}}{p_1} = \frac{82.2}{85.7} = 0.959 \quad \frac{p_{t2n}}{p_{t1n}} = \frac{60.2}{62.8} = 0.959$$

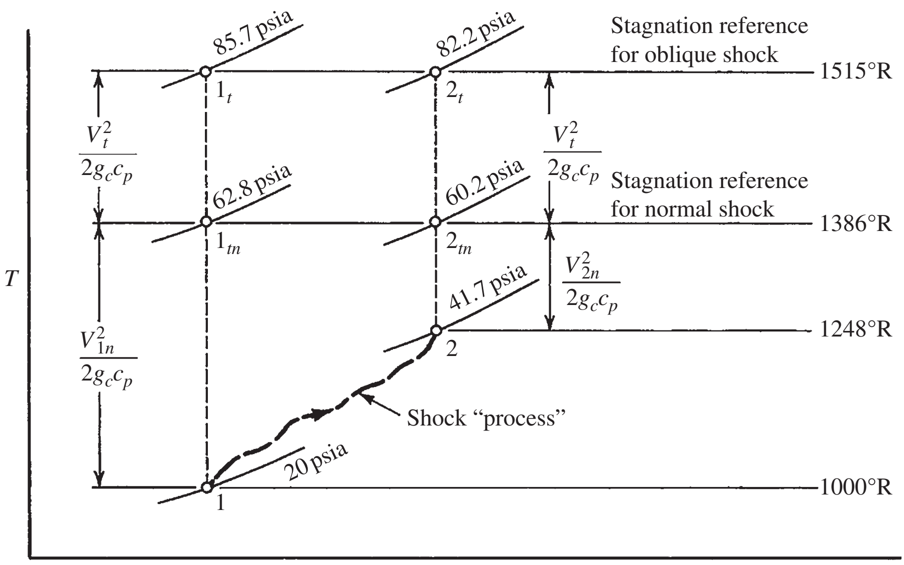


Figure E7.4 *T–s* diagram for oblique shock (showing the included normal shock).

Is this a coincidence? No! Remember that the stagnation pressure *ratio* is a measure of the loss across the shock. Superposition of a tangential velocity onto a normal shock does not affect the actual shock process, so the losses remain the same. Thus, although one cannot use the stagnation pressures from the normal-shock problem, one can use the stagnation pressure *ratio* (which is listed in the tables). Be careful! These conclusions do *not* apply to the moving normal shock, which was discussed in Section 7.3.

7.5 OBLIQUE-SHOCK ANALYSIS: PERFECT GAS

In Section 7.4 we saw how an oblique shock could be viewed as the result of a normal shock with a tangential velocity. If the initial conditions *and the shock angle* are known, the problem can be solved through application of the normal-shock equations and table. Frequently, however, the shock angle is *not* known and thus we seek a new approach to the problem. The oblique shock with its components and angles is shown again in Figure 7.8.

Our objective will be to relate the deflection angle (δ) to the shock angle (θ) and the entering Mach number. We start by applying the continuity equation to a unit area at the shock:

$$\rho_1 V_{1n} = \rho_2 V_{2n} \tag{7.9}$$

or

$$\frac{\rho_2}{\rho_1} = \frac{V_{1n}}{V_{2n}} \tag{7.10}$$

From Figure 7.8 we see that

$$V_{1n} = V_t \tan\theta \quad \text{and} \quad V_{2n} = V_t \tan(\theta - \delta) \tag{7.11}$$

Thus, from equations (7.10) and (7.11),

$$\frac{\rho_2}{\rho_1} = \frac{V_{1n}}{V_{2n}} = \frac{V_t \tan\theta}{V_t \tan(\theta-\delta)} = \frac{\tan\theta}{\tan(\theta-\delta)} \tag{7.12}$$

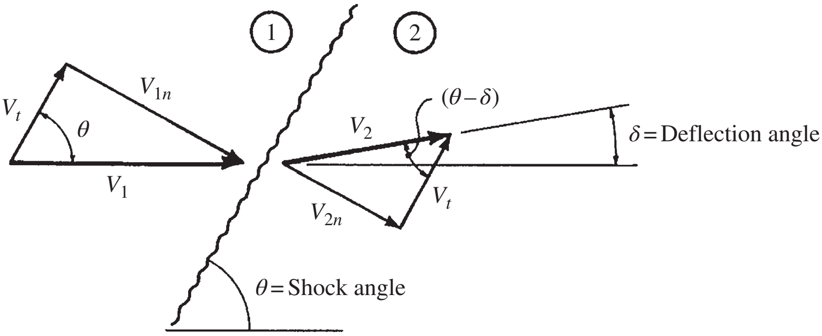


Figure 7.8 Oblique shock.

From the normal-shock relations that we derived in Chapter 6, property ratios across the shock were developed as a function of the approaching (normal) Mach number. Specifically, the density ratio was given in equation (6.26) as

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\,M_{1n}^2}{(\gamma-1)\,M_{1n}^2+2} \tag{6.26}$$

Note that we have added subscripts to the Mach numbers to indicate that these are normal to the shock. Equating (7.12) and (6.26) yields

$$\frac{\tan\theta}{\tan(\theta-\delta)} = \frac{(\gamma+1)\,M_{1n}^2}{(\gamma-1)\,M_{1n}^2+2} \tag{7.13}$$

But

$$M_{1n} = M_1\sin\theta \tag{7.5}$$

Hence

$$\frac{\tan\theta}{\tan(\theta-\delta)} = \frac{(\gamma+1)\,M_1^2\sin^2\theta}{(\gamma-1)\,M_1^2\sin^2\theta+2} \tag{7.14}$$

and we have succeeded in relating the shock angle, deflection angle, and entering Mach number. Unfortunately, equation (7.14) cannot be solved for θ as an explicit function of M, δ, and γ, but we can obtain an explicit solution for

$$\delta = f(M,\theta,\gamma)$$

which is

$$\boxed{\tan\delta = 2(\cot\theta)\left(\frac{M_1^2\sin^2\theta-1}{M_1^2(\gamma+\cos 2\theta)+2}\right)} \tag{7.15}$$

It is interesting to examine equation (7.15) for the extreme values of θ that might accompany any given Mach number.

For $\theta = \theta_{max} = \pi/2$ equation (7.15) yields $\tan\delta = 0$, or $\delta = 0$, which we know to be true for the normal shock.

For $\theta = \theta_{min} = \sin^{-1}(1/M_1)$ equation (7.15) again yields $\tan\delta = 0$ or $\delta = 0$, which we know to be true for the limiting case of the Mach wave or no shock. Thus the relationship developed for the oblique shock includes as special cases the strongest shock possible (normal shock) and the weakest shock possible (Mach wave) as

well as all other intermediate-strength shocks. Note that for the given deflection angle of $\delta = 0°$, there are two possible shock angles for any given Mach number. In the next section we see that this holds true for any deflection angle.

7.6 OBLIQUE-SHOCK TABLE AND CHARTS

Equations (7.14) and (7.15) provide relationships among the shock angle, deflection angle, and entering Mach number. Our motivation to obtain this relationship was to solve problems in which the shock angle (θ) is the unknown, but we found that an explicit solution $\theta = f(M, \delta, \gamma)$ is not possible. The next best thing is to plot equation (7.14) or (7.15). This can be done in several ways, but it is perhaps most instructive to look at a plot of shock angle (θ) versus entering Mach number (M_1) for various deflection angles (δ). This is shown in Figure 7.9.

One can quickly visualize from the figure all possible shocks for any entering Mach number. For example, the dashed vertical line at any arbitrary Mach number starts at the top of the plot with the normal shock ($\theta = 90°$, $\delta = 0°$), which is the strongest possible shock. As we move downward, the shock angle decreases continually to $\theta_{min} = \mu$ (Mach angle), which means that the shock strength is decreasing continually. Why is this so? What is the *normal Mach number* doing as we move down this vertical line?

It is interesting to note that as the shock angle decreases, the deflection angle at first increases from $\delta = 0$ to $\delta = \delta_{max}$, and then the deflection angle decreases back to zero. Thus for any given Mach number and deflection angle, two shock

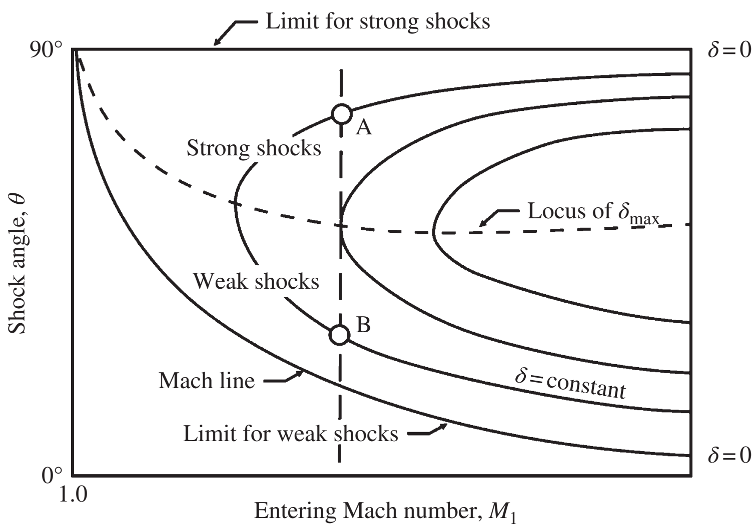


Figure 7.9 Skeletal oblique shock relations among θ, M_1, and δ. (See Appendix D for several detailed working charts.)

situations are possible (assuming that $\delta < \delta_{max}$). Two such points are labeled A and B. One of these (A) is associated with a higher shock angle and thus has a higher normal Mach number, which means that it is a stronger shock with a resulting higher pressure ratio. The other (B) has a lower shock angle and thus is a weaker shock with a lower pressure rise across the shock.

All of the *strong shocks* (above the δ_{max} points) result in *subsonic flow* after passage through the shock wave. In general, nearly all the region of *weak shocks* (below δ_{max}) result in *supersonic flow* after the shock, although there is a very small region just below δ_{max} where M_2 is still subsonic. This is clearly shown on the *detailed working chart* in Figure AD.1. In practice, we find the weak shock solution occurring more frequently, although this is entirely dependent on the boundary conditions that are imposed. This issue, along with several applications of oblique shocks, is the subject of the next two sections. In many problems, explicit knowledge of the shock angle θ is not necessary. In Appendix D you will find two additional charts which may be helpful. The first of these depicts the Mach number after the oblique shock M_2 as a function of M_1 and δ (Figure AD.2). The second shows the static pressure ratio across the shock p_2/p_1 as a function of M_1 and δ (Figure AD.3). One can also use detailed oblique-shock tables such as those by Keenan and Kaye (Ref. 31). Another possibility is to use equation (7.15) with standard computer software as discussed in Section 7.11—the *Gasdynamics Calculator* has a section on oblique shocks. The use of equation (7.15) yields higher accuracies, which are essential for some problems.

Example 7.5

Observation of an oblique shock in air (Figure E7.5) reveals that a Mach 2.2 flow at 550 K and 2 bar abs. is deflected by 14°. What are the conditions after the shock? Assume that the weak solution prevails.

We enter the chart (in Appendix D) with $M_1 = 2.2$ and $\delta = 14°$ and we find that $\theta = 40°$ and 83°. Knowing that the weak solution exists, we select $\theta = 40°$.

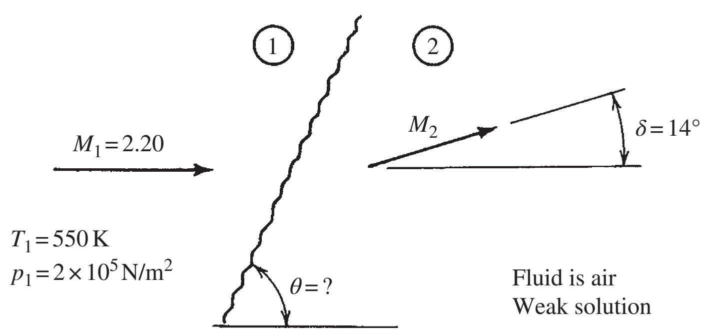


Figure E7.5

$$M_{1n} = M_1 \sin\theta = 2.2 \sin 40^\circ = 1.414$$

Enter the normal-shock table at M_{1n} = 1.414 and interpolate:

$$M_{2n} = 0.7339 \quad \frac{T_2}{T_1} = 1.2638 \quad \frac{p_2}{p_1} = 2.1660$$

$$T_2 = \frac{T_2}{T_2} T_2 = (1.2638)(550) = 695 \text{ K} \quad (\text{or } 1251^\circ\text{R})$$

$$p_2 = \frac{p_2}{p_1} p_1 = (2.166)\left(2 \times 10^5\right) = 4.33 \times 10^5 \text{ N/m}^2 \quad (\text{or } 62.82\,\text{psia})$$

$$M_2 = \frac{M_{2n}}{\sin(\theta - \delta)} = \frac{0.7339}{\sin(40 - 14)} = 1.674$$

We could have found M_2 and p_2 more directly using the other charts in Appendix D. Using the value of $M_{2n} \approx 0.75$ in Appendix H or using Figure AD.3, p_2 is found as

$$p_2 = \frac{p_2}{p_1} p_1 \approx (2)\left(2 \times 10^5\right) = 4 \times 10^5 \text{ N/m}^2 \quad (\text{or } 58.0\,\text{psia})$$

7.7 BOUNDARY CONDITION OF FLOW DIRECTION

We have seen that one of the characteristics of oblique shocks is that they change the flow direction. In fact, this is *one of only two methods* by which a supersonic flow direction can be changed. (The other method is discussed in Chapter 8.) Consider supersonic flow over a wedge-shaped object as shown in Figure 7.10. This figure, for example, could represent the leading edge of a supersonic airfoil. In this case, the flow is forced to change direction to *meet the boundary condition of flow*

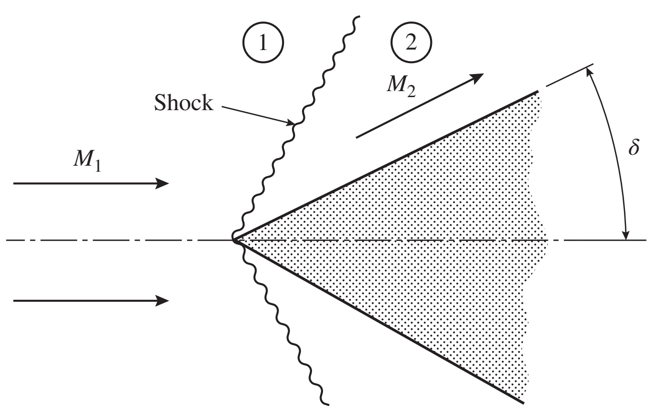


Figure 7.10 Supersonic flow over a wedge.

tangency along the wall, and this can be done only through the mechanism of an oblique shock. The example in Section 7.6 was just such a situation. (Recall that a flow of $M = 2.2$ was deflected by 14°.) Now, for any given Mach number and deflection angle there are two possible shock angles. Thus a question naturally arises as to which solution will occur, the *strong* one or the *weak* one. Here is where the surrounding gas pressure must be considered. Recall that the strong shock occurs at the higher shock angle and results in a large pressure change. For this solution to occur, a physical situation must exist that can sustain the necessary pressure differential. It is conceivable that such a case might exist in an *internal* flow situation. However, for *external* flow situations such as around the airfoil, no means are available to support the greater pressure difference required by the strong shock. Thus, for external flow problems (flow around objects), we always find the weak shock solution.

Looking back at Figure 7.9, you may notice that there is a maximum deflection angle (δ_{max}) associated with any given Mach number. Does this mean that the flow cannot turn through an angle greater than this? This is true if we limit ourselves to the simple oblique shock that is *attached* to the object as shown in Figure 7.10. But what happens if we build a wedge with a half angle greater than δ_{max}? Or suppose we ask the flow to pass over a blunt object? The resulting flow pattern is shown in Figure 7.11.

Here a *detached shock* is seen to form which has a curved wave front. Behind this wave we find all possible shock solutions associated with the initial Mach number M_1. At the center a normal shock exists, with subsonic flow resulting.

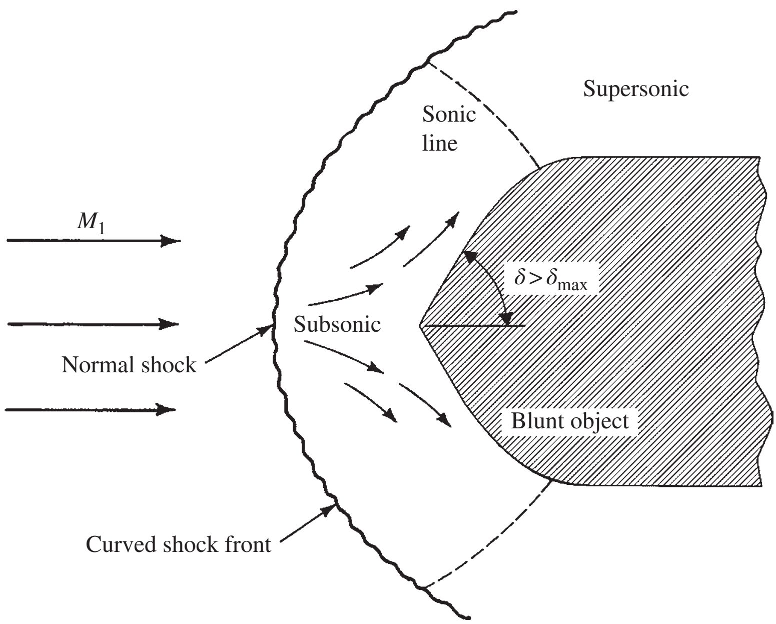


Figure 7.11 Detached shock caused by $\delta > \delta_{max}$.

Subsonic flow has no difficulty adjusting to the large deflection angle required. As the wave front curves around, the shock angle decreases continually, with a resultant decrease in shock strength. Eventually, we reach a point where supersonic flow may exist after the shock front. Although Figures 7.10 and 7.11 illustrate external flow over objects, the same patterns result from internal flow along a wall, or *corner flow,* shown in Figure 7.12. The significance of δ_{max} is again seen to be the maximum deflection angle for which the shock can remain *attached* to the turning corner.

A very practical situation involving a detached shock is caused when a pitot tube is installed in a supersonic tunnel (see Figure 7.13). The tube will reflect the total pressure after the shock front, which at this location is a normal shock. An additional tap off the side of the tunnel can pick up the static pressure ahead of the shock. Consider the ratio

$$\frac{p_{t2}}{p_1} = \frac{p_{t2}}{p_{t1}}\frac{p_{t1}}{p_1}$$

p_{t2}/p_{t1} is the total pressure ratio across the shock and is a function of M_1 only [see equation (6.28)]. p_{t2}/p_1 is also a function of M_1 only [see equation (5.40)]. Thus the ratio p_{t2}/p_1 is only a function of the initial Mach number (and γ) and can be found as a parameter shown in the normal-shock table (Appendix H).

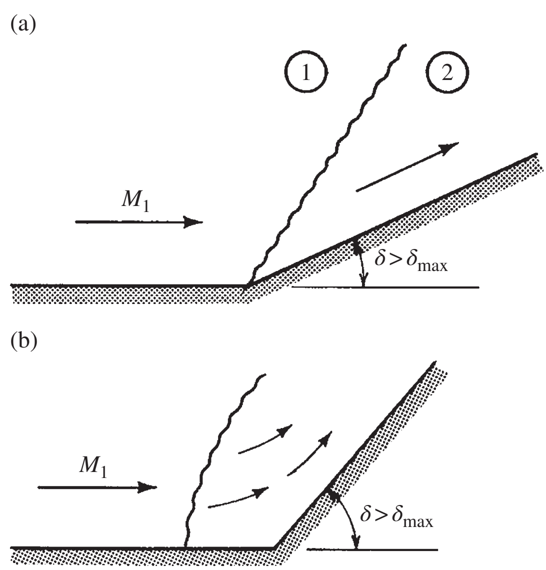


Figure 7.12 Supersonic flow in a corner.

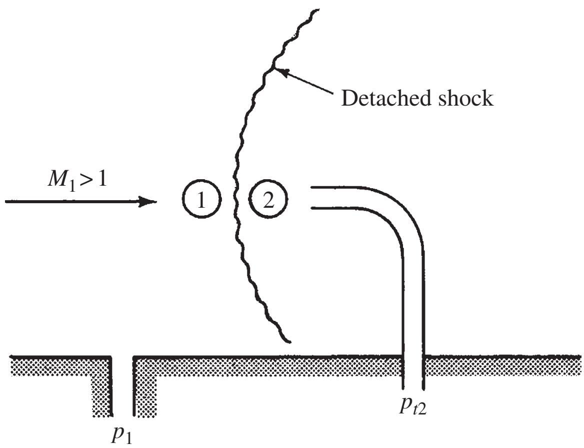


Figure 7.13 Supersonic pitot tube installation.

Example 7.6
A supersonic pitot tube indicates a total pressure of 30 psig and a static pressure of zero gage. Determine the free-stream velocity if the temperature of the air is 450°R.

$$\frac{p_{t2}}{p_1} = \frac{30 + 14.7}{0 + 14.7} = \frac{44.7}{14.7} = 3.041$$

From the shock table we find that $M_1 = 1.398$.

$$a_1 = [(1.4)(32.2)(53.3)(450)]^{1/2} = 1040 \text{ ft/sec}$$

$$V_1 = M_1 a_1 = (1.398)(1040) = 1454 \text{ ft/sec} \quad (\text{or } 443.2\,\text{m/s})$$

So far, we have discussed oblique shocks that are caused by flow deflections. Another case of this event is found in engine inlets of supersonic aircraft. Figure 7.14 shows a sketch of an aircraft that is an excellent example of this situation. As aircraft and missile speeds increase, we usually see two directional changes with their accompanying shock systems, as shown in Figure 7.15. The losses that occur across the series of shocks shown are less than those which would occur across a single normal shock at the same incoming Mach number. A warning must be given here concerning the application of our results to inlets with circular cross sections. These will have conical *spikes* for flow deflection, which cause *conical-shock* fronts to form. This type of shock has also been analyzed and is covered in Section 7.9. The design of supersonic diffusers for propulsion systems is discussed further in Chapter 12.

In problems such as the multiple-shock inlet and the supersonic airfoil, we are generally not interested in the shock angle itself but are concerned with the resulting Mach numbers and pressures downstream of each oblique shock. As already mentioned, for flows that are *not* conical the charts in Appendix D show these exact variables as a function of M_1 and the turning angle δ. The stagnation pressure ratio can be inferred from these using the proper relations. Alternatively, you can use the *Gasdynamics Calculator*.

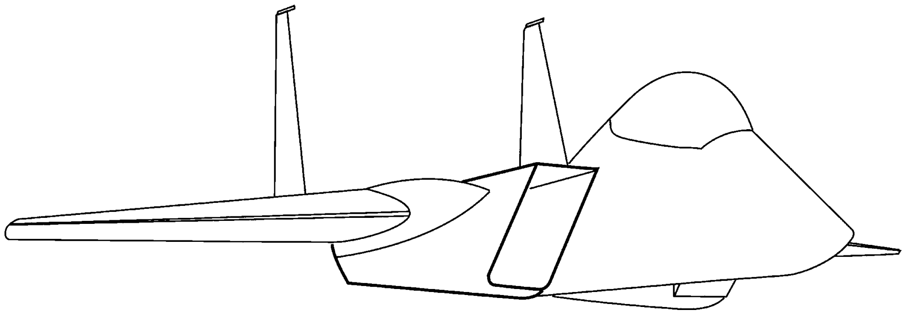


Figure 7.14 Sketch of a rectangular engine inlet. [Similar images can be found by searching the web for: “aircraft engine inlets rectangular images”]

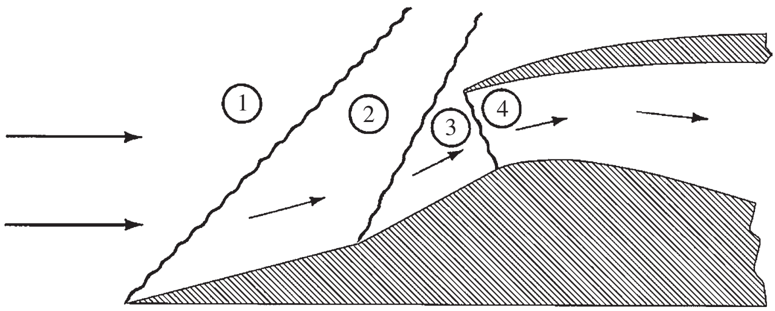


Figure 7.15 Multiple-shock inlet for supersonic aircraft.

7.8 BOUNDARY CONDITION OF PRESSURE EQUILIBRIUM

Now let us consider a case where the existing pressure conditions cause an oblique shock to form. Recall our friend the converging–diverging nozzle. When it is operating at its second critical point, a normal shock is located at the exit plane. The pressure rise that occurs across this shock is exactly that which is required to go from the low pressure that exists within the nozzle up to the higher receiver pressure that has been imposed on the system. We again emphasize that *the existing operating pressure ratio is what causes the shock to be located at this particular position*. (If you have forgotten these details, review Section 6.6.)

We now ask: What happens when the operating pressure ratio is between the second and third critical points? A normal shock is too strong to meet the required pressure rise. What is needed is a compression process that is weaker than a normal shock, and our oblique shock is precisely the mechanism for the job. *No matter what pressure rise is required,* the shock can form at an angle that will produce any desired pressure rise from that of a normal shock on down to the third critical condition, which requires no pressure change. Figure 7.16 shows a typical weak

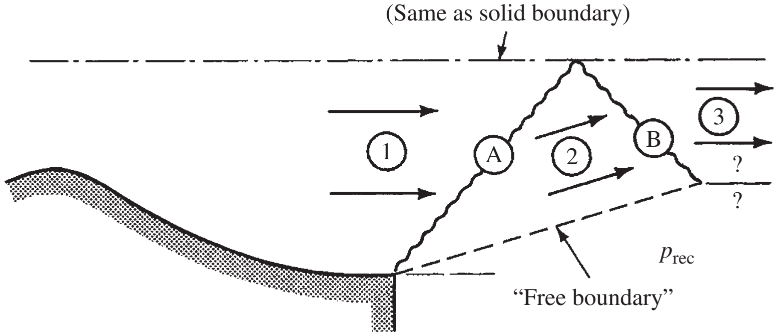


Figure 7.16 Supersonic nozzle operating between second and third critical points.

oblique shock at the lip of a two-dimensional nozzle. We have shown only half the picture, as symmetry considerations force the upper half to be the same. This also permits an alternative viewpoint—thinking of the central streamline as though it were a solid boundary.

The flow in region 1 is parallel to the centerline and as per the design conditions for the nozzle (i.e., the flow is supersonic and $p_1 < p_{rec}$). The weak oblique shock A forms at the appropriate angle such that the pressure rise that occurs is just sufficient to meet the boundary condition of $p_2 = p_{rec}$. There is a *free boundary* between the jet and the surroundings as opposed to a *physical boundary*. Now remember that the flow is also turned away from the normal and thus will have the direction indicated in region 2.

This presents a problem since the flow in region 2 cannot cross the centerline. Something must occur where wave A meets the centerline, and that something must turn the flow parallel to the centerline. Here it is the boundary condition of flow direction that causes another oblique shock B to form, which not only changes the flow direction but also increases the pressure still further. Since $p_2 = p_{rec}$ and $p_3 > p_2$, $p_3 > p_{rec}$ and pressure equilibrium does not exist between region 3 and the receiver.

Obviously, some type of an expansion is needed which emanates from the point where wave B intersects the free boundary. An *expansion shock* would be just the thing, but we know that such an animal cannot exist. Do you recall why not? We shall have to study another phenomenon before we can complete the story of a supersonic nozzle operating between the second and third critical points, and we do that in Chapter 8.

Example 7.7

A converging–diverging nozzle (Figure E7.7) with an area ratio of 5.9 is fed by air from a chamber with a stagnation pressure of 100 psia. Exhaust is to the atmosphere at 14.7 psia. Show that this nozzle is operating between the second and third critical points and determine the conditions after the first shock.

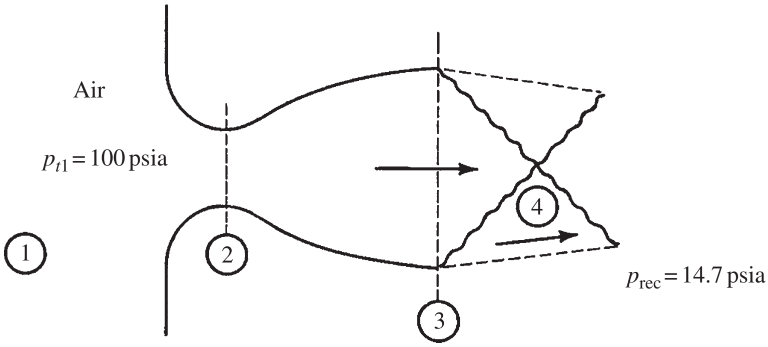


Figure E7.7

Third critical:

$$\frac{A_3}{A_3^*} = \frac{A_3}{A_2}\frac{A_2}{A_2^*}\frac{A_2^*}{A_3^*} = (5.9)(1)(1) = 5.9$$

$$M_3 = 3.35 \text{ and } \frac{p_3}{p_{t3}} = 0.01625$$

$$\frac{p_3}{p_{t1}} = \frac{p_3}{p_{t3}}\frac{p_{t3}}{p_{t1}} = (0.01625)(1) = 0.01625$$

Second critical: normal shock at

$$M_3 = 3.35 \quad \text{and} \quad \frac{p_4}{p_3} = 12.9263$$

$$\frac{p_4}{p_{t1}} = \frac{p_4}{p_3}\frac{p_3}{p_{t1}} = (12.9263)(0.01625) = 0.2100$$

Since our operating pressure ratio (14.7/100 = 0.147) lies between that of the second and third critical points, an oblique shock must form as shown. Remember, under these conditions the nozzle itself operates internally as if it were at the third critical point. Thus the required pressure ratio across the oblique shock is

$$\frac{p_4}{p_3} = \frac{p_{\text{rec}}}{p_3} = \frac{14.7}{1.625} = 9.046$$

From the normal-shock table we see that this pressure ratio requires that M_{3n} = 2.81 and M_{4n} = 0.4875:

$$\sin\theta = \frac{M_{3n}}{M_3} = \frac{2.81}{3.35} = 0.8388 \quad \theta = 57^\circ$$

From the oblique-shock chart, δ = 34° and

$$M_4 = \frac{M_{4n}}{\sin(\theta - \delta)} = \frac{0.4875}{\sin(57 - 34)} = 1.25$$

Thus to match the receiver pressure, an oblique shock forms at 57°. The flow in region 4 is deflected 34° and is still supersonic at a Mach number of 1.25.

7.9 CONICAL SHOCKS

We include here the subject of conical shocks because of its practical importance in many design problems. For example, many supersonic aircraft have diffusers with conical *spikes* at their air inlets. Figure 7.17 shows the YF-12 aircraft, which is an excellent example of this case. In addition to inlets of this type, the forebodies of missiles and supersonic aircraft fuselages are largely conical in shape. Although detailed analysis of such flows is beyond the scope of this book, the results bear much similarity to flows associated with planar (wedge-generated) oblique shocks. As we examine conical flows at zero angle of attack, for the continuity equation in axisymmetric (three-dimensional) flows to be satisfied, the streamlines can no longer be parallel to the cone surface but must curve. After the conical shock, the static pressure increases as we approach the surface of the cone, and this increase is isentropic. *Conical shocks are weak shocks,* and there is no counterpart to the strong oblique shock of wedge flow. If the angle of the cone is too high for

Figure 7.17 YF-12 plane showing conical air inlets. (Lockheed Martin photo.)

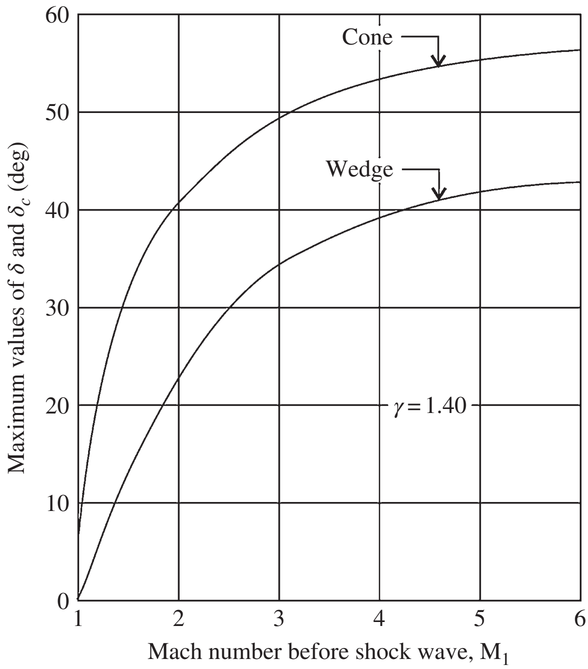


Figure 7.18 Comparison between oblique- and conical-shock flow limits for attached shocks. (Ref. 20.)

an approaching Mach number to turn, the flow will detach in a fashion similar to the two-dimensional oblique shock (see Figure 7.11). A comparison of the detached flow limits between these two types of shocks is shown in Figure 7.18. The cone can sustain a higher flow turning angle because it represents less "blockage" to the flow. Thus it also produces a weaker compression or flow disturbance in comparison to the two-dimensional oblique shock at the same Mach number. Note also that the flow variables (M, T, p, etc.) remain constant along any given ray.

In Figure 7.19 we show the relevant geometry of a conical shock on a symmetrical cone at zero angle of attack. In this section the subscript c will refer to the conical analysis and the *subscript s to the values of the variables at the cone's surface*. (Those interested in the details of conical flow away from the cone's surface should consult Ref. 32 or Ref. 33.) The counterpart to Figure 7.19 is Figure 7.20 or Figure AE.1, which shows the shock wave angle θ_c as a function of the approaching Mach number M_1 for various cone half-angles, δ_c. Notice that only weak shock solutions are indicated. In Appendix E you will find additional charts which give the downstream conditions on the surface of the cone (Figures AE.2 and AE.3). Notice also that we are only depicting the *surface* Mach number and *surface* static pressure downstream of the conical shock because these variables are not the same across the flow inside the cone.

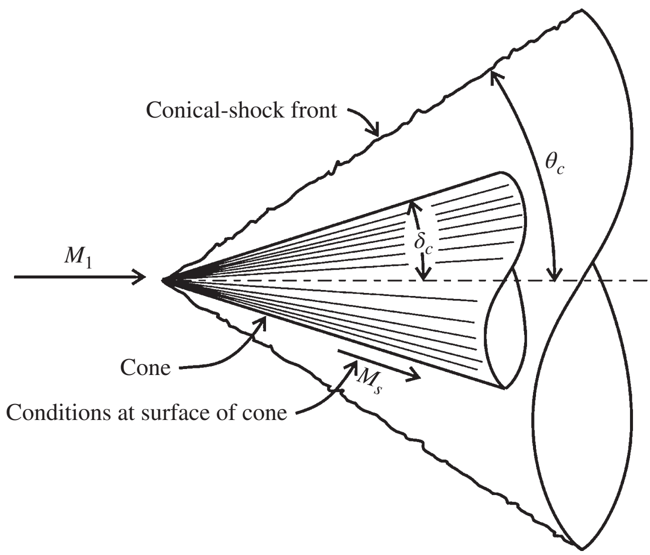


Figure 7.19 Conical shock with angle definitions

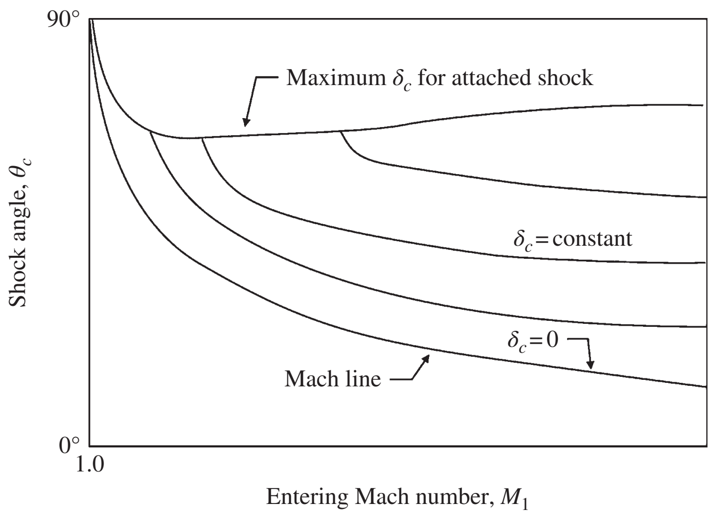


Figure 7.20 Skeletal conical-shock relations among θ_c, M_1, and δ_c. (See Appendix E for detailed working charts.)

Example 7.8
Air approaches a 27° conical diffuser at M_1 = 3.0 and p_1 = 0.404 psia. Find the conical-shock angle and the surface pressure.

We enter the chart in Appendix E with $M_1 = 3.0$ and $\delta_c = 13.5°$ and obtain $\theta_c \approx 25°$. Also from the appendix we get $p_c/p_1 \approx 1.9$, so that

$$p_c = (p_1)/(p_c/p_1) = (1.9)(0.404) = 0.768 \text{ psia} \quad (\text{or } 5.29\,\text{kPa}).$$

Unlike wedge flow, after passing through a conical shock, most of the flow is not locally one-dimensional even at zero angle of attack. With the charts in Appendix E we can only compute conditions on the surface of the cone and therefore cannot conclude anything about engine inlet performance as we can for wedge flows (see Problems 7.13 and 7.19). In reality, actual aircraft inlet engine designs are a compromise between maximizing the stagnation pressure recovery and minimizing the drag at all angles of attack.

7.10 THE SHOCK TUBE

We now return to the subject of normal-shock waves which are moving by looking briefly at an important application. Shock tubes have been a common tool for analysis of transient shock wave phenomena and, when properly instrumented, they are used in experiments to expose, for relatively short time periods, certain gases to uniform high temperatures as well as material samples to high temperatures and pressures. In Section 7.3 we discussed the topic of normal velocity superposition, which allows some analysis of normal shocks moving at constant velocity, but most other aspects of transient wave phenomena are beyond the scope of this book. However, equations (6.23) and (6.25) in Chapter 6 do represent the static relation between the temperatures and the pressures across stationary or shocks moving at constant speed in implicit form. By elimination of M_1 between them we get

$$\frac{T_2}{T_1} = \left(\frac{1 + \left(\frac{\gamma - 1}{\gamma + 1}\right)\frac{p_2}{p_1}}{1 + \left(\frac{\gamma - 1}{\gamma + 1}\right)\frac{p_1}{p_2}} \right) \tag{7.16}$$

and when linking the densities across the shock to the pressure ratio using equations (6.25) and (6.26) by eliminating M_1 we arrive at the Rankine–Hugoniot relation (try it!).

Whereas standing normal shocks relate conditions at an upstream region (1) where $M_1 > 1.0$ to those at (2) where $M_2 < 1.0$ (see Figure 6.1), shock tube description requires the insertion of intermediate regions, going from (4) the *driver gas* to (1) the *driven gas*, each at its original condition before shock initiation. Shock action results from the pressure difference $p_4 \gg p_1$ [regions (3) and (2) will be described below equation (7.17)]. The equation relating p_4 and p_1 is shown here in terms of the initial shock Mach number M_s and two values of γ as the gases in the driver and

driven sections may differ in composition (a good derivation of this equation can be found in Ref. 18). We have

$$\frac{p_4}{p_1} = \left(\frac{p_4}{p_3}\right)\left(\frac{p_3}{p_2}\right)\left(\frac{p_2}{p_1}\right) \text{ where } p_3 = p_2 \text{ and } V_3 = V_2 \text{ and } T_2 > T_1 \approx T_4 > T_3$$

need : $\dfrac{p_4}{p_3}$ isentropic expansion equation (5.40), $\dfrac{p_2}{p_1}$ shock equation (6.25), continuity equation (6.11)

$$\frac{p_4}{p_1} = \left(\frac{2\gamma_1}{\gamma_1+1}M_S^2 - \frac{\gamma_1-1}{\gamma_1+1}\right)\left[1 - \frac{\gamma_4-1}{\gamma_1+1}\left(\frac{a_1}{a_4}\right)\frac{\left(M_S^2-1\right)}{M_S}\right]^{-\frac{2\gamma_4}{\gamma_4-1}} \tag{7.17}$$

Shock tubes are mounted horizontally because they are usually long and heavy-walled, made to contain high-pressure and high-temperature gases; the higher the initial pressure differences and the longer the path that the initial shock wave can travel the better. A physical partition or diaphragm is needed to separate the high- and low-pressure regions (which may or may not contain the same gas) and this diaphragm must be rapidly pierced or ruptured in order to start the experiment. Figure 7.21 displays a shock tube, pictured as the ordinate with the time scale shown as the abscissa, and only the first set of waves are indicated. The types of gases used together with their temperatures govern their local sound speed "*a*" according to equation (4.5) or (4.10). Looking at the initial set of time wave displays in Figure 7.21, region (1) contains the *driven gas* at its original conditions; in region (2) the *driven gas* has been compressed and heated by the shock; region (3) represents the *driver gas* at a modified high pressure but at its original temperature. As shown, the *contact surface* physically separates the driver and driven gases. Lastly, at the bottom end, the *driver gas* expands isentropically from its original high pressure (4). As time increases, the isentropic expansion reflects as yet another fan from the lower end whereas the shock wave reflects as yet another shock from the upper end as shown in Figure 7.21.

Since shock tube experiments are by nature of very short duration, it is expedient to find a longest sample exposure time location for examining the behavior of gases or materials being tested under high pressures and temperatures. When the test object is positioned at location (**x**) shown in Figure 7.21, it experiences uniform conditions for a maximum time interval (usually milliseconds—but enough for measurements for appropriately small samples and with proper instrumentation), see Problem 7.22. Shock tubes can be used as wind tunnels to provide a practical means for testing gases at high enthalpies something which is of interest in hypersonic flows.

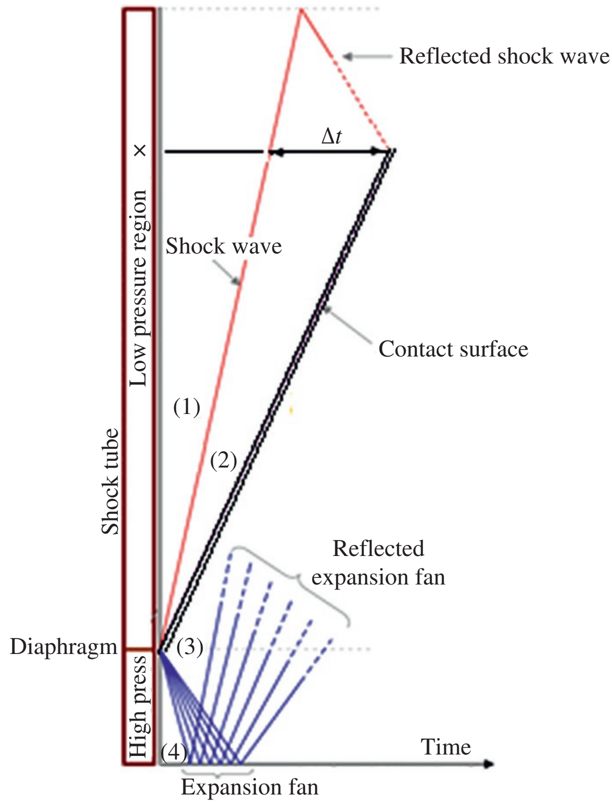


Figure 7.21 Shock tube diagram indicating time increasing to the right. The apparatus is shown vertically here to highlight its finite span. Location (**x**) denotes the shock tube test sample position with longest time increment (Δt) with uniform high-temperature and -pressure conditions. Only the first few wave interactions are of interest during most experiments. [Shock tube images can be found by searching the web for: "shock tube images"]

Below we elaborate on the different numbered regions in Figure 7.21 shortly after the diaphragm separating the high-pressure gas from the low-pressure region is burst open:

1. Region (1) is being traversed by a finite pressure pulse propagating as a normal shock toward the driven-end section of the tube. The shock speed in this region, V_s (or V_1'), can be analyzed as in Section 7.3 with M_S from equation (7.17).
2. The reflected shock wave bounding region (2) (initial pressure pulse returning from driven-shock tube end) can be analyzed as in Section 7.3, Figures 7.3 and 7.4.

3. A contact surface separates regions (2) and (3). It is not a wave but an interface that follows behind the shock wave separating the *driven gas* compressed by the shock from the *driver gas* which is being cooled by expansion. Because diffusion is relatively very slow here, gases do not mix appreciably across this surface (and if the driver and driven gases are of different composition, they remain different). Behind this contact surface, the pressure stays the same but the temperature drops.
4. An expansion fan separates regions (3) and (4). It is a region of reduced pressure propagating toward the end of the high-pressure region. In time, this expansion fan reflects back from the driver-end creating many complex wave interactions.

7.11 (OPTIONAL) BEYOND THE TABLES

As first illustrated in Chapter 5, one can eliminate a lot of interpolation and get accurate answers for any ratio of the specific heats γ and/or any Mach number by using a computer utility such as MAPLE. We return here to two-dimensional (wedge-type) oblique shocks. Since the dependence on γ is the same as that for normal shocks, we are not presenting such curves in this chapter. But one unique difficulty with oblique-shock problems is that the value of θ needs to be quite accurate, and often the charts (such as those in Appendix D) are not precise enough to permit this. Therefore, one is often motivated to solve equation (7.15) (or its equivalent) by direct means. The MAPLE program below actually works with equation (7.14), in which θ shows implicitly. The program requires the entering Mach number (M_1), the wedge half-angle (δ), and the ratio of specific heats (γ). Because there are usually two values of θ for every value of M_1 and δ, we need to introduce an index (m) to make the computer look for either the weak or the strong shock solution. Furthermore, we need to be careful because these regions are not separated by a unique value of m or θ. Moreover, there are certain δ and M combinations for which no solution exists (i.e., when the shock must detach, as shown in Figure 7.11). Beyond $M = 1.75$, the weak-shock solution is obtained with $m \leq 1.13$ (which is the radians equivalent of 65°; see the chart in Appendix D) and the strong shock solution is found with $m > 1.13$. This value has to be refined for the lower Mach numbers because there the weak shock region becomes more dominant. Note that MAPLE makes calculations with all angles in radians.

Example 7.9
For a two-dimensional oblique shock in air where $M_1 = 2.0$ and the deflection angle is 10°, calculate the two possible shock angles in degrees.

Start with equation (7.14):

$$\frac{\tan\theta}{\tan(\theta-\delta)} = \frac{(\gamma+1)\,M_1^2\sin^2\theta}{(\gamma-1)\,M_1^2\sin^2\theta+2} \tag{7.14}$$

Let

$g \equiv \gamma$, a parameter (the ratio of specific heats)
$d \equiv \delta$, a parameter (the turning angle in radians)
$X \equiv$ the independent variable (which in this case is M_1)
$Y \equiv$ the dependent variable (which in this case is θ)

Listed below are the precise inputs and program that you use in the computer.
First, the weak shock solution:

```
> g := 1.4 : x : = 2.0 :  m  : = 1.0 :
> del  : = 10*Pi/180 :
> fsolve ((tan(Y))/(tan(Y-del)) = ((g + 1)* (X* sin(Y))^2)/
  ((g-1)* ((X* sin(Y))^2) + 2), Y, 0..m);
                              .6861575526
> evalf (0.68615526* 180 / Pi);
                              39.31380048
```

Next, the strong shock solution:

```
> m := 1.5 :
> fsolve ((tan(Y))/(tan(Y-del)) = ((g + 1)* (X* sin(Y))^2)/
  ((g-1)* ((X* sin(Y))^2) + 2), Y, 0..m);
                              1.460541987
> evalf (1.46084* 180 / Pi);
                                    83.69996652
```

Since MAPLE only works with radians, we have the need to convert the answer back to degrees for both weak- and strong-shock solutions. This yields the Y unknowns in degrees (i.e., $\theta = 39.3°$ or $83.7°$) without knowing θ as an explicit function of M_1, γ, and δ.

The *Gasdynamics Calculator* is also programmed to solve equation (7.15) to the desired accuracy and may be used to solve Example 7.9 for other values of γ as well.

7.12 SUMMARY

We have seen how a standing normal shock can be made into a moving normal shock by superposition of a constant velocity (normal to the shock front) on the entire flow field. Similarly, the superposition of a constant velocity tangent to the shock front turns a normal shock into an oblique shock. Since velocity superposition does not change the static conditions in a flow fluid, the normal-shock table may be used to solve oblique-shock problems if we deal with the *normal Mach number*. However, to avoid trial-and-error solutions, oblique-shock charts and calculators are available. The following is a significant relation among the variables in an oblique shock:

$$\tan\delta = 2(\cot\theta)\left(\frac{M_1^2\sin^2\theta - 1}{M_1^2(\gamma+\cos 2\theta)+2}\right) \tag{7.15}$$

Another helpful relation is

$$M_2 = \frac{M_{2n}}{\sin(\theta-\delta)} \tag{7.5a}$$

We summarize the important characteristics of oblique shocks as follows:

1. The flow is always turned *away* from the normal.
2. For given values of M_1 and δ, two values of θ may result.
   a. If a large pressure ratio is available (or required), a strong shock at the higher θ will occur and M_2 will be subsonic.
   b. If a small pressure ratio is available (or required), a weak shock at the lower θ will occur and M_2 will be supersonic (except for a small region near δ_{max}).
3. A maximum value of δ exists for any given Mach number. If δ is physically greater than δ_{max}, a *detached* shock will form.

It is important to realize that oblique shocks are caused for two reasons:

1. To meet a physical boundary condition that causes the flow to change direction, or
2. To meet a free boundary condition of pressure equilibrium.

An alternative way of stating this is to say that the flow must be tangent to any boundary, whether it is a physical wall or a *free boundary*. If it is a free boundary, pressure equilibrium must also exist across the flow boundary.

Conical shocks (three-dimensional) are introduced as similar in nature to oblique shocks (two-dimensional) but more complicated in their solution. A discussion of the shock tube is presented since it is a common experimental tool.

PROBLEMS

7.1. A normal shock is traveling into still air (14.7 psia and 520°R) at a velocity of 1800 ft/sec.

(a) Determine the temperature, pressure, and velocity that exist after passage of the shock wave.

(b) What is the entropy change experienced by the air?

7.2. The velocity of a certain atomic explosion blast wave has been determined to be approximately 46,000 m/s relative to the ground. Assume that it is moving into still air at 300 K and 1 bar. Estimate the static and stagnation temperatures and pressures that exist after the blast wave passes. (*Hint:* You will have to resort to equations or the *Gasdynamics Calculator*, as Appendix H does not detail hypersonic Mach numbers.)

7.3. Air flows in a duct, and a valve is quickly closed. A normal shock is observed to propagate back through the duct at a speed of 1010 ft/sec. The gas that has been brought to rest has a temperature and pressure of 600°R and 30 psia, respectively. What were the original temperature, pressure, and velocity of the air flow before the valve was closed?

7.4. Oxygen at 100°F and 20 psia is flowing at 450 ft/sec in a duct. A valve is quickly shut, causing a normal shock to travel back through the duct.

(a) Determine the speed of the traveling shock wave.

(b) What are the temperature and pressure of the oxygen that is brought to rest?

7.5 A closed tube contains nitrogen at 20°C and a pressure of 1×10^4 N/m^2 (Figure P7.5). A shock wave progresses through the tube at a speed of 380 m/s.

(a) Calculate the conditions that exist immediately after the shock wave passes a given point. (The fact that this is inside a tube should not bother you, as it is merely a normal shock moving into a gas at rest.)

(b) When the shock wave hits the end wall, it is reflected back. What are the temperature and pressure of the gas between the wall and the reflected shock? At what speed is the reflected shock traveling? (This is just like the sudden closing of a valve in a duct.)

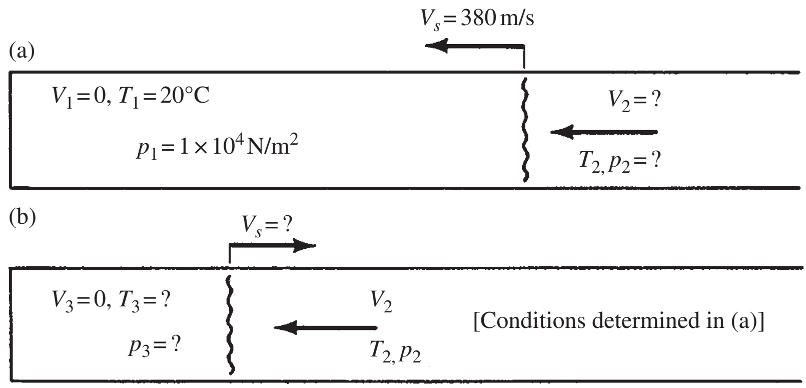


Figure P7.5

7.6. An oblique shock forms in air at an angle of $\theta = 30°$. Before passing through the shock, the air has a temperature of 60°F, a pressure of 10 psia, and is traveling at $M = 2.6$.

(a) Compute the normal and tangential velocity components before and after the shock.

(b) Determine the temperature and pressure after the shock.

(c) What is the deflection angle?

7.7. Conditions before a shock are $T_1 = 40°C$, $p_1 = 1.2$ bar, and $M_1 = 3.0$. An oblique shock is observed at 45° to the approaching air flow.

(a) Determine the Mach number and flow direction after the shock.

(b) What are the temperature and pressure after the shock?

(c) Is this a weak or a strong shock?

7.8. Air at 800°R and 15 psia is flowing at a Mach number of $M = 1.8$ and is deflected through a 15° angle. The directional change is accompanied by an oblique shock.

(a) What are the possible shock angles?

(b) For each shock angle, compute the temperature and pressure after the shock.

7.9. The supersonic flow of a gas ($\gamma = 1.4$) approaches a wedge with a half-angle of 24° ($\delta = 24°$).

(a) What Mach number will put the shock on the verge of detaching?

(b) Is this value a minimum or a maximum?

7.10. A simple wedge with a total included angle of 28° is used to measure the Mach number of supersonic flows. When inserted into a wind tunnel and aligned with the flow, oblique shocks are observed at 50° angles to the free stream (similar to Figure 7.10).

(a) What is the Mach number in the wind tunnel?

(b) Through what range of Mach numbers could this wedge be useful? (*Hint:* Would it be of any value if a detached shock were to occur?)

7.11. A pitot tube is installed in a wind tunnel in the manner shown in Figure 7.13. The tunnel air temperature is 500°R and the static tap (p_1) indicates a pressure of 14.5 psia.

(a) Determine the tunnel air velocity if the stagnation probe (p_{t2}) indicates 65 psia.

(b) Suppose that $p_{t2} = 26$ psia. What is the tunnel velocity under this condition?

7.12. A converging–diverging nozzle is designed to produce an exit Mach number of 3.0 when $\gamma = 1.4$. When operating at its second critical point, the shock angle is 90° and the deflection angle is zero. Call p_{exit} the pressure at the exit plane of the nozzle just before the shock. As the receiver pressure is lowered, both θ and δ change. For the range between the second and third critical points:

(a) Plot θ versus $p_{\text{rec}}/p_{\text{exit}}$.

(b) Plot δ versus $p_{\text{rec}}/p_{\text{exit}}$.

7.13. Pictured in Figure P7.13 is the air inlet to a jet aircraft. The plane is operating at 50,000 ft, where the pressure is 243 psia and the temperature is 392°R. Assume that the flight speed is $M_0 = 2.5$.

(a) What are the conditions of the air (temperature, pressure, and entropy change) just after it passes through the normal shock?

(b) Draw a reasonably detailed T–s diagram for the air inlet. Start the diagram at the free stream and end it at the subsonic diffuser entrance to the compressor.

(c) If the single 15° wedge is replaced by a double wedge of 7° and 8° (see Figure 7.15), determine the conditions of the air after it enters the diffuser.

(d) Compare the losses for parts (a) and (c).

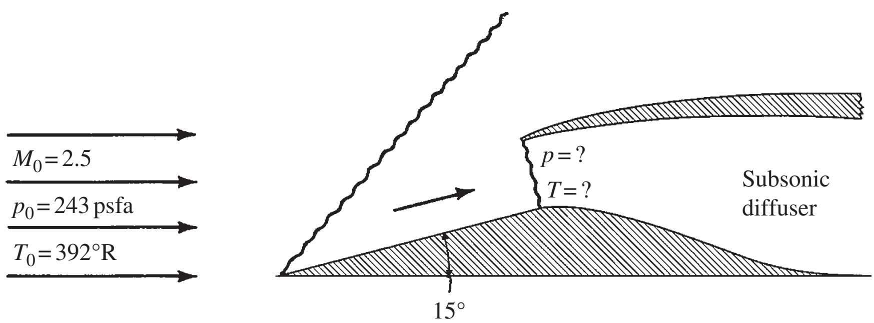


Figure P7.13

7.14. A converging–diverging nozzle is operating between the second and third critical points as shown in Figure 7.16. $M_1 = 2.5$, $T_1 = 150$ K, $p_1 = 0.7$ bar, the receiver pressure is 1 bar, and the fluid is nitrogen.

(a) Compute the Mach number, temperature, and flow deflection in region 2.

(b) Through what angle is the flow deflected as it passes through shock wave B?

(c) Determine the conditions in region 3.

7.15. For the flow situation shown in Figure P7.15, $M_1 = 1.8$, $T_1 = 600°R$, $p_1 = 15$ psia, and $\gamma = 1.4$.

(a) Find conditions in region 2 assuming that they are supersonic.

(b) What must occur along the dashed line?

(c) Find the conditions in region 3.

(d) Find the value of T_2, p_2, and M_2 if $p_{t2} = 71$ psia.

(e) How would the problem change if the flow in region 2 were subsonic?

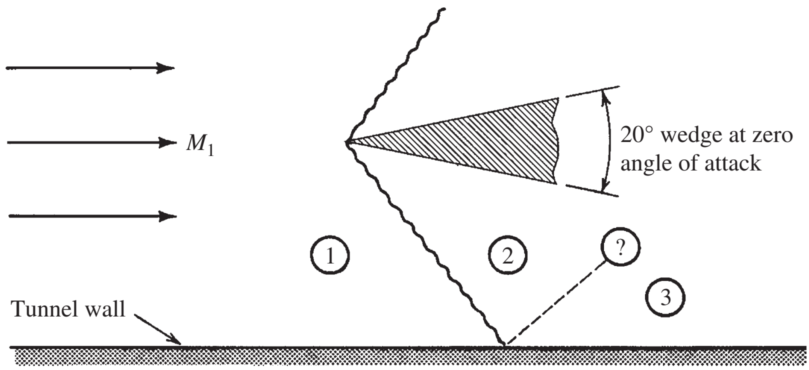


Figure P7.15

7.16. Carbon monoxide flows in the duct shown in Figure P7.16. The first shock, which turns the flow 15°, is observed to form at a 40° angle. The flow is known to be supersonic in regions 1 and 2 and subsonic in region 3.

(a) Determine M_3 and the angle β.

(b) Determine the pressure ratios p_3/p_1 and p_{t3}/p_{t1}.

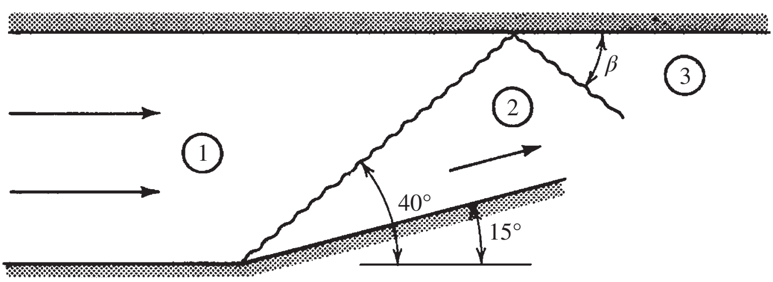


Figure P7.16

7.17. A uniform flow of air has a Mach number of 3.3. The bottom of the duct is bent upward at a 25° angle. At the point where the shock intersects the upper wall, the boundary is bent 5° upward as shown in Figure P7.17. Assume that the flow is supersonic throughout the system. Compute M_3, p_3/p_1, and T_3/T_1, and the angle β.

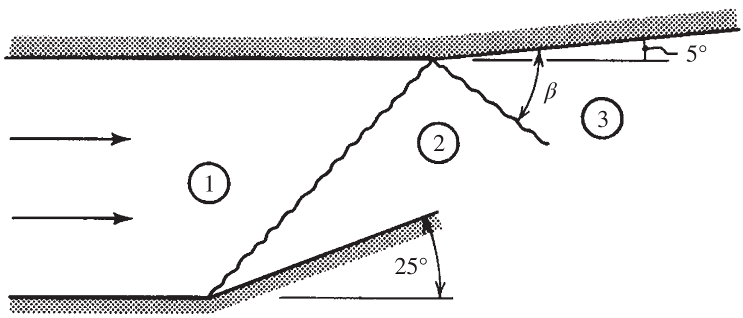


Figure P7.17

7.18. A round-nosed projectile travels through air at a temperature of −15°C and a pressure of 1.8 × 10^4 N/m^2. The stagnation pressure on the nose of the projectile is measured at 2.1 × 10^5 N/m^2.

(a) At what speed (m/s) is the projectile traveling?

(b) What is the temperature on the projectile's nose?

(c) Now assume that the nose tip is shaped like a cone. What is the maximum cone angle for the shock to remain attached?

7.19. Work Problem 7.13(a) for a conical shock of the same half-angle and compare results (these will only be valid at the surface of the cone).

7.20. In a shock tube that uses the same gas in the driver and driven sections and for the same initial T_4/T_1 ($T_1 = T_4 = 300$ K) and $p_4/p_1 = 1000$, which gas, *argon or air*, would result in higher T_2-temperatures? Are the corresponding shock Mach numbers for these two different gases equal for same given conditions? Consult Figure 7.21 for subscripts used.

7.21. **(a)** Calculate the pressure ratio needed (p_4/p_1) to operate a shock tube that produces in a shock Mach number $M_S = 4.0$. Air at 20°C is used as the medium on both sides of the diaphragm. Consult Figure 7.21 for the subscripts that apply to shock tubes.

(b) Referring to Figures 7.1 through 7.4, calculate the initial shock speed (V_1' or V_S), the contact surface speed (V_2 or V_C), and the reflected shock speed (V_2' or V_R) with the pressure ratio calculated above. (*Hint*: Apply concepts from the four items listed at the end on Section 7.10 to calculate the needed speeds.)

7.22. (Optional) Figure 7.21 shows noteworthy shock tube regions, where region (2) has the most interest for testing samples. Shock tube experiments require special test object locations and high speed instrumentation. At location (**x**) in Figure 7.21 the test object is exposed to high temperatures for a maximum Δt; it is found at the first intersection of the contact surface and the shock wave that reflects from the closed

end of the tube. Using the same conditions given in Problem 7.21 (and those resulting wave velocities), calculate the total length of the low-pressure region when the test object is placed at a distance L_1 downstream from the diaphragm for a test time $\Delta t = 1.0$ ms.

CHECK TEST

You should be able to complete this test without reference to material in the chapter.

7.1. By velocity superposition the moving shock picture shown in Figure CT7.1 can be transformed into the stationary shock problem shown. Select the statements below which are true.

(a) $p_1 = p_1' \quad p_1 < p_1' \quad p_1 > p_1' \quad p_1' = p_2'$

(b) $T_{t1}' > T_{t2}' \quad T_{t1}' = T_{t2}' \quad T_{t1}' < T_{t2}' \quad T_{t1}' = T_{t1}'$

(c) $\rho_1 > \rho_2 \quad \rho_1 = \rho_1 \quad \rho_1' < \rho_2 \quad \rho_1' > \rho_2'$

(d) $u_2' > u_1' \quad u_2' = u_1' \quad u_2' < u_1' \quad u_2' = u_2$

$(u \equiv \text{internal energy})$

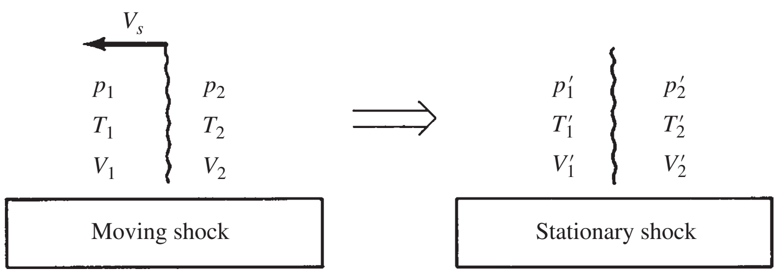


Figure CT7.1

7.2. Fill in the blanks from the choices indicated.

(a) A blast wave will travel through standard air (14.7 psia and 60°F) at a speed (less than, equal to, greater than) _____ approximately 1118 ft/sec.

(b) If an oblique shock is broken down into components that are normal and tangent to the wave front:

(i) The normal Mach number (increases, decreases, remains constant) _____ as the flow passes through the wave.

(ii) The tangential Mach number (increases, decreases, remains constant) _____ as the flow passes through the wave. (*Careful!* You are dealing with Mach number, not velocity.)

7.3. List the conditions that cause an oblique shock to form.

7.4. Describe the general results of oblique-shock analysis by drawing a plot of shock angle versus deflection angles.

7.5. Sketch the resulting flow pattern over the nose of the object shown in Figure CT7.5. The figure depicts a two-dimensional wedge.

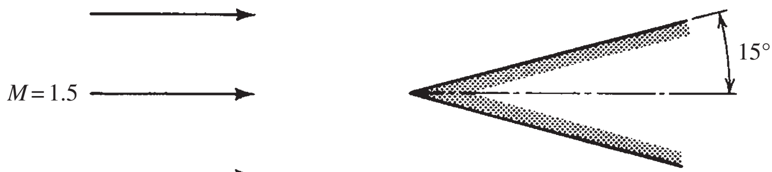


Figure CT7.5

7.6. A normal-shock wave travels at 2500 ft/sec into still air at 520°R and 14.7 psia. What velocity exists just after the wave passes?

7.7. Oxygen at 5 psia and 450°R is traveling at $M = 2.0$ and leaves a duct as shown in Figure CT7.7. The receiver conditions are 14.1 psia and 600°R.

(a) At what angle will the first shocks form? By how much is the flow deflected?

(b) What are the temperature, pressure, and Mach number in region 2?

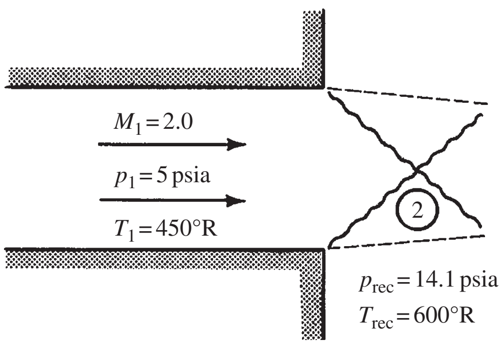


Figure CT7.7

7.8. Can oblique shocks be considered to be one-dimensional? What about conical shocks?

Chapter 8

Prandtl–Meyer Flow

8.1 INTRODUCTION

This chapter begins with an examination of weak shocks in supersonic flows. We show that for very weak oblique shocks the pressure change is related to the *first* power of the deflection angle, whereas the entropy change is related to the *third* power of the deflection angle. This will enable us to explain how a smooth turn can be accomplished isentropically—a situation known as *Prandtl–Meyer Flow*. Being reversible, such flows may represent expansions or compressions, depending on the circumstances.

A detailed analysis of Prandtl–Meyer flow is made for the case of a perfect gas and, as usual, a tabular entry is developed to aid in problem solution. Typical flow fields involving Prandtl–Meyer flow are discussed. In particular, the entire performance of a converging–diverging nozzle can now be fully explained. We include a discussion of the flow around supersonic airfoils and introduce the aerospike nozzle.

8.2 OBJECTIVES

After completing this chapter successfully, you should be able to:

1. State how entropy and pressure changes vary with deflection angles for weak oblique shocks.
2. Explain how finite turns (with finite pressure ratios) can be accomplished isentropically in supersonic flow.

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

3. Describe and sketch what occurs as fluid flows supersonically past a smooth concave corner and a smooth convex corner.
4. Show Prandtl–Meyer flow (both expansions and compressions) on a *T–s* diagram.
5. (*Optional*) Develop the differential relation between Mach number (M) and flow turning angle (ν) for Prandtl–Meyer flow.
6. Given the equation for the Prandtl–Meyer function (8.48), show how tabular entries can be developed for Prandtl–Meyer flow. Explain the significance of the angle ν.
7. Explain the governing boundary conditions and show the results when shock waves and Prandtl–Meyer waves reflect off both physical and free boundaries.
8. Draw the wave forms created by supersonic flows over rounded and/or wedge-shaped wings as the angle of attack changes. Be able to solve for the flow properties in each region.
9. Demonstrate the ability to solve typical Prandtl–Meyer flow problems by the use of the appropriate equations and tables.

8.3 ARGUMENT FOR ISENTROPIC TURNING FLOW

Pressure Change for Normal Shocks

Let us first investigate some special characteristics of any *normal shock*. Throughout this section we assume that the medium is a perfect gas, and this will enable us to develop some precise relations. We begin by recalling equation (6.25):

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}M_1^2 - \frac{\gamma-1}{\gamma+1} \tag{6.25}$$

Subtracting 1 from both sides, we get

$$\frac{p_2}{p_1} - 1 = \frac{2\gamma}{\gamma+1}M_1^2 - \frac{2\gamma}{\gamma+1} \tag{8.1}$$

The left-hand side is readily seen to be the pressure difference across the normal shock divided by the initial pressure. Now express the right side over a common denominator and this becomes

$$\boxed{\frac{p_2 - p_1}{p_1} = \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right)} \tag{8.2}$$

This relation shows that the pressure rise *across a normal shock* is directly proportional to the quantity $\left(M_1^2-1\right)$. We return to this fact later when we apply it to weak shocks at very small Mach numbers.

Entropy Changes for Normal Shocks

The entropy change for any process with a perfect gas can be expressed in terms of the specific volumes and pressures by equation (1.42). It is a simple matter to change the ratio of specific volumes to a density ratio and to introduce γ from equations (1.39) and (1.40):

$$\frac{s_2-s_1}{R}=\frac{\gamma}{\gamma-1}\ln\left(\frac{\rho_1}{\rho_2}\right)+\frac{1}{\gamma-1}\ln\left(\frac{p_2}{p_1}\right) \tag{8.3}$$

Since we are after the entropy change across a normal shock purely in terms of M_1, γ, and R, it is somewhat easier to use equations (5.25) and (5.28). These equations express the pressure ratio and density ratio across the shock as a function of the entropy rise Δs as well as the Mach number and γ. To get our desired result, we manipulate equations (5.25) and (5.28) as follows:

By taking logs on both sides of equation (5.25) we obtain

$$\ln\left(\frac{p_2}{p_1}\right)=\frac{\gamma}{\gamma-1}\ln\left[\frac{1+[(\gamma-1)/2]\,M_1^2}{1+[(\gamma-1)/2]\,M_2^2}\right]-\frac{\Delta s}{R} \tag{8.4}$$

And from equation (5.28) we obtain:

$$\gamma\ln\left(\frac{\rho_1}{\rho_2}\right)=-\frac{\gamma}{\gamma-1}\ln\left[\frac{1+[(\gamma-1)/2]\,M_1^2}{1+[(\gamma-1)/2]\,M_2^2}\right]+\gamma\frac{\Delta s}{R} \tag{8.5}$$

We can now subtract equation (8.5) from (8.4) to cancel out the bracketed term and solve for the entropy change. *Show* that after rearranging this result may be written as

$$\frac{s_2-s_1}{R}=\ln\left[\left(\frac{p_2}{p_1}\right)^{1/(\gamma-1)}\left(\frac{\rho_2}{\rho_1}\right)^{-\gamma/(\gamma-1)}\right] \tag{8.6}$$

Now equation (8.2) (in a slightly modified form) can be substituted for the pressure ratio and similarly, equation (6.26) for the density ratio, with the following result:

$$\frac{s_2 - s_1}{R} = \ln\left\{\left[1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right)\right]^{1/(\gamma-1)}\left[\frac{(\gamma+1)\,M_1^2}{(\gamma-1)\,M_1^2 + 2}\right]^{-\gamma/(\gamma-1)}\right\} \tag{8.7}$$

To aid in simplification, let

$$m \equiv M_1^2 - 1 \tag{8.8}$$

and thus, also,

$$M_1^2 = m + 1 \tag{8.9}$$

Introduce equations (8.8) and (8.9) into (8.7) and *show* that this becomes

$$\frac{s_2 - s_1}{R} = \ln\left\{\left(1 + \frac{2\gamma m}{\gamma+1}\right)^{1/(\gamma-1)}(1+m)^{-\gamma/(\gamma-1)}\left[1 + \frac{(\gamma-1)m}{\gamma+1}\right]^{\gamma/(\gamma-1)}\right\} \tag{8.10}$$

Now each of the terms in equation (8.10) is of the form $(1 + x)$, and we can take advantage of the expansion

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{8.11}$$

Put equation (8.10) into the proper form to expand each bracket according to (8.11). Be careful to retain all terms up to and including the *third* power. If you have not made any mistakes, you will find that all terms involving m and m^2 cancel out and you are left with

$$\frac{s_2 - s_1}{R} = \frac{2\gamma m^3}{3(\gamma+1)^2} + \text{higher-order terms in } m \tag{8.12}$$

Or we can say that the entropy rise *across a normal shock* is proportional to the *third power* of the quantity $\left(M_1^2 - 1\right)$ plus higher-order terms (HOT).

$$\boxed{\frac{s_2 - s_1}{R} = \frac{2\gamma\left(M_1^2 - 1\right)^3}{3(\gamma+1)^2} + \text{HOT}} \tag{8.13}$$

Note that as we want to consider *weak* shocks for which $M_1 \rightarrow 1$ or $m \rightarrow 0$, we can legitimately neglect all higher-order terms.

Pressure and Entropy Changes versus Deflection Angles for Weak Oblique Shocks

The developments made earlier in this section were for normal shocks and thus apply equally to the *normal component* of an oblique shock. Since

$$M_{1n} = M_1 \sin\theta \tag{7.5}$$

we can rewrite equation (8.2) as

$$\frac{p_2 - p_1}{p_1} = \frac{2\gamma}{\gamma + 1}\left(M_1^2 \sin^2\theta - 1\right) \tag{8.14}$$

and equation (8.13) becomes

$$\frac{s_2 - s_1}{R} = \frac{2\gamma\left(M_1^2 \sin^2\theta - 1\right)^3}{3(\gamma + 1)^2} + \text{HOT} \tag{8.15}$$

We shall proceed to relate the quantity $\left(M_1^2 \sin^2\theta - 1\right)$ to the deflection angle for the case of very weak oblique shocks. For this case,

1. δ will be very small and $(\tan\delta) \approx \delta$; and
2. θ will be approaching the Mach angle μ.

Thus we may write equation (7.15) as

$$\delta \approx 2(\cot\mu)\left(\frac{M_1^2 \sin^2\theta - 1}{M_1^2(\gamma + \cos 2\mu) + 2}\right) \tag{8.16}$$

Now for a given M_1, μ_1 is known (why?), and equation (8.16) may be written as

$$\delta \approx \text{const}\left(M_1^2 \sin^2\theta - 1\right) \tag{8.17}$$

Remember, equation (8.17) is valid only for *very weak* oblique shocks that are associated with *very small* deflection angles. But this will be *exactly* the case under

consideration in the next section. If we introduce (8.17) into (8.14) and (8.15) (omitting the higher-order terms), we get the following relations:

$$\frac{p_2 - p_1}{p_1} \approx \frac{2\gamma}{\gamma + 1}(\text{const})\delta \tag{8.18}$$

$$\frac{s_2 - s_1}{R} \approx \frac{2\gamma}{3(\gamma + 1)^2}[(\text{const})\delta]^3 \tag{8.19}$$

Let us now pause for a moment to interpret these results. They really say that for *very weak* oblique shocks at any arbitrary set of initial conditions,

$$\boxed{\Delta p \propto \delta} \tag{8.20}$$

$$\boxed{\Delta s \propto \delta^3} \tag{8.21}$$

These are important results that should be memorized.

Isentropic Turns from Infinitesimal Shocks

We have laid the groundwork to show a remarkable phenomenon. Figure 8.1 shows a finite turn divided into n equal segments of δ each. The total turning angle will be indicated by δ_{total} or δ_T and thus

$$\delta_T = n\delta \tag{8.22}$$

Each segment of the turn causes a shock wave to form with an appropriate change in Mach number, pressure, temperature, entropy, and so on. As we

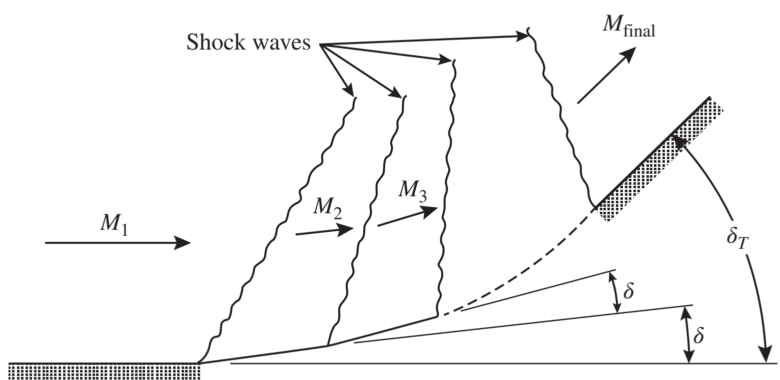


Figure 8.1 Finite turn composed of many small turns.

increase the number of segments n, δ becomes very small, which means that each shock will become a very weak oblique shock and the earlier results in this section are applicable. Thus, for each segment we may write

$$\Delta p' \propto \delta \tag{8.23}$$

$$\Delta s' \propto \delta^3 \tag{8.24}$$

where $\Delta p'$ and $\Delta s'$ are the pressure and entropy changes across each segment. Now for the total turn,

$$\text{Total } \Delta p = \sum \Delta p' \propto n\delta \tag{8.25}$$

$$\text{Total } \Delta s = \sum \Delta s' \propto n\delta^3 \tag{8.26}$$

But from (8.22) we can express $\delta = \delta_T/n$.

We now also take the limit as $n \to \infty$:

$$\text{Total } \Delta p \propto \lim_{n\to\infty} n\left(\frac{\delta_T}{n}\right) \propto \delta_T \tag{8.27}$$

$$\text{Total } \Delta s \propto \lim_{n\to\infty} n\left(\frac{\delta_T}{n}\right)^3 \to 0 \tag{8.28}$$

In the limit as $n \to \infty$, we conclude that:

1. The wall makes a smooth turn through angle δ_T.
2. The shock waves approach Mach waves.
3. The Mach number changes continuously.
4. There is a finite pressure change.
5. There is *no* entropy change.

The final result is shown in Figure 8.2. Note that as the turn progresses, the Mach number is decreasing, and thus the Mach waves are at ever-increasing angles. (Also, μ_2 is measured from an increasing baseline.) Hence, we observe an envelope of Mach lines that forms a short distance from the wall. The Mach waves eventually coalesce to form an oblique shock inclined at the proper angle (θ), corresponding to the initial Mach number and the overall deflection angle δ_T.

We return to the flow in the neighborhood of the wall, as this is a region of great interest. Here we have an infinite number of infinitesimal compression waves. We have achieved a decrease in Mach number and an increase in pressure *without any change in entropy*. Since we are dealing with adiabatic flow ($ds_e = 0$), an

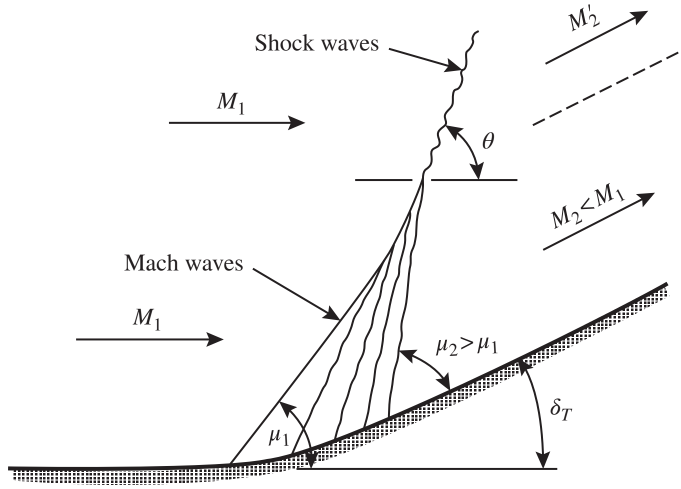


Figure 8.2 Smooth turn. Note the isentropic compression near the wall.

isentropic process ($ds = 0$) indicates that there are no losses ($ds_i = 0$) (i.e., *the process is reversible*!). The opposite process (an infinite number of infinitesimal expansion waves) is shown in Figure 8.3. Here we have a smooth turn in the other direction from that discussed previously. In this case, as the turn progresses, the Mach number increases. Thus the Mach angles are decreasing and the Mach waves will never intersect. If the corner were sharp, all of the *expansion waves* would emanate from the corner as illustrated in Figure 8.4. This is called a *centered expansion fan*. Figures 8.3 and 8.4 depict the same overall result provided that the wall is turned through the same angle.

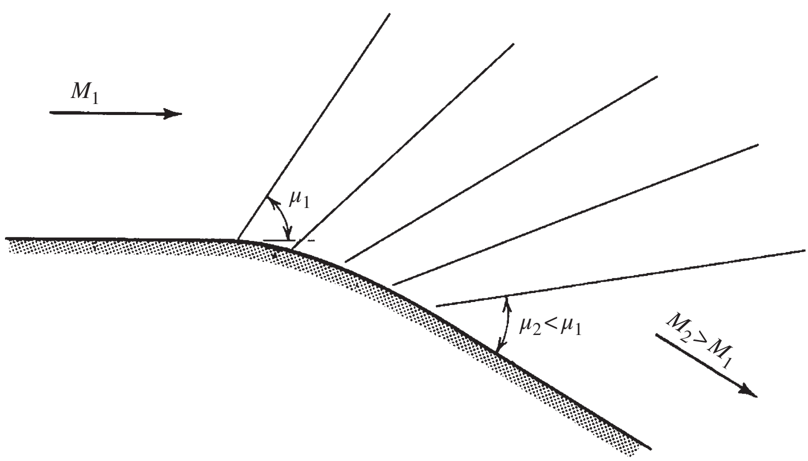


Figure 8.3 Smooth turn. Note the isentropic expansion.

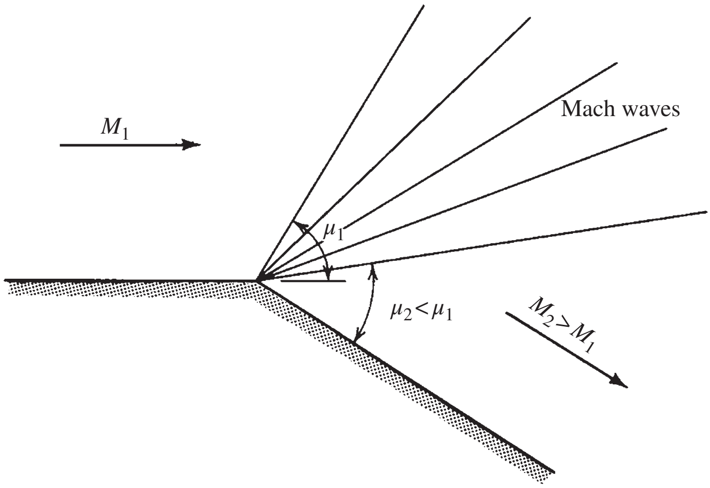


Figure 8.4 Isentropic expansion around sharp corner.

Each of the isentropic flows above is called a *Prandtl–Meyer flow*. At a smooth concave wall (Figure 8.2) we have a Prandtl–Meyer compression; flows of this type are not too realistic or important since boundary layers and other real gas effects greatly interfere with the isentropic region near the wall. At a smooth convex wall (Figure 8.3) or at a sharp convex turn (Figure 8.4) we have Prandtl–Meyer expansions. These expansions are quite prevalent in supersonic flow, as the examples given later in this chapter will show. Incidentally, you have now discovered the second means by which the flow direction of a supersonic stream may be changed. What was the first?

8.4 ANALYSIS OF PRANDTL–MEYER FLOW

We have already established that the flow is isentropic through a Prandtl–Meyer compression or expansion. If we know the final Mach number, we can use the isentropic equations and table to compute the final thermodynamic state for any given set of initial conditions. Thus our objective in this section is to relate the changes in Mach number to the turning angle in Prandtl–Meyer flow. Figure 8.5 shows a single Mach wave caused by turning the flow through an infinitesimal angle $d\nu$. It is more convenient to measure ν positive in the direction shown, which corresponds to an expansion wave. The pressure difference across the wave front causes a momentum change and hence a velocity change *perpendicular* to the wave front. There is no mechanism by which the tangential velocity component can be changed. In this respect the situation is similar to that of an oblique shock. A detail of this velocity relationship is shown in Figure 8.6.

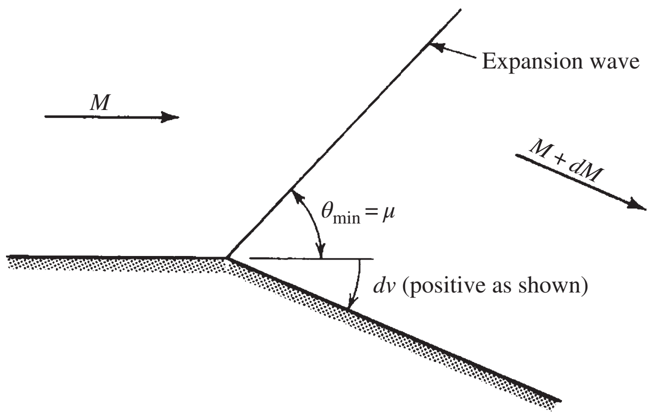


Figure 8.5 Infinitesimal Prandtl–Meyer expansion.

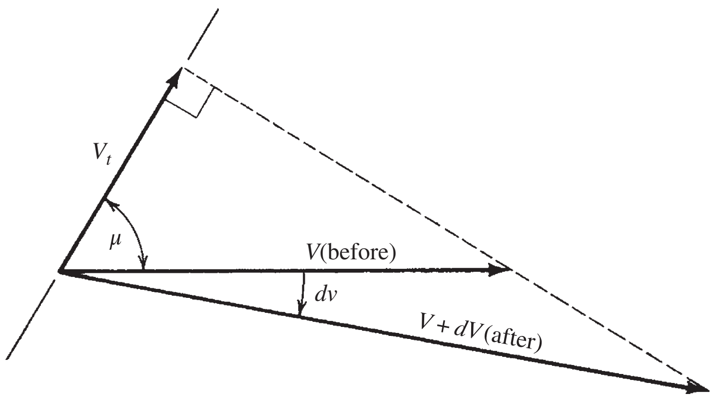


Figure 8.6 Velocities in an infinitesimal Prandtl–Meyer expansion.

In Figure 8.6, V represents the magnitude of the velocity before the expansion wave and $V + dV$ is the magnitude after the wave. In both cases the tangential component of the velocity is V_t. From the velocity triangles we see that

$$V_t = V \cos \mu \tag{8.29}$$

and

$$V_t = (V + dV)\cos(\mu + dv) \tag{8.30}$$

Equating these, we obtain

$$V \cos \mu = (V + dV)(\cos \mu + dv) \tag{8.31}$$

If we expand the $\cos(\mu + dv)$ this becomes

$$V\cos\mu = (V + dV)(\cos\mu\cos dv - \sin\mu\sin dv) \tag{8.32}$$

But dv is a very small angle; thus

$$\cos dv \approx 1 \quad \text{and} \quad \sin dv \approx dv$$

and equation (8.32) becomes

$$V\cos\mu = (V + dV)(\cos\mu - dv\sin\mu) \tag{8.33}$$

By writing each term on the right side, we get

$$\cancel{V\cos\mu} = \cancel{V\cos\mu} - V\,dv\sin\mu + dV\cos\mu - \overset{\text{HOT}}{\cancel{dV\,dv}}\sin\mu \tag{8.34}$$

Canceling like terms and dropping the higher-order term yields

$$dv = \frac{\cos\mu}{\sin\mu}\frac{dV}{V}$$

or

$$dv = \cot\mu\frac{dV}{V} \tag{8.35}$$

Now the cotangent of μ can readily be obtained in terms of the Mach number. We know that $\sin\mu = 1/M$. From the triangle shown in Figure 8.7, we see that

$$\cot\mu = \sqrt{M^2 - 1} \tag{8.36}$$

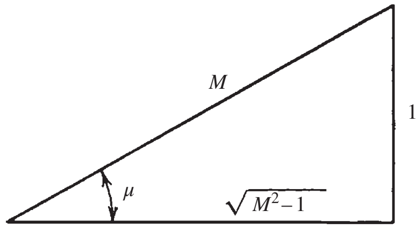


Figure 8.7

Substitution of equation (8.36) into (8.35) yields

$$\boxed{d\nu = \sqrt{M^2 - 1}\,\frac{dV}{V}} \tag{8.37}$$

Recall that our objective is to obtain a relationship between the Mach number (M) and the turning angle ($d\nu$). Thus we seek a means of expressing dV/V as a function of Mach number. To obtain an explicit expression, we shall assume that the fluid is a perfect gas. From equations (4.10) and (4.11), we know that

$$V = Ma = M\sqrt{\gamma g_c RT} \tag{8.38}$$

Hence

$$dV = dM\sqrt{\gamma g_c RT} + \frac{M}{2}\sqrt{\frac{\gamma g_c R}{T}}\,dT \tag{8.39}$$

Show that

$$\frac{dV}{V} = \frac{dM}{M} + \frac{dT}{2T} \tag{8.40}$$

Knowing that

$$T_t = T\left(1 + \frac{\gamma - 1}{2}M^2\right) \tag{4.18}$$

then

$$dT_t = dT\left(1 + \frac{\gamma - 1}{2}M^2\right) + T(\gamma - 1)MdM \tag{8.41}$$

But since there is no heat or shaft work transferred to or from the fluid as it passes through the expansion wave, the stagnation enthalpy (h_t) remains constant. For our perfect gas this means that the total temperature remains fixed. Thus

$$T_t = \text{constant} \quad \text{or} \quad dT_t = 0 \tag{8.42}$$

From equations (8.41) and (8.42) we solve for

$$\frac{dT}{T} = -\frac{(\gamma - 1)MdM}{1 + [(\gamma - 1)/2]M^2} \tag{8.43}$$

If we insert this result for dT/T into equation (8.40), we have

$$\frac{dV}{V}=\frac{dM}{M}-\frac{(\gamma-1)MdM}{2(1+[(\gamma-1)/2]M^2)} \tag{8.44}$$

Show that this can be written as

$$\frac{dV}{V}=\frac{1}{1+[(\gamma-1)/2]M^2}\frac{dM}{M} \tag{8.45}$$

We can now accomplish our objective by substitution of equation (8.45) into (8.37) with the following result:

$$\boxed{dv=\frac{(M^2-1)^{1/2}}{1+[(\gamma-1)/2]M^2}\frac{dM}{M}} \tag{8.46}$$

This is a significant relation, for it says that

$$dv=f(M,\gamma)$$

For a given fluid, γ is fixed and equation (8.46) can be integrated to yield

$$v+\text{const}=\left(\frac{\gamma+1}{\gamma-1}\right)^{1/2}\tan^{-1}\left[\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)\right]^{1/2}-\tan^{-1}\left(M^2-1\right)^{1/2} \tag{8.47}$$

If we set $v = 0$ when $M = 1$, the constant will be zero and we have

$$\boxed{v=\left(\frac{\gamma+1}{\gamma-1}\right)^{1/2}\tan^{-1}\left[\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)\right]^{1/2}-\tan^{-1}\left(M^2-1\right)^{1/2}} \tag{8.48}$$

Establishing the constant as zero in the manner described above attaches a special significance to the angle v. *This is the angle, measured from the* ***flow direction*** *where* $M = 1.0$, *through which the flow has been turned (by an isentropic process) to reach the Mach number indicated*. The expression (8.48) is called the *Prandtl–Meyer function*.

8.5 PRANDTL–MEYER FUNCTION

Equation (8.48) is the basis for solving all problems involving Prandtl–Meyer expansions or compressions. When the Mach number and γ are known, it is relatively easy to solve for the turning angle. However, in a typical problem the

turning angle might be prescribed and no explicit solution is available for the Mach number. Fortunately, none is required, for the Prandtl–Meyer function can be calculated in advance and tabulated or dedicated computer software can come to the rescue. Remember that this type of flow is isentropic; therefore, the function (ν) has been included as a column of the *isentropic table*. The following examples illustrate how rapidly the problems of this type are solved.

Example 8.1

The wall in Figure E8.1 turns an angle of 28° with a sharp corner. The fluid, which is initially at $M = 1.0$, must follow the wall and in so doing executes a Prandtl–Meyer expansion at the corner. Recall that ν always represents the angle (measured from the flow direction where $M = 1.0$) through which the flow has turned. Since M_1 is unity, then $\nu_2 = 28°$.

From the isentropic table (Appendix G), we see that this Prandtl–Meyer function corresponds to $M_2 \approx 2.06$.

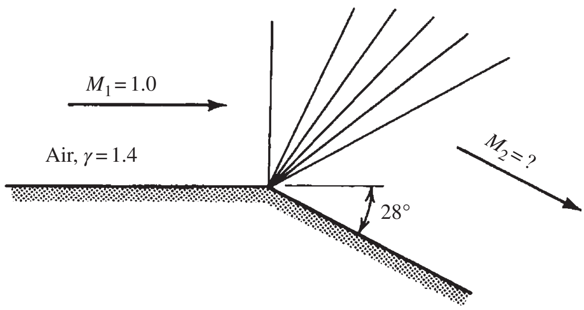


Figure E8.1 Prandtl–Meyer expansion from Mach = 1.0.

Example 8.2

Now consider flow at a Mach number of 2.06, which expands through a turning angle of 12°. Figure E8.2 shows such a situation, and we want to determine the final Mach number M_2.

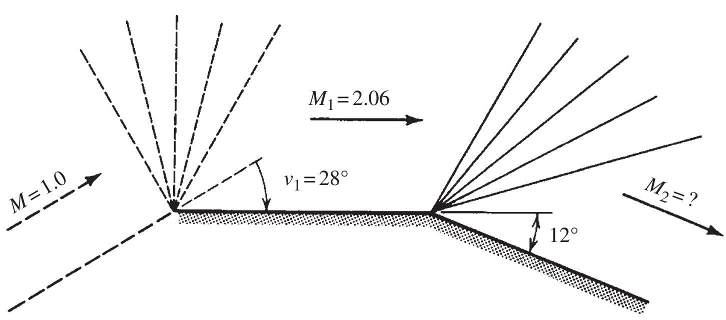


Figure E8.2 Prandtl–Meyer expansion from Mach $\neq 1.0$.

Now regardless of how the flow with $M_1 = 2.06$ came into existence, we know that *it could have been obtained* by expanding a flow at $M = 1.0$ around a corner of 28°. This is shown by dashed lines in the figure. It is easy to see that the flow in region 2 *could have been obtained* by taking a flow at $M = 1.0$ and turning it through an angle of 28° + 12°, or

$$\nu_2 = 28^\circ + 12^\circ = 40^\circ$$

From the isentropic table we find that this corresponds to a flow at $M_2 \approx 2.54$.

From the examples above, we see a general rule for Prandtl–Meyer flows:

$$\boxed{\nu_2 = \nu_1 + \Delta\nu} \tag{8.49}$$

where ν_1 corresponds to M_1 and $\Delta\nu \equiv$ the turning angle.

Note that for an expansion (as shown in Figures E8.1 and E8.2) $\Delta\nu$ is positive and thus both the Prandtl–Meyer function and the Mach number increase. Once the final Mach number is obtained, all properties may be determined easily since it is isentropic flow.

For a turn in the opposite direction, $\Delta\nu$ will be negative, which leads to a Prandtl–Meyer compression. In this case both the Prandtl–Meyer function and the Mach number will decrease. An example of this case follows.

Example 8.3

Air at $M_1 = 2.40$, $T_1 = 325$ K, and $p_1 = 1.5$ bar approaches a smooth concave turn of 20° as shown in Figure E8.3. We have previously discussed how in ideal flow the region close to the wall will be an isentropic compression. We seek the properties in the flow after the turn.

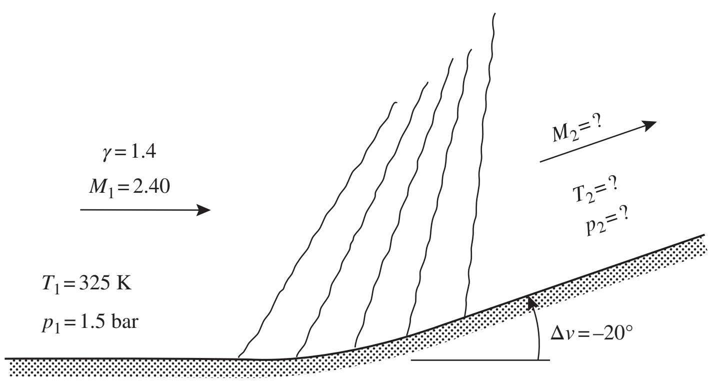


Figure E8.3 Prandtl–Meyer compression.

From the table, $\nu_1 = 36.7465^\circ$. Remember that $\Delta\nu$ is negative.

$$\nu_2 = \nu_1 + \Delta\nu = 36.7465^\circ + (-20^\circ) = 16.7465^\circ$$

Again, from the table we see that this corresponds to a Mach number of

$$M_2 = 1.664$$

Since the flow is adiabatic, with no shaft work, and a perfect gas, we know that the stagnation temperature is constant ($T_{t1} = T_{t2}$). In addition, there are no losses, and thus the stagnation pressure remains constant ($p_{t1} = p_{t2}$). Can you verify these statements with the appropriate equations?

To continue with this example, we solve for the temperature and pressure in the usual fashion:

$$p_2 = \frac{p_2}{p_{t2}} \frac{p_{t2}}{p_{t1}} \frac{p_{t1}}{p_1} p_1 = (0.2139)(1)\left(\frac{1}{0.0684}\right)\left(1.5 \times 10^5\right) = 4.69 \times 10^5 \ \mathrm{N/m^2} \quad \text{(or 68.0 psia)}$$

$$T_2 = \frac{T_2}{T_{t2}} \frac{T_{t2}}{T_{t1}} \frac{T_{t1}}{T_1} T_1 = (0.6436)(1)\left(\frac{1}{0.4647}\right)(325) = 450 \ \mathrm{K} \quad (\text{or } 810^\circ\mathrm{R})$$

As we move away from the wall we know that the Mach waves will coalesce and form an oblique shock. At what angle will the *shock* be to deflect the flow by 20°? What will the temperature and pressure be after the *shock*? If you work out this oblique-shock problem, you should obtain $\theta = 44^\circ$, $M_{1n} = 1.667$, $p_2' = 4.61 \times 10^5 \ \mathrm{N/m^2}$, and $T_2' = 466$ K Since pressure equilibrium does not exist across this *free boundary*, another wave formation must emanate from the region where the compression waves coalesce into the shock. Further discussion of this problem is beyond the scope of this book, but interested readers are referred to Chapter 16 of Shapiro (Ref. 19). Also, any viscous boundary layers present would affect all the calculations above.

8.6 OVEREXPANDED AND UNDEREXPANDED NOZZLES

We have now obtained the knowledge to complete the analysis of a converging–diverging nozzle. Previously, we discussed its isentropic operation, both in the subsonic (venturi) regime and its supersonic design operation (Section 5.7). Non-isentropic operation with a normal shock standing in the diverging portion was also covered (Section 6.6). In Section 7.8 we saw that with operating pressure ratios below second critical point, oblique shocks come into play, but we were unable to complete the picture.

Figure 8.8 shows an *overexpanded* nozzle; it is operating someplace between its second and third critical points. Recall from the summary of Chapter 7 that there are two types of boundary conditions that must be met. One of these concerns flow direction and the other concerns pressure equilibrium.

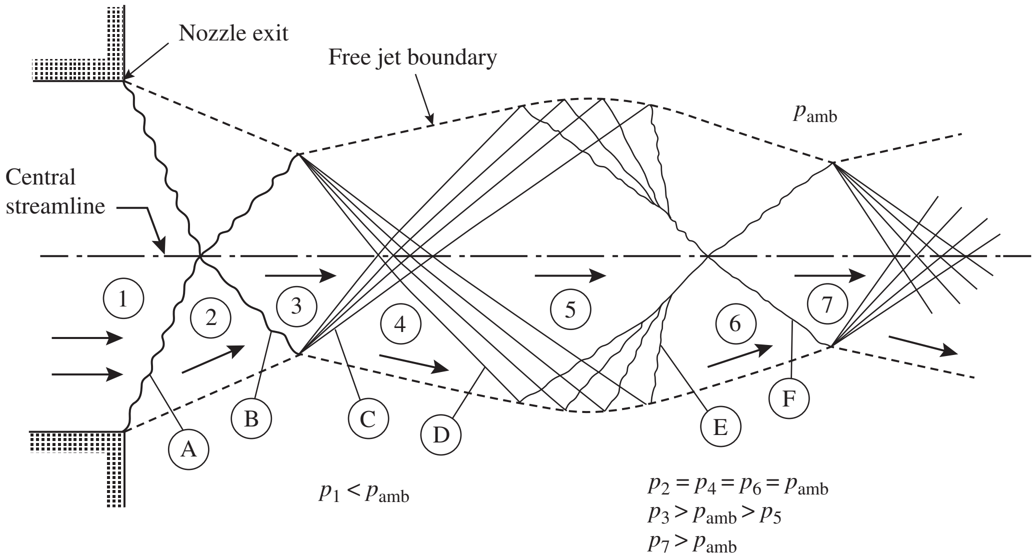


Figure 8.8 Overexpanded nozzle for weak oblique shocks.

1. From symmetry aspects we know that a central streamline exists. Any fluid touching this boundary must have a velocity that is tangent to the streamline. In this respect it is identical to a physical boundary.
2. Once the jet leaves the nozzle, there is an outer surface that is in contact with the surrounding ambient fluid. Since this is a *free* or unrestrained boundary, pressure equilibrium must exist across this surface.

We can now follow from region to region, and by matching the appropriate boundary condition, determine the flow pattern that must exist.

Since the nozzle is operating with a pressure ratio between the second and third critical points, it is obvious that we need a compression process at the exit in order for the flow to end up at the ambient pressure. However, a normal shock at the exit will produce too strong a compression. What is needed is a shock process that is weaker than a normal shock, and the oblique shock has been shown to be just this. Thus, at the exit we observe oblique shock A at the appropriate angle so that $p_2 = p_{amb}$.

Before proceeding we must distinguish two subdivisions of the flow between the second and third critical. If the oblique shock is strong (see Figure 7.9), then the resulting flow will be subsonic and no more waves will be possible or necessary at region 2. The pressure at region 2 is matched to that of the receiver, and subsonic flow can turn without waves to avoid any centerline problems. On the other hand, if the oblique shock is weak, supersonic flow will prevail (although attenuated) and additional waves will be needed to turn the flow as described below. The exact boundary between strong and weak shocks is close but not

the same as the line representing the minimum M_1 for attached oblique shocks shown in Appendix D. Rather, it is the line shown as $M_2 = 1.0$.

We recall that the flow across an oblique shock is always deflected away from a normal to the shock front, and thus the flow in region 2 is no longer parallel to the centerline. Wave front B must deflect the flow back to its original axial direction. This can easily be accomplished by another oblique shock. (A Prandtl–Meyer expansion would turn the flow in the wrong direction.) An alternative way of viewing this is that the oblique shocks from both the upper and lower lips of the nozzle *pass through each other* when they meet at the centerline. If one adopts this philosophy, one should realize that the waves must be slightly altered or *bent* in the process of traveling through one another.

Now, since $p_2 = p_{amb}$, passage of the flow through oblique shock B will make $p_3 > p_{amb}$ and region 3 cannot have a free surface in contact with the surroundings. Consequently, a wave formation must emanate from the point where wave B meets the free boundary, and the pressure must decrease across this wave. We now realize that wave form C must be a Prandtl–Meyer expansion so that $p_4 = p_{amb}$.

However, passage of the flow through the expansion fan, C, causes it to turn away from the centerline, and the flow in region 4 is no longer parallel to the centerline. Thus, as each wave of the Prandtl–Meyer expansion fan meets the centerline, a wave form must emanate to turn the flow parallel to the axis again. If wave D were a compression, in which direction would the flow turn? We see that to meet the boundary condition of flow direction, wave D must be another Prandtl–Meyer expansion. Thus, the pressure in region 5 is less than ambient.

Can you now reason that to get from regions 5 to 6 and meet the boundary condition imposed by the free boundary, E must consist of Prandtl–Meyer compression waves? Depending on the pressures involved, these usually coalesce into an oblique shock, as shown. Then F is another oblique shock to turn the flow from region 6 to match the direction of the *wall*. Now is p_7 equal to, greater than, or less than p_{amb}? You should recognize that conditions in region 7 are similar to those in region 3, and so the cycle of waves repeats.

Next let us examine an *underexpanded* nozzle. This means that we have an operating pressure ratio *below* the third critical point or design condition. Figure 8.9 shows such a situation. The flow leaving this nozzle has a pressure greater than ambient, and the flow is parallel to the axis. Think about it and you will realize that this condition is exactly the same as region 3 in the overexpanded nozzle (see Figure 8.8). Thus the flow patterns are identical from this point on. Figures 8.8 and 8.9 represent ideal behavior. The general wave forms described can be seen by special flow visualization techniques such as Schlieren photography. Eventually, the large velocity difference that exists over the free boundary causes a turbulent shear layer, which rather quickly dissipates the wave

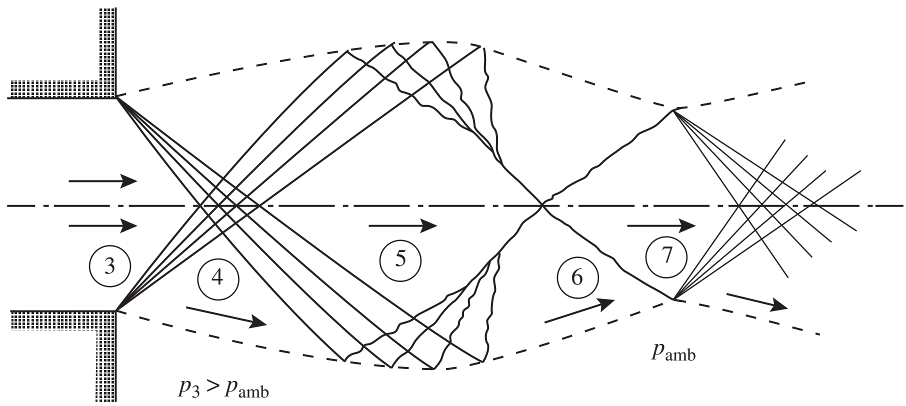


Figure 8.9 Underexpanded nozzle.

patterns. This can be seen in Figure 8.10, which shows actual Schlieren photographs of a converging–diverging nozzle operating at various pressure ratios.

Example 8.4

Nitrogen issues from a nozzle at a Mach number of 2.5 and a pressure of 10 psia. The ambient pressure is 5 psia. What is the Mach number, and through what angle is the flow turned after passing through the first Prandtl–Meyer expansion fan?

With reference to Figure 8.9, we know that $M_3 = 2.5$, $p_3 = 10$ psia, and $p_4 = p_{amb} =$ 5 psia.

$$\frac{p_4}{p_{t4}} = \frac{p_4}{p_3}\frac{p_3}{p_{t3}}\frac{p_{t3}}{p_{t4}} = \left(\frac{5}{10}\right)(0.0585)(1) = 0.0293$$

Thus

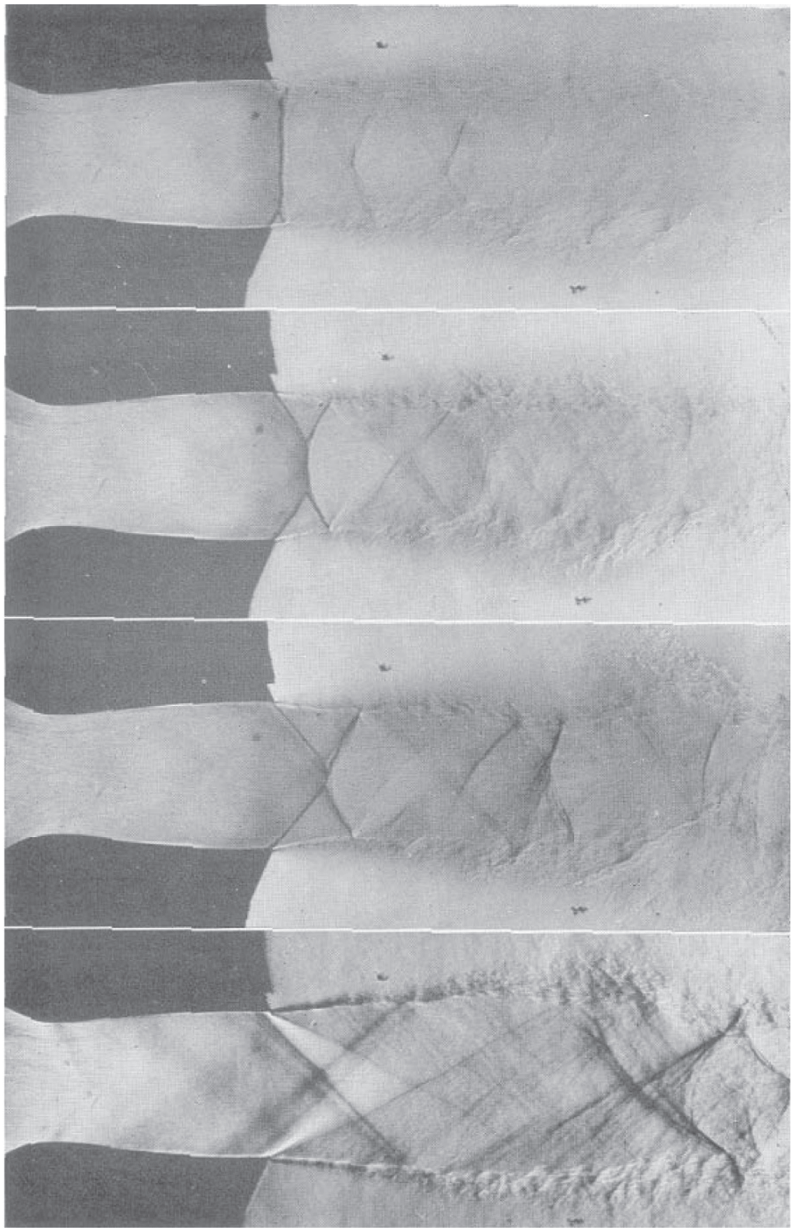
$$M_4 = 2.952$$

$$\Delta\nu = \nu_4 - \nu_3 = 48.8226 - 39.1236 \approx 9.7°$$

Wave Reflections

From the discussions above we have not only learned about the details of nozzle jets when operating at off-design conditions, but we have also been looking at *wave reflections,* although we have not called them such. We could think of the waves as *bouncing* or *reflecting* off the free boundary. Similarly, if the central streamline had been visualized as a solid boundary, we could have thought of the waves as reflecting off that boundary. In retrospect, we may draw some general conclusions about wave reflections.

1. Reflections from a physical or pseudo-physical boundary (where the boundary condition concerns the flow direction) are of the *same family*. That is,



$\frac{p_e}{p_{amb}} = 0.4$

$\frac{p_e}{p_{amb}} = 0.6$

$\frac{p_e}{p_{amb}} = 0.8$

$\frac{p_e}{p_{amb}} = 1.5$

Figure 8.10 Nozzle performance: flow from a converging–diverging nozzle at different back-pressures. (p_e = pressure just ahead of exit). (© Crown Copyright 2001. Reproduced by permission of the Controller of HMSO.) [Similar images can be found by searching the web for: "supersonic nozzle over under expansion images"]

shocks reflect as shocks, compression waves reflect as compression waves, and expansion waves reflect as expansion waves.

2. Reflections from a free boundary (where pressure equalization exists) are of the *opposite family* (i.e., compression waves reflect as expansion waves, and expansion waves reflect as compression waves).

Warning: Care should be taken in viewing these waves as reflections. Not only is their character sometimes changed (case 2 above) but the angle of reflection is not quite the same as the angle of incidence. Also, the *strength* of the wave changes somewhat. This can be shown clearly by considering the case of an oblique shock *reflecting* off a solid boundary.

Example 8.5

Air at Mach = 2.2 passes through an oblique shock at a 35° angle. The shock runs into a physical boundary as shown in Figure E8.5. Find β, the angle of *reflection*, and compare the strengths of the two shock waves.

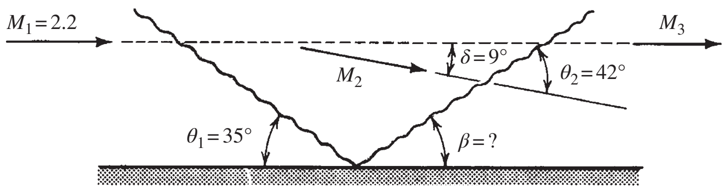


Figure E8.5

From the oblique shock chart at $M_1 = 2.2$ and $\theta_1 = 35°$, we find that $\delta_1 = 9°$.

$$M_{1n} = 2.2 \sin 35° = 1.262 \quad \text{thus} \quad M_{2n} = 0.806$$

$$M_2 = \frac{M_{2n}}{\sin(\theta - \delta)} = \frac{0.806}{\sin(35 - 9)} = 1.839$$

The reflected shock must turn the flow back parallel to the wall. Thus, from the chart at $M_2 = 1.839$ and $\delta_2 = 9°$, we find that $\theta_2 = 42°$.

$$\beta = 42° - 9° = 33°$$

$$M_{2n} = 1.839 \sin 42° = 1.230$$

Notice that the *angle of incidence* (35°) is not the same as the *angle of reflection* (33°). Also, the normal Mach number, which indicates the strength of the wave, has decreased from 1.262 to 1.230.

8.7 SUPERSONIC AIRFOILS

Airfoils designed for *subsonic* flight have rounded leading edges to prevent flow separation. The use of an airfoil of this type at purely *supersonic* speeds would cause a detached shock to form in front of the leading edge (see Section 7.7).

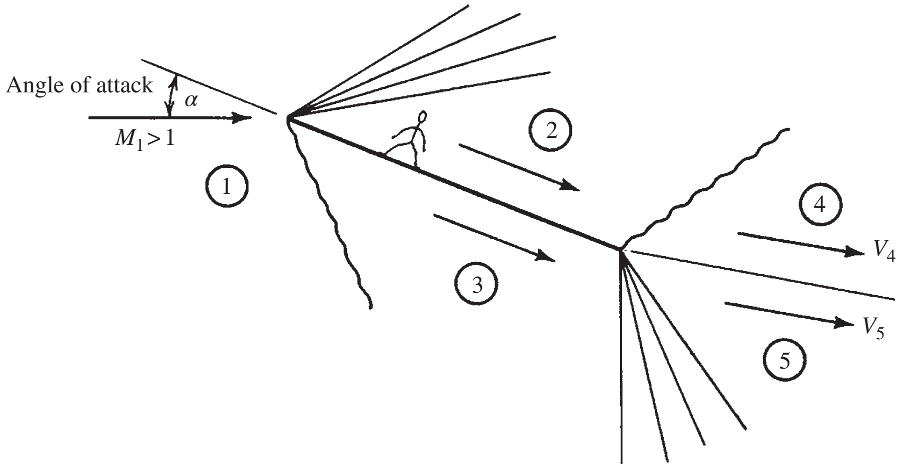


Figure 8.11 Flat-plate airfoil.

Consequently, all supersonic airfoil shapes have sharp leading edges. Also, to provide good aerodynamic characteristics, supersonic foils are very thin. The obvious limiting case of a thin foil with a sharp leading edge is the flat-plate airfoil shown in Figure 8.11. Although impractical from structural considerations, it provides for an interesting study and has characteristics that are typical of all supersonic airfoils.

Using the foil as a frame of reference yields a steady flow picture. When operating at an angle of attack (α) the flow must change direction to pass over the foil surface. You should have no trouble recognizing that to pass along the upper surface requires a Prandtl–Meyer expansion through angle α at the leading edge. Thus the pressure in region 2 is less than atmospheric. To pass along the lower surface necessitates an oblique shock which will be of the weak variety for the required deflection angle α. (Why is it impossible for the strong solution to occur? See Section 7.7.) The pressure in region 3 is greater than atmospheric and the pressure difference acting on the airfoil generates a lift force (see Example 8.6).

Now consider what happens at the trailing edge. Pressure equilibrium must exist between regions 4 and 5 as there is no surface to maintain a pressure difference. Thus a compression must occur off the upper surface and an expansion is necessary on the lower surface. The corresponding wave patterns are indicated in the diagram: an oblique shock from 2 to 4 and a Prandtl–Meyer expansion from 3 to 5. Note that the flows in regions 4 and 5 are not necessarily parallel to those of region 1 nor are the pressures p_4 and p_5 necessarily atmospheric. *The boundary conditions that must be met are flow tangency and pressure equilibrium,* or

$$V_4 \text{ parallel to } V_5 \quad \text{and} \quad p_4 = p_5$$

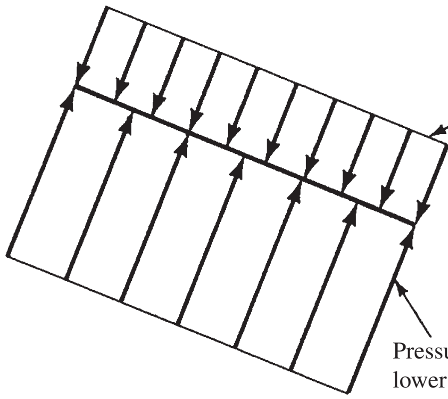


Figure 8.12 Pressure distribution over flat-plate airfoil.

The solution at the trailing edge is a trial-and-error type since neither the final flow direction nor the final pressure is known, and many other wave interactions (not shown) are involved in the wake of the airfoil.

A sketch of the pressure distribution is given in Figure 8.12. One can easily see that the *center of pressure* is at the middle or midchord position. If the angle of attack were changed, the values of the pressures over the upper and lower surfaces would change, but the center of pressure would remain at the midchord. Students of aeronautics, who are familiar with the term *aerodynamic center,* will appreciate that this important location is also found at the midchord. This is approximately true of all *supersonic* airfoils since they are quite thin and generally operate at small angles of attack. (The aerodynamic center of an airfoil section is defined as the point about which the pitching moment is independent of angle of attack. For *subsonic airfoils*, this is approximately at the one-quarter chord point, or 25% of the chord measured from the leading edge back toward the trailing edge.)

Example 8.6

Compute the lift per unit span of a flat-plate airfoil with a chord of 2 m when flying at $M = 1.8$ and an angle of attack of 5°. Ambient air pressure is 0.4 bar. Use Figure 8.11 for the identification of regions.

The flow over the top is turned 5° by a Prandtl–Meyer expansion.

$$\nu_2 = \nu_1 + \Delta\nu = 20.7251 + 5 = 25.7251^\circ$$

Thus

$$M_2 = 1.976 \quad \text{and} \quad \frac{p_2}{p_{t2}} = 0.1327$$

The flow under the bottom is turned 5° by an oblique shock. From the chart at $M = 1.8$ and $\delta = 5°$, we find that $\theta = 38.5°$. (Compare this value to what would be obtained using the relevant figure in Appendix D.)

$$M_{1n} = 1.8 \sin 38.5° = 1.20 \quad \text{and} \quad \frac{p_3}{p_1} = 1.2968$$

From Appendix D we get p_3/p_1 = _____.

The lift force is defined as that component which is perpendicular to the free stream. Thus, the lift force per unit span will be

$$L = (p_3 - p_2)(\text{chord})(\cos\alpha) = (0.5187 - 0.3051)\left(10^5\right)\cdot 2(\cos 5°)$$

$$L = 4.26 \times 10^4 \text{ N/unit span} \quad (\text{or } 9576 \text{ lbf/unit span})$$

A more practical design of a supersonic airfoil is shown in Figure 8.13. Here, the wave formation depends on whether or not the angle of attack is less than or

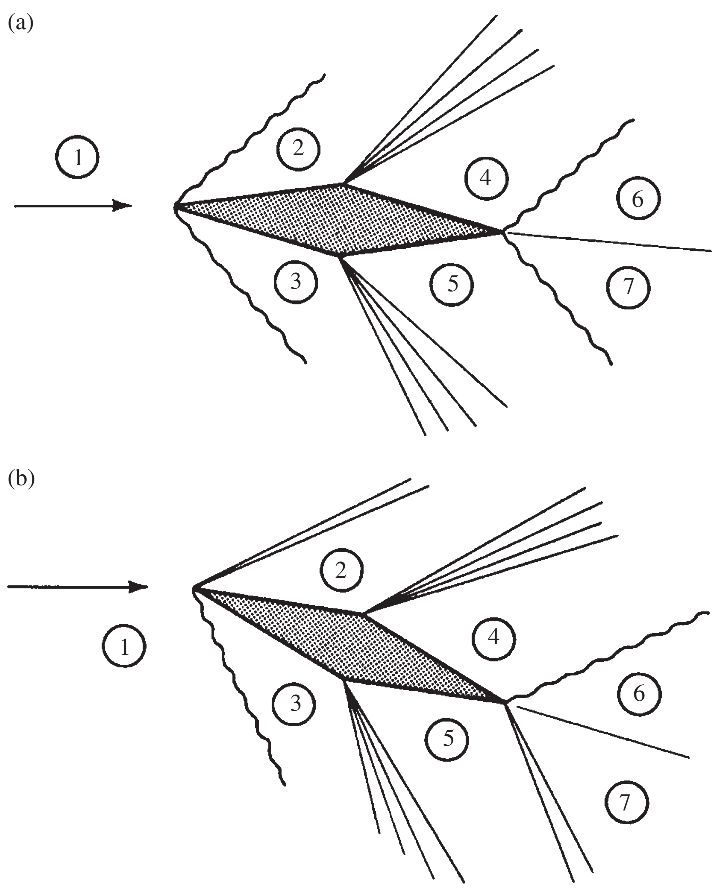


Figure 8.13 Double-wedge airfoil. (a) Low angle of attack. (b) High angle of attack.

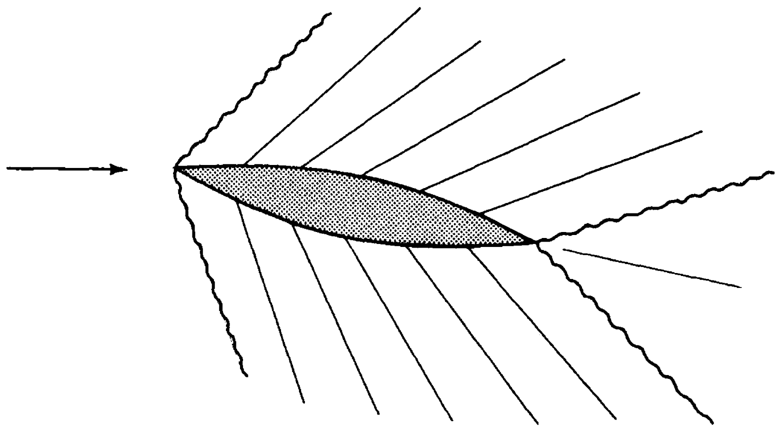


Figure 8.14 Biconvex airfoil at low angle of attack.

greater than the half angle of the wedge at the leading edge. In either case, straightforward solutions exist on all surfaces up to the trailing edge. A trial-and-error solution is required only if one is interested in regions 6 and 7. Fortunately, these regions are only of academic interest, as they have no effect on the pressure distribution over the foil. Modifications of the double-wedge airfoil with sections of constant thickness in the center are frequently found in practice.

Another widely used supersonic airfoil shape is the *biconvex,* which is shown in Figure 8.14. This is generally constructed of circular or parabolic arcs. The wave formation is quite similar to that on the double wedge in that the type of waves found at the leading (and trailing) edge is dependent on the angle of attack. Also, in the case of the biconvex, the expansions are spread out over the entire upper and lower surface.

Example 8.7

It has been suggested that a supersonic airfoil be designed as an isosceles triangle with 10° equal angles and an 8-ft chord. When operating at a 5° angle of attack the air flow appears as shown in Figure E8.7. Find the pressures on the various surfaces and the lift and drag forces when flying at $M = 1.5$ through air with a pressure of 8 psia.

From the oblique shock chart at $M_1 = 1.5$ and $\delta = 5°$, $\theta = 48°$:

$$M_{1n} = M_1 \sin\theta = 1.5(\sin 48°) = 1.115$$

From the shock table,

$$M_{2n} = 0.900 \quad \text{and} \quad \frac{p_2}{p_1} = 1.2838$$

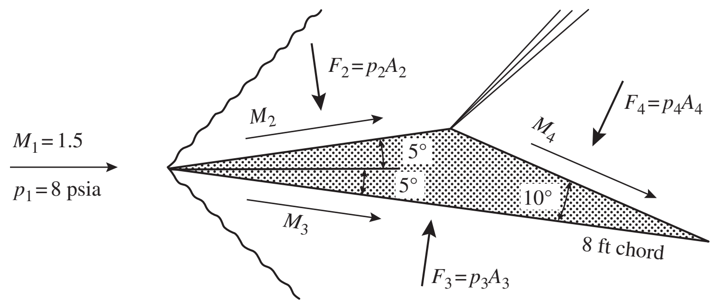


Figure E8.7

The Prandtl–Meyer expansion turns the flow by 20°:

$$\nu_4 = \nu_2 + 20 = 6.7213 + 20 = 26.7213 \quad \text{and} \quad M_4 = 2.012$$

Note that for this problem conditions in region 3 are identical with region 2. We next need to find the pressures in regions 2, 3, and 4 and the areas over which they act to calculate F_2, F_3, and F_4 (use Appendixes G and H). The lift force L (perpendicular to the free stream) will be

$$L = F_3 \cos 5° - F_2 \cos 5° - F_4 \cos 15°$$

Show that the lift per unit span will be 3715 lbf/ft (or 54.210 kN/m).

The drag is that force which is parallel to the free-stream velocity. *Show* that the drag force per unit span is 999 lbf (or 4.443 kN). (Compare the oblique shock results above with those obtained using the relevant charts in Appendix D).

8.8 AEROSPIKE NOZZLE

The fact that in overexpanded nozzles the flow contracts as it exits (as shown in Figure 8.8) and in underexpanded nozzles it expands (as shown in Figure 8.9) has a practical application in rocket-propelled space launch vehicles. To place payloads into earth orbit or into space, rockets must traverse the atmosphere whose pressure acting at the nozzle exit continuously decreases with altitude from its sea-level high to a vacuum. As might be evident in Figure 5.9 and as discussed in Section 12.8, variable exit-area nozzles have to date been impractical (see Ref. 24), and presently, all rocket propulsion nozzles operate with fixed *supersonic area ratios*, which along with fixed chamber propellant-flow conditions result in unchanging nozzle exit pressures (why?)—and because each rocket subunit (or stage) is designed for "optimum" performance at some given altitude, the nozzle exit will operate first overexpanded and then underexpanded as the rocket rises in the atmosphere, both conditions performing less than optimum. Presently, two

or more booster stages each with different fixed bell-shaped contour nozzles are used to minimize this problem but this turns out to be very complicated and costly.

If, on the other hand, we could realize variable jet-exit areas without any moving hardware, which first contract like region 2 in Figure 8.8 and then, as the external pressure drops, expand like region 4 in Figure 8.9, and if changes in the nozzle-discharge outer boundaries would automatically match the local ambient pressure, then we could get optimum expansions throughout the rocket's flight. In order to do this and *not* lengthen or change the nozzle by means of additional hardware, the *linear aerospike nozzle* concept has been explored, which incorporates a fixed exit "center body" that shapes the exhaust jet and transmits thrust to the vehicle. This design is a newer version of the *plug nozzle*, which has been tested for a NASA vehicle called the X-33—an aerospike nozzle schematic together with a pair of "thruster chambers or modules" is shown in Figure 8.15.

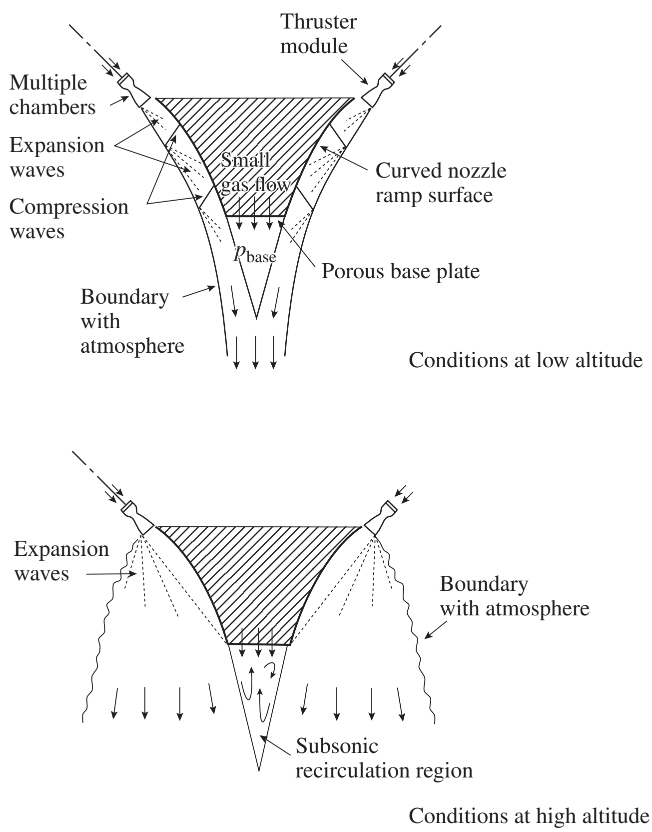


Figure 8.15 Truncated aerospike nozzle sectional view showing the resulting exit-area contraction at low altitudes and expansion at high altitudes. Because exit area changes are continuously driven by the surrounding atmospheric pressure, the overall nozzle arrangement operates at nearly optimum conditions throughout its flight. (Adapted from Ref. 24). [Aerospike nozzle images can be found by searching the web for: "aerospike nozzle images"]

Here, a "truncated" center body design is shown where propellant gas expansion inside each thruster module remains fixed (not shown is the fact that each nozzle cross section is modified to go from a circular throat to a rectangular exit). The vehicle contains multiple individual rocket chambers that are arranged linearly on each side of the center-body ramp, each exhausting supersonically a portion of the total propellant flow. To withstand the high heat transfer rates, the center body is truncated at its tail-end and must be cooled even when made up of high-temperature materials. The shape of this center body is such that it directs the individual exhaust jets, bending them toward the axial direction thereby transmitting the thrust force to the vehicle while leaving the jet outer regions free to contract or expand and thus react to changing atmospheric pressure forces (continuously making $p_3 = p_{\rm rec}$ in the nomenclature of Figure 5.10).

Because there are no moving nozzle hardware components and because this arrangement allows the rocket engine to operate near optimum conditions at all altitudes, this is an attractive scheme that may save substantial amounts of fuel. Moreover, the entire X-33 vehicle concept is single-stage-to-orbit and reusable (it would return to earth like the now-retired Space Shuttle) so it should be considerably less expensive to operate than disposable multistage counterparts. Also, a linear, truncated aerospike configuration is more compact and allows for certain rocket maneuvers without moving any engine parts.

8.9 WHEN γ IS NOT EQUAL TO 1.4

As indicated earlier, the Prandtl–Meyer function is tabulated *within* the isentropic table for $\gamma = 1.4$. The behavior of this function for $\gamma = 1.13$, 1.4, and 1.67 is given in Figure 8.16 up to $M = 5$. Here we can see that the dependence on γ is rather noticeable except perhaps for $M \le 1.2$. Thus, below this Mach number, the tabulations in Appendix G can be used with little error for any γ. The *Gasdynamics Calculator* will work in the range $1.0 < \gamma \le 1.67$. The appendix tabulation indicates that the value of ν eventually saturates as $M \to \infty$, but we do not show this limit because, among other things, it is not realistic for any value of γ. However, the calculation is not difficult and is demonstrated in Problem 8.15.

Strictly speaking, the curves on Figure 8.16 are only representative for cases where γ variations are *negligible within the flow*. However, they offer hints as to what magnitude of changes is to be expected in other cases. Flows where γ variations are *not negligible within the flow* are treated in Chapter 11.

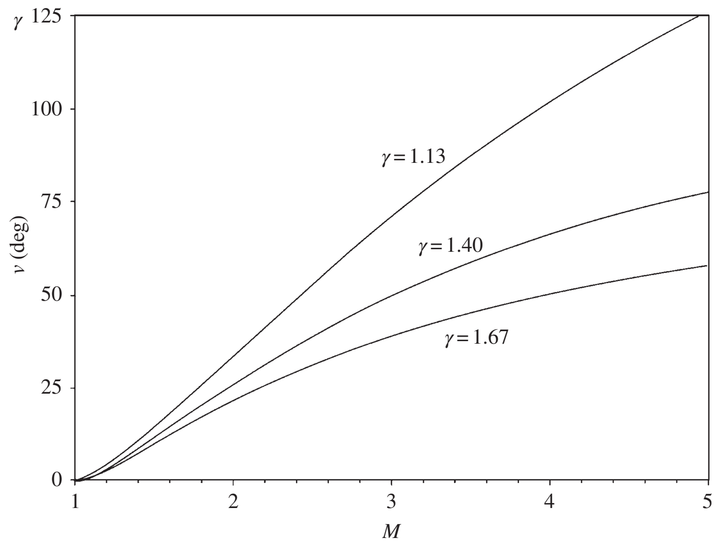


Figure 8.16 Prandtl–Meyer function versus Mach number for various values of γ.

8.10 (OPTIONAL) BEYOND THE TABLES

As illustrated in Chapter 5, one can eliminate a lot of interpolation and get accurate answers for any ratio of the specific heats γ and/or any Mach number by using a computer utility such as MAPLE. The calculation of the Prandtl–Meyer function can readily be obtained from the example below. We use here equation (8.48) which for your convenience is repeated in the example below. This procedure allows the solution for different values of γ as well as the calculation of M given γ and ν.

Example 8.8
Calculate the function ν for γ = 1.4 and M = 3.0.

We begin with equation (8.48):

$$\nu = \left(\frac{\gamma+1}{\gamma-1}\right)^{1/2} \tan^{-1}\left[\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)\right]^{1/2} - \tan^{-1}\left(M^2-1\right)^{1/2} \tag{8.48}$$

Let

$$g \equiv \gamma, \text{a parameter (the ratio of specific heats)}$$

$$X \equiv \text{the independent variable (which in this case is } M)$$

$$Y \equiv \text{the dependent variable (which in this case is } \nu)$$

Listed below are the precise inputs and program that you use in the computer.

```
[>g:=1.4:  X:=3.0:
[>Y:=sqrt(((g+1)/(g-1)))*arctan(sqrt(((g-1)/(g+1))
    *(X^2-1))) - arctan(sqrt(X^2-1));
              Y:=.868429529
```

We now need to convert from radians to degrees as follows:

```
[>evalf(Y*(180/Pi));
```

which gives the desired answer, $v = 49.76°$.

Example 8.8 can be worked with the *Gasdynamics Calculator* as well.

8.11 SUMMARY

A detailed examination of very weak oblique shocks (with small deflection angles) shows that

$$\Delta p \propto \delta \quad \text{and} \quad \Delta s \propto \delta^3 \tag{8.20), (8.21}$$

This enables us to reason that a smooth *concave* turn can be negotiated isentropically by a supersonic stream, although a typical oblique shock will form at some distance from the wall. Of even greater significance is the fact that this is a reversible process and supersonic flow turns of a *convex* nature can be accomplished by isentropic expansions.

The phenomena above are called Prandtl–Meyer flows. An analysis for a perfect gas reveals that the turning angle can be related to the change in Mach number by

$$dv = \frac{(M^2-1)^{1/2}}{1+[(\gamma-1)/2]M^2}\frac{dM}{M} \tag{8.46}$$

which when integrated yields the Prandtl–Meyer function:

$$\nu = \left(\frac{\gamma+1}{\gamma-1}\right)^{1/2} \tan^{-1}\left[\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)\right]^{1/2} - \tan^{-1}\left(M^2-1\right)^{1/2} \tag{8.48}$$

In establishing equation (8.48), ν was set equal to zero at $M = 1.0$, meaning that ν represents the angle, measured from the direction where $M = 1.0$, through which the flow has been turned (isentropically) to reach the indicated Mach number. The relation above has been tabulated in the isentropic table, which permits easy problem solutions according to the relation

$$\nu_2 = \nu_1 + \Delta\nu \tag{8.49}$$

where $\Delta\nu$ is the turning angle. Remember that $\Delta\nu$ will be positive for expansions and negative for compressions.

It must be understood that Prandtl–Meyer expansions and compressions are caused by the same two situations that govern the formation of oblique shocks (i.e., the flow must be tangent to a boundary, and pressure equilibrium must exist along the edge of a free boundary). Consideration of these boundary conditions together with any given physical situation should enable you to determine the resulting flow patterns rather quickly.

Waves may sometimes be thought of as *reflecting* off boundaries, in which case it is helpful to remember that:

1. Reflections from *physical* boundaries are of the same *family*.
2. Reflections from *free* boundaries are of the opposite *family*.

Remember that all isentropic relations and the isentropic table (Appendix G) may be used when dealing with Prandtl–Meyer flows.

PROBLEMS

8.1. Air approaches a sharp 15° convex corner (see Figure 8.4) with a Mach number of 2.0, temperature of 520°R, and pressure of 14.7 psia. Determine the Mach number, static and stagnation temperature, and static and stagnation pressure of the air after it has expanded around the corner.

8.2. A Schleiren photo of the flow around a corner reveals the edges of the expansion fan to be indicated by the angles shown in Figure P8.2. Assume that $\gamma = 1.4$.

(a) Determine the Mach number before and after the corner.

(b) Through what angle was the flow turned, and what is the angle of the expansion fan (θ_3)?

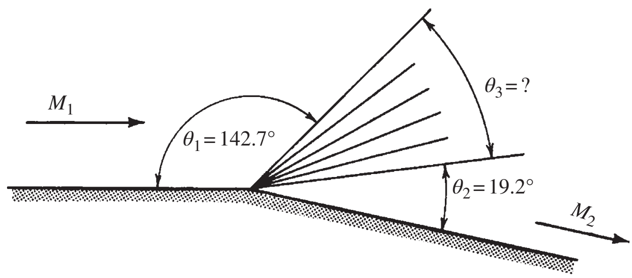


Figure P8.2

8.3. A supersonic flow of air has a pressure of 1×10^5 N/m^2 and a temperature of 350 K. After expanding through a 35° turn, the Mach number is 3.5.

(a) What are the final temperature and pressure?

(b) Make a sketch similar to Figure P8.2 and determine angles θ_1, θ_2, and θ_3.

8.4. In a problem similar to Problem 8.2, θ_1 is unknown, but $\theta_2 = 15.90°$ and $\theta_3 = 82.25°$. Can you determine the initial Mach number?

8.5. Nitrogen at 25 psia and 850°R is flowing at a Mach number of 2.54. After expanding around a smooth convex corner, the velocity of the nitrogen is found to be 4000 ft/sec. Through how many degrees did the flow turn?

8.6. A smooth concave turn similar to that shown in Figure 8.2 turns the flow through a 30° angle. The fluid is oxygen, and it approaches the turn at $M_1 = 4.0$.

(a) Compute M_2, T_2/T_1, and p_2/p_1 via the Prandtl–Meyer compression, which occurs close to the wall.

(b) Compute M_2', T_2'/T_1, and p_2'/p_1 via the oblique shock that forms away from the wall. Assume that this flow is also deflected by 30°.

(c) Draw a *T–s* diagram showing each process.

(d) Can these two regions coexist next to one another?

8.7. A simple flat-plate airfoil has a chord of 8 ft and is flying at $M = 1.5$ and a 10° angle of attack. Ambient air pressure is 10 psia, and the temperature is 450°R.

(a) Determine the pressures above and below the airfoil.

(b) Calculate the lift and drag forces per unit span.

(c) Determine the pressure and flow direction as the air leaves the trailing edge (regions 4 and 5 in Figure 8.11).

8.8. The symmetrical diamond-shaped airfoil shown in Figure P8.8 is operating at a 3° angle of attack. The flight speed is $M = 1.8$, and the air pressure equals 8.5 psia.

(a) Compute the pressure on each surface.

(b) Calculate the lift and drag forces.

(c) Repeat the problem with a 10° angle of attack.

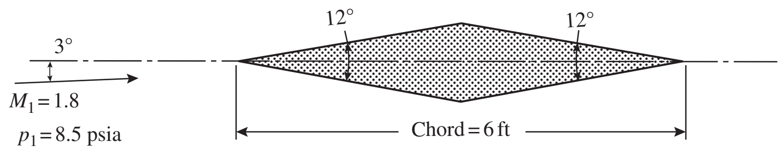


Figure P8.8

8.9. A biconvex airfoil (see Figure 8.14) is constructed of circular arcs. We shall approximate the curve on the upper surface by 10 straight-line segments, as shown in Figure P8.9.

(a) Determine the pressure immediately after the oblique shock at the leading edge.

(b) Determine the Mach number and pressure on each segment.

(c) Compute the contribution to the lift and drag from each segment.

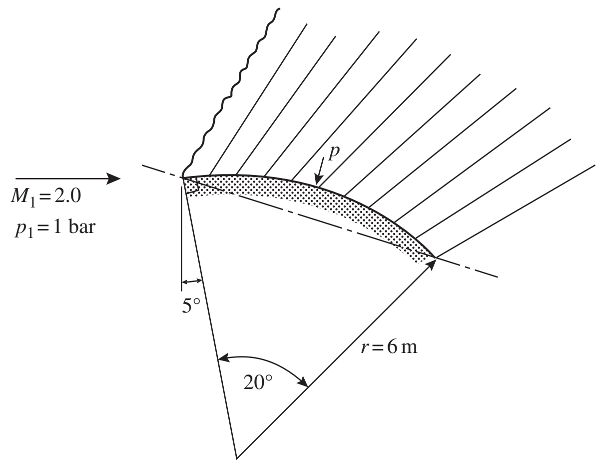


Figure P8.9

8.10. Properties of the flow are given at the exit plane of the two-dimensional duct shown in Figure P8.10. The receiver pressure is 12 psia.

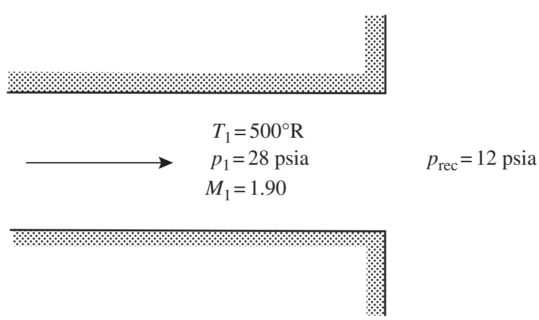


Figure P8.10

(a) Determine the Mach number and temperature just past the exit (after the flow has passed through the first wave formation). Assume that $\gamma = 1.4$.

(b) Make a sketch showing the flow direction, wave angles, and so on.

8.11. Stagnation conditions in a large reservoir are 7 bar and 420 K. A converging-only nozzle delivers nitrogen from this reservoir into a receiver where the pressure is 1 bar.

(a) Sketch the first wave formation that will be seen as the nitrogen leaves the nozzle.

(b) Find the conditions (T, p, V) that exist after the nitrogen has passed through this wave formation.

8.12. Air flows through a converging-diverging nozzle that has an area ratio of 3.5. The nozzle is operating at its third critical point (design condition). The jet stream strikes a two-dimensional wedge with a total wedge angle of 40° as shown in Figure P8.12.

(a) Make a sketch to show the initial wave pattern that results from the jet stream striking the wedge.

(b) Show the additional wave pattern formed by the interaction of the initial wave system with the free boundary. Mark the flow direction in the region following each wave form and show what happens to the free boundary.

(c) Compute the Mach number and direction of flow after the air jet passes through each system of waves.

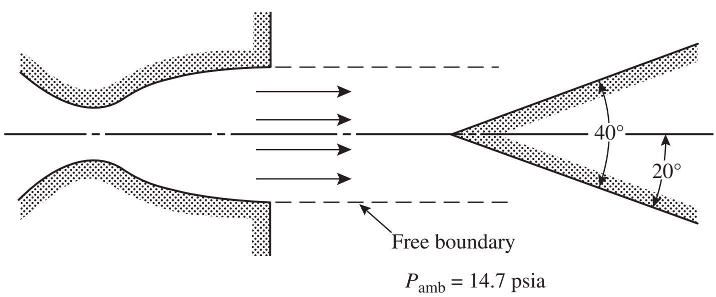


Figure P8.12

8.13. Air flows in a two-dimensional channel and exhausts to the atmosphere as shown in Figure P8.13. Note that the oblique shock just touches the upper corner.

(a) Find the deflection angle.

(b) Determine M_2 and p_2 (in terms of p_{amb}).

(c) What is the nature of the wave form emanating from the upper corner and dividing regions 2 and 3?

(d) Compute M_3, p_3, and T_3 (in terms of T_1). Show the flow direction in region 3.

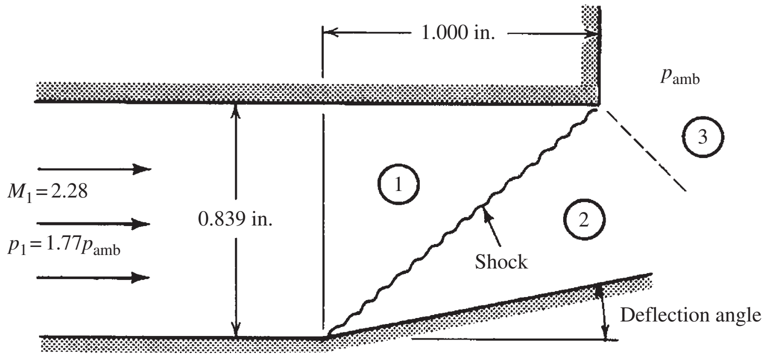


Figure P8.13

8.14. A supersonic nozzle produces a flow of nitrogen at $M_1 = 2.0$ and $p_1 = 0.7$ bar. This discharges into an ambient pressure of 1.0 bar, producing the flow pattern shown in Figure 8.8.

(a) Compute the pressures, Mach numbers, and flow directions in regions 2, 3, and 4.

(b) Make a sketch of the exit jet showing all angles to scale (streamlines, shock lines, and Mach lines).

8.15. Consider the expression for the Prandtl–Meyer function that is given in equation (8.48).

(a) Show that the maximum possible value for ν is

$$\nu_{max} = \frac{\pi}{2}\left(\sqrt{\frac{\gamma+1}{\gamma-1}} - 1\right)$$

(b) At what Mach number does this occur?

(c) If $\gamma = 1.4$, what are the maximum turning angles for accelerating flows with initial Mach numbers of 1.0, 2.0, 5.0, and 10.0?

(d) If a flow of air at $M = 2.0$, $p = 100$ psia, and $T = 600°R$ expands through its maximum turning angle, what is the velocity?

8.16. Flow, initially at a Mach number of unity, expands around a corner through angle ν and reaches Mach number M_2 (see Figure P8.16). Lengths L_1 and L_2 are measured perpendicular to the wall and measure the distance out to the same streamline as shown.

(a) Derive an equation for the ratio $L_2/L_1 = f(M_2, \gamma)$. (*Hints:* What fundamental concept must be obeyed? What kind of process is this?)

(b) If $M_1 = 1.0$, $M_2 = 1.79$, and $\gamma = 1.67$, compute the ratio L_2/L_1.

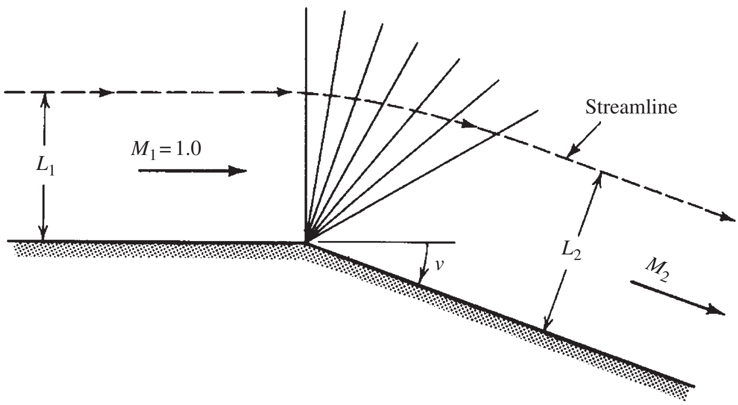


Figure P8.16

8.17. Nitrogen flows along a horizontal surface at M_1 = 2.5. Calculate and sketch the constant-slope surface orientation angles (with respect to the horizontal) that would cause a change by Prandtl–Meyer flow to

(a) M_{2a} = 2.9 and

(b) M_{2b} = 2.1.

(c) Should these changes be equal? State why or why not.

8.18. An experimental drone aircraft in the shape of a flat-plate wing flies at an angle of attack α. It operates at a Mach number of 3.0.

(a) Find the maximum α consistent with *both* an attached oblique shock on the airfoil and a Mach number over the airfoil not exceeding 5.

(b) Find the ratio of lift to (wave) drag forces on this airfoil at the α of part (a). You may assume an arbitrary chord length c.

8.19. In Figure 8.15, the total *width* of the exit jet in the *aerospike nozzle* is seen to increase by a about factor of five. Ignoring all losses and assuming that the upper diagram matches a sea-level exit pressure, estimate the corresponding matched exit pressure for the lower diagram and a corresponding local altitude. Assume that all nozzles are chocked and operate with a chamber pressure of 10.0 MPa with γ = 1.26 and that the effective overall rectangular nozzle exit area has a common constant *length L*. (In order to get the resulting exit pressure as an altitude, you will need to use a Standard Atmosphere table.)

8.20. Rework Problem 8.1 for helium, water vapor, and carbon dioxide using the *Gasdynamics Calculator* and γ entries in Appendix A or B.

CHECK TEST

You should be able to complete this test without reference to material in the chapter.

8.1. For very weak oblique shocks, state how entropy changes and pressure changes are related to deflection angles.

8.2. Explain what the Prandtl–Meyer function represents. (That is, if someone were to say that $\nu = 36.8°$, what would this mean to you?)

8.3. State the rules for wave reflection.

(a) Waves reflect off physical boundaries as _____.

(b) Waves reflect off free boundaries as _____.

8.4. A flow at $M_1 = 1.5$ and $p_1 = 2 \times 10^5$ N/m^2 approaches a sharp turn. After negotiating the turn, the pressure is 1.5×10^5 N/m^2. Determine the deflection angle if the fluid is oxygen.

8.5. Compute the net force (per square foot of area) acting on the flat-plate airfoil shown in Figure CT8.5.

Figure CT8.5

8.6. **(a)** Sketch the waveforms that you might expect to find over the airfoil shown in Figure CT8.6.

(b) Identify all wave forms by name.

(c) State the boundary conditions that must be met as the flow comes off the trailing edge of the airfoil.

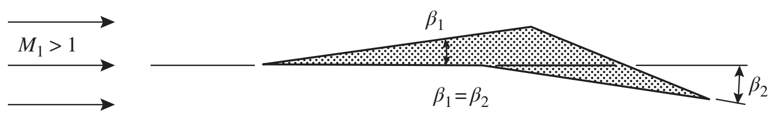


Figure CT8.6

8.7. Figure CT8.7 is a representation of a Schlieren photo showing a converging–diverging nozzle in operation. Indicate whether the pressures in regions a, b, c, d, and e are equal to, greater than, or less than the receiver pressure.

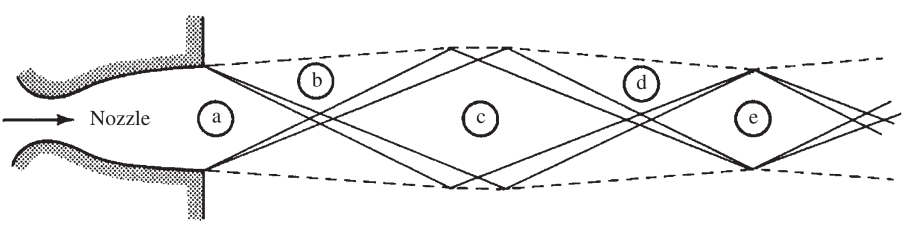


Figure CT8.7

Chapter 9

Fanno Flow

9.1 INTRODUCTION

At the start of Chapter 5 we mentioned that area changes, friction, and heat transfer are the most important factors affecting the properties in a flow system. Up to this point we have considered only one of these factors, that of variations in area. However, we have also discussed the various mechanisms by which a flow adjusts to meet imposed boundary conditions of either flow direction or pressure equalization. We now wish to take a look at the subject of friction losses.

In order to focus only on the effects of friction, we analyze flows in constant-area ducts without heat transfer. This corresponds to many practical flow situations that involve relatively short ducts. We consider first the flow of an arbitrary fluid and discover that its behavior follows definite patterns which depend notably on whether the flow is in the subsonic or supersonic regime. Working equations are developed for the case of a perfect gas, and the introduction of a reference point allows tabulation to be constructed. As before, such table permits rapid solutions to many cases of this type, which are called *Fanno flow* problems.

9.2 OBJECTIVES

After completing this chapter successfully, you should be able to:

1. List the assumptions made in the analysis of Fanno flow.
2. (*Optional*) Simplify the general equations of continuity, energy, and momentum to obtain basic relations valid for any fluid in Fanno flow.

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

3. Sketch a Fanno line in the *h–v* and the *h–s* planes. Identify the sonic point and regions of subsonic and supersonic flow.
4. Describe the variation of static and stagnation pressure, static and stagnation temperature, static density, and velocity as flow progresses along a Fanno line. Do this for both subsonic and supersonic flow.
5. (*Optional*) Starting with basic principles of continuity, energy, and momentum, derive expressions for property ratios such as T_2/T_1, p_2/p_1, and so on, in terms of Mach number (M) and specific heat ratio (γ) for Fanno flow with a perfect gas.
6. Describe (including a *T–s* diagram) how the Fanno table is developed with the use of a * reference location.
7. *Define friction factor, equivalent diameter, absolute and relative roughness, absolute and kinematic viscosity, and Reynolds number, and know how to determine each.*
8. Compare similarities and differences between Fanno flow and normal shocks. Sketch an *h–s* diagram showing a typical Fanno line together with a normal shock for the same mass velocity.
9. Explain what is meant by *friction choking*.
10. (*Optional*) Describe some possible consequences of adding duct in a choked Fanno flow situation (for both subsonic and supersonic flow).
11. Demonstrate the ability to solve typical Fanno flow problems by use of the appropriate tables and equations.

9.3 ANALYSIS FOR A GENERAL FLUID

We first consider the general behavior of an arbitrary fluid. To isolate the effects of friction, we make the following assumptions:

Steady one-dimensional flow	
Adiabatic	$\delta q = 0$ or $ds_e = 0$
No shaft work	$\delta w_s = 0$
Neglect potential	$dz = 0$
Constant area	$dA = 0$

We proceed by applying the basic concepts of continuity, energy, and momentum.

Continuity

$$\dot{m} = \rho A V = \text{const} \tag{2.30}$$

but since the flow area is constant, this reduces to

$$\rho V = \text{const} \tag{9.1}$$

We assign a new symbol G to this constant (the quantity ρV), which is referred to as the *mass velocity*, and thus

$$\boxed{\rho V = G = \text{const}} \tag{9.2}$$

What are the typical units of G?

Energy

We start with

$$h_{t1} + \cancel{q} = h_{t2} + \cancel{w_s} \tag{3.19}$$

For adiabatic and no work, this becomes

$$h_{t1} = h_{t2} = h_t = \text{const} \tag{9.3}$$

If we neglect the potential energy term, this means that

$$h_t = h + \frac{V^2}{2g_c} = \text{const} \tag{9.4}$$

Substitute for the velocity from equation (9.2) and *show* that

$$\boxed{h_t = h + \frac{G^2}{\rho^2 2g_c} = \text{const}} \tag{9.5}$$

Now for any given flow, the constant h_t and G are known. Thus equation (9.5) establishes a unique relationship between h and ρ. Figure 9.1 is a plot of this equation in the h–v plane for various values of G (but all for the same h_t). Each curve is called a *Fanno line* and represents flow at a particular *mass velocity*. Note carefully that this is constant G and not constant $\dot{m}$. Ducts of various sizes could pass the same mass flow rate but would have different mass velocities.

Once the fluid is known, one can also plot lines of constant entropy on the h–v diagram. Typical curves of s = constant are shown as dashed lines in the figure. It is much more instructive to plot these Fanno lines in the familiar h–s plane. Such a

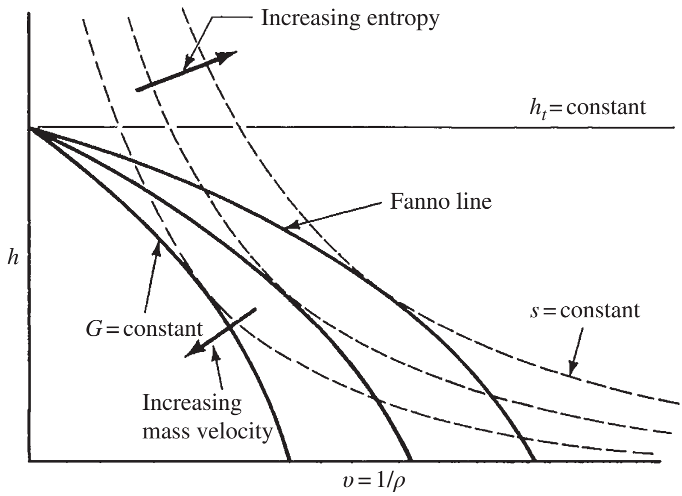


Figure 9.1 Fanno lines in *h–v* plane.

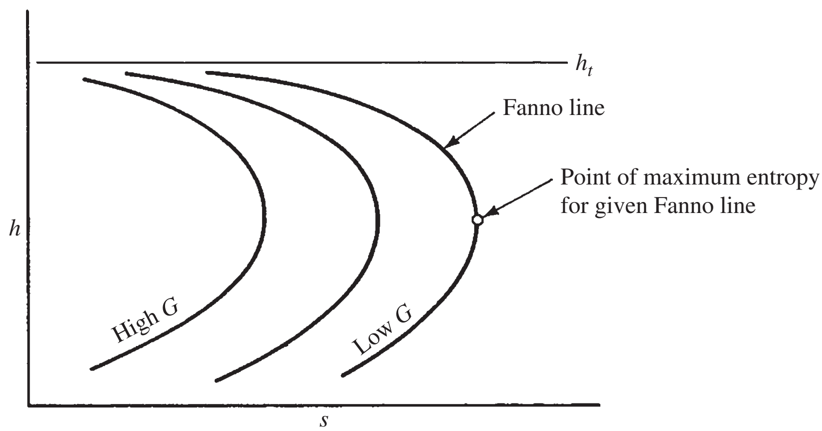


Figure 9.2 Fanno lines in *h–s* plane.

diagram is shown in Figure 9.2. At this point, a significant fact becomes quite clear. Since we have assumed that there is no heat transfer ($ds_e = 0$), the *only* way that entropy can be changed is through irreversibilities (ds_i). Thus *the flow can only progress toward increasing values of entropy*! Why? Can you locate the points of maximum entropy for each Fanno line in Figure 9.1?

Let us examine one Fanno line in greater detail. Figure 9.3 shows a given Fanno line together with typical pressure lines. All points on this line represent states with the same mass flow rate per unit area (mass velocity) and the same stagnation enthalpy. Due to the irreversible nature of the frictional effects, the flow can only proceed to the right. Thus the Fanno line is divided into two distinct parts, an upper and a lower branch, which are separated by a limiting point of maximum entropy.

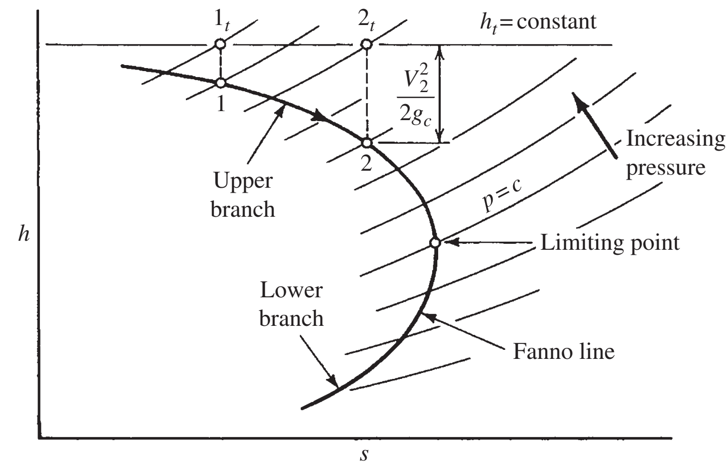


Figure 9.3 Two branches of a Fanno line.

What does intuition tell us about adiabatic flow in a constant-area duct? We normally feel that frictional effects will show up as an internal generation of "heat" with a corresponding reduction in density of the fluid. To pass the same flow rate (with constant area), continuity then forces the velocity to increase. This increase in kinetic energy must cause a decrease in enthalpy, since the stagnation enthalpy remains constant. As can be seen in Figure 9.3, this agrees with flow along the *upper branch* of the Fanno line. It is also clear that in this case both the static and stagnation pressure are decreasing.

But what about flow along the *lower branch?* Mark two points on the lower branch and draw an arrow to indicate proper movement along the Fanno line. What is happening to the enthalpy? To the density [see equation (9.5)]? To the velocity [see equation (9.2)]? From the figure, what is happening to the static pressure? The stagnation pressure? Fill in Table 9.1 with *increase, decrease,* or *remains constant.*

Table 9.1 Analysis of Fanno Flow for Figure 9.3

Property	Upper Branch	Lower Branch
Enthalpy		
Density		
Velocity		
Pressure (static)		
Pressure (stagnation)		

Notice that on the lower branch, properties do not vary in the manner predicted by *intuition*. Thus this must be a flow regime with which we are not very familiar. Before we investigate the limiting point that separates these two flow regimes, let us note that these flows do have one thing in common. Recall the stagnation pressure energy equation from Chapter 3.

$$\frac{dp_t}{\rho_t} + \cancel{ds_e}(T_t - T) + T_t ds_i + \cancel{\delta w_s} = 0 \tag{3.25}$$

For Fanno flow, $ds_e = \delta w_s = 0$.

Thus any frictional effect must cause a decrease in the total or stagnation pressure! Figure 9.3 implies this for any flow along either the upper or lower branches of the Fanno line.

Limiting Point

From the energy equation we had developed,

$$h_t = h + \frac{V^2}{2g_c} = \text{constant} \tag{9.4}$$

Differentiating, we obtain

$$dh_t = dh + \frac{VdV}{g_c} = 0 \tag{9.6}$$

From continuity we had found that

$$\rho V = G = \text{constant} \tag{9.2}$$

Differentiating this, we obtain

$$\rho dV + V d\rho = 0 \tag{9.7}$$

which can be solved for

$$dV = -V\frac{d\rho}{\rho} \tag{9.8}$$

Introduce equation (9.8) into (9.6) and *show* that

$$dh = \frac{V^2 d\rho}{g_c \rho} \tag{9.9}$$

Now recall the property relation

$$Tds = dh - vdp \tag{1.31}$$

which can be written as

$$Tds = dh - \frac{dp}{\rho} \tag{9.10}$$

Substituting for dh from equation (9.9) yields

$$\boxed{Tds = \frac{V^2 d\rho}{g_c \rho} - \frac{dp}{\rho}} \tag{9.11}$$

We hasten to point out that this expression is valid for *any* fluid and between two differentially separated points *anyplace* along the Fanno line. Now let's apply equation (9.11) to two adjacent points that surround the limiting point of maximum entropy. At this location s = const; thus $ds = 0$, and (9.11) becomes

$$\frac{V^2 d\rho}{g_c} = dp \quad \text{at limit point} \tag{9.12}$$

or

$$V^2 = g_c \left(\frac{dp}{d\rho}\right)_{\text{at limit point}} = g_c \left(\frac{\partial p}{\partial \rho}\right)_{s=\text{const}} \tag{9.13}$$

This should be a familiar expression [see equation (4.5)] and we recognize that *the velocity is sonic at the limiting point*. The upper branch can now be more significantly called the *subsonic branch,* and the lower branch is seen to be the *supersonic branch*.

Now we begin to see a reason for the failure of our intuition to predict behavior on the lower branch of the Fanno line. From our studies in Chapter 5 we saw that fluid behavior in supersonic flow is frequently contrary to our expectations. This results from the fact that most of us live our lives only "subsonically," and moreover our acquaintance with fluid phenomena comes mainly from experiences with incompressible fluids. It should be apparent that we cannot use our intuition to guess at what might be happening, particularly in the supersonic flow regime. We must become religious and put faith in our carefully derived relations.

Momentum

The foregoing analysis was made using only the continuity and energy relations. We now proceed to apply momentum concepts to the control volume shown in Figure 9.4. The x-component of the momentum equation for steady, one-dimensional flow is

$$\sum F_x = \frac{\dot{m}}{g_c}(V_{\text{out}_x} - V_{\text{in}_x}) \tag{3.46}$$

From Figure 9.4 we see that the force summation is

$$\sum F_x = p_1 A - p_2 A - F_f \tag{9.14}$$

where F_f represents the total wall frictional force on the fluid (arising from the wetted surface) between sections 1 and 2. Thus the momentum equation in the direction of flow becomes

$$(p_1 - p_2)A - F_f = \frac{\dot{m}}{g_c}(V_2 - V_1) = \frac{\rho A V}{g_c}(V_2 - V_1) \tag{9.15}$$

Show that equation (9.15) can be written as

$$p_1 - p_2 - \frac{F_f}{A} = \frac{\rho_2 V_2^2}{g_c} - \frac{\rho_1 V_1^2}{g_c} \tag{9.16}$$

or

$$\boxed{\left(p_1 + \frac{\rho_1 V_1^2}{g_c}\right) - \frac{F_f}{A} = p_2 + \frac{\rho_2 V_2^2}{g_c}} \tag{9.17}$$

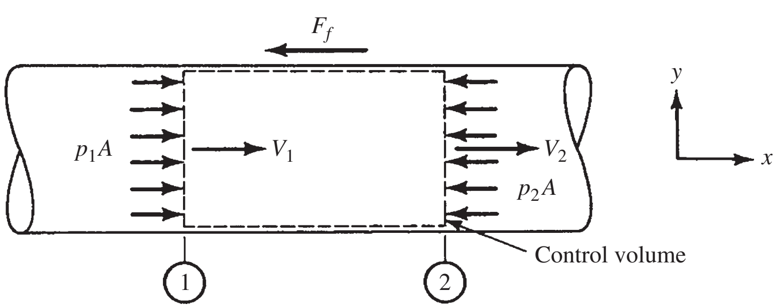


Figure 9.4 Momentum analysis for Fanno flow.

In this form the equation is useful because it brings out the significant fact that for *steady, one-dimensional, constant-area flows of any fluid,* the value of $p + \rho V^2/g_c$ *cannot* remain constant when significant frictional forces are present. This fact will be recalled later in the chapter when Fanno flow is compared with normal shocks.

Before leaving this section on fluids in general, we might say a few words about Fanno flow at low Mach numbers. A glance at Figure 9.3 shows that the upper branch is asymptotically approaching the horizontal line of constant total enthalpy. Thus, the extreme left end of the Fanno line will be nearly horizontal. This indicates that flow at very low Mach numbers will have almost constant velocity. This validates our previous work, which indicated that we could treat gases as incompressible fluids when the Mach numbers were very small.

9.4 WORKING EQUATIONS FOR PERFECT GASES

We have discovered the general trend of property variations that occur in Fanno flow, both in the subsonic and supersonic flow regime. Now we wish to develop some specific working equations for the case of a perfect gas. Recall that these are to be relations between properties at arbitrary sections of a flow system written in terms of Mach numbers and the specific heat ratio.

Energy

We start with the energy equation developed in Section 9.3 since this leads immediately to a temperature ratio:

$$h_{t1} = h_{t2} \tag{9.3}$$

But for a perfect gas, enthalpy is a function of temperature only. Therefore,

$$T_{t1} = T_{t2} \tag{9.18}$$

Now for a perfect gas with constant specific heats,

$$T_t = T\left(1 + \frac{\gamma - 1}{2}M^2\right) \tag{4.18}$$

Hence the energy equation for Fanno flow can be written as

$$T_1\left(1 + \frac{\gamma - 1}{2}M_1^2\right) = T_2\left(1 + \frac{\gamma - 1}{2}M_2^2\right) \tag{9.19}$$

or

$$\boxed{\frac{T_2}{T_1} = \frac{1 + [(\gamma - 1)/2]\, M_1^2}{1 + [(\gamma - 1)/2]\, M_2^2}} \tag{9.20}$$

Continuity

From Section 9.3 we have

$$\rho V = G = \text{const} \tag{9.2}$$

or

$$\rho_1 V_1 = \rho_2 V_2 \tag{9.21}$$

If we introduce the perfect gas equation of state

$$p = \rho R T \tag{1.16}$$

the definition of Mach number

$$V = Ma \tag{4.11}$$

and sonic speed for a perfect gas

$$a = \sqrt{\gamma g_c R T} \tag{4.10}$$

equation (9.21) can be solved for

$$\frac{p_2}{p_1} = \frac{M_1}{M_2}\left(\frac{T_2}{T_1}\right)^{1/2} \tag{9.22}$$

Can you obtain this expression? Now introduce the temperature ratio from (9.20) and you will have the following working relation for static pressure:

$$\boxed{\frac{p_2}{p_1} = \frac{M_1}{M_2}\left(\frac{1 + [(\gamma - 1)/2]\, M_1^2}{1 + [(\gamma - 1)/2]\, M_2^2}\right)^{1/2}} \tag{9.23}$$

The density relation can easily be obtained from equation (9.20), (9.23), and the perfect gas law:

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{M_1}{M_2}\left(\frac{1+[(\gamma-1)/2]\,M_2^2}{1+[(\gamma-1)/2]\,M_1^2}\right)^{1/2}} \tag{9.24}$$

Entropy Change

We start with an expression for entropy change that is valid between any two points:

$$\Delta s_{1-2} = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1} \tag{1.43}$$

Equation (4.15) can be used to substitute for c_p and we nondimensionalize the equation to

$$\frac{s_2-s_1}{R} = \frac{\gamma}{\gamma-1}\ln\frac{T_2}{T_1} - \ln\frac{p_2}{p_1} \tag{9.25}$$

If we now utilize the expressions just developed for the temperature ratio (9.20) and the pressure ratio (9.23), the entropy change becomes

$$\frac{s_2-s_1}{R} = \frac{\gamma}{\gamma-1}\ln\left(\frac{1+[(\gamma-1)/2]\,M_1^2}{1+[(\gamma-1)/2]\,M_2^2}\right) - \ln\frac{M_1}{M_2}\left(\frac{1+[(\gamma-1)/2]\,M_1^2}{1+[(\gamma-1)/2]\,M_2^2}\right)^{1/2} \tag{9.26}$$

Show that this entropy change between two points in Fanno flow can be written as

$$\frac{s_2-s_1}{R} = \ln\frac{M_2}{M_1}\left(\frac{1+[(\gamma-1)/2]\,M_1^2}{1+[(\gamma-1)/2]\,M_2^2}\right)^{(\gamma+1)/2(\gamma-1)} \tag{9.27}$$

Now recall that in Section 4.5 we integrated the stagnation pressure–energy equation for adiabatic no-work flow of a perfect gas, with the result

$$\frac{p_{t2}}{p_{t1}} = e^{-\Delta s/R} \tag{4.28}$$

Thus, from equations (4.28) and (9.27) we obtain a simple expression for the stagnation pressure ratio:

$$\boxed{\frac{p_{t2}}{p_{t1}} = \frac{M_1}{M_2}\left(\frac{1+[(\gamma-1)/2]\,M_2^2}{1+[(\gamma-1)/2]\,M_1^2}\right)^{(\gamma+1)/2(\gamma-1)}} \tag{9.28}$$

We now have the means to obtain all the properties at a downstream point 2 if we know all the properties at some upstream point 1 *and* the Mach number at point 2. However, in many situations one does not know both Mach numbers. A typical problem would be to predict the final Mach number, given the initial conditions and information on duct length, material, and so on. Thus our next job must be to relate the change in Mach number to the friction losses.

Momentum

We turn to the differential form of the momentum equation that was developed in Chapter 3:

$$\frac{dp}{\rho} + f\frac{V^2 dx}{2g_c D_e} + \frac{g}{g_c}dz + \frac{dV^2}{2g_c} = 0 \tag{3.63}$$

Our objective is to get all this equation in terms of Mach number. If we introduce the perfect gas equation of state together with expressions for Mach number and sonic speed, we obtain

$$\frac{dp}{p}(RT) + f\frac{dx}{D_e}\frac{M^2\gamma g_c RT}{2g_c} + \frac{g}{g_c}dz + \frac{dM^2\gamma g_c RT + M^2\gamma g_c R dT}{2g_c} = 0 \tag{9.29}$$

or

$$\boxed{\frac{dp}{p} + f\frac{dx}{D_e}\frac{\gamma}{2}M^2 + \frac{g\,dz}{g_c RT} + \frac{\gamma}{2}dM^2 + \frac{\gamma}{2}M^2\frac{dT}{T} = 0} \tag{9.30}$$

Equation (9.30) is boxed since it is a useful form of the momentum equation that is valid for *all* steady flow problems involving a perfect gas. We now proceed to apply this to Fanno flow. From (9.18) and (4.18) we know that

$$T_t = T\left(1 + \frac{\gamma-1}{2}M^2\right) = \text{const} \tag{9.31}$$

Taking natural logarithms

$$\ln T + \ln\left(1 + \frac{\gamma - 1}{2}M^2\right) = \ln(\text{const}) \tag{9.32}$$

and then differentiating, we obtain

$$\frac{dT}{T} + \frac{d(1 + [(\gamma - 1)/2]M^2)}{1 + [(\gamma - 1)/2]M^2} = 0 \tag{9.33}$$

which can be used to substitute for *dT/T* in (9.30).

The continuity relation [equation (9.2)] put in terms of a perfect gas becomes

$$\frac{pM}{\sqrt{T}} = \text{const} \tag{9.34}$$

By logarithmic differentiation (take the natural logarithm and then differentiate) *show* that

$$\frac{dp}{p} + \frac{dM}{M} - \frac{1}{2}\frac{dT}{T} = 0 \tag{9.35}$$

We can introduce equation (9.33) to eliminate *dT/T*, with the result that

$$\frac{dp}{p} = -\frac{dM}{M} - \frac{1}{2}\frac{d(1 + [(\gamma - 1)/2]M^2)}{1 + [(\gamma - 1)/2]M^2} \tag{9.36}$$

which can be used to substitute for *dp/p* in (9.30).

Make the indicated substitutions for *dp/p* and *dT/T* in the momentum equation, neglect the potential term, and *show* that equation (9.30) can be put into the following form:

$$f\frac{dx}{D_e} = \frac{d(1 + [(\gamma - 1)/2]M^2)}{1 + [(\gamma - 1)/2]M^2} - \frac{dM^2}{M^2} + \frac{2}{\gamma}\frac{dM}{M^3} + \frac{1}{\gamma M^2}\frac{d(1 + [(\gamma - 1)/2]M^2)}{1 + [(\gamma - 1)/2]M^2} \tag{9.37}$$

The last term can be simplified for integration by noting that

$$\frac{1}{\gamma M^2}\frac{d(1 + [(\gamma - 1)/2]M^2)}{1 + [(\gamma - 1)/2]M^2} = \frac{(\gamma - 1)}{2\gamma}\frac{dM^2}{M^2} - \frac{(\gamma - 1)}{2\gamma}\frac{d(1 + [(\gamma - 1)/2]M^2)}{1 + [(\gamma - 1)/2]M^2} \tag{9.38}$$

The momentum equation may now be written as

$$f\frac{dx}{D_e}=\frac{\gamma+1}{2\gamma}\frac{d(1+[(\gamma-1)/2]M^2)}{1+[(\gamma-1)/2]M^2}+\frac{2}{\gamma}\frac{dM}{M^3}-\frac{\gamma+1}{2\gamma}\frac{dM^2}{M^2} \tag{9.39}$$

Equation (9.39) is restricted to steady, one-dimensional flow of a perfect gas, with no heat or work transfer, constant area, and negligible potential changes. We can now integrate this equation between two points in the flow and obtain

$$\boxed{\frac{f(x_2-x_1)}{D_e}=\frac{\gamma+1}{2\gamma}\ln\frac{1+[(\gamma-1)/2]\,M_2^2}{1+[(\gamma-1)/2]\,M_1^2}-\frac{1}{\gamma}\left(\frac{1}{M_2^2}-\frac{1}{M_1^2}\right)-\frac{\gamma+1}{2\gamma}\ln\frac{M_2^2}{M_1^2}} \tag{9.40}$$

Note that in performing the integration we have held the friction factor constant. Some comments will be made on this in a later section. If you have forgotten the concept of equivalent diameter, you may want to review the last part of Section 3.8 and equation (3.61).

9.5 REFERENCE STATE AND FANNO TABLE

The equations developed in Section 9.4 provide the means of computing the properties at one location in terms of those given at some other location. The key to problem solution is predicting the Mach number at the new location through the use of equation (9.40). The solution of this equation for the unknown M_2 presents a messy task, as no explicit relation is possible. Thus we turn to a technique similar to that used with isentropic flow in Chapter 5.

We introduce *another* * reference state, which is defined in the same manner as before (i.e., "that thermodynamic state which would exist if the fluid reached a Mach number of unity *by a particular process*"). In this case we imagine that we continue *by Fanno flow* (i.e., more duct is added) until the velocity reaches Mach 1.0. Figure 9.5 shows a physical system together with its *T–s* diagram for a subsonic Fanno flow. We know that if we continue along the Fanno line (remember that we always move to the right), we will eventually reach the limiting point where the sonic speed exists. The dashed lines show a hypothetical duct of sufficient length to enable the flow to traverse the remaining portion of the upper branch and reach the limit point. This is the * *reference point for Fanno flow*.

The locus of *isentropic* * reference points has also been included on the *T–s* diagram to emphasize the fact that the Fanno * reference is a totally different thermodynamic state. One other fact should be mentioned: If there is any entropy difference between two points (such as points 1 and 2), their isentropic * reference

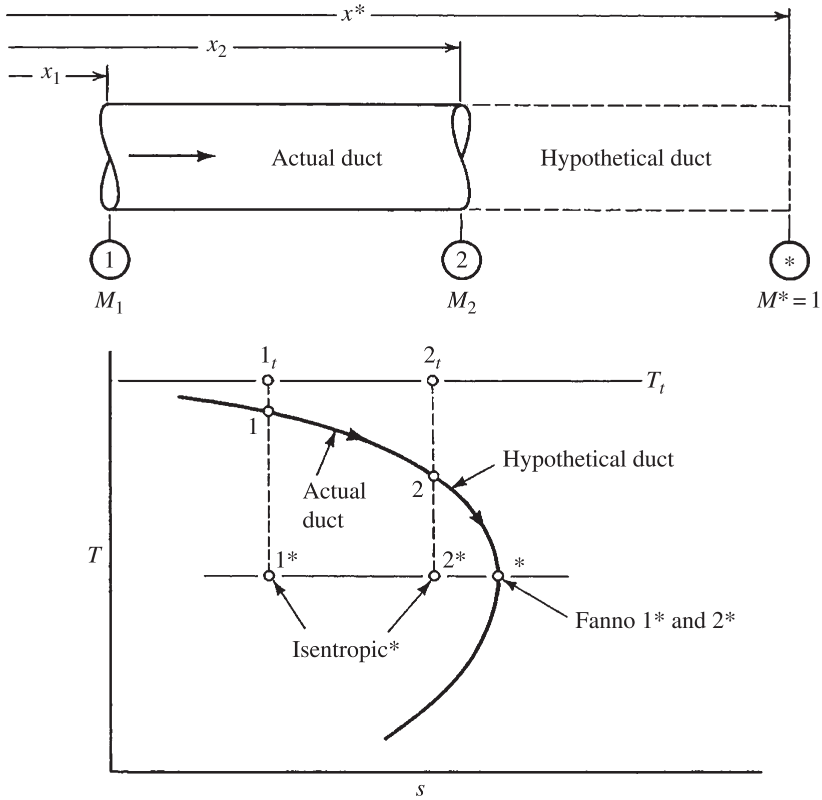


Figure 9.5 The * reference for Fanno flow.

conditions are not the same and we have always taken great care to label them separately as 1^* and 2^*(as shown in Figure 9.5).

However, proceeding from either point 1 or point 2 *by Fanno flow* will ultimately lead to the same place when Mach 1.0 is reached. Thus we do not have to talk of 1^* or 2^* but merely * in the case of Fanno flow. Incidentally, why are all three * reference points shown on the same horizontal line in Figure 9.5? (You may need to review Section 4.6.)

We now rewrite the working equations in terms of the Fanno flow * reference condition. Consider first

$$\frac{T_2}{T_1} = \frac{1 + [(\gamma - 1)/2]\, M_1^2}{1 + [(\gamma - 1)/2]\, M_2^2} \tag{9.20}$$

Let point 2 be an arbitrary point in the flow system and let its Fanno * condition be point 1. Then

$$T_2 \Rightarrow T \quad M_2 \Rightarrow M(\text{any value})$$
$$T_1 \Rightarrow T^* \quad M_1 \Rightarrow 1$$

and equation (9.20) becomes

$$\frac{T}{T^*} = \frac{(\gamma+1)/2}{1+[(\gamma-1)/2]M^2} = f(M,\gamma) \tag{9.41}$$

We see that $T/T^* = f(M, \gamma)$ and we can readily construct a table giving values of T/T^* versus M for a particular γ. Equation (9.23) can be treated in a similar fashion. In this case

$$p_2 \Rightarrow p \quad M_2 \Rightarrow M(\text{any value})$$
$$p_1 \Rightarrow p^* \quad M_1 \Rightarrow 1$$

and equation (9.23) becomes

$$\frac{p}{p^*} = \frac{1}{M}\left(\frac{(\gamma+1)/2}{1+[(\gamma-1)/2]M^2}\right)^{1/2} = f(M,\gamma) \tag{9.42}$$

The density ratio can be obtained as a function of Mach number and γ from equation (9.24). This is particularly useful since it also represents a velocity ratio. Why?

$$\frac{\rho}{\rho^*} = \frac{V^*}{V} = \frac{1}{M}\left(\frac{1+[(\gamma-1)/2]M^2}{(\gamma+1)/2}\right)^{1/2} = f(M,\gamma) \tag{9.43}$$

Apply the same techniques to equation (9.28) and *show* that

$$\frac{p_t}{p_t^*} = \frac{1}{M}\left(\frac{1+[(\gamma-1)/2]M^2}{(\gamma+1)/2}\right)^{(\gamma+1)/2(\gamma-1)} = f(M,\gamma) \tag{9.44}$$

We now perform the same type of transformation on equation (9.40); that is, let

$$x_2 \Rightarrow x \quad M_2 \Rightarrow M \quad (\text{any value})$$
$$x_1 \Rightarrow x^* \quad M_1 \Rightarrow 1$$

with the following result:

$$\frac{f(x-x^*)}{D_e} = \left(\frac{\gamma+1}{2\gamma}\right)\ln\left(\frac{1+[(\gamma-1)/2]M^2}{(\gamma+1)/2}\right) - \frac{1}{\gamma}\left(\frac{1}{M^2}-1\right) - \frac{\gamma+1}{2\gamma}\ln M^2 \tag{9.45}$$

But a glance at the physical diagram in Figure 9.5 shows that $(x - x^*)$ will always be a negative quantity; thus it is more convenient to change all signs in equation (9.45) and simplify it to

$$\frac{f(x^* - x)}{D_e} = \left(\frac{\gamma + 1}{2\gamma}\right) \ln \left(\frac{[(\gamma + 1)/2]M^2}{1 + [(\gamma - 1)/2]M^2}\right) + \frac{1}{\gamma}\left(\frac{1}{M^2} - 1\right) = f(M, \gamma) \tag{9.46}$$

The quantity $(x^* - x)$ represents the amount of duct that would have to be added to cause the flow to reach the Fanno * reference condition. It can alternatively be viewed as the maximum duct length that may be added without changing a given flow condition. Thus, the expression

$$\frac{f(x^* - x)}{D_e} \quad \text{is called} \quad \frac{fL_{\max}}{D_e}$$

and is listed in Appendix I along with the other Fanno flow parameters: T/T^*, p/p^*, V/V^*, and p_t/p_t^*. In the next section we shall see how this table greatly simplifies the solution of Fanno flow problems. But first, some words about the determination of friction factors.

Dimensional analysis (see Sections 1.3 and 9.9) of this fluid flow problem shows that the friction factor may be expressed as

$$f = f(\text{Re}, \varepsilon/D) \tag{9.47}$$

where Re is the *Reynolds number* based on the tube diameter D and the mass average velocity V

$$\text{Re} \equiv \frac{\rho V D}{\mu g_c} \tag{9.48}$$

and

$$\varepsilon/D \equiv \textit{relative roughness} \text{ (for wetted surface)}$$

Typical values of ε, the *absolute roughness* or average (effective) height of wall irregularities, are shown in Table 9.2.

The relationship among f, Re, and ε/D has been determined experimentally and plotted on a chart similar to Figure 9.6, which is called a *Moody diagram*; here the use of dimensionless quantities greatly generalizes experimental data. A larger working chart appears in Appendix C. If the flow rate is known together with the duct size and its wall material, the Reynolds number and relative roughness can easily be calculated and the value of the friction factor then found from the

Table 9.2 Absolute Roughness of Common Materials

Material	ε (ft)
Glass, brass, copper, lead	smooth < 0.00001
Steel, wrought iron	0.00015
Galvanized iron	0.0005
Cast iron	0.00085
Riveted steel	0.03

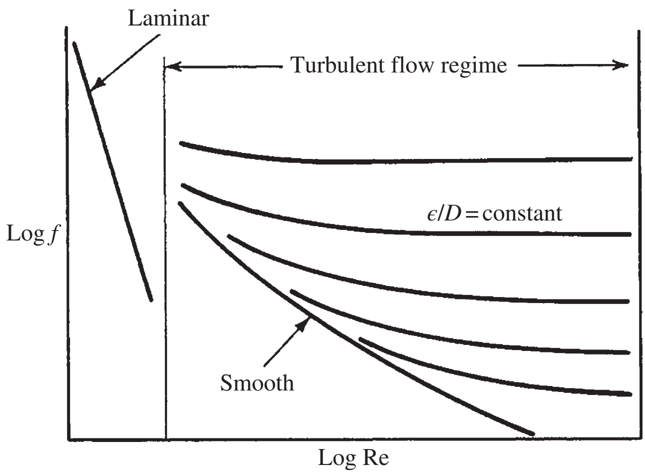


Figure 9.6 Moody diagram for friction factor in circular ducts. (See Appendix C for working chart in the turbulent flow regime.)

diagram. The curve for the laminar flow region is not one-dimensional (see Figure 2.1) and has been omitted from Appendix C; it is well represented by

$$f = \frac{64}{\text{Re}} \quad (\text{laminar fully-developed tube flow}) \tag{9.49}$$

For noncircular cross sections the *equivalent diameter* as described in Section 3.8 can be used.

$$D_e \equiv \frac{4A}{P} \tag{3.61}$$

This equivalent diameter may be used in the determination of relative roughness and Reynolds number, and hence the friction factor. However, care must be taken

to work with the *actual* mass average velocity (V) in all computations. Experience has shown that the use of an equivalent diameter works quite well in the turbulent zone. In the laminar flow region this concept is *not* sufficient and consideration must also be given to the aspect ratio of the duct.

In some problems the flow rate is not known and thus a trial-and-error solution results. As long as the duct size is given, the problem is not too difficult; an excellent approximation to the friction factor can be made by taking the value corresponding to where the ε/D curve begins to *level off*. This converges rapidly to the final answer, as most engineering problems are well into the turbulent range.

The last entry in Appendix I is S_{max}/R. In order to tabulate equation (9.27) for entropy changes along the Fanno line we need to represent the entropy maximum at Mach 1.0 by introducing the * reference condition. Take equation (9.27) and define location 2 at this maximum with location 1 kept variable.

Let $s_2 = s$ (at $M = 1.0$) and $s_1 = s(M)$, the table notation is $S_{max} \equiv s^* - s$,

$$\frac{S_{max}}{R} = \frac{s^* - s}{R} = \ln \frac{1}{M}\left(\frac{1+[(\gamma-1)/2]M^2}{1+(\gamma-1)/2}\right)^{(\gamma+1)/2(\gamma-1)} \tag{9.50}$$

Equation (9.50) is tabulated in Appendix I for air at moderate temperatures ($\gamma = 1.4$) becoming $S_{max}/R = [\ln(0.8333 + 0.1667M^2)^3/M]$.

9.6 APPLICATIONS

The following steps are recommended to develop a good problem-solving technique:

1. Sketch the physical situation (including the hypothetical * reference point).
2. Label sections where conditions are known or desired.
3. List all given information with consistent units.
4. Compute the equivalent diameter, relative roughness, and Reynolds number.
5. Find the friction factor from the Moody diagram.
6. Determine the unknown Mach number.
7. Calculate the additional properties desired.

The procedure above may have to be altered depending on what type of information is given and, occasionally, trial-and-error solutions may be required. You should have no difficulty incorporating these features once the basic straightforward solution has been mastered. In complicated flow systems that involve more than just Fanno flow, a T–s diagram is frequently helpful in solving problems.

For the following examples we are dealing with the steady one-dimensional flow of air ($\gamma = 1.4$), which can be treated as a perfect gas. Assume that $Q = W_s = 0$ and negligible potential changes. The cross-sectional area of the duct remains constant. Figure E9.1 is common to Examples 9.1 through 9.3.

Example 9.1

Given $M_1 = 1.80$, $p_1 = 40$ psia, and $M_2 = 1.20$, find p_2 and $f\Delta x/D$.

Since both Mach numbers are known, we can solve immediately for

$$p_2 = \frac{p_2}{p^*}\frac{p^*}{p_1}p_1 = (0.8044)\left(\frac{1}{0.4741}\right)(40) = 67.9\,\text{psia}$$

Check Figure E9.1 to see that

$$\frac{f\Delta x}{D} = \frac{fL_{1\max}}{D} - \frac{fL_{2\max}}{D} = 0.2419 - 0.0336 = 0.208$$

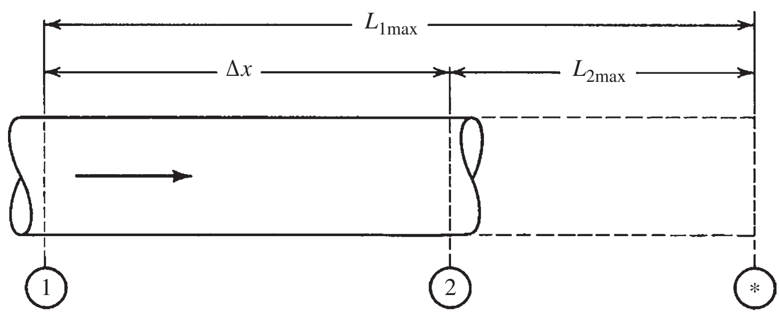


Figure E9.1

Example 9.2

Given $M_2 = 0.94$, $T_1 = 400$ K, and $T_2 = 350$ K, find M_1 and p_2/p_1.

To determine conditions at section 1 in Figure E9.1, we must establish the ratio

$$\frac{T_1}{T^*} = \underbrace{\frac{T_1}{T_2}}_{\text{Given}}\underbrace{\frac{T_2}{T^*}}_{\text{From Fanno table at } M=0.94} = \left(\frac{400}{350}\right)(1.0198) = 1.1655$$

Look up $T/T^* = 1.1655$ in the Fanno table (Appendix I) and determine that $M_1 = 0.385$. Thus

$$\frac{p_2}{p_1} = \frac{p_2}{p^*}\frac{p^*}{p_1} = (1.0743)\left(\frac{1}{2.8046}\right) = 0.383$$

Notice that these examples confirm previous statements concerning static pressure changes. In subsonic flow the static pressure decreases, whereas in supersonic flow the static pressure increases. *Compute* the stagnation pressure ratio and show that the friction losses cause p_{t2}/p_{t1} to decrease in each case.

For Example 9.1:

$$\frac{p_{t2}}{p_{t1}} = \qquad\qquad (p_{t2}/p_{t1} = 0.716)$$

For Example 9.2:

$$\frac{p_{t2}}{p_{t1}} = \qquad\qquad (p_{t2}/p_{t1} = 0.611)$$

Example 9.3

Air flows in a 6-in.-diameter, insulated, galvanized iron duct. Initial conditions are p_1 = 20 psia, T_1 = 70°F, and V_1 = 406 ft/sec. After 70 ft, determine the final Mach number, temperature, and pressure.

Since the duct is circular we do not have to compute an equivalent diameter. From Table 9.2 the absolute roughness ε is 0.0005. Thus the relative roughness

$$\frac{\varepsilon}{D} = \frac{0.0005}{0.5} = 0.001$$

We compute the Reynolds number at section 1 (Figure E9.1) since this is the only location where information is known.

$$\rho_1 = \frac{p_1}{RT_1} = \frac{(20)(144)}{(53.3)(530)} = 0.102 \text{ lbm/ft}^3$$

$$\mu_1 = 3.8 \times 10^{-7} \text{ lbf-sec/ft}^2 \text{ (from table in Appendix A)}$$

Thus

$$\text{Re}_1 = \frac{\rho_1 V_1 D_1}{\mu_1 g_c} = \frac{(0.102)(406)(0.5)}{(3.8 \times 10^{-7})(32.2)} = 1.69 \times 10^6$$

From the Moody diagram (in Appendix C) at Re = 1.69×10^6 and $\varepsilon/D = 0.001$, we determine that the friction factor is $f = 0.0198$. To use the Fanno table (or equations), we need information on Mach numbers.

$$a_1 = (\gamma g_c R T_1)^{1/2} = [(1.4)(32.2)(53.3)(530)]^{1/2} = 1128 \text{ ft/sec}$$

$$M_1 = \frac{V_1}{a_1} = \frac{406}{1128} = 0.36$$

From the Fanno table (Appendix I) at $M_1 = 0.36$, we find that

$$\frac{p_1}{p^*} = 3.0042 \quad \frac{T_1}{T^*} = 1.1697 \quad \frac{fL_{1\max}}{D} = 3.1801$$

The key to completing the problem is in establishing the Mach number at the outlet, and this is done through the *friction length*:

$$\frac{f\,\Delta x}{D} = \frac{(0.0198)(70)}{0.5} = 2.772$$

Looking at the physical sketch it is apparent (since f and D are constants) that

$$\frac{fL_{2\max}}{D} = \frac{fL_{1\max}}{D} - \frac{f\,\Delta x}{D} = 3.1801 - 2.772 = 0.408$$

We enter the Fanno table with this friction length and find that

$$M_2 = 0.623 \quad \frac{p_2}{p^*} = 1.6939 \quad \frac{T_2}{T^*} = 1.1136$$

Thus

$$p_2 = \frac{p_2}{p^*}\frac{p^*}{p_1}p_1 = (1.6939)\left(\frac{1}{3.0042}\right)(20) = 11.28\,\text{psia}\ \ (\text{or } 77.75\,\text{kPa})$$

and

$$T_2 = \frac{T_2}{T^*}\frac{T^*}{T_1}T_1 = (1.1136)\left(\frac{1}{1.1697}\right)(530) = 505°\text{R}\ \ (\text{or } 280.5\,\text{K})$$

In the example above, the friction factor was assumed constant. In fact, this assumption was made when equation (9.39) was integrated to obtain (9.40), and with the introduction of the * reference state, this became equation (9.46), which is listed in the Fanno table. Is this a reasonable assumption? Friction factors are functions of Reynolds numbers, which in turn depend on velocity and density—both of which can change quite rapidly in Fanno flow. Calculate the velocity at the outlet in Example 9.3 and compare it with that at the inlet. ($V_2 = 686$ ft/sec and $V_1 = 406$ ft/sec.)

But don't despair! From continuity, we know that the product of ρV in Fanno flow is constant, and thus the only variable in Reynolds number is the viscosity. Extremely large temperature variations are required to change the gas viscosity significantly, and thus variations in the Reynolds number are small in Fanno problems when $M < 5.0$. We are also fortunate in that most engineering problems are well into the turbulent range where the friction factor is relatively insensitive to

Reynolds number. A greater potential error is involved in the estimation of the duct roughness, which has a more significant effect on the friction factor.

Example 9.4

A converging-diverging nozzle with an area ratio of 5.42 connects to an 8-ft-long constant-area rectangular duct (see Figure E9.4). The duct is 8 × 4 in. in cross section and has a friction factor of $f = 0.02$. What is the minimum stagnation pressure feeding the nozzle if the flow is supersonic throughout the entire duct and it exhausts to 14.7 psia?

$$D_e = \frac{4A}{P} = \frac{(4)(32)}{24} = 5.334\,\text{in.}$$

$$\frac{f\Delta x}{D} = \frac{(0.02)(8)(12)}{5.334} = 0.36$$

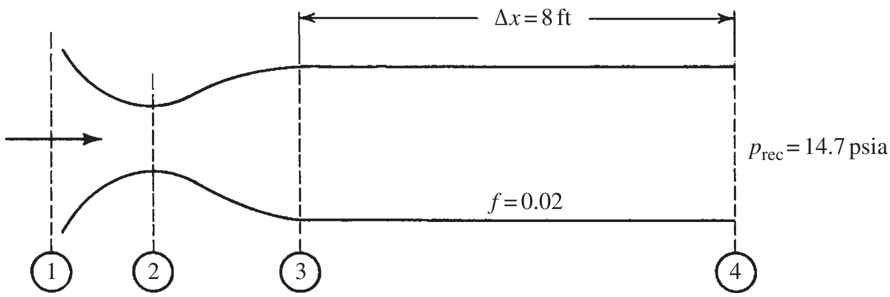


Figure E9.4

To be supersonic with $A_3/A_2 = 5.42$, $M_3 = 3.26$, $p_3/p_{t3} = 0.0185$, $p_3/p^* = 0.1901$, and $fL_{3\,\max}/D = 0.5582$,

$$\frac{fL_{4\max}}{D} = \frac{fL_{3\max}}{D} - \frac{f\Delta x}{D} = 0.5582 - 0.36 = 0.1982$$

Thus

$$M_4 = 1.673 \quad \text{and} \quad \frac{p_4}{p^*} = 0.5243$$

and

$$p_{t1} = \frac{p_{t1}}{p_{t3}}\frac{p_{t3}}{p_3}\frac{p_3}{p^*}\frac{p^*}{p_4}p_4 = (1)\left(\frac{1}{0.0185}\right)(0.1901)\left(\frac{1}{0.5243}\right)(14.7) = 228\,\text{psia (or 1.57 MPa)}$$

Any pressure above 288 psia will maintain the flow system as specified but with expansion waves outside the duct. (Recall an underexpanded nozzle.) Can you envision what would happen if the inlet stagnation pressure fell below 288 psia? (Recall the operation of an over-expanded nozzle.)

9.7 CORRELATION WITH SHOCKS

As you have progressed through this chapter you may have noticed some similarities between Fanno flow and normal shocks. Let us summarize some pertinent information.

The points just before and after a normal shock represent states with the same mass flow per unit area, the same value of $p + \rho V^2/g_c$, and the same stagnation enthalpy. These facts are the result of applying the basic concepts of continuity, momentum, and energy to any arbitrary fluid. This analysis resulted in equations (6.2), (6.3), and (6.9).

A Fanno line represents states with the same mass flow per unit area and the same stagnation enthalpy. This is confirmed by equations (9.2) and (9.5). To move *along* a Fanno line requires friction. At the end of Section 9.3 [see equation (9.17)] it was pointed out that it is this very friction which causes the value of $p + \rho V^2/g_c$ to change.

The variation of the quantity $p + \rho V^2/g_c$ along a Fanno line is quite interesting. Such a plot is shown in Figure 9.7. You will notice that for every point on the supersonic branch of the Fanno line there is a corresponding point on the subsonic branch with the same value of $p + \rho V^2/g_c$. Thus these two points satisfy all three conditions for the end points of a normal shock and could be connected by such a shock.

Now we can imagine a supersonic Fanno flow leading into a normal shock. If this is followed by additional duct, subsonic Fanno flow would occur. Such a situation is shown in Figure 9.8a. Note that the shock merely causes the flow to jump from the supersonic branch to the subsonic branch of the *same* Fanno line. [See Figure 9.8b.]

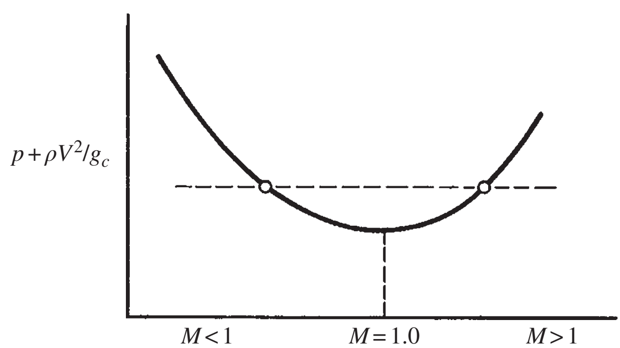


Figure 9.7 Variation of $p + \rho V^2/g_c$ in Fanno flow.

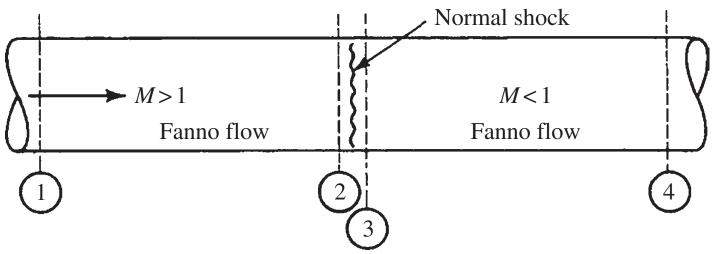


Figure 9.8*a* Combination of Fanno flow and normal shock (physical system).

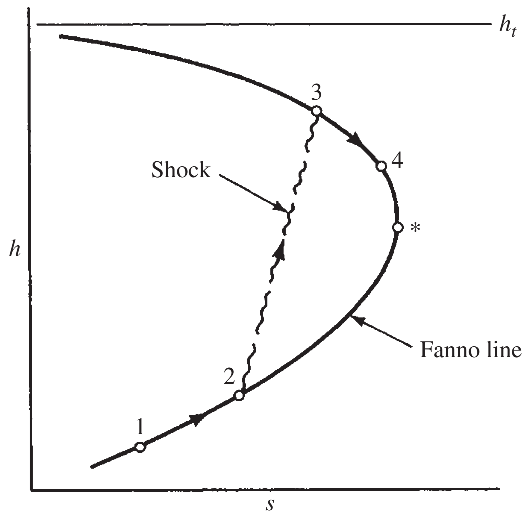


Figure 9.8*b* Combination of Fanno flow and normal shock.

Example 9.5

A large chamber contains air at a temperature of 300 K and a pressure of 8 bar abs (Figure E9.5). The air enters a converging–diverging nozzle with an area ratio of 2.4. A constant-area duct is attached to the nozzle and a normal shock stands at the exit plane. Receiver pressure is 3 bar abs. Assume the entire system to be adiabatic and neglect friction in the nozzle. Compute the $f\Delta x/D$ for the duct.

For a shock to occur as specified, the duct flow must be supersonic, which means that the nozzle is operating at its third critical point. The inlet conditions and nozzle area ratio fix conditions at location 3. We can then find p^* at the tip of the Fanno line. Then the ratio p_5/p^* can be computed and the Mach number after the shock is found from the Fanno table. This solution probably would not have occurred to us had we not drawn the T–s diagram and recognized that point 5 is on the same Fanno line as 3, 4, and *.

For $A_3/A_2 = 2.4$, $M_3 = 2.4$ and $p_3/p_{t3} = 0.06840$. We proceed immediately to compute p_5/p^*:

$$\frac{p_5}{p^*} = \frac{p_5}{p_{t1}}\frac{p_{t1}}{p_{t3}}\frac{p_{t3}}{p_3}\frac{p_3}{p^*} = \left(\frac{3}{8}\right)(1)\left(\frac{1}{0.0684}\right)(0.3111) = 1.7056$$

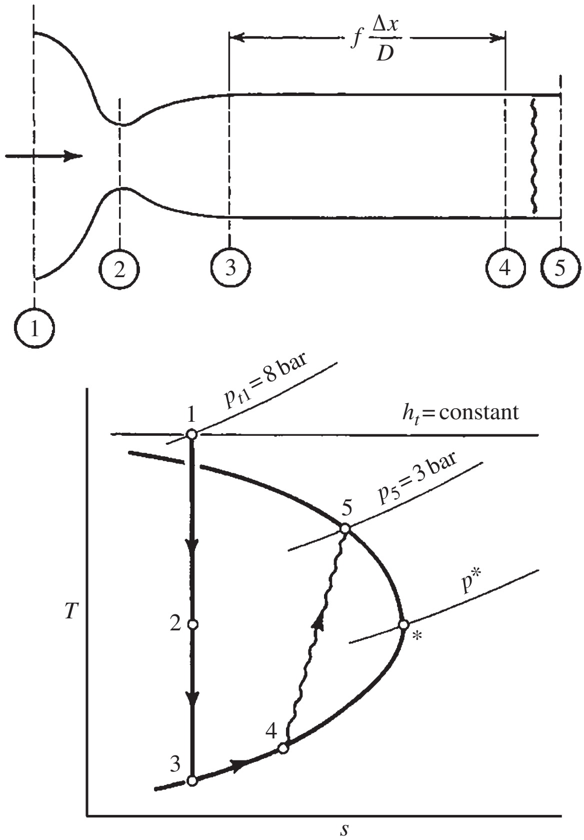


Figure E9.5

From the Fanno table we find that $M_5 = 0.619$, and then from the shock table, $M_4 = 1.789$. Returning to the Fanno table, $fL_{3\ max}/D = 0.4099$ and $fL_{4\ max}/D = 0.2382$. Thus

$$\frac{f\Delta x}{D} = \frac{fL_{3\,\max}}{D} - \frac{fL_{4\,\max}}{D} = 0.4099 - 0.2382 = 0.172$$

9.8 FRICTION CHOKING

In Chapter 5 we discussed the operation of nozzles that were fed by constant stagnation inlet conditions (see Figures 5.6 and 5.8). We found that at first as the receiver pressure was lowered, the flow through the nozzle increased. When the *operating pressure ratio* reached a certain value, the section of minimum area developed a Mach number of unity. The nozzle was then said to be *choked*. Further reduction in the pressure ratio did not increase the flow rate. This was an example of *area choking*.

The subsonic Fanno flow situation is quite similar. Figure 9.9 shows a given length of duct fed by a large tank and converging nozzle. If the receiver pressure is below the tank pressure, flow will occur, producing a T–s diagram shown as path 1–2–3 in Figure 9.10. Note that we have isentropic flow at the entrance to the duct and then we move down along a Fanno line. As the receiver pressure is lowered still more, the flow rate and exit Mach number continue to increase while the system moves to Fanno lines of higher mass velocities (shown as path 1–2′–3′). It is important to recognize that the receiver pressure (or more properly, the operating pressure ratio) is controlling the flow. This is because in subsonic flow the pressure at the duct exit must equal that of the receiver.

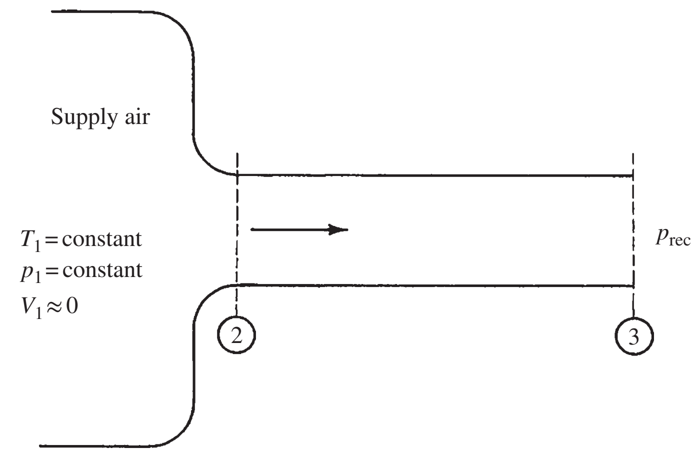


Figure 9.9 Converging nozzle and constant-area duct combination.

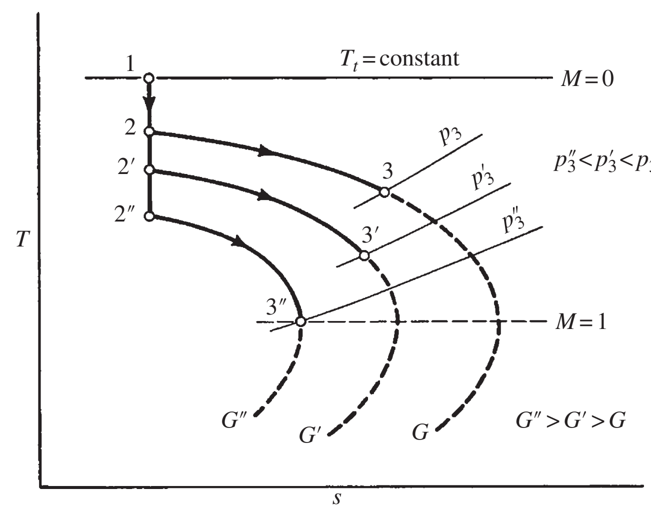


Figure 9.10 T–s diagram for nozzle–duct combination.

Eventually, when a certain pressure ratio is reached, the Mach number at the duct exit will be unity (shown as path 1–2″–3″). This is called *friction choking* and any further reduction in receiver pressure would not affect the flow conditions *inside* the system. What would need to occur as the flow leaves the duct and enters a region of reduced pressure?

Let us consider this last case of choked flow with the exit pressure equal to the receiver pressure. Now *suppose that the receiver pressure is maintained at this value* but more duct is added to the system. (Nothing can physically prevent us from doing this.) What happens? We know that we cannot move *around the Fanno line,* yet somehow we must reflect the added friction losses. This is done by moving to a new Fanno line at a *decreased* flow rate. The *T–s* diagram for this is shown as path 1–2‴–3‴–4 in Figure 9.11. Note that pressure equilibrium is still maintained at the exit but the system is no longer choked, although the flow rate has decreased. What would occur if the receiver pressure were now lowered?

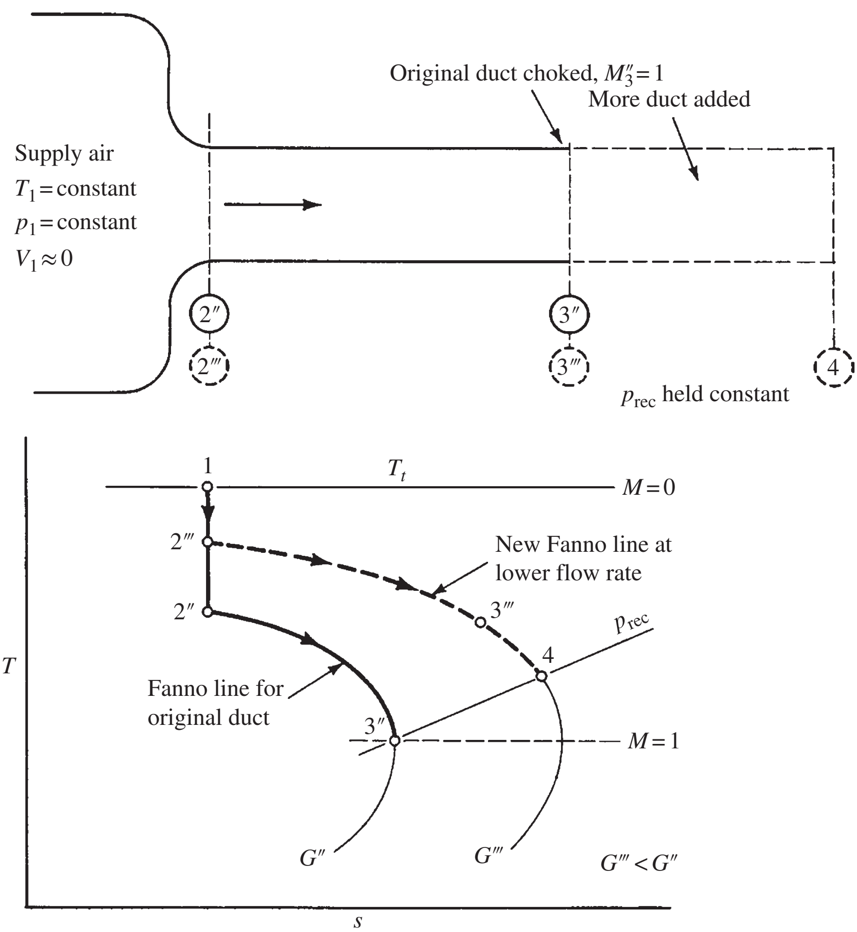


Figure 9.11 Addition of more duct when subsonic flow is choked.

In summary, when a *subsonic* Fanno flow has become *friction choked* and more duct is added to the system, the flow rate must decrease. Just how much it decreases and whether or not the exit velocity remains sonic depends on how much duct is added and the receiver pressure imposed on the system.

Now suppose that we are dealing with *supersonic* Fanno flow that is *friction choked*. In this case the addition of more duct causes a normal shock to form inside the duct. The resulting subsonic flow can in most cases accommodate the increased duct length at the same flow rate. For example, Figure 9.12 shows a Mach 2.18 flow that has an fL_{max}/D value of 0.356. If a normal shock were to occur at this point, the Mach number after the shock would be about 0.550, which corresponds to an fL_{max}/D value of 0.728. Thus, in this case, the appearance of the shock permits over twice the duct length to the choke point. This difference becomes even greater as higher Mach numbers are reached.

The shock location is determined by the amount of duct added. As more duct is added, the shock moves upstream and occurs at a higher Mach number. Eventually, the shock will move into that portion of the system that precedes the constant-area duct. (Most likely, a converging–diverging nozzle was used to produce the supersonic flow.) If sufficient friction length is added, the entire system will become subsonic and then the flow rate will decrease. Whether or not the exit velocity remains sonic will again depend on the receiver pressure.

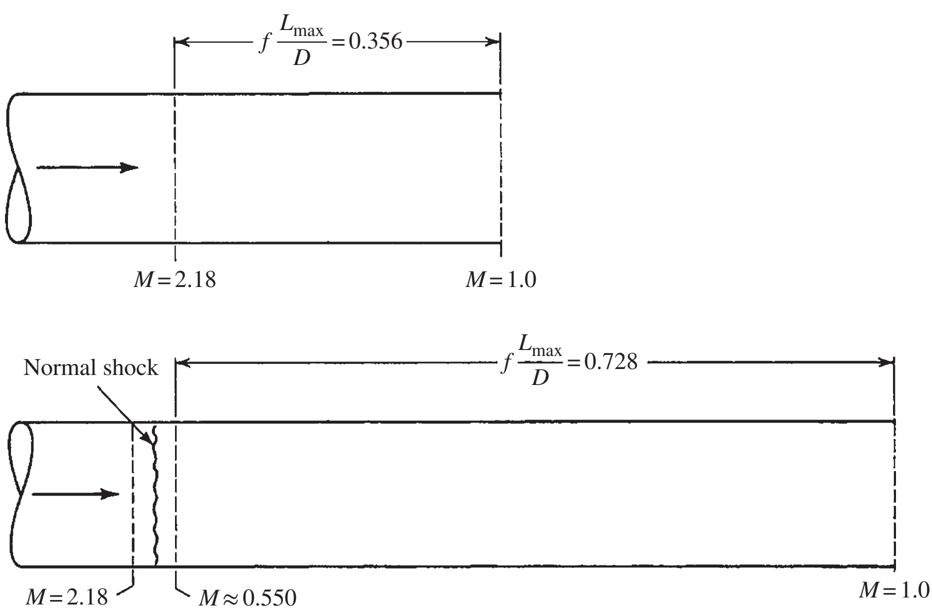


Figure 9.12 Influence of shock on maximum duct length.

9.9 (OPTIONAL) HOW THE LEFT-HAND-SIDE OF EQUATION (9.40) AROSE

Internal flows with significant amounts of friction are affected by both the length in the flow-direction and the flow enclosure's cross-section, and specifying the relevant dimensions gets complicated because the flow can be either "entrance flow" or "fully developed." Furthermore, the latter case can be either laminar or turbulent depending on the Reynolds number (Re) based on tube diameter. As we first introduced the shear stress at the wall (τ_w) in equation (3.59), it is multiplied by a differential area Pdx (where P is the conduit's perimeter and dx a flow length-increment which is part of the total length L) so as to represent the differential friction force acting in internal flows. Resistance to fluid flow depends on the "wetted area" which is the total surface that the fluid contacts, or πDL for circular tubes [we extend this situation to non-circular cross sections by defining an equivalent diameter in equation (3.61)]. Because two independent lengths that have the same fundamental dimension are involved it is not clear which one should be used as reference to non-dimensionalize equation (3.59). One might legitimately ask why we have chosen D to arrive at the dimensionless form of the Reynolds number, equation (9.48) (and of the absolute pipe roughness ratio ε/D) instead of choosing $L = x_2 - x_1$ for the reference length?

As defined in equation (3.60), the friction factor depends solely on the Reynolds number. It is traditional to use Re based on diameter in fully-developed *internal* flows, whereas for *external* and *developing internal* flows we use Re_L (i.e., based on length along the flow). There is excellent empirical data available for Fanno flow problems in terms of ($\rho VD/\mu$, ε/D) in the Moody diagram so using D is somewhat a matter of convenience. Instead of working directly with relevant areas like πDL, in equation (9.40) we work with the equivalent product of two dimensionless numbers, a friction factor f and $(x_2 - x_1)/D_e$ to properly model internal frictional forces and, as defined, f is unambiguously related to the Reynolds number as in Appendix C. This technique results in a practical method that well represents nearly all gases and liquids.

9.10 WHEN γ IS NOT EQUAL TO 1.4

As indicated earlier, the Fanno flow table in Appendix I is only for $\gamma = 1.4$. The behavior of $fL_{\max}/D$, the friction function, is shown in Figure 9.13 for $\gamma = 1.13$, 1.4, and 1.67 for Mach numbers up to $M = 5$. Here we can see that the dependence on γ is rather noticeable for $M \geq 1.4$. Thus below this Mach number, the tabulations in Appendix I can be used only approximately for any γ. This means that for subsonic flows, where most Fanno flow problems occur, there are only minor differences between the various gases. The desired accuracy of results will govern

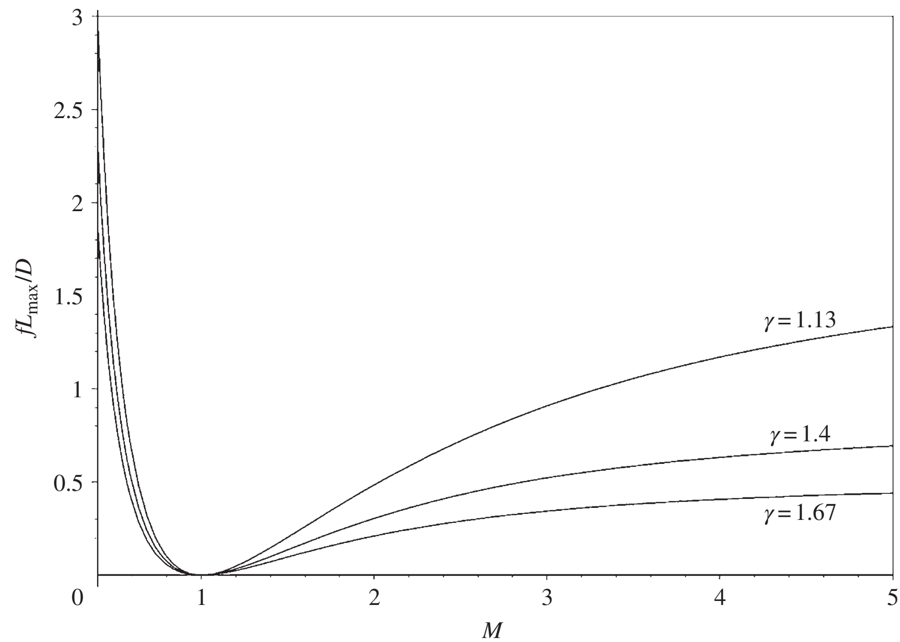


Figure 9.13 Fanno flow fL_{max}/D, versus Mach number for various values of γ.

how far you want to carry this approximation into the supersonic region. The *Gasdynamics Calculator* will work in the range $1.0 < \gamma \leq 1.67$.

Strictly speaking, the curves in Figure 9.13 are only representative for cases where γ variations are *negligible within the flow*. However, they offer hints as to what magnitude of changes are to be expected in other cases. Flows where γ variations are *not negligible within the flow* are treated in Chapter 11.

9.11 (OPTIONAL) BEYOND THE TABLES

As pointed out in Chapter 5, one can eliminate a lot of interpolation and get accurate answers for any ratio of the specific heats γ and/or any Mach number by using a computer utility such as MAPLE. This utility is useful in the evaluation of equation (9.46). Example 9.6 is one such application.

Example 9.6
Let us rework Example 9.3 without using the Fanno table. For $M_1 = 0.36$, calculate the value of fL_{max}/D. The procedure follows equation (9.46):

$$\frac{f(x^* - x)}{D_e} = \left(\frac{\gamma+1}{2\gamma}\right) \ln \frac{[(\gamma+1)/2]M^2}{1 + [(\gamma-1)/2]M^2} + \frac{1}{\gamma}\left(\frac{1}{M^2} - 1\right) \tag{9.46}$$

Let

$$g \equiv \gamma, \text{a parameter (the ratio of specific heats)}$$

$$X \equiv \text{the independent variable(which in this case is } M_1)$$

$$Y \equiv \text{the dependent variable(which in this case is } fL_{\max}/D)$$

Listed below are the precise inputs and program that you use in the computer.

```
[>g:=1.4:  X:=0.36:
[>Y:=((g+1)/(2*g))*log(((g+1)*(X^2)/2)/(1+(g-1)*(X^2)+(1/g)*((1/X^2)-1);
                                Y:=3.180117523
```

We can proceed to find the Mach number at station 2. The new value of Y is 3.1801 – 2.772 = 0.408. Now we use the same equation (9.46) but solve for M_2 as shown below. Note that since M is implicit in the equation, we are going to utilize "fsolve." Let

$$g \equiv \gamma, \text{a parameter (the ratio of specific heats)}$$

$$X \equiv \text{the dependent variable(which in this case is } M_2)$$

$$Y \equiv \text{the independent variable(which in this case is } fL_{\max}/D)$$

Listed below are the precise inputs and program that you use in the computer.

```
[>g2:=1.4: Y2:=0.408:
[>fsolve(Y2=((g2+1)/(2*g2))*log(((g2+1)*(X2^2)/2)/(1+
(g2-1)*(X2^2)/2))+(1/g2)*((1/X2^2)-1), X2, 0..1);
                    .6227097475
```

The answer of $M_2 = 0.6227$ is consistent with that obtained in Example 9.3. We can now proceed to calculate the required static properties, but this will be left as an exercise for the reader.

9.12 SUMMARY

We have analyzed flow in constant-area ducts with friction but without heat transfer. The fluid properties change in a predictable manner dependent on the flow regime as shown in Table 9.3. The property variations in subsonic Fanno flow follow an intuitive pattern but we note that the supersonic flow behavior is

Table 9.3 Fluid Property Variation for Fanno Flow

Property	Subsonic	Supersonic
Velocity	Increases	Decreases
Mach number	Increases	Decreases
Enthalpy[a]	Decreases	Increases
Stagnation enthalpy[a]	Constant	Constant
Pressure	Decreases	Increases
Density	Decreases	Increases
Stagnation pressure	Decreases	Decreases

[a]Also temperature if the fluid is a perfect gas.

completely different. The only common occurrence is the decrease in stagnation pressure, which in Fanno flow reflects the losses arising from internal entropy increases originating from frictional dissipation.

Perhaps the most significant equations are those that apply to all fluids:

$$\rho V = G = \text{constant} \tag{9.2}$$

$$h_t = h + \frac{G^2}{\rho^2 2g_c} = \text{constant} \tag{9.5}$$

Along with these equations you should keep in mind the appearance of Fanno lines in the h–v and T–s diagrams (see Figures 9.1 and 9.2). Remember that each Fanno line represents points with the same mass velocity (G) and stagnation enthalpy (h_t), and a normal shock can connect two points on opposite branches of a Fanno line which have the same value of $p + \rho V^2/g_c$. *Families* of Fanno lines could represent:

1. Different values of G for the same h_t (such as those in Figure 9.10), or
2. The same G for different values of h_t (see Problem 10.17).

Detailed working equations were developed for perfect gases, and the introduction of the * reference point enabled the construction of a Fanno table which simplifies problem solution. The * condition for Fanno flow has no relation to the one used previously in isentropic flow (except for denoting where $M = 1.0$). All Fanno flows must proceed toward a limiting point, namely, Mach 1.0. Friction choking of a flow passage is possible in Fanno flow just as area choking occurs in varying-area isentropic flow. An h–s (or the more useful T–s) diagram is of great help in the analysis of a complicated flow system. *Get into the habit of drawing these diagrams.*

Note: many subsonic flows with friction in very *long ducts* tend to become *isothermal* because there is ample opportunity for heat exchange with the environment. These *non-adiabatic* flows are common in natural gas pipe lines and can be modelled in a manner similar to Fanno flows; such analysis can be found in References 16, 19, and 20, among others.

PROBLEMS

In the problems that follow you may assume that all systems are completely adiabatic (being short in length, and we neglect any heating due to friction). Also, all ducts are of constant area unless otherwise indicated. You may neglect friction in the varying-area sections. You may also assume that the friction factor shown in Appendix C applies to noncircular cross sections when the equivalent diameter concept is used and the flow is turbulent.

9.1. Conditions at the entrance to a duct are $M_1 = 3.0$ and $p_1 = 8 \times 10^4$ N/m^2. After a certain length the flow has reached $M_2 = 1.5$. Determine p_2 and $f\Delta x/D$ if $\gamma = 1.4$.

9.2. A flow of nitrogen is discharged from a duct with $M_2 = 0.85$, $T_2 = 500°R$, and $p_2 = 28$ psia. The temperature at the inlet is 560°R. Compute the pressure at the inlet and the mass velocity (G).

9.3. Air enters a circular duct with a Mach number of 3.0. The friction factor is 0.01.

(a) How long a duct (measured in diameters) is required to reduce the Mach number to 2.0?

(b) What is the percentage change in temperature, pressure, and density?

(c) Determine the entropy increase of the air.

(d) Assume the same length of duct as computed in part (a), but the initial Mach number is 0.5. Compute the percentage change in temperature, pressure, density, and the entropy increase for this case. Compare the changes in the same length duct for subsonic and supersonic flow.

9.4. Oxygen enters a 6-in.-diameter duct with T_1, = 600°R, $p_1 = 50$ psia, and $V_1 = 600$ ft/sec. The friction factor is $f = 0.02$.

(a) What is the maximum length of duct permitted that will not change any of the conditions at the inlet?

(b) Determine T_2, p_2, and V_2 for the maximum duct length found in part (a).

9.5. Air flows in an 8-cm-inside diameter pipe that is 4 m long. The air enters with a Mach number of 0.45 and a temperature of 300 K.

(a) What friction factor would cause sonic speed at the exit?

(b) If the pipe is made of cast iron, estimate the inlet pressure.

9.6. At one section in a constant-area duct the stagnation pressure is 66.8 psia and the Mach number is 0.80. At another section the pressure is 60 psia and the temperature is 120°F.

(a) Compute the temperature at the first section and the Mach number at the second section if the fluid is air.

(b) Which way is the air flowing?

(c) What is the friction length ($f\Delta x/D$) of the duct?

9.7. A 50×50 cm duct is 10 m in length. Nitrogen enters at $M_1 = 3.0$ and leaves at $M_2 = 1.7$, with $T_2 = 280$ K and $p_2 = 7 \times 10^4$ N/m^2.

(a) Find the static and stagnation conditions at the entrance.

(b) What is the friction factor of the duct?

9.8. A duct of 2 ft × 1 ft cross section is made of riveted steel and is 500 ft long. Air enters with a velocity of 174 ft/sec, $p_1 = 50$ psia, and $T_1 = 100°$F.

(a) Determine the temperature, pressure, and velocity at the exit.

(b) Compute the pressure drop assuming the flow to be incompressible. Use the entering conditions and equation (3.29). Note that equation (3.64) can easily be integrated to evaluate

$$\int T ds_i = f \frac{\Delta x}{D_e} \frac{V^2}{2g_c}$$

(c) How do the results of parts (a) and (b) compare? Did you expect this?

9.9. Air enters a duct with a mass flow rate of 35 lbm/sec at $T_1 = 520°$R and $p_1 = 20$ psia. The duct is square and has an area of 0.64 ft^2. The outlet Mach number is unity.

(a) Compute the temperature and pressure at the outlet.

(b) Find the length of the duct if it is made of steel.

9.10. Consider the flow of a perfect gas along a Fanno line. Show that the pressure at the * reference state is given by the relation

$$p^* = \frac{\dot{m}}{A} \left[\frac{2RT_t}{\gamma g_c(\gamma + 1)} \right]^{1/2}$$

9.11. A 10-ft duct 12 in. in diameter contains oxygen flowing at the rate of 80 lbm/sec. Measurements at the inlet give $p_1 = 30$ psia and $T_1 = 800°$R. The pressure at the outlet is $p_2 = 23$ psia.

(a) Calculate M_1, M_2, V_2, T_{t2}, and p_{t2}.

(b) Determine the friction factor and estimate the absolute roughness of the duct material.

9.12. At the outlet of a 25-cm-diameter duct, air is traveling at sonic speed with a temperature of 16°C and a pressure of 1 bar. The duct is very smooth and is 15 m long. There are two possible conditions that could exist at the entrance to the duct.

(a) Find the static and stagnation temperature and pressure for each entrance condition.

(b) Assuming the surrounding air to be at 1 bar pressure, how much horsepower is necessary to get ambient air into the duct for each case? (You may assume no losses in the work process.)

9.13. Ambient air at 60°F and 14.7 psia accelerates isentropically into a 12-in.-diameter duct. After 100 ft the duct transitions into an 8×8 in. square section where the Mach number is 0.50. Neglect all frictional effects except in the constant-area duct, where $f = 0.04$.

(a) Determine the Mach number at the duct entrance.

(b) What are the temperature and pressure in the square section?

(c) How much 8×8 in. square duct could be added before the flow chokes? (Assume that $f = 0.04$ in this duct also.)

9.14. Nitrogen with $p_t = 7 \times 10^5$ N/m^2 and $T_t = 340$ K enters a frictionless converging–diverging nozzle having an area ratio of 4.0. The nozzle discharges supersonically into a constant-area duct that has a friction length $f\Delta x/D = 0.355$. Determine the temperature and pressure at the exit of the duct.

9.15. Conditions before a normal shock are $M_1 = 2.5$, $p_{t1} = 67$ psia, and $T_{t1} = 700$°R. This is followed by a length of Fanno flow and a converging nozzle as shown in Figure P9.15. The area change is such that the system is choked. It is also known that $p_4 = p_{amb} = 14.7$ psia.

(a) Draw a T–s diagram for the system.

(b) Find M_2 and M_3.

(c) What is $f\Delta x/D$ for the duct?

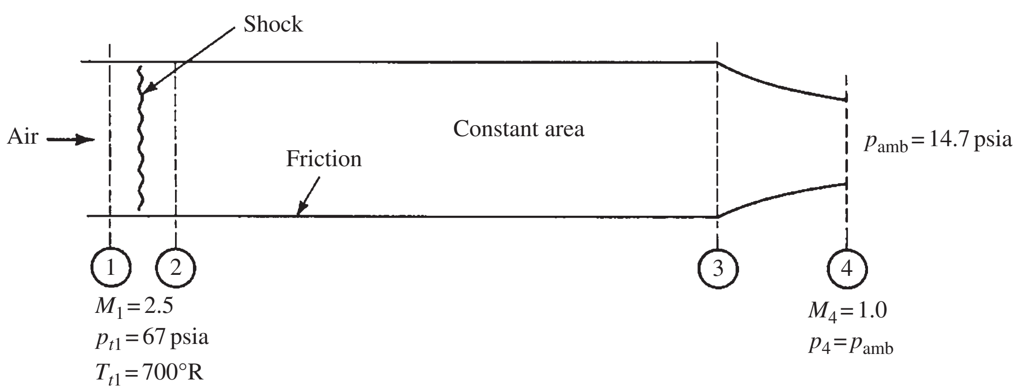


Figure P9.15

9.16. A converging–diverging nozzle (Figure P9.16) has an area ratio of 3.0. The stagnation conditions of the inlet air are 150 psia and 550°R. A constant-area duct with a length of 12 diameters is attached to the nozzle outlet. The friction factor in the duct is 0.025.

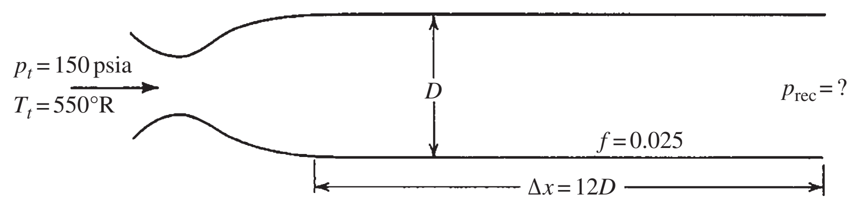


Figure P9.16

(a) compute the receiver pressure that would place a shock
  (i) in the nozzle throat;
  (ii) at the nozzle exit;
  (iii) at the duct exit.

(b) What receiver pressure would cause supersonic flow throughout the duct with no shocks within the system (or after the duct exit)?

(c) Make a sketch similar to Figure 6.3 showing the pressure distribution for the various operating points of parts (a) and (b).

9.17. For a nozzle–duct system similar to that of Problem 9.16, the nozzle is designed to produce a Mach number of 2.8 with $\gamma = 1.4$. The inlet conditions are $p_{t1} = 10$ bar and $T_{t1} = 370$ K. The duct is 8 diameters in length, but the duct friction factor is unknown. The receiver pressure is fixed at 3 bar and a normal shock has formed at the duct exit.

(a) Sketch a T–s diagram for the system.

(b) Determine the friction factor of the duct.

(c) What is the total change in entropy for the system?

9.18. A large chamber contains air at 65 bar pressure and 400 K. The air passes through a converging-only nozzle and then into a constant-area duct. The friction length of the duct is $f\Delta x/D = 1.067$ and the Mach number at the duct exit is 0.96.

(a) Draw a T–s diagram for the system.

(b) Determine conditions at the duct entrance.

(c) What is the pressure in the receiver? (*Hint:* How is this related to the duct exit pressure?)

(d) If the length of the duct is doubled and the chamber and receiver conditions remain unchanged, what are the new Mach numbers at the entrance and exit of the duct?

9.19. A constant-area duct is fed by a converging-only nozzle as shown in Figure P9.19. The nozzle receives oxygen from a large chamber at $p_1 = 100$ psia and $T_1 = 1000°$R. The duct has a friction length of 5.3 and it is choked at the exit. The receiver pressure is exactly the same as the pressure at the duct exit.

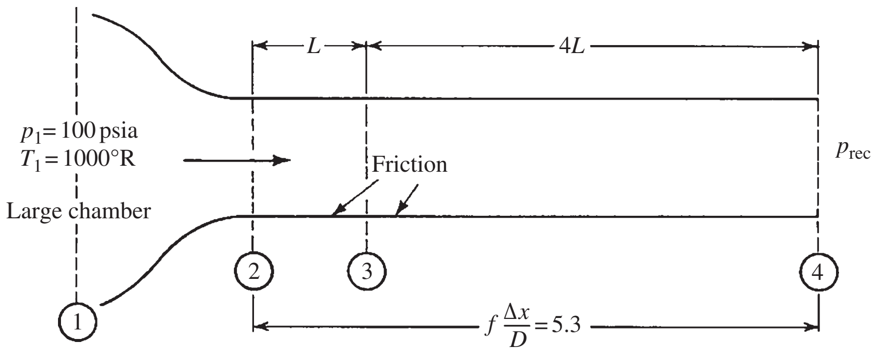


Figure P9.19

(a) What is the pressure at the end of the duct?

(b) Four-fifths of the duct is removed. (The end of the duct is now at 3.) The chamber pressure, receiver pressure, and friction factor remain unchanged. Now what is the pressure at the exit of the duct?

(c) Sketch both of the cases above on the same *T–s* diagram.

9.20. **(a)** Plot a Fanno line to scale in the *T–s* plane for air entering a duct with a Mach number of 0.20, a static pressure of 100 psia, and a static temperature of 540°R. Indicate the Mach number at various points along the curve.

(b) On the same diagram, plot another Fanno line for a flow with the same total enthalpy, the same entering entropy, but double the mass velocity.

9.21. Which, if any, of the ratios tabulated in the Fanno table (T/T^*, p/p^*, p_t/p_t^*, etc.) could also be listed in the Isentropic table with the same numerical values?

9.22. A contractor is to connect an air supply from a compressor to test apparatus 21 ft away. The exit diameter of the compressor is 2 in. and the entrance to the test equipment has a l-in.-diameter pipe. The contractor has the choice of putting a reducer at the compressor followed by 1-in. tubing or using 2-in. tubing and putting the reducer at the entrance to the test equipment. Since smaller tubing is cheaper and less obtrusive, the contractor is leaning toward the first possibility, but just to be sure, he sends the problem to the engineering personnel. The air coming out of the compressor is at 520°R and the pressure is 40 psia. The flow rate is 0.7 lbm/sec. Consider that each size of tubing has an effective $f = 0.02$. What would be the conditions at the entrance to the test equipment for each tubing size? (You may assume isentropic flow everywhere but in the 21 ft of tubing.)

9.23. (*Optional*) Verify that the relation for S_{max}/R given under equation (9.50) does correspond to $\gamma = 1.4$ by calculating several values of $(s^* - s)/R$ versus Mach number and comparing your results with the Fanno table listings in Appendix I.

9.24. Rework Problem 9.4 for helium, water vapor, and carbon dioxide using the *Gas-dynamics Calculator* and γ and R entries in Appendix A.

CHECK TEST

You should be able to complete this test without reference to material in the chapter.

9.1. Sketch a Fanno line in the h–v plane. Include enough additional information as necessary to locate the sonic point and then identify the regions of subsonic and supersonic flow.

9.2. Fill in the blanks in Table CT9.2 to indicate whether the quantities *increase, decrease*, or *remain constant* in the case of Fanno flow.

Table CT9.2 Analysis of Fanno Flow

Property	Subsonic Regime	Supersonic Regime
Velocity		
Temperature		
Pressure		
Thrust function ($p + \rho V^2/g_c$)		

9.3. In the system shown in Figure CT9.3, the friction length of the duct is $f\Delta x/D = 12.40$ and the Mach number at the exit is 0.8. $A_3 = 1.5\ \text{in}^2$ and $A_4 = 1.0\ \text{in}^2$. What is the air pressure in the tank if the receiver is at 15 psia?

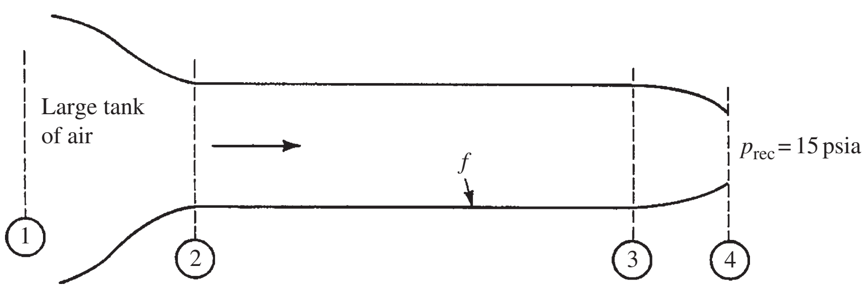


Figure CT9.3

9.4. Over what range of receiver pressures will normal shocks occur someplace within the system shown in Figure CT9.4? The area ratio of the nozzle is $A_3/A_2 = 2.403$ and the duct $f\Delta x/D = 0.30$.

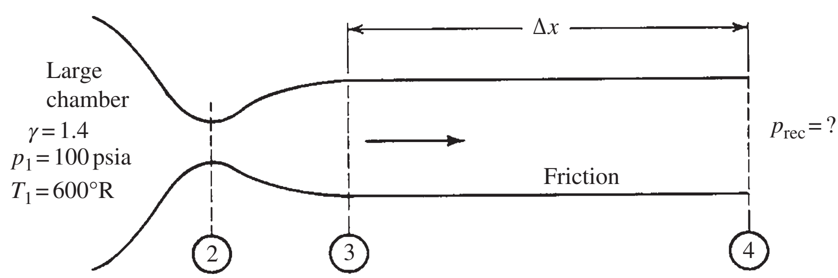


Figure CT9.4

9.5. There is no friction in the system shown in Figure CT9.5 except in the constant-area ducts from 3 to 4 and from 6 to 7. Sketch the *T*–*s* diagram for the entire system.

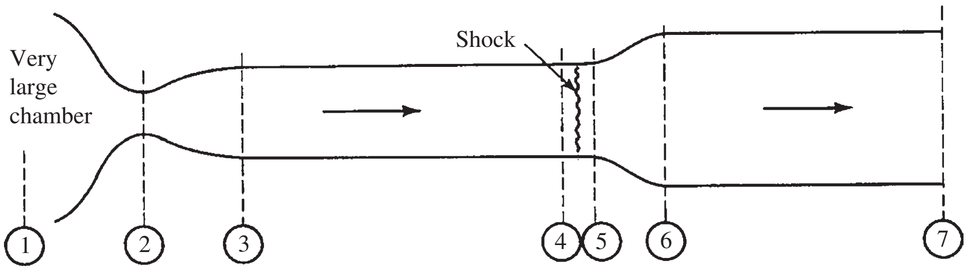


Figure CT9.5

9.6. Starting with the basic principles of continuity, energy, and so on, derive an expression for the property ratio p_2/p_1 in terms of Mach numbers and the specific heat ratio for Fanno flow with a perfect gas.

9.7. Work Problem 9.18.

Chapter 10

Rayleigh Flow

10.1 INTRODUCTION

In this chapter we consider the consequences of heat crossing the boundaries of a system. To isolate the effects of heat transfer from the other major factors we assume flow in a constant-area duct without friction. At first this may seem to be an unrealistic situation, but actually it is a good first approximation to many real problems, as most heat exchangers have constant-area flow passages. It is also a simple and reasonably equivalent process for a constant-area combustion chamber. Naturally, in these actual systems, frictional effects are present, and what we really are saying is the following:

In systems where *high rates of heat transfer* occur relative to their flow-passage length, the entropy change caused by the heat transfer can be much greater than that caused by friction or

$$ds_e \gg ds_i \tag{10.1}$$

Thus

$$ds \approx ds_e \tag{10.2}$$

and the frictional effects may be neglected. There are obviously some situations for which this assumption is not reasonable and other methods must be used to obtain more accurate predictions for these systems.

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

We first examine the general behavior of an arbitrary fluid and again find that property variations follow different patterns in the subsonic and supersonic regimes. The flow of a perfect gas is considered next with the now familiar end result of constructing a table. The heat transfer situation we study is called the *Rayleigh flow* problem.

10.2 OBJECTIVES

After completing this chapter successfully, you should be able to:

1. State the assumptions made in the analysis of Rayleigh flow.
2. (*Optional*) Simplify the general equations of continuity, energy, and momentum to obtain basic relations valid for any fluid in Rayleigh flow.
3. Sketch a Rayleigh line in the p–v plane together with lines of constant entropy and constant temperature (for a typical gas). Indicate directions of increasing entropy and temperature.
4. Sketch a Rayleigh line in the h–s plane. Also sketch the corresponding stagnation curves. Identify the sonic point and regions of subsonic and supersonic flow.
5. Describe the variations in fluid properties that occur as flow progresses along a Rayleigh line for the case of heating and for cooling. Do this for both subsonic and supersonic flow.
6. (*Optional*) Starting with basic principles of continuity, energy, and momentum, derive expressions for property ratios such as T_2/T_1, p_2/p_1, and so on, in terms of Mach number (M) and specific heat ratio (γ) for Rayleigh flow with a perfect gas.
7. Describe (include a T–s diagram) how a Rayleigh table is developed with the aid of the * reference location.
8. Compare similarities and differences between Rayleigh flow and normal shocks. Sketch an h–s diagram showing a typical Rayleigh line and a normal shock for the same mass velocity. Show that the end points of a normal shock are represented by the two intersections of the Fanno and Rayleigh lines with the entropy increasing along the shock path.
9. Explain what is meant by *thermal choking*.
10. (*Optional*) Describe some possible consequences of adding more heat in a choked Rayleigh flow situation (for both subsonic and supersonic flow).
11. Demonstrate the ability to solve typical Rayleigh flow problems by the use of appropriate tables and equations.

10.3 ANALYSIS FOR A GENERAL FLUID

We shall first consider the general behavior of an arbitrary fluid. To isolate the effects of heat transfer we make the following assumptions

Steady one-dimensional flow	
Negligible friction	$ds_i \approx 0$
No shaft work	$\delta w_s = 0$
Neglect potential	$dz = 0$
Constant area	$dA = 0$

We proceed by applying the basic concepts of continuity, energy, and momentum.

Continuity

$$\dot{m} = \rho A V = \text{const} \tag{2.30}$$

but since the flow area is constant, this reduces to

$$\rho V = \text{const} \tag{10.3}$$

From our work in Chapter 9 we know that this constant is G, the *mass velocity,* and thus

$$\boxed{\rho V = G = \text{const}} \tag{10.4}$$

Energy

We start with

$$h_{t1} + q = h_{t2} + w_s \tag{3.19}$$

which for no shaft work becomes

$$\boxed{h_{t1} + q = h_{t2}} \tag{10.5}$$

Warning! This is the first major flow category for which the total enthalpy has *not* remained constant. By now you have accumulated a store of knowledge—all based on flows for which h_t = constant. So examine carefully any information that you retrieve from your memory bank!

Momentum

We now proceed to apply the momentum equation to the control volume shown in Figure 10.1. The x-component of the momentum equation for steady, one-dimensional flow is

$$\sum F_x = \frac{\dot{m}}{g_c}(V_{\text{out}_x} - V_{\text{in}_x}) \tag{3.46}$$

From Figure 10.1 we see that this becomes

$$p_1 A - p_2 A = \frac{\rho A V}{g_c}(V_2 - V_1) \tag{10.6}$$

Canceling the area, we have

$$p_1 - p_2 = \frac{\rho\, V}{g_c}(V_2 - V_1) = \frac{G}{g_c}(V_2 - V_1) \tag{10.7}$$

Show that this can be written as

$$p + \frac{GV}{g_c} = \text{const} \tag{10.8}$$

Alternative forms of equation (10.8) are

$$\boxed{p + \frac{G^2}{g_c \rho} = \text{const}} \tag{10.9a}$$

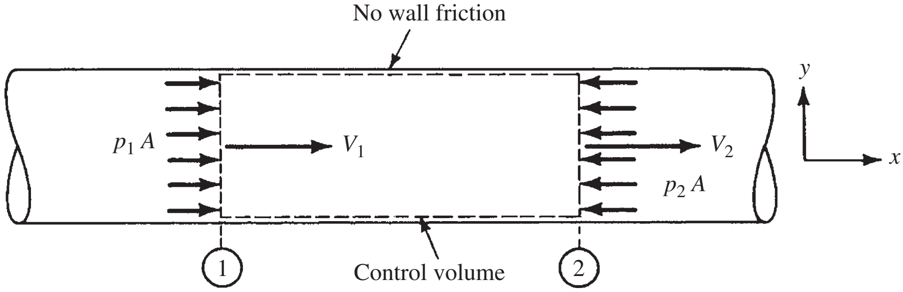


Figure 10.1 Momentum analysis for Rayleigh flow.

$$\boxed{p + \frac{G^2}{g_c} v = \text{const}} \tag{10.9b}$$

As an aside we might note here that this is the same relation that holds across a standing normal shock. Recall that for the normal shock:

$$p + \rho \frac{V^2}{g_c} = \text{const} \tag{6.9}$$

In both cases we are led to equivalent results since both analyses deal with constant area and assume negligible friction.

If we multiply either equation (6.9) or (10.8) by the constant area, we obtain

$$pA + \frac{(\rho A V)V}{g_c} = \text{const} \tag{10.10}$$

or

$$\boxed{pA + \frac{\dot{m}V}{g_c} = \text{const}} \tag{10.11}$$

The constant in equation (10.11) is called the *impulse function* or *thrust function* by various authors. We shall see a reason for these names when we study some propulsion devices in Chapter 12. For now let us merely note *that the thrust function remains constant for Rayleigh flow and across a normal shock.*

We return now to equation (10.9b), which will plot as a straight line in the p–v plane (see Figure 10.2). Such a line is called a *Rayleigh line* and represents flow at a particular mass velocity (G). If the fluid is known, one can also plot lines of constant temperature on the same diagram. Typical isothermals can be obtained easily by assuming the perfect gas equation of state. Some of these $pv = \text{const}$ lines are also shown in Figure 10.2.

Does the information depicted by this plot make sense? Normally, we would expect the effects of simple heating to increase the temperature and decrease the density. This appears to be in agreement with a process from point 1 to 2 as marked in Figure 10.2. If we add more heat, we move farther along the Rayleigh line and the temperature increases more. Soon point 3 is reached where the temperature is a maximum. Is this a limiting point of some sort? Have we reached some kind of a *choked* condition?

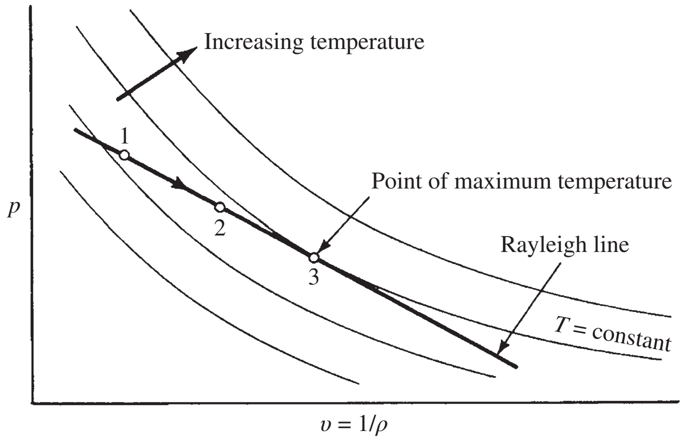


Figure 10.2 Rayleigh line in *p–v* plane.

To answer these questions, we must turn elsewhere. Recall that the addition of heat causes the entropy of the fluid to increase since

$$ds_e = \frac{\delta q}{T} \tag{3.10}$$

From our basic assumption of negligible friction,

$$ds \approx ds_e \tag{10.2}$$

Thus, it appears that the real limiting condition involves entropy (as usual). We can continue to add heat until the fluid reaches a state of maximum entropy. It might be that this point of maximum entropy is reached before the point of maximum temperature, in which case we would never be able to reach point 3 (of Figure 10.2). We must investigate the shape of constant entropy lines in the *p–v* diagram. This can easily be done for the case of a perfect gas that will serve to illustrate the general trend.

For a T = constant line,

$$pv = RT = \text{const} \tag{10.12}$$

Differentiating yields

$$pdv + vdp = 0 \tag{10.13}$$

and

$$\frac{dp}{dv} = -\frac{p}{v} \tag{10.14}$$

For an s = constant line,

$$pv^{\gamma} = \text{const} \tag{10.15}$$

Differentiating yields

$$v^{\gamma} dp + p\gamma v^{\gamma-1} dv = 0 \tag{10.16}$$

and

$$\frac{dp}{dv} = -\gamma \frac{p}{v} \tag{10.17}$$

Comparing equations (10.14) and (10.17) and noting that γ is always greater than 1.0, we see that the isentropic line has the *greater* negative slope and thus these lines will plot as shown in Figure 10.3. (Actually, this should come as no great surprise since they were shown this way in Figure 1.2; but did you really believe it then?)

We now see that not only can we reach the point of maximum temperature, but we can add more heat to take us beyond this point. If desired, we can move (by heating) all the way to the maximum entropy point. It may seem odd that in the region from point 3 to 4, we add heat to the system and its temperature decreases. Let us reflect further on the phenomenon occurring. In a previous discussion, we

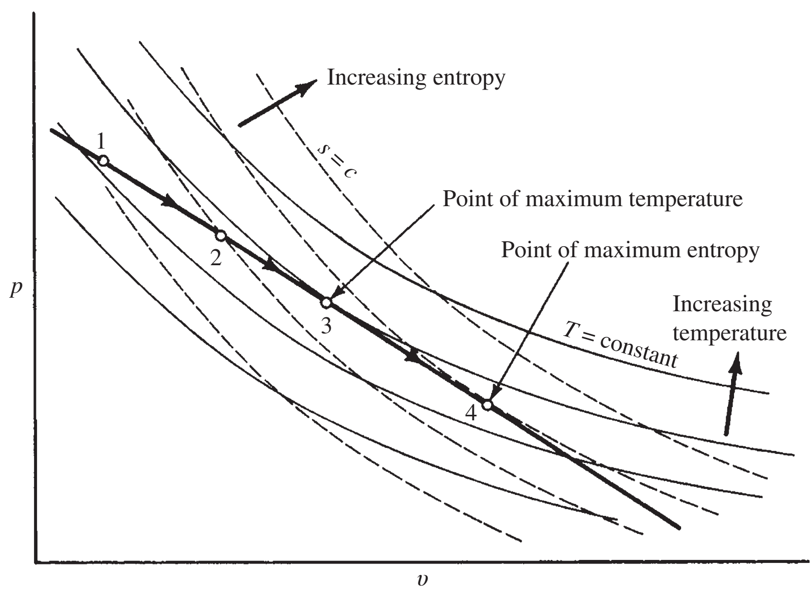


Figure 10.3 Rayleigh line in p–v plane.

noted that the effects of heat addition are normally thought of as causing the fluid density to decrease. This requires the velocity to increase since ρV = constant by continuity. This velocity increase automatically boosts the kinetic energy of the fluid by a certain amount. Thus the chain of events caused by heat addition forces a definite increase in kinetic energy. Some of the heat that is added to the system is converted into this increase in kinetic energy of the fluid, with the heat energy in excess of this amount being available to increase the enthalpy of the fluid.

Noting that kinetic energy is proportional to the square of velocity, we realize that as higher velocities are reached, the addition of more heat is accompanied by much greater increases in kinetic energy. Eventually, we reach a point where *all* of the heat energy added is required for the kinetic energy increase. At this point there is no heat energy left over and the system is at a point of maximum enthalpy (maximum temperature for a perfect gas). Further addition of heat causes the kinetic energy to increase by an amount *greater* than the heat energy being added. Thus, from this point on, the enthalpy must decrease to provide the proper energy balance.

Perhaps the foregoing discussion would be clearer if the Rayleigh lines were plotted in the h–s plane. For any given fluid this could easily be done, and the typical result is shown in Figure 10.4, along with lines of constant pressure. All points on this Rayleigh line represent states with the same mass flow rate per unit area (mass velocity) and the same impulse (or thrust) functions. *For heat addition, the entropy must increase and the flow moves to the right.* Thus it appears that the Rayleigh line, like the Fanno line, is divided into two distinct branches that are separated by a limiting point of maximum entropy.

We have been discussing a *familiar* heating process along the upper branch. What about the lower branch? Mark two points along the lower branch and draw

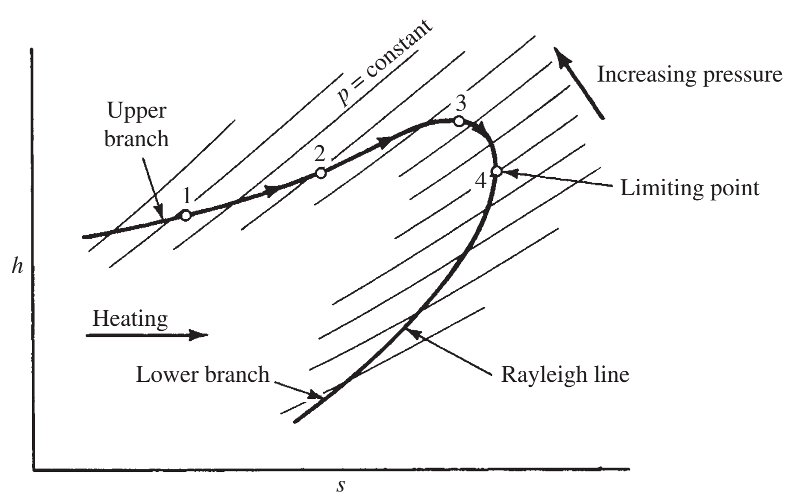


Figure 10.4 Rayleigh line in h–s plane.

Table 10.1 Analysis of Rayleigh Flow for Heating

Property	Upper Branch	Lower Branch
Enthalpy		
Density		
Velocity		
Pressure (static)		
Pressure (stagnation)		

an arrow to indicate the proper movement for a heating process. What is happening to the enthalpy? The static pressure? The density? The velocity? The stagnation pressure? Use the information available in the figures together with any equations that have been developed and fill in Table 10.1 with *increases, decreases,* or *remains constant*.

As was the case for Fanno flow, notice that flow along the lower branch of a Rayleigh line appears to be a regime with which we are not very familiar. The point of maximum entropy is some sort of a limiting point that separates these two flow regimes.

Limiting Point

Let's start with the equation of a Rayleigh line in the form

$$p + \frac{G^2}{g_c \rho} = \text{const} \tag{10.9a}$$

Differentiating gives us

$$dp + \frac{G^2}{g_c}\left(-\frac{d\rho}{\rho^2}\right) = 0 \tag{10.18}$$

Upon introduction of equation (10.4), this becomes

$$\frac{dp}{d\rho} = \frac{G^2}{g_c \rho^2} = \frac{V^2}{g_c} \tag{10.19}$$

Thus we have for an *arbitrary* fluid that

$$V^2 = g_c \frac{dp}{d\rho} \tag{10.20}$$

which is valid *anyplace* along the Rayleigh line. Now for a differential movement at the limit point of maximum entropy, $ds = 0$ or $s = \text{const}$. Thus, at this point equation (10.20) becomes

$$V^2 = g_c \left(\frac{\partial p}{\partial \rho} \right)_{s=c} \quad \text{(at the limit point)} \tag{10.21}$$

This should be immediately recognized as the sonic speed. The upper branch of the Rayleigh line, where property variations appear *reasonable,* is seen to be a region of subsonic flow and the lower branch is for supersonic flow. Once again we notice that occurrences in supersonic flow are frequently contrary to our expectations.

Another interesting fact can be shown to be true at the limit point. From equation (10.19) we have

$$dp = \frac{V^2}{g_c} d\rho \tag{10.22}$$

Differentiating equation (10.4), we can show that

$$d\rho = -\rho \frac{dV}{V} \tag{10.23}$$

Combining equations (10.22) and (10.23), we obtain

$$dp = -\rho \frac{V}{g_c} dV \tag{10.24}$$

This can be introduced into the property relation

$$T ds = dh - \frac{dp}{\rho} \tag{1.31}$$

to obtain

$$T ds = dh + \frac{V dV}{g_c} \tag{10.25}$$

At the limit point where $M = 1.0$, $ds = 0$, and (10.25) becomes

$$0 = dh + \frac{V dV}{g_c} \quad \text{(at the limit point)} \tag{10.26}$$

If we neglect potentials, our definition of stagnation enthalpy is

$$h_t = h + \frac{V^2}{2g_c} \tag{3.18}$$

which when differentiated becomes

$$dh_t = dh + \frac{VdV}{g_c} \tag{10.27}$$

Therefore, comparing equations (10.26) and (10.27), we see that equation (10.26) really tells us that

$$dh_t = 0 \quad \text{(at the limit point)} \tag{10.28}$$

and thus the limit point is seen to be a point of maximum *stagnation* enthalpy. This is easily confirmed by looking at equation (10.5). The stagnation enthalpy increases as long as heat can be added. At the point of maximum entropy, no more heat can be added and thus h_t must be a maximum at this location.

We have not talked very much of stagnation enthalpy except to note that it is changing. Figure 10.5 shows the Rayleigh line (which represents the locus of static states) together with the corresponding stagnation reference lines. Remember that for a perfect gas this *h–s* diagram is equivalent to a *T–s* diagram. Notice that there are *two* stagnation curves, one for subsonic flow and the other for

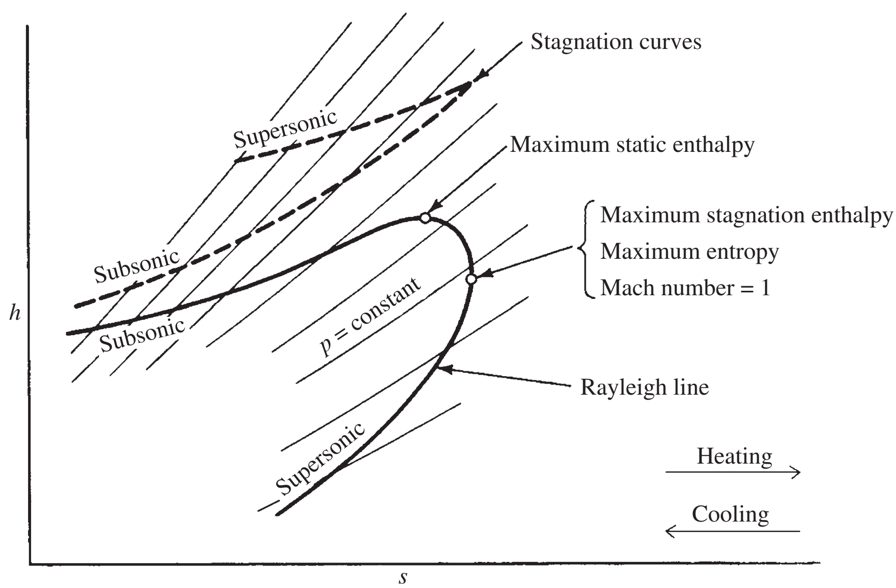


Figure 10.5 Rayleigh line in *h–s* plane (including stagnation curves).

supersonic flow. You might ask now how we knew that the supersonic stagnation curve is the top one. We can show this by starting with the differential form of the energy equation:

$$\delta q = \cancel{\delta w_s} + dh_t \tag{3.20}$$

or

$$\delta q = dh_t \tag{10.29}$$

Knowing that

$$\delta q = T ds_e \tag{3.10}$$

and

$$ds_e \approx ds \tag{10.2}$$

we have for Rayleigh flow

$$dh_t = T ds_e = T ds \tag{10.30}$$

or

$$\boxed{\frac{dh_t}{ds} = T} \tag{10.31}$$

Note that equation (10.31) gives the slope of the stagnation curve in terms of the *static* temperature.

Now draw a constant-entropy line on Figure 10.5. This line will cross the subsonic branch of the (static) Rayleigh line at a higher temperature than where it crosses the supersonic branch. Consequently, the *slope* of the subsonic stagnation reference curve will be greater than that of the supersonic stagnation curve. Since both stagnation curves must come together at the point of maximum entropy, this means that the supersonic stagnation curve is a separate curve lying *above* the subsonic one. In Section 10.7 we will see another reason why this must be so.

In which direction does a *cooling* process move along the subsonic branch of the Rayleigh line? Along the supersonic branch? From Figure 10.5 it would appear that the *stagnation pressure* will *increase* during a cooling process. This can be substantiated from the stagnation pressure–energy equation:

$$\frac{dp_t}{\rho_t} + ds_e(T_t - T) + T_t\, ds_i + \delta w_s = 0 \tag{3.25}$$

With the assumptions made for Rayleigh flow, this reduces to

$$\frac{dp_t}{\rho_t} + ds_e(T_t - T) = 0 \tag{10.32}$$

Now $(T_t - T)$ is always positive. Thus, the sign of dp_t can be seen to depend only on ds_e.

For heating,

$$ds_e+; \text{thus } dp_t-, \text{or } p_t \text{ decreases}$$

For cooling,

$$ds_e-; \text{thus } dp_t+, \text{or } p_t \text{ increases}$$

In practice, the latter condition is difficult to achieve because any friction that is inevitably present introduces a greater drop in stagnation pressure than the rise created by the cooling process, *unless* the cooling is done by vaporization of an injected liquid. (See "The Aerothermopressor: A Device for Improving the Performance of a Gas Turbine Power Plant" by A. H. Shapiro et al., *Transactions of the ASME*, April 1956.)

10.4 WORKING EQUATIONS FOR PERFECT GASES

By this time you should have a good idea of how property changes occur in both subsonic and supersonic Rayleigh flow. Remember that we can progress along a Rayleigh line in *either* direction, depending on whether the heat is being added to or removed from the system. We now proceed to develop relations between properties at arbitrary sections. Recall that we want these working equations to be expressed in terms of Mach numbers and a specific heat ratio. To obtain explicit relations, we assume the fluid to be a perfect gas.

Momentum

We start with the momentum equation developed in Section 10.3 since this will lead directly to a pressure ratio:

$$p + \frac{GV}{g_c} = \text{const} \tag{10.8}$$

or from (10.4) this can be written as

$$p + \frac{\rho V^2}{g_c} = \text{const} \tag{10.33}$$

Substitute for density from the equation of state:

$$\rho = \frac{p}{RT} \tag{10.34}$$

and for the velocity from equations (4.9) and (4.11):

$$V^2 = M^2 a^2 = M^2 \gamma g_c RT \tag{10.35}$$

Show that equation (10.33) becomes

$$p\left(1 + \gamma M^2\right) = \text{const} \tag{10.36}$$

If we apply this between two arbitrary points, we have

$$p_1\left(1 + \gamma M_1^2\right) = p_2\left(1 + \gamma M_2^2\right) \tag{10.37}$$

which can be solved for

$$\boxed{\frac{p_2}{p_1} = \frac{1 + \gamma\, M_1^2}{1 + \gamma\, M_2^2}} \tag{10.38}$$

Continuity

From Section 10.3 we have

$$\rho V = G = \text{constant} \tag{10.4}$$

Again, if we introduce the perfect gas equation of state together with the definition of Mach number and sonic speed, equation (10.4) can be expressed as

$$\frac{pM}{\sqrt{T}} = \text{constant} \tag{10.39}$$

Written between two points, this gives us

$$\frac{p_1 M_1}{\sqrt{T_1}} = \frac{p_2 M_2}{\sqrt{T_2}} \tag{10.40}$$

which can be solved for the temperature ratio:

$$\frac{T_2}{T_1} = \frac{p_2^2 M_2^2}{p_1^2 M_1^2} \tag{10.41}$$

The introduction of the pressure ratio from (10.38) results in the following working equation for static temperatures:

$$\boxed{\frac{T_2}{T_1} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2 \frac{M_2^2}{M_1^2}} \tag{10.42}$$

The density relation can equally be obtained from equations (10.38) and (10.42) and the perfect gas equation of state:

$$\frac{\rho_2}{\rho_1} = \frac{M_1^2}{M_2^2}\left(\frac{1+\gamma M_2^2}{1+\gamma M_1^2}\right) \tag{10.43}$$

Does this also represent something else besides the density ratio? [See equation (10.4).]

Stagnation Conditions

This is the first type of flow that we have examined in which the stagnation enthalpy does not remain constant. Thus we must seek a stagnation temperature ratio for use with perfect gases. We know that

$$T_t = T\left(1 + \frac{\gamma-1}{2}M^2\right) \tag{4.18}$$

If we write this for each location and then divide one equation by the other, we will have

$$\frac{T_{t2}}{T_{t1}} = \frac{T_2}{T_1}\left(\frac{1+[(\gamma-1)/2]\,M_2^2}{1+[(\gamma-1)/2]\,M_1^2}\right) \tag{10.44}$$

Since we already have solved for the static temperature ratio (10.42), this can now be written as

$$\boxed{\frac{T_{t2}}{T_{t1}} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2 \frac{M_2^2}{M_1^2}\left(\frac{1+[(\gamma-1)/2]\,M_2^2}{1+[(\gamma-1)/2]\,M_1^2}\right)} \tag{10.45}$$

Similarly, we can obtain an expression for the stagnation pressure ratio, since we know that

$$p_t = p\left(1+\frac{\gamma-1}{2}M^2\right)^{\gamma/(\gamma-1)} \tag{4.21}$$

which means that

$$\frac{p_{t2}}{p_{t1}} = \frac{p_2}{p_1}\left(\frac{1+[(\gamma-1)/2]\,M_2^2}{1+[(\gamma-1)/2]\,M_1^2}\right)^{\gamma/(\gamma-1)} \tag{10.46}$$

Substitution for the pressure ratio from equation (10.38) yields

$$\boxed{\frac{p_{t2}}{p_{t1}} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)\left(\frac{1+[(\gamma-1)/2]\,M_2^2}{1+[(\gamma-1)/2]\,M_1^2}\right)^{\gamma/(\gamma-1)}} \tag{10.47}$$

Incidentally, can this stagnation pressure ratio be related to the entropy change in the previous manner?

$$\frac{p_{t2}}{p_{t1}} \stackrel{?}{=} e^{-\Delta s/R} \tag{4.28}$$

What assumptions were used to develop equation (4.28)? Are these the same assumptions that were made for Rayleigh flow? If not, how would you go about determining the entropy change between two points? Would the same *method* used in Chapter 9 for Fanno flow be applicable here? [Examine equations (9.25) to (9.27).]

In summary, we have developed the means to solve for all properties at one location (2) if we know all the properties at some other location (1) and the Mach number at point (2). Actually, any piece of information about point (2) would suffice. For example, we might be given the pressure at (2). The Mach number at (2) could then be found from equation (10.38) and the solution for the other properties could be obtained in the usual manner.

There are also some types of problems in which nothing is known at the downstream section and our job is to predict the final Mach number given the initial

conditions and information on the heat transferred to or from the system. For this we turn to the fundamental relation that involves heat transfer.

Energy

From Section 10.3 we have

$$h_{t1} + q = h_{t2} \tag{10.5}$$

For perfect gases we express enthalpy as

$$h = c_p T \tag{1.38}$$

which can also be applied to the stagnation conditions

$$h_t = c_p T_t \tag{10.48}$$

Thus the energy equation can be written as

$$c_p T_{t1} + q = c_p T_{t2} \tag{10.49}$$

or

$$\boxed{q = c_p (T_{t2} - T_{t1})} \tag{10.50}$$

Note carefully that subscripts matter

$$q = c_p \Delta T_t \neq c_p \Delta T \tag{10.51}$$

In all of the developments above, we have not only introduced the perfect gas equation of state but have made the usual assumption of constant specific heats. In some cases where heat transfer rates are extremely high and large temperature changes result, c_p may vary enough to warrant using an average value of c_p. If, in addition, significant variations in γ result, it will be necessary to return to the basic equations and derive new working relations by treating γ as a variable. See Chapter 11 on methods to apply to the analysis of such real gases.

10.5 REFERENCE STATE AND THE RAYLEIGH TABLE

The equations developed in Section 10.4 provide the means of predicting properties at one location if sufficient information is known concerning a Rayleigh flow system. Although the relations are straightforward, their use is frequently

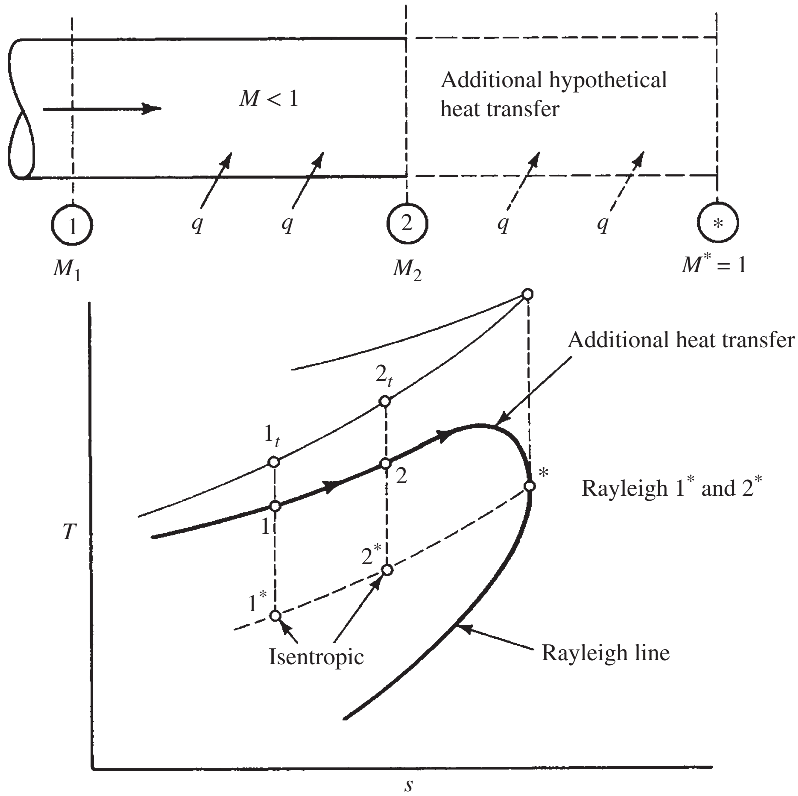


Figure 10.6 The * reference for Rayleigh flow.

cumbersome and thus we turn to techniques used previously that greatly simplify problem solution.

We introduce yet *another* * reference state defined as before, where *the Mach number of unity must be reached by some particular process*. In the current case, we imagine that the Rayleigh flow is continued (i.e., more heat is added) until the velocity reaches sonic level. Figure 10.6 shows a T–s diagram for subsonic Rayleigh flow with heat addition. A sketch of the physical system is also shown. If we imagine that more heat is added, the entropy continues to increase and we will eventually reach the limiting point where sonic speed exists. The dashed lines show a hypothetical duct in which the additional heat transfer takes place. At the end of the process, we reach the * reference point *for Rayleigh flow*.

The locus of *isentropic* * reference points has also been included (the dashed mid-line in Figure 10.6) on the T–s diagram to emphasize the fact that the Rayleigh * reference is a completely different thermodynamic state from those encountered before. Also, we note that proceeding from either point 1 or 2 *by Rayleigh flow* will ultimately lead to the same state when Mach 1.0 is reached. Thus we do not have to write 1^* or 2^* but simply * in the case of Rayleigh flow. (Recall that this was true for Fanno flow also, *but* you should also realize that the * reference for Rayleigh flow has nothing to do with the * reference used

Figure 10.7 Supersonic cooling in Rayleigh flow.

for Fanno or isentropic flows.) Notice in Figure 10.6 that the various * locations are *not* on a horizontal line as they were for Fanno flow (see Figure 9.5). Why is this so?

In Figure 10.6 an example of subsonic heating has been depicted. Consider now a case of *cooling* in the *supersonic* regime. Figure 10.7 shows such a physical duct. Locate points 1 and 2 on the accompanying *T–s* diagram. Also show the hypothetical duct and the * reference point on the physical system.

We proceed now to rewrite the working equations in terms of the Rayleigh flow * reference condition. Consider first

$$\frac{p_2}{p_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2} \tag{10.38}$$

Let point 2 be any arbitrary point in the flow system and let its Rayleigh * condition be point 1. Then

$$p_2 \Rightarrow p \quad M_2 \Rightarrow M(\text{any value})$$
$$p_1 \Rightarrow p^* \quad M_1 \Rightarrow 1$$

and equation (10.38) becomes

$$\frac{p}{p^*} = \frac{1+\gamma}{1+\gamma M^2} = f(M,\gamma) \tag{10.52}$$

We see that $p/p^* = f(M, \gamma)$, and thus a table can be computed for p/p^* versus M for any particular γ. By now, this scheme is quite familiar and you should have no difficulty in showing that

$$\frac{T}{T^*} = \frac{M^2(1+\gamma)^2}{(1+\gamma M^2)^2} = f(M,\gamma) \tag{10.53}$$

$$\frac{\rho}{\rho^*} = \frac{1+\gamma M^2}{(1+\gamma)M^2} = f(M,\gamma) \tag{10.54}$$

$$\frac{T_t}{T_t^*} = \frac{2(1+\gamma)M^2}{(1+\gamma M^2)^2}\left(1+\frac{\gamma-1}{2}M^2\right) = f(M,\gamma) \tag{10.55}$$

$$\frac{p_t}{p_t^*} = \frac{1+\gamma}{1+\gamma M^2}\left(\frac{1+[(\gamma-1)/2]M^2}{(\gamma+1)/2}\right)^{\gamma/(\gamma-1)} = f(M,\gamma) \tag{10.56}$$

Values for the functions represented in equations (10.52) through (10.56) are listed in the Rayleigh table in Appendix J. Examples of the use of this table are given in the next section. Equation (10.53) has a notable peak or maximum (shown as "Maximum static enthalpy" in Figure 10.5) which occurs at $M = \sqrt{1/\gamma}$ where $(T/T^*)_{max} = (1+\gamma)^2/4\gamma$.

Now for Rayleigh flow the last entry in Appendix J is S_{max}/R. In order to tabulate it we follow the method of approach in Fanno flow [*caution*, equation (4.28) does not apply here, see Problem 10.22 for the Rayleigh line equivalent to equation (9.27)]. To represent the entropy maximum at Mach 1.0 we introduce our new * reference condition. Take equation (9.25) and define location 2 at this maximum with location 1 kept as variable.

Let $s_2 = s$ (at $M = 1.0$) and $s_1 = s(M)$, the table notation is $S_{max} \equiv s^* - s$,

$$\frac{S_{max}}{R} = \frac{s^*-s}{R} = \frac{\gamma}{\gamma-1}\ln\frac{1}{M^2} + \left(\frac{\gamma+1}{\gamma-1}\right)\ln\left(\frac{1+\gamma M^2}{1+\gamma}\right) \tag{10.57}$$

10.6 APPLICATIONS

The procedure for solving Rayleigh flow problems is quite similar to the approach used for Fanno flow except that the tie between any two locations in Rayleigh flow is determined by heat transfer considerations rather than by duct friction. Also, the entropy may increase or decrease along the flow depending on the direction of the heat transfer. The recommended steps are, therefore, as follows:

1. Sketch the physical situation (including the hypothetical * reference point).
2. Label sections where conditions are known or desired.
3. List all given information with consistent units.
4. Determine the unknown Mach number.
5. Calculate the additional properties desired.

Variations in the above procedure are frequently involved at step 4, depending on what information is known. For example, the amount of heat transferred may be given and a prediction of the downstream Mach number might be desired. On the other hand, one of the downstream properties may be known, and we could be

asked to compute the heat transfer. In flow systems that involve a combination of Rayleigh flow with other phenomena (such as shocks, nozzles, etc.), the *T–s* diagram is usually a great aid to problem solution.

For the following examples, we are dealing with the steady one-dimensional flow of air ($\gamma = 1.4$), which can be treated as a perfect gas. Assume that $w_s = 0$, negligible friction, constant area, and negligible potential changes. Figure E10.1 is common to Examples 10.1 and 10.2.

Example 10.1
For Figure E10.1, given $M_1 = 1.5$, $p_1 = 10$ psia, and $M_2 = 3.0$, find p_2 and the direction of heat transfer.

Since both Mach numbers are known, we can solve immediately for

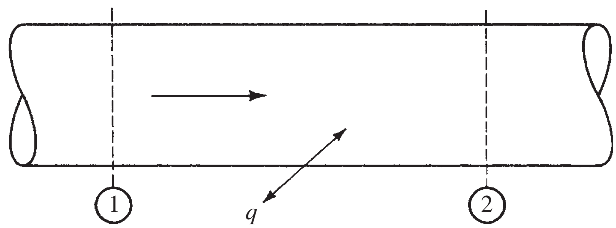


Figure E10.1

$$p_2 = \frac{p_2}{p^*}\frac{p^*}{p_1}p_1 = (0.1765)\left(\frac{1}{0.5783}\right)(10) = 3.05 \text{ psia} \quad (\text{or } 21 \text{ kPa})$$

The flow is getting more supersonic or moving away from the * reference point. A look at Figure 10.5 should confirm that the entropy is decreasing and thus heat is being removed from the system. Alternatively, we could compute the ratio T_{t2}/T_{t1}.

$$\frac{T_{t2}}{T_{t1}} = \frac{T_{t2}}{T_t^*}\frac{T_t^*}{T_{t1}} = (0.6540)\left(\frac{1}{0.9093}\right) = 0.719$$

Since this ratio is less than 1, it indicates a cooling process.

Example 10.2
Given $M_2 = 0.93$, $T_{t2} = 300°\text{C}$, and $T_{t1} = 100°\text{C}$, find M_1 and p_2/p_1.

To determine conditions at section 1 in Figure E10.1 we must establish the ratio

$$\frac{T_{t1}}{T_t^*} = \frac{T_{t1}}{T_{t2}}\frac{T_{t2}}{T_t^*} = \left(\frac{273+100}{273+300}\right)(0.9963) = 0.6486$$

Look up $T_t/T_t^* = 0.6486$ in the Rayleigh table and determine that $M_1 = 0.472$. Thus

$$\frac{p_2}{p_1} = \frac{p_2}{p^*}\frac{p^*}{p_1} = (1.0856)\left(\frac{1}{1.8294}\right) = 0.593$$

Example 10.3
A constant-area combustion chamber is supplied air at 400°R and 10.0 psia (Figure E10.3). The air stream has a velocity of 402 ft/sec. Determine the exit conditions if 50 Btu/lbm is added in the combustion process and the chamber handles the maximum amount of air possible.

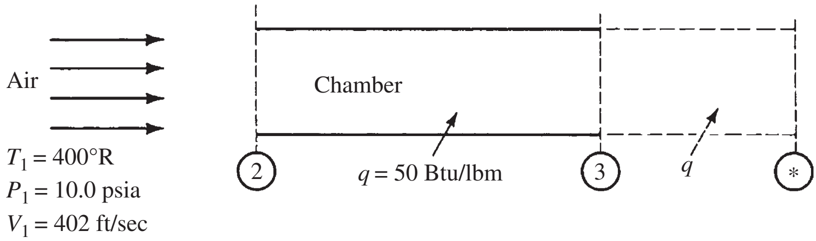


Figure E10.3

For the chamber to handle the maximum amount of air, there will be no *spillover* at the entrance and conditions at 2 will be the same as those of the free stream.

$$T_2 = T_1 = 400°\text{R} \quad p_2 = p_1 = 10.0 \text{ psia} \quad V_2 = V_1 = 402 \text{ ft/sec}$$

$$a_2 = \sqrt{\gamma g_c R T_2} = [(1.4)(32.2)(53.3)(400)]^{1/2} = 980 \text{ ft/sec}$$

$$M_2 = \frac{V_2}{a_2} = \frac{402}{980} = 0.410$$

$$T_{t2} = \frac{T_{t2}}{T_2} T_2 = \left(\frac{1}{0.9675}\right)(400) = 413°\text{R}$$

From the Rayleigh table at $M_2 = 0.41$, we find that

$$\frac{T_{t2}}{T_t^*} = 0.5465 \quad \frac{T_2}{T^*} = 0.6345 \quad \frac{p_2}{p^*} = 1.9428$$

To determine conditions at the end of the chamber, we must work through the heat transfer that fixes the outlet stagnation temperature:

$$\Delta T_t = \frac{q}{c_p} = \frac{500}{0.24} = 208°\text{R}$$

Thus

$$T_{t3} = T_{t2} + \Delta T_t = 413 + 208 = 621°\text{R} \quad (\text{or } 229.4 \text{ K})$$

and

$$\frac{T_{t3}}{T_t^*} = \frac{T_{t3}}{T_{t2}} \frac{T_{t2}}{T_t^*} = \left(\frac{621}{413}\right)(0.5465) = 0.8217$$

We enter the Rayleigh table with this value of T_t/T_t^* and find that

$$M_3 = 0.603 \quad \frac{T_3}{T^*} = 0.9196 \quad \frac{p_3}{p^*} = 1.5904$$

Thus

$$p_3 = \frac{p_3}{p^*}\frac{p^*}{p_2}p_2 = (1.5904)\left(\frac{1}{1.9428}\right)(10.0) = 8.19 \text{ psia} \quad (\text{or } 56.5 \text{ kPa})$$

and

$$T_3 = \frac{T_3}{T^*}\frac{T^*}{T_2}T_2 = (0.9196)\left(\frac{1}{0.6345}\right)(400) = 580°\text{R} \quad (\text{or } 332.2 \text{ K})$$

Example 10.4
In Example 10.3, let us ask another question: How much more heat (fuel) could be added without changing conditions at the entrance to the duct? We know that as more heat is added, we move along the Rayleigh line until the point of maximum entropy is reached. Thus M_3 will now have to equal 1.0 (Figure E10.4).

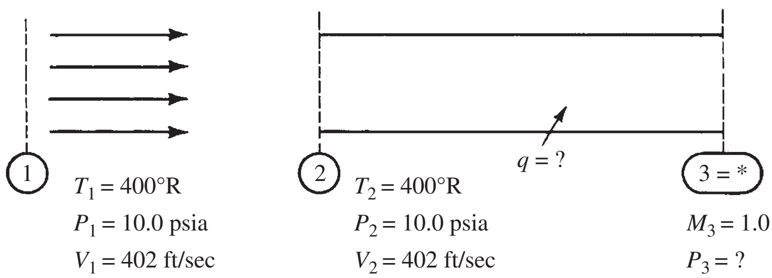


Figure E10.4

From Example 10.3 we have $M_2 = 0.41$ and $T_{t2} = 413°\text{R}$. Then

$$T_{t3} = T_t^* = \frac{T_t^*}{T_{t2}}T_{t2} = \left(\frac{1}{0.5465}\right)(413) = 756°\text{R}$$

$$p_3 = p^* = \frac{p^*}{p_2}p_2 = \left(\frac{1}{1.9428}\right)(10.0) = 5.15 \text{ psia}$$

and

$$q = c_p \Delta T_t = (0.24)(756 - 413) = 82.3 \text{ Btu/lbm} \quad (\text{or } 191.4 \text{ kJ/kg})$$

or 32.3 Btu/lbm more than the original 50 Btu/lbm.

In these last two examples it has been assumed that the outlet pressure is maintained at the values calculated. Actually, in Example 10.4 the receiver pressure could be anywhere below 5.15 psia, since sonic speed would be present at the exit.

10.7 CORRELATION WITH SHOCKS

At several places in this chapter we have pointed out some similarities between Rayleigh flow and normal shocks. Let us review these points carefully:

1. The end points before and after a normal shock represent states with the same mass flow per unit area, the same impulse function, and the same stagnation enthalpy.
2. A Rayleigh line represents states with the same mass flow per unit area and the same impulse function. All points on a Rayleigh line do *not* have the same stagnation enthalpy because of the heat transfer involved. To move *along* a Rayleigh line requires heat transfer to take place.

For confirmation of the above, compare equations (6.2), (6.3), and (6.9) for a normal shock with equations (10.4), (10.5), and (10.9) for Rayleigh flow. Now check Figure 10.8 and you will notice that for every point on the supersonic branch of the Rayleigh line there is a corresponding point on the subsonic branch with the same stagnation enthalpy. Thus these two points satisfy all three conditions for the end points of a normal shock and could be connected by such a shock.

We can now picture a supersonic Rayleigh flow followed by a normal shock, with additional heat transfer taking place subsonically. Such a situation is shown in Figure 10.9. Note that the shock merely jumps the flow from the supersonic branch to the subsonic branch of the *same* Rayleigh line. This also brings to light

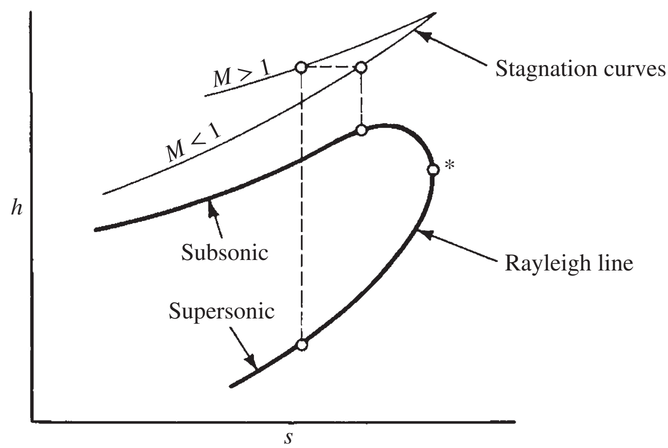


Figure 10.8 Static and stagnation curves for Rayleigh flow.

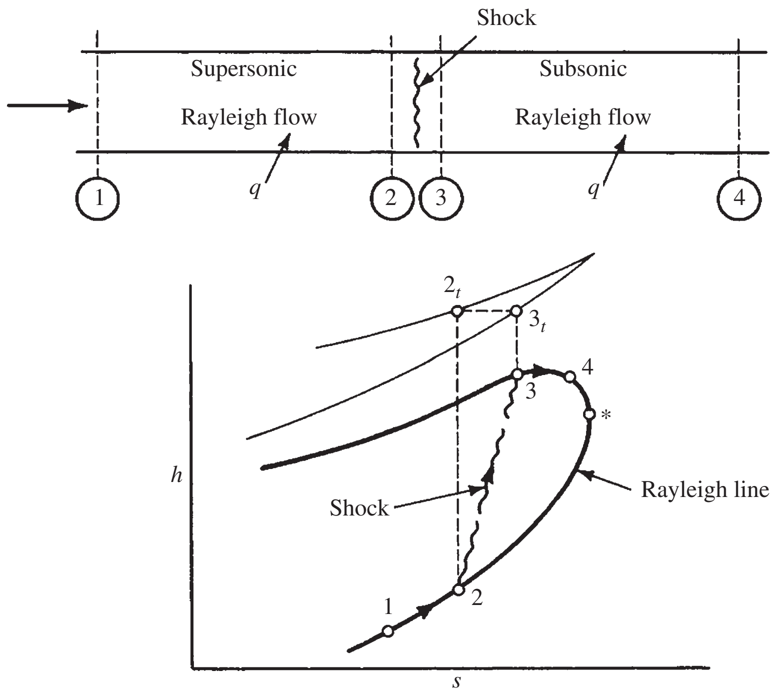


Figure 10.9 Combination of Rayleigh flow and normal shock.

another reason why the supersonic stagnation curve must lie above the subsonic stagnation curve. If this were not so, a shock would exhibit a decrease in entropy, which is not physically correct.

If you recall the information from Section 9.7 dealing with the correlation of Fanno flow and shocks, it should now be apparent that *the end points of a normal shock can represent the intersection of a Fanno line and a Rayleigh line* as shown in Figure 10.10. Remember that these Fanno and Rayleigh lines are for the same mass velocity (mass flow per unit area).

Example 10.5
Air flowing inside a constant-area duct enters a normal shock at $M_1 = 2.65$. Across the shock, we neglect any friction and heat transfer present as in Chapter 6. Using equations (9.50) and (10.57) for the entropy, *show* that two intersections of the Fanno and Rayleigh lines describe the path of the shock itself. After "matching," plot the entropy lines as a function of Mach number.

From Appendix H, we know that $M_2 \approx 0.5$ and that the ratio p_{t2}/p_{t1} must decrease reflecting an entropy increase [equation (4.28)]. Because the reference points for the Fanno and Rayleigh lines *differ*, we need to match them in order to plot those curves in the same graph and this can be conveniently done by subtracting equations (9.50) and (10.57) from their corresponding entropy values at $M_2 = 2.65$ (thus generating a common zero). Figure E10.5 shows the resulting plot where it is evident that the shock

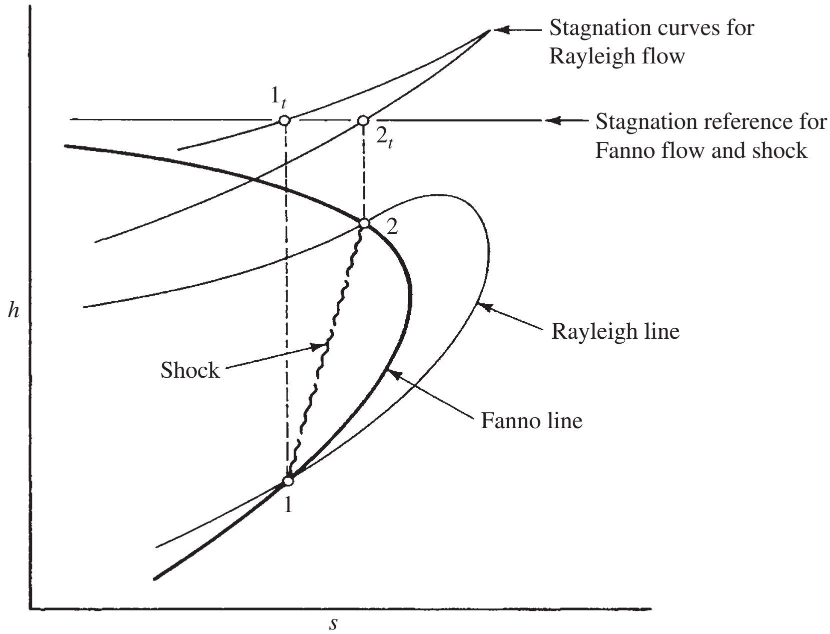


Figure 10.10 Correlation of Fanno flow and Rayleigh flow with a normal shock for the same mass velocity.

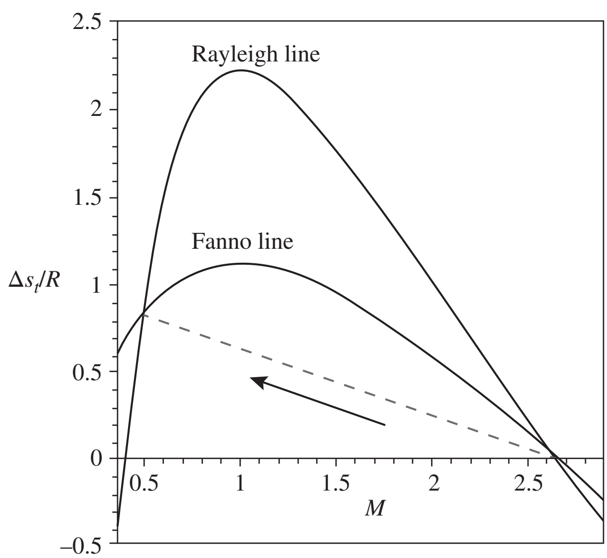


Figure E10.5 Intersection of Fanno and Rayleigh lines representing a normal shock after setting the entropies of the incoming air to zero at $M_1 = 2.65$ for this comparison. Note in the figure that, for the shock, $M_2 \approx 0.5$ and $\Delta s_t/R = \ln(p_{t1}/p_{t2}) \approx 0.82$ in agreement with values in Table H.

end points fall at the intersections of the Fanno and Rayleigh lines. Furthermore, the resulting values correspond precisely to those in the shock table. Because shocks going from subsonic to supersonic conditions have never been observed, such plot further emphasizes that Δs_i always increases in the direction of an irreversible process.

Example 10.6

Air enters a constant-area duct with a Mach number of 1.6, a temperature of 200 K, and a pressure of 0.56 bar (Figure E10.6). After some heat transfer, a normal shock occurs, whereupon the cross-sectional area is reduced as shown. At the exit, the Mach number is found to be 1.0 and the pressure is 1.20 bar. Compute the amount and direction of heat transfer.

It is not known whether a heating or cooling process is involved. We construct the T–s diagram under the assumption that cooling takes place and will find out if this is correct. The flow from 3 to 4 is assumed to be isentropic; thus

$$p_{t3} = p_{t4} = \frac{p_{t4}}{p_4} p_4 = \left(\frac{1}{0.5283}\right)(1.20) = 2.2714 \text{ bar}$$

Note that point 3 is on the same Rayleigh line as point 1 and this permits us to compute M_2 through the use of the Rayleigh table. This approach might not have occurred to us had we not drawn the T–s diagram.

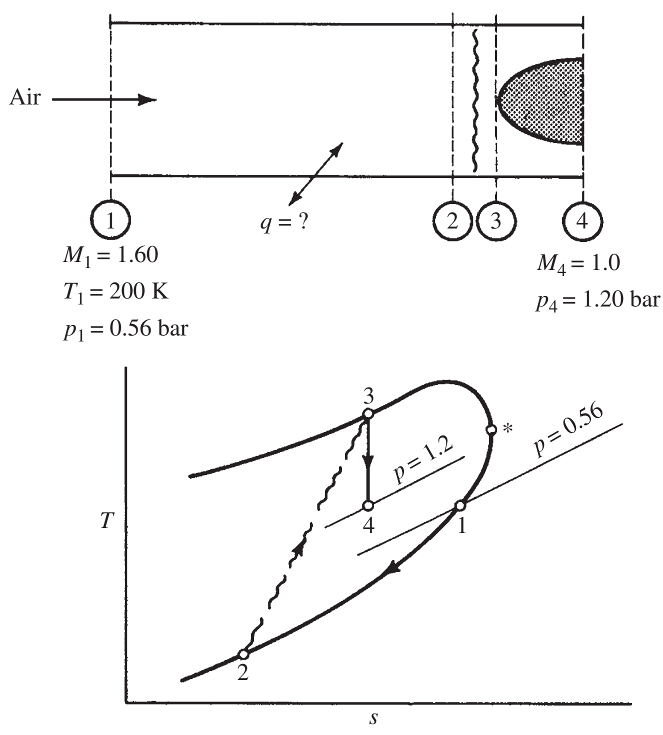


Figure E10.6

$$\frac{p_{t3}}{p_t^*} = \frac{p_{t3}}{p_1}\frac{p_1}{p_{t1}}\frac{p_{t1}}{p_t^*} = \left(\frac{2.2714}{0.56}\right)(0.2353)(1.1756) = 1.1220$$

From the Rayleigh table, we find $M_3 = 0.481$ and from the shock table, $M_2 = 2.906$.

Now we can compute the stagnation temperatures:

$$T_{t1} = \frac{T_{t1}}{T_1}T_1 = \left(\frac{1}{0.6614}\right)(200) = 302\ \mathrm{K}$$

$$T_{t2} = \frac{T_{t2}}{T_t^*}\frac{T_t^*}{T_{t1}}T_{t1} = (0.6629)\left(\frac{1}{0.8842}\right)(302) = 226\ \mathrm{K}$$

and the heat transfer:

$$q = c_p(T_{t2} - T_{t1}) = (1000)(226 - 302) = -7.6 \times 10^4\ \mathrm{J/kg}\quad (\text{or} -32.68\ \mathrm{Btu/lbm})$$

The minus sign indicates a cooling process that is consistent with the Mach number's increase from 1.60 to 2.906.

10.8 THERMAL CHOKING DUE TO HEATING

In Section 5.7 we discussed *area choking,* and in Section 9.8, *friction choking*. In Fanno flow we recall that once sufficient duct was added, or the receiver pressure was lowered far enough, we reached a Mach number of unity at the end of the duct. Further reduction of the receiver pressure could not affect conditions in the flow system. The addition of any more duct caused the flow to move along a new Fanno line at a reduced flow rate. You might wish to review Figure 9.11, which shows this physical situation along with the corresponding *T–s* diagram.

Subsonic Rayleigh flows are quite similar. Figure 10.11 shows a given duct fed by a large tank and converging nozzle. Once sufficient heat has been *added*, we reach Mach 1.0 at the end of the duct. The *T–s* diagram for this is shown as path 1–2–3. This is called *thermal choking*. It is assumed that the receiver pressure is at p_3 or below. Reduction of the receiver pressure below p_3 would not affect the flow conditions inside the system. However, any more addition of heat will change the original flow conditions.

Now suppose that we do add more heat to the system. This would usually be done by increasing the heat transfer rate through the walls of the original duct. However, it is more intuitive to indicate the additional heat transfer at the original rate in an extra piece of duct, as shown in Figure 10.11. The only way that the system can reflect the required additional entropy change is to move to a new Rayleigh line at a *decreased* flow rate. This is shown as path 1–2′–3′–4 on the *T–s* diagram. Whether or not the exit velocity arrives at the sonic condition depends on how much extra heat is added and on the receiver pressure imposed on the system.

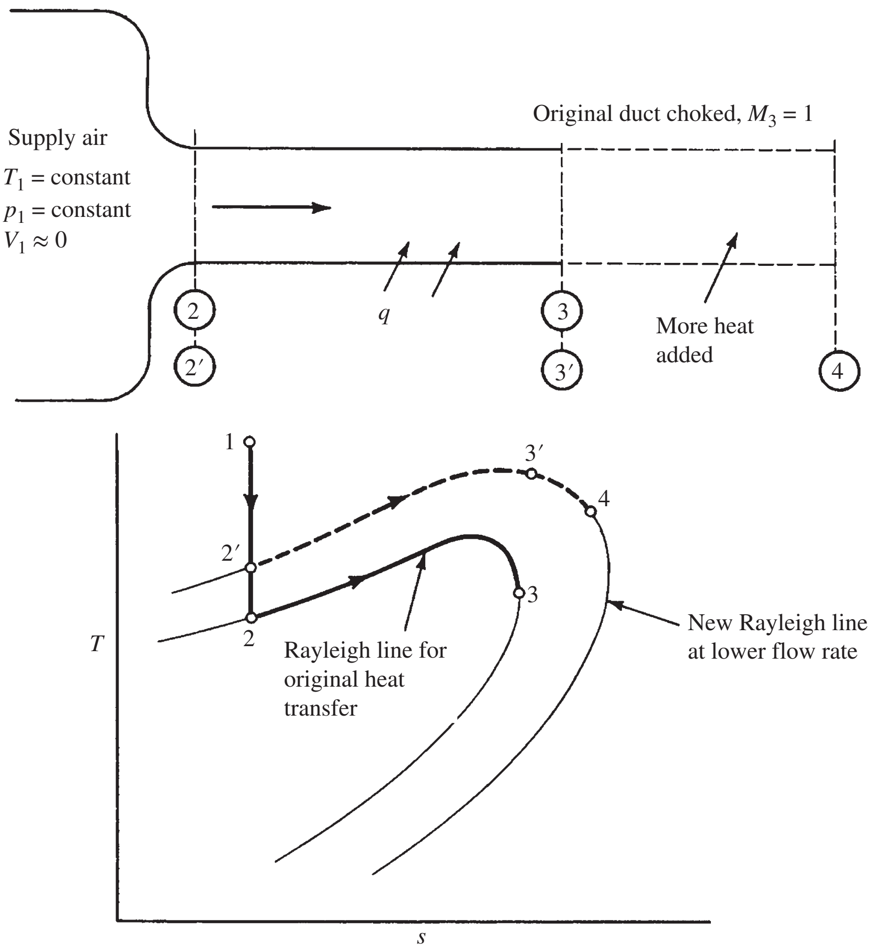


Figure 10.11 Addition of more heat when choked.

As a specific example of choked flow, we return to the combustion chamber of Example 10.4, which had the maximum amount of heat addition possible, assuming that the free-stream airflow entered the chamber with no change in velocity. We now consider what happens as more fuel (heat) is added.

Example 10.7
Continuing with Example 10.4, let us add sufficient fuel to raise the outlet stagnation temperature to 3000°R. Assume that the receiver pressure is very low so that a sonic speed still exists at the exit. The additional entropy generated by the extra fuel can only be accommodated by moving to a new Rayleigh line at a decreased flow rate, which lowers the inlet Mach number. If the chamber is fed by the same air stream, some *spillage* must occur at the entrance. This produces a region of external diffusion, as shown in Figure E10.7, which is isentropic. We would like to know the Mach number at the inlet and the pressure at the exit.

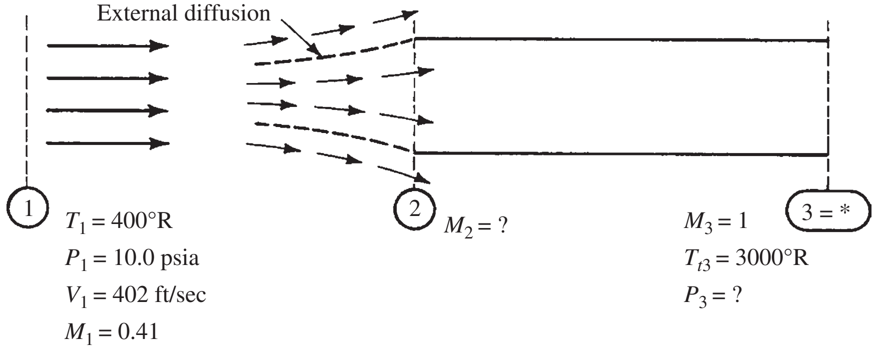


Figure E10.7

Since it is isentropic from the free stream to the inlet, we know that

$$T_{t2} = T_{t1} = 413°\text{R}$$

and since $M_3 = 1$, we know that $T_{t3} = T_t^*$.

Thus we can determine conditions at 2 by computing

$$\frac{T_{t2}}{T_t^*} = \frac{T_{t2}}{T_{t3}}\frac{T_{t3}}{T_t^*} = \left(\frac{413}{3000}\right)(1) = 0.1377$$

and from the Rayleigh table, $M_2 = 0.176$ and $p_2/p^* = 2.3002$.

To find the pressure at the outlet we need to use both the isentropic table and the Rayleigh table.

First

$$p_2 = \frac{p_2}{p_{t2}}\frac{p_{t2}}{p_{t1}}\frac{p_{t1}}{p_1}p_1 = (0.9786)(1)\left(\frac{1}{0.89907}\right)(10.0) = 10.99 \text{ psia}$$

then

$$p_3 = \frac{p_3}{p^*}\frac{p^*}{p_2}p_2 = (1)\left(\frac{1}{2.3002}\right)(10.99) = 4.78 \text{ psia} \quad (\text{or } 32.95 \text{ kPa})$$

Note that to maintain sonic speed at the chamber exit, the pressure in the receiver must be reduced to at least 4.78 psia.

Suppose that in Example 10.7 we were unable to lower the receiver pressure to 4.78 psia. Assume that as fuel was added to raise the stagnation temperature to 3000°R, the pressure in the receiver was maintained at its previous value of 5.15 psia. This would lower the flow rate even further as we move to another Rayleigh line with a lower mass velocity, *and* this time the exit velocity would not be

quite sonic. Although both M_2 and M_3 are unknown, two pieces of information are given at the exit. Two simultaneous equations could be written, but it is easier to use tables and a trial-and-error solution. The important thing to remember here is that once a *subsonic* flow is *thermally choked,* the addition of more heat causes the flow rate to decrease. Just how much it decreases and whether or not the exit remains sonic depends on the pressure that exists after the exit.

The parallel between choked Rayleigh and Fanno flow does not quite extend into the *supersonic* regime. Recall that for choked Fanno flow, the addition of more duct introduced a shock in the duct, which permitted considerably more friction length to the sonic point (see Figure 9.12). Figure 10.12 shows a Mach 3.53 flow that has $T_t/T_t^* = 0.6139$. For a given total temperature at this section, the value of T_t/T_t^* is a direct indication of the amount of heat that can be added to the choke point. If a normal shock were to occur at this point, the Mach number after the shock would be 0.450, which also has $T_t/T_t^* = 0.6139$ (as depicted in Figure 10.10). Thus the heat added after the shock must be exactly the same as it would be without the shock.

The situation above is not surprising since heat transfer is a function of stagnation temperature, which does not change across a shock (see Problem 10.11). To allow for more added heat, the shock must occur at some location *preceding* the Rayleigh flow. Perhaps this would be observed in a converging–diverging nozzle, which produces the supersonic flow. Or if this were a situation similar to Example 10.4 (only supersonic), a detached shock would occur in the free stream ahead of the duct. In either case, the resulting subsonic flow could accommodate additional heat transfer.

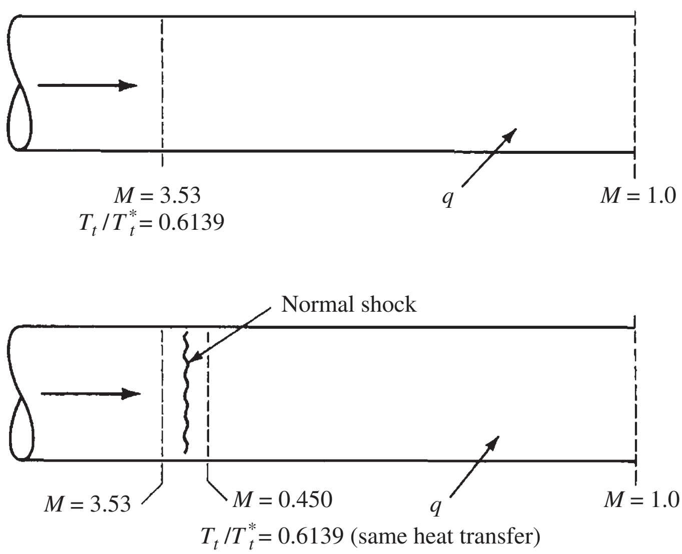


Figure 10.12 Influence of shock on maximum heat transfer.

10.9 WHEN γ IS NOT EQUAL TO 1.4

As mentioned earlier, the Rayleigh flow table in Appendix J is for $\gamma = 1.4$. The behavior of T_t/T_t^*, the dominant heating function, for $\gamma = 1.13$, 1.4, and 1.67 is given in Figure 10.13 up to $M = 5$. Here we can see that the dependence on γ becomes rather noticeable for $M \geq 1.4$. Thus below this Mach number, the tabulations in Appendix J can be used only approximately for any γ. This means that for subsonic flows, where most Rayleigh flow problems occur, there are only minor differences between the various gases. The desired accuracy of results will govern how far you want to carry this approximation into the supersonic region. The *Gas-dynamics Calculator* will work in the range $1.0 < \gamma \leq 1.67$.

Strictly speaking, these curves are only representative for cases where γ variations are *negligible within the flow*. However, they offer hints as to what magnitude of changes are to be expected in other cases. Flows where γ variations are *not negligible within the flow* are treated in Chapter 11.

10.10 (OPTIONAL) BEYOND THE TABLES

As illustrated in Chapter 5, one can eliminate a lot of interpolation and get accurate answers for any ratio of the specific heats γ and/or any Mach number by using a computer utility such as MAPLE. The calculation of equation (10.55) is well suited to this section.

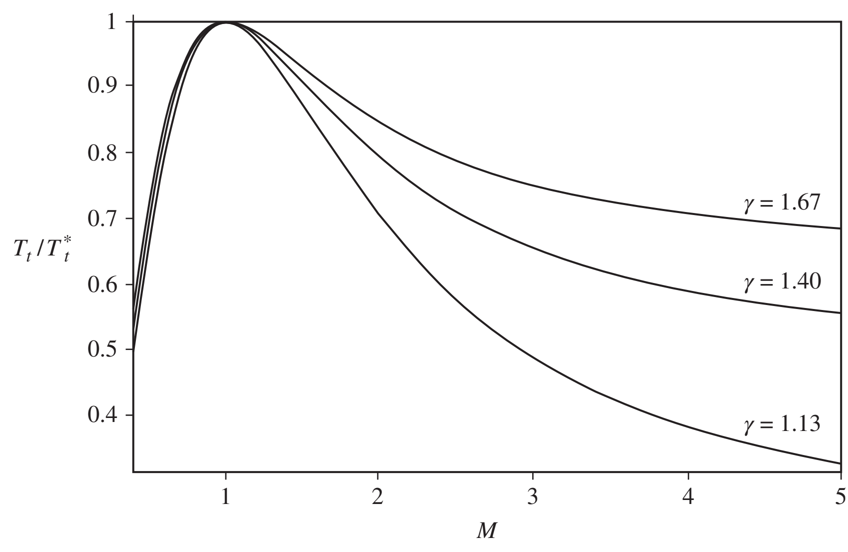


Figure 10.13 Rayleigh flow T_t/T_t^* versus Mach number for various values of γ.

Example 10.8

Let us rework some aspects of Example 10.3 without using the Rayleigh table. For M_2 = 0.41, calculate the value of T_t/T_t^*. The procedure follows equation (10.55):

$$\frac{T_t}{T_t^*} = \frac{2(1+\gamma)M^2}{(1+\gamma M^2)^2}\left(1+\frac{\gamma-1}{2}M^2\right) \tag{10.55}$$

Let

$$\begin{aligned}
&g \equiv \gamma, \text{ a parameter (the ratio of specific heats)}\\
&X \equiv \text{the independent variable (which in this case is } M_2)\\
&Y \equiv \text{the dependent variable (which in this case is } T_t/T_t^*)
\end{aligned}$$

Listed below are the precise inputs and program that you use in the computer.

```
[>g2:=1.4: X2:=0.41:
[>Y2:=(((2*(1+g2)*X2^2)/(1+g2*X2^2)^2))*((1+(g2-1)*(X2^2)/2));
                    Y2:=.5465084066
```

Now we can proceed to find the new Mach number at station 3. The new value of Y is (621)(0.5465)/(413) = 0.827. Now we use equation (**10.55**) but solve for M_3 as shown below. Note that since M is implicit in the equation, we are going to utilize "fsolve." Let

$$\begin{aligned}
&g \equiv \gamma, \text{ a parameter (the ratio of specific heats)}\\
&X \equiv \text{the independent variable (which in this case is } M_3)\\
&Y \equiv \text{the dependent variable (which in this case is } T_t/T_t^*)
\end{aligned}$$

Listed below are the precise inputs and program that you use in the computer.

```
[>g3:=1.4: Y3:=0.8217:
[>fsolve(Y3=(((2*(1+g3)*X3^2)/(1+g3*X3^2)^2))*((1+(g3-1)*(X3^2)/2)),X3,0..1);
                    .6025749883
```

The answer of $M_3 = 0.6026$ is consistent with that obtained in Example 10.3. We can now proceed to calculate the required static properties, but this will be left as an exercise for the reader.

10.11 SUMMARY

We have analyzed steady one-dimensional flow in a constant-area duct with heat transfer but with negligible friction. Fluid properties can vary in a number of ways, depending on whether the flow is subsonic or supersonic, plus consideration of the

Table 10.2 Fluid Property Variation for Rayleigh Flow

	Heating		Cooling	
Property	$M < 1$	$M > 1$	$M < 1$	$M > 1$
Velocity	Increase	Decrease	Decrease	Increase
Mach number	Increase	Decrease	Decrease	Increase
Enthalpy[a]	Increase/decrease	Increase	Increase/decrease	Decrease
Stagnation enthalpy[a]	Increase	Increase	Decrease	Decrease
Pressure	Decrease	Increase	Increase	Decrease
Density	Decrease	Increase	Increase	Decrease
Stagnation pressure	Decrease	Decrease	Increase	Increase
Entropy	Increase	Increase	Decrease	Decrease

[a]Also temperature if the fluid is a perfect gas.

direction of heat transfer. However, these variations are easily predicted and are summarized in Table 10.2.

As we might expect, the property variations that occur in subsonic Rayleigh flow follow an intuitive pattern, but we find that the behavior of a supersonic system is quite different. Notice that even in the absence of friction, heating causes the stagnation pressure to drop. On the other hand, a cooling process predicts an increase in p_t. This is difficult to achieve in practice (except by latent cooling), due to frictional effects that are inevitably present.

Perhaps the most significant equations in this unit are the general ones:

$$\rho V = G \tag{10.4}$$

$$h_{t1} + q = h_{t2} \tag{10.5}$$

$$p + \frac{GV}{g_c} = \text{const} \tag{10.8}$$

An alternative way of expressing the latter equation is to say that the *impulse function* remains constant:

$$pA + \frac{\dot{m}}{g_c} = \text{constant} \tag{10.11}$$

Along with these equations, you should keep in mind the appearance of Rayleigh lines in the *p–v* and *h–s* diagrams (see Figures 10.2 and 10.4) as well as the stagnation reference curves (see Figure 10.5). Remember that each Rayleigh line represents points with the same mass velocity and impulse function, and a normal shock can connect two points on opposite branches of a Rayleigh line, which have the same stagnation enthalpy.

Working equations for perfect gases were developed and then simplified with the introduction of the * reference point. This permitted the production of tables that help immeasurably in problem solving. Do not forget that the * condition (i.e., where $M = 1.0$) for Rayleigh flow is *not* the same as that used for either isentropic or Fanno flow. *Thermal choking* occurs in heat addition problems, and the reaction of a choked system to the addition of more heat is quite similar to the way that a choked Fanno system reacts to the addition of more duct. Remember: Drawing a good T–s diagram can help clarify your thinking on any given problem.

PROBLEMS

In the problems that follow, you may assume that all ducts are of constant area unless specifically indicated otherwise. In these constant-area ducts, you may neglect friction where heat transfer is involved, and you may neglect heat transfer where friction is indicated. You may also neglect both heat transfer and friction in sections of varying area.

10.1. Air enters a constant-area duct with $M_1 = 2.95$ and $T_1 = 500°\text{R}$. Heat transfer decreases the outlet Mach number to $M_2 = 1.60$.

(a) Compute the exit static and stagnation temperatures.

(b) Find the amount and direction of heat transfer.

10.2. At the beginning of a duct the nitrogen pressure is 1.5 bar, the stagnation temperature is 280 K, and the Mach number is 0.80. After some heat transfer the static pressure is 2.5 bar. Determine the direction and amount of heat transfer.

10.3. Air flows at the rate of 39.0 lbm/sec with a Mach number of 0.30, a pressure of 50 psia, and a temperature of 650°R. The duct has a 0.5-ft^2 cross-sectional area. Find the final Mach number, the stagnation temperature ratio T_{t2}/T_{t1}, and the density ratio ρ_2/ρ_1, if heat is added at the rate of 290 Btu/lbm of air.

10.4. In a flow of air $\rho_1 = 1.35 \times 10^5$ N/m^2, $T_1 = 500$ K, and $V_1 = 540$ m/s. Heat transfer occurs in a constant-area duct until the ratio $T_{t2}/T_{t1} = 0.639$.

(a) Compute the final conditions M_2, p_2, and T_2.

(b) What is the entropy change for the air?

10.5. At some point in a flow system of oxygen $M_1 = 3.0$, $T_{t1} = 800°\text{R}$, and $p_1 = 35$ psia. At a section farther along in the duct, the Mach number has been reduced to $M_2 = 1.5$ by heat transfer.

(a) Find the static and stagnation temperatures and pressures at the downstream section.

(b) Determine the direction and amount of heat transfer that took place between these two sections.

10.6. Show that for a constant-area, frictionless, steady, one-dimensional flow of a perfect gas, the maximum amount of heat that can be added to such system is given by the expression

$$\frac{q_{\max}}{c_p T_1} = \frac{\left(M_1^2 - 1\right)^2}{2 M_1^2 (\gamma + 1)}$$

10.7. Starting with equation (10.53), show that the peak or maximum (static) temperature in Rayleigh flow occurs when the Mach number is $\sqrt{1/\gamma}$. (*Hint*: see Problem 5.21). [Note this is the same limit that arises in *isothermal flow with friction*, e.g., see Ref. 16.]

10.8. Air enters a 15-cm-diameter duct with a velocity of 120 m/s. The pressure is 1 atm and the temperature is 100°C.

(a) How much heat may be added to the flow to create the maximum (static) temperature?

(b) Determine the final temperature and pressure for the conditions of part (a).

10.9. The 12-in.-diameter duct shown in Figure P10.9 has a friction factor of 0.02 and no heat transfer from section 1 to 2. There is negligible friction from 2 to 3. Sufficient heat is added in the latter portion to just choke the flow at the exit. The fluid is nitrogen.

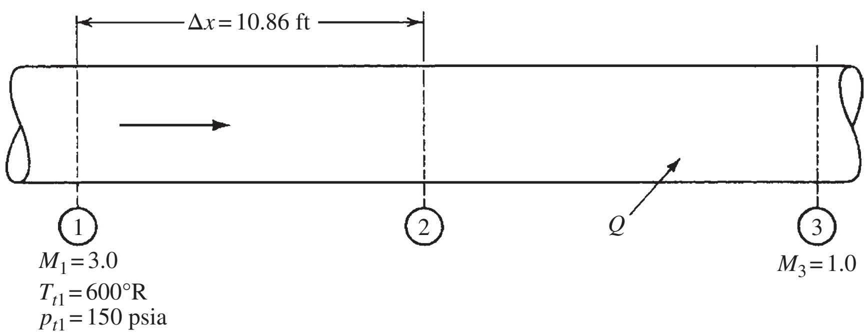


Figure P10.9

(a) Draw a *T–s* diagram for the system, showing the complete Fanno and Rayleigh lines involved.

(b) Determine the Mach number and stagnation conditions at section 2.

(c) Determine the static and stagnation conditions at section 3.

(d) How much heat was added to the flow?

10.10. Conditions just prior to a standing normal shock in air are $M_1 = 3.53$, with a temperature of 650°R and a pressure of 12 psia.

(a) Compute the conditions that exist just after the shock.

(b) Show that these two points lie on the same Fanno line.

(c) Show that these two points lie on the same Rayleigh line.

10.11. Air enters a duct with a Mach number of 2.0, and the temperature and pressure are 170 K and 0.7 bar, respectively. Heat transfer takes place while the flow proceeds down the duct. A converging section ($A_2/A_3 = 1.45$) is attached to the outlet as shown in Figure P10.11, and the exit Mach number is 1.0. Assume that the inlet conditions and exit Mach number remain fixed. Find the amount and direction of heat transfer:

(a) If there are no shocks in the system.

(b) If there is a normal shock someplace in the duct.

(c) For part (b), does it make any difference where the shock occurs?

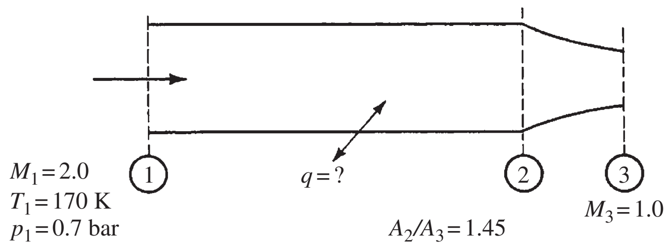


Figure P10.11

10.12. In the system shown in Figure P10.12, friction exists only from 2 to 3 and from 5 to 6. Heat is removed between 7 and 8. The Mach number at section 9 is unity. Draw the *T–s* diagram for the system, showing both the static and stagnation curves. Are points 4 and 9 on the same horizontal level?

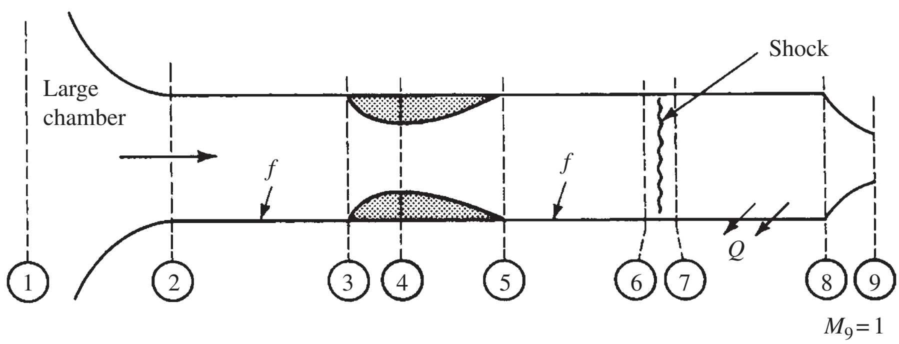


Figure P10.12

10.13. Oxygen is stored in a large tank where the pressure is 40 psia and the temperature is 500°R. A DeLaval nozzle with an area ratio of 3.5 is attached to the tank and discharges into a constant-area duct where heat is transferred. The pressure at the duct exit is equal to 15 psia. Determine the amount and direction of heat transfer if a normal shock stands where the nozzle is attached to the duct.

10.14. Air enters a converging-diverging nozzle with stagnation conditions of 35×10^5 N/m^2 and 450 K. The area ratio of the nozzle is 4.0. After passing through the nozzle, the flow enters a duct where heat is added. At the end of the duct there is a normal shock, after which the static temperature is found to be 560 K.

(a) Draw a T–s diagram for the system.

(b) Find the Mach number after the shock.

(c) Determine the amount of heat added in the duct.

10.15. A converging-only nozzle feeds a constant-area duct in a system similar to that shown in Figure 10.11. Conditions in the nitrogen supply chamber are $p_1 =$ 100 psia and $T_1 = 600°$R. Sufficient heat is added to choke the flow ($M_3 = 1.0$) and the Mach number at the duct entrance is $M_2 = 0.50$. The pressure at the exit is equal to that of the receiver.

(a) Compute the receiver pressure.

(b) How much heat is transferred?

(c) Assume that the receiver pressure remains fixed at the value calculated in part (a) as more heat is added in the duct. The flow rate must decrease and the flow moves to a new Rayleigh line, as indicated in Figure 10.11. Is the Mach number at the exit still unity, or is it less than 1? (*Hint:* Assume any lower Mach number at section 2. From this you can compute a new p^* which should help answer the question. You can then compute the heat transferred and show this to be greater than the initial value. A T–s diagram might also help.)

10.16. Draw the stagnation curves for both Rayleigh lines shown in Figure 10.11.

10.17. Recall the expression $p_t A^* =$ const [see equation (5.35)].

(a) State whether the following equations are true or false for the system shown in Figure P10.17.

(i) $p_{t1}A_1^* = pt_3A_3^*$

(ii) $pt_3A_3^* = pt_5A_5^*$

(b) Draw a T–s diagram for the system shown in Figure P10.17. Include both static and stagnation curves. Are the flows from 1 to 2 and from 4 to 5 on the same Fanno line?

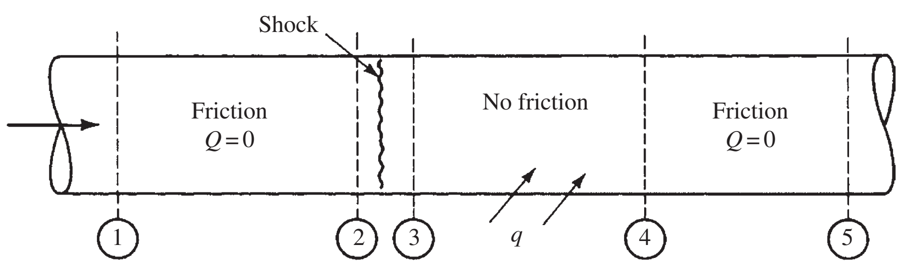


Figure P10.17

10.18. In Figure P10.18, points 1 and 2 represent flows on the same Rayleigh line (same mass flow rate, same area, same impulse function) and are located such that $s_1 = s_2$ as shown. Now imagine that we take the fluid under conditions at 1 and isentropically expand to 3. Further, let's imagine that the fluid at 2 undergoes an isentropic compression to 4.

(a) If 3 and 4 are coincident state points (same T and s), prove that A_3 is greater than, equal to, or less than A_4.

(b) Now suppose that points 3 and 4 are not necessarily coincident but it is known that the Mach number is unity at each point (i.e., $3 \equiv 1_s^*$ and $4 \equiv 2_s^*$).

(i) Is V_3 equal to, greater than, or less than V_4?

(ii) Is A_3 equal to, greater than, or less than A_4?

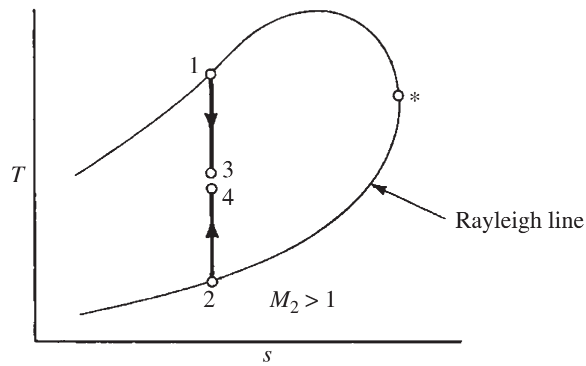


Figure P10.18

10.19. **(a)** Plot a Rayleigh line to scale in the T–s plane for air entering a duct with a Mach number of 0.25, a static pressure of 100 psia, and a static temperature of 400°R. Indicate the Mach number at various points along the curve.

(b) Add the stagnation curve to the T–s diagram.

10.20. Shown in Figure P10.20 is a portion of a T–s diagram for a system that has steady, one-dimensional flow of a perfect gas with no friction. Heat is added to subsonic flow in the constant-area duct from 1 to 2. Isentropic, variable-area flow occurs from 2 to 3. More heat is added in a constant-area duct from 3 to 4. There are no shocks in the system.

(a) Complete the diagram of the physical system. (*Hint:* To do this, you must prove that A_3 is greater than, equal to, or less than A_2.)

(b) Sketch the entire flow system in the p–v plane.

(c) Complete the T–s diagram for the system.

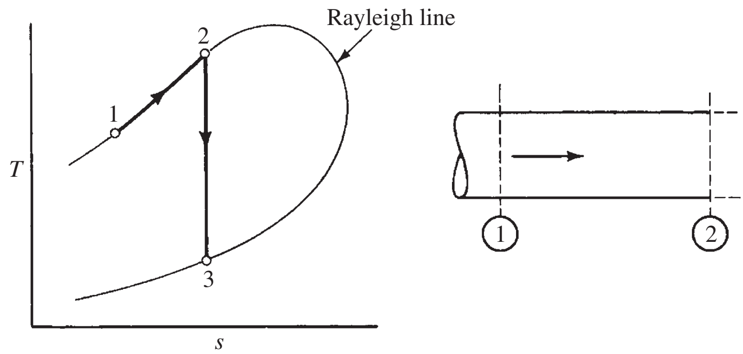


Figure P10.20

10.21. Consider steady one-dimensional flow of a perfect gas through a horizontal duct of infinitesimal length (dx) with a constant area (A) and perimeter (P). The flow is known to be isothermal and *has heat transfer as well as friction*. Starting with the fundamental momentum equation in the form

$$\sum F_x = \frac{\dot{m}}{g_c}(V_{\text{out}_x} - V_{\text{in}_x})$$

examine the infinitesimal length of the duct and (introducing basic definitions as required) show that

$$\frac{dp}{p} + \frac{\gamma M^2 f}{2}\frac{dx}{D_e} + \frac{\gamma M^2}{2}\frac{dV^2}{V^2} = 0$$

10.22. (*Optional*)

(a) By the method of approach used in Section 9.4 [see equations (9.25) through (9.27)], show that the entropy change between two points in Rayleigh flow can be represented by the following expression if the fluid is a perfect gas:

$$\frac{s_2 - s_1}{R} = \ln\left(\frac{M_2}{M_1}\right)^{2\gamma/(\gamma-1)}\left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^{(\gamma+1)/(\gamma-1)}$$

(b) Introduce the * reference condition and obtain the expression for $(s^* - s)/R$ given as equation (10.57).

(c) Specialize the expression developed in part (b) for $(s^* - s)/R$ for $\gamma = 1.4$ and compute at several Mach numbers. Check your results with those listed in Appendix J.

10.23. Rework Problem 10.4 for helium, water vapor, and carbon dioxide using the *Gasdynamics Calculator* and γ and R entries in Appendix B.

CHECK TEST

You should be able to complete this test without reference to material in the chapter.

10.1. A Rayleigh line represents the locus of points that have the same _____ and _____.

10.2. Fill in the blanks in Table CT10.2 to indicate whether the properties *increase, decrease,* or *remain constant* in the case of Rayleigh flow.

Table CT10.2 Fluid Property Variation for Rayleigh Flow

	Heating		Cooling	
Property	$M < 1$	$M > 1$	$M < 1$	$M > 1$
Mach number				
Density				
Entropy				
Stagnation pressure				

10.3. Sketch a Rayleigh line in the p–v plane, together with lines of constant entropy and constant temperature (for a typical perfect gas). Indicate directions of increasing entropy and temperature. Show regions of subsonic and supersonic flow.

10.4. Air flows in the system shown in Figure CT10.4.

(a) Find the temperature in the large chamber at location 3.

(b) Compute the amount and direction of heat transfer.

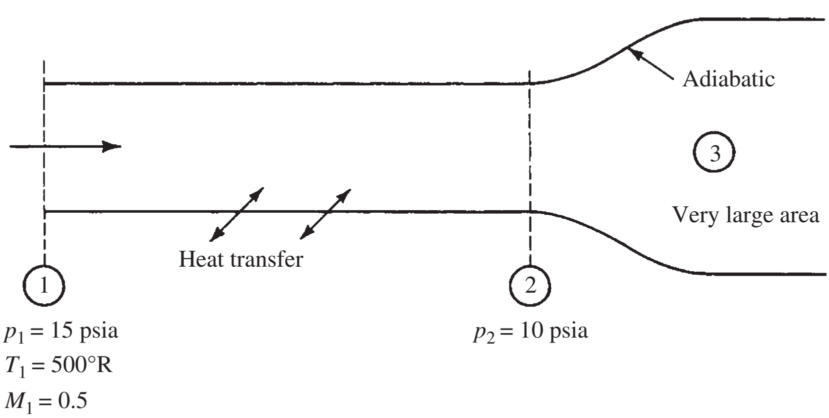


Figure CT10.4

10.5. Sketch the T–s diagram for the system shown in Figure CT10.5. Include in the diagram both the static and stagnation curves.

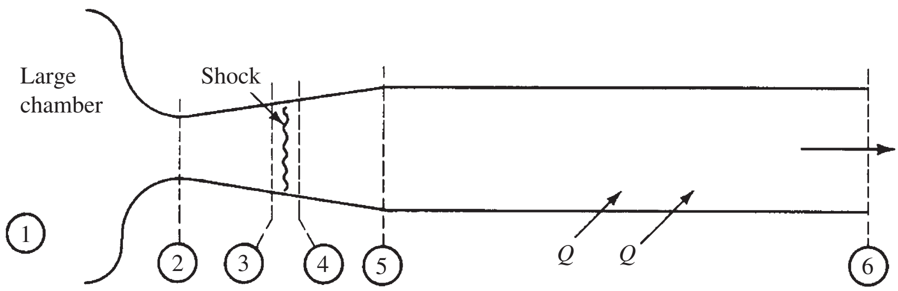


Figure CT10.5

10.6. Work Problem 10.14.

Chapter 11

Real Gas Effects

11.1 INTRODUCTION

The control-volume equations for steady, one-dimensional flow introduced in previous chapters are summarized below for two arbitrary flow locations. These equations are given here in their more general form, before being specialized to perfect gases with constant specific heats.

We first include relations from the 0^2 law.

State:

$$p = Z\rho RT \tag{1.16 modified}$$

$$du = c_v dT \quad \text{and} \quad dh = c_p dT \tag{1.33, 1.34}$$

We then write down the equations for mass and energy conservation as well as the momentum equation.

Continuity:

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \tag{2.30}$$

Energy:

$$h_{t1} + q_{1-2} = h_{t2} \quad [\text{from (3.19) without shaft work}]$$

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

Momentum:

$$\sum \mathbf{F} = \frac{\dot{m}}{g_c}(\mathbf{V}_{\text{out}} - \mathbf{V}_{\text{in}}) \tag{3.45}$$

Note that equation (1.16) has been modified by the introduction of Z, the compressibility factor, which up to now has been implicitly assumed to be 1.0. The second law is not listed because it often does not appear explicitly: rather, having an implicit effect on the direction of irreversible processes.

The set of equations above is the starting point for a study of gas dynamics with real gas effects. What needs to be done first is to account for any deviations from perfect gas behavior that may occur. This is often accomplished through a dependence of the factor Z on temperature and pressure, as discussed in Section 11.5. Moreover, one needs to find the enthalpies from the integration of equation (1.34) because even for gases that obey equation (1.16), the specific heats may vary with temperature when those temperature changes are large enough. This has been done in the development of *gas tables* that include real gas effects such as those by Keenan and Kay (Ref. 31).

We begin this chapter with a brief description of the microscopic structure of gases, to explain why monatomic gases have a different γ than diatomic gases (such as air), and why polyatomic gases have yet a different ratio of specific heats. Next, we introduce the concept of the nonperfect or real gas and elaborate on how temperature may govern the behavior of the heat capacities. In this book we restrict ourselves to situations where there is no dissociation (the breakup of molecules) and where the flow remains below the hypersonic regime. As a result, the sole contribution to heat capacity variations will result from the temperature activation of vibrational internal energies in diatomic and polyatomic molecules. We finish up with a discussion of how to work with the equations presented at the start of this section for nonperfect gases.

11.2 OBJECTIVES

After completing this chapter successfully, you should be able to:

1. Identify which microscopic properties are responsible for the macroscopic (observable) characteristics of temperature and pressure.
2. Describe three categories of molecular motion that contribute to the heat capacities.
3. Identify which of these categories of motion are present in monoatomic, diatomic, and polyatomic molecules.
4. Define

   a. relative pressure and relative volume.
   b. reduced pressure and reduced temperature.
5. Make simple process calculations (such as $s = \text{const}$, $p = \text{const}$) with the aid of gas tables that include semiperfect gas effects.
6. Compute entropy, enthalpy, and internal energy changes for any process with the aid of gas tables that include real gas effects.
7. Given the pressure and temperature, determine the volume of a given quantity of gas by using the generalized compressibility chart.
8. Analyze the supersonic nozzle problem with real gas effects utilizing "Method I" when all conditions at the plenum are given along with either the exit temperature, exit pressure, or exit Mach number.
9. (*Optional*) Be able to work the normal shock problem with real gas effects utilizing "Method I" when all properties upstream of the shock are known.

11.3 WHAT'S REALLY GOING ON

Up to now, we have assumed that the specific heats never change and thus that γ remains constant during any and all flow processes. This has yielded useful, closed-form equations for perfect gases where $Z \approx 1.0$. We are now ready to explore what results from γ-variations within the flow, as these represent more accurately many practical situations (especially those in a jet engine or a rocket motor). There are several reasons why γ may change, and they may be related to changes in the chemical composition of the gas atoms or gas molecules as well as to the level of temperature and to some extent pressure of operation. In addition, the kinetics of how the flow in question approaches its equilibrium condition can affect γ changes, and thus, this problem turns out to be relatively complicated. *Theoretically*, γ can never equal or be less than 1 and can never exceed $\frac{5}{3}$ (see Reference 26). *In practice*, changes in γ are limited to between about 1.1 and 1.7 for nearly all gases of interest. However, this seemingly narrow range of values is very significant because, as we have seen, γ is often encountered as an exponent.

Microscopic Model of Gases

Up to now we have taken the macroscopic approach (as mentioned in Chapter 1) by dealing with observable and/or measurable properties. This leads to the axiomatic approach of thermodynamics, which is found in the important thermodynamic laws and corollaries. But ordinary gases really consist of myriads of atoms and/or molecules that are in *continuous random motion* with respect to one another, in addition to any mean-mass motion that they may have with respect to any given frame of reference.

The kinetic energy of this *random motion* forms the basis for the property that we call the temperature. Thus random particle motion makes up the *static* temperature of the gas, whereas the kinetic energy of the mean-mass motion is the sole contributor to the difference between the static and stagnation temperatures. Gas molecules are also continuously changing direction as they collide and exchange momentum with one another and, as particles collide with a physical surface, their momentum exchanges give rise to the property that we call the pressure.

Because the above effects are distributed among a vast number of constituent particles, we can only observe averages, which under equilibrium conditions tend to be quite predictable. However, the concept of temperature becomes considerably more complicated under nonequilibrium conditions since internal molecular degrees of freedom have different relaxation times. We give an example of this in Section 11.8.

Molecular Structure

Monatomic gases consist of only one individual atom per particle. These gases are well represented by the inert gases (such as helium, neon, and argon) at standard conditions. They exhibit constant γ over a very wide range of temperatures and normal pressures. (We note here that diatomic and polyatomic gases may yield monatomic constituents at sufficiently high temperatures and low enough pressures when dissociation takes place.)

Diatomic gases have molecules that consist of two atoms. They are the most common type of gases, with oxygen and nitrogen (the main constituents of air) as best examples. Diatomic gases are more complicated than monatomic because they have an active internal structure and are typically internally rotating, and as their temperatures increase they start vibrating in addition to translating and rotating. (Reference 27 includes some rigorous discussions of diatomic gas thermodynamics.)

Polyatomic gases consist of three or more atoms per molecule (e.g., carbon dioxide). These share the same attributes as diatomic gases except for extra vibrational modes that depend on the structure and number of atoms present in each molecule.

Thus, as a minimum, there are three categories of molecular *degrees of freedom*: *translation, rotation*, and *vibration*. Each contributes to the heat capacities because each acts as a storage mode of energy for the gas. This is another way of saying that each degree of freedom contributes to the molecule's ability to absorb and release energy, thus affecting the eventual gas temperature. Figure 11.1 illustrates these internal degrees of freedom for a diatomic molecule. Single atoms are not subject to vibrational activation, and molecules consisting of three or more atoms have more than one vibrational degree of freedom. (Additional information is presented in Refs. 28, 29, and 30.)

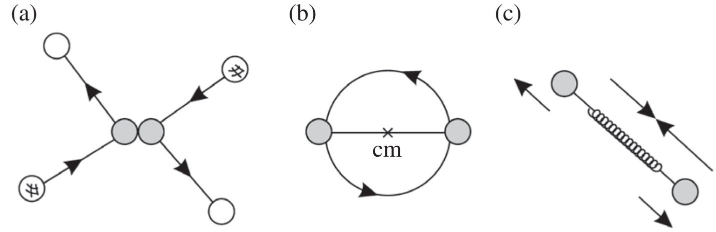


Figure 11.1 (a) Translation, (b) rotation, and (c) vibration for a diatomic molecule. During translation the internal structure of molecules is insignificant but it becomes most relevant during vibration.

Nonequilibrium Effects in Gas Dynamics

As the Mach number becomes supersonic inside a nozzle, the overall temperature and pressure can drop significantly and nonequilibrium effects may start to become apparent. We are referring here to a lag in certain property changes, such as a *time delay* or *inertia* of the specific heat capacities to follow local temperature changes instantaneously. This lag will affect the behavior of property changes in expansions through sufficiently short nozzles because γ may remain practically unchanged inside such nozzles. In these cases (when γ remains constant), the analysis is referred to as the *frozen-composition-flow limit*, which is considerably easier to calculate than *shifting equilibrium flows* where the properties react instantly according to the local static temperature and pressure profiles. Criteria governing when to expect frozen composition flow depend on activation, relaxation, or reaction times compared with travel times through nozzles and other flow devices and can be found in the literature (e.g., Ref. 26).

For example, in the field of rocket propulsion, all preliminary calculations are made using the frozen-flow limit because of its simplicity. According to Sutton and Biblarz (Ref. 24), this method tends to *underestimate* the performance of typical rockets. On the other hand, *instantaneous chemical equilibrium limits* (labelled *shifting equilibrium*), which are a great deal more complex, tend to *overestimate* the performance of typical rockets. Since the assumptions of isentropic flow in ideal systems (i.e., no flow separation, friction, shocks, or major instabilities) may carry an inherent error of up to ±10%, frozen-flow analyses are the preferred approach. Noncombustion systems such as electrically heated rockets and hypersonic wind tunnels behave in ways similar to chemical rockets; because at their high initial temperatures, air dissociates and begins to react chemically. Nonequilibrium effects in flows are sometimes desirable, as in the gas dynamic laser (GDL) introduced later in Section 11.8, and are present in nearly all hypersonic situations.

Figures 6.9 and 6.10 show the effect of γ variations in normal shocks using Chapter 6 formulations. Note that the variability of the pressure ratio with γ for

a given Mach number is considerably less than that of the temperature ratio across the shock. It should be understood, however, that all property changes across a shock front are anticipated to reflect the upstream γ entering the shock. Adjustments to temperature changes are not likely to take place within the narrow extent of a shock but in a relaxation region downstream of it. That is, the flow through the shock front itself is *frozen* in γ value and composition. However, the gas properties will finally approach their equilibrium values in a small region behind the shock. The same arguments hold for oblique shocks.

On the other hand, Prandtl–Meyer expansions are much less prone to nonequilibrium because the flow always starts and ends supersonic. This means that the temperature swings are quite restricted and, more importantly, the gas is typically cold enough so that its molecules are not vibrationally activated to begin with.

11.4 SEMIPERFECT GAS BEHAVIOR AND DEVELOPMENT OF THE GAS TABLES

A semiperfect gas is a gas that can be described with the perfect gas equation of state but with an allowance made for variation of the specific heats with temperature. These are also called *thermally perfect gases* or *imperfect gases* in the literature, and unfortunately, there is no consistency among the various authors. Figure 11.2 shows the semiperfect-gas variation of c_p and γ for a diatomic gas mixture (air) and a polyatomic gas (CO_2) as a function of temperature. The different plateaus in γ depend on the activation of the rotational and vibrational modes of energy storage. Vibrational modes are the most critical since they only manifest themselves at the higher temperatures. For example, even below room temperature, air molecules (which are mostly nitrogen) have fully active translational and rotational degrees of freedom, but only at temperatures above about 1000 K (1800°R) does vibration begin to change the value of γ significantly (because of its relatively higher molecular activation energy).

In summary, diatomic and polyatomic gases can change their molecular behavior substantially as both the temperature and pressure decrease, such as in the flow through a supersonic nozzle. Molecular structure most often changes as a result of chemical reactions in combustion chambers and from dissociation at hypersonic conditions. Also, the effects on γ from vibrational excitation and from dissociation often counteract each other in complicated ways, as shown in References 29 and 30. Moreover, when flow kinetic effects are manifest, as in hypersonic flows, such problems can only be solved with the aid of computers. We feel, however, that a *constant* or an effective *average-γ* approach can be useful, and preliminary analysis of many propulsion systems is often based on such an approach. We shall introduce the *average-γ* approach in Section 11.6.

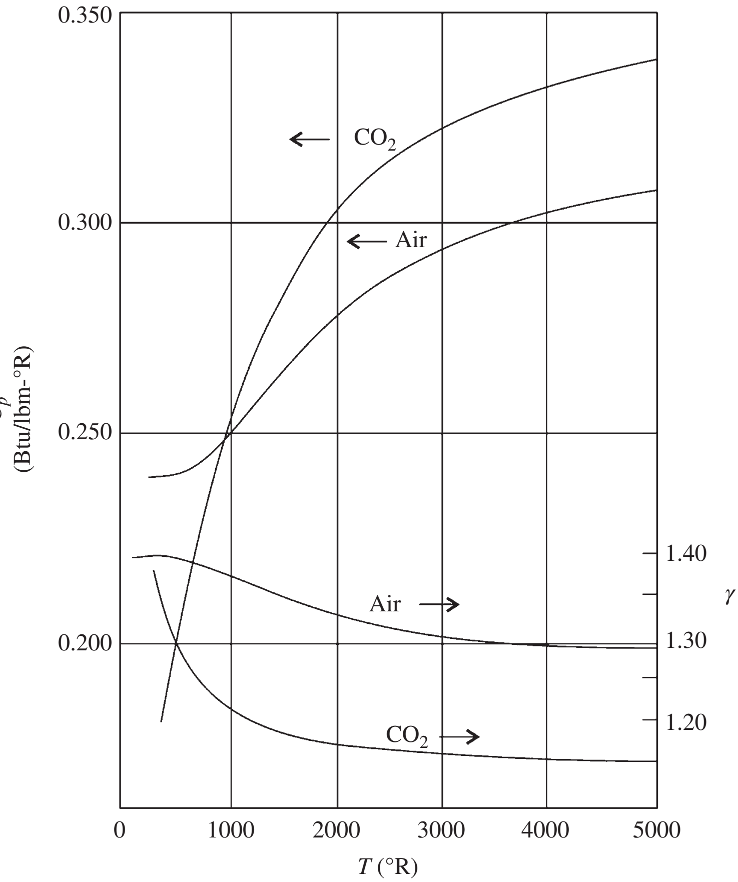


Figure 11.2 Specific heat at constant pressure and specific heat ratio for two common gases.

Gas Tables

The perfect gas equation of state is reasonably accurate and can be used over a wide range of temperatures. However, the semiperfect gas approach is unavoidable in combustion-driven propulsion systems. A table in Appendix L (Table 2 in Ref. 31) shows values of c_p and c_v for air at low pressures as a function of a wide temperature range.

Recall that as long as we can say that $p = \rho RT$, the internal energy and enthalpy are functions of temperature *only*. From Chapter 1, we then have

$$du = c_v dT \quad \text{and} \quad dh = c_p dT \tag{1.33, 1.34}$$

Arbitrarily assigning $u = 0$ and $h = 0$ when $T = 0$, we can obtain integrals for u and h:

$$u = \int_0^T c_v dT \quad \text{and} \quad h = \int_0^T c_p dT \tag{11.1, 11.2}$$

Now, when the temperature changes are sufficiently large, we must obtain the functional relationships between the specific heats and temperature and perform the integration. This has been done for commonly used gases, with the results tabulated in the *Gas Tables* (Ref. 31), see also Appendix K for air.

Once the table entries have been constructed for a particular gas, we can obtain values of u and h directly at any desired temperature within the tabulated range. But how do we compute entropy changes? Consider that for any substance

$$Tds = dh - vdp \tag{1.31}$$

and if the substance obeys the perfect gas law we know that

$$dh = c_p dT \tag{1.34}$$

Show that the entropy change can be written as

$$ds = c_p \frac{dT}{T} - R\frac{dp}{p}$$

We integrate each term:

$$\int_1^2 ds = \int_1^2 c_p \frac{dT}{T} - \int_1^2 R\frac{dp}{p}$$

If we now define ϕ as

$$\phi \equiv \int_0^T c_p \frac{dT}{T} \tag{11.3}$$

then

$$\boxed{\Delta s_{1-2} = \phi_2 - \phi_1 - R\ln\frac{p_2}{p_1}} \tag{11.4}$$

Note that since c_p is a known function of temperature, the integration indicated above can be performed *once*, and the result (being a function of temperature only)

added as a column in our gas table. Tabulations of u, h, and ϕ versus temperature for air at low pressures can be found in Appendix K.

Example 11.1
Air at 40 psia and 500 °F undergoes an irreversible process with heat transfer to 20 psia and 1000°F. Calculate the entropy change.

From the air table (Appendix K), we obtain

$$\phi_1 = 0.7403\,\text{Btu/lbm-°R at } 500\text{°F} \quad \text{and} \quad \phi_2 = 0.8470\,\text{Btu/lbm-°R at } 1000\text{°F}$$

Thus

$$\Delta s_{1-2} = 0.8470 - 0.7403 - \frac{53.3}{778} \ln \frac{20}{40}$$
$$\Delta s_{1-2} = 0.1067 + 0.0685 \ln 2 = 0.1542\,\text{Btu/lbm-°R}$$

Let us now consider an *isentropic process*. Equation (11.4) becomes

$$\Delta s_{1-2} = 0 = \phi_2 - \phi_1 - R \ln \frac{p_2}{p_1}$$

or

$$\boxed{\phi_2 - \phi_1 = R \ln \frac{p_2}{p_1}} \tag{11.5}$$

Depending on the information given, many isentropic processes can be solved directly using equation (11.5). For instance:

1. Given p_1, p_2, and T_1, solve for ϕ_2 and look up T_2.
2. Given T_1, T_2, and p_1, solve directly for p_2.

However, some problems are not this simple. If we knew v_1, v_2, and T_1, solving for T_2 would be a trial-and-error problem. Let's devise a better method. We establish a reference point as shown in Figure 11.3. Now, for the isentropic process from 0 to 1, we have from equation (11.5),

$$\phi_1 - \phi_0 = R \ln \frac{p_1}{p_0} \tag{11.6}$$

But, from (11.3),

$$\phi_0 = \int_0^{T_0} c_p \frac{dT}{T} = f(T_0) \tag{11.7}$$

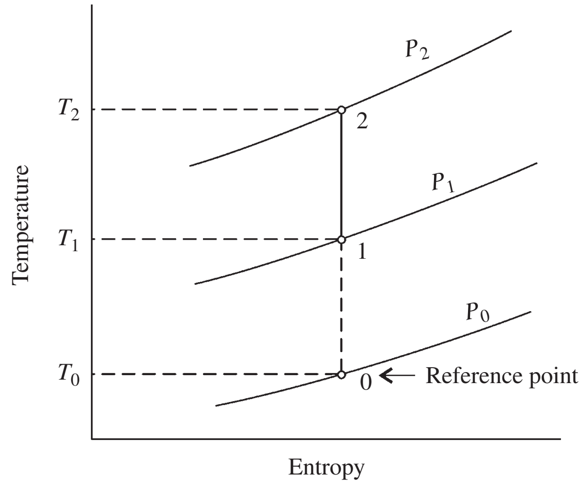


Figure 11.3 *T–s* diagram showing reference point.

Once the reference point has been chosen, ϕ_0 is a known constant and equation (11.6) can be thought of as

$$\phi_1 - \text{const} = R \ln \frac{p_1}{p_0} \tag{11.8}$$

Since ϕ_1 is a known function of T_1, equation (11.8) is really telling us that the ratio p_1/p_0 is also a function only of temperature T_1 for this process. We call this ratio the *relative pressure*. In general,

$$\text{relative pressure} \equiv p_r \equiv \frac{p}{p_0} \tag{11.9}$$

These relative pressures can be computed and introduced as another column in the gas table.

What have we gained with the introduction of the relative pressures? Notice that

$$\frac{p_2}{p_1} = \frac{p_2/p_0}{p_1/p_0} = \frac{p_{r_2}}{p_{r_1}}$$

or

$$\frac{p_2}{p_1} = \frac{p_{r2}}{p_{r1}} \tag{11.10}$$

Equation (11.10) together with the gas table may now be used for *isentropic* processes.

Example 11.2
Air undergoes an isentropic compression from 50 psia and 500°R to 150 psia. Determine the final temperature.

From the air table in Appendix K, we have

$$p_{r1} = 1.0590 \text{ at } 500°\text{R}$$

From (11.10),

$$p_{r2} = p_{r1}\left(\frac{p_2}{p_1}\right) = (1.0590)\left(\frac{150}{50}\right) = 3.177$$

From the table opposite $p_r = 3.177$, we find that $T_2 = 684°\text{R}$.

We can follow a similar chain of reasoning to develop a *relative volume*, which is a unique function of temperature only and this can also be tabulated:

$$\text{relative volume} \equiv v_r \equiv \frac{v}{v_0} \tag{11.11}$$

Also note that

$$\frac{v_2}{v_1} = \frac{v_{r2}}{v_{r1}} \tag{11.12}$$

Relative volumes may be used to solve isentropic processes quickly in exactly the same manner as with relative pressures.

In summary, we now have tabulations for the following variables as unique functions of temperature only: h, u, ϕ, p_r, and v_r.

1. h, u, and ϕ may be used for *any* process.
2. p_r and v_r may *only* be used for *isentropic* processes.

More complete tables for air and other gases may be found in *Gas Tables* by Keenan and Kaye (Ref. 31); only an abridged table for air has been given in Appendix K. This table shows the variation of h, p_r, u, v_r, and ϕ for air between 200 and 6500°R. The use of such tables is adequate for air-breathing engines since the composition of the products of combustion differs little from that of the original air. But for other gases and for certain gas dynamic relations, which are lacking in these tables, such as Mach numbers and isentropic area ratios, more needs to be done and this is addressed in Section 11.6.

Properties in Equation Form

Operating from tables and charts is very convenient when working simple problems. However, when more complicated problems are involved, one frequently employs a digital computer for solutions. In this case it is nice to have simple equations for the fluid properties. For instance, a group of polynomials for the most common properties of *air* follow:

c_p from 180 to 2430°R:

$$c_p = 0.242333 - (2.15256\text{E}-5)T + (3.65\text{E}-8)T^2 - (8.43996\text{E}-12)T^3$$

c_v from 300 to 3600°R:

$$c_v = 0.164435 + (7.69284\text{E-}6)T + (1.21419\text{E-}8)T^2 - (2.61289\text{E-}12)T^3$$

γ from 198 to 3420°R:

$$\gamma = 1.42616 - (4.21505\text{E-}5)T - (7.93962\text{E-}9)T^2 + (2.40318\text{E-}12)T^3$$

h from 200 to 2400°R:

$$h = (0.239788)T - (6.71311\text{E-}6)T^2 + (9.69339\text{E-}9)T^3 - (1.60794\text{E-}12)T^4$$

u from 200 to 2400°R:

$$u = (0.171225)T - (6.68651\text{E-}6)T^2 + (9.67706\text{E-}9)T^3 - (1.60477\text{E-}12)T^4$$

ϕ from 200 to 2400°R:

$$\phi = 0.232404 + (8.56494\text{E-}4)T - (4.08016\text{E-}7)T^2 + (7.64068\text{E-}11)T^3$$

Exponential notation has been used in the equations above; for example, E-7 means $\times 10^{-7}$.

All of the equations above are in English Engineering units, and absolute temperature is used throughout. These equations were obtained from a report by J. R. Andrews and O. Biblarz, "Gas Properties Computational Procedure Suitable for Electronic Calculators", *NPS-57Zi740701A*, July 1, 1974. A later expanded version by the *same authors* is in SI units, "Temperature Dependence of Gas Properties in Polynomial Form", NPS67-81-001, January 1981. It is available on the Web: https://core.ac.uk/download/pdf/36723064.pdf

11.5 REAL GAS BEHAVIOR, EQUATIONS OF STATE AND, COMPRESSIBILITY FACTORS

Gases can be said to exist in three distinct forms: vapors, perfect gases, and supercritical fluids. This distinction can be made more rigorous as necessary (refer to Figure 11.4, which depicts a pressure–volume diagram with the various phases of a typical pure substance). Vapors exist close to the condensation or two-phase dome region, and supercritical fluids inhabit the high-pressure region above the two-phase dome. *Perfect gases are represented by any gas at sufficiently high temperature and sufficiently low pressure to exist away from the previous two regions*. Thus, while certainly substantial, the occurrence of perfect gas operation is not the whole story.

Equations of State

Once we enter regions where the perfect gas equation is no longer valid, we must resort to other, more complicated relations among properties. One of the earliest expressions to be used was the van der Waals equation, which was introduced in 1873:

$$\left(p+\frac{a}{v^2}\right)(v-b)=RT \tag{11.13}$$

The constants a and b are unique for each gas, and tables giving these values can be found in many texts (see, e.g., Ref. 6). The term a/v^2 is an attempt to correct for

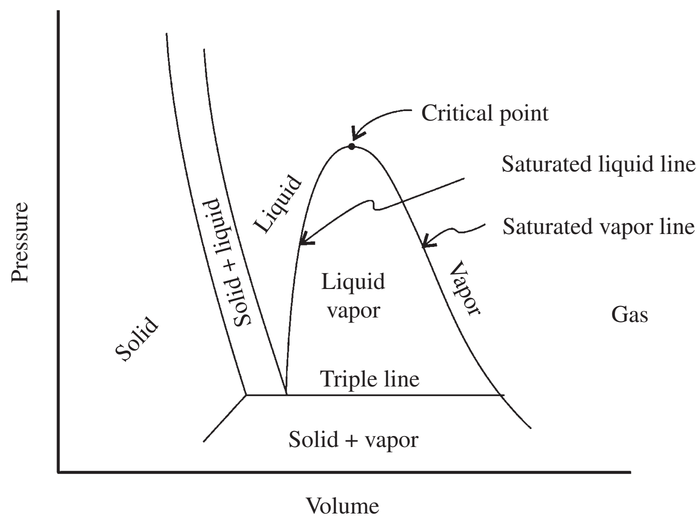


Figure 11.4 Two-phase dome for a typical pure substance.

the attractive forces among molecules. At high pressures the term a/v^2 is small relative to p and can be neglected. The constant b is an attempt to account for the volume occupied by the molecules. At low pressures, one may omit b from the term containing the specific volume. The fact that only two new constants are involved makes the van der Waals equation relatively easy to use. However, as discussed by Obert, it begins to lose accuracy as the density increases.

Attempts to gain accuracy are found in other forms of the equation of state. Perhaps the most general of these is the *virial equation of state*, which is of the form

$$\frac{pv}{RT} = 1 + \frac{B}{v} + \frac{C}{v^2} + \frac{D}{v^3} + \cdots \tag{11.14}$$

Constants B, C, D, and so on, are called *virial coefficients*, which are postulated to be functions of temperature alone. What would these virials be for a perfect gas? The virial equation was introduced around 1901 and is quite accurate at densities below the critical point.

There are *many* other equations of state, and no attempt is made here to cover these. Our main purpose is to indicate that over restricted regions of the p–v–T surface we can always write down the expressions accurate enough to satisfy the 0^2 law. If you are interested in this subject, Reference 6 has an excellent chapter entitled "The pvT Relationships."

Compressibility Charts

Is there another way to approach the equation-of-state problem? Can these property relations be represented in a simple but general manner? Look at the right side of equation (11.14). For any given state point (of a given gas), the entire right side represents some value that has been given the symbol Z and labeled the *compressibility factor*:

$$p = Z\rho RT \tag{1.16 modified}$$

Individual plots for various gases are available showing the compressibility factor as a function of temperature and pressure. However, it is possible to represent all gases on one plot through the concept of *reduced properties* with little sacrifice of accuracy. Let us define here a *new* set of dimensionless variables denoted as p_r and T_r

$$\text{reduced pressure} \equiv p_r \equiv \frac{p}{p_c} \tag{11.15}$$

$$\text{reduced temperature} \equiv T_r \equiv \frac{T}{T_c} \tag{11.16}$$

where for any given substance

$$p_c \equiv \text{critical pressure}$$
$$T_c \equiv \text{critical temperature}$$

Note that the reduced pressure above and the relative pressure from Section 11.4 share the same symbol. But this is the way it is usually done and hopefully will cause no confusion. The compressibility factor can now be plotted against reduced temperature and reduced pressure, with a result similar to that shown in Figure 11.5. It turns out that this diagram is so nearly identical for most gases that an average diagram can be used for most known gases.

Generalized compressibility charts can be found in most engineering thermodynamics texts (and in Appendix F). These are least accurate near the critical point, where the averaging procedure introduces some error as the value at the critical point, denoted as Z_c, varies from 0.23 to 0.33 for different gases—in Appendix F, $Z_c = 0.27$; critical values of temperature and pressure for several gases are given in Appendixes A and B. (It should be pointed out here that for steam and a few other gases, empirically derived tables are available, which are more accurate than the compressibility chart.) We may now define the *perfect gas region* as $0.95 \le Z \le 1.05$. (Is this what you would expect?)

Atmospheric air is a mixture of 79% N_2, 20% O_2, and other trace gases. Perfect gas behavior in air (i.e., when Z remains within ±5% of unity) without dissociation or recombination may be expected up to 4100 psia (279 atm) for temperatures

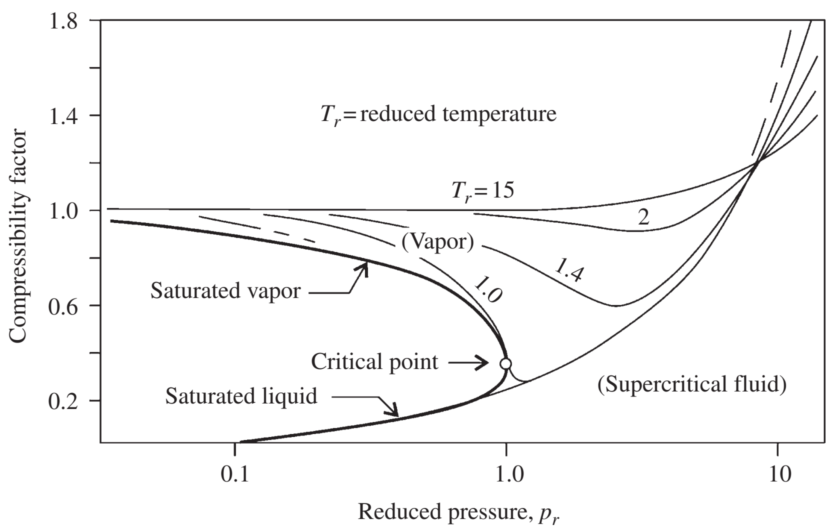


Figure 11.5 Skeletal generalized compressibility chart. (See Appendix F for a working chart that has been modified to highlight the gaseous region.)

above 20°F (480°R). At temperatures as low as −160 °F (300°R), we can expect perfect gas behavior in air up to about 1000 psia (74 atm). These values of pressure and temperature vary considerably for other gases, but as can be seen, perfect gas behavior in air can be a very common occurrence.

Example 11.3
Determine the specific volume of air at 227°R and 9.3 atm. Use the generalized compressibility chart in Appendix F and compare to the perfect gas calculations. The pseudo-critical constants for air are $T_c = 239°\text{R}$ and $p_c = 37.2$ atm (546 psia) from Appendix A.

$$T_r = \frac{227}{239} = 0.95$$
$$p_r = \frac{9.3}{37.2} = 0.25$$

From the compressibility chart, $Z = 0.889$.

$$v = \frac{ZRT}{p} = \frac{(0.889)(53.3)(227)}{(9.3)(14.7)(144)} = 0.546 \text{ ft}^3/\text{lbm}$$

If the perfect gas equation of state is used directly:

$$v = \frac{RT}{p} = \frac{(53.3)(227)}{(9.3)(14.7)(144)} = 0.615 \text{ ft}^3/\text{lbm}$$

So the perfect gas equation of state turns out to be accurate for many situations of interest in gas dynamics. It is fortuitous that in many applications high pressures are usually associated with relatively high temperatures and low temperatures are usually associated with relatively low pressures so that gaseous condensation, for example, is rare. Also, the gas molecules remain on the average far from each other. Supersonic nozzles feeding from combustion chambers are in this category. Wind tunnels, jet engines, and rocket engines may also be analyzed with the semiperfect gas approach, which uses the perfect gas equation of state augmented by the variation of the heat capacities with temperature and gas composition. Thus, for many practical examples, deviations from perfect gas behavior can largely be neglected, and we let $Z \approx 1.0$. When Z is not sufficiently close to 1.0, iterative calculations are performed starting with $Z = 1.0$, which often converge rather quickly. Here the information in tabular or graphical form is most commonly used. (See Refs. 30 and 32 for additional information.)

11.6 VARIABLE-γ VARIABLE-AREA FLOWS

Isentropic Calculations

Isentropic γ variability from the formulations in Chapter 5 is shown in Figure 5.14*a*, *b*, *c*. There we show constant γ results, but the possible effects of γ variations can be inferred from the spread of the different constant γ curves. For example, the p/p_t curves are relatively insensitive to the values of γ for Mach numbers up to about 2.5 (less than 10% change for air). This means that for variable γ, calculations involving pressure (in this range of Mach numbers) are essentially the same as those assuming constant γ. The temperature ratios, on the other hand, show considerable variability beyond $M = 1.0$, so that calculations involving temperature are more restricted in their independence on γ changes. The density ratio sensitivity falls between temperature and pressure. The A/A^* ratios are not strongly dependent on γ below $M = 1.5$. Recall that under our models, monoatomic gases do not exhibit γ variations because they lack internal vibrational modes. So only diatomic and polyatomic gases require the techniques outlined below.

Several methods have been developed to handle variable-γ variable-area problems. The method of choice depends on the information that you are managing and on the required accuracy of the results. Here, we discuss two methods. The first one is based on rather simple extensions of the material in earlier chapters. The other method is more rigorous. As presented, neither method allows for deviations from $Z \approx 1.0$.

Method I: Average γ approach. This assumes perfect gas relations throughout but works with an *average* γ (or $\bar{\gamma}$) appropriately inserted in the stagnation enthalpy and stagnation pressure equations.

Method II: Real gas approach. This assumes a semiperfect gas in that the perfect gas equation of state is used but property values are taken from the gas table or equivalent calculations (these account for variable specific heats).

Both methods are iterative in nature, but Method I can be considerably easier and faster. It may work sufficiently well for preliminary design purposes, having been verified with numerous examples in gases flowing through supersonic nozzles. It is based on the following equations with an appropriately modified c_p:

$$c_p = \frac{\gamma R}{\gamma - 1} \tag{4.15}$$

$$h = \int_0^T c_p dT \approx c_p T = \left(\frac{\gamma R}{\gamma - 1}\right) T \tag{11.17}$$

$$T_t = T\left(1 + \frac{\gamma - 1}{2}M^2\right) \tag{4.18}$$

$$p_t = p\left(1 + \frac{\gamma - 1}{2}M^2\right)^{\gamma/(\gamma-1)} \tag{4.21}$$

$$\dot{m} = pAM\sqrt{\frac{\gamma g_c}{RT}} = \text{const} \tag{4.13}$$

Although these equations are strictly valid for perfect gases (because of the constant heat capacities), we introduce the modified/average $\bar{\gamma}$ to obtain more accurate solutions. We pose first a customary isentropic nozzle problem with section locations defined in Figure 11.6.
In our first discussion we assume the following information:

Given: The gas composition, $T_{t1} \approx T_1$, $p_{t1} \approx p_1$, and p_3.
Find: a. The temperature and Mach number at the exit (T_3 and M_3).
b. The required area ratio to produce these conditions (A_3/A_2).

Solution:

1. Assume T_3 from the perfect gas, constant-γ solution.
2. Find γ_3 from Appendix L (γ is only a function of the static temperature). As an alternative, we can bypass this step by assuming a low enough temperature at T_3 so that no vibrational modes remain activated. For air, this means that $\gamma_3 \approx$ 1.4 (otherwise, at the higher temperatures, $\gamma \to 1.3$). For other gases, Appendix A gives low-temperature values.

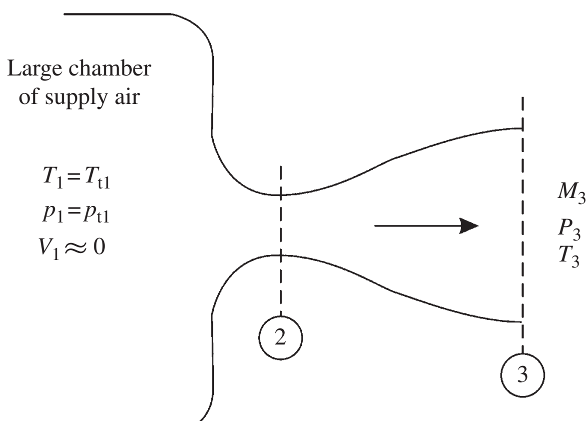


Figure 11.6 Supersonic nozzle.

3. Compute the average γ (i.e., an effective nozzle $\bar{\gamma}$ value *up to* station 3) from

$$\bar{\gamma}_3 = \frac{\gamma_3 + \gamma_1}{2} \tag{11.18}$$

4. Now, since $h_{t3} = h_{t1}$ from the energy equation (10.48),

$$\begin{gathered} \bar{c}_{p3} T_{t3} \approx c_{p1} T_{t1} \\ \left(\frac{\bar{\gamma}_3 \cancel{R}}{\bar{\gamma}_3 - 1} \right) T_{t3} \approx \left(\frac{\gamma_1 \cancel{R}}{\gamma_1 - 1} \right) T_{t1} \\ T_{t3} \approx T_{t1} \left(\frac{\gamma_1 (\bar{\gamma}_3 - 1)}{\bar{\gamma}_3 (\gamma_1 - 1)} \right) \end{gathered} \tag{11.19}$$

This allows us to find our first estimate of T_{t3} in the supersonic region.

5. We continue to use the average γ for properties at station 3 as long as they are not locally given (i.e., when they need to be found from upstream values). We use equation (4.21) to get an estimate for M_3. (Remember that the stagnation pressure remains constant because the expansion is isentropic, see equation (4.28).)

$$M_3 \approx \sqrt{\frac{2}{\bar{\gamma}_3 - 1} \left[\left(\frac{p_{t1}}{p_3} \right)^{(\bar{\gamma}_3 - 1)/(\bar{\gamma}_3)} - 1 \right]} \tag{11.20}$$

6. Knowing M_3 and T_{t3}, we can compute T_3 from equation (4.18).
7. Examine the value of T_3 computed in step 6 and see how closely it compares to the value assumed in step 1.
8. We can now reevaluate γ_3 at the new T_3 value and see whether it differs appreciably from the value assumed originally. Observe that γ remains nearly constant as long as we are in the low-temperature plateaus shown in Figure 11.2.
9. If there is a need to improve the value of γ_3, do so and go back to step 3; otherwise, take the newly calculated value of T_3 as acceptable and proceed.

Now, for the area ratio, write equation (4.13) at stations 2 and 3. For supersonic flow at station 3, $M_2 = 1.0$ and in isentropic flow, $T_{t1} = T_{t2}$, $p_{t1} = p_{t2}$ [equation (4.28)], and $A_1^* = A_2^* = A_3^*$ [equation (5.35)]. Also, in the subsonic and sonic regions p/p_t. T/T_t and A/A^* are relatively insensitive to γ varying between 1.13 and 1.67 (see Figure 5.14). This means that for most gases between stations 1 and 2, we may use values from our isentropic table for $\gamma = 1.4$ (Appendix G) without introducing significant error, and also $\gamma_1 = \gamma_2$.

$$p_2 A_2 M_2 \sqrt{\frac{\gamma_2 g_c}{RT_2}} = p_3 A_3 M_3 \sqrt{\frac{\gamma_3 g_c}{RT_3}} \text{ or } \frac{A_3}{A_2} = \frac{1}{M_3} \frac{p_2}{p_3} \sqrt{\frac{\gamma_2 T_3}{\gamma_3 T_2}}$$

$$p_2 = \left(\frac{p_2}{p_{t2}}\right)\left(\frac{p_{t2}}{p_{t1}}\right)\left(\frac{p_{t1}}{p_1}\right) p_1 \approx (0.52828) p_1$$

$$T_2 = \left(\frac{T_2}{T_{t2}}\right)\left(\frac{T_{t2}}{T_{t1}}\right)\left(\frac{T_{t1}}{T_1}\right) T_1 \approx (0.83333) T_1$$

10. Substituting these values into the rearranged equation (4.13) we get a useful approximate relation for the nozzle area ratio for variable γ flows:

$$\frac{A_3}{A_2} = \frac{A_3}{A_3^*} \approx \frac{0.579}{M_3} \left(\frac{p_1}{p_3}\right) \sqrt{\frac{\gamma_1 T_3}{\gamma_3 T_1}} \tag{11.21}$$

where for the expansion in Figure 11.6 $T_{t1} = T_1$ and $p_{t1} = p_1$.

Example 11.4

Air expands isentropically through a supersonic nozzle from stagnation conditions $p_1 = 455$ psia and $T_1 = 2400°\text{R}$ to an exit pressure of $p_3 = 3$ psia. Calculate the exit Mach number, the area ratio of the nozzle, and the exit temperature using the perfect gas results and Method I, then compare to Method II.

By now, the perfect gas solution should be easy for you. We begin with those results.

$$M_3 = 4, \quad A_3/A_3^* = 10.72, \quad \text{and} \quad T_3 = 571°\text{R}.$$

We proceed by applying Method I.

1. Assume that $T_3 \approx 571°\text{R}$.
2. From Appendix L (or Figure 11.2), we get $\gamma_3 = 1.3995$ and $\gamma_1 = 1.317$.
3. Now

$$\bar{\gamma}_3 = \frac{\gamma_3 + \gamma_1}{2} = \frac{1.3995 + 1.317}{2} = 1.35825$$

4. $$T_{t3} \approx T_{t1} \left(\frac{\gamma_1}{\bar{\gamma}_3}\right)\left(\frac{\bar{\gamma}_3 - 1}{\gamma_1 - 1}\right) = (2400)\left[\left(\frac{1.317}{1.35825}\right)\left(\frac{1.35825 - 1}{1.317 - 1}\right)\right]$$

$$= (2400)(1.0958) = 2629.93°\text{R}$$

5. The Mach number

$$M_3 \approx \sqrt{\frac{2}{\gamma_3 - 1}\left[\left(\frac{p_{t3}}{p_3}\right)^{(\gamma_3-1)/\gamma_3} - 1\right]} = \left(\frac{2}{1.3995-1}\right)\left[\left(\frac{455}{3}\right)^{0.285459} - 1\right] = 3.9983$$

Here, we used equation (4.21) locally at 3.

6. So, now we are ready to recalculate T_3

$$T_3 = \frac{T_{t3}}{\left[1 + \frac{\gamma_3 - 1}{2} M_3^2\right]} = \frac{2629.93}{\left[1 + \frac{1.3995-1}{2}(3.9983)^2\right]} = 627.16°\text{R}$$

Note that the value of γ_3 remains the same (to three significant figures) at this new value of T_3. A second iteration yields $T_{t2} = 2627°\text{R}$, $M_3 = 3.9965$, and $T_3 = 628.1°\text{R}$.

Next, we work with Method II, for which we utilize the air table from Appendix K as in Example 11.2. We calculate [from (11.10)],

$$p_{r3} = p_{r1}\frac{p_3}{p_1} = (367.6)\left(\frac{3}{455}\right) = 2.424$$

which yields $T_3 = 635.5°\text{R}$, a slightly higher value than above. We still have to calculate A_3/A_3^*, but the air table is not helpful here. So we proceed with step 10 of Method I and obtain

$$\frac{A_3}{A_2} = \frac{A_3}{A_3^*} \approx \frac{0.579 p_1}{M_3 \; p_3}\sqrt{\frac{\gamma_1 T_3}{\gamma_3 T_1}} = \left(\frac{0.579}{3.9965}\right)\left(\frac{455}{3}\right)\sqrt{\frac{(1.317)(628.1)}{(1.3995)(2400)}}$$
$$\approx 10.904$$

We now compare the results. The static temperature calculation at station 3 compares well between Methods I and II (within 2%) but not so well between the perfect gas result and Method II (within 10%). Since Method II is based on the air table, its results are the most accurate, and in this example we can see why the perfect gas results do need improvement.

When the pressure ratio across the nozzle is not known, but rather the exit temperature (T_3) or exit Mach number (M_3), or when the nozzle area ratio (A_3/A_2) is given, Method I as introduced above can still be made applicable. For instance, we might have:

Given: The gas composition, $T_{t1} = T_1$, $p_{t1} = p_1$, and T_3.

Find: a. The pressure and Mach number at the exit (p_3 and M_3).

b. The required area ratio to produce these conditions (A_3/A_2).

Since T_3 is given, there is no requirement to iterate because γ_3 is obtainable directly. We may proceed from step 2 of Method I. After finding T_{t3} from step 4, we may calculate M_3 from equation (4.18):

$$M_3 \approx \sqrt{\frac{2}{\gamma_3 - 1}\left(\frac{T_{t3}}{T_3} - 1\right)} \tag{4.18}$$

Now the static pressure can be calculated from the same equation as step 5, equation (11.20), *but* with $\bar{\gamma}$ because we use the stagnation pressures at station 1 (upstream):

$$p_{t1} \approx p_3 \left(1 + \frac{\bar{\gamma}_3 - 1}{2} M_3^2\right)^{\bar{\gamma}_3/(\bar{\gamma}_3 - 1)}$$

Finally, the area ratio may be estimated from the equation of step 10 in Method I. The technique is basically the same but without the initial uncertainty of the value of γ at 3 and the need to use $\bar{\gamma}$ for properties such as the pressure found from upstream values.Another common type of problem is:

Given: The gas composition, $T_{t1} = T_1$, $p_{t1} = p_1$, and M_3.

Find: a. The pressure and temperature at the exit (p_3 and T_3).

b. The required area ratio to produce these conditions (A_3/A_2).

This type of problem calls for an iterative technique because of the unknown temperature at the nozzle exit. We shall use Method I and compare it with Method II using a detailed example found in Zucrow and Hoffman (pp. 183–187 of Ref. 20) which here becomes Example 11.5 given below.

Example 11.5

The problem is to deliver air at Mach 6 in an isentropic, blow-down wind tunnel with plenum conditions of 2000 K and 3.5 MPa as seen in Figure E11.5. Assume that the air properties are related by the perfect gas equation of state ($Z \approx 1.0$) but have variable specific heats. Determine conditions at the throat and at the exit, including the area ratio.

The procedure begins with the usual calculation for the perfect gas. For Method I we start at step 1 and proceed to obtain T_{t3} from step 4. Now step 5 differs because we use equation (4.18) to solve for T_3 since M_3 is given. The exit pressure p_3 may be calculated from equation (4.21) using the given p_{t1} upstream. There is a great deal of detail in this example that is not reproduced here. In particular, the calculations for the values at the throat (station 2) will not be shown because we assume that they are well represented by the perfect gas calculations with $\gamma_2 \approx \gamma_1 = 1.30$.

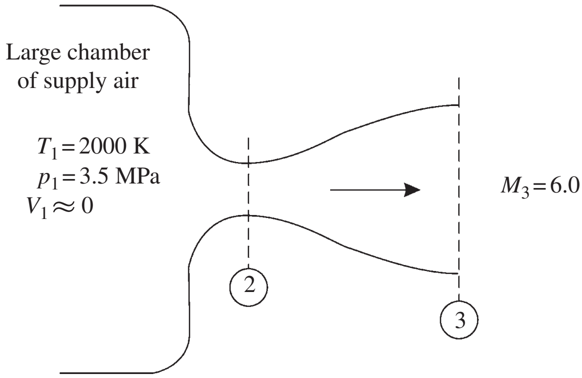


Figure E11.5

1. Assume that $T_3 = 243.9$ K, the perfect gas value.
2. For air we can surmise the ratio of specific heats to be $\gamma_3 = 1.401$, $\gamma_1 = 1.298$.
3. The average

$$\bar{\gamma}_3 = \frac{1.401 + 1.298}{2} = 1.3495$$

4. Now the value of T_{t3} can be estimated:

$$T_{t3} \approx T_{t1}\left(\frac{\gamma_1}{\bar{\gamma}_3}\right)\left(\frac{\bar{\gamma}_3 - 1}{\gamma_1 - 1}\right) = (2000)\left(\frac{1.298}{1.3495}\right)\left(\frac{1.3495 - 1}{1.298 - 1}\right) = 2256.123\,\text{K}$$

5. With M_3 and p_{t3} we calculate p_3:

$$p_3 \approx \frac{p_{t1}}{\left(1 + \frac{\bar{\gamma}_3 - 1}{2} M_3^2\right)^{\bar{\gamma}_3/(\bar{\gamma}_3 - 1)}} = \frac{3.5 \times 10^6}{\left[1 + \left(\frac{1.3495 - 1}{2}\right)(6)^2\right]^{3.8612}}$$
$$= 1.63173 \times 10^3\,\text{N/m}^2$$

6. With M_3 and T_{t3} we may proceed to find T_3:

$$T_3 = \frac{T_{t3}}{1 + \frac{\gamma_3 - 1}{2} M_3^2} = \frac{2256.123}{1 + \left(\frac{1.401 - 1}{2}\right)(6)^2} = 274.53\text{ K}$$

The first guess for γ_3 is sufficiently accurate, so we may proceed with the area ratio calculation.

7. Here we use the area ratio equation (11.21) as was been developed:

$$\frac{A_3}{A_2} = \frac{A_3}{A_3^*} = \frac{0.579}{M_3}\frac{p_1}{p_3}\sqrt{\frac{\gamma_1 T_3}{\gamma_3 T_1}} = \left(\frac{0.579}{6.0}\right)\left(\frac{3.5 \times 10^6}{1.63173 \times 10^3}\right)\sqrt{\frac{(1.298)(274.53)}{(1.401)(2000)}} = 73.82$$

Table 11.1 Summary of Calculations for Example Problem 11.5

Property	Units	Perfect Gas	Method I	Method II (Ref. 20)
p_2	MPa	1.9101	1.92	1.9073
T_2	K	1739.1	1739	1738.3
ρ_2	kg/m^3	3.8263	3.83	3.8225
V_2	m/s	805.57	806	806.52
G_2	kg/s-m^2	3082.4	3090	3082.9
p_3	N/m^2	2216.8	1631.73	1696.4
T_3	K	243.9	274.53	273.23
ρ_3	kg/m^3	0.031664	0.02068	0.02163
V_3	m/s	1878.4	1992.64	1989.0
G_3	kg/s-m^2	59.478	41.29	43.022
A_3/A_2		53.18	73.82	71.659

Table 11.1 gives results from Zucrow and Hoffman (Method II) for these calculations along with the perfect gas or constant specific heats solution and Method I as described above. Interested readers can view many of the details of the Method II calculations by consulting Reference 20.

A close look at the results in Table 11.1 leads to the following conclusions for this type of problem:

1. In the convergent section of the nozzle (where the flow is subsonic), the perfect gas solution is quite adequate.
2. In the diverging section of the nozzle (where the flow is supersonic), semiperfect gas effects must be considered.
3. Method I produces somewhat quick and satisfactory results for the pressure, temperature, and area ratio at the exit.

In reviewing Examples 11.4 and 11.5, you will notice that when we apply the equation that relates static to stagnation pressure we sometimes use the average γ [i.e., equation (11.20)] and other times the local γ [i.e., equation (4.21)]. The reasoning is simply that whenever we have available *local values* for pressure or temperature at station 3, we use γ_3 as in the case in Example 11.4. In Example 11.5, we need to calculate the exit pressure given the exit Mach number and the upstream pressure (nonlocal). This may seem rather artificial, but remember that this is an empirical method that has been developed to better account for γ variations on the temperature and the pressure. Note also the consistent use of γ_3 in calculating T_3 with local values.

The last case that we discuss, when A_3/A_2 together with the chamber (stagnation) properties is given, is also commonly found. At station 2 you should have little difficulty with the calculations but here no local gas properties are given at

station 3, so all of them must be inferred from upstream values. Of the various possibilities available, perhaps using the average γ from equation (11.18) of Method I directly into the isentropic equations of Section 5.6 is the simplest. This case is not detailed here but you can do this on your own by examining Example 11.7 and/or working Problem 11.13.

It should be recalled here that at sufficiently high exit Mach numbers and for relatively short supersonic nozzles, *kinetic lag effects* become increasingly real so that eventually the flow may be treated as if it were *frozen* in composition and in γ; this effect is exploited in gas dynamic lasers, see Section 11.8. Knowing the plenum properties accurately in a combustion chamber and using frozen-flow analysis, one can obtain good engineering estimates for adiabatic nozzle flows (Ref. 24). The only difference here is that the value of γ will be that of the hot gases, which is lower than the usual value of 1.4.

11.7 VARIABLE-γ CONSTANT-AREA FLOWS

Shocks

For shocks, both normal and oblique, we specialize here the set of equations given in Section 11.1 for adiabatic flow, with constant area and no friction. These are really the equations first assembled in Chapter 6 [i.e., equations (6.2), (6.4), and (6.9)] together with the modified equation of state (1.16 m). The shock problem becomes considerably more complicated when Z depends on T and p according to the compressibility charts and when c_p may vary with temperature (see, e.g., Ref. 32). In air without dissociation and below Mach 5, the perfect gas calculations fall within about 10% of the real gas values and may be used as a good first estimate.

As shown in Figure 6.10, the pressure ratio across the shock is the least sensitive to variations of γ, and perfect gas calculations turn out to be reasonable for determining the pressure. Now to improve calculations for the temperature, we can resort to the average γ concept given in equation (11.18). A useful technique involves introduction of the mass velocity $G = \rho v = \text{const}$, and equation (11.17) into equations (6.2), (6.4), and (6.9), to arrive at

$$h_2 = h_1 + \frac{G^2}{2g_c}\left(\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2}\right) \tag{11.22}$$

and

$$T_2 = \frac{\bar{\gamma}_2 - 1}{\bar{\gamma}_2}\left[\frac{\gamma_1}{\gamma_1 - 1}T_1 + \frac{G^2}{2Rg_c}\left(\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2}\right)\right] \tag{11.23}$$

A simple scheme when *all conditions* at location 1 are known is, then:

1. Obtain ρ_2 and T_2 from the perfect gas solution.
2. Find γ_1 and γ_2 (use Appendix L for air) and calculate $\bar{\gamma}_2$ from equation (11.18).
3. Compute G from the information given at 1.
4. From equation (11.23), obtain T_2 using $\bar{\gamma}_2$. This new value of T_2 should be more accurate than the perfect gas result.
5. If desired, now calculate an improved estimate of ρ_2 using the new T_2 in the perfect gas law. Assume that p_2 remains as found from the perfect gas shock.

Example 11.6

Let us apply the technique outlined above to Example 7.7 in Zucrow and Hoffman (pp. 353–356 of Ref. 20). Air flows at $M_1 = 6.2691$ and undergoes a normal shock. The other upstream static properties are $T_1 = 216.65$ K and $p_1 = 12{,}112$ N/m^2. Find the properties downstream of the shock assuming no dissociation. Because of the low temperatures, $\gamma_1 = 1.402$.

The perfect gas results are $T_2 = 1859.6$ K, $p_2 = 0.5534$ MPa, $\rho_2 = 1.0366$ kg/m^3, and $V_2 = 347.57$ m/s. Next, we find γ_2 as 1.301, based on the perfect gas temperature.

$$\bar{\gamma}_2 = \frac{1.301 + 1.402}{2} = 1.3515$$

Now the mass flow rate

$$\begin{aligned} G = \rho_1 V_1 &= p_1 M_1 \sqrt{\frac{\gamma_1}{RT_1}} = (12,112)(6.2691)\sqrt{\frac{1.402}{(287)(216)}} \\ &= 361\,\mathrm{kg/s{\cdot}m^3} \end{aligned}$$

The new value of the temperature can now be calculated from equation (11.23) as

$$\begin{aligned} T_2 &= \frac{\bar{\gamma}_2 - 1}{\bar{\gamma}_2}\left[\frac{\gamma_1}{\gamma_1 - 1}T_1 + \frac{G^2}{2Rg_c}\left(\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2}\right)\right] \\ &= \left(\frac{1.36-1}{1.36}\right)\left[\frac{1.4}{1.4-1}T_1 + \frac{(360)^2}{(2)(287)}(26.2 - 0.925)\right] = 1710\text{ K} \end{aligned}$$

This result is within 1% of the answer from Reference 20 ($T_2 = 1701$ K) so that further refinements are not needed. To calculate the improved estimate of the density, we have

$$\rho_2 = \frac{p_2}{RT_2} = \frac{5.51 \times 10^5}{(287)(1710)} = 1.12\,\mathrm{kg/m^3}$$

which compares within 4% of the value from Reference 20 ($\rho_2 = 1.1614$ kg/m^3).

When real gas effects are significant, the calculations become considerably more complicated, as information from compressibility charts or from tables will be necessary. In such cases the reader should consult References 20 or 32 for details.

Fanno Flows

For Fanno flow we specialize the set of equations given in Section 11.1 for adiabatic flow in constant-area ducts with friction as shown in Figure 9.4. In Figure 9.13 Fanno flow curves for various γ values show little variability in the subsonic range, which is typically the most common range for constant-area flows with friction.

Rayleigh Flows

For Rayleigh flow we specialize the set of equations given in Section 11.1 for constant-area flow without friction but with heat transfer. Rayleigh flow in ordinary combustors is typically constant area and subsonic. Note that property variations strongly depend on the chemical reactions taking place; as the flow equilibrates in a burner or other heating chamber, gas composition reaches a given unique equilibrium value, which then yields gas properties such as pressure and temperature. Rayleigh flow curves for various γ values, such as those shown in Figure 10.13, indicate only a small dependence of the stagnation temperature to γ variations in the subsonic flow regime.

We may conclude that for Fanno and Rayleigh flows, the constant-γ approach is satisfactory as long as these flows remain below Mach 1.0. Fortunately, most present applications for these flows are in the subsonic region. What remains to be done is to establish the appropriate value of γ for each gas under study.

11.8 HIGH-ENERGY GAS LASERS

Several high-energy gas lasers of practical interest are based on continuous-laser-output (or CW) gas-flow-devices, of which the *gas dynamic laser* (GDL) and the *electric discharge laser* (EDL) are prime examples. In the GDL the flow is supersonically whereas the EDL flows only subsonically (the lasing medium flows only for the purpose of cooling it). As the name implies, electric lasers derive their energy inputs from an electric discharge whereas the GDL relies on the relatively fast supersonic expansion of a special gas mixture that has been previously compressed and heated (either from a combustion source, electric discharge, or other means). Both of these high-energy lasers typically use CO_2 as the lasing medium and output in the infrared; they have been used for material processing (i.e., cutting and welding) and for military applications. Lasers produce very pure

monochromatic light outputs and require specially tailored energy inputs for the "selective pumping of the lasing medium's upper energy level." Lasers are inherently nonadiabatic devices and, as the acronym presently implies, their energy output is an intense, narrowly focused electromagnetic photon beam with more useful and unique properties than natural radiant sources such as combustion flames or the sun.

Because the gas dynamic laser utilizes a rather unusual nozzle design to accomplish its supersonic gas expansion and because its nonequilibrium comes from the same molecular properties that govern the behavior of the specific-heats as presented in Sections 11.1 and 11.4, we will focus on this type laser. The GDL's fast supersonic expansion is accomplished through the specially designed short-nozzle array shown in Figure 11.7. Infrared lasers, in general, depend on an internal nonequilibrium between internal molecular degrees of freedom, which here correspond to vibrational modes (one shown in Figure 11.1 for a diatomic molecule like N_2). Hot nitrogen is a major gas mixture component because it efficiently transfers energy (during collisions) to certain vibrational modes of the CO_2 (a triatomic molecule with 3 vibrational degrees of freedom), thus, during lasing, efficiently populating the desired upper vibrational level. After lasing, CO_2 at lower vibrational energies is produced so traces of H_2O-vapor are added to the gas mixture to decrease CO_2's lower laser level lifetime and bring it to its *ground state*. In essence, the fast supersonic expansion in the GDL "freezes out"

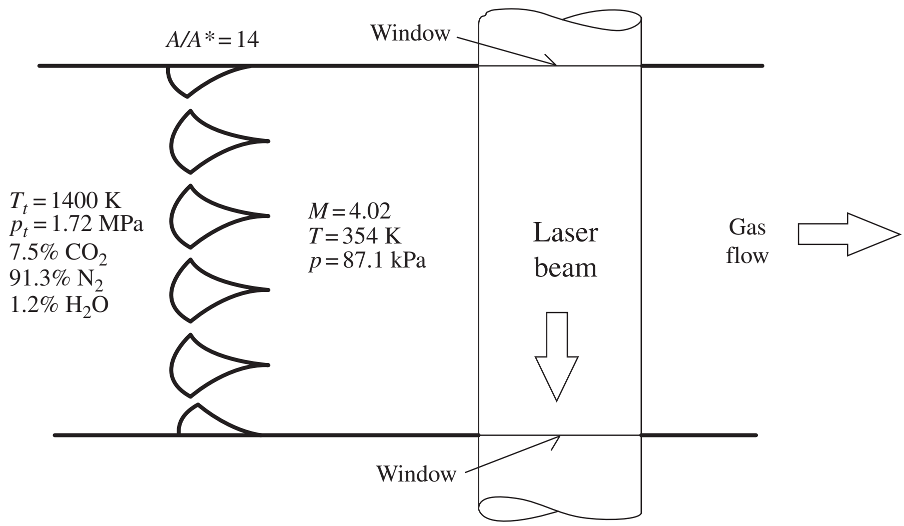


Figure 11.7 Gas dynamic laser showing a top view of a portion of its fast-expansion supersonic nozzle array (note the discontinuity at the throats). High-density photons exit the chamber as a beam of light through the laser windows. The gas flows internally through this beam and then goes into a diffuser (not shown) for exit to the atmosphere. The included data are from E. T. Gerry (IEEE Spectrum, Nov. 1970). Also not shown are the laser's optical elements. [Gas dynamic laser images can be found by searching the web for: "gas dynamic laser images"]

the upper laser levels of the coupled N_2/CO_2 molecular populations at their original gas temperature before the expansion cooling, and these excited molecules are subsequently *stimulated* to equilibrate by radiating energy in the laser cavity; only with the help of H_2O collisions, does the CO_2 gas mixture finally equilibrate to the lower cavity temperatures. Laser gases are recycled in most flow lasers but not in the original GDL designs.

Because we are dealing with microscopic phenomena here, only certain molecular states are able to selectively interact with each other for lasing and those states are relatively few in the GDL gas mixture. While laser power outputs can be the high, their thermal efficiencies are low (the electric the CO_2 lasers can approach 30% but for the GDL it is less than 10%), but because of their high coherence and high power they were widely used until high-power CW *fiber lasers* emerged. There are other molecular lasers that utilize chemical reactions more directly (called *chemical lasers*) and thus more efficiently; they use similar flow expansion principles as those depicted in Figure 11.7.

Example 11.7

Estimate and compare the parameters that result from the fast expansion nozzles depicted in Figure 11.7. Assume gas property values given are before the laser energy output and that the losses in the system are negligible. Use the given stagnation properties and area ratio.

We will assume the isentropic equations from Section 5.6 can be directly applied here and make allowance for the large temperature change using throughout the average $\bar{\gamma}$ approach (Method I, see Section 11.6). The gas mixture is mostly N_2 so the values for air from Appendix L or from polynomial Properties from Equations (above Section 11.5), or from Figure 11.2 could be used as a first approximation (and this yields $\bar{\gamma} \approx 1.36$). A more accurate technique is to use Reference 31 with the given compositions and temperatures. For the mixture described, we find

$$\gamma(1400\ \text{K}) = 1.305 \quad \text{and} \quad \gamma(354\ \text{K}) = 1.387, \quad \text{so that} \quad \bar{\gamma} = (1.305 + 1.387)/2 = 1.35$$

Applying $A/A^* = 14.0$, the Method I isentropic flow relations give $M = 4.08$, $p/p_t = 0.0051$, and $T/T_t = 0.255$, which compare well with the given $M = 4.02$, $p/p_t = 0.0051$, and $T/T_t = 0.253$.

It is conceivable that that the information in Figure 11.7 represents gas conditions *after lasing* but since the energy-conversion efficiencies are low this would not much affect our estimate. Note that if we had just used $\gamma = 1.4$, the results (from Appendix G) would be considerably different.

11.9 SUMMARY

We have discussed how microscopic behavior and molecular structure affect the gas dynamics. As the temperatures of operation of gases such as air and CO_2 rise much above room temperature, we see how some complicated microscopic

behavior becomes more apparent because of the activation of molecular vibrational modes. In monatomic gases, the equations arrived at in previous chapters remain applicable, but they must be modified for diatomic and polyatomic gases. Also, subtle nonequilibrium effects may come into play as Mach numbers reach high supersonic values.

Semiperfect gases follow the perfect gas law but have variable specific heats. Remember that as long as $p = \rho RT$ is valid, the enthalpy and internal energy are functions of temperature only. We arbitrarily assign $u = 0$ and $h = 0$ at $T = 0$ so that

$$u = \int_0^T c_v dT \quad \text{and} \quad h = \int_0^T c_p dT \tag{11.1, 11.2}$$

Entropy changes can be computed by

$$\Delta s_{1-2} = \phi_2 - \phi_1 - R \ln \frac{p_2}{p_1} \tag{11.4}$$

where

$$\phi \equiv \int_0^T c_p \frac{dT}{T} \tag{11.3}$$

Isentropic problems are more easily solved with the aid of

$$\text{relative pressure} \equiv p_r \equiv \frac{p}{p_0} \tag{11.9}$$

$$\text{relative volume} \equiv v_r \equiv \frac{v}{v_0} \tag{11.11}$$

All these functions, being unique functions of temperature, can be computed in advance and tabulated (see Appendix K). Remember:

1. h, u, and ϕ may be used for *any* process.
2. p_r and v_r may *only* be used for *isentropic* processes.

Many other equations of state have been developed for use when the perfect gas law is not accurate enough. In general, the more complicated expressions have a larger region of validity. But most lose accuracy near the critical point.

A useful means of handling the problem of deviations from perfect gas behavior involves use of the compressibility factor (remember Z is a pure number):

$$p = Z\rho RT \tag{1.16 modified}$$

Unless extreme accuracy is desired near the critical point, a single *generalized* compressibility chart may be used for all gases. In that case, Z is a function of

$$\text{reduced pressure} \equiv p_r \equiv \frac{p}{p_c} \tag{11.15}$$

$$\text{reduced temperature} \equiv T_r \equiv \frac{T}{T_c} \tag{11.16}$$

(What do p_c and T_c represent?)

Complicated equations of state can be handled readily with the help of digital computer software. For such a task, simple polynomials are available for nearly all properties of common gases for restricted temperature ranges. When available, use of the property tables (such as the steam table) is recommended because being largely experimental, they are more accurate in the vapor and supercritical fluid regimes.

The traditional isentropic nozzle problem gets modified as γ variations become significant at high plenum temperatures. The most important modification is that of the stagnation and static pressures and temperatures, and here one can either use the *Gas Tables* (Ref. 31) or the equations of Method I. At the nozzle exit, station 3 of Figure 11.6,

$$T_{t3} \approx T_{t1} \left(\frac{\gamma_1}{\bar{\gamma}_3}\right)\left(\frac{\bar{\gamma}_3 - 1}{\gamma_1 - 1}\right) \tag{11.19}$$

where

$$\bar{\gamma}_3 = \frac{\gamma_3 + \gamma_1}{2} \tag{11.18}$$

together with other equations from Method I, such as

$$M_3 \approx \sqrt{\frac{2}{\bar{\gamma}_3 - 1}\left[\left(\frac{p_{t1}}{p_3}\right)^{(\bar{\gamma}_3 - 1)/\bar{\gamma}_3} - 1\right]} \tag{11.20}$$

and

$$\frac{A_3}{A_2} \approx \frac{0.579}{M_3}\frac{p_1}{p_3}\sqrt{\frac{\gamma_1}{\gamma_3}\frac{T_3}{T_1}} \tag{11.21}$$

Care must be taken to properly use the *given data* in the equations above. Normal shocks are also treatable using Method I, but here the accuracy of perfect gas calculations is often satisfactory. Fanno and Rayleigh flows are commonly subsonic and quite amenable to the perfect gas treatment of Chapters 9 and 10 with the appropriate value of γ. When γ values differ from the tables, the use of the *Gasdynamics Calculator* is recommended.

PROBLEMS

11.1. Beginning at a temperature of 60°F and a volume of 10 ft^3, 2 lbm of air undergoes a constant-pressure process. The air is then heated to a temperature of 1000°F, and there is no shaft work. Using the air tables (Appendix K), find the work, the change of internal energy and of enthalpy, and the entropy change for this process.

11.2. In a two-step set of processes, a quantity of air is heated reversibly at constant pressure until the volume is doubled, and then, it is heated reversibly at constant volume until the pressure is doubled. If the air is initially at 70°F, find the total work, total heat transfer, and total entropy change to the end state. Use the air tables (Appendix K).

11.3. Compute the values of c_p, c_v, h, and u for air at 2000°R using the equations in Section 11.4. Check your values of specific heats and the enthalpy and internal energy values with the air tables in Appendix K.

11.4. Air at 2500°R and 150 psia is expanded through an isentropic turbine to a pressure of 20 psia. Determine the final temperature and the change of enthalpy. (Assume semiperfect gas behavior.)

11.5. Air at 1000°R and 100 psia undergoes a heat addition process to 1500°R and 80 psia. Compute the entropy change. If no work is done, also compute the heat added. (Assume semiperfect gas behavior.)

11.6. Compute γ for air at 300°R using the equations given at the end of Section 11.4.

11.7. For a gas that follows the perfect gas equation of state but has variable specific heats, the equation

$$s_2 - s_1 = \int_1^2 c_p \frac{dT}{T}$$

applies to which of the following?

(a) Any reversible process.

(b) Any constant-pressure process.

(c) An irreversible process only.

(d) Any constant-volume process.

(e) The equation is never correct.

11.8. Find the density of air at 360°R and 1000 psia using the compressibility chart. (The pseudo-critical point for air is often taken to be 238.7°R and 37.2 atm.)

11.9. Oxygen exists at 100 atm and 150°R. Compute its specific volume by use of the compressibility chart and by the perfect gas law.

11.10. The chemical formula for propane gas is C_3H_8, which corresponds to a molecular mass of 44.094. Determine the specific volume of propane at 1200 psia and 280°F using the generalized compressibility chart and compare to the result for a perfect gas. Propane has a critical temperature of 665.9°R and a critical pressure of 42 atm.

11.11. Calculate p_3 in Example 11.4 when $p_{t1} = 455$ psia, $T_{t1} = 2400$°R, and T_3 is given as 640°R.

11.12. Calculate p_3 in Example 11.4 when $p_{t1} = 455$ psia, $T_{t1} = 2400$°R, and M_3 is given as 3.91.

11.13. Calculate p_3 in Example 11.4 when $p_{t1} = 455$ psia, $T_{t1} = 2400$°R, and A_3/A_2 is given as 11.17. (*Hint*: use the *Gasdymics Calculator* with $\bar{\gamma}_3$ and the given area ratio to find M_3.)

11.14. Work out Example 11.4 in its entirety for argon instead of air with $p_{t1} = 3.0$ MPa, $T_{t1} = 1500$ K, and $p_3 = 0.02$ MPa.

11.15. Consider the nozzle in Example 11.5 operating at the second critical point (i.e., there is a normal shock at the exit). Calculate the properties after the shock when $M_1 = 6.0$, $T_1 = 272$ K, and $p_1 = 1696$ N/m^2.

11.16. For a supersonic nozzle array similar to the one shown in Figure 11.7, examine the variation in M_3 that result from equation (4.21) using the same p_1/p_3 in terms of possible ranges of γ. Take for example $p_1/p_3 = 200$ with $\gamma = 1.67$ (helium, any temperature), 1.40 (nitrogen and CO_2, room temperature). Then compare your results to using equation (11.20) in a hot expansion with $\gamma = 1.28$ (nitrogen, high temperature) and 1.19 (CO_2, high temperature). [All temperatures used should be below dissociation or ionization.]

CHECK TEST

11.1. What internal degree of freedom in diatomic and polyatomic gases is responsible for the observable variation in heat capacities with temperature and thus for semi-perfect gas behavior (under the assumptions made in this chapter)?

11.2. State the three distinct gaseous forms of matter and describe the possible microscopic reasons for real gas behavior (i.e., when Z is not equal to 1.0).

11.3. Calculate the enthalpy change for air when it is heated from 460°R to 3000°R at constant pressure. Use both the gas tables (Appendix K) and the perfect gas relations. What is the nature of the discrepancy, if any?

11.4. True or False: The concepts of the relative pressure (p_r) and the relative specific volume (v_r) are valid for any semiperfect gas undergoing any process whatsoever.

11.5. Find the density of water vapor at 500°F and 500 psia using the compressibility chart and perfect gas relations. The steam tables answer is 1.008 lbm/ft^3; how does it compare to your answer?

11.6. Work out the subsonic portion of Example 11.4 for both argon and carbon dioxide and compare all answers.

11.7. (*Optional*) Work Problem 11.12.

11.8. The pressure is a dynamical quantity and the temperature a thermodynamical quantity, and the perfect gas equation of state connects them through the gas density and an individual gas constant. Semiperfect-gas formulations introduce more coefficients but follow the same pattern. Why don't we use momentum principles to dynamically relate pressure to the average kinetic energy of molecules (which represents the temperature)?

Chapter 12

Propulsion Systems

12.1 INTRODUCTION

All craft that move through a fluid medium must operate with some form of propulsion system. We do not attempt to discuss all types of such systems but do concentrate on those used for aircraft or missile propulsion and popularly thought of as *jet propulsion devices*. Working with these systems permits a practical application of your knowledge in the field of gas dynamics. These engines can be classified as either *air-breathers* (such as the turbojet, turbofan, turboprop, ramjet, and pulsejet) or *non–air-breathers*, which are called *rockets*. Many schemes for rocket propulsion have been proposed, but we discuss only thermal rockets, of which the chemical rocket is the most common. The reader should consult References 21 to 25 for further details on propulsion systems in general.

Many air-breathing engines operate with the same basic thermodynamic cycle. Thus we first examine the Brayton cycle to discover its relevant features. Each propulsion system is then briefly described, and some of its operating characteristics discussed. We then apply momentum principles to an arbitrary propulsive device to develop a general relationship for net propulsive force or thrust. Other significant performance parameters, such as power and efficiency criteria, are also defined and discussed. The chapter closes with an interesting analysis of fixed-geometry supersonic air inlets.

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

12.2 OBJECTIVES

After completing this chapter successfully, you should be able to:

1. Make a schematic of the Brayton cycle and draw h–s diagrams for both ideal and real power plants.
2. Analyze both the ideal and real Brayton cycles. Compute all work and heat quantities as well as cycle efficiency.
3. State the distinguishing feature of the Brayton cycle that makes it well suited for turbomachinery. Explain why machine efficiencies are so critical in this cycle.
4. Discuss the difference between an open and a closed cycle.
5. Draw a schematic and an h–s diagram (where appropriate) and describe the operation of any three of the following propulsion systems: turbojet, turbofan, turboprop, ramjet, pulsejet, and rocket.
6. Compute all state points in a turbojet or ramjet cycle when given appropriate operating parameters, component efficiencies, and so on.
7. State the normal operating regimes for various types of propulsion systems.
8. (*Optional*) Develop the expression for the net propulsive thrust of an arbitrary propulsion system.
9. (*Optional*) Define or give expressions for input power, propulsive power, thrust power, thermal efficiency, propulsive efficiency, overall efficiency, and specific fuel consumption.
10. Compute the significant performance parameters for an air-breathing propulsion system when given appropriate velocities, areas, pressures, and so on.
11. (*Optional*) Derive an expression for the ideal propulsive efficiency of an air-breathing engine in terms of the speed ratio ν.
12. For rockets, define or give expressions for the effective exhaust velocity, the specific impulse, and the thrust coefficient.
13. Compute the significant performance parameters for a rocket when given appropriate velocities, areas, pressures, and so on.
14. (*Optional*) Derive an expression for the ideal propulsive efficiency of a rocket engine in terms of its speed ratio ν.
15. Explain why *fixed*-geometry converging–diverging diffusers are not used for air inlets on supersonic aircraft.

12.3 BRAYTON CYCLE

Basic Closed Cycle

Many small power plants and most air-breathing jet propulsion systems operate on a cycle that was developed over 100 years ago by George B. Brayton. Although

his first model was a reciprocating engine, this cycle had certain features that destined it to become the basic cycle for all gas turbine plants. We first consider the basic ideal closed cycle in order to develop some of its characteristic operating parameters. A schematic of the components of this cycle is shown in Figure 12.1 and includes a compression process from 1 to 2 with work input designated as w_c, a constant pressure heat addition from 2 to 3 with the heat added denoted by q_a, an expansion process from 3 to 4 with the work output designated as w_t, and a constant pressure heat rejection from 4 to 1 with the heat rejected denoted by q_r.

For our initial analysis we shall assume no pressure drops in the heat exchangers, no heat loss in the compressor or turbine, and all reversible processes. Our cycle then consists of

1. two reversible adiabatic processes and
2. two reversible constant-pressure processes.

An h–s diagram for this cycle is shown in Figure 12.2. Keep in mind that the working medium for this cycle is in a gaseous form, and thus this h–s diagram is similar to a T–s diagram. In fact, for perfect gases the diagrams are identical except for the vertical scale.

We shall proceed to make a steady flow analysis of each portion of the cycle. *Turbine:*

$$h_{t3} + \cancel{q} = h_{t4} + w_s \tag{12.1}$$

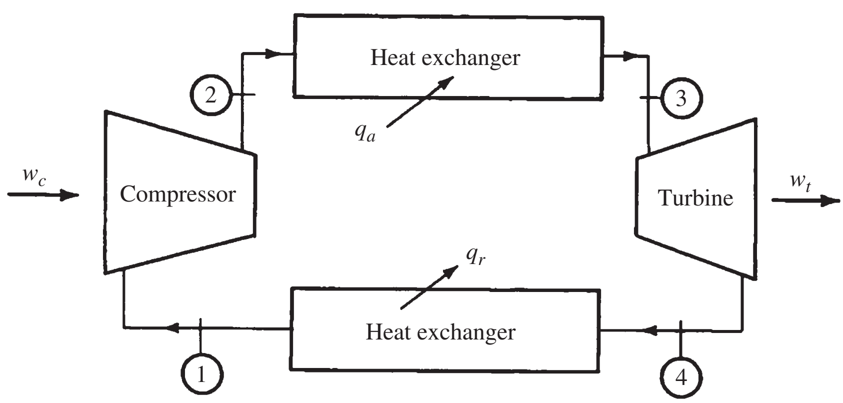


Figure 12.1 Schematic of a basic Brayton cycle.

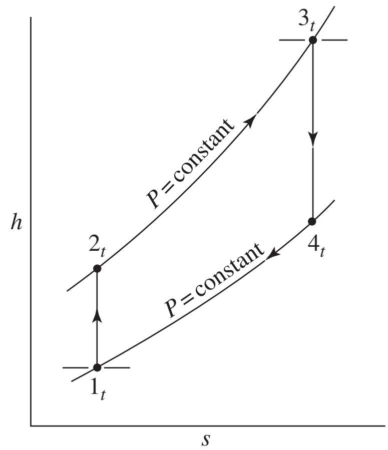


Figure 12.2 *h–s* diagram for ideal Brayton cycle.

Thus

$$\boxed{w_t \equiv w_s = h_{t3} - h_{t4}} \tag{12.2}$$

Compressor:

$$h_{t1} + \cancel{q} = h_{t2} + w_s \tag{12.3}$$

Designating w_c as the (*positive*) quantity of work that the compressor puts into the system, we have

$$\boxed{w_c \equiv -w_s = h_{t2} - h_{t1}} \tag{12.4}$$

The *net* work output is

$$w_n \equiv w_t - w_c = (h_{t3} - h_{t4}) - (h_{t2} - h_{t1}) \tag{12.5}$$

Heat Added:

$$h_{t2} + q = h_{t3} + \cancel{w_s} \tag{12.6}$$

Thus

$$\boxed{q_a \equiv q = h_{t3} - h_{t2}} \tag{12.7}$$

Heat Rejected:

$$h_{t4} + q = h_{t1} + \cancel{w_s} \tag{12.8}$$

Denoting q_r as the (*positive*) quantity of heat that is rejected from the system, we have

$$\boxed{q_r \equiv -q = h_{t4} - h_{t1}} \tag{12.9}$$

The *net* heat added is

$$q_n \equiv q_a - q_r = (h_{t3} - h_{t2}) - (h_{t4} - h_{t1}) \tag{12.10}$$

The thermodynamic efficiency of the cycle is defined as

$$\boxed{\eta_{\text{th}} \equiv \frac{\text{net work output}}{\text{heat input}} = \frac{w_n}{q_a}} \tag{12.11}$$

For the Brayton cycle, this becomes

$$\begin{aligned} \eta_{\text{th}} &= \frac{(h_{t3} - h_{t4}) - (h_{t2} - h_{t1})}{h_{t3} - h_{t2}} = \frac{(h_{t3} - h_{t2}) - (h_{t4} - h_{t1})}{h_{t3} - h_{t2}} \\ \eta_{\text{th}} &= 1 - \frac{h_{t4} - h_{t1}}{h_{t3} - h_{t2}} = 1 - \frac{q_r}{q_a} \end{aligned} \tag{12.12}$$

We observe that the efficiency can be expressed solely in terms of the heat quantities. The latter result can be arrived at much quicker by noting that for any cycle,

$$w_n = q_n \qquad (1.17 \text{ on a unit mass basis})$$

and the cycle efficiency can be rewritten as

$$\eta_{\text{th}} = \frac{w_n}{q_a} = \frac{q_n}{q_a} = \frac{q_a - q_r}{q_a} = 1 - \frac{q_r}{q_a} \tag{12.13}$$

If the working medium is assumed to be a perfect gas, additional relationships can be brought into play. For instance, all of the heat and work quantities above can be expressed in terms of temperature differences since

$$\Delta h = c_p \Delta T \tag{1.36}$$

and similarly,

$$\Delta h_t = c_p \Delta T_t \tag{12.14}$$

Equation (12.12) can thus be written as

$$\eta_{\text{th}} = 1 - \frac{c_p(T_{t4} - T_{t1})}{c_p(T_{t3} - T_{t2})} = 1 - \frac{T_{t4} - T_{t1}}{T_{t3} - T_{t2}} \tag{12.15}$$

With a little more manipulation this can be put into a relatively simple and significant form. Let us digress for a moment to show how this can be done.

Looking at Figure 12.2, we notice that the entropy change calculated between points 2 and 3 will be the same as that calculated between points 1 and 4. Now the entropy change between any two points, say A and B, can be computed by

$$\Delta s_{A-B} = c_p \ln \frac{T_B}{T_A} - R \ln \cancel{\frac{p_B}{p_A}} \tag{1.43}$$

If we are dealing with a constant-pressure process, the last term is zero, and the resulting simple expression is applicable between 2 and 3 as well as between 1 and 4. Thus

$$\Delta s_{2-3} = \Delta s_{1-4} \tag{12.16}$$

$$c_p \ln \frac{T_{t3}}{T_{t2}} = c_p \ln \frac{T_{t4}}{T_{t1}} \tag{12.17}$$

and if c_p is considered constant [as it was in order to derive equation (1.43)],

$$\frac{T_{t3}}{T_{t2}} = \frac{T_{t4}}{T_{t1}} \tag{12.18}$$

Show that under the condition expressed by (12.18), we can write

$$\frac{T_{t4} - T_{t1}}{T_{t3} - T_{t2}} = \frac{T_{t1}}{T_{t2}} \tag{12.19}$$

and the cycle efficiency (12.15) can be expressed as

$$\eta_{\text{th}} = 1 - \frac{T_{t1}}{T_{t2}} \tag{12.20}$$

Now since the compression process between 1 and 2 is isentropic, the temperature ratio can be related to a pressure ratio. If we designate the *pressure* ratio of the compression process as r_p,

$$r_p \equiv \frac{p_{t2}}{p_{t1}} \tag{12.21}$$

the *ideal* Brayton cycle efficiency for a perfect gas becomes [by equation (1.47)]

$$\boxed{\eta_{\text{th}} = 1 - \left(\frac{1}{r_p}\right)^{(\gamma-1)/\gamma}} \tag{12.22}$$

Remember that this relation is valid only for an ideal cycle and when the working medium may be considered a perfect gas. Equation (12.22) is plotted in Figure 12.3 and shows the influence of the compressor pressure ratio on cycle efficiency. *Even for real power plants, the pressure ratio remains as the most significant basic parameter.*

Normally in closed cycles, all velocities in the flow ducts (stations 1, 2, 3, and 4) are relatively small and may be neglected. Thus all enthalpies, temperatures, and pressures in the equations above represent static as well as stagnation quantities. However, this is *not* true for open cycles, which are used for propulsion systems.

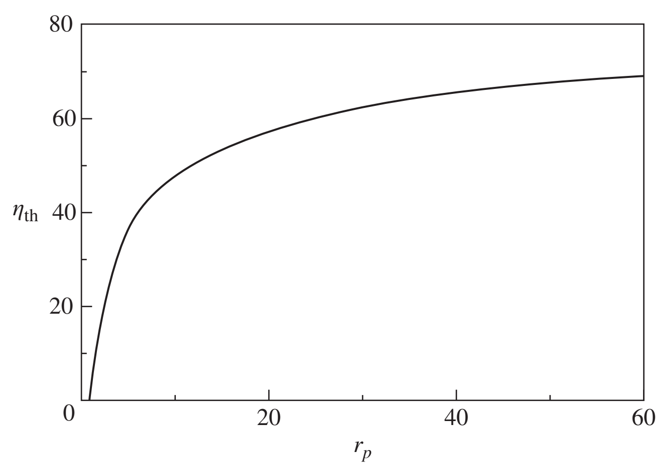


Figure 12.3 Thermodynamic efficiency of ideal Brayton cycle (γ = 1.4).

The modifications required for the analysis of various propulsion engines are discussed in Section 12.4.

Example 12.1

Air enters the compressor at 15 psia and 550°R. The pressure ratio is 10. The maximum allowable cycle temperature is 2000°R (Figure E12.1). Consider an ideal cycle with negligible velocities and treat the air as a perfect gas with constant specific heats. Determine the turbine and compressor work and cycle efficiency. Since velocities are negligible, we use static conditions in all equations.Thus, using equation (1.43),

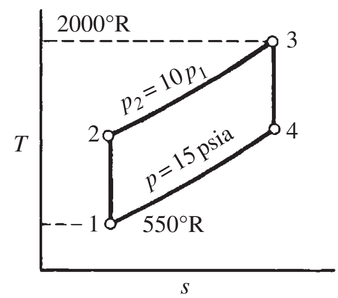


Figure E12.1

$$T_2 = (1.931)(550) = 1062°\text{R}$$

and similarly,

$$T_4 = \frac{2000}{1.931} = 1036°\text{R}$$

$$w_t = c_p(T_3 - T_4) = (0.24)(2000 - 1036) = 231\,\text{Btu/lbm} \quad (\text{or } 537\,\text{kJ/kg})$$

$$w_c = c_p(T_2 - T_1) = (0.24)(1062 - 550) = 123\,\text{Btu/lbm} \quad (\text{or } 286\,\text{kJ/kg})$$

$$w_n = w_t - w_c = 231 - 123 = 108\,\text{Btu/lbm}$$

$$q_a = c_p(T_3 - T_2) = (0.24)(2000 - 1062) = 225\,\text{Btu/lbm}$$

$$\eta_{\text{th}} = \frac{w_n}{q_a} = \frac{108}{225} = 48\%$$

Notice that even in an ideal cycle, the *net* work is a rather small proportion of the turbine work. By comparison, in the Rankine cycle (which is used for steam power plants), over 95% of the turbine work remains as useful work. This radical difference is accounted for by the fact that in the Rankine cycle the working medium is compressed as a liquid and in the Brayton cycle the working fluid is *always* a gas.

This large proportion of *back work* accounts for the basic characteristics of the Brayton cycle.

1. Large volumes of gas must be handled to obtain reasonable work capacities. For this reason, the cycle is particularly suitable for use with turbomachinery.
2. Machine efficiencies are extremely critical to economical operation. In fact, some low efficiency that could be tolerated in other cycles would reduce the net output of a Brayton cycle to zero. (See Example 12.2.)

The latter point highlights the stumbling block, which for years prevented exploitation of this cycle, particularly for purposes of aircraft and missile propulsion. Efficient, lightweight, high-pressure ratio compressors did not become available until about 1950. Another problem concerns the temperature limitation where the gas enters the turbine. Turbine blading must be able to *continuously* withstand this temperature while operating under high-stress conditions and new materials needed to be introduced.

Cycle Improvements

The basic Brayton cycle performance can be improved by several techniques. If the turbine outlet temperature T_4 is sufficiently higher than the compressor outlet temperature T_2, some of the heat that would normally be rejected can be used to furnish part of the heat added. This is called *regeneration*, and it reduces the heat that must be supplied. The net result is considerable efficiency improvement. Could a regenerator be used in Example 12.1?

The compression process can also be done in stages with *intercooling* (heat removal between each stage). This reduces the amount of compressor work. Similarly, the expansion can take place in stages with *reheat* (heat addition between stages). This increases the amount of turbine work. Unfortunately, this type of staging slightly decreases the cycle efficiency, but this can be tolerated in order to increase the *net work produced per unit mass of fluid flowing*. This parameter is called *specific output* and is an indication of the size of unit required to produce a given amount of power. Because the techniques of regeneration and staging with intercooling or reheating are only of use in stationary power plants, they are not discussed further here. Those interested in more details on these topics may wish to consult any recent text on gas turbine power plants or Volume II of Zucrow (Ref. 25).

Real Cycles

The thermodynamic efficiency of 48% calculated in Example 12.1 is quite high because the cycle was assumed to be ideal. To obtain more meaningful results, we

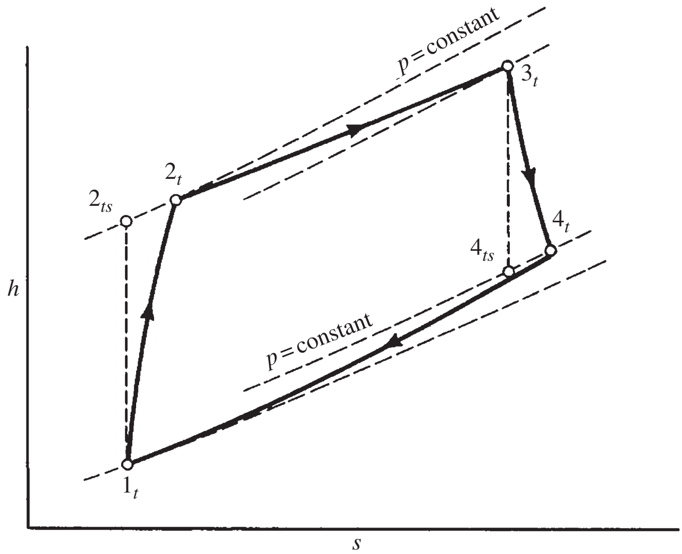


Figure 12.4 *h–s* diagram for an "exaggerated" real Brayton cycle.

must consider flow losses. We have already mentioned the importance of having high machine efficiencies. Relatively speaking, this is not too difficult to accomplish in the turbine, where an expansion process takes place, but it is quite a task to build an efficient compressor. In addition, pressure drops are involved in all ducts and heat exchangers (burners, intercoolers, reheaters, regenerators, etc.). An *h–s* diagram for a real Brayton cycle is given in Figure 12.4, which somewhat exaggerates the effects of machine efficiencies and pressure drops (making the net work output w_s appear negative). Note that the irreversible effects cause entropy increases in both the compressor and turbine.

Turbine efficiency, assuming negligible heat loss, becomes

$$\eta_t \equiv \frac{\text{actual work output}}{\text{ideal work output}} = \frac{h_{t3} - h_{t4}}{h_{t3} - h_{t4s}} \tag{12.23}$$

For a perfect gas with constant specific heats, this can also be represented in terms of temperatures:

$$\eta_t \equiv \frac{c_p(T_{t3} - T_{t4})}{c_p(T_{t3} - T_{t4s})} = \frac{T_{t3} - T_{t4}}{T_{t3} - T_{t4s}} \tag{12.24}$$

Note that the actual and ideal turbines operate between the same pressures.

The *compressor efficiency* similarly becomes

$$\eta_c \equiv \frac{\text{ideal work input}}{\text{actual work input}} = \frac{h_{t2s} - h_{t1}}{h_{t2} - h_{t1}} \tag{12.25}$$

$$\eta_c = \frac{T_{t2s} - T_{t1}}{T_{t2} - T_{t1}} \tag{12.26}$$

Again, note that the actual and ideal machines operate between the same pressures (see Figure 12.4).

Example 12.2
Assume the same information as given in Example 12.1 except that the compressor and turbine efficiencies are both 80%. Neglect any pressure drops in the heat exchangers. Thus the results only show the effect of low machine efficiencies on the Brayton cycle. We take the ideal values that were calculated in Example 12.1.

$$\begin{aligned}
T_1 &= 550°\text{R} \;\; T_3 = 2000°\text{R} \;\; \eta_t = \eta_c = 0.8 \\
T_{2s} &= 1062°\text{R} \;\; T_{4s} = 1036°\text{R} \\
w_t &= (0.8)(0.24)(2000 - 1036) = 185.1 \text{ Btu/lbm} \;\; (\text{or } 431 \text{ kJ/kg}) \\
w_c &= \frac{(0.24)(1062 - 550)}{0.8} = 153.6 \text{ Btu/lbm} \qquad (\text{or } 357 \text{ kJ/kg}) \\
w_n &= 185.1 - 153.6 = 31.5 \text{ Btu/lbm} \\
T_2 &= 550 + \frac{153.6}{0.24} = 1190°\text{R} \\
q_a &= (0.24)(2000 - 1190) = 194.4 \text{ Btu/lbm} \\
\eta_{\text{th}} &= \frac{w_n}{q_a} = \frac{31.5}{194.4} = 16.2\%
\end{aligned}$$

Note that the introduction of 80% machine efficiencies drastically reduces the net work and cycle efficiency, to values about 29% and 34% of their respective ideal ones. What would the net work and cycle efficiency be if the machine efficiencies were 75%?

Open Brayton Cycle for Propulsion Systems

Most stationary gas turbine power plants operate on the *closed cycle* illustrated in Figure 12.1. Gas turbine engines used for aircraft and missile propulsion operate on an *open cycle*; that is, the process of heat rejection (from the turbine exit to the compressor inlet) does not physically take place within the engine, but occurs in the atmosphere. Thermodynamically speaking, the open and closed cycles are identical, but there are a number of significant differences in the actual hardware.

1. The air enters the system at high velocity and thus must be diffused before being allowed to pass into the compressor. A significant portion of the compression occurs in this diffuser. If flight speeds are supersonic, increases in pressure also occur across any shock system that forms at the front of the inlet.
2. The heat addition is carried out by an internal combustion process within a burner or combustion chamber. Thus the products of combustion together with the unburned air pass through the remainder of the system.
3. After passing through the turbine, the air mixture leaves the system by further expanding through a nozzle. This increases the kinetic energy of the exhaust gases, which aids in producing thrust.
4. Although the compression and expansion processes generally occur in stages (most particularly with axial compressors), no intercooling is normally involved. Any thrust augmentation with the use of an *afterburner* could be considered as a form of reheat between the last turbine stage and the nozzle expansion. The use of regenerators is considered impractical for flight propulsion systems.

The division of the compression process between the diffuser and compressor and amount of expansion that takes place within the turbine and the exit nozzle vary greatly depending on the type of propulsion system involved. This is discussed in greater detail in the next section, where we describe a number of common propulsion engines.

12.4 PROPULSION ENGINES

Turbojet

Although the first jet engine patent was issued in 1922, the building of practical turbojets did not take place until a decade later. Development work was started in both England and Germany in 1930, with the British realizing the first operable engine in 1937. However, it was not used to power an airplane until 1941. The thrust of this engine was about 850 lbf (3781 N). The Germans managed to achieve the first actual flight of a turbojet plane in 1939, with an engine of 1100 lbf (4893 N) thrust. (Historical notes on various engines were obtained from Reference 25.)

Figure 12.5 shows a cutaway picture of a typical turbojet. Although this looks rather formidable, the schematic shown in Figure 12.6 helps identify the basic parts. Figure 12.6 also shows the important section locations necessary for engine analysis. Air enters the diffuser and is somewhat compressed as its velocity is decreased. The amount of compression that takes place in the diffuser depends

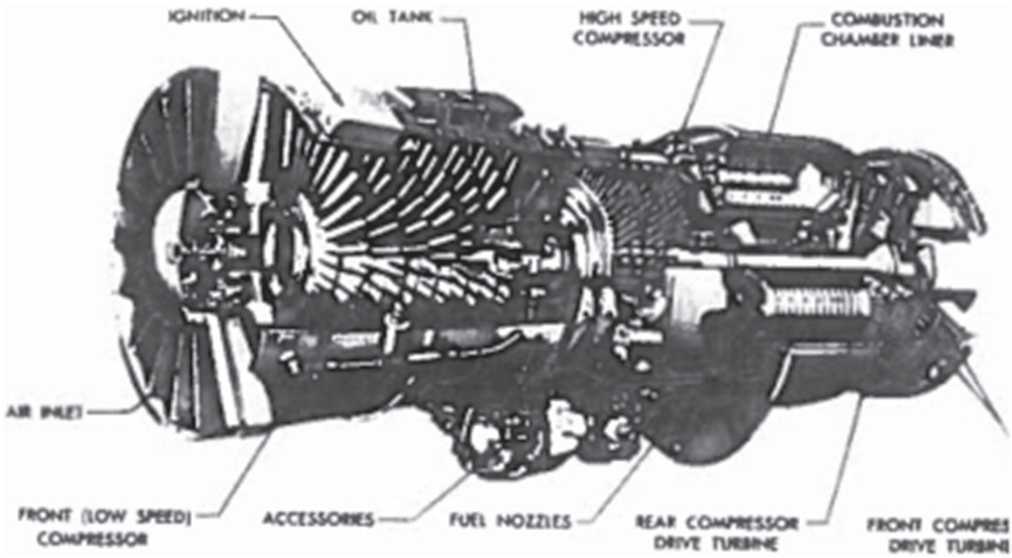


Figure 12.5 Cutaway view of a turbojet engine. (Courtesy of Pratt & Whitney Aircraft.) [Additional images can be found by searching the Web for: "turbojet turbofan turboprop images"]

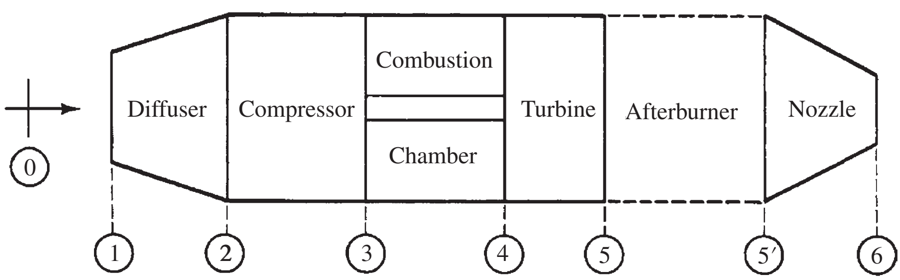


Figure 12.6 Basic parts of a turbojet engine.

on the flight speed of the vehicle. The greater the flight speed, the greater the pressure rise within the diffuser.

After passing through the diffuser, the air enters an adiabatic compressor, where the remainder of the pressure rise occurs. The early turbojets used *centrifugal* compressors, as these were the most efficient type available. Since then a great deal more has been learned about compressor aerodynamics and this has enabled the rapid development of efficient *axial-flow* compressors, which are now widely used in jet engines.

A portion of the air then enters the combustion chamber for the heat addition by internal combustion, which is ideally carried out at constant pressure. Combustion

chambers come in several configurations; some are annular chambers, but most consist of a number of small chambers surrounding the central shaft. The remainder of the air is used to cool the chamber, and eventually, all excess air is mixed with the products of combustion to cool them before entering the turbine. Here is the most critical temperature in the entire engine since most turbine blading have reduced strength at elevated temperatures while operating at high stress levels. As better materials are developed, the maximum allowable turbine inlet temperature have been raised, which results in more efficient engines. Also, methods of blade cooling have helped alleviate this problem.

The gas is *not* expanded back to atmospheric pressure within the turbine. It is only expanded enough to produce sufficient shaft work to run the compressor plus engine auxiliaries. This expansion is essentially adiabatic. In most jet engines the gases are then exhausted to the atmosphere through a nozzle. Here, the expansion permits conversion of enthalpy into kinetic energy and the resulting high velocities produce the thrust. Normally, converging-only nozzles are used and they operate in a choked exit condition.

Many jet engines used for military aircraft have a section between the turbine and the exhaust nozzle, which includes an *afterburner*. Since the flowing gases contain a large amount of excess air, additional fuel can be added in this section. The temperature can be raised quite high since all surrounding material here operates at a low stress levels. The use of an afterburner enables much greater exhaust velocities to be obtained from the nozzle with higher resultant thrusts. However, this increase in thrust is obtained at the expense of an extremely high rate of fuel consumption.

An *h–s* diagram for a turbojet is shown in Figure 12.7, which for the sake of simplicity indicates all processes as ideal. The station numbers refer to those marked in the schematic of Figure 12.6. The diagram represents *static* values. The free stream exists at nearly state 0 and has a high velocity (relative to the engine). These same conditions may or may not exist at the actual inlet to the engine. An external diffusion with spillage or an external shock system would cause the thermodynamic state at 1 to differ from that of the free stream. Notice that point 1 does not even appear on the *h–s* diagram. This is because the performance of an air inlet is usually given with respect to free-stream conditions, enabling one directly to compute properties at section 2.

Operation both with and without an afterburner is shown on Figure 12.7, the process from 5 to $5'$ indicating the use of an afterburner, with $5'$ to $6'$ representing subsequent flow through the exhaust nozzle. In this case a nozzle with a variable exit area is required to accommodate the flow when in the afterburning mode. Since the converging nozzle is usually choked, we have indicated point 6 (and $6'$) at a pressure greater than atmospheric. High velocities exist at the inlet and outlet (0, 1, and 6 or $6'$), and relatively low velocities exist at all other sections. Thus points 2 through 5 (and $5'$) also represent approximate stagnation values.

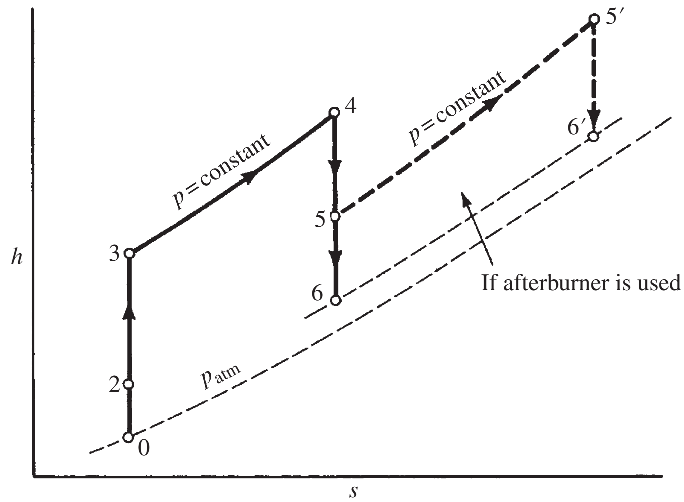


Figure 12.7 *h–s* diagram for ideal turbojet. (For schematic, see Figure 12.6.)

(These internal velocities may not always be negligible, especially in the afterburner region.) A detailed analysis of a turbojet is identical with that of the primary air passing through a turbofan engine. A problem related to this case is worked out in Example 12.3.

A turbojet engine has high fuel consumption because it creates thrust by accelerating a relatively small amount of air through a large velocity differential. In a later section we shall see that this creates low propulsion efficiency unless the flight velocity is very high. Thus a profitable application of the turbojet is for flight speed ranges from $M_0 = 1.0$ up to about $M_0 = 2.5$ or 3.0. At flight speeds above approximately $M_0 = 3.0$, the ramjet can be shown to be more desirable. In the subsonic flight speed range, other variations of the turbojet are more economical, and these are discussed next.

Turbofan

The concept here is to move a great deal more air through a smaller velocity differential, thus increasing the propulsion efficiency at low flight speeds. This is accomplished by adding a large shrouded fan to the engine. Figure 12.8 shows a cutaway picture of a typical turbofan engine. The schematic in Figure 12.9 helps to identify the basic parts and indicate the important section locations necessary for the engine analysis.

The flow through the central portion, or basic gas generator (0–1–2–3–4–5–6), is identical to that discussed previously for the pure jet (without an afterburner). Additional air, often called *secondary* or *bypass air* (1-2-3′-4′), is drawn in through the diffuser and passed to the fan section, where it is compressed through

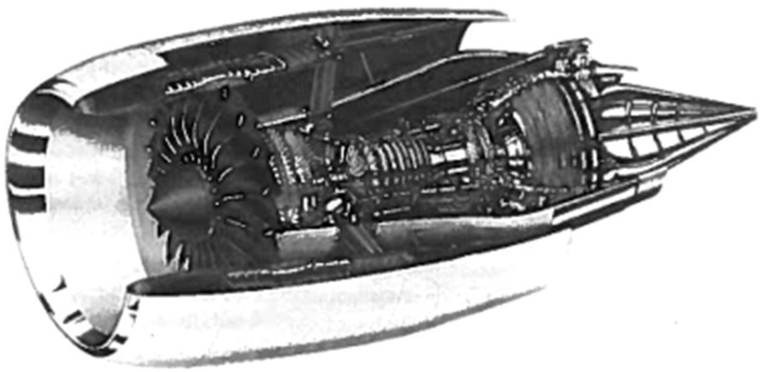


Figure 12.8 Cutaway view of a turbofan engine. (Courtesy of General Electric Aircraft Engines.) [Additional images can be found by searching the web for: "turbojet turbofan turboprop images"]

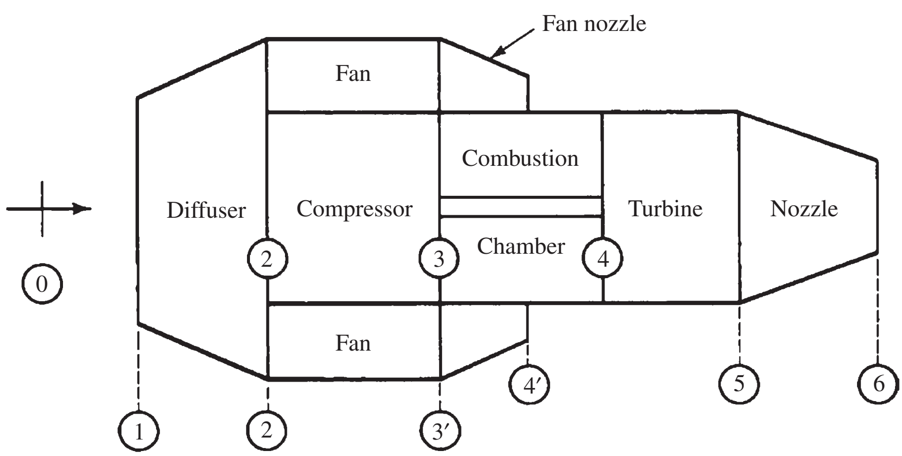


Figure 12.9 Basic parts of a turbofan engine.

a relatively low pressure ratio. The bypass air flow is then exhausted through a nozzle to the atmosphere. Many variations of this configuration are found. Some fans are located near the rear with their own inlet and diffuser. In some models the bypass air flow from the fan is mixed with the main air from the turbine, and the total air flow exits through a common nozzle.

The *bypass ratio* is defined as

$$\beta \equiv \frac{\dot{m}'_a}{\dot{m}_a} \tag{12.27}$$

where

$\dot{m}_a \equiv$ mass flow rate of primary air (through compressor)

$\dot{m}'_a \equiv$ mass flow rate of secondary air (through fan)

An h–s diagram for the primary air is shown in Figure 12.10 and for the secondary air in Figure 12.11. In these diagrams, both the actual and ideal processes are shown so that a more accurate picture of the losses can be obtained. These diagrams are for the configuration shown in Figure 12.9, in which a common diffuser is used for all entering air and separate nozzles are used for the fan and turbine exhaust.

The analysis of a fanjet is identical to that of a pure jet, with the exception of sizing the turbine. In the fanjet the turbine must produce enough work to run both the compressor and the fan:

$$\begin{aligned} \text{turbine work} &= \text{compressor work} + \text{fan work} \\ \dot{m}_a(h_{t4}-h_{t5}) &= \dot{m}_a(h_{t3}-h_{t2}) + \dot{m}'_a(h_{t3'}-h_{t2}) \end{aligned} \tag{12.28}$$

If we divide by $\dot{m}_a$ and introduce the bypass ratio β [see equation (12.27)], this becomes

$$(h_{t4}-h_{t5}) = (h_{t3}-h_{t2}) + \beta(h_{t3'}-h_{t2}) \tag{12.29}$$

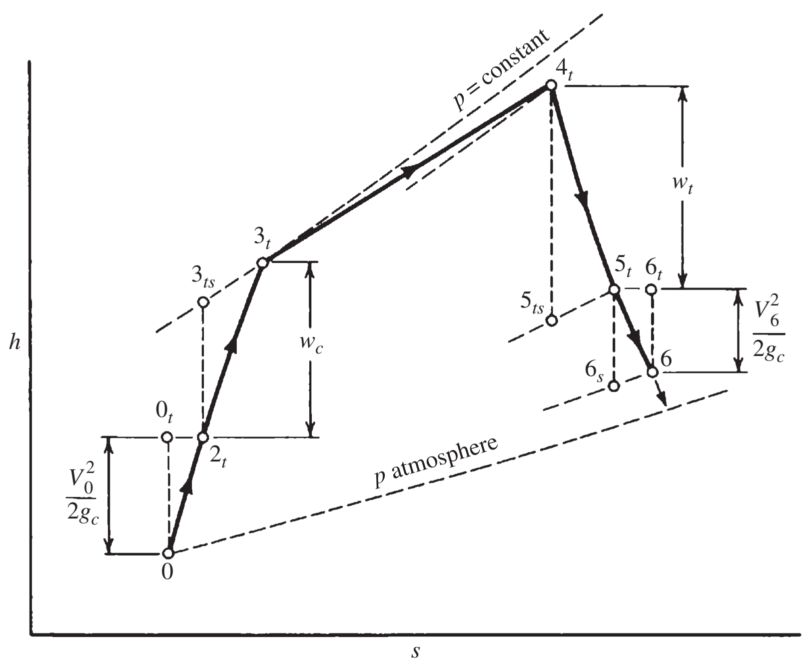


Figure 12.10 h–s diagram for primary air flow of turbofan. (For schematic see Figure 12.9.)

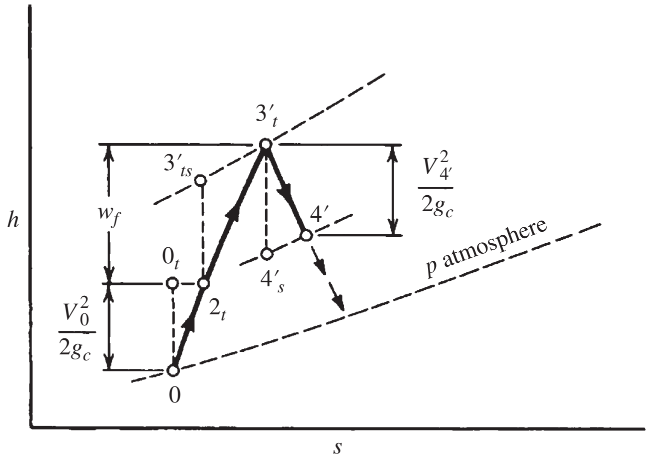


Figure 12.11 *h–s* diagram for secondary air flow of turbofan. (For schematic see Figure 12.9.)

Note that the mass flow rate of the fuel has been neglected in computing the turbine work. This is quite realistic since the air amount bled from the compressor for cabin pressurization and air-conditioning plus operation of auxiliary power amounts to approximately the mass of fuel that is added in the burner.

The following example serves to illustrate the method of analysis for turbojet and turbofan engines. Some simplification is made in that the working medium is treated as a perfect gas with constant specific heats. These assumptions would actually yield fairly satisfactory results if *two* different values of c_p (and γ) were used: one for the cold section (diffuser, compressor, fan, and fan nozzle) and another one for the hot section (turbine and turbine nozzle). For the sake of simplicity we shall use only one value of c_p (and γ) in the example that follows. If more accurate results were desired (as we did in Chapter 11), we could resort to gas tables, which give precise enthalpy versus temperature relations not only for the entering air but also for the particular products of combustion that pass through the turbine and other parts (see Ref. 31).

Example 12.3

A turbofan engine is operating at Mach 0.9 at an altitude of 33,000 ft, where the temperature and pressure are 400°R and 546 psfa (3.79 psia). The engine has a bypass ratio of 3.0 and the primary air flow is 50 lbm/sec. Exit nozzles for both the main and bypass flow are converging-only. Propulsion workers generally use the stagnation-pressure recovery factor versus efficiency for calculating component performance, but in this example, we use the following efficiencies:

$$\eta_c = 0.88 \quad \eta_f = 0.90 \quad \eta_b = 0.96 \quad \eta_t = 0.94 \quad \eta_n = 0.95$$

The total-pressure recovery factor of the diffuser (related to the free stream) is $\eta_r = 0.98$, the compressor total-pressure ratio is 15, the fan total-pressure ratio is 2.5, the maximum allowable turbine inlet temperature is 2500°R, the total-pressure loss in the combustor is 3%, and the heating value of the fuel is 18,900 Btu/lbm. Assume the working medium to have the characteristics of atmospheric air and treat it as a perfect gas with constant-specific heats. Compute the properties at each section (see Figure 12.9 for section numbers). Later in Example 12.5, air is treated as a real gas and the results are compared.

Diffuser:

$$\begin{aligned}
M_0 &= 0.9 \quad T_0 = 400°\text{R} \quad p_0 = 546 \text{ psfa} \\
a_0 &= \sqrt{(1.4)(32.2)(53.3)(400)} = 980 \text{ ft/sec} \\
V_0 &= M_0 a_0 = (0.9)(980) = 882 \text{ ft/sec} \\
p_{t0} &= \frac{p_{t0}}{p_0} p_0 = \left(\frac{1}{0.5913}\right)(546) = 923 \text{ psfa} \\
T_{t0} &= \frac{T_{t0}}{T_0} T_0 = \left(\frac{1}{0.8606}\right)(400) = 465°\text{R} = T_{t2}
\end{aligned}$$

It is common practice to base the performance of an air inlet on the free-stream conditions.

$$p_{t2} = \eta_r p_{t0} = (0.98)(923) = 905 \text{ psfa}$$

Compressor:

$$p_{t3} = 15 p_{t2} = (15)(905) = 13{,}575 \text{ psfa}$$

$$\frac{T_{t3s}}{T_{t2}} = \left(\frac{p_{t3}}{p_{t2}}\right)^{(\gamma-1)/\gamma} = (15)^{0.286} = 2.170$$

$$T_{t3s} = (2.17)(465) = 1009°\text{R}$$

$$\eta_c = \frac{h_{t3s} - h_{t2}}{h_{t3} - h_{t2}} = \frac{T_{t3s} - T_{t2}}{T_{t3} - T_{t3}}$$

Thus

$$T_{t3} - T_{t2} = \frac{1009 - 465}{0.88} = 618°\text{R}$$

and

$$T_{t3} = T_{t2} + 618 = 465 + 618 = 1083°\text{R}$$

Fan:

$$p_{t3'} = 2.5 p_{t2} = (2.5)(905) = 2263 \text{ psfa}$$

$$\frac{T_{t3's}}{T_{t2}} = \left(\frac{p_{t3'}}{p_{t2}}\right)^{(\gamma-1)/\gamma} = (2.5)^{0.286} = 1.300$$

$$T_{t3's} = (1.3)(465) = 604°\text{R}$$

$$T_{t3'} - T_{t2} = \frac{T_{t3's} - T_{t2}}{\eta_f} = \frac{604 - 465}{0.90} = 154.4°\text{R}$$

and

$$T_{t3'} = T_{t2} + 154.4 = 465 + 154.4 = 619°\text{R}$$

Burner:

$$p_{t4} = 0.97 p_{t3} = (0.97)(13,575) = 13,168 \text{ psfa}$$

$$T_{t4} = 2500°\text{R} \,(\text{max. allowable})$$

An energy analysis of the burner reveals

$$\left(\dot{m}_f + \dot{m}_a\right) h_{t3} + \eta_b (\text{HV}) \dot{m}_f = \left(\dot{m}_f + \dot{m}_a\right) h_{t4} \tag{12.30}$$

where

$$\text{HV} \equiv \text{heating value of the fuel}$$

$$\eta_b \equiv \text{combustion efficieny}$$

Let $f \equiv \dot{m}_f / \dot{m}_a$ denote the fuel–air ratio. Then

$$\eta_b (\text{HV}) f = (1 + f) c_p (T_{t4} - T_{t3}) \tag{12.31}$$

or

$$f = \frac{1}{\dfrac{\eta_b (\text{HV})}{c_p (T_{t4} - T_{t3})} - 1} = \frac{1}{\dfrac{(0.96)(18,900)}{(0.24)(2500 - 1083)} - 1} = 0.0191$$

Turbine: If we neglect the mass of fuel added, we have from equation (12.29) (for constant-specific heats):

$$(T_{t4} - T_{t5}) = (T_{t3} - T_{t2}) + \beta(T_{t3'} - T_{t2})$$
$$T_{t4} - T_{t5} = (1083 - 465) + (3)(619 - 465) = 1080°\text{R}$$

and

$$T_{t5} = T_{t4} - 1080 = 2500 - 1080 = 1420°\text{R}$$

and

$$\eta_t = \frac{h_{t4} - h_{t5}}{h_{t4} - h_{t5s}} = \frac{T_{t4} - T_{t5}}{T_{t4} - T_{t5s}}$$
$$T_{t4} - T_{t5s} = \frac{1080}{0.94} = 1149°\text{R}$$

and

$$T_{t5s} = T_{t4} - 1149 = 2500 - 1149 = 1351°\text{R}$$
$$\frac{p_{t4}}{p_{t5}} = \left(\frac{T_{t4}}{T_{t5s}}\right)^{\gamma/(\gamma-1)} = \left(\frac{2500}{1351}\right)^{3.5} = 8.62$$
$$p_{t5} = \frac{p_{t4}}{8.62} = \frac{13,168}{8.62} = 1528 \text{ psfa}$$

Turbine nozzle: The operating pressure ratio for the nozzle will be

$$\frac{p_0}{p_{t5}} = \frac{546}{1528} = 0.357 < 0.528$$

which means that the nozzle is choked and has sonic speed at the exit.

$$T_{t6} = T_{t5} = 1420°\text{R} \quad M_6 = 1 \quad \text{and thus} \quad \frac{T_6}{T_{t6}} = 0.8333$$
$$T_6 = (0.8333)(1420) = 1183°\text{R}$$
$$V_6 = a_6\sqrt{(1.4)(32.2)(53.3)(1183)} = 1686 \text{ ft/sec}$$
$$\eta_n = \frac{h_{t5} - h_6}{h_{t5} - h_{6s}} = \frac{T_{t5} - T_6}{T_{t5} - T_{6s}}$$

Thus

$$T_{t5} - T_{6s} = \frac{1420 - 1183}{0.95} = \frac{237}{0.95} = 249°\text{R}$$

and

$$T_{6s} = T_{t5} - 249 = 1420 - 249 = 1171°\text{R}$$

$$\frac{p_{t5}}{p_{6s}} = \left(\frac{T_{t5}}{T_{6s}}\right)^{\gamma/(\gamma-1)} = \left(\frac{1420}{1171}\right)^{3.5} = 1.964$$

$$p_6 = p_{6s} = \frac{p_{t5}}{1.964} = \frac{1528}{1.964} = 778 \text{ psfa}$$

Fan nozzle:

$$\frac{p_0}{p_{t3'}} = \frac{546}{2263} = 0.241 < 0.528 \quad \text{(nozzle is choked)}$$

$$T_{t4'} = T_{t3'} = 619°\text{R}$$

$$M_{4'} = 1 \quad T_{4'} = (0.8333)(619) = 516°\text{R}$$

$$V_{4'} = a_{4'} = \sqrt{(1.4)(32.2)(53.3)(516)} = 1113 \text{ ft/sec}$$

$$T_{t3'} - T_{4's} = \frac{T_{t3'} - T_{4'}}{\eta_n} = \frac{619 - 516}{0.95} = 108°\text{R}$$

$$T_{4's} = 619 - 108 = 511°\text{R}$$

$$\frac{p_{t3'}}{p_{4's}} = \left(\frac{T_{t3'}}{T_{4's}}\right)^{\gamma/(\gamma-1)} = \left(\frac{619}{511}\right)^{3.5} = 1.956$$

$$p_{4'} = p_{4's} = \frac{2263}{1.956} = 1157 \text{ psfa}$$

In Section 12.6 we continue this example to determine the thrust and other performance parameters of the engine. In Section 12.7, we compare our results with computer software.

Turboprop

Figure 12.12 shows a cutaway picture of a typical tuboprop engine. The schematic in Figure 12.13 helps identify the basic parts and indicates the important section locations. It is quite similar to the turbofan engine except for the following:

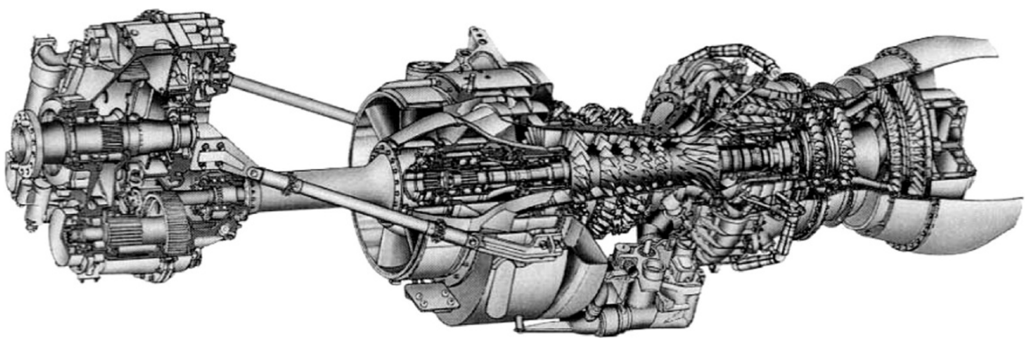


Figure 12.12 Cutaway view of a turboprop engine. (Courtesy of General Electric Aircraft Engines.) [Additional images can be found by searching the Web for: "turbojet turbofan turboprop images"]

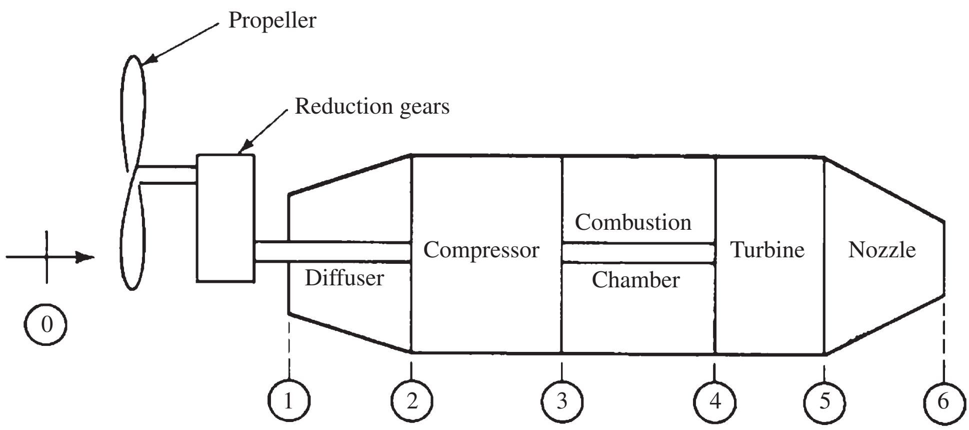


Figure 12.13 Basic parts of a turboprop engine.

1. As much power as possible is developed in the turbine, and thus more power is available to operate the propeller. In essence, the engine is operating as a stationary power plant—but on an open cycle.
2. The propeller operates through reduction gears at relatively low rpm (revolutions per minute) value compared with a fan.

As a result of extracting so much power from the turbine, very little expansion can take place in the nozzle, and consequently, the exit gas velocity is relatively low. Thus little thrust (about 10–20% of the total) is obtained from the jet.

On the other hand, the propeller accelerates very large quantities of air (compared with the turbofan and turbojet) through a very small velocity differential. This results in an extremely efficient propulsion device for the lower subsonic

flight regime. Another operating characteristic of a propeller-driven aircraft is that of high thrust and power available for takeoff. The turboprop engine is both considerably smaller in diameter and lighter in weight than a reciprocating engine of comparable power output.

Ramjet

The ramjet cycle is basically the same as that of the turbojet. Air enters the diffuser and most of its kinetic energy is converted into a pressure rise. If the flight speed is supersonic, part of this compression actually occurs across a shock system that precedes the inlet (see Figure 7.15). Because flight speeds must be high, sufficient compression can be attained at the inlet and in the diffusing section, and thus a compressor is not needed. Once the compressor is eliminated, the turbine is no longer required and can also be omitted. The result is a ramjet engine, which is shown schematically in Figure 12.14.

The combustion region in a ramjet is generally a large single chamber, similar to an afterburner. Since the cross-sectional area is relatively small, velocities are much higher in the combustion zone than are experienced in a turbojet. Thus *flame holders* (similar to those used in afterburners) must be introduced to stabilize the flame and prevent blowouts. Some experimental work has been carried out with solid-fuel ramjets. Supersonic combustion would simplify the diffuser (and eliminate much loss), but results to date have not been fruitful.

Although a ramjet engine can operate at speeds as low as $M_0 = 0.2$, the fuel consumption is poor at these low velocities. The operation of a ramjet does not become competitive with that of a turbojet until speeds of about $M_0 = 2.5$ or above are reached. Another disadvantage of a ramjet is that it cannot operate at zero flight speed and thus requires some auxiliary means of take-off; it needs to be dropped from a carrier plane or launched by rocket assist. A suitable combination of turbojet and ramjet engines for high-speed piloted craft would solve the launch problem as well as the inefficient operation at low speeds. The YF12, shown in

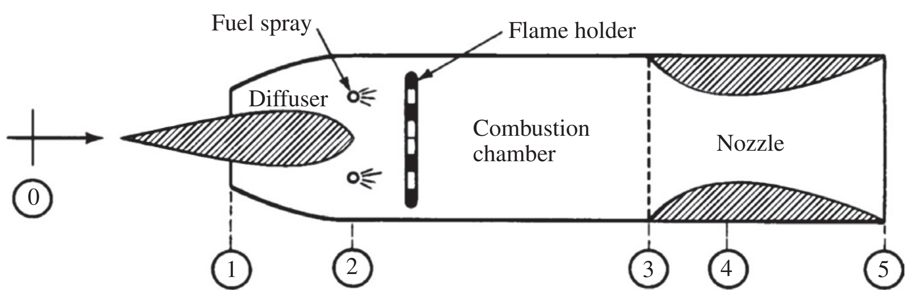


Figure 12.14 Basic components of a ramjet engine. [Ramjet images can be found by searching the web for: “ramjet pulsejet images”]

Figure 7.17, acts as a ramjet by bypassing the incoming air to the afterburner at $M_0 > 2.0$.

The ramjet was invented in 1913 by a Frenchman named Lorin. Various other patents were obtained in England and Germany in the 1920s. The first plane to be powered by a ramjet was designed in France by Leduc in 1938, but its construction was delayed by World War II and it did not fly until 1949. Ramjets are very simple and lightweight and thus are ideally suited as expendable engines for high-speed target drones or guided missiles.

Example 12.4

A ramjet has a flight speed of $M_0 = 1.8$ at an altitude of 13,000 m, where the temperature is 218 K and the pressure is 1.7×10^4 N/m^2 (17.0 kPa). Assume a two-dimensional inlet with a deflection angle of 10° (Figure E12.4). Neglect frictional losses in the diffuser and combustion chamber. The inlet area is $A_1 = 0.2$ m^2; sufficient fuel is added to increase the total temperature to 2225 K. The heating value of the fuel is 4.42×10^7 J/kg with $\eta_b = 0.98$. The nozzle expands to atmospheric pressure for maximum thrust with $\eta_n = 0.96$. The velocity entering the combustion chamber is to be kept as large as possible but not greater than $M_2 = 0.25$.

Assume the fluid to be air and treat it as a perfect gas with $\gamma = 1.4$. Compute significant properties at each section, mass flow rate, fuel–air ratio, and diffuser total-pressure recovery factor.

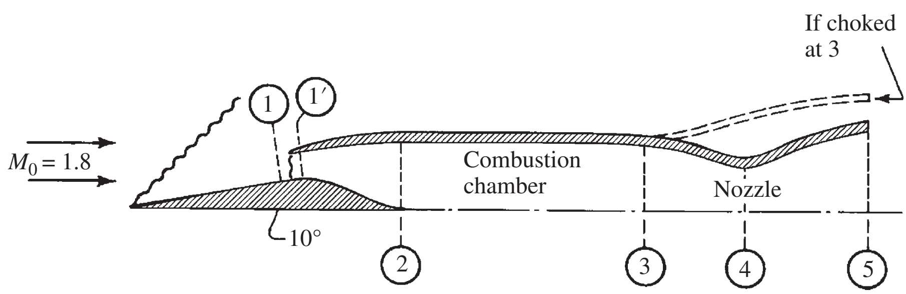


Figure E12.4

Oblique shock: For $M_0 = 1.8$, $\delta = 10°$ with $\theta = 44°$:

$$M_{0n} = M_0 \quad \sin\theta = 1.8 \quad \sin 44° = 1.250$$

$$M_{1n} = 0.8126 \quad \frac{p_1}{p_0} = 1.6562 \quad \frac{T_1}{T_0} = 1.1594$$

$$M_1 = \frac{M_{1n}}{\sin(\theta - \delta)} = \frac{0.8126}{\sin(44 - 10)} = 1.453$$

Normal shock: For $M_1 = 1.453$:

$$M_{1'} = 0.7184 \quad \frac{p_{1'}}{p_1} = 2.2964 \quad \frac{T_{1'}}{T_1} = 1.2892$$

$$p_{1'} = \frac{p_{1'}}{p_1}\frac{p_1}{p_0}p_0 = (2.2964)(1.6562)p_0 = 3.803p_0$$

$$T_{t2} = T_{t0} = T_0\frac{T_{t0}}{t_0} = (218)\left(\frac{1}{0.6068}\right) = 359.3 \text{ K}$$

Rayleigh flow: If $M_2 = 0.25$:

$$T_t^* = T_{t2}\frac{T_t^*}{T_{t2}} = (359.3)\left(\frac{1}{0.2568}\right) = 1399 \text{ K}$$

Thus, adding fuel to make $T_{t3} = 2225$ K means that the flow is choked ($M_3 = 1.0$) and $M_2 < 0.25$. We proceed to find M_2.

$$\frac{T_{t2}}{T_t^*} = \frac{T_{t2}}{T_{t3}}\frac{T_{t3}}{T_t^*} = \left(\frac{359.3}{2225}\right)(1) = 0.1615$$

$$M_2 = 0.192$$

Diffuser:

$$p_2 = \frac{p_2}{p_{t2}}\frac{p_{t2}}{p_{t1'}}\frac{p_{t1'}}{p_{1'}}p_{1'} = (0.9746)(1)\left(\frac{1}{0.7091}\right)(3.803p_0) = 5.227\,p_0$$

$$T_2 = \frac{T_2}{T_{t2}}T_{t2} = (0.9927)(359.3) = 356.7 \text{ K}$$

Combustion chamber:

$$p_3 = p^* = \frac{p^*}{p_2}p_2 = \left(\frac{1}{2.2822}\right)(5.227p_0) = 2.29p_0$$

$$T_3 = T_{t3}\frac{T_3}{T_{t3}} = (2225)(0.8333) = 1854\,\text{K}$$

Nozzle: Since $M_3 = 1.0$, the nozzle diverges immediately.

$$T_{5s} = T_3\left(\frac{p_3}{p_{5s}}\right)^{(1-\gamma)/\gamma} = (1854)\left(\frac{2.29p_0}{p_0}\right)^{(1-1.4)/1.4} = 1463\,\text{K}$$

$$T_5 = T_3 - \eta_n(T_3 - T_{5s}) = 1854 - (0.96)(1854 - 1463) = 1479\,\text{K}$$

$$\frac{T_5}{T_{t5}} = \frac{1479}{2225} = 0.6647 \quad \text{and} \quad M_5 = 1.588$$

Flow rate:

$$p_1 = \frac{p_1}{p_0} p_0 = (1.6562)\left(1.7 \times 10^4\right) = 2.846 \times 10^4 \ \mathrm{N/m^2}$$

$$T_1 = \frac{T_1}{T_0} T_0 = (1.1594)(218) = 253 \ \mathrm{K}$$

$$\rho_1 = \frac{p_1}{RT_1} = \frac{2.816 \times 10^4}{(287)(253)} = 0.388 \ \mathrm{kg/m^3}$$

$$V_1 = M_1 a_1 = (1.453)[(1.4)(1)(287)(253)]^{1/2} = 463 \ \mathrm{m/s}$$

$$\dot{m} = \rho_1 A_1 V_1 = (0.388)(0.2)(463) = 35.9 \ \mathrm{kg/s}$$

Fuel–air ratio:

$$f = \frac{1}{\dfrac{\eta_b(\mathrm{HV})}{c_p(T_{t3} - T_{t2})} - 1} = \frac{1}{\dfrac{(0.98)\left(4.42 \times 10^7\right)}{(1000)(2225 - 359.3)} - 1} = 0.0450$$

Total-pressure recovery factor:

$$\eta_r = \frac{p_{t2}}{p_{t0}} = \frac{p_{t2}}{p_2}\frac{p_2}{p_0}\frac{p_0}{p_{t0}} = \left(\frac{1}{0.9746}\right)\left(\frac{5.227 p_0}{p_0}\right)(0.17404) = 0.933$$

In Section 12.6 we continue with this example to determine the thrust and other performance parameters.

Pulsejet

The turbojet, turbofan, turboprop, and ramjet all operate on variations of the Brayton cycle. The pulsejet is a totally different device and is shown in Figure 12.15.

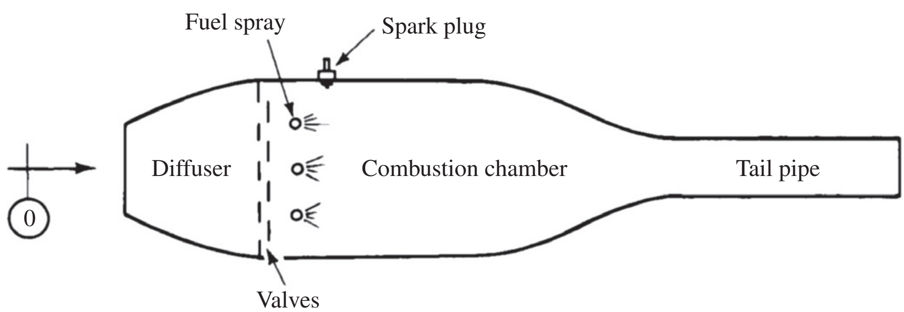


Figure 12.15 Basic components of a pulsejet engine. [Pulsejet images can be found by searching the web for: "ramjet pulsejet images"]

A key feature in the design of the pulsejet is a bank of spring-loaded check valves that forms the wall between the diffuser and the combustion chamber. These valves are normally closed, but when a predetermined pressure differential appears, they will open to permit high-pressure air from the diffusing section to pass into the combustion chamber. They cannot allow flow from the chamber back into the diffuser. A spark plug initiates combustion, which occurs at a rate approaching a *constant-volume process*. The resultant high temperature and pressure cause the gases to flow out the tail pipe at high velocity. The inertia of the exhaust gases creates a slight vacuum in the combustion chamber. This vacuum combined with the ram pressure developed at the inlet diffuser causes sufficient pressure differential to reopen the check valves. A new volume of air enters the chamber and the cycle repeats. The frequency of the cycle above depends on the size of the engine, and the dynamic characteristics of the check valves must be matched carefully to this frequency. Small engines operate as high as 300–400 cycles per second, and large engines have been built that operate as low as 40 cycles per second.

The idea of a pulsejet originated in France in 1906, but the modern configuration was not developed until the early 1930s in Germany. Perhaps the most famous pulsejet was the V-1 engine that powered the German "buzz bombs" during World War II. The speed range of pulsejets is limited to the subsonic regime since the large frontal area required (because the air is admitted intermittently) causes high drag. Its extreme noise and vibration levels render it useless for piloted craft. However, its ability to develop thrust at zero speed gives it a clear-cut advantage over the ramjet.

Rocket

All the propulsion systems discussed so far belong to the category of air-breathing engines. As such, their application is limited to earth-altitudes below about 100,000 ft. On the other hand, because rockets carry on board oxidizer as well as fuel, they can function within and/or outside the atmosphere. Schematics of common rocket engines are shown in Figure 12.16. The chemical rocket propellants at the combustion chamber are either *solid* or *liquid*, or a combination of the two. In a liquid system the fuel and oxidizer are separately stored and are injected under high pressure (about 300–800 psia) into the combustion chamber for burning to take place. When solid propellants are used, both fuel and oxidizer are premixed in the propellant grain and burning takes place at the propellant surface. In this figure the *combustion chamber* continually increases in volume as propellant burning takes place. Most solid propellants are *internal burning*, as shown in Figure 12.16*a*, whereas some simpler ones are *end burning* (like a cigarette). Solid propellants develop chamber pressures of from about 500 to 3000 psia.

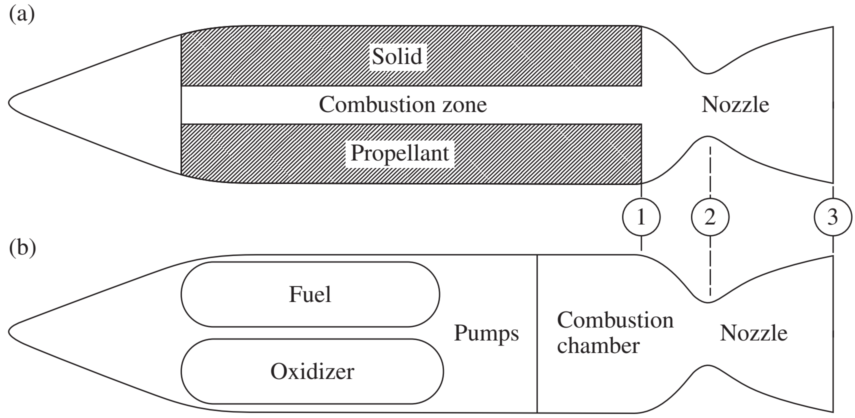


Figure 12.16 Basic components of a rocket engine: (a) Solid-propellant rocket. (b) Liquid-propellant rocket. (See also Figures 5.9 and 5.10) [Rocket engine images can be found by searching the Web for: "rockets solid liquid propellant images"]

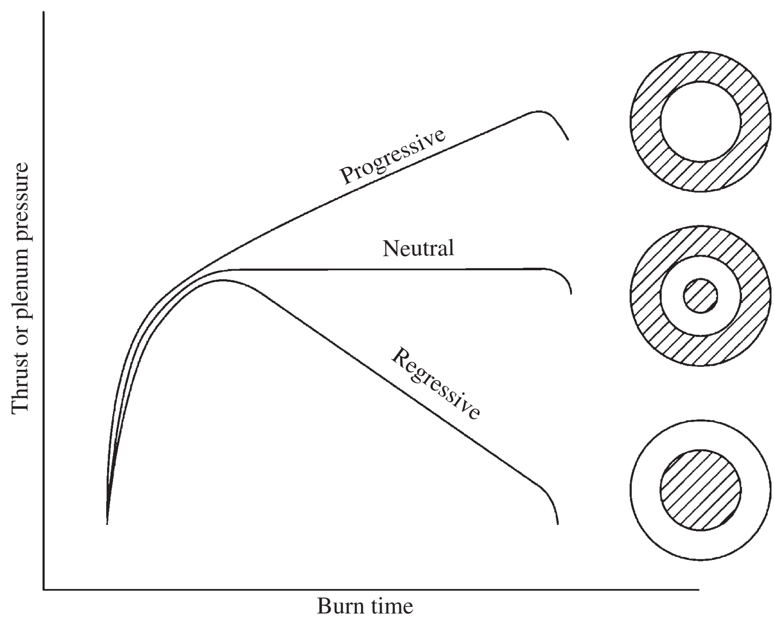


Figure 12.17 Typical thrust profiles and corresponding cross sections for solid propellants.

Figure 12.17 shows some common thrust profiles that can result from the solid propellants configurations depicted. *Neutral burning* is based on constant-burning area, which is closely accomplished with certain specific propellant geometries. Similarly, both *progressive* and *regressive burning* depend on propellant cross

section. All such burning profiles affect the mass flow rate and thus the acceleration of the rocket so that any ultimate mission must be designed into the propellant grain configuration. The products of combustion are exhausted through a converging–diverging nozzle with exit velocities ranging from about 5000 to 10,000 ft/sec. The high temperatures reached during combustion plus the high rates of fuel consumption limit rocket motor applications to short times (on the order of seconds or minutes).

Liquid propellant rocket engines can be throttled, and this is of great importance to certain missions, particularly manned missions to space. Liquids significantly outperform solids and can range in thrust from micropounds (or μN) to megapounds (or MN). They tend to be very complex and must be carefully checked out prior to operation but their exhaust gases can be nontoxic and some hardware is reusable. Solids, on the other hand, are considerably less expensive than liquids and are preferred in "throwaway missions" such as sounding rockets and military rockets. Although some thrust variation is possible with solids, this must always be preprogrammed, and in general, once the solid is started, accidentally or otherwise, it cannot be shut off. Solids have been successfully designed to remain in storage for several years, which is a great advantage over cryogenic liquid propellants that can only be fueled prior to their use. Also, solid motor systems can be packaged very compactly for less drag and can be quickly activated if necessary.

The invention of solid propellant rockets is generally attributed to the Chinese in around the year 1000, although there is some evidence that rockets may have been used by the Greeks as much as 500 years earlier. The father of modern liquid rockets is generally considered to be an American named Robert Goddard. His experiments started in 1915 and extended well into the 1930s. Some of the first successful American rockets were the JATO (jet-assisted take-off) units used during WWII (solid in 1941 and liquid in 1942). Also famous was the V-2 rocket developed by Wernher von Braun in Germany. This first flew in 1942 and had a liquid propulsion system that developed 56,000 lbf of thrust. The first rocket-propelled aircraft was the German ME-163. Incidentally, one thing that rockets do have in common with air-breathing engines is that heat addition in the combustion chamber takes place at constant pressure.

12.5 GENERAL PERFORMANCE PARAMETERS, THRUST, POWER, AND EFFICIENCY

In this section we analyze propulsion systems in order to develop a general expression for their net propulsive thrust. We then continue by examining some significant performance parameters, such as power and efficiency.

Thrust Considerations

Consider an airplane or missile that is traveling to the left at a constant velocity V_0 as shown in Figure 12.18. The thrust force results from the interaction between the fluid and the propulsive device. The fluid pushes on the propulsive device and provides thrust to the left, or in the direction of motion, whereas the propulsive device pushes on the fluid opposite to the direction of flight.

Analysis of Fluid

We start by analyzing the fluid as it passes through the propulsive device. We define a control volume that surrounds all the fluid inside the propulsion system (see Figure 12.19). Velocities are shown relative to the device, which is used as a frame of reference in order to make a steady-flow picture. The x-component of the momentum equation for *steady flow* is [from equation (3.42)]

$$\sum F_x = \int_{\text{cs}} \frac{\rho V_x}{g_c} (\mathbf{V} \cdot n) dA \tag{12.32}$$

and for *one-dimensional flow* this becomes

$$\sum F_x = \frac{\dot{m}_2 V_{2x}}{g_c} - \frac{\dot{m}_1 V_{1x}}{g_c} \tag{12.33}$$

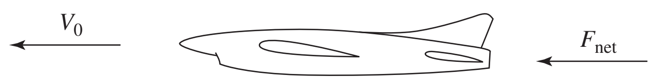


Figure 12.18 Direction of flight and net propulsive force in air breathing systems.

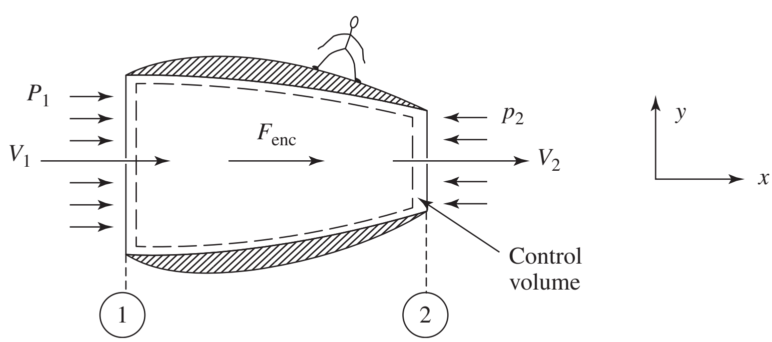


Figure 12.19 Forces *on the fluid* inside the propulsion system.

We define an *enclosure force* as the vector sum of the friction forces and the pressure forces of the wall on the fluid within the control volume. We shall designate F_{enc} as the x-component of this enclosure force on the fluid inside the control volume. Then

$$\sum F_x = F_{\text{enc}} + p_1 A_1 - p_2 A_2 \tag{12.34}$$

and

$$p_1 A_1 - p_2 A_2 + F_{\text{enc}} = \frac{\dot{m}_2 V_2}{g_c} - \frac{\dot{m}_1 V_1}{g_c} \tag{12.35}$$

or

$$F_{\text{enc}} = \left(p_2 A_2 + \frac{\dot{m}_2 V_2}{g_c} \right) - \left(p_1 A_1 + \frac{\dot{m}_1 V_1}{g_c} \right) \tag{12.36}$$

Notice that the newly defined enclosure force, which is an extremely complicated summation of internal pressure and friction forces, may be easily expressed in terms of known quantities at the inlet and exit. This shows the great capability of the momentum equation. You may recall from Chapter 10 [see equation (10.11)] that the combination of variables found in equation (12.36) is called the *impulse or thrust function*. Perhaps now you can see a reason for this name.

Analysis of Enclosure

We now analyze the forces on the enclosure or the propulsive device. If the enclosure is pushing on the fluid with a force of magnitude F_{enc} to the right, the fluid must be pushing on the enclosure with a force of equal magnitude to the left. This is the internal reaction of the fluid and is shown in Figure 12.20 as F_{int}:

$$\begin{gathered} F_{\text{int}} = \text{positive thrust on enclosure from internal forces} \\ |F_{\text{int}}| = |F_{\text{enc}}| \end{gathered} \tag{12.37}$$

In Figure 12.20 we have indicated the external forces as being ambient pressure over the entire enclosure. At first you might think that this is incorrect since the pressure is not constant over the external surface. Furthermore, we have not shown any friction forces over the external surface. The answer is that these differences are accounted for when the drag forces are computed, since the drag force includes

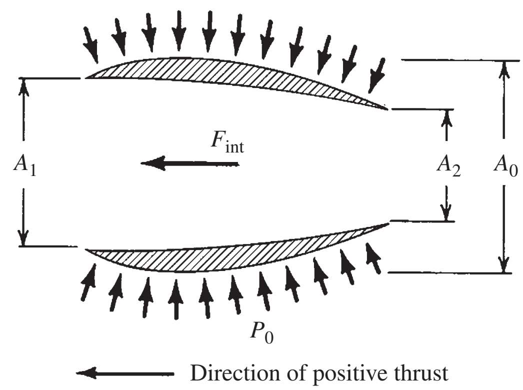


Figure 12.20 Forces *on the propulsion device.*

an integration of the shear stresses along the surface and also a pressure drag term, which is normally put in the following form:

$$\text{Pressure drag} = \int_1^2 (p - p_0)\, dA_x \tag{12.38}$$

In equation (12.38) the integration is carried out over the entire external surface of the device and dA_x represents the projection of the increment of area on a plane perpendicular to the x-axis.

We define F_{ext} as the positive thrust that arises from the external forces pushing on the enclosure:

$$F_{\text{ext}} \equiv \text{positive thrust on enclosure from external forces}$$

Since this has been represented as a constant pressure, the integration of these forces is quite simple:

$$F_{\text{ext}} = p_0(A_0 - A_2) - p_0(A_0 - A_1) = p_0(A_1 - A_2) \tag{12.39}$$

The first term in the center expression represents *positive* thrust from the pressure forces over the rear portion of the propulsive device. The second term represents *negative* thrust from the pressure forces acting over the forward portion.

The *net positive thrust* on the propulsive device is the sum of the internal and external forces:

$$F'_{\text{net}} = F_{\text{int}} + F_{\text{ext}} \tag{12.40}$$

Show that the net positive thrust can be expressed as

$$F'_{\text{net}} = \left(p_2 A_2 + \frac{\dot{m}_2 V_2}{g_c} \right) - \left(p_1 A_1 + \frac{\dot{m}_1 V_1}{g_c} \right) + p_0(A_1 - A_2) \tag{12.41}$$

or

$$\boxed{F'_{\text{net}} = \frac{\dot{m}_2 V_2}{g_c} - \frac{\dot{m}_1 V_1}{g_c} + A_2(p_2 - p_0) - A_1(p_1 - p_0)} \tag{12.42}$$

Notice that equation (12.42) has been left in a general form and as such can apply to all cases (i.e., $\dot{m}_2$ can be different from $\dot{m}_1$ if it is desired to account for the fuel added, p_2 may be different than p_0 for the case of sonic or supersonic exhausts, and p_1 may not be the same as p_0). If $p_1 \neq p_0$, then $V_1 \neq V_0$. An example of this is shown for subsonic flight in Figure 12.21. Here, the flow system is choked and an external diffusion with flow *spill-over* occurs. The fluid that actually enters the engine is said to be contained within the *pre-entry streamtube*.

It is customary in the field of propulsion to work with the free-stream conditions (p_0 and V_0) that exist far ahead of the actual inlet. Thus, by applying equation (12.42) between sections 0 and 2 (instead of between 1 and 2), we obtain a simpler expression which is much more convenient to use. We call this the *net propulsive thrust*:

$$\boxed{F_{\text{net}} = \frac{\dot{m}_2 V_2}{g_c} - \frac{\dot{m}_0 V_0}{g_c} + A_2(p_2 - p_0)} \tag{12.43}$$

It should be clearly noted that equations (12.42) and (12.43) are *not* equal since the last one, in effect, considers the region from zero to 1 as a part of the propulsive device. Thus, this equation includes the *pre-entry thrust*, or the propulsive force that the internal fluid exerts on the boundary of the pre-entry streamtube. This error will be compensated for when the drag is computed since the pressure drag must now be integrated from 0 to 2 as follows:

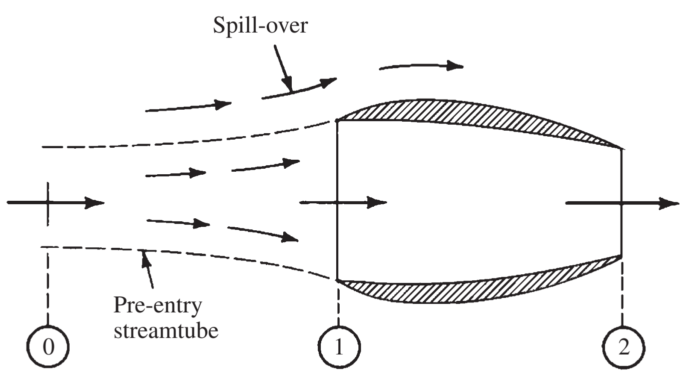


Figure 12.21 ExternaÙl diffusion prior to inlet for air-breathing systems.

$$\text{pressure drag} = \int_0^1 (p-p_0)dA_x + \int_1^2 (p-p_0)dA_x \tag{12.44}$$

The integral from 0 to 1, called the *pre-entry drag* or *additive drag*, exactly balances the pre-entry thrust if the flow is as pictured in Figure 12.21.

Power Considerations

There are three different measures of power connected with propulsion systems:

1. Input power
2. Propulsive power
3. Thrust power

Consideration of these power quantities enables us to separate the performance of the thermodynamic cycle from that of the propulsion element. The general relationship among these various power quantities is shown in Figure 12.22. The thermodynamic cycle is concerned with input power and propulsive power, whereas the propulsive device is the link between the propulsive power and the thrust power.

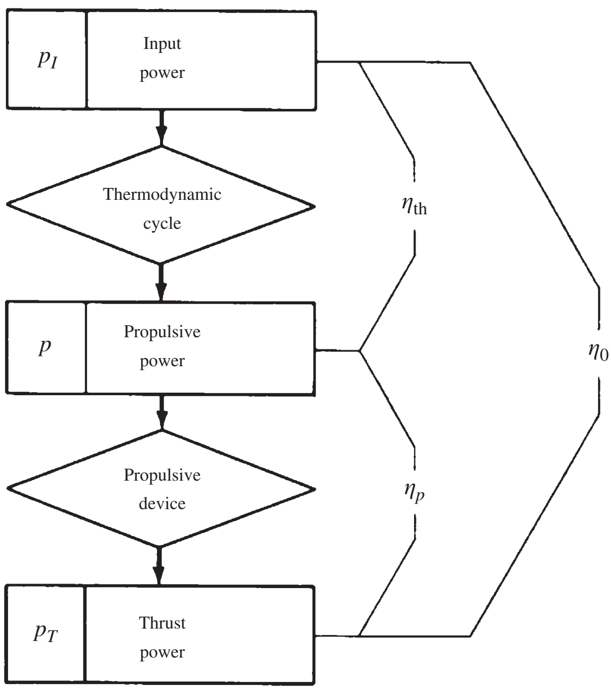


Figure 12.22 Power quantities of a propulsion system.

The *power input* to the working fluid, designated as P_I is the rate at which heat or chemical energy is supplied to the system (HV = fuel *heating value*). This energy is the input to the thermodynamic cycle:

$$P_I = \dot{m}_f(\mathrm{HV}) \tag{12.45}$$

The output of the cycle is the input to the propulsion element and is designated as P and called *propulsive power*. In the case of propeller-driven systems, the propulsive power is easily visualized, as it is the shaft power supplied to the propeller. For other systems, the propulsive power can be viewed as the change in kinetic energy rate of the working medium as it passes through the system:

$$P = \Delta\dot{\mathrm{KE}} = \frac{\dot{m}_2 V_2^2}{2g_c} - \frac{\dot{m}_0 V_0^2}{2g_c} \tag{12.46}$$

The *thrust power* output of the propulsive device is the actual rate of doing useful propulsion work and is designated as P_T:

$$P_T = F_{\mathrm{net}} V_0 \tag{12.47}$$

It is generally easier to compute the propulsive power by noting that the difference between the propulsive power and the thrust power is the *lost power*, P_L, or

$$P = P_T + P_L \tag{12.48}$$

A major loss here is the *absolute* kinetic energy of the exit jet, and this is an unavoidable loss, even for a flawless propulsion system. In addition to this, some other energy may be unavailable for thrust purposes. For instance, the exhaust jet may not all be directed axially, or it may have a swirl component. In any event, the *minimum* power loss can be computed as follows:

$$\begin{aligned} V_2 - V_0 &= \text{absolute velocity of exit jet} \\ P_{L\,\min} &= \frac{\dot{m}_2}{2g_c}(V_2 - V_0)^2 \end{aligned} \tag{12.49}$$

Efficiency Considerations

The identification of the power quantities P_I, P, and P_T permits various efficiency factors to be defined. These are also indicated in Figure 12.22.

Thermal efficiency:

$$\eta_{\text{th}} \equiv \frac{P}{P_I} \tag{12.50}$$

Propulsive efficiency:

$$\eta_p \equiv \frac{P_T}{P} = \frac{P_T}{P_T + P_L} \tag{12.51}$$

Overall efficiency:

$$\eta_0 \equiv \frac{P_T}{P_I} = \eta_{\text{th}}\eta_p \tag{12.52}$$

The *thermal efficiency* indicates how well the thermodynamic cycle converts the chemical energy of the fuel into work that is available for propulsion. The *propulsive efficiency* indicates how well this work is actually utilized by the thrust device to propel the vehicle. An alternative form of propulsive efficiency is shown in terms of the lost power. The *overall efficiency* is a performance index for the entire propulsion system. Be careful to always use consistent units when computing any of these efficiency factors, which are always dimensionless.

12.6 AIR-BREATHING PROPULSION SYSTEMS PERFORMANCE PARAMETERS

We start with the basic thrust equation

$$F_{\text{net}} = \frac{\dot{m}_2 V_2}{g_c} - \frac{\dot{m}_0 V_0}{g_c} + A_2(p_2 - p_0) \tag{12.43}$$

For the purposes of examining the characteristics of air-breathing jet engines, we can make two simplifying assumptions:

1. Most operate at low fuel–air ratios, and some of the high-pressure air is bled off to run the auxiliaries. Thus, we may assume that the flow rates $\dot{m}_2$ and $\dot{m}_0$ are approximately equal.
2. For most systems, the pressure thrust term $A_2(p_2 - p_0)$ is a small portion of the overall net thrust and may be dropped.

Under these assumptions, the net thrust becomes

$$\boxed{F_{\text{net}} = \frac{\dot{m}}{g_c}(V_2 - V_0)} \tag{12.53}$$

This form of the thrust equation reveals an interesting characteristic of all air-breathing propulsion systems. As their flight speed approaches the exhaust velocity, the thrust goes to zero. Even long before reaching this point, the thrust drops below the drag force (which increases rapidly with flight speed). Because of this, *no air-breathing propulsion system can ever fly faster than its exit jet*!

This equation also helps explain the natural operating speed range of various engines. Recall that the turboprop provides a small velocity change to a very large mass of air. Thus its exit jet has quite a low velocity, which limits the system to low-speed operation. At the other end of the spectrum, we have the turbojet (or pure jet), which provides a large velocity increment to a relatively small mass of air. Therefore, this device operates at much higher flight speeds.

We return now to the basic thrust equation [see equation (12.43)]. Using equation (12.47), the thrust power can be written as

$$P_T = F_{\text{net}} V_0 = \left[\frac{\dot{m}_2 V_2}{g_c} - \frac{\dot{m}_0 V_0}{g_c} + A_2(p_2 - p_0) \right] V_0 \tag{12.54}$$

Let us examine an *ideal* jet-propulsion system, one in which there are *no unavoidable losses*. As before, we neglect the difference between $\dot{m}_0$ and $\dot{m}_2$ and drop the pressure contribution to the thrust. Equation (12.54) then becomes

$$P_T = \frac{\dot{m}_0 V_0}{g_c}(V_2 - V_0) \tag{12.55}$$

Looking at equation (12.55), we can see that the thrust power of an air-breather is zero when the flight speed is either zero or equal to V_2. In the former case, we have a high thrust but no motion, thus no thrust power. In the latter case the thrust is reduced to zero.

Somewhere between these extremes there must be a point of maximum thrust power. To find this condition, we differentiate equation (12.55) with respect to V_0, keeping V_2 constant. Setting this equal to zero reveals that *maximum thrust power* results when

$$V_2 = 2V_0$$

From equations (12.51), (12.49), and (12.47), the propulsive efficiency becomes

$$\eta_p = \frac{P_T}{P_T + P_L} = \frac{\left[\dfrac{\dot{m}_2 V_2}{g_c} - \dfrac{\dot{m}_0 V_0}{g_c} + A_2(p_2 - p_0)\right] V_0}{\left[\dfrac{\dot{m}_2 V_2}{g_c} - \dfrac{\dot{m}_0 V_0}{g_c} + A_2(p_2 - p_0)\right] V_0 + \dfrac{\dot{m}_2}{2g_c}(V_2 - V_0)^2} \tag{12.56}$$

We again neglect the difference between $\dot{m}_0$ and $\dot{m}_2$ and drop the pressure term. With these assumptions the propulsive efficiency becomes

$$\eta_p = \frac{V_0}{V_0 + \dfrac{1}{2}(V_2 - V_0)} \tag{12.57}$$

This relation can be further simplified with the introduction of the speed ratio ν:

$$\nu \equiv \frac{V_0}{V_2} \tag{12.58}$$

Show that under these conditions equation (12.57) can be written as

$$\boxed{\eta_p = \frac{2\nu}{1 + \nu}} \tag{12.59}$$

This shows that the propulsive efficiency for air-breathers continually increases with flight speed, reaching the value of 1.0 when $\nu = 1.0$ [or when $V_0 = V_2$ which cannot be exceeded, see equation (12.53)]. This is quite reasonable since under this condition the absolute velocity of the exit jet is zero and there is no exit loss [see equation (12.49)].

At this point you can begin to see some of the problems involved in optimizing air-breathing jet propulsion systems. We showed previously that maximum thrust power is attained when $V_2 = 2V_0$. Now we see that maximum propulsive efficiency is attained when $V_2 = V_0$, but unfortunately, for the latter case the thrust is zero. Remember that the relations in this section apply only to air-breathing propulsion systems. Equation (12.59) further confirms the natural operating speed range of the various turbojet engines. Recall that a pure jet provides a large velocity change to a relatively small mass of air. Thus, as stated earlier, to have a high propulsive efficiency ($\nu \to 1$) it must fly at high speeds. The fanjet provides a moderate velocity increment to a larger mass of air. Thus, it is more efficient at medium flight speeds. By providing a small velocity increment to a very large mass of air, the turboprop is well suited to low-speed operation.

Specific Fuel Consumption

Specific fuel consumption is a useful overall performance indicator for air-breathing engines. For a propeller-driven engine it is based on shaft power and is called *brake-specific fuel consumption* (bsfc):

$$\text{bsfc} \equiv \frac{\text{lbm fuel per hour}}{\text{shaft horsepower}} = \frac{\text{lbm}}{\text{hp-hr}} \left(\text{or } \frac{\text{kg}}{\text{N}-\text{h}}\right) \tag{12.60}$$

For other air-breathers it is based on thrust and is called *thrust-specific fuel consumption* (tsfc).

$$\text{tsfc} \equiv \frac{\text{lbm fuel per hour}}{\text{lbf thrust}} = \frac{\text{lbm}}{\text{lbf-hr}} \left(\text{or } \frac{\text{kg}}{\text{N}-\text{h}}\right) \tag{12.61}$$

or

$$\text{tsfc} = \frac{\dot{m}_f(3600)}{F_{\text{net}}} \tag{12.62}$$

By comparing equation (12.62) with (12.52) and (12.45) we see that the thrust-specific fuel consumption also can be written as

$$\text{tsfc} = \frac{V_0(3600)}{\eta_0(\text{HV})} \tag{12.63}$$

and is a direct indication of the overall efficiency. Thus it is not surprising to find that tsfc *is the primary economic parameter for any air-breathing propulsion system*. Equation (12.63) also shows that as we increase flight speeds, we must develop more efficient propulsion schemes or the fuel consumption will become unbearable.

Example 12.5

We continue with Example 12.3 and compute the thrust and other performance parameters of the turbofan engine. The following pertinent information is repeated here for convenience:

$$\begin{array}{lll}
\dot{m}_a = 50 \text{ lbm/sec} & \dot{m}'_a = 150 \text{ lbm/sec} & \\
f = 0.0191 & \text{HV} = 18{,}900 \text{ Btu/lbm} & \\
V_0 = 882 \text{ ft/sec} & p_0 = 546 \text{ psfa} & T_0 = 400°\text{R} \\
V_{4'} = 1113 \text{ ft/sec} & p_{4'} = 1157 \text{ psfa} & T_{4'} = 516°\text{R} \\
V_6 = 1686 \text{ ft/sec} & p_6 = 778 \text{ psfa} & T_6 = 1183°\text{R}
\end{array}$$

We now compute the exit densities and areas.

$$\rho_{4'} = \frac{p_{4'}}{RT_{4'}} = \frac{1157}{(53.3)(516)} = 0.0421 \text{ lbm/ft}^3$$

$$A_{4'} = \frac{\dot{m}'_a}{\rho_{4'} V_{4'}} = \frac{150}{(0.0421)(1113)} = 3.20 \text{ ft}^2$$

$$\rho_6 = \frac{p_6}{RT_6} = \frac{778}{(53.3)(1183)} = 0.01234 \text{ lbm/ft}^3$$

$$A_6 = \frac{\dot{m}_a}{\rho_6 V_6} = \frac{50}{(0.01234)(1686)} = 2.40 \text{ ft}^2$$

Note that to calculate the net propulsive thrust, we must include contributions from both the primary jet and the fan.

$$F_{\text{net}} = \frac{\dot{m}_a V_6}{g_c} + A_6(p_6 - p_0) + \frac{\dot{m}'_a V_{4'}}{g_c} + A_{4'}(p_{4'} - p_0) - \left(\dot{m}_a + \dot{m}'_a\right)\frac{V_0}{g_c}$$

$$= \frac{(50)(1686)}{32.2} + (2.40)(778 - 546) + \frac{(150)(1113)}{32.2} + (3.20)(1157 - 546) - (50 + 150)\frac{882}{32.2}$$

$$F_{\text{net}} = 4840 \text{ lbf}$$

The thrust horsepower is [by (12.47)]

$$P_T = F_n V_0 = \frac{(4840)(882)}{550} = 7760 \text{ hp}$$

The input horsepower is [by (12.45)]

$$P_I = \dot{m}_f(\text{HV}) = \dot{m}_a(f)(\text{HV}) = \frac{(50)(0.0191)(18{,}900)(778)}{550} = 25{,}530 \text{ hp}$$

The overall efficiency is [by (12.52)]

$$\eta_0 = \frac{P_T}{P_I} = \frac{7760}{25{,}530} = 30.4\%$$

Thrust-specific fuel consumption is [by (12.62)]

$$\text{tsfc} = \frac{\dot{m}_f(3600)}{F_n} = \frac{(50)(0.0191)(3600)}{4840} = 0.71 \frac{\text{lbm}}{\text{lbf-hr}}$$

This specific fuel consumption is slightly low, even for a fanjet engine. Had we simply changed to a higher value of specific heat in the hot sections (turbine and turbine nozzle), two effects would result:

1. The fuel–air ratio would increase because the enthalpy entering the turbine would increase.
2. The thrust would rise due to an increased exhaust velocity and exit pressure.

The increase in thrust would be small compared with the increase in fuel–air ratio, and the net effect would be to raise the tsfc to about 0.8.

Example 12.6
We continue and compute the performance parameters for the ramjet of Example 12.4. The following pertinent information is repeated here for convenience:

$$\dot{m}_a = 35.9 \text{ kg/s} \quad f = 0.04550 \quad \text{HV} = 4.42 \times 10^{-7} \text{ J/kg}$$

$$M_0 = 1.8 \quad T_0 = 218 \text{ K} \quad M_5 = 1.588 \quad T_5 = 1479 \text{ K}$$

$$V_0 = M_0 a_0 = (1.8)[(1.4)(1)(287)]^{1/2} = 533 \text{ m/s}$$

$$V_5 = M_5 a_5 = (1.588)\left[(1.4)(1)(287)(1479)^{1/2}\right] = 1224 \text{ m/s}$$

If we neglect the mass of fuel added together with the pressure term, the net propulsive thrust is

$$F_{\text{net}} = \frac{\dot{m}}{g_c}(V_5 - V_0) = \left(\frac{35.9}{1}\right)(1224 - 533) = 24,800 \text{ N}$$

The thrust-specific fuel consumption is

$$\text{tsfc} = \frac{\dot{m}_f(3600)}{F_n} = \frac{(0.0450)(35.9)(3600)}{24,800} = 0.235 \frac{\text{kg}}{\text{N·h}}$$

This is equivalent to tsfc = 2.3 lbm/lbf-hr, which is quite high in comparison with the fanjet of Example 12.5. This fact illustrates the uneconomical operation of ramjets at low flight speeds.

12.7 AIR-BREATHING PROPULSION SYSTEMS INCORPORATING REAL GAS EFFECTS

A computer program called *Gas Turb GmbH* has been available for some time (now in version 12 © 2015) [http://www.gasturb.de/Gtb12Manual/GasTurb12.pdf], called "Design and Off-Design Performance of Gas Turbines." This program uses

to advantage the capabilities of modern desktop computers to calculate the performance of turbojets, turboprops, turbofans, and ramjets. The calculations assume that the specific heats are a function of temperature but not of pressure. This is the same assumption that we presented in Section 11.4 with respect to the high-temperature γ behavior of a semiperfect gas. Extensive use is made here of polynomial fits for the temperature dependencies.

The program is quite elaborate and is not described here but we report on some calculations for the turbofan engine used in Examples 12.3 and 12.5. In our example, the result is a net thrust of 4840 lbf and the program outputs 5460 lbf, but bear in mind that the latter are real-gas machine calculations. This and other comparisons are indicated in Table 12.1, where it can be seen that perfect gas results compare somewhat reasonably, within about 11%. From these results, we may conclude that in the cold regions, estimates with $\gamma = 1.4$ are satisfactory. However, in the hot regions (and at high Mach numbers), perfect gas results deviate noticeably from *Gas Turb*, particularly at the nozzle exit (this is consistent with what we found in the examples of Section 11.6).

Table 12.1 Perfect Gas versus Real Gas for Turbofan

Location	Variable (units)	Perfect Gas Examples 12.3 and 12.5	Real Gas *Gas Turb* Program
Diffuser exit	T_{t2} (°R)	465	466
	p_{t2} (psia)	6.29	6.30
Compressor exit	T_{t3} (°R)	1083	1082
	p_{t3} (psia)	94.3	94.5
	Flow (lbm/sec)	50	50.2
Fan exit	$T_{t3'}$ (°R)	619	621
	$p_{t3'}$ (psia)	15.71	15.75
	Flow (lbm/sec)	150	150.6
Combustion chamber exit	T_{t4} (°R)	2500	2500
	p_{t4} (psia)	91.4	91.6
Turbine exit	T_{t5} (°R)	1420	1614
	p_{t5} (psia)	10.6	12.76
Nozzle exit	T_6 (°R)	1183	1400
	p_6 (psia)	5.4	5.0
	V_6 (ft/sec)	1686	
Net thrust	F_{net} (lbf)	4840	5460
SFC	lbm/lbf-hr	0.71	0.75

12.8 ROCKET PROPULSION SYSTEMS PERFORMANCE PARAMETERS

Recall here our basic thrust equation

$$F_{\text{net}} = \frac{\dot{m}_2 V_2}{g_c} - \frac{\dot{m}_0 V_0}{g_c} + A_2(p_2 - p_0) \tag{12.43}$$

This may be applied to rockets by observing that for them there is *no atmospheric air inlet*. Moreover, for rockets we need to change notation to more appropriate nozzle-location designators—those shown in Figures 5.10 and 12.16, where $p_{\text{rec}} \equiv p_0$ now represent the surrounding or ambient pressure and V_3, A_3, and p_3, the velocity, area and pressure at the nozzle-exit, respectively. Location 2 denotes the nozzle throat. Dropping the inflow term from equation (12.43) with our new notation, we arrive at

$$F_{\text{net}} = \frac{\dot{m} V_3}{g_c} + A_3(p_3 - p_0) \tag{12.64}$$

Note that here there is only one mass flow rate (that originating from the propellants, which are stored inside the rocket) and that the propulsive thrust is independent of vehicle flight speed (V_0), and thus a rocket can fly faster than its exit jet.

Effective Exhaust Velocity

In rocket propulsion systems, the nozzle exit pressure (p_3) may be noticeably less or greater than the ambient (p_0) and the pressure term in equation (12.64) cannot be ignored as it can amount to measurable thrust. In order to account for this pressure thrust term, we typically use a different exhaust velocity, one that would produce the same net thrust. So a *fictitious* velocity called the *effective exhaust velocity* (also called the *equivalent exhaust velocity*) is commonly defined which includes both terms on the right hand side of equation (12.64) and is given the symbol V_e (it equals V_3 only when $p_3 = p_0$):

$$\frac{\dot{m} V_3}{g_c} + A_3(p_3 - p_0) \equiv \frac{\dot{m} V_e}{g_c} \tag{12.65}$$

Introducing this definition permits writing rocket thrust, equation (12.64), in a more compact form:

$$\boxed{F_{\text{net}} = \frac{\dot{m}V_e}{g_c}} \tag{12.66}$$

and using the rocket vehicle speed V_0, the thrust power [by equation (12.47)] becomes

$$\boxed{P_T = F_{\text{net}}V_0 = \frac{\dot{m}}{g_c}V_eV_0} \tag{12.67}$$

Here, no maximum is reached, as thrust power increases continually with flight speed.

The propulsive efficiency of a rocket can be found by substituting equations (12.49) and (12.67) into (12.51) using the new nozzle subscript introduced above:

$$\eta_p = \frac{\dfrac{\dot{m}}{g_c}V_eV_0}{\dfrac{\dot{m}}{g_c}V_eV_0 + \dfrac{\dot{m}}{2g_c}(V_3 - V_0)^2} \tag{12.68}$$

To gain greater insight into the propulsive efficiency of a rocket, we examine the same simplifying case for the air breather (i.e., where no significant thrust is obtained from the pressure term; hence $V_e \approx V_3$). Making this substitution and introducing a new speed ratio $\nu \equiv V_0/V_3$, equation (12.68) for rockets becomes

$$\boxed{\eta_p = \frac{2\nu}{1+\nu^2}} \tag{12.69}$$

Unlike equation (12.59) for air-breathing engines, this expression is a maximum when $\nu = 1.0$ and, in the case of rockets, this condition is actually attainable. This propulsion efficiency is most significant during flights within the Earth's atmosphere because *in space* the jet left behind the rocket has no meaningful relative velocity and there are no eddy losses.

Specific Impulse

Since the thrust of any engine is dependent on its size, the use of thrust alone as a performance criterion is relatively meaningless. Moreover, unlike air breathers, chemical rockets must accelerate with both oxidizer and fuel on board and these

usually represent most of the rocket mass. A useful performance indicator that reflects needed propellant usage to achieve a given thrust is the net thrust per unit mass flow rate, which is called *specific thrust* or *specific impulse*, and is given the symbol I_{sp} being defined as follows:

$$\boxed{I_{sp} \equiv \frac{\text{thrust}}{(\text{mass flow rate})g_0} = \frac{F_{net}g_c}{\dot{m}g_0}} \tag{12.70}$$

where g_0 is a constant (the gravity at the Earth's surface, 32.174 ft/sec^2 or 9.807 m/s^2). This definition is similar but an inverse of the "tsfc" indicator given in equation (12.62).

The use of the multiplier $1/g_0$ is purely arbitrary but its effect is to change the units of I_{sp} to sec (s); note that equation (12.70) is independent of the rocket's location in a gravity field. Introducing F_{net} from equation (12.66) yields

$$I_{sp} = \frac{\dot{m}V_e}{g_c}\frac{1}{\dot{m}}\frac{g_c}{g_0}$$

or simplifying

$$\boxed{I_{sp} = \frac{V_e}{g_0}} \tag{12.71}$$

Other countries use the effective exhaust velocity V_e itself as the performance criterion for rockets since it is related to the specific impulse by a constant [as shown in equation (12.71)]. For typical rocket propulsion systems, representative values of specific impulse are shown in Table 12.2 (from Ref. 24). Monopropellants, as the name implies, operate with a single propellant like a pure gas or a decomposing chemical gas, bipropellants burn a fuel with an oxidizer, solid propellants have the fuel and oxidizer premixed in storage, and electromagnetic use a variety of electrical energy inputs.

Nozzle calculations for rocket performance are usually based on our ideal, constant γ analysis developed in the first 10 chapters. However, while propellant flow

Table 12.2 Performance of Rockets

Type of Rocket	Monopropellant	Liquid Bipropellant	Solid	Electromagnetic (nonthermal)
Specific Impulse (sec)	180–220	278–410	192–266	700–5000

in the nozzle can be assumed to be "frozen in composition," an appropriate ratio of the specific heats at the high chamber temperatures, as discussed as in Chapter 11, is needed to reflect the propellant working temperatures (see Ref. 24). All rocket nozzles are relatively short, and their flow is supersonic and, except for very brief startup and shutdown transients, their operation is well represented by steady-state conditions without internal shocks. Tactical missiles operate within the atmosphere where p_0 generally remains constant. For launches into space, rockets must fly through decreasing ambient pressures and are typically designed for a specified atmospheric pressure operation; this "design condition" reflects a matching of the nozzle exhaust pressure p_3 to the ambient or back pressure at a given design altitude, and it also represents the *optimum* thrust as described below (i.e., when $V_e \approx V_3$ for each active rocket stage). Because rocket propulsion systems primarily operate with a fixed chamber pressure and flow rate, maximum thrust is always found where the ambient pressure is negligible, as in the outer layers of the atmosphere and in the vacuum of space. (Is this what equation (12.64) says?)

A convenient way to analyze ideal nozzle performance in rockets is based on the definition of the *thrust coefficient* C_F (see Refs. 19, 20, 21, 23, and 24). If we take equation (12.64) and divide F_{net} by the product p_tA^* (where p_t is the chamber stagnation pressure and A^* the throat cross section at $M_2 = 1.0$), we obtain a dimensionless number, which may be thought of as a "magnification factor"—the thrust modification that isentropic supersonic nozzles may provide the combustion products leaving the chamber. We need here to express the change of momentum in the nozzle [the first term in equation (12.64)] for isentropic flow in terms of the pressures, and this is done by using equations (1.47), (3.18), and (5.44b) (for details see the references listed above). Proceeding with the following notation: $p_{\text{rec}} = p_0$, $p_1 = p_t$, and $A_2 = A^*$, we finally obtain C_F as

$$C_F \equiv \frac{F_{\text{net}}}{p_tA^*} = \sqrt{\frac{2\gamma^2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}\left[1-\left(\frac{p_3}{p_t}\right)^{(\gamma-1)/\gamma}\right]} + \left(\frac{p_3}{p_t}-\frac{p_0}{p_t}\right)\frac{A_3}{A^*} \tag{12.72}$$

Because it represents thrust force, for any fixed A_3/A^* the *maximum* value of C_F is attained in space as mentioned above. But we can arrive at an *optimum* value of C_F when $p_3 = p_0$ because for a given γ, the values of p_3/p_t and A_3/A^* are related through M_3 [see equations (5.37) and (5.40)]. Another way to see this is to hold p_3/pt constant and vary A_3/A^* in equation (12.72)—remember that this formulation only applies to isentropic flows. The optimum value obtained here is called C_{Fopt} and is a function that could also be tabulated in our isentropic table (with the notation below being consistent with that in Section 5.6). Note that equation (12.73) C_{Fopt}, although correct for any M, can only represent the net thrust of a rocket at station 3. (Why? See Problem 12.25.)

$$C_{\text{Fopt}} = \sqrt{\frac{2\gamma^2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}\left[1-\left(\frac{p}{p_t}\right)^{(\gamma-1)/\gamma}\right]} = \sqrt{\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}\frac{\gamma^2 M^2}{1+\frac{\gamma-1}{2}M^2}} \tag{12.73}$$

The second version of equation (12.73) was obtained by careful substitution of equations (1.47), (4.10), (4.11), and (5.44b) into equation (12.64) divided by $p_t A^*$. Try it!

As shown in the five references quoted earlier, C_{Fopt} typically has values greater than 1.0, which is why rockets use supersonic nozzles exclusively to enhance the thrust. Furthermore, when graphing C_F some *real gas effects* can be introduced, as in Figure 4.39 of Reference 20 or Figure 3.7 of Reference 24 where *flow separation* regions arising from gas overexpansion within the nozzle's divergent portion are shown (recall our Figure 5.11 and Section 6.6). As rockets lift through the atmosphere, p_0 decreases and since p_t, propellant flow rate, and nozzle configuration remain fixed each rocket stage design must avoid flow separation inside the nozzle because of its dire consequences.

Example 12.7

A liquid rocket engine has a pressure and temperature of 400 psia and 5000°R, respectively, in the combustion chamber and is operating at an altitude where the ambient pressure is 200 psfa. The gases exit through an isentropic converging–diverging nozzle at a Mach number of 4.0. Approximate the exhaust gases by taking $\gamma = 1.4$ and a molecular mass of 20, but assume perfect gas behavior otherwise. Determine C_{Fopt}, C_F, and I_{sp}. Here, we denote the nozzle exit as section 3 as per Figures 5.10 and 12.16.

Looking at the given information it is appropriate to start with equation (12.73) in the form shown below.

$$C_{\text{Fopt}} = \sqrt{\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}\frac{\gamma^2 M^2}{1+\frac{\gamma-1}{2}M^2}} = \sqrt{\left(\frac{2}{2.4}\right)^{2.4/0.4}\frac{(1.4^2)(4.0^2)}{1+\frac{0.4}{2}4.0^2}} = 1.581$$

At $M_3 = 4.0$, $p/p_t = 0.00659$ and $T/T_t = 0.2381$ from the isentropic tables so

$$p_3 = (0.00659)(400)(144) = 389 \text{ psfa} \quad \text{and} \quad T_3 = (0.2381)(5000) = 1190°\text{R}$$

Because $p_3 > p_0$, we have to include the pressure thrust in the total thrust coefficient. We proceed using equation (12.72) together with equation (11.21) for the nozzle area ratio:

$$C_F = 1.581 + \left(\frac{p_3}{p_1} - \frac{p_0}{p_1}\right)\frac{A_3}{A_3^*} \approx 1.581 + \left(\frac{p_3}{p_1} - \frac{p_0}{p_1}\right)\left(\frac{0.579 p_1}{M_3\, p_3}\sqrt{\frac{T_3}{T_1}}\right) = 1.581 + 0.033 = 1.614$$

To determine the I_{sp} we can use C_F above but we need the mass flow rate from equation (5.44*b*)

$$I_{sp} = \frac{F_n g_c}{\dot{m} g_0} = \frac{C_F p_1 A^* g_c}{\dot{m} g_0} = \frac{C_F g_c / g_0}{\sqrt{\dfrac{2 g_c}{R T_t}\left(\dfrac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}}} = \frac{1.614}{\sqrt{\dfrac{(1.4)(32.2)(20)}{(1545)(5000)}\left(\dfrac{2}{2.4}\right)^{(2.4)/(0.4)}}} = 258.2\,\text{sec}$$

In Example 12.7, if we had used a more correct value of $\gamma = 1.25$ for the products of rocket combustion at station 1, to attain $M_3 = 4.0$ would require a nozzle of much larger area ratio resulting in $p/p_t = 0.00412$ and $T/T_t = 0.3333$. The new specific impulse would also be higher, namely, $I_{sp} = 285$ sec. (*Hint*: use the *Gasdynamics Calculator*.)

12.9 SUPERSONIC DIFFUSERS

The deceleration of the inlet stream in air-breathing propulsion systems causes special problems at *supersonic* flight speeds. If a *subsonic diffuser* (a diverging section) is used, a normal shock will occur at the inlet with an associated loss in stagnation pressure. This loss is small if flight speeds are low, say $M_0 < 1.4$. At speeds between $1.4 < M_0 < 2.0$, an oblique-shock inlet is required (similar to the one used on the ramjet in Example 12.4). Above $M_0 = 2.0$, two oblique shocks, as shown in Figure 7.15, are necessary.

The requirement to be met in each case is to keep the total-pressure recovery factor (η_r) as high as possible. A value of $\eta_r = 0.95$ is considered satisfactory at low supersonic speeds, but this requirement becomes increasingly critical as flight speeds increase. Two oblique shocks plus one normal shock are inadequate at speeds above approximately $M_0 = 2.5$. See Zucrow (pp. 421–427 of Vol. I of Ref. 25) for the effects of multiple conical shocks. From our studies of varying-area flow, we might assume that a converging–diverging section would make a good supersonic diffuser—and indeed it would. Recall that this configuration was used for the exhaust section of a supersonic wind tunnel in Chapter 6. However, there are some practical operating difficulties involved in using a *fixed-geometry* converging–diverging section for a *supersonic* air inlet.

Suppose that we design the inlet diffuser for an airplane that will fly at about $M = 1.86$. From the isentropic table, we see that the area ratio corresponding to this Mach number is 1.507. For simplicity, we construct the diffuser with an area ratio (inlet area to throat area) of 1.50. The design operation of this diffuser is shown in Figure 12.23. In the discussion below, we follow the operation of this diffuser as the aircraft takes off and accelerates to its design speed.

Note that as the flight speed reaches approximately $M_0 = 0.43$, the diffuser becomes choked with $M = 1.0$ in the throat. (Check the subsonic portion of the

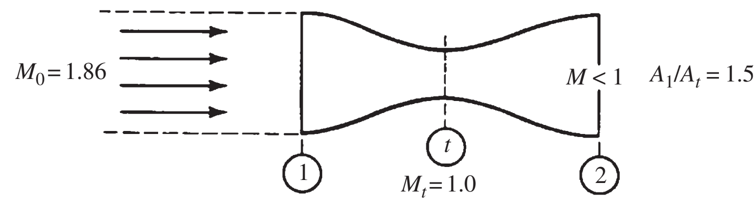


Figure 12.23 Desired operation of converging–diverging diffuser.

isentropic table for the above area ratio.) This condition is shown in Figure 12.24*a*. Now increase the flight speed to, say, $M_0 = 0.6$. *Spillage* or external diffusion occurs, as indicated in Figure 12.24*b*. As M_0 is increased to 1.0, there is a further decrease in the *capture area* (area of the flow at the free-stream Mach number that actually enters the diffuser; see Figure 12.24*c*).

As we increase M_0 to supersonic speeds, a *detached* shock wave forms in front of the inlet. Spillage still occurs as shown in Figure 12.24*d*. Note that at higher flight speeds, less external diffusion is necessary to produce the required $M =$ 0.43 at the inlet. Thus, the shock moves closer to the inlet as speeds increase (see Figure 12.24*e*). Also note that it is necessary to fly at approximately $M_0 = 4.19$ in order for the shock to become attached to the inlet. (Check the shock table to substantiate this.) This condition, indicated in Figure 12.24*f*, is far above the design flight speed.

If we now increase M_0 to 4.2, the shock moves very rapidly into the diffuser and locates itself in the divergent section downstream of the throat. This is referred to as *swallowing the shock*, and the diffuser is said to be *started* (see Figure 12.24*g*). Under these conditions we no longer have Mach 1.0 in the throat. (What Mach number does exist in the throat?) We can now slowly decrease the flight speed to the design condition of $M = 1.86$, and the shock will move to a position just downstream of the throat and occur at the Mach number of just slightly greater than 1.0. Thus we have a very weak shock and negligible losses, as shown in Figure 12.24*h*.

Two comments can now be made on the performance described above.

1. To *start* the diffuser, which was designed for $M_0 = 1.86$, it is necessary to *overspeed* the vehicle to a Mach number of 4.2.
2. If the vehicle slows down just slightly below its design speed (or perhaps minor air disturbances might cause M_0 to drop below 1.86), the shock will pop out in front of the inlet and the diffuser must be *started* all over again.

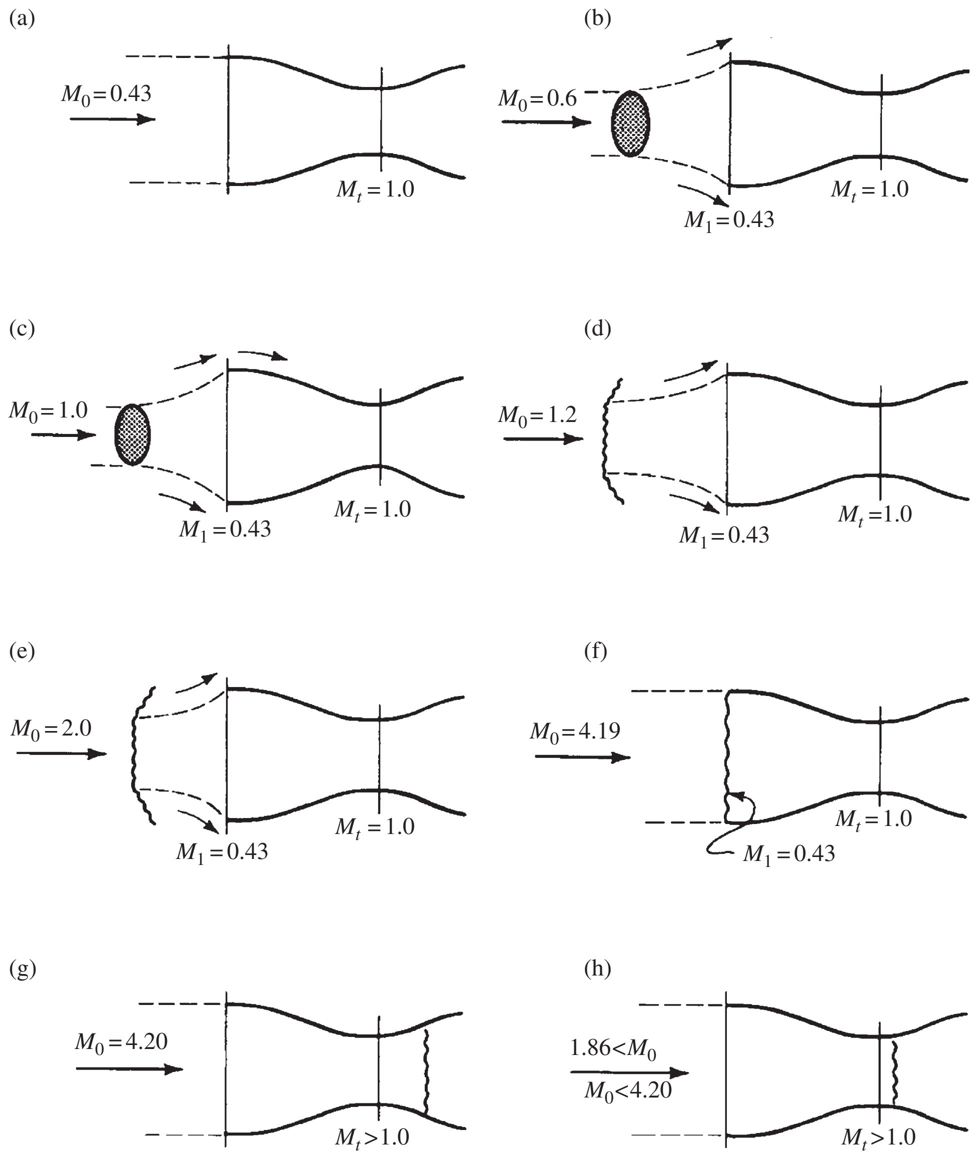


Figure 12.24 Starting a fixed-geometry supersonic diffuser (area ratio = 1.5).

The behavior of fixed-geometry supersonic diffusers can be summarized conveniently in a chart similar to Figure 12.25.

It should be obvious that the operation described above could not be tolerated, and for this reason one does not see *fixed-geometry* converging–diverging diffusers used for air inlets. At flight speeds above $M_0 \approx 2.0$, a combination of oblique shocks and a *variable-geometry* converging–diverging diffuser is required for efficient pressure recovery.

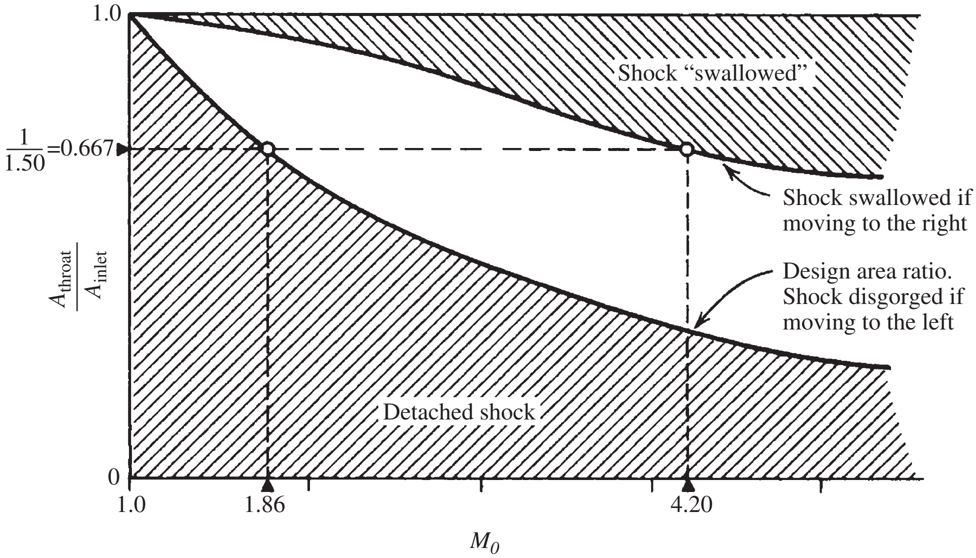


Figure 12.25 Performance of fixed-geometry supersonic diffusers.

12.10 SUMMARY

An analysis of the ideal Brayton cycle revealed that its thermodynamic efficiency is only a function of the pressure ratio and given by

$$\eta_{\text{th}} = 1 - \left(\frac{1}{r_p}\right)^{(\gamma-1)/\gamma} \tag{12.22}$$

Perhaps a most significant feature of this cycle is that the work input is a large percentage of the work output. Because of this, machine efficiencies are most critical in any power plant operating on the Brayton cycle. Also, to produce a reasonable quantity of net work, large amounts of air must be handled, which makes this cycle particularly suitable for turbomachinery.

In discussing the various types of jet propulsion systems, it was noted that pure jets move a relatively small amount of air through a large velocity change. On the other hand, propeller systems move a relatively large amount of air through a small velocity increment. Fanjets occupy a middle ground on both criteria.

The net thrust of any propulsive device was found to be (refer to Figure 12.21)

$$F_{\text{net}} = \frac{\dot{m}_2 V_2}{g_c} - \frac{\dot{m}_0 V_0}{g_c} + A_2(p_2 - p_0) \tag{12.43}$$

You should learn this equation, as it is probably the most important relation in this chapter. Also, you should not overlook the various power and efficiency parameters discussed in Section 12.5. Perhaps the most interesting of these is the

propulsive efficiency, since this is a measure of what the propulsive device is accomplishing, exclusive of the energy producer.

For air-breathers, in terms of the speed ratio $\nu = V_0/V_2$,

$$\eta_p = \frac{2\nu}{1+\nu} \tag{12.59}$$

Equation (12.59) explains why pure jets operate more efficiently at high speeds, whereas fanjets and propjets fare better at progressively lower speeds. We can also show that for air-breathers, highest efficiency occurs at minimum thrust.

Rockets (see Figure 12.16) are not subject to this dilemma and their thrust and propulsive efficiency are (where V_e is from equation (12.65) and $\nu \equiv V_0/V_3$)

$$F_{\text{net}} = \frac{\dot{m}V_e}{g_c} \text{ and } \eta_p = \frac{2\nu}{1+\nu^2} \tag{12.66 and 12.69}$$

Other important performance indicators are, for air-breathers:

$$\text{tsfc} = \frac{\text{lbm fuel per hr}}{\text{lbf thrust}} = \frac{\dot{m}_f(3600)}{F_{\text{net}}} \tag{12.61, 12.62}$$

for rockets, C_F (equation (12.73)) identifies the effects of a supersonic nozzle on the thrust, and the specific impulse is

$$I_{\text{sp}} = \frac{\text{thrust}}{(\text{mass flow rate})g_0} = \frac{V_e}{g_0} \tag{12.70, 12.71}$$

Air inlets for supersonic vehicles need to have total-pressure recovery factors of 0.95 or above. At the lower supersonic speeds, one uses a subsonic diffuser preceded by ramps or a spike to induce one or more oblique shocks before the normal shock. At high supersonic flight speeds, variable-geometry features are also required.

PROBLEMS

In the problems that follow you may assume perfect gas behavior and constant-specific heats unless otherwise specified, even though the temperature range may be rather high in some cases. Also, neglect any effects of gas dissociation and assume that, unless otherwise stated, all propellants have the properties of air.

12.1. Conditions entering the compressor of an ideal Brayton cycle are 520°R and 5 psia. The compressor pressure ratio is 12 and the maximum allowable cycle temperature is 2400°R. Assume that air has negligible velocities in the ducting.

(a) Determine w_t, w_c, w_n, q_a, and η_{th}.

(b) What flow rate is required for a net output of 5000 hp?

12.2. Rework Problem 12.1 with a compressor efficiency of 89% and a turbine efficiency of 92%.

12.3. A stationary power plant produces 1×10^7 W output when operating under the following conditions: Compressor inlet is 0°C and 1 bar abs, turbine inlet is 1250 K, cycle pressure ratio is 10, and fluid is air with negligible velocities. The turbine and compressor efficiencies are both 90%. Determine the cycle efficiency and the mass flow rate.

12.4. Assume that all data given in Problem 12.3 remain the same except that the turbine and compressor are 80% efficient.

(a) Determine the cycle efficiency.

(b) Compare the net work output and cycle efficiency with that of Problem 12.3.

(c) What value of machine efficiency (assuming that $\eta_t = \eta_c$) will cause zero net work output from this cycle?

12.5. Consider an ideal Brayton cycle as shown in Figure 12.2. Let

$$\alpha = \frac{T_{t3}}{T_{t1}} \quad \text{the cycle temperature ratio}$$

$$\theta = \left(\frac{p_{t2}}{p_{t1}}\right)^{(\gamma-1)/\gamma} \quad \text{the cycle pressure ratio parameter}$$

(a) Show that the net work output can be expressed as

$$w_n = c_p T_{t1} \frac{\theta - 1}{\theta}(\alpha - \theta)$$

(b) Show that for a given α the maximum net work occurs when $\theta = \sqrt{\alpha}$.

(c) On the same *T–s* diagram, sketch cycles for a given temperature ratio but for different pressure ratios. Which one is most efficient? Which produces the most net work?

12.6. An airplane is traveling at 550 mph at an altitude where the ambient pressure is 6.5 psia. The exit area of the jet engine is 1.65 ft^2 and the exit jet has a relative velocity of 1500 ft/sec. The pressure at the exit plane is found to be 10 psia. Air flow is measured at 175 lbm/sec. You may neglect the weight of fuel added. What is the net propulsive thrust of this engine?

12.7. The air flow through a jet engine is 30 kg/s, and the fuel flow is 1 kg/s. The exhaust gases leave with a relative velocity of 610 m/s. Pressure equilibrium exists over the exit plane. Compute the velocity of the airplane if the thrust power is 1.12×10^6 W.

12.8. A twin-engine jet aircraft requires a total net propulsive thrust of 6000 lbf. Each engine consumes air at the rate of 120 lbm/sec when traveling at 650 ft/sec. Fuel is added in each engine at the rate of 3.0 lbm/sec. Assume that pressure equilibrium exists across the exit plane and compute the velocity of the exhaust gases relative to the plane.

12.9. A boat is propelled by an hydraulic jet. The inlet scoop has an area of 0.5 ft^2, and the area of the exit duct is 0.20 ft^2. Since the exit velocity will always be subsonic, pressure equilibrium exists over the exit plane. No spillage occurs at the inlet when the boat is moving through fresh water at 50 mph.

(a) Compute the net propulsive force being developed.

(b) What is the propulsive efficiency?

(c) How much energy is added to the water as it passes through the device? (Assume no losses.)

12.10. It is proposed to power a monorail car by a pulsejet. A net propulsive thrust of 5350 N is required when traveling at a speed of 210 km/h. The gases leave the engine with an average velocity of 350 m/s. Assume that pressure equilibrium exists at the outlet plane and neglect the weight of fuel added.

(a) Compute the mass flow rate required.

(b) What inlet area is necessary, assuming that no spillage occurs? (Assume 16°C and 1 atm.)

(c) What is the thrust power?

(d) What is the propulsive efficiency?

(e) How much energy is added to the air as it passes through the engine if the outlet temperature is 980°C?

12.11. A ramjet flies at $M_0 = 4.0$ at 30,000 ft altitude where $T_0 = 411°R$ and $p_0 = 628$ psfa. The exhaust nozzle exit diameter is 18 in. The exhaust jet has a velocity of 5000 ft/sec relative to the missile and is at 1800°R and 850 psfa. Neglect the fuel added.

(a) Determine the net propulsive thrust.

(b) How much thrust power is developed?

12.12. An example of a fanjet engine analysis was given in Sections 12.4 and 12.6. Remove the fan from this engine. Read just the turbine expansion to produce the appropriate compressor work. Assume that all component efficiencies remain unchanged. Compute the net propulsive thrust and thrust-specific fuel consumption for the pure jet engine and compare with that of the fanjet.

12.13. It has been suggested that an afterburner be added to the fanjet engine used in the examples in Sections 12.4 and 12.6. Assume that the gas leaves the turbine with a velocity of 400 ft/sec. Enough fuel is added in the afterburner to raise the stagnation temperature to 3500°R with a combustion efficiency of $\eta_{ab} = 0.85$. Determine the cross-sectional area of the afterburner, the conditions at the exit of the afterburner (assume Raleigh flow), the new conditions at the nozzle exit, the required exit area, and the resultant effect on the performance parameters of the engine. (Neglect the mass of the fuel.)

12.14. A ramjet is designed to operate at $M_0 = 3.0$ at an altitude of 40,000 ft where the temperature and pressure are 390°R and 400 psfa. The total-pressure recovery factor for the inlet is $\eta_r = p_{t2}/p_{t0} = 0.85$. The velocity is reduced to 300 ft/sec before entering the combustion chamber, where the total temperature is raised to 4000°R. Combustion efficiency is $\eta_b = 0.96$, and the heating value of the fuel is 18,500 Btu/lbm. The exit nozzle has an efficiency of $\eta_n = 0.95$ and expands the flow through a converging–diverging section to the same area as the combustion chamber (similar to that shown in Figure 12.14). Compute the net propulsive thrust per unit area and the thrust specific fuel consumption. (You may neglect the mass of fuel added.)

12.15. A rocket sled used for test purposes requires a thrust of 20,000 lbf. The specific impulse is 240 sec.

(a) What is the flow rate?

(b) Compute the exhaust velocity if the nozzle expands the gases to ambient pressure.

12.16. The German V-2 had a sea-level thrust of 249,000 N, a propellant flow rate of 125 kg/s, and exhaust velocity of 1995 m/s, and the nozzle outlet size was 74 cm in diameter.

(a) Compute the specific impulse.

(b) Calculate the pressure at the nozzle outlet.

12.17. An ideal rocket nozzle was originally *designed* to expand the exhaust gases to ambient pressure when at sea level and operating with a combustion chamber pressure of 400 psia and a temperature of 5000°R. The rocket is now used to propel a missile fired from an airplane at 38,000 ft, where the pressure is 3.27 psia.

(a) Determine the exit area required to produce a thrust of 1000 lbf at 38,000 ft.

(b) Compute the exit velocity, effective exhaust velocity, and specific impulse.

12.18. The combustion chamber of a rocket has stagnation conditions of 22 bar and 2500 K. Assume that the nozzle is ideal and expands the flow to the ambient pressure of 0.25 bar.

(a) Determine the nozzle area ratio and exit velocity.

(b) What is the specific impulse?

12.19. A rocket nozzle operates supersonically with a constant chamber pressure of 500 psia exhausting to 14.7 psia. Find the ratio of the thrust at sea level to the thrust in space (0 psia). Assume that the chamber temperature is 2500°R, that $\gamma = 1.4$, and that $R = 20$ ft-lbf/lbm-°R.

12.20. It turns out that for a given pressure ratio across the nozzle, the ideal thrust from a rocket does *not* explicitly depend on temperature. Show this by taking the thrust equation (12.64) for a rocket at the design condition (pressure equilibrium at the exit) and manipulating the parameters. On what actual physical entities does the ideal thrust depend (e.g., areas, pressures, specific heat ratio)? (*Hint*: examine C_F.)

12.21. Compare the total-pressure recovery factors for the air inlets described in Problem 7.13.

12.22. Sketch a supersonic inlet that has one oblique shock followed by a normal shock attached to the entrance of a subsonic diffuser. Draw streamlines and identify the capture area (that portion of the free stream that actually enters the diffuser). Now vary the wedge angle and cause the oblique shock to form at a different angle. Again, determine the capture area. Show that maximum flow enters the inlet when the oblique shock just touches the outer lip of the diffuser.

12.23. Figure 12.25 illustrates the peculiar operating conditions associated with fixed-geometry supersonic diffusers. Unfortunately, this figure was not drawn to scale and therefore cannot be used as a working plot.

(a) Construct an accurate version of Figure 12.25.

(b) If the design flight speed is $M_0 = 1.5$, to what velocity must the vehicle be overspeeded in order to start the diffuser?

(c) Suppose the design speed is $M_0 = 2.0$. How fast must the vehicle go to start the diffuser?

12.24. A converging–diverging supersonic inlet is to be designed with a variable area. The idea is to swallow the shock when the vehicle has just reached its design flight speed. Then, the diffuser area ratio will be changed to operate properly without any shock. Thus, the inlet does not have to be overspeeded to start. Calculate the maximum and minimum area ratios that would be required to operate in the manner described above if the flight speed is $M_0 = 2.80$.

12.25. At the combustion-chamber exit of a liquid rocket engine the pressure is 2.76 MPa and temperature 2778 K. Combustion products then feed into an isentropic converging–diverging nozzle as shown in Figure 5.10 or 12.16. The gases entering the nozzle have molecular mass of 20.3 with $\gamma = 1.25$. Consider flow composition to be "frozen" in the nozzle and assume isentropic perfect gas behavior. Calculate C_{Fopt} using equation (12.73) at three locations: the nozzle entrance ($M = 0.2$), the throat, and the exit ($M = 4.0$). Also determine *engine thrust* for a "nozzle throat diameter" = 18.5 cm at the condition $p_3 = p_0$.

CHECK TEST

You should be able to complete this test without much reference to material in the chapter.

12.1. We wish to build an electric generator for use at a ski lodge. To keep this small and lightweight, we have decided to use an open Brayton cycle as shown in Figure CT12.1. Write an expression (in terms of properties at 1, 2, 3, and 4) that will represent for each pound mass flowing:

(a) The compressor work input.

(b) The turbine work output.

(c) The cycle thermodynamic efficiency.

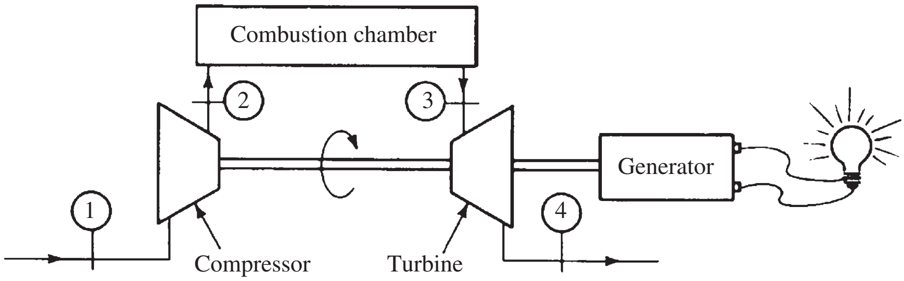


Figure CT12.1

12.2. If the machine efficiencies are not fairly high, the thermodynamic efficiency of a Brayton cycle will be extremely poor. What basic characteristic of the Brayton cycle accounts for this fact?

12.3. Conditions entering a turbine are $T_t = 1060°\text{C}$ and $p_t = 6.5$ bar. The turbine efficiency is $\eta_t = 90\%$, and the mass flow rate is 45 kg/s. Compute the turbine outlet stagnation conditions if the turbine produces 2.08×10^7 W of work. Neglect any heat transfer.

12.4. Draw an h–s diagram for the secondary (fan) air of a turbofan engine (a real engine – not an ideal one).

(a) Indicate static and stagnation points if they are significantly different.

(b) Indicate pertinent velocities, work quantities, and so on.

12.5. State whether each of the following statements is true or false.

(a) Thrust power output can be viewed as the change in kinetic energy of the working medium.

(b) If the exhaust gases leave a rocket at a speed of 7000 ft/sec relative to the rocket, it would be impossible for the rocket to be traveling at 8000 ft/sec relative to the ground.

(c) It is possible to operate a ramjet at 100% propulsive efficiency and develop thrust.

(d) One would expect that a turbofan engine will have a higher tsfc than a ramjet engine.

12.6. A rocket is traveling at 4500 ft/sec at an altitude of 20,000 ft, where the temperature and pressure are 447°R and 972 psfa, respectively. The exit diameter of the nozzle is 24 in. and the exhaust jet has the following characteristics: $T = 1500°\text{R}$, $p = 1200$ psfa, and $V = 6600$ ft/sec (relative to the rocket).

(a) Compute the flow rate and net propulsive thrust.

(b) What is the effective exhaust velocity?

(c) Compute the specific impulse and thrust power.

12.7. A fixed-geometry converging–diverging supersonic diffuser is contemplated for a vehicle having a design Mach number of $M_0 = 1.65$. How fast must the plane fly to *start* this diffuser?

Appendix A

Summary of the English Engineering (EE) System of Units

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

Force	pound force	lbf
Mass	pound mass	lbm
Length	foot	ft
Time	second	sec
Temperature	Rankine	°R

NEVER say pound, as this is ambiguous! It is either a pound force (lbf) or a pound mass (lbm).

A 1-pound force will give a 1-pound mass an acceleration of 32.174 feet/second^2.

$$F = \frac{ma}{g_c}$$

$$1\,(\text{ lbf}) = \frac{1\,(\text{lbm})\cdot 32.174(\text{ft}/\text{sec}^2)}{g_c}$$

Thus

$$g_c = 32.174\ \text{lbm-ft}/\text{lbf-sec}^2$$

Temperature	T (°R) = T (°F) + 459.67
Gas constant	R = 1545/M.M*. ft-lbf/1bm-°R
Pressure	1 atm = 2116.2 lbf/ft^2
Heat to work	1 Btu = 778.2 ft-lbf
Power	1 hp = 550 ft-lbf/sec
Standard gravity	g_0 = 32.174 ft/sec^2

* M.M., molecular mass.

USEFUL CONVERSION FACTORS

To convert from:		To:	Multiply by:
meter		foot	3.281
meter		inch	3.937×10
newton		lbf	2.248×10^{-1}
kilogram		lbm	2.205
K		°R	1.800
joule	(Q)	Btu	9.479×10^{-4}
kWh	(Q)	Btu	3.413×10^{3}
joule	(W)	ft-lbf	7.375×10^{-1}
watt		horsepower	1.341×10^{-3}
m/s	(V)	ft/sec	3.281
m/s	(V)	mph	2.237
km/h	(V)	mph	6.215×10^{-1}
N/m^2	(p)	atmosphere	9.872×10^{-6}
N/m^2	(p)	lbf/in^2	1.450×10^{-4}
N/m^2	(p)	lbf/ft^2	2.089×10^{-2}
kg/m^3	(ρ)	lbm/ft^3	6.242×10^{-2}
$N{\cdot}s/m^2$	(μ)	$lbf\text{-}sec/ft^2$	2.089×10^{-2}
m^2/s	(ν)	ft^2/sec	1.076×10
J/kg·K	(c_p)	Btu/lbm-°R	2.388×10^{-4}
N·m/kg·K	(R)	ft-lbf/lbm-°R	1.858×10^{-1}

Source: "The International System of Units," *NASA SP-7012*, 1973.

PROPERTIES OF GASES—ENGLISH ENGINEERING (EE) SYSTEM[a]

Gas	Symbol	Molecular Mass	$\gamma = \frac{c_p}{c_v}$	Gas Constant R ft-lbf/lbm-°R	Specific Heats Btu/lbm-°R c_p	c_v	Viscosity μ lbf-sec/ft^2	Critical Point T_c °R	p_c psia
Air		28.97	1.40	53.3	0.240	0.171	3.8×10^{-7}	239	546
Ammonia	NH_3	17.03	1.32	90.74	0.523	0.396	2.1×10^{-7}	739.7	1636
Argon	Ar	39.94	1.67	38.7	0.124	0.074	4.7×10^{-7}	272	705
Carbon dioxide	CO_2	44.01	1.29	35.1	0.203	0.157	3.1×10^{-7}	547.5	1071
Carbon monoxide	CO	28.01	1.40	55.2	0.248	0.177	3.7×10^{-7}	240	507
Helium	He	4.00	1.67	386	1.25	0.750	4.2×10^{-7}	9.5	33.2
Hydrogen	H_2	2.02	1.41	766	3.42	2.43	1.9×10^{-7}	59.9	188.1
Methane	CH_4	16.04	1.32	96.4	0.532	0.403	2.3×10^{-7}	343.9	673
Nitrogen	N_2	28.02	1.40	55.1	0.248	0.177	3.6×10^{-7}	227.1	492
Oxygen	O_2	32.00	1.40	48.3	0.218	0.156	4.2×10^{-7}	278.6	736
Water vapor	H_2O	18.02	1.33	85.7	0.445	0.335	2.2×10^{-7}	1165.3	3204

[a] Values for γ, R, c_p, c_v, and μ are for normal room temperature and pressure.

Appendix B

Summary of the International System (SI) of Units

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

Force	newton	N
Mass	kilogram	kg
Length	meter	m
Time	second	s
Temperature	kelvin	K

$$\boxed{\text{A 1} - \text{Newton force will give a 1} - \text{kilogram mass an acceleration of 1 meter/second}^2.}$$

$$F = \frac{ma}{g_c}$$

$$1(\mathrm{N}) = \frac{1(\mathrm{kg}) \cdot 1(\mathrm{m/s^2})}{g_c}$$

Thus

$$g_c = 1\ \mathrm{kg{\cdot}m/N{\cdot}s^2}$$

Temperature	T (K) = T (°C) + 273.15
Gas constant	$R = 8314/\mathrm{M.M.}^*$ N·m/kg·K
Pressure	1 atm = 1.013×10^5 N/m^2
	1 pascal (Pa) = 1 N/m^2
	1 bar (bar) = 1×10^5 N/m^2
	1 MPa = 1×10^6 N/m^2
Heat to work	1 joule (J) = 1 N·m
Power	1 watt (W) = 1 J/s
Standard gravity	$g_0 = 9.81$ m/s^2

* M.M., molecular mass.

USEFUL CONVERSION FACTORS

To convert from:		To:	Multiply by:
foot		meter	3.048×10^{-1}
inch		meter	2.54×10^{-2}
lbf		newton	4.448
lbm		kilogram	4.536×10^{-1}
°R		K	5.555×10^{-1}
Btu	(Q)	joule	1.055×10^{3}
Btu	(Q)	kWh	2.930×10^{-4}
ft-lbf	(W)	joule	1.356
horsepower		watt	7.457×10^{2}
ft/sec	(V)	m/s	3.048×10^{-1}
mph	(V)	m/s	4.470×10^{-1}
mph	(V)	km/h	1.609
atmosphere	(p)	N/m^2	1.013×10^{5}
lbf/in^2	(p)	N/m^2	6.895×10^{3}
lbf/ft^2	(p)	N/m^2	4.788×10
lbm/ft^3	(ρ)	kg/m^3	1.602×10
$lbf\text{-}sec/ft^2$	(μ)	$N \cdot s/m^2$	4.788×10
ft^2/sec	(ν)	m^2/s	9.290×10^{-2}
Btu/lbm-°R	(c_p)	J/kg·K	4.187×10^{3}
ft-lbf/lbm-°R	(R)	N·m/kg·K	5.381

Source: "The International System of Units," *NASA SP-7012*, 1973.

PROPERTIES OF GASES—INTERNATIONAL SYSTEM (SI)[a]

Gas	Symbol	Molecular Mass	$\gamma = \frac{c_p}{c_v}$	Gas Constant R N·m/kg·K	Specific Heats J/kg·K c_p	Specific Heats J/kg·K c_v	Viscosity μ N·s/m^2	Critical Point T_c K	Critical Point p_c MPa
Air		28.97	1.40	287	1,000	716	1.8×10^{-5}	132.8	3.76
Ammonia	NH_3	17.03	1.32	488	2,175	1,648	1.0×10^{-5}	405	11.36
Argon	Ar	39.94	1.67	208	519	310	2.3×10^{-5}	151.1	4.86
Carbon dioxide	CO_2	44.01	1.29	189	850	657	1.5×10^{-5}	304.1	7.38
Carbon monoxide	CO	28.01	1.40	297	1,040	741	1.8×10^{-5}	133.3	3.49
Helium	He	4.00	1.67	2,080	5,230	3,140	2.0×10^{-5}	5.28	0.229
Hydrogen	H_2	2.02	1.41	4,120	14,300	10,200	9.1×10^{-5}	33.3	1.30
Methane	CH_4	16.04	1.32	519	2,230	1,690	1.1×10^{-5}	191.0	4.64
Nitrogen	N_2	28.02	1.40	296	1,040	741	1.7×10^{-5}	126.2	3.39
Oxygen	O_2	32.00	1.40	260	913	653	2.0×10^{-5}	154.8	5.07
Water vapor	H_2O	18.02	1.33	461	1,860	1,400	1.1×10^{-5}	647.3	22.09

[a] Values for γ, R, c_p, c_v, and μ are for normal room temperature and pressure.

Appendix C

Friction-Factor Chart

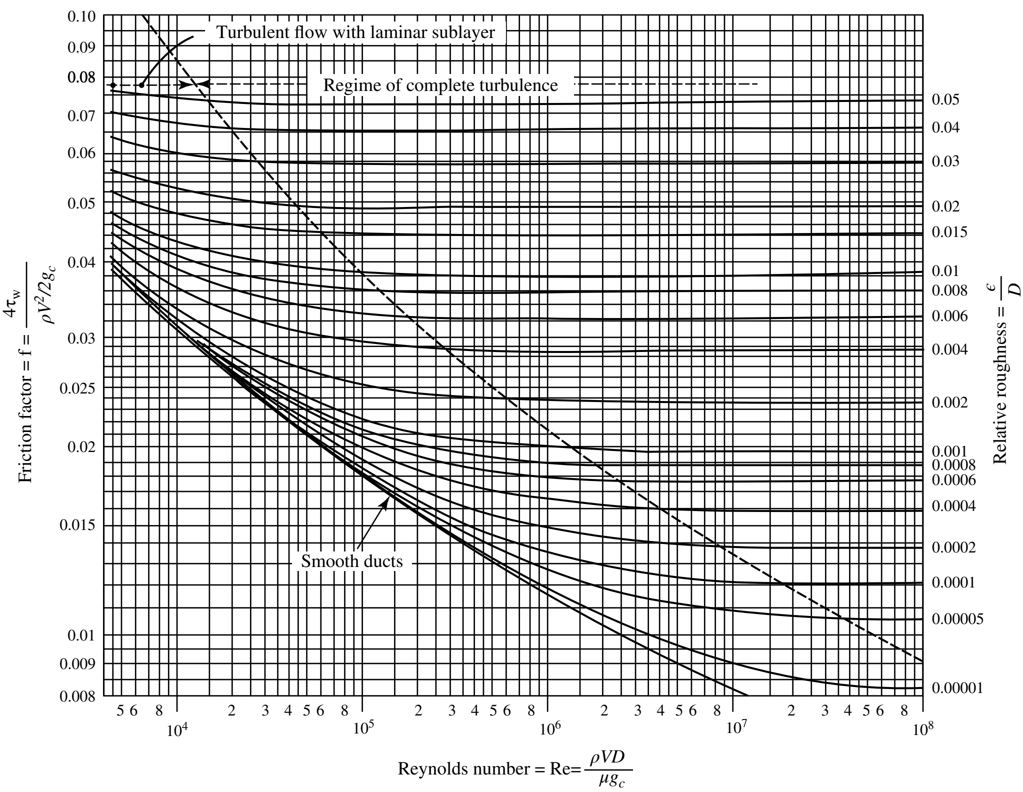


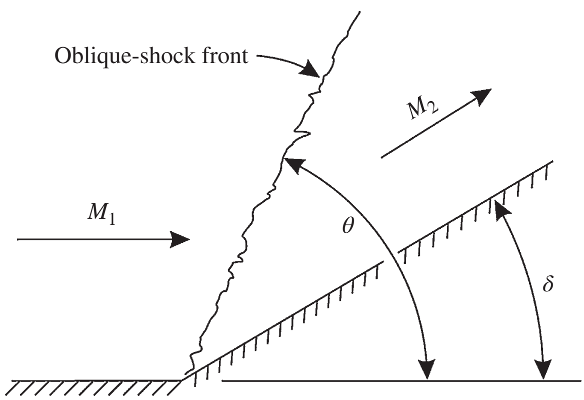
Figure AC.1 Moody diagram for determination of friction factor with laminar region omitted. (Adapted with permission from L. F. Moody, Friction factors for pipe flow, *Transactions of ASME*, Vol. 66, 1944.)

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

Appendix D

Oblique-Shock Charts (γ = 1.4) (Two-Dimensional)



Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

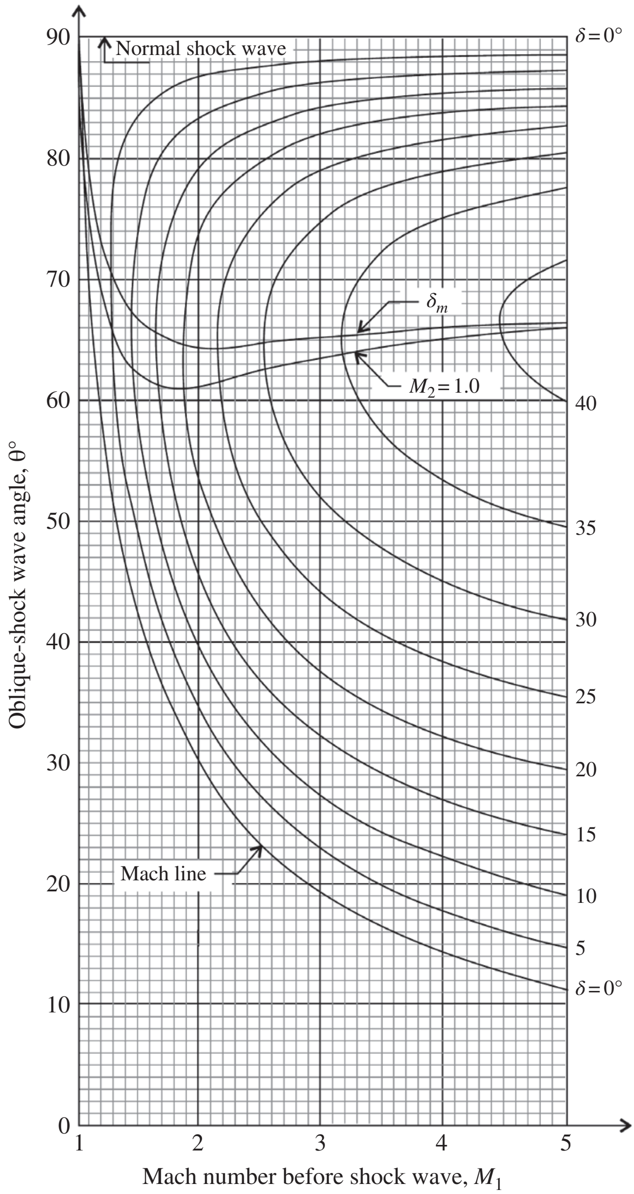


Figure AD.1 Shock-wave angle θ as a function of the initial Mach number M_1 for different values of the flow deflection angle δ for $\gamma = 1.4$. (Adapted with permission from M. J. Zucrow and J. D. Hoffman, *Gas Dynamics*, Vol. I, copyright 1976, John Wiley & Sons, New York.)

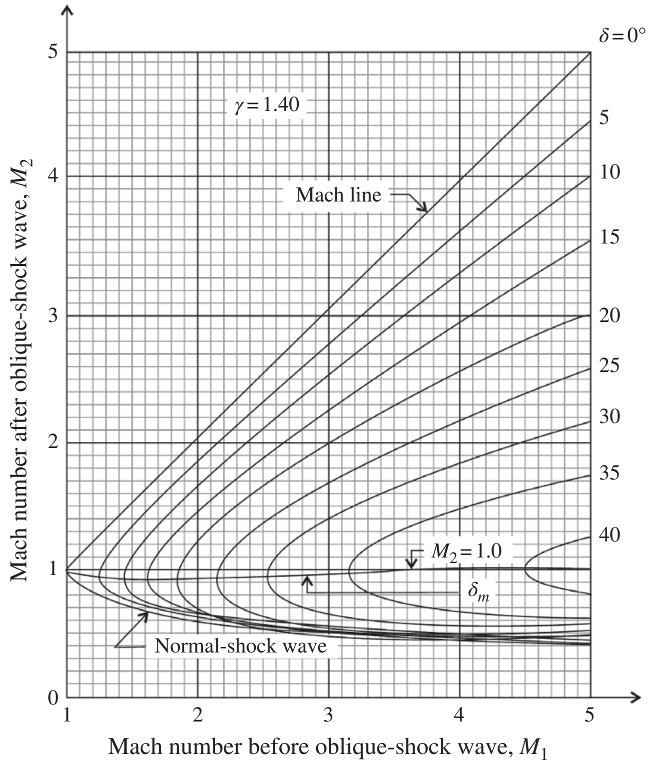


Figure AD.2 Mach number downstream M_2 for an oblique-shock wave as a function of the initial Mach number M_1 for different values of the flow deflection angle δ for $\gamma = 1.4$. (Adapted with permission from M. J. Zucrow and J. D. Hoffman, *Gas Dynamics*, Vol. I, copyright 1976, John Wiley & Sons, New York.)

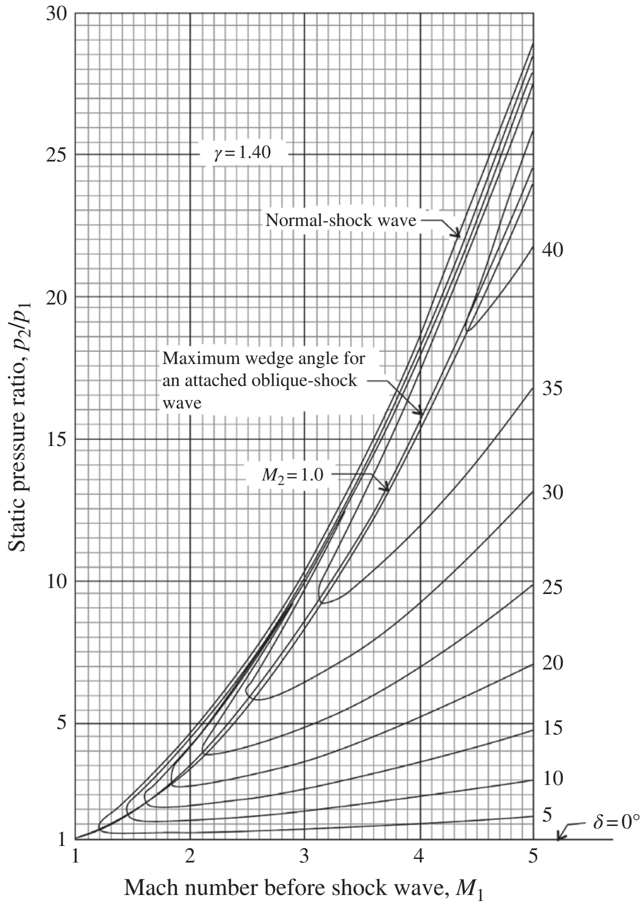
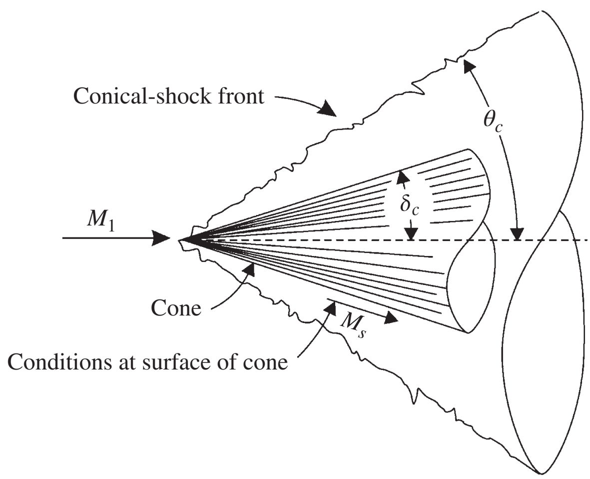


Figure AD.3 Static pressure ratio p_2/p_1 across an oblique-shock wave as a function of the initial Mach number M_1 for different values of the flow deflection angle δ for $\gamma = 1.40$. (Adapted with permission from M. J. Zucrow and J. D. Hoffman, *Gas Dynamics*, Vol. I, copyright 1976, John Wiley & Sons, New York.)

Appendix E

Conical-Shock Charts ($\gamma = 1.4$) (Three-Dimensional)



Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

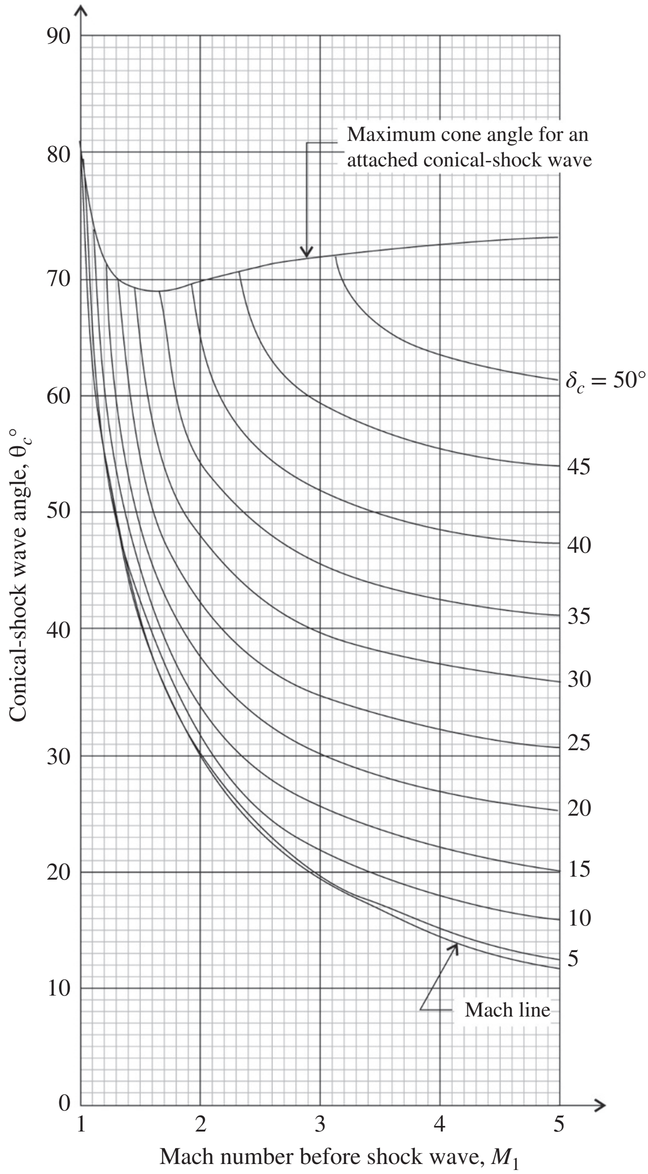


Figure AE.1 Shock wave angle θ_c for a conical-shock wave as a function of the initial Mach number M_1 for different values of the cone angle δ_c for $\gamma = 1.40$. (Adapted with permission from M. J. Zucrow and J. D. Hoffman, *Gas Dynamics*, Vol. I, copyright 1976, John Wiley & Sons, New York.)

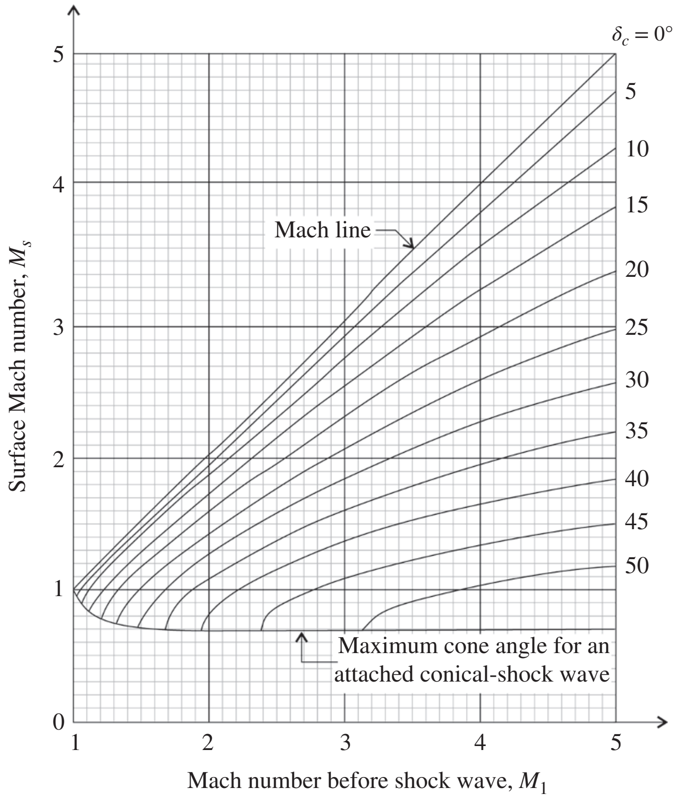


Figure AE.2 Surface Mach number M_s for a conical-shock wave as a function of the initial Mach number M_1 for different values of the cone angle δ_c for $\gamma = 1.40$. (Adapted with permission from M. J. Zucrow and J. D. Hoffman, *Gas Dynamics*, Vol. I, copyright 1976, John Wiley & Sons, New York.)

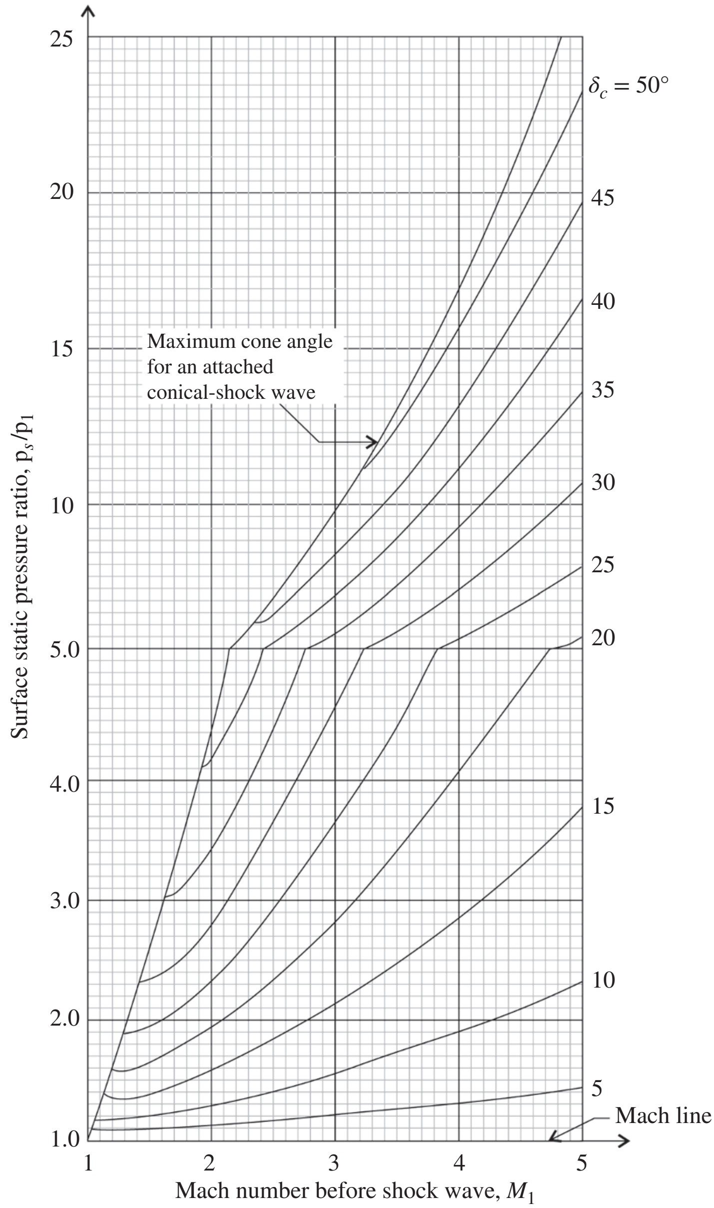


Figure AE.3 Surface static pressure ratio p_s/p_1 for a conical-shock wave as a function of the initial Mach number M_1 for different values of the cone angle δ_c for $\gamma = 1.40$. (Adapted with permission from M. J. Zucrow and J. D. Hoffman, *Gas Dynamics*, Vol. I, copyright 1976, John Wiley & Sons, New York.)

Appendix F

Generalized Compressibility Factor Chart

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

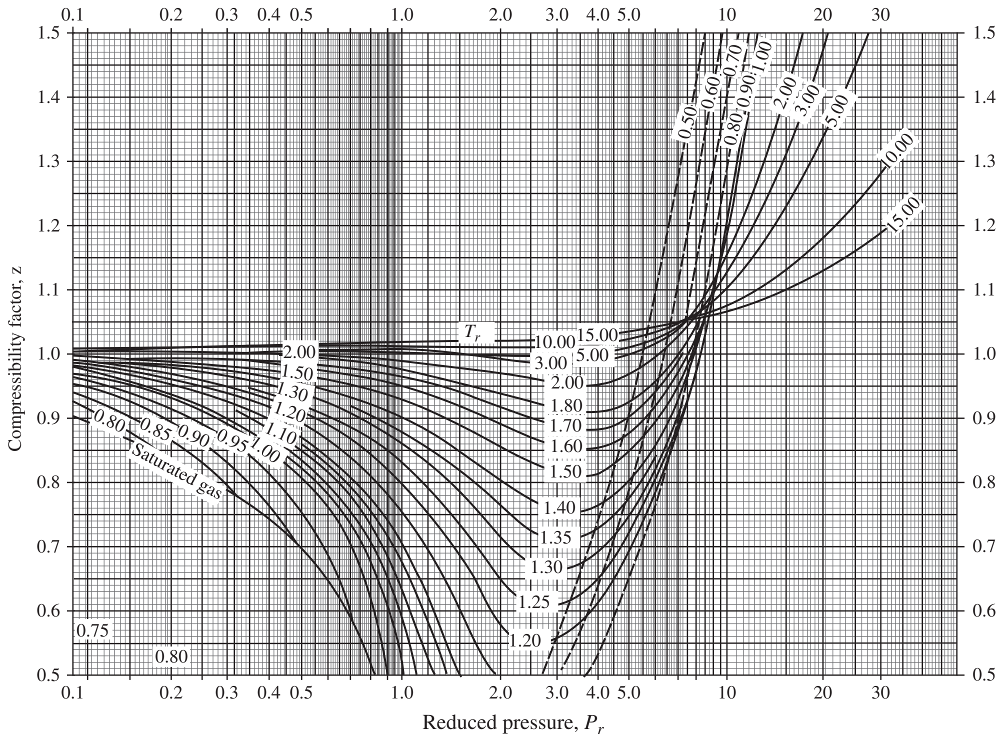


Figure AF.1 Generalized compressibility factors ($Z_c = 0.27$). (Adapted with permission from R. E. Sontag, C. Borgnakke, and C. J. Van Wylen, *Fundamentals of Thermodynamics*, 5th ed., copyright 1997, John Wiley & Sons, New York.)

Appendix G

Isentropic Flow Parameters ($\gamma = 1.4$) (Including Prandtl–Meyer Function)

M	p/p_t	T/T_t	A/A^*	pA/p_tA^*	ν	μ
0.0	1.00000	1.00000	∞	∞		
0.01	0.99993	0.99998	57.87384	57.86979		
0.02	0.99972	0.99992	28.94213	28.93403		
0.03	0.99937	0.99982	19.30054	19.28839		
0.04	0.99888	0.99968	14.48149	14.46528		
0.05	0.99825	0.99950	11.59144	11.57118		
0.06	0.99748	0.99928	9.66591	9.64159		
0.07	0.99658	0.99902	8.29153	8.26315		
0.08	0.99553	0.99872	7.26161	7.22917		
0.09	0.99435	0.99838	6.46134	6.42484		
0.10	0.99303	0.99800	5.82183	5.78126		
0.11	0.99158	0.99759	5.29923	5.25459		
0.12	0.98998	0.99713	4.86432	4.81560		
0.13	0.98826	0.99663	4.49686	4.44406		
0.14	0.98640	0.99610	4.18240	4.12552		
0.15	0.98441	0.99552	3.91034	3.84937		
0.16	0.98228	0.99491	3.67274	3.60767		
0.17	0.98003	0.99425	3.46351	3.39434		

(*Continued*)

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

M	p/p_t	T/T_t	A/A^*	pA/p_tA^*	ν	μ
0.18	0.97765	0.99356	3.27793	3.20465		
0.19	0.97514	0.99283	3.11226	3.03487		
0.20	0.97250	0.99206	2.96352	2.88201		
0.21	0.96973	0.99126	2.82929	2.74366		
0.22	0.96685	0.99041	2.70760	2.61783		
0.23	0.96383	0.98953	2.59681	2.50290		
0.24	0.96070	0.98861	2.49556	2.39750		
0.25	0.95745	0.98765	2.40271	2.30048		
0.26	0.95408	0.98666	2.31729	2.21089		
0.27	0.95060	0.98563	2.23847	2.12789		
0.28	0.94700	0.98456	2.16555	2.05078		
0.29	0.94329	0.98346	2.09793	1.97896		
0.30	0.93947	0.98232	2.03507	1.91188		
0.31	0.93554	0.98114	1.97651	1.84910		
0.32	0.93150	0.97993	1.92185	1.79021		
0.33	0.92736	0.97868	1.87074	1.73486		
0.34	0.92312	0.97740	1.82288	1.68273		
0.35	0.91877	0.97609	1.77797	1.63355		
0.36	0.91433	0.97473	1.73578	1.58707		
0.37	0.90979	0.97335	1.69609	1.54308		
0.38	0.90516	0.97193	1.65870	1.50138		
0.39	0.90043	0.97048	1.62343	1.46179		
0.40	0.89561	0.96899	1.59014	1.42415		
0.41	0.89071	0.96747	1.55867	1.38833		
0.42	0.88572	0.96592	1.52890	1.35419		
0.43	0.88065	0.96434	1.50072	1.32161		
0.44	0.87550	0.96272	1.47401	1.29049		
0.45	0.87027	0.96108	1.44867	1.26073		
0.46	0.86496	0.95940	1.42463	1.23225		
0.47	0.85958	0.95769	1.40180	1.20495		
0.46	0.85413	0.95595	1.38010	1.17878		
0.49	0.84861	0.95418	1.35947	1.15365		
0.50	0.84302	0.95238	1.33984	1.12951		
0.51	0.83737	0.95055	1.32117	1.10630		
0.52	0.83165	0.94869	1.30339	1.08397		
0.53	0.82588	0.94681	1.28645	1.06246		
0.54	0.82005	0.94489	1.27032	1.04173		
0.55	0.81417	0.94295	1.25495	1.02173		
0.56	0.80823	0.94098	1.24029	1.00244		
0.57	0.80224	0.93898	1.22633	0.98381		
0.58	0.79621	0.93696	1.21301	0.96580		
0.59	0.79013	0.93491	1.20031	0.94840		
0.60	0.78400	0.93284	1.18820	0.93155		

M	p/p_t	T/T_t	A/A^*	pA/p_tA^*	ν	μ
0.61	0.77784	0.93073	1.17665	0.91525		
0.62	0.77164	0.92861	1.16565	0.89946		
0.63	0.76540	0.92646	1.15515	0.88416		
0.64	0.75913	0.92428	1.14515	0.86932		
0.65	0.75283	0.92208	1.13562	0.85493		
0.66	0.74650	0.91986	1.12654	0.84096		
0.67	0.74014	0.91762	1.11789	0.82739		
0.68	0.73376	0.91535	1.10965	0.81422		
0.69	0.72735	0.91306	1.10182	0.80141		
0.70	0.72093	0.91075	1.09437	0.78896		
0.71	0.71448	0.90841	1.08729	0.77685		
0.72	0.70803	0.90606	1.08057	0.76507		
0.73	0.70155	0.90369	1.07419	0.75360		
0.74	0.69507	0.90129	1.06814	0.74243		
0.75	0.68857	0.89888	1.06242	0.73155		
0.76	0.68207	0.89644	1.05700	0.72095		
0.77	0.67556	0.89399	1.05188	0.71061		
0.78	0.66905	0.89152	1.04705	0.70053		
0.79	0.66254	0.88903	1.04251	0.69070		
0.80	0.65602	0.88652	1.03823	0.68110		
0.81	0.64951	0.88400	1.03422	0.67173		
0.82	0.64300	0.88146	1.03046	0.66259		
0.83	0.63650	0.87890	1.02696	0.65366		
0.84	0.63000	0.87633	1.02370	0.64493		
0.85	0.62351	0.87374	1.02067	0.63640		
0.86	0.61703	0.87114	1.01787	0.62806		
0.87	0.61057	0.86852	1.01530	0.61991		
0.88	0.60412	0.86589	1.01294	0.61193		
0.89	0.59768	0.86324	1.01080	0.60413		
0.90	0.59126	0.86059	1.00886	0.59650		
0.91	0.58486	0.85791	1.00713	0.58903		
0.92	0.57848	0.85523	1.00560	0.58171		
0.93	0.57211	0.85253	1.00426	0.57455		
0.94	0.56578	0.84982	1.00311	0.56753		
0.95	0.55946	0.84710	1.00215	0.56066		
0.96	0.55317	0.84437	1.00136	0.55392		
0.97	0.54691	0.84162	1.00076	0.54732		
0.98	0.54067	0.83887	1.00034	0.54085		
0.99	0.53446	0.83611	1.00008	0.53451		
1.00	0.52828	0.83333	1.00000	0.52828	0.0	90.0000
1.01	0.52213	0.83055	1.00008	0.52218	0.04472	81.9307

(Continued)

M	p/p_t	T/T_t	A/A^*	pA/p_tA^*	ν	μ
1.02	0.51602	0.82776	1.00033	0.51619	0.12569	78.6351
1.03	0.50994	0.82496	1.00074	0.51031	0.22943	76.1376
1.04	0.50389	0.82215	1.00131	0.50454	0.35098	74.0576
1.05	0.49787	0.81934	1.00203	0.49888	0.48741	72.2472
1.06	0.49189	0.81651	1.00291	0.49332	0.63669	70.6300
1.07	0.48595	0.81368	1.00394	0.48787	0.79729	69.1603
1.08	0.48005	0.81085	1.00512	0.48250	0.96804	67.8084
1.09	0.47418	0.80800	1.00645	0.47724	1.14795	66.5534
1.10	0.46835	0.80515	1.00793	0.47207	1.33620	65.3800
1.11	0.46257	0.80230	1.00955	0.46698	1.53210	64.2767
1.12	0.45682	0.79944	1.01131	0.46199	1.73504	63.2345
1.13	0.45111	0.79657	1.01322	0.45708	1.94448	62.2461
1.14	0.44545	0.79370	1.01527	0.45225	2.15996	61.3056
1.15	0.43983	0.79083	1.01745	0.44751	2.38104	60.4082
1.16	0.43425	0.78795	1.01978	0.44284	2.60735	59.5497
1.17	0.42872	0.78506	1.02224	0.43825	2.83852	58.7267
1.18	0.42322	0.78218	1.02484	0.43374	3.07426	57.9362
1.19	0.41778	0.77929	1.02757	0.42930	3.31425	57.1756
1.20	0.41238	0.77640	1.03044	0.42493	3.55823	56.4427
1.21	0.40702	0.77350	1.03344	0.42063	3.80596	55.7354
1.22	0.40171	0.77061	1.03657	0.41640	4.05720	55.0520
1.23	0.39645	0.76771	1.03983	0.41224	4.31173	54.3909
1.24	0.39123	0.76481	1.04323	0.40814	4.56936	53.7507
1.25	0.38606	0.76190	1.04675	0.40411	4.82989	53.1301
1.26	0.38093	0.75900	1.05041	0.40014	5.09315	52.5280
1.27	0.37586	0.75610	1.05419	0.39622	5.35897	51.9433
1.28	0.37083	0.75319	1.05810	0.39237	5.62720	51.3752
1.29	0.36585	0.75029	1.06214	0.38858	5.89768	50.8226
1.30	0.36091	0.74738	1.06630	0.38484	6.17029	50.2849
1.31	0.35603	0.74448	1.07060	0.38116	6.44488	49.7612
1.32	0.35119	0.74158	1.07502	0.37754	6.72133	49.2509
1.33	0.34640	0.73867	1.07957	0.37396	6.99953	48.7535
1.34	0.34166	0.73577	1.08424	0.37044	7.27937	48.2682
1.35	0.33697	0.73287	1.08904	0.36697	7.56072	47.7946
1.36	0.33233	0.72997	1.09396	0.36355	7.84351	47.3321
1.37	0.32773	0.72707	1.09902	0.36018	8.12762	46.8803
1.38	0.32319	0.72418	1.10419	0.35686	8.41297	46.4387
1.39	0.31869	0.72128	1.10950	0.35359	8.69946	46.0070
1.40	0.31424	0.71839	1.11493	0.35036	8.98702	45.5847
1.41	0.30984	0.71550	1.12048	0.34717	9.27556	45.1715
1.42	0.30549	0.71262	1.12616	0.34403	9.56502	44.7670
1.43	0.30118	0.70973	1.13197	0.34093	9.85531	44.3709
1.44	0.29693	0.70685	1.13790	0.33788	10.14636	43.9830

M	p/p_t	T/T_t	A/A^*	pA/p_tA^*	ν	μ
1.45	0.29272	0.70398	1.14396	0.33486	10.43811	43.6028
1.46	0.28856	0.70110	1.15015	0.33189	10.73050	43.2302
1.47	0.28445	0.69824	1.15646	0.32896	11.02346	42.8649
1.48	0.28039	0.69537	1.16290	0.32606	11.31694	42.5066
1.49	0.27637	0.69251	1.16947	0.32321	11.61087	42.1552
1.50	0.27240	0.68966	1.17617	0.32039	11.90521	41.8103
1.51	0.26848	0.68680	1.18299	0.31761	12.19990	41.4718
1.52	0.26461	0.68396	1.18994	0.31487	12.49489	41.1395
1.53	0.26078	0.68112	1.19702	0.31216	12.79014	40.8132
1.54	0.25700	0.67828	1.20423	0.30949	13.08559	40.4927
1.55	0.25326	0.67545	1.21157	0.30685	13.38121	40.1778
1.56	0.24957	0.67262	1.21904	0.30424	13.67696	39.8683
1.57	0.24593	0.66980	1.22664	0.30167	13.97278	39.5642
1.58	0.24233	0.66699	1.23438	0.29913	14.26865	39.2652
1.59	0.23878	0.66418	1.24224	0.29662	14.56452	38.9713
1.60	0.23527	0.66138	1.25023	0.29414	14.86035	38.6822
1.61	0.23181	0.65858	1.25836	0.29170	15.15612	38.3978
1.62	0.22839	0.65579	1.26663	0.28928	15.45180	38.1181
1.63	0.22501	0.65301	1.27502	0.28690	15.74733	37.8428
1.64	0.22168	0.65023	1.28355	0.28454	16.04271	37.5719
1.65	0.21839	0.64746	1.29222	0.28221	16.33789	37.3052
1.66	0.21515	0.64470	1.30102	0.27991	16.63284	37.0427
1.67	0.21195	0.64194	1.30996	0.27764	16.92755	36.7842
1.68	0.20879	0.63919	1.31904	0.27540	17.22198	36.5296
1.69	0.20567	0.63645	1.32825	0.27318	17.51611	36.2789
1.70	0.20259	0.63371	1.33761	0.27099	17.80991	36.0319
1.71	0.19956	0.63099	1.34710	0.26883	18.10336	35.7885
1.72	0.19656	0.62827	1.35674	0.26669	18.39643	35.5487
1.73	0.19361	0.62556	1.36651	0.26457	18.68911	35.3124
1.74	0.19070	0.62285	1.37643	0.26248	18.98137	35.0795
1.75	0.18782	0.62016	1.38649	0.26042	19.27319	34.8499
1.76	0.18499	0.61747	1.39670	0.25837	19.56456	34.6235
1.77	0.18219	0.61479	1.40705	0.25636	19.85544	34.4003
1.78	0.17944	0.61211	1.41755	0.25436	20.14584	34.1802
1.79	0.17672	0.60945	1.42819	0.25239	20.43571	33.9631
1.80	0.17404	0.60680	1.43898	0.25044	20.72506	33.7490
1.81	0.17140	0.60415	1.44992	0.24851	21.01387	33.5377
1.82	0.16879	0.60151	1.46101	0.24661	21.30211	33.3293
1.83	0.16622	0.59888	1.47225	0.24472	21.58977	33.1237
1.84	0.16369	0.59626	1.48365	0.24286	21.87685	32.9207
1.85	0.16119	0.59365	1.49519	0.24102	22.16332	32.7204

(*Continued*)

M	p/p_t	T/T_t	A/A^*	pA/p_tA^*	ν	μ
1.86	0.15873	0.59104	1.50689	0.23920	22.44917	32.5227
1.87	0.15631	0.58845	1.51875	0.23739	22.73439	32.3276
1.88	0.15392	0.58586	1.53076	0.23561	23.01896	32.1349
1.89	0.15156	0.58329	1.54293	0.23385	23.30288	31.9447
1.90	0.14924	0.58072	1.55526	0.23211	23.58613	31.7569
1.91	0.14695	0.57816	1.56774	0.23038	23.86871	31.5714
1.92	0.14470	0.57561	1.58039	0.22868	24.15059	31.3882
1.93	0.14247	0.57307	1.59320	0.22699	24.43178	31.2072
1.94	0.14028	0.57054	1.60617	0.22532	24.71226	31.0285
1.95	0.13813	0.56802	1.61931	0.22367	24.99202	30.8519
1.96	0.13600	0.56551	1.63261	0.22203	25.27105	30.6774
1.97	0.13390	0.56301	1.64608	0.22042	25.54935	30.5050
1.98	0.13184	0.56051	1.65972	0.21882	25.82691	30.3347
1.99	0.12981	0.55803	1.67352	0.21724	26.10371	30.1664
2.00	0.12780	0.55556	1.68750	0.21567	26.37976	30.0000
2.01	0.12583	0.55309	1.70165	0.21412	26.65504	29.8356
2.02	0.12389	0.55064	1.71597	0.21259	26.92955	29.6730
2.03	0.12197	0.54819	1.73047	0.21107	27.20328	29.5123
2.04	0.12009	0.54576	1.74514	0.20957	27.47622	29.3535
2.05	0.11823	0.54333	1.75999	0.20808	27.74837	29.1964
2.06	0.11640	0.54091	1.77502	0.20661	28.01973	29.0411
2.07	0.11460	0.53851	1.79022	0.20516	28.29028	28.8875
2.08	0.11282	0.53611	1.80561	0.20371	28.56003	28.7357
2.09	0.11107	0.53373	1.82119	0.20229	28.82896	28.5855
2.10	0.10935	0.53135	1.83694	0.20088	29.09708	28.4369
2.11	0.10766	0.52898	1.85289	0.19948	29.36438	28.2899
2.12	0.10599	0.52663	1.86902	0.19809	29.63085	28.1446
2.13	0.10434	0.52428	1.88533	0.19672	29.89649	28.0008
2.14	0.10273	0.52194	1.90184	0.19537	30.16130	27.8585
2.15	0.10113	0.51962	1.91854	0.19403	30.42527	27.7177
2.16	0.09956	0.51730	1.93544	0.19270	30.68841	27.5785
2.17	0.09802	0.51499	1.95252	0.19138	30.95070	27.4406
2.18	0.09649	0.51269	1.96981	0.19008	31.21215	27.3043
2.19	0.09500	0.51041	1.98729	0.18879	31.47275	27.1693
2.20	0.09352	0.50813	2.00497	0.18751	31.73250	27.0357
2.21	0.09207	0.50586	2.02286	0.18624	31.99139	26.9035
2.22	0.09064	0.50361	2.04094	0.18499	32.24943	26.7726
2.23	0.08923	0.50136	2.05923	0.18375	32.50662	26.6430
2.24	0.08785	0.49912	2.07773	0.18252	32.76294	26.5148
2.25	0.08648	0.49689	2.09644	0.18130	33.01841	26.3878
2.26	0.08514	0.49468	2.11535	0.18010	33.27301	26.2621
2.27	0.08382	0.49247	2.13447	0.17890	33.52676	26.1376
2.28	0.08251	0.49027	2.15381	0.17772	33.77963	26.0144

M	p/p_t	T/T_t	A/A^*	pA/p_tA^*	ν	μ
2.29	0.08123	0.48809	2.17336	0.17655	34.03165	25.8923
2.30	0.07997	0.48591	2.19313	0.17539	34.28279	25.7715
2.31	0.07873	0.48374	2.21312	0.17424	34.53307	25.6518
2.32	0.07751	0.48158	2.23332	0.17310	34.78249	25.5332
2.33	0.07631	0.47944	2.25375	0.17198	35.03103	25.4158
2.34	0.07512	0.47730	2.27440	0.17086	35.27871	25.2995
2.35	0.07396	0.47517	2.29528	0.16975	35.52552	25.1843
2.36	0.07281	0.47305	2.31638	0.16866	35.77146	25.0702
2.37	0.07168	0.47095	2.33771	0.16757	36.01653	24.9572
2.38	0.07057	0.46885	2.35928	0.16649	36.26073	24.8452
2.39	0.06948	0.46676	2.38107	0.16543	36.50406	24.7342
2.40	0.06840	0.46468	2.40310	0.16437	36.74653	24.6243
2.41	0.06734	0.46262	2.42537	0.16332	36.98813	24.5154
2.42	0.06630	0.46056	2.44787	0.16229	37.22886	24.4075
2.43	0.06527	0.45851	2.47061	0.16126	37.46872	24.3005
2.44	0.06426	0.45647	2.49360	0.16024	37.70772	24.1945
2.45	0.06327	0.45444	2.51683	0.15923	37.94585	24.0895
2.46	0.06229	0.45242	2.54031	0.15823	38.18312	23.9854
2.47	0.06133	0.45041	2.56403	0.15724	38.41952	23.8822
2.48	0.06038	0.44841	2.58801	0.15626	38.65507	23.7800
2.49	0.05945	0.44642	2.61224	0.15529	38.88974	23.6786
2.50	0.05853	0.44444	2.63672	0.15432	39.12356	23.5782
2.51	0.05762	0.44247	2.66146	0.15337	39.35652	23.4786
2.52	0.05674	0.44051	2.68645	0.15242	39.58862	23.3799
2.53	0.05586	0.43856	2.71171	0.15148	39.81987	23.2820
2.54	0.05500	0.43662	2.73723	0.15055	40.05026	23.1850
2.55	0.05415	0.43469	2.76301	0.14963	40.27979	23.0888
2.56	0.05332	0.43277	2.78906	0.14871	40.50847	22.9934
2.57	0.05250	0.43085	2.81538	0.14780	40.73630	22.8988
2.58	0.05169	0.42895	2.84197	0.14691	40.96329	22.8051
2.59	0.05090	0.42705	2.86884	0.14602	41.18942	22.7121
2.60	0.05012	0.42517	2.89598	0.14513	41.41471	22.6199
2.61	0.04935	0.42329	2.92339	0.14426	41.63915	22.5284
2.62	0.04859	0.42143	2.95109	0.14339	41.86275	22.4377
2.63	0.04784	0.41957	2.97907	0.14253	42.08551	22.3478
2.64	0.04711	0.41772	3.00733	0.14168	42.30744	22.2586
2.65	0.04639	0.41589	3.03588	0.14083	42.52852	22.1702
2.66	0.04568	0.41406	3.06472	0.13999	42.74877	22.0824
2.67	0.04498	0.41224	3.09385	0.13916	42.96819	21.9954
2.68	0.04429	0.41043	3.12327	0.13834	43.18678	21.9090
2.69	0.04362	0.40863	3.15299	0.13752	43.40454	21.8234

(Continued)

M	p/p_t	T/T_t	A/A^*	pA/p_tA^*	ν	μ
2.70	0.04295	0.40683	3.18301	0.13671	43.62148	21.7385
2.71	0.04229	0.40505	3.21333	0.13591	43.83759	21.6542
2.72	0.04165	0.40328	3.24395	0.13511	44.05288	21.5706
2.73	0.04102	0.40151	3.27488	0.13432	44.26735	21.4876
2.74	0.04039	0.39976	3.30611	0.13354	44.48100	21.4053
2.75	0.03978	0.39801	3.33766	0.13276	44.69384	21.3237
2.76	0.03917	0.39627	3.36952	0.13199	44.90586	21.2427
2.77	0.03858	0.39454	3.40169	0.13123	45.11708	21.1623
2.78	0.03799	0.39282	3.43418	0.13047	45.32749	21.0825
2.79	0.03742	0.39111	3.46699	0.12972	45.53709	21.0034
2.80	0.03685	0.38941	3.50012	0.12897	45.74589	20.9248
2.81	0.03629	0.38771	3.53358	0.12823	45.95389	20.8469
2.82	0.03574	0.38603	3.56737	0.12750	46.16109	20.7695
2.83	0.03520	0.38435	3.60148	0.12678	46.36750	20.6928
2.84	0.03467	0.38268	3.63593	0.12605	46.57312	20.6166
2.85	0.03415	0.38102	3.67072	0.12534	46.77794	20.5410
2.86	0.03363	0.37937	3.70584	0.12463	46.98198	20.4659
2.87	0.03312	0.37773	3.74131	0.12393	47.18523	20.3914
2.88	0.03263	0.37610	3.77711	0.12323	47.38770	20.3175
2.89	0.03213	0.37447	3.81327	0.12254	47.58940	20.2441
2.90	0.03165	0.37286	3.84977	0.12185	47.79031	20.1713
2.91	0.03118	0.37125	3.88662	0.12117	47.99045	20.0990
2.92	0.03071	0.36965	3.92383	0.12049	48.18982	20.0272
2.93	0.03025	0.36806	3.96139	0.11982	48.38842	19.9559
2.94	0.02980	0.36647	3.99932	0.11916	48.58626	19.8852
2.95	0.02935	0.36490	4.03760	0.11850	48.78333	19.8149
2.96	0.02891	0.36333	4.07625	0.11785	48.97965	19.7452
2.97	0.02848	0.36177	4.11527	0.11720	49.17520	19.6760
2.98	0.02805	0.36022	4.15466	0.11655	49.37000	19.6072
2.99	0.02764	0.35868	4.19443	0.11591	49.56405	19.5390
3.00	0.02722	0.35714	4.23457	0.11528	49.75735	19.4712
3.01	0.02682	0.35562	4.27509	0.11465	49.94990	19.4039
3.02	0.02642	0.35410	4.31599	0.11403	50.14171	19.3371
3.03	0.02603	0.35259	4.35728	0.11341	50.33277	19.2708
3.04	0.02564	0.35108	4.39895	0.11279	50.52310	19.2049
3.05	0.02526	0.34959	4.44102	0.11219	50.71270	19.1395
3.06	0.02489	0.34810	4.48347	0.11158	50.90156	19.0745
3.07	0.02452	0.34662	4.52633	0.11098	51.08969	19.0100
3.08	0.02416	0.34515	4.56959	0.11039	51.27710	18.9459
3.09	0.02380	0.34369	4.61325	0.10979	51.46378	18.8823
3.10	0.02345	0.34223	4.65731	0.10921	51.64974	18.8191
3.11	0.02310	0.34078	4.70178	0.10863	51.83499	18.7563
3.12	0.02276	0.33934	4.74667	0.10805	52.01952	18.6939

M	p/p_t	T/T_t	A/A^*	pA/p_tA^*	ν	μ
3.13	0.02243	0.33791	4.79197	0.10748	52.20333	18.6320
3.14	0.02210	0.33648	4.83769	0.10691	52.38644	18.5705
3.15	0.02177	0.33506	4.88383	0.10634	52.56884	18.5094
3.16	0.02146	0.33365	4.93039	0.10578	52.75053	18.4487
3.17	0.02114	0.33225	4.97739	0.10523	52.93153	18.3884
3.18	0.02083	0.33085	5.02481	0.10468	53.11182	18.3285
3.19	0.02053	0.32947	5.07266	0.10413	53.29143	18.2691
3.20	0.02023	0.32808	5.12096	0.10359	53.47033	18.2100
3.21	0.01993	0.32671	5.16969	0.10305	53.64855	18.1512
3.22	0.01964	0.32534	5.21887	0.10251	53.82609	18.0929
3.23	0.01936	0.32398	5.26849	0.10198	54.00294	18.0350
3.24	0.01908	0.32263	5.31857	0.10145	54.17910	17.9774
3.25	0.01880	0.32129	5.36909	0.10093	54.35459	17.9202
3.26	0.01853	0.31995	5.42008	0.10041	54.52941	17.8634
3.27	0.01826	0.31862	5.47152	0.09989	54.70355	17.8069
3.28	0.01799	0.31729	5.52343	0.09938	54.87703	17.7508
3.29	0.01773	0.31597	5.57580	0.09887	55.04983	17.6951
3.30	0.01748	0.31466	5.62865	0.09837	55.22198	17.6397
3.31	0.01722	0.31336	5.68196	0.09787	55.39346	17.5847
3.32	0.01698	0.31206	5.73576	0.09737	55.56428	17.5300
3.33	0.01673	0.31077	5.79003	0.09688	55.73445	17.4756
3.34	0.01649	0.30949	5.84479	0.09639	55.90396	17.4216
3.35	0.01625	0.30821	5.90004	0.09590	56.07283	17.3680
3.36	0.01602	0.30694	5.95577	0.09542	56.24105	17.3147
3.37	0.01579	0.30568	6.01201	0.09494	56.40862	17.2617
3.38	0.01557	0.30443	6.06873	0.09447	56.57556	17.2090
3.39	0.01534	0.30318	6.12596	0.09399	56.74185	17.1567
3.40	0.01512	0.30193	6.18370	0.09353	56.90751	17.1046
3.41	0.01491	0.30070	6.24194	0.09306	57.07254	17.0529
3.42	0.01470	0.29947	6.30070	0.09260	57.23694	17.0016
3.43	0.01449	0.29824	6.35997	0.09214	57.40071	16.9505
3.44	0.01428	0.29702	6.41976	0.09168	57.56385	16.8997
3.45	0.01408	0.29581	6.48007	0.09123	57.72637	16.8493
3.46	0.01388	0.29461	6.54092	0.09078	57.88828	16.7991
3.47	0.01368	0.29341	6.60229	0.09034	58.04957	16.7493
3.48	0.01349	0.29222	6.66419	0.08989	58.21024	16.6997
3.49	0.01330	0.29103	6.72664	0.08945	58.37030	16.6505
3.50	0.01311	0.28936	6.78962	0.08902	58.52976	16.6015
3.51	0.01293	0.28868	6.85315	0.08858	58.68861	16.5529
3.52	0.01274	0.28751	6.91723	0.08815	58.84685	16.5045
3.53	0.01256	0.28635	6.98186	0.08773	59.00450	16.4564

(*Continued*)

M	p/p_t	T/T_t	A/A^*	pA/p_tA^*	ν	μ
3.54	0.01239	0.28520	7.04705	0.08730	59.16155	16.4086
3.55	0.01221	0.28405	7.11281	0.08688	59.31801	16.3611
3.56	0.01204	0.28291	7.17912	0.08646	59.47387	16.3139
3.57	0.01188	0.28177	7.24601	0.08605	59.62914	16.2669
3.58	0.01171	0.28064	7.31346	0.08563	59.78383	16.2202
3.59	0.01155	0.27952	7.38150	0.08522	59.93793	16.1738
3.60	0.01138	0.27840	7.45011	0.08482	60.09146	16.1276
3.61	0.01123	0.27728	7.51931	0.08441	60.24440	16.0817
3.62	0.01107	0.27618	7.58910	0.08401	60.39677	16.0361
3.63	0.01092	0.27507	7.65948	0.08361	60.54856	15.9907
3.64	0.01076	0.27398	7.73045	0.08322	60.69978	15.9456
3.65	0.01062	0.27289	7.80203	0.08282	60.85044	15.9008
3.66	0.01047	0.27180	7.87421	0.08243	61.00052	15.8562
3.67	0.01032	0.27073	7.94700	0.08205	61.15005	15.8119
3.66	0.01018	0.26965	8.02040	0.08166	61.29902	15.7678
3.69	0.01004	0.26858	8.09442	0.08128	61.44742	15.7239
3.70	0.00990	0.26752	8.16907	0.08090	61.59527	15.6803
3.71	0.00977	0.26647	8.24433	0.08052	61.74257	15.6370
3.72	0.00963	0.26542	8.32023	0.08014	61.88932	15.5939
3.73	0.00950	0.26437	8.39676	0.07977	62.03552	15.5510
3.74	0.00937	0.26333	8.47393	0.07940	62.18118	15.5084
3.75	0.00924	0.26230	8.55174	0.07904	62.32629	15.4660
3.76	0.00912	0.26127	8.63020	0.07867	62.47086	15.4239
3.77	0.00899	0.26024	8.70931	0.07831	62.61490	15.3819
3.78	0.00887	0.25922	8.78907	0.07795	62.75840	15.3402
3.79	0.00875	0.25821	8.86950	0.07759	62.90136	15.2988
3.80	0.00863	0.25720	8.95059	0.07723	63.04380	15.2575
3.81	0.00851	0.25620	9.03234	0.07688	63.18571	15.2165
3.82	0.00840	0.25520	9.11477	0.07653	63.32709	15.1757
3.83	0.00828	0.25421	9.19788	0.07618	63.46795	15.1351
3.84	0.00817	0.25322	9.28167	0.07584	63.60829	15.0948
3.85	0.00806	0.25224	9.36614	0.07549	63.74811	15.0547
3.86	0.00795	0.25126	9.45131	0.07515	63.88741	15.0147
3.87	0.00784	0.25029	9.53717	0.07481	64.02620	14.9750
3.88	0.00774	0.24932	9.62373	0.07447	64.16448	14.9355
3.89	0.00763	0.24836	9.71100	0.07414	64.30225	14.8962
3.90	0.00753	0.24740	9.79897	0.07381	64.43952	14.8572
3.91	0.00743	0.24645	9.88766	0.07348	64.57628	14.8183
3.92	0.00733	0.24550	9.97707	0.07315	64.71254	14.7796
3.93	0.00723	0.24456	10.06720	0.07282	64.84829	14.7412
3.94	0.00714	0.24362	10.15806	0.07250	64.98356	14.7029
3.95	0.00704	0.24269	10.24965	0.07217	65.11832	14.6649
3.96	0.00695	0.24176	10.34197	0.07185	65.25260	14.6270

M	p/p_t	T/T_t	A/A^*	pA/p_tA^*	ν	μ
3.97	0.00686	0.24084	10.43504	0.07154	65.38638	14.5893
3.98	0.00676	0.23992	10.52886	0.07122	65.51968	14.5519
3.99	0.00667	0.23900	10.62343	0.07091	65.65249	14.5146
4.00	0.00659	0.23810	10.71875	0.07059	65.78482	14.4775
4.10	0.00577	0.22925	11.71465	0.06758	67.08200	14.1170
4.20	0.00506	0.22085	12.79164	0.06475	68.33324	13.7741
4.30	0.00445	0.21286	13.95490	0.06209	69.54063	13.4477
4.40	0.00392	0.20525	15.20987	0.05959	70.70616	13.1366
4.50	0.00346	0.19802	16.56219	0.05723	71.83174	12.8396
4.60	0.00305	0.19113	18.01779	0.05500	72.91915	12.5559
4.70	0.00270	0.18457	19.58283	0.05290	73.97012	12.2845
4.80	0.00239	0.17832	21.26371	0.05091	74.98627	12.0247
4.90	0.00213	0.17235	23.06712	0.04903	75.96915	11.7757
5.00	0.00189	0.16667	25.00000	0.04725	76.92021	11.5370
5.10	0.00168	0.16124	27.06957	0.04556	77.84087	11.3077
5.20	0.00150	0.15605	29.28333	0.04396	78.73243	11.0875
5.30	0.00134	0.15110	31.64905	0.04244	79.59616	10.8757
5.40	0.00120	0.14637	34.17481	0.04100	80.43323	10.6719
5.50	0.00107	0.14184	36.86896	0.03963	81.24479	10.4757
5.60	0.000964	0.13751	39.74018	0.03832	82.03190	10.2866
5.70	0.000866	0.13337	42.79743	0.03708	82.79558	10.1042
5.80	0.000779	0.12940	46.05000	0.03589	83.53681	9.9282
5.90	0.000702	0.12560	49.50747	0.03476	84.25649	9.7583
6.00	0.000633	0.12195	53.17978	0.03368	84.95550	9.5941
6.10	0.000572	0.11846	57.07718	0.03265	85.63467	9.4353
6.20	0.000517	0.11510	61.21023	0.03167	86.29479	9.2818
6.30	0.000468	0.11188	65.58987	0.03073	86.93661	9.1332
6.40	0.000425	0.10879	70.22736	0.02982	87.56084	8.9893
6.50	0.000385	0.10582	75.13431	0.02896	88.16816	8.8499
6.60	0.000350	0.10297	80.32271	0.02814	88.75922	8.7147
6.70	0.000319	0.10022	85.80487	0.02734	89.33463	8.5837
6.80	0.000290	0.09758	91.59351	0.02658	89.89499	8.4565
6.90	0.000265	0.09504	97.70169	0.02586	90.44084	8.3331
7.00	0.000242	0.09259	104.14286	0.02516	90.97273	8.2132
7.50	0.000155	0.08163	141.84148	0.02205	93.43967	7.6623
8.00	0.000102	0.07246	190.10937	0.01947	95.62467	7.1808
8.50	0.0000690	0.06472	251.086167	0.01732	97.57220	6.7563
9.00	0.0000474	0.05814	327.189300	0.01550	99.31810	6.3794
9.50	0.0000331	0.05249	421.131373	0.01396	100.89148	6.0423
10.00	0.0000236	0.04762	535.937500	0.01263	102.31625	5.7392
∞	0.0	0.0	∞	0.0	130.4541	0.0

Appendix H

Normal-Shock Parameters ($\gamma = 1.4$)

M_1	M_2	p_2/p_1	T_2/T_1	$\Delta V/a_1$	p_{t2}/p_{t1}	p_{t2}/p_1
1.00	1.00000	1.00000	1.00000	0.0	1.00000	1.89293
1.01	0.99013	1.02345	1.00664	0.01658	1.00000	1.91521
1.02	0.98052	1.04713	1.01325	0.03301	0.99999	1.93790
1.03	0.97115	1.07105	1.01981	0.04927	0.99997	1.96097
1.04	0.96203	1.09520	1.02634	0.06538	0.99992	1.98442
1.05	0.95313	1.11958	1.03284	0.08135	0.99985	2.00825
1.06	0.94445	1.14420	1.03931	0.09717	0.99975	2.03245
1.07	0.93598	1.16905	1.04575	0.11285	0.99961	2.05702
1.08	0.92771	1.19413	1.05217	0.12840	0.99943	2.08194
1.09	0.91965	1.21945	1.05856	0.14381	0.99920	2.10722
1.10	0.91177	1.24500	1.06494	0.15909	0.99893	2.13285
1.11	0.90408	1.27078	1.07129	0.17425	0.99860	2.15882
1.12	0.89656	1.29680	1.07763	0.18929	0.99821	2.18513
1.13	0.88922	1.32305	1.08396	0.20420	0.99777	2.21178
1.14	0.88204	1.34953	1.09027	0.21901	0.99726	2.23877
1.15	0.87502	1.37625	1.09658	0.23370	0.99669	2.26608
1.16	0.86816	1.40320	1.10287	0.24828	0.99605	2.29372
1.17	0.86145	1.43038	1.10916	0.26275	0.99535	2.32169

(*Continued*)

M_1	M_2	p_2/p_1	T_2/T_1	$\Delta V/a_1$	p_{t2}/p_{t1}	p_{t2}/p_1
1.18	0.85488	1.45780	1.11544	0.27712	0.99457	2.34998
1.19	0.84846	1.48545	1.12172	0.29139	0.99372	2.37858
1.20	0.84217	1.51333	1.12799	0.30556	0.99280	2.40750
1.21	0.83601	1.54145	1.13427	0.31963	0.99180	2.43674
1.22	0.82999	1.56980	1.14054	0.33361	0.99073	2.46628
1.23	0.82408	1.59838	1.14682	0.34749	0.98958	2.49613
1.24	0.81830	1.62720	1.15309	0.36129	0.98836	2.52629
1.25	0.81264	1.65625	1.15937	0.37500	0.98706	2.55676
1.26	0.80709	1.68553	1.16566	0.38862	0.98568	2.58753
1.27	0.80164	1.71505	1.17195	0.40217	0.98422	2.61860
1.28	0.79631	1.74480	1.17825	0.41562	0.98268	2.64996
1.29	0.79108	1.77478	1.18456	0.42901	0.98107	2.68163
1.30	0.78596	1.80500	1.19087	0.44231	0.97937	2.71359
1.31	0.78093	1.83545	1.19720	0.45553	0.97760	2.74585
1.32	0.77600	1.86613	1.20353	0.46869	0.97575	2.77840
1.33	0.77116	1.89705	1.20988	0.48177	0.97382	2.81125
1.34	0.76641	1.92820	1.21624	0.49478	0.97182	2.84438
1.35	0.76175	1.95958	1.22261	0.50772	0.96974	2.87781
1.36	0.75718	1.99120	1.22900	0.52059	0.96758	2.91152
1.37	0.75269	2.02305	1.23540	0.53339	0.96534	2.94552
1.38	0.74829	2.05513	1.24181	0.54614	0.96304	2.97981
1.39	0.74396	2.08745	1.24825	0.55881	0.96065	3.01438
1.40	0.73971	2.12000	1.25469	0.57143	0.95819	3.04924
1.41	0.73554	2.15278	1.26116	0.58398	0.95566	3.08438
1.42	0.73144	2.18580	1.26764	0.59648	0.95306	3.11980
1.43	0.72741	2.21905	1.27414	0.60892	0.95039	3.15551
1.44	0.72345	2.25253	1.28066	0.62130	0.94765	3.19149
1.45	0.71956	2.28625	1.28720	0.63362	0.94484	3.22776
1.46	0.71574	2.32020	1.29377	0.64589	0.94196	3.26431
1.47	0.71198	2.35438	1.30035	0.65811	0.93901	3.30113
1.48	0.70829	2.38880	1.30695	0.67027	0.93600	3.33823
1.49	0.70466	2.42345	1.31357	0.68238	0.93293	3.37562
1.50	0.70109	2.45833	1.32022	0.69444	0.92979	3.41327
1.51	0.69758	2.49345	1.32688	0.70646	0.92659	3.45121
1.52	0.69413	2.52880	1.33357	0.71842	0.92332	3.48942
1.53	0.69073	2.56438	1.34029	0.73034	0.92000	3.52791
1.54	0.68739	2.60020	1.34703	0.74221	0.91662	3.56667
1.55	0.68410	2.63625	1.35379	0.75403	0.91319	3.60570
1.56	0.68087	2.67253	1.36057	0.76581	0.90970	3.64501
1.57	0.67768	2.70905	1.36738	0.77755	0.90615	3.68459
1.58	0.67455	2.74580	1.37422	0.78924	0.90255	3.72445
1.59	0.67147	2.78278	1.38108	0.80089	0.89890	3.76457
1.60	0.66844	2.82000	1.38797	0.81250	0.89520	3.80497

M_1	M_2	p_2/p_1	T_2/T_1	$\Delta V/a_1$	p_{t2}/p_{t1}	p_{t2}/p_1
1.61	0.66545	2.85745	1.39488	0.82407	0.89145	3.84564
1.62	0.66251	2.89513	1.40182	0.83560	0.88765	3.88658
1.63	0.65962	2.93305	1.40879	0.84709	0.88381	3.92780
1.64	0.65677	2.97120	1.41578	0.85854	0.87992	3.96928
1.65	0.65396	3.00958	1.42280	0.86995	0.87599	4.01103
1.66	0.65119	3.04820	1.42985	0.88133	0.87201	4.05305
1.67	0.64847	3.08705	1.43693	0.89266	0.86800	4.09535
1.68	0.64579	3.12613	1.44403	0.90397	0.86394	4.13791
1.69	0.64315	3.16545	1.45117	0.91524	0.85985	4.18074
1.70	0.64054	3.20500	1.45833	0.92647	0.85572	4.22383
1.71	0.63798	3.24478	1.46552	0.93767	0.85156	4.26720
1.72	0.63545	3.28480	1.47274	0.94884	0.84736	4.31083
1.73	0.63296	3.32505	1.47999	0.95997	0.84312	4.35473
1.74	0.63051	3.36553	1.48727	0.97107	0.83886	4.39890
1.75	0.62809	3.40625	1.49458	0.98214	0.83457	4.44334
1.76	0.62570	3.44720	1.50192	0.99318	0.83024	4.48804
1.77	0.62335	3.48838	1.50929	1.00419	0.82589	4.53301
1.78	0.62104	3.52980	1.51669	1.01517	0.82151	4.57825
1.79	0.61875	3.57145	1.52412	1.02612	0.81711	4.62375
1.80	0.61650	3.61333	1.53158	1.03704	0.81268	4.66952
1.81	0.61428	3.65545	1.53907	1.04793	0.80823	4.71555
1.82	0.61209	3.69780	1.54659	1.05879	0.80376	4.76185
1.83	0.60993	3.74038	1.55415	1.06963	0.79927	4.80841
1.84	0.60780	3.78320	1.56173	1.08043	0.79476	4.85524
1.85	0.60570	3.82625	1.56935	1.09122	0.79023	4.90234
1.86	0.60363	3.86953	1.57700	1.10197	0.78569	4.94970
1.87	0.60158	3.91305	1.58468	1.11270	0.78112	4.99732
1.88	0.59957	3.95680	1.59239	1.12340	0.77655	5.04521
1.89	0.59758	4.00078	1.60014	1.13408	0.77196	5.09336
1.90	0.59562	4.04500	1.60792	1.14474	0.76736	5.14178
1.91	0.59368	4.08945	1.61573	1.15537	0.76274	5.19046
1.92	0.59177	4.13413	1.62357	1.16597	0.75812	5.23940
1.93	0.58988	4.17905	1.63144	1.17655	0.75349	5.28861
1.94	0.58802	4.22420	1.63935	1.18711	0.74884	5.33808
1.95	0.58618	4.26958	1.64729	1.19765	0.74420	5.38782
1.96	0.58437	4.31520	1.65527	1.20816	0.73954	5.43782
1.97	0.58258	4.36105	1.66328	1.21865	0.73488	5.48808
1.98	0.58082	4.40713	1.67132	1.22912	0.73021	5.53860
1.99	0.57907	4.45345	1.67939	1.23957	0.72555	5.58939
2.00	0.57735	4.50000	1.68750	1.25000	0.72087	5.64044
2.01	0.57565	4.54678	1.69564	1.26041	0.71620	5.69175

(*Continued*)

M_1	M_2	p_2/p_1	T_2/T_1	$\Delta V/a_1$	p_{t2}/p_{t1}	p_{t2}/p_1
2.02	0.57397	4.59380	1.70382	1.27079	0.71153	5.74333
2.03	0.57231	4.64105	1.71203	1.28116	0.70685	5.79517
2.04	0.57068	4.68853	1.72027	1.29150	0.70218	5.84727
2.05	0.56906	4.73625	1.72855	1.30183	0.69751	5.89963
2.06	0.56747	4.78420	1.73686	1.31214	0.69284	5.95226
2.07	0.56589	4.83238	1.74521	1.32242	0.68817	6.00514
2.08	0.56433	4.88080	1.75359	1.33269	0.68351	6.05829
2.09	0.56280	4.92945	1.76200	1.34294	0.67885	6.11170
2.10	0.56128	4.97833	1.77045	1.35317	0.67420	6.16537
2.11	0.55978	5.02745	1.77893	1.36339	0.66956	6.21931
2.12	0.55829	5.07680	1.78745	1.37358	0.66492	6.27351
2.13	0.55683	5.12638	1.79601	1.38376	0.66029	6.32796
2.14	0.55538	5.17620	1.80459	1.39393	0.65567	6.38268
2.15	0.55395	5.22625	1.81322	1.40407	0.65105	6.43766
2.16	0.55254	5.27653	1.82188	1.41420	0.64645	6.49290
2.17	0.55115	5.32705	1.83057	1.42431	0.64185	6.54841
2.18	0.54977	5.37780	1.83930	1.43440	0.63727	6.60417
2.19	0.54840	5.42878	1.84806	1.44448	0.63270	6.66019
2.20	0.54706	5.48000	1.85686	1.45455	0.62814	6.71648
2.21	0.54572	5.53145	1.86569	1.46459	0.62359	6.77303
2.22	0.54441	5.58313	1.87456	1.47462	0.61905	6.82983
2.23	0.54311	5.63505	1.88347	1.48464	0.61453	6.88690
2.24	0.54182	5.68720	1.89241	1.49464	0.61002	6.94423
2.25	0.54055	5.73958	1.90138	1.50463	0.60553	7.00182
2.26	0.53930	5.79220	1.91040	1.51460	0.60105	7.05967
2.27	0.53805	5.84505	1.91944	1.52456	0.59659	7.11778
2.28	0.53683	5.89813	1.92853	1.53450	0.59214	7.17616
2.29	0.53561	5.95145	1.93765	1.54443	0.58771	7.23479
2.30	0.53441	6.00500	1.94680	1.55435	0.58329	7.29368
2.31	0.53322	6.05878	1.95599	1.56425	0.57890	7.35283
2.32	0.53205	6.11280	1.96522	1.57414	0.57452	7.41225
2.33	0.53089	6.16705	1.97448	1.58401	0.57015	7.47192
2.34	0.52974	6.22153	1.98378	1.59387	0.56581	7.53185
2.35	0.52861	6.27625	1.99311	1.60372	0.56148	7.59205
2.36	0.52749	6.33120	2.00249	1.61356	0.55718	7.65250
2.37	0.52638	6.38638	2.01189	1.62338	0.55289	7.71321
2.38	0.52528	6.44180	2.02134	1.63319	0.54862	7.77419
2.39	0.52419	6.49745	2.03082	1.64299	0.54437	7.83542
2.40	0.52312	6.55333	2.04033	1.65278	0.54014	7.89691
2.41	0.52206	6.60945	2.04988	1.66255	0.53594	7.95867
2.42	0.52100	6.66560	2.05947	1.67231	0.53175	8.02068
2.43	0.51996	6.72238	2.06910	1.68206	0.52758	8.08295
2.44	0.51894	6.77920	2.07876	1.69180	0.52344	8.14549

M_1	M_2	p_2/p_1	T_2/T_1	$\Delta V/a_1$	p_{t2}/p_{t1}	p_{t2}/p_1
2.45	0.51792	6.83625	2.08846	1.70153	0.51931	8.20828
2.46	0.51691	6.89353	2.09819	1.71125	0.51521	8.27133
2.47	0.51592	6.95105	2.10797	1.72095	0.51113	8.33464
2.48	0.51493	7.00880	2.11777	1.73065	0.50707	8.39821
2.49	0.51395	7.06678	2.12762	1.74033	0.50303	8.46205
2.50	0.51299	7.12500	2.13750	1.75000	0.49901	8.52614
2.51	0.51203	7.18345	2.14742	1.75966	0.49502	8.59049
2.52	0.51109	7.24213	2.15737	1.76931	0.49105	8.65510
2.53	0.51015	7.30105	2.16737	1.77895	0.48711	8.71996
2.54	0.50923	7.36020	2.17739	1.78858	0.48318	8.78509
2.55	0.50831	7.41958	2.18746	1.79820	0.47928	8.85048
2.56	0.50741	7.47920	2.19756	1.80781	0.47540	8.91613
2.57	0.50651	7.53905	2.20770	1.81741	0.47155	8.98203
2.58	0.50562	7.59913	2.21788	1.82700	0.46772	9.04820
2.59	0.50474	7.65945	2.22809	1.83658	0.46391	9.11462
2.60	0.50387	7.72000	2.23834	1.84615	0.46012	9.18131
2.61	0.50301	7.78078	2.24863	1.85572	0.45636	9.24825
2.62	0.50216	7.84180	2.25896	1.86527	0.45263	9.31545
2.63	0.50131	7.90305	2.26932	1.87481	0.44891	9.38291
2.64	0.50048	7.96453	2.27972	1.88434	0.44522	9.45064
2.65	0.49965	8.02625	2.29015	1.89387	0.44156	9.51862
2.66	0.49883	8.08820	2.30063	1.90338	0.43792	9.58685
2.67	0.49802	8.15038	2.31114	1.91289	0.43430	9.65535
2.68	0.49722	8.21280	2.32168	1.92239	0.43070	9.72411
2.69	0.49642	8.27545	2.33227	1.93188	0.42714	9.79312
2.70	0.49563	8.33833	2.34289	1.94136	0.42359	9.86240
2.71	0.49485	8.40145	2.35355	1.95083	0.42007	9.93193
2.72	0.49408	8.46480	2.36425	1.96029	0.41657	10.00173
2.73	0.49332	8.52838	2.37498	1.96975	0.41310	10.07178
2.74	0.49256	8.59220	2.38576	1.97920	0.40965	10.14209
2.75	0.49181	8.65625	2.39657	1.98864	0.40623	10.21266
2.76	0.49107	8.72053	2.40741	1.99807	0.40283	10.28349
2.77	0.49033	8.78505	2.41830	2.00749	0.39945	10.35457
2.78	0.48960	8.84980	2.42922	2.01691	0.39610	10.42592
2.79	0.48888	8.91478	2.44018	2.02631	0.39277	10.49752
2.80	0.48817	8.98000	2.45117	2.03571	0.38946	10.56939
2.81	0.48746	9.04545	2.46221	2.04511	0.38618	10.64151
2.82	0.48676	9.11113	2.47328	2.05449	0.38293	10.71389
2.83	0.48606	9.17705	2.48439	2.06387	0.37969	10.78653
2.84	0.48538	9.24320	2.49554	2.07324	0.37649	10.85943
2.85	0.48469	9.30958	2.50672	2.08260	0.37330	10.93258

(*Continued*)

M_1	M_2	p_2/p_1	T_2/T_1	$\Delta V/a_1$	p_{t2}/p_{t1}	p_{t2}/p_1
2.86	0.48402	9.37620	2.51794	2.09196	0.37014	11.00600
2.87	0.48335	9.44305	2.52920	2.10131	0.36700	11.07967
2.88	0.48269	9.51013	2.54050	2.11065	0.36389	11.15361
2.89	0.48203	9.57745	2.55183	2.11998	0.36080	11.22780
2.90	0.48138	9.64500	2.56321	2.12931	0.35773	11.30225
2.91	0.48073	9.71278	2.57462	2.13863	0.35469	11.37695
2.92	0.48010	9.73080	2.58607	2.14795	0.35167	11.45192
2.93	0.47946	9.84905	2.59755	2.15725	0.34867	11.52715
2.94	0.47884	9.91753	2.60908	2.16655	0.34570	11.60263
2.95	0.47821	9.98625	2.62064	2.17585	0.34275	11.67837
2.96	0.47760	10.05520	2.63224	2.18514	0.33982	11.75438
2.97	0.47699	10.12438	2.64387	2.19442	0.33692	11.83064
2.98	0.47638	10.19380	2.65555	2.20369	0.33404	11.90715
2.99	0.47578	10.26345	2.66726	2.21296	0.33118	11.98393
3.00	0.47519	10.33333	2.67901	2.22222	0.32834	12.06096
3.01	0.47460	10.40345	2.69080	2.23148	0.32553	12.13826
3.02	0.47402	10.47380	2.70263	2.24073	0.32274	12.21581
3.03	0.47344	10.54438	2.71449	2.24997	0.31997	12.29362
3.04	0.47287	10.61520	2.72639	2.25921	0.31723	12.37169
3.05	0.47230	10.68625	2.73833	2.26844	0.31450	12.45002
3.06	0.47174	10.75753	2.75031	2.27767	0.31180	12.52860
3.07	0.47118	10.82905	2.76233	2.28689	0.30912	12.60745
3.08	0.47063	10.90080	2.77438	2.29610	0.30646	12.68655
3.09	0.47008	10.97278	2.78647	2.30531	0.30383	12.76591
3.10	0.46953	11.04500	2.79860	2.31452	0.30121	12.84553
3.11	0.46899	11.11745	2.81077	2.32371	0.29862	12.92540
3.12	0.46846	11.19013	2.82298	2.33291	0.29605	13.00554
3.13	0.46793	11.26305	2.83522	2.34209	0.29350	13.08593
3.14	0.46741	11.33620	2.84750	2.35127	0.29097	13.16659
3.15	0.46689	11.40958	2.85982	2.36045	0.28846	13.24750
3.16	0.46637	11.48320	2.87218	2.36962	0.28597	13.32866
3.17	0.46586	11.55705	2.88458	2.37879	0.28350	13.41009
3.18	0.46535	11.63113	2.89701	2.38795	0.28106	13.49178
3.19	0.46485	11.70545	2.90948	2.39710	0.27863	13.57372
3.20	0.46435	11.78000	2.92199	2.40625	0.27623	13.65592
3.21	0.46385	11.85478	2.93454	2.41539	0.27384	13.73838
3.22	0.46336	11.92980	2.94713	2.42453	0.27148	13.82110
3.23	0.46288	12.00505	2.95975	2.43367	0.26914	13.90407
3.24	0.46240	12.08053	2.97241	2.44280	0.26681	13.98731
3.25	0.46192	12.15625	2.98511	2.45192	0.26451	14.07080
3.26	0.46144	12.23220	2.99785	2.46104	0.26222	14.15455
3.27	0.46097	12.30838	3.01063	2.47016	0.25996	14.23856
3.28	0.46051	12.38480	3.02345	2.47927	0.25771	14.32283

M_1	M_2	p_2/p_1	T_2/T_1	$\Delta V/a_1$	p_{t2}/p_{t1}	p_{t2}/p_1
3.29	0.46004	12.46145	3.03630	2.48837	0.25548	14.40735
3.30	0.45959	12.53833	3.04919	2.49747	0.25328	14.49214
3.31	0.45913	12.61545	3.06212	2.50657	0.25109	14.57718
3.32	0.45868	12.69280	3.07509	2.51566	0.24892	14.66248
3.33	0.45823	12.77038	3.08809	2.52475	0.24677	14.74804
3.34	0.45779	12.84820	3.10114	2.53383	0.24463	14.83385
3.35	0.45735	12.92625	3.11422	2.54291	0.24252	14.91992
3.36	0.45691	13.00453	3.12734	2.55198	0.24043	15.00626
3.37	0.45648	13.08305	3.14050	2.56105	0.23835	15.09285
3.38	0.45605	13.16180	3.15370	2.57012	0.23629	15.17969
3.39	0.45562	13.24078	3.16693	2.57918	0.23425	15.26680
3.40	0.45520	13.32000	3.18021	2.58824	0.23223	15.35417
3.41	0.45478	13.39945	3.19352	2.59729	0.23022	15.44179
3.42	0.45436	13.47913	3.20687	2.60634	0.22823	15.52967
3.43	0.45395	13.55905	3.22026	2.61538	0.22626	15.61781
3.44	0.45354	13.63920	3.23369	2.62442	0.22431	15.70620
3.45	0.45314	13.71958	3.24715	2.63345	0.22237	15.79486
3.46	0.45273	13.80020	3.26065	2.64249	0.22045	15.88377
3.47	0.45233	13.88105	3.27420	2.65151	0.21855	15.97294
3.48	0.45194	13.96213	3.28778	2.66054	0.21667	16.06237
3.49	0.45154	14.04345	3.30139	2.66956	0.21480	16.15206
3.50	0.45115	14.12500	3.31505	2.67857	0.21295	16.24200
3.51	0.45077	14.20678	3.32875	2.68758	0.21111	16.33220
3.52	0.45038	14.28880	3.34248	2.69659	0.20929	16.42266
3.53	0.45000	14.37105	3.35625	2.70559	0.20749	16.51338
3.54	0.44962	14.45353	3.37006	2.71460	0.20570	16.60436
3.55	0.44925	14.53625	3.38391	2.72359	0.20393	16.69559
3.56	0.44887	14.61920	3.39780	2.73258	0.20218	16.78709
3.57	0.44850	14.70238	3.41172	2.74157	0.20044	16.87884
3.58	0.44814	14.78580	3.42569	2.75056	0.19871	16.97085
3.59	0.44777	14.86945	3.43969	2.75954	0.19701	17.06311
3.60	0.44741	14.95333	3.45373	2.76852	0.19531	17.15564
3.61	0.44705	15.03745	3.46781	2.77749	0.19363	17.24842
3.62	0.44670	15.12180	3.48192	2.78646	0.19197	17.34146
3.63	0.44635	15.20638	3.49608	2.79543	0.19032	17.43476
3.64	0.44600	15.29120	3.51027	2.80440	0.18869	17.52831
3.65	0.44565	15.37625	3.52451	2.81336	0.18707	17.62213
3.66	0.44530	15.46153	3.53878	2.82231	0.18547	17.71620
3.67	0.44496	15.54705	3.55309	2.83127	0.18388	17.81053
3.68	0.44462	15.63280	3.56743	2.84022	0.18230	17.90512
3.69	0.44428	15.71878	3.58182	2.84916	0.18074	17.99996

(*Continued*)

M_1	M_2	p_2/p_1	T_2/T_1	$\Delta V/a_1$	p_{t2}/p_{t1}	p_{t2}/p_1
3.70	0.44395	15.80500	3.59624	2.85811	0.17919	18.09507
3.71	0.44362	15.89145	3.61071	2.86705	0.17766	18.19043
3.72	0.44329	15.97813	3.62521	2.87599	0.17614	18.28605
3.73	0.44296	16.06505	3.63975	2.88492	0.17464	18.38192
3.74	0.44263	16.15220	3.65433	2.99385	0.17314	18.47806
3.75	0.44231	16.23958	3.66894	2.90278	0.17166	18.57445
3.76	0.44199	16.32720	3.68360	2.91170	0.17020	18.67110
3.77	0.44167	16.41505	3.69829	2.92062	0.16875	18.76801
3.78	0.44136	16.50313	3.71302	2.92954	0.16731	18.86518
3.79	0.44104	16.59145	3.72779	2.93846	0.16588	18.96260
3.80	0.44073	16.68000	3.74260	2.94737	0.16447	19.06029
3.81	0.44042	16.76878	3.75745	2.95628	0.16307	19.15823
3.82	0.44012	16.85780	3.77234	2.96518	0.16168	19.25642
3.83	0.43981	16.94705	3.78726	2.97409	0.16031	19.35488
3.84	0.43951	17.03653	3.80223	2.98299	0.15895	19.45359
3.85	0.43921	17.12625	3.81723	2.99188	0.15760	19.55257
3.86	0.43891	17.21620	3.83227	3.00078	0.15626	19.65180
3.87	0.43862	17.30638	3.84735	3.00967	0.15493	19.75128
3.88	0.43832	17.39680	3.86246	3.01856	0.15362	19.85103
3.89	0.43803	17.48745	3.87762	3.02744	0.15232	19.95103
3.90	0.43774	17.57833	3.89281	3.03632	0.15103	20.05129
3.91	0.43746	17.66945	3.90805	3.04520	0.14975	20.15181
3.92	0.43717	17.76080	3.92332	3.05408	0.14848	20.25259
3.93	0.43689	17.85238	3.93863	3.06296	0.14723	20.35362
3.94	0.43661	17.94420	3.95398	3.07183	0.14598	20.45491
3.95	0.43633	18.03625	3.96936	3.08070	0.14475	20.55646
3.96	0.43605	18.12853	3.98479	3.08956	0.14353	20.65827
3.97	0.43577	18.22105	4.00025	3.09843	0.14232	20.76034
3.98	0.43550	18.31380	4.01575	3.10729	0.14112	20.86266
3.99	0.43523	18.40678	4.03130	3.11614	0.13993	20.96524
4.00	0.43496	18.50000	4.04687	3.12500	0.13876	21.06808
4.10	0.43236	19.44500	4.20479	3.21341	0.12756	22.11065
4.20	0.42994	20.41333	4.36657	3.30159	0.11733	23.17899
4.30	0.42767	21.40500	4.53221	3.38953	0.10800	24.27311
4.40	0.42554	22.42000	4.70171	3.47727	0.09948	25.39300
4.50	0.42355	23.45833	4.87509	3.56481	0.09170	26.53867
4.60	0.42168	24.52000	5.05233	3.65217	0.08459	27.71010
4.70	0.41992	25.60500	5.23343	3.73936	0.07809	28.90729
4.80	0.41826	26.71333	5.41842	3.82639	0.07214	30.13026
4.90	0.41670	27.84500	5.60727	3.91327	0.06670	31.37898
5.00	0.41523	29.00000	5.80000	4.00000	0.06172	32.65347
5.10	0.41384	30.17833	5.99660	4.08660	0.05715	33.95373
5.20	0.41252	31.38000	6.19709	4.17308	0.05297	35.27974

M_1	M_2	p_2/p_1	T_2/T_1	$\Delta V/a_1$	p_{t2}/p_{t1}	p_{t2}/p_1
5.30	0.41127	32.60500	6.40144	4.25943	0.04913	36.63152
5.40	0.41009	33.85333	6.60968	4.34568	0.04560	38.00906
5.50	0.40897	35.12500	6.82180	4.43182	0.04236	39.41235
5.60	0.40791	36.42000	7.03779	4.51786	0.03938	40.84141
5.70	0.40690	37.73833	7.25767	4.60380	0.03664	42.29622
5.80	0.40594	39.08000	7.48143	4.68966	0.03412	43.77679
5.90	0.40503	40.44500	7.70907	4.77542	0.03179	45.28312
6.00	0.40416	41.83333	7.94059	4.86111	0.02965	46.81521
6.10	0.40333	43.24500	8.17599	4.94672	0.02767	48.37305
6.20	0.40254	44.68000	8.41528	5.03226	0.02584	49.95665
6.30	0.40179	46.13833	8.65845	5.11772	0.02416	51.56600
6.40	0.40107	47.62000	8.90550	5.20312	0.02259	53.20111
6.50	0.40038	49.12500	9.15643	5.28846	0.02115	54.86198
6.60	0.39972	50.65333	9.41126	5.37374	0.01981	56.54860
6.70	0.39909	52.20500	9.66996	5.45896	0.01857	58.26097
6.80	0.39849	53.78000	9.93255	5.54412	0.01741	59.99910
6.90	0.39791	55.37833	10.19903	5.62923	0.01634	61.76299
7.00	0.39736	57.00000	10.46939	5.71429	0.01535	63.55263
7.50	0.39491	65.45833	11.87948	6.13889	0.01133	72.88713
8.00	0.39289	74.50000	13.38672	6.56250	0.00849	82.86547
8.50	0.39121	84.12500	14.99113	6.98529	0.00645	93.48763
9.00	0.38980	94.33333	16.69273	7.40741	0.00496	104.75360
9.50	0.38860	105.12500	18.49152	7.82895	0.00387	116.66339
10.00	0.38758	116.50000	20.38750	8.25000	0.00304	129.21697
∞	0.37796	∞	∞	∞	0.0	∞

Appendix I

Fanno Flow Parameters ($\gamma = 1.4$)

M	T/T^*	p/p^*	p_t/p_t^*	V/V^*	fL_{max}/D	S_{max}/R
0.0	1.20000	∞	∞	0.0	∞	∞
0.01	1.19998	109.54342	57.87384	0.01095	7134.40454	4.05827
0.02	1.19990	54.77006	28.94213	0.02191	1778.44988	3.36530
0.03	1.19978	36.51155	19.30054	0.03286	787.08139	2.96013
0.04	1.19962	27.38175	14.48149	0.04381	440.35221	2.67287
0.05	1.19940	21.90343	11.59144	0.05476	280.02031	2.45027
0.06	1.19914	18.25085	9.66591	0.06570	193.03108	2.26861
0.07	1.19833	15.64155	8.29153	0.07664	140.65501	2.11523
0.08	1.19847	13.68431	7.26161	0.08758	106.71822	1.98260
0.09	1.19806	12.16177	6.46134	0.09851	83.49612	1.86584
0.10	1.19760	10.94351	5.82183	0.10944	66.92156	1.76161
0.11	1.19710	9.94656	5.29923	0.12035	54.68790	1.66756
0.12	1.19655	9.11559	4.86432	0.13126	45.40796	1.58193
0.13	1.19596	8.41230	4.49686	0.14217	38.20700	1.50338
0.14	1.19531	7.80932	4.18240	0.15306	32.51131	1.43089
0.15	1.19462	7.28659	3.91034	0.16395	27.93197	1.36363
0.16	1.19389	6.82907	3.67274	0.17482	24.19783	1.30094
0.17	1.19310	6.42525	3.46351	0.18569	21.11518	1.24228

(*Continued*)

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

M	T/T^*	p/p^*	p_t/p_t^*	V/V^*	fL_{max}/D	S_{max}/R
0.18	1.19227	6.06618	3.27793	0.19654	18.54265	1.18721
0.19	1.19140	5.74480	3.11226	0.20739	16.37516	1.13535
0.20	1.19048	5.45545	2.96352	0.21822	14.53327	1.08638
0.21	1.18951	5.19355	2.82929	0.22904	12.95602	1.04003
0.22	1.18850	4.95537	2.70760	0.23984	11.59605	0.99606
0.23	1.18744	4.73781	2.59681	0.25063	10.41609	0.95428
0.24	1.18633	4.53829	2.49556	0.26141	9.38648	0.91451
0.25	1.18519	4.35465	2.40271	0.27217	8.48341	0.87660
0.26	1.18399	4.18505	2.31729	0.28291	7.68757	0.84040
0.27	1.18276	4.02795	2.23847	0.29364	6.98317	0.80579
0.28	1.18147	3.88199	2.16555	0.30435	6.35721	0.77268
0.29	1.18015	3.74602	2.09793	0.31504	5.79891	0.74095
0.30	1.17878	3.61906	2.03507	0.32572	5.29925	0.71053
0.31	1.17737	3.50022	1.97651	0.33637	4.85066	0.68133
0.32	1.17592	3.38874	1.92185	0.34701	4.44674	0.65329
0.33	1.17442	3.28396	1.87074	0.35762	4.08205	0.62634
0.34	1.17288	3.18529	1.82288	0.36822	3.75195	0.60042
0.35	1.17130	3.09219	1.77797	0.37879	3.45245	0.57547
0.36	1.16968	3.00422	1.73578	0.38935	3.18012	0.55146
0.37	1.16802	2.92094	1.69609	0.39988	2.93198	0.52832
0.38	1.16632	2.84200	1.65870	0.41039	2.70545	0.50603
0.39	1.16457	2.76706	1.62343	0.42087	2.49828	0.48454
0.40	1.16279	2.69582	1.59014	0.43133	2.30849	0.46382
0.41	1.16097	2.62801	1.55867	0.44177	2.13436	0.44384
0.42	1.15911	2.56338	1.52890	0.45218	1.97437	0.42455
0.43	1.15721	2.50171	1.50072	0.46257	1.82715	0.40594
0.44	1.15527	2.44280	1.47401	0.47293	1.69152	0.38798
0.45	1.15329	2.38648	1.44867	0.48326	1.56643	0.37065
0.46	1.15128	2.33256	1.42463	0.49357	1.45091	0.35391
0.47	1.14923	2.28089	1.40180	0.50385	1.34413	0.33775
0.48	1.14714	2.23135	1.38010	0.51410	1.24534	0.32215
0.49	1.14502	2.18378	1.35947	0.52433	1.15385	0.30709
0.50	1.14286	2.13809	1.33984	0.53452	1.06906	0.29255
0.51	1.14066	2.09415	1.32117	0.54469	0.99041	0.27852
0.52	1.13843	2.05187	1.30339	0.55483	0.91742	0.26497
0.53	1.13617	2.01116	1.28645	0.56493	0.84962	0.25189
0.54	1.13387	1.97192	1.27032	0.57501	0.78663	0.23927
0.55	1.13154	1.93407	1.25495	0.58506	0.72805	0.22709
0.56	1.12918	1.89755	1.24029	0.59507	0.67357	0.21535
0.57	1.12678	1.86228	1.22633	0.60505	0.62287	0.20402
0.56	1.12435	1.82820	1.21301	0.61501	0.57568	0.19310
0.59	1.12189	1.79525	1.20031	0.62492	0.53174	0.18258
0.60	1.11940	1.76336	1.18820	0.63481	0.49082	0.17244

M	T/T^*	p/p^*	p_t/p_t^*	V/V^*	fL_{max}/D	S_{max}/R
0.61	1.11688	1.73250	1.17665	0.64466	0.45271	0.16267
0.62	1.11433	1.70261	1.16565	0.65448	0.41720	0.15328
0.63	1.11175	1.67364	1.15515	0.66427	0.38412	0.14423
0.64	1.10914	1.64556	1.14515	0.67402	0.35330	0.13553
0.65	1.10650	1.61831	1.13562	0.68374	0.32459	0.12718
0.66	1.10383	1.59187	1.12654	0.69342	0.29785	0.11915
0.67	1.10114	1.56620	1.11789	0.70307	0.27295	0.11144
0.68	1.09842	1.54126	1.10965	0.71268	0.24978	0.10405
0.69	1.09567	1.51702	1.10182	0.72225	0.22820	0.09696
0.70	1.09290	1.49345	1.09437	0.73179	0.20814	0.09018
0.71	1.09010	1.47053	1.08729	0.74129	0.18948	0.08369
0.72	1.08727	1.44823	1.08057	0.75076	0.17215	0.07749
0.73	1.08442	1.42652	1.07419	0.76019	0.15605	0.07157
0.74	1.08155	1.40537	1.06814	0.76958	0.14112	0.06592
0.75	1.07865	1.38478	1.06242	0.77894	0.12728	0.06055
0.76	1.07573	1.36470	1.05700	0.78825	0.11447	0.05543
0.77	1.07279	1.34514	1.05188	0.79753	0.10262	0.05058
0.78	1.06982	1.32605	1.04705	0.80677	0.09167	0.04598
0.79	1.06684	1.30744	1.04251	0.81597	0.08158	0.04163
0.80	1.06383	1.28928	1.03823	0.82514	0.07229	0.03752
0.81	1.06080	1.27155	1.03422	0.83426	0.06376	0.03365
0.82	1.05775	1.25423	1.03046	0.84335	0.05593	0.03001
0.83	1.05469	1.23732	1.02696	0.85239	0.04878	0.02660
0.84	1.05160	1.22080	1.02370	0.86140	0.04226	0.02342
0.85	1.04849	1.20466	1.02067	0.87037	0.03633	0.02046
0.86	1.04537	1.18888	1.01787	0.87929	0.03097	0.01771
0.87	1.04223	1.17344	1.01530	0.88818	0.02613	0.01518
0.88	1.03907	1.15835	1.01294	0.89703	0.02179	0.01286
0.89	1.03589	1.14358	1.01080	0.90583	0.01793	0.01074
0.90	1.03270	1.12913	1.00886	0.91460	0.01451	0.00882
0.91	1.02950	1.11499	1.00713	0.92332	0.01151	0.00711
0.92	1.02627	1.10114	1.00560	0.93201	0.00891	0.00558
0.93	1.02304	1.08758	1.00426	0.94065	0.00669	0.00425
0.94	1.01978	1.07430	1.00311	0.94925	0.00482	0.00310
0.95	1.01652	1.06129	1.00215	0.95781	0.00328	0.00214
0.96	1.01324	1.04854	1.00136	0.96633	0.00206	0.00136
0.97	1.00995	1.03604	1.00076	0.97481	0.00113	0.00076
0.98	1.00664	1.02379	1.00034	0.98325	0.00049	0.00034
0.99	1.00333	1.01178	1.00008	0.99165	0.00012	0.00008
1.00	1.00000	1.00000	1.00000	1.00000	0.00000	0.00000
1.01	0.99666	0.98844	1.00008	1.00831	0.00012	0.00008

(*Continued*)

M	T/T^*	p/p^*	p_t/p_t^*	V/V^*	$fL_{\max}/D$	$S_{\max}/R$
1.02	0.99331	0.97711	1.00033	1.01658	0.00046	0.00033
1.03	0.98995	0.96598	1.00074	1.02481	0.00101	0.00074
1.04	0.98658	0.95507	1.00131	1.03300	0.00177	0.00130
1.05	0.98320	0.94435	1.00203	1.04114	0.00271	0.00203
1.06	0.97982	0.93383	1.00291	1.04925	0.00384	0.00290
1.07	0.97642	0.92349	1.00394	1.05731	0.00513	0.00393
1.08	0.97302	0.91335	1.00512	1.06533	0.00658	0.00511
1.09	0.96960	0.90338	1.00645	1.07331	0.00819	0.00643
1.10	0.96618	0.89359	1.00793	1.08124	0.00994	0.00789
1.11	0.96276	0.88397	1.00955	1.08913	0.01182	0.00950
1.12	0.95932	0.87451	1.01131	1.09699	0.01382	0.01125
1.13	0.95589	0.86522	1.01322	1.10479	0.01595	0.01313
1.14	0.95244	0.85608	1.01527	1.11256	0.01819	0.01515
1.15	0.94899	0.84710	1.01745	1.12029	0.02053	0.01730
1.16	0.94554	0.83826	1.01978	1.12797	0.02298	0.01959
1.17	0.94208	0.82958	1.02224	1.13561	0.02552	0.02200
1.18	0.93861	0.82103	1.02484	1.14321	0.02814	0.02454
1.19	0.93515	0.81263	1.02757	1.15077	0.03085	0.02720
1.20	0.93168	0.80436	1.03044	1.15828	0.03364	0.02999
1.21	0.92820	0.79623	1.03344	1.16575	0.03650	0.03289
1.22	0.92473	0.78822	1.03657	1.17319	0.03943	0.03592
1.23	0.92125	0.78034	1.03983	1.18057	0.04242	0.03906
1.24	0.91777	0.77258	1.04323	1.18792	0.04547	0.04232
1.25	0.91429	0.76495	1.04675	1.19523	0.04858	0.04569
1.26	0.91080	0.75743	1.05041	1.20249	0.05174	0.04918
1.27	0.90732	0.75003	1.05419	1.20972	0.05495	0.05277
1.28	0.90383	0.74274	1.05810	1.21690	0.05820	0.05647
1.29	0.90035	0.73556	1.06214	1.22404	0.06150	0.06028
1.30	0.89686	0.72848	1.06630	1.23114	0.06483	0.06420
1.31	0.89338	0.72152	1.07060	1.23819	0.06820	0.06822
1.32	0.88989	0.71465	1.07502	1.24521	0.07161	0.07234
1.33	0.88641	0.70789	1.07957	1.25218	0.07504	0.07656
1.34	0.88292	0.70122	1.08424	1.25912	0.07850	0.08088
1.35	0.87944	0.69466	1.08904	1.26601	0.08199	0.08529
1.36	0.87596	0.68818	1.09396	1.27286	0.08550	0.08981
1.37	0.87249	0.68180	1.09902	1.27968	0.08904	0.09441
1.38	0.86901	0.67551	1.10419	1.28645	0.09259	0.09911
1.39	0.86554	0.66931	1.10950	1.29318	0.09615	0.10391
1.40	0.86207	0.66320	1.11493	1.29987	0.09974	0.10879
1.41	0.85860	0.65717	1.12048	1.30652	0.10334	0.11376
1.42	0.85514	0.65122	1.12616	1.31313	0.10694	0.11882
1.43	0.85168	0.64536	1.13197	1.31970	0.11056	0.12396
1.44	0.84822	0.63958	1.13790	1.32623	0.11419	0.12919

M	T/T^*	p/p^*	p_t/p_t^*	V/V^*	fL_{max}/D	S_{max}/R
1.45	0.84477	0.63387	1.14396	1.33272	0.11782	0.13450
1.46	0.84133	0.62825	1.15015	1.33917	0.12146	0.13989
1.47	0.83788	0.62269	1.15646	1.34558	0.12511	0.14537
1.48	0.83445	0.61722	1.16290	1.35195	0.12875	0.15092
1.49	0.83101	0.61181	1.16947	1.35828	0.13240	0.15655
1.50	0.82759	0.60648	1.17617	1.36458	0.13605	0.16226
1.51	0.82416	0.60122	1.18299	1.37083	0.13970	0.16805
1.52	0.82075	0.59602	1.18994	1.37705	0.14335	0.17391
1.53	0.81734	0.59089	1.19702	1.38322	0.14699	0.17984
1.54	0.81393	0.58583	1.20423	1.38936	0.15063	0.18584
1.55	0.81054	0.58084	1.21157	1.39546	0.15427	0.19192
1.56	0.80715	0.57591	1.21904	1.40152	0.15790	0.19807
1.57	0.80376	0.57104	1.22664	1.40755	0.16152	0.20428
1.58	0.80038	0.56623	1.23438	1.41353	0.16514	0.21057
1.59	0.79701	0.56148	1.24224	1.41948	0.16875	0.21692
1.60	0.79365	0.55679	1.25023	1.42539	0.17236	0.22333
1.61	0.79030	0.55216	1.25836	1.43127	0.17595	0.22981
1.62	0.78695	0.54759	1.26663	1.43710	0.17954	0.23636
1.63	0.78361	0.54308	1.27502	1.44290	0.18311	0.24296
1.64	0.78027	0.53862	1.28355	1.44866	0.18667	0.24963
1.65	0.77695	0.53421	1.29222	1.45439	0.19023	0.25636
1.66	0.77363	0.52986	1.30102	1.46008	0.19377	0.26315
1.67	0.77033	0.52556	1.30996	1.46573	0.19729	0.27000
1.68	0.76703	0.52131	1.31904	1.47135	0.20081	0.27690
1.69	0.76374	0.51711	1.32825	1.47693	0.20431	0.28386
1.70	0.76046	0.51297	1.33761	1.48247	0.20780	0.29088
1.71	0.75718	0.50887	1.34710	1.48798	0.21128	0.29795
1.72	0.75392	0.50482	1.35674	1.49345	0.21474	0.30508
1.73	0.75067	0.50082	1.36651	1.49889	0.21819	0.31226
1.74	0.74742	0.49686	1.37643	1.50429	0.22162	0.31949
1.75	0.74419	0.49295	1.38649	1.50966	0.22504	0.32678
1.76	0.74096	0.48909	1.39670	1.51499	0.22844	0.33411
1.77	0.73774	0.48527	1.40705	1.52029	0.23182	0.34149
1.78	0.73454	0.48149	1.41755	1.52555	0.23519	0.34893
1.79	0.73134	0.47776	1.42819	1.53078	0.23855	0.35641
1.80	0.72816	0.47407	1.43898	1.53598	0.24189	0.36394
1.81	0.72498	0.47042	1.44992	1.54114	0.24521	0.37151
1.82	0.72181	0.46681	1.46101	1.54626	0.24851	0.37913
1.83	0.71866	0.46324	1.47225	1.55136	0.25180	0.38680
1.84	0.71551	0.45972	1.48365	1.55642	0.25507	0.39450
1.85	0.71238	0.45623	1.49519	1.56145	0.25832	0.40226

(*Continued*)

M	T/T^*	p/p^*	p_t/p_t^*	V/V^*	fL_{max}/D	S_{max}/R
1.86	0.70925	0.45278	1.50689	1.56644	0.26156	0.41005
1.87	0.70614	0.44937	1.51875	1.57140	0.26478	0.41789
1.88	0.70304	0.44600	1.53076	1.57633	0.26798	0.42576
1.89	0.69995	0.44266	1.54293	1.58123	0.27116	0.43368
1.90	0.69686	0.43936	1.55526	1.58609	0.27433	0.44164
1.91	0.69379	0.43610	1.56774	1.59092	0.27748	0.44964
1.92	0.69073	0.43287	1.58039	1.59572	0.28061	0.45767
1.93	0.68769	0.42967	1.59320	1.60049	0.28372	0.46574
1.94	0.68465	0.42651	1.60617	1.60523	0.28681	0.47385
1.95	0.68162	0.42339	1.61931	1.60993	0.28989	0.48200
1.96	0.67861	0.42029	1.63261	1.61460	0.29295	0.49018
1.97	0.67561	0.41724	1.64608	1.61925	0.29599	0.49840
1.98	0.67262	0.41421	1.65972	1.62386	0.29901	0.50665
1.99	0.66964	0.41121	1.67352	1.62844	0.30201	0.51493
2.00	0.66667	0.40825	1.68750	1.63299	0.30500	0.52325
2.01	0.66371	0.40532	1.70165	1.63751	0.30796	0.53160
2.02	0.66076	0.40241	1.71597	1.64201	0.31091	0.53998
2.03	0.65783	0.39954	1.73047	1.64647	0.31384	0.54839
2.04	0.65491	0.39670	1.74514	1.65090	0.31676	0.55683
2.05	0.65200	0.39388	1.75999	1.65530	0.31965	0.56531
2.06	0.64910	0.39110	1.77502	1.65967	0.32253	0.57381
2.07	0.64621	0.38834	1.79022	1.66402	0.32538	0.58234
2.08	0.64334	0.38562	1.80561	1.66833	0.32822	0.59090
2.09	0.64047	0.38292	1.82119	1.67262	0.33105	0.59949
2.10	0.63762	0.38024	1.83694	1.67687	0.33385	0.60810
2.11	0.63478	0.37760	1.85289	1.68110	0.33664	0.61674
2.12	0.63195	0.37498	1.86902	1.68530	0.33940	0.62541
2.13	0.62914	0.37239	1.88533	1.68947	0.34215	0.63411
2.14	0.62633	0.36982	1.90184	1.69362	0.34489	0.64282
2.15	0.62354	0.36728	1.91854	1.69774	0.34760	0.65157
2.16	0.62076	0.36476	1.93544	1.70183	0.35030	0.66033
2.17	0.61799	0.36227	1.95252	1.70589	0.35298	0.66912
2.18	0.61523	0.35980	1.96981	1.70992	0.35564	0.67794
2.19	0.61249	0.35736	1.98729	1.71393	0.35828	0.68677
2.20	0.60976	0.35494	2.00497	1.71791	0.36091	0.69563
2.21	0.60704	0.35255	2.02286	1.72187	0.36352	0.70451
2.22	0.60433	0.35017	2.04094	1.72579	0.36611	0.71341
2.23	0.60163	0.34782	2.05923	1.72970	0.36869	0.72233
2.24	0.59895	0.34550	2.07773	1.73357	0.37124	0.73128
2.25	0.59627	0.34319	2.09644	1.73742	0.37378	0.74024
2.26	0.59361	0.34091	2.11535	1.74125	0.37631	0.74922
2.27	0.59096	0.33865	2.13447	1.74504	0.37881	0.75822
2.28	0.58833	0.33641	2.15381	1.74882	0.38130	0.76724

M	T/T^*	p/p^*	p_t/p_t^*	V/V^*	fL_{max}/D	S_{max}/R
2.29	0.58570	0.33420	2.17336	1.75257	0.38377	0.77628
2.30	0.58309	0.33200	2.19313	1.75629	0.38623	0.78533
2.31	0.58049	0.32983	2.21312	1.75999	0.38867	0.79440
2.32	0.57790	0.32767	2.23332	1.76366	0.39109	0.80349
2.33	0.57532	0.32554	2.25375	1.76731	0.39350	0.81260
2.34	0.57276	0.32342	2.27440	1.77093	0.39589	0.82172
2.35	0.57021	0.32133	2.29528	1.77453	0.39826	0.83085
2.36	0.56767	0.31925	2.31638	1.77811	0.40062	0.84001
2.37	0.56514	0.31720	2.33771	1.78166	0.40296	0.84917
2.38	0.56262	0.31516	2.35928	1.78519	0.40529	0.85835
2.39	0.56011	0.31314	2.38107	1.78869	0.40760	0.86755
2.40	0.55762	0.31114	2.40310	1.79218	0.40989	0.87676
2.41	0.55514	0.30916	2.42537	1.79563	0.41217	0.88598
2.42	0.55267	0.30720	2.44787	1.79907	0.41443	0.89522
2.43	0.55021	0.30525	2.47061	1.80248	0.41668	0.90447
2.44	0.54777	0.30332	2.49360	1.80587	0.41891	0.91373
2.45	0.54533	0.30141	2.51683	1.80924	0.42112	0.92300
2.46	0.54291	0.29952	2.54031	1.81258	0.42332	0.93229
2.47	0.54050	0.29765	2.56403	1.81591	0.42551	0.94158
2.48	0.53810	0.29579	2.58801	1.81921	0.42768	0.95089
2.49	0.53571	0.29394	2.61224	1.82249	0.42984	0.96021
2.50	0.53333	0.29212	2.63672	1.82574	0.43198	0.96954
2.51	0.53097	0.29031	2.66146	1.82898	0.43410	0.97887
2.52	0.52862	0.28852	2.68645	1.83219	0.43621	0.98822
2.53	0.52627	0.28674	2.71171	1.83538	0.43831	0.99758
2.54	0.52394	0.28498	2.73723	1.83855	0.44039	1.00695
2.55	0.52163	0.28323	2.76301	1.84170	0.44246	1.01632
2.56	0.51932	0.28150	2.78906	1.84483	0.44451	1.02571
2.57	0.51702	0.27978	2.81538	1.84794	0.44655	1.03510
2.58	0.51474	0.27808	2.84197	1.85103	0.44858	1.04450
2.59	0.51247	0.27640	2.86884	1.85410	0.45059	1.05391
2.60	0.51020	0.27473	2.89598	1.85714	0.45259	1.06332
2.61	0.50795	0.27307	2.92339	1.86017	0.45457	1.07274
2.62	0.50571	0.27143	2.95109	1.86318	0.45654	1.08217
2.63	0.50349	0.26980	2.97907	1.86616	0.45850	1.09161
2.64	0.50127	0.26818	3.00733	1.86913	0.46044	1.10105
2.65	0.49906	0.26658	3.03588	1.87208	0.46237	1.11050
2.66	0.49687	0.26500	3.06472	1.87501	0.46429	1.11996
2.67	0.49469	0.26342	3.09385	1.87792	0.46619	1.12942
2.68	0.49251	0.26186	3.12327	1.88081	0.46808	1.13888
2.69	0.49035	0.26032	3.15299	1.88368	0.46996	1.14835

(*Continued*)

M	T/T^*	p/p^*	p_t/p_t^*	V/V^*	fL_{max}/D	S_{max}/R
2.70	0.48820	0.25878	3.18301	1.88653	0.47182	1.15783
2.71	0.48606	0.25726	3.21333	1.88936	0.47367	1.16731
2.72	0.48393	0.25575	3.24395	1.89218	0.47551	1.17679
2.73	0.48182	0.25426	3.27488	1.89497	0.47733	1.18628
2.74	0.47971	0.25278	3.30611	1.89775	0.47915	1.19577
2.75	0.47761	0.25131	3.33766	1.90051	0.48095	1.20527
2.76	0.47553	0.24985	3.36952	1.90325	0.48273	1.21477
2.77	0.47345	0.24840	3.40169	1.90598	0.48451	1.22427
2.78	0.47139	0.24697	3.43418	1.90868	0.48627	1.23378
2.79	0.46933	0.24555	3.46699	1.91137	0.48803	1.24329
2.80	0.46729	0.24414	3.50012	1.91404	0.48976	1.25280
2.81	0.46526	0.24274	3.53358	1.91669	0.49149	1.26231
2.82	0.46323	0.24135	3.56737	1.91933	0.49321	1.27183
2.83	0.46122	0.23998	3.60148	1.92195	0.49491	1.28135
2.64	0.45922	0.23861	3.63593	1.92455	0.49660	1.29087
2.85	0.45723	0.23726	3.67072	1.92714	0.49828	1.30039
2.86	0.45525	0.23592	3.70584	1.92970	0.49995	1.30991
2.87	0.45328	0.23459	3.74131	1.93225	0.50161	1.31943
2.88	0.45132	0.23326	3.77711	1.93479	0.50326	1.32896
2.89	0.44937	0.23195	3.81327	1.93731	0.50489	1.33849
2.90	0.44743	0.23066	3.84977	1.93981	0.50652	1.34801
2.91	0.44550	0.22937	3.88662	1.94230	0.50813	1.35754
2.92	0.44358	0.22809	3.92383	1.94477	0.50973	1.36707
2.93	0.44167	0.22682	3.96139	1.94722	0.51132	1.37660
2.94	0.43977	0.22556	3.99932	1.94966	0.51290	1.38612
2.95	0.43788	0.22431	4.03760	1.95208	0.51447	1.39565
2.96	0.43600	0.22307	4.07625	1.95449	0.51603	1.40518
2.97	0.43413	0.22185	4.11527	1.95688	0.51758	1.41471
2.98	0.43226	0.22063	4.15466	1.95925	0.51912	1.42423
2.99	0.43041	0.21942	4.19443	1.96162	0.52064	1.43376
3.00	0.42857	0.21822	4.23457	1.96396	0.52216	1.44328
3.01	0.42674	0.21703	4.27509	1.96629	0.52367	1.45280
3.02	0.42492	0.21585	4.31599	1.96861	0.52516	1.46233
3.03	0.42310	0.21467	4.35728	1.97091	0.52665	1.47185
3.04	0.42130	0.21351	4.39895	1.97319	0.52813	1.48137
3.05	0.41951	0.21236	4.44102	1.97547	0.52959	1.49088
3.06	0.41772	0.21121	4.48347	1.97772	0.53105	1.50040
3.07	0.41595	0.21008	4.52633	1.97997	0.53249	1.50991
3.08	0.41418	0.20895	4.56959	1.98219	0.53393	1.51942
3.09	0.41242	0.20783	4.61325	1.98441	0.53536	1.52893
3.10	0.41068	0.20672	4.65731	1.98661	0.53678	1.53844
3.11	0.40894	0.20562	4.70178	1.98879	0.53818	1.54794
3.12	0.40721	0.20453	4.74667	1.99097	0.53958	1.55744

M	T/T^*	p/p^*	p_t/p_t^*	V/V^*	fL_{max}/D	S_{max}/R
3.13	0.40549	0.20344	4.79197	1.99313	0.54097	1.56694
3.14	0.40378	0.20237	4.83769	1.99527	0.54235	1.57644
3.15	0.40208	0.20130	4.88383	1.99740	0.54372	1.58593
3.16	0.40038	0.20024	4.93039	1.99952	0.54509	1.59542
3.17	0.39870	0.19919	4.97739	2.00162	0.54644	1.60490
3.18	0.39702	0.19814	5.02481	2.00372	0.54778	1.61439
3.19	0.39536	0.19711	5.07266	2.00579	0.54912	1.62387
3.20	0.39370	0.19608	5.12096	2.00786	0.55044	1.63334
3.21	0.39205	0.19506	5.16969	2.00991	0.55176	1.64281
3.22	0.39041	0.19405	5.21887	2.01195	0.55307	1.65228
3.23	0.38878	0.19304	5.26849	2.01398	0.55437	1.66174
3.24	0.38716	0.19204	5.31857	2.01599	0.55566	1.67120
3.25	0.38554	0.19105	5.36909	2.01799	0.55694	1.68066
3.26	0.38394	0.19007	5.42008	2.01998	0.55822	1.69011
3.27	0.38234	0.18909	5.47152	2.02196	0.55948	1.69956
3.28	0.38075	0.18812	5.52343	2.02392	0.56074	1.70900
3.29	0.37917	0.18716	5.57580	2.02587	0.56199	1.71844
3.30	0.37760	0.18621	5.62865	2.02781	0.56323	1.72787
3.31	0.37603	0.18526	5.68196	2.02974	0.56446	1.73730
3.32	0.37448	0.18432	5.73576	2.03165	0.56569	1.74672
3.33	0.37293	0.18339	5.79003	2.03356	0.56691	1.75614
3.34	0.37139	0.18246	5.84479	2.03545	0.56812	1.76555
3.35	0.36986	0.18154	5.90004	2.03733	0.56932	1.77496
3.36	0.36833	0.18063	5.95577	2.03920	0.57051	1.78436
3.37	0.36682	0.17972	6.01201	2.04106	0.57170	1.79376
3.38	0.36531	0.17882	6.06873	2.04290	0.57287	1.80315
3.39	0.36381	0.17793	6.12596	2.04474	0.57404	1.81254
3.40	0.36232	0.17704	6.18370	2.04656	0.57521	1.82192
3.41	0.36083	0.17616	6.24194	2.04837	0.57636	1.83129
3.42	0.35936	0.17528	6.30070	2.05017	0.57751	1.84066
3.43	0.35789	0.17441	6.35997	2.05196	0.57865	1.85002
3.44	0.35643	0.17355	6.41976	2.05374	0.57978	1.85938
3.45	0.35498	0.17270	6.48007	2.05551	0.58091	1.86873
3.46	0.35353	0.17185	6.54092	2.05727	0.58203	1.87808
3.47	0.35209	0.17100	6.60229	2.05901	0.58314	1.88742
3.48	0.35066	0.17016	6.66419	2.06075	0.58424	1.89675
3.49	0.34924	0.16933	6.72664	2.06247	0.58534	1.90608
3.50	0.34783	0.16851	6.78962	2.06419	0.58643	1.91540
3.51	0.34642	0.16768	6.85315	2.06589	0.58751	1.92471
3.52	0.34502	0.16687	6.91723	2.06759	0.58859	1.93402
3.53	0.34362	0.16606	6.98186	2.06927	0.58966	1.94332

(*Continued*)

M	T/T^*	p/p^*	p_t/p_t^*	V/V^*	fL_{max}/D	S_{max}/R
3.54	0.34224	0.16526	7.04705	2.07094	0.59072	1.95261
3.55	0.34086	0.16446	7.11281	2.07261	0.59178	1.96190
3.56	0.33949	0.16367	7.17912	2.07426	0.59282	1.97118
3.57	0.33813	0.16288	7.24601	2.07590	0.59387	1.98045
3.58	0.33677	0.16210	7.31346	2.07754	0.59490	1.98972
3.59	0.33542	0.16132	7.38150	2.07916	0.59593	1.99898
3.60	0.33408	0.16055	7.45011	2.08077	0.59695	2.00823
3.61	0.33274	0.15979	7.51931	2.08238	0.59797	2.01747
3.62	0.33141	0.15903	7.58910	2.08397	0.59898	2.02671
3.63	0.33009	0.15827	7.65948	2.08556	0.59998	2.03594
3.64	0.32877	0.15752	7.73045	2.08713	0.60098	2.04517
3.65	0.32747	0.15678	7.80203	2.08870	0.60197	2.05438
3.66	0.32616	0.15604	7.87421	2.09026	0.60296	2.06359
3.67	0.32487	0.15531	7.94700	2.09180	0.60394	2.07279
3.68	0.32358	0.15458	8.02040	2.09334	0.60491	2.08199
3.69	0.32230	0.15385	8.09442	2.09487	0.60588	2.09118
3.70	0.32103	0.15313	8.16907	2.09639	0.60684	2.10035
3.71	0.31976	0.15242	8.24433	2.09790	0.60779	2.10953
3.72	0.31850	0.15171	8.32023	2.09941	0.60874	2.11869
3.73	0.31724	0.15100	8.39676	2.10090	0.60968	2.12785
3.74	0.31600	0.15030	8.47393	2.10238	0.61062	2.13699
3.75	0.31475	0.14961	8.55174	2.10386	0.61155	2.14613
3.76	0.31352	0.14892	8.63020	2.10533	0.61247	2.15527
3.77	0.31229	0.14823	8.70931	2.10679	0.61339	2.16439
3.78	0.31107	0.14755	8.78907	2.10824	0.61431	2.17351
3.79	0.30985	0.14687	8.86950	2.10968	0.61522	2.18262
3.80	0.30864	0.14620	8.95059	2.11111	0.61612	2.19172
3.81	0.30744	0.14553	9.03234	2.11254	0.61702	2.20081
3.82	0.30624	0.14487	9.11477	2.11395	0.61791	2.20990
3.83	0.30505	0.14421	9.19788	2.11536	0.61879	2.21897
3.84	0.30387	0.14355	9.28167	2.11676	0.61968	2.22804
3.85	0.30269	0.14290	9.36614	2.11815	0.62055	2.23710
3.86	0.30151	0.14225	9.45131	2.11954	0.62142	2.24615
3.87	0.30035	0.14161	9.53717	2.12091	0.62229	2.25520
3.88	0.29919	0.14097	9.62373	2.12228	0.62315	2.26423
3.89	0.29803	0.14034	9.71100	2.12364	0.62400	2.27326
3.90	0.29688	0.13971	9.79897	2.12499	0.62485	2.28228
3.91	0.29574	0.13908	9.88766	2.12634	0.62569	2.29129
3.92	0.29460	0.13846	9.97707	2.12767	0.62653	2.30029
3.93	0.29347	0.13784	10.06720	2.12900	0.62737	2.30928
3.94	0.29235	0.13723	10.15806	2.13032	0.62819	2.31827
3.95	0.29123	0.13662	10.24965	2.13163	0.62902	2.32724
3.96	0.29011	0.13602	10.34197	2.13294	0.62984	2.33621

M	T/T^*	p/p^*	p_t/p_t^*	V/V^*	fL_{max}/D	S_{max}/R
3.97	0.28900	0.13541	10.43504	2.13424	0.63065	2.34517
3.98	0.28790	0.13482	10.52886	2.13553	0.63146	2.35412
3.99	0.28681	0.13422	10.62343	2.13681	0.63227	2.36306
4.00	0.28571	0.13363	10.71875	2.13809	0.63306	2.37199
4.10	0.27510	0.12793	11.71465	2.15046	0.64080	2.46084
4.20	0.26502	0.12257	12.79164	2.16215	0.64810	2.54879
4.30	0.25543	0.11753	13.95490	2.17321	0.65499	2.63583
4.40	0.24631	0.11279	15.20987	2.18368	0.66149	2.72194
4.50	0.23762	0.10833	16.56219	2.19360	0.66763	2.80712
4.60	0.22936	0.10411	18.01779	2.20300	0.67345	2.89136
4.70	0.22148	0.10013	19.58283	2.21192	0.67895	2.97465
4.80	0.21398	0.09637	21.26371	2.22038	0.68417	3.05700
4.90	0.20683	0.09281	23.06712	2.22842	0.68911	3.13841
5.00	0.20000	0.08944	25.00000	2.23607	0.69380	3.21888
5.10	0.19349	0.08625	27.06957	2.24334	0.69826	3.29841
5.20	0.18727	0.08322	29.28333	2.25026	0.70249	3.37702
5.30	0.18132	0.08034	31.64905	2.25685	0.70652	3.45471
5.40	0.17564	0.07761	34.17481	2.26313	0.71035	3.53149
5.50	0.17021	0.07501	36.86896	2.26913	0.71400	3.60737
5.60	0.16502	0.07254	39.74018	2.27484	0.71748	3.68236
5.70	0.16004	0.07018	42.79743	2.28030	0.72080	3.75648
5.80	0.15528	0.06794	46.05000	2.28552	0.72397	3.82973
5.90	0.15072	0.06580	49.50747	2.29051	0.72699	3.90212
6.00	0.14634	0.06376	53.17978	2.29528	0.72988	3.97368
6.10	0.14215	0.06181	57.07718	2.29984	0.73264	4.04440
6.20	0.13812	0.05994	61.21023	2.30421	0.73528	4.11431
6.30	0.13426	0.05816	65.58987	2.30840	0.73780	4.18342
6.40	0.13055	0.05646	70.22736	2.31241	0.74022	4.25174
6.50	0.12698	0.05482	75.13431	2.31626	0.74254	4.31928
6.60	0.12356	0.05326	80.32271	2.31996	0.74477	4.38605
6.70	0.12026	0.05176	85.80487	2.32351	0.74690	4.45208
6.80	0.11710	0.05032	91.59351	2.32691	0.74895	4.51736
6.90	0.11405	0.04894	97.70169	2.33019	0.75091	4.58192
7.00	0.1111	0.04762	104.14286	2.33333	0.75280	4.64576
7.50	0.09796	0.04173	141.84148	2.34738	0.76121	4.95471
8.00	0.08696	0.03686	190.10937	2.35907	0.76819	5.24760
8.50	0.07767	0.03279	251.08617	2.36889	0.77404	5.52580
9.00	0.06977	0.02935	327.18930	2.37722	0.77899	5.79054
9.50	0.06299	0.02642	421.13137	2.38433	0.78320	6.04294
10.00	0.05714	0.02390	535.93750	2.39046	0.78683	6.28402
∞	0.0	0.0	∞	2.4495	0.82153	∞

Appendix J

Rayleigh Flow Parameters ($\gamma = 1.4$)

M	T_t/T_t^*	T/T^*	p/p^*	p_t/p_t^*	V/V^*	S_{max}/R
0.0	0.0	0.0	2.40000	1.26790	0.0	∞
0.01	0.00048	0.00058	2.39966	1.26779	0.00024	26.98422
0.02	0.00192	0.00230	2.39866	1.26752	0.00096	22.13471
0.03	0.00431	0.00517	2.39698	1.26708	0.00216	19.30065
0.04	0.00765	0.00917	2.39464	1.26646	0.00383	17.29274
0.05	0.01192	0.01430	2.39163	1.26567	0.00598	15.73828
0.06	0.01712	0.02053	2.38796	1.26470	0.00860	14.47123
0.07	0.02322	0.02784	2.38365	1.26356	0.01168	13.40303
0.08	0.03022	0.03621	2.37869	1.26226	0.01522	12.48081
0.09	0.03807	0.04562	2.37309	1.26078	0.01922	11.67046
0.10	0.04678	0.05602	2.36686	1.25915	0.02367	10.94870
0.11	0.05630	0.06739	2.36002	1.25735	0.02856	10.29890
0.12	0.06661	0.07970	2.35257	1.25539	0.03388	9.70879
0.13	0.07768	0.09290	2.34453	1.25329	0.03962	9.16904
0.14	0.08947	0.10695	2.33590	1.25103	0.04578	8.67240
0.15	0.10196	0.12181	2.32671	1.24863	0.05235	8.21311
0.16	0.11511	0.13743	2.31696	1.24608	0.05931	7.78653
0.17	0.12888	0.15377	2.30667	1.24340	0.06666	7.38886

(*Continued*)

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

M	T_t/T_t^*	T/T^*	p/p^*	p_t/p_t^*	V/V^*	S_{max}/R
0.18	0.14324	0.17078	2.29586	1.24059	0.07439	7.01694
0.19	0.15814	0.18841	2.28454	1.23765	0.08247	6.66813
0.20	0.17355	0.20661	2.27273	1.23460	0.09091	6.34018
0.21	0.18943	0.22533	2.26044	1.23142	0.09969	6.03118
0.22	0.20574	0.24452	2.24770	1.22814	0.10879	5.73946
0.23	0.22244	0.26413	2.23451	1.22475	0.11821	5.46359
0.24	0.23948	0.28411	2.22091	1.22126	0.12792	5.20232
0.25	0.25684	0.30440	2.20690	1.21767	0.13793	4.95454
0.26	0.27446	0.32496	2.19250	1.21400	0.14821	4.71926
0.27	0.29231	0.34573	2.17774	1.21025	0.15876	4.49561
0.28	0.31035	0.36667	2.16263	1.20642	0.16955	4.28281
0.29	0.32855	0.38774	2.14719	1.20251	0.18058	4.08016
0.30	0.34686	0.40887	2.13144	1.19855	0.19183	3.88703
0.31	0.36525	0.43004	2.11539	1.19452	0.20329	3.70283
0.32	0.38369	0.45119	2.09908	1.19045	0.21495	3.52706
0.33	0.40214	0.47228	2.08250	1.18632	0.22678	3.35922
0.34	0.42056	0.49327	2.06569	1.18215	0.23879	3.19888
0.35	0.43894	0.51413	2.04866	1.17795	0.25096	3.04565
0.36	0.45723	0.53482	2.03142	1.17371	0.26327	2.89915
0.37	0.47541	0.55529	2.01400	1.16945	0.27572	2.75904
0.38	0.49346	0.57553	1.99641	1.16517	0.28828	2.62500
0.39	0.51134	0.59549	1.97866	1.16088	0.30095	2.49673
0.40	0.52903	0.61515	1.96078	1.15658	0.31373	2.37397
0.41	0.54651	0.63448	1.94278	1.15227	0.32658	2.25645
0.42	0.56376	0.65346	1.92468	1.14796	0.33951	2.14394
0.43	0.58076	0.67205	1.90649	1.14366	0.35251	2.03622
0.44	0.59748	0.69025	1.88822	1.13936	0.36556	1.93306
0.45	0.61393	0.70804	1.86989	1.13508	0.37865	1.83429
0.46	0.63007	0.72538	1.85151	1.13082	0.39178	1.73970
0.47	0.64589	0.74228	1.83310	1.12659	0.40493	1.64912
0.48	0.66139	0.75871	1.81466	1.12238	0.41810	1.56239
0.49	0.67655	0.77466	1.79622	1.11820	0.43127	1.47935
0.50	0.69136	0.79012	1.77778	1.11405	0.44444	1.39985
0.51	0.70581	0.80509	1.75935	1.10995	0.45761	1.32374
0.52	0.71990	0.81955	1.74095	1.10588	0.47075	1.25091
0.53	0.73361	0.83351	1.72258	1.10186	0.48387	1.18121
0.54	0.74695	0.84695	1.70425	1.09789	0.49696	1.11453
0.55	0.75991	0.85987	1.68599	1.09397	0.51001	1.05076
0.56	0.77249	0.87227	1.66778	1.09011	0.52302	0.98977
0.57	0.78468	0.88416	1.64964	1.08630	0.53597	0.93148
0.58	0.79648	0.89552	1.63159	1.08256	0.54887	0.87577
0.59	0.80789	0.90637	1.61362	1.07887	0.56170	0.82255
0.60	0.81892	0.91670	1.59574	1.07525	0.57447	0.77174

M	T_t/T_t^*	T/T^*	p/p^*	p_t/p_t^*	V/V^*	S_{max}/R
0.61	0.82957	0.92653	1.57797	1.07170	0.58716	0.72323
0.62	0.83983	0.93584	1.56031	1.06822	0.59978	0.67696
0.63	0.84970	0.94466	1.54275	1.06481	0.61232	0.63284
0.64	0.85920	0.95298	1.52532	1.06147	0.62477	0.59078
0.65	0.86833	0.96081	1.50801	1.05821	0.63713	0.55073
0.66	0.87708	0.96816	1.49083	1.05503	0.64941	0.51260
0.67	0.88547	0.97503	1.47379	1.05193	0.66158	0.47634
0.68	0.89350	0.98144	1.45688	1.04890	0.67366	0.44187
0.69	0.90118	0.98739	1.44011	1.04596	0.68564	0.40913
0.70	0.90850	0.99290	1.42349	1.04310	0.69751	0.37807
0.71	0.91548	0.99796	1.40701	1.04033	0.70928	0.34861
0.72	0.92212	1.00260	1.39069	1.03764	0.72093	0.32072
0.73	0.92843	1.00682	1.37452	1.03504	0.73248	0.29433
0.74	0.93442	1.01062	1.35851	1.03253	0.74392	0.26940
0.75	0.94009	1.01403	1.34266	1.03010	0.75524	0.24587
0.76	0.94546	1.01706	1.32696	1.02777	0.76645	0.22370
0.77	0.95052	1.01970	1.31143	1.02552	0.77755	0.20283
0.78	0.95528	1.02198	1.29606	1.02337	0.78853	0.18324
0.79	0.95975	1.02390	1.28086	1.02131	0.79939	0.16486
0.80	0.96395	1.02548	1.26582	1.01934	0.81013	0.14767
0.81	0.96787	1.02672	1.25095	1.01747	0.82075	0.13162
0.82	0.97152	1.02763	1.23625	1.01569	0.83125	0.11668
0.83	0.97492	1.02823	1.22171	1.01400	0.84164	0.10280
0.84	0.97807	1.02853	1.20734	1.01241	0.85190	0.08995
0.85	0.98097	1.02854	1.19314	1.01091	0.86204	0.07810
0.86	0.98363	1.02826	1.17911	1.00951	0.87207	0.06722
0.87	0.98607	1.02771	1.16524	1.00820	0.88197	0.05727
0.88	0.98828	1.02689	1.15154	1.00699	0.89175	0.04822
0.89	0.99028	1.02583	1.13801	1.00587	0.90142	0.04004
0.90	0.99207	1.02452	1.12465	1.00486	0.91097	0.03270
0.91	0.99366	1.02297	1.11145	1.00393	0.92039	0.02618
0.92	0.99506	1.02120	1.09842	1.00311	0.92970	0.02044
0.93	0.99627	1.01922	1.08555	1.00238	0.93889	0.01547
0.94	0.99729	1.01702	1.07285	1.00175	0.94797	0.01124
0.95	0.99814	1.01463	1.06030	1.00122	0.95693	0.00771
0.96	0.99883	1.01205	1.04793	1.00078	0.96577	0.00488
0.97	0.99935	1.00929	1.03571	1.00044	0.97450	0.00271
0.98	0.99971	1.00636	1.02365	1.00019	0.98311	0.00119
0.99	0.99993	1.00326	1.01174	1.00005	0.99161	0.00029
1.00	1.00000	1.00000	1.00000	1.00000	1.00000	0.00000
1.01	0.99993	0.99659	0.98841	1.00005	1.00828	0.00029

(*Continued*)

M	T_t/T_t^*	T/T^*	p/p^*	p_t/p_t^*	V/V^*	S_{max}/R
1.02	0.99973	0.99304	0.97698	1.00019	1.01645	0.00114
1.03	0.99940	0.98936	0.96569	1.00044	1.02450	0.00254
1.04	0.99895	0.98554	0.95456	1.00078	1.03246	0.00447
1.05	0.99838	0.98161	0.94358	1.00122	1.04030	0.00690
1.06	0.99769	0.97755	0.93275	1.00175	1.04804	0.00983
1.07	0.99690	0.97339	0.92206	1.00238	1.05567	0.01324
1.08	0.99601	0.96913	0.91152	1.00311	1.06320	0.01711
1.09	0.99501	0.96477	0.90112	1.00394	1.07063	0.02143
1.10	0.99392	0.96031	0.89087	1.00486	1.07795	0.02618
1.11	0.99275	0.95577	0.88075	1.00588	1.08518	0.03135
1.12	0.99148	0.95115	0.87078	1.00699	1.09230	0.03692
1.13	0.99013	0.94645	0.86094	1.00821	1.09933	0.04288
1.14	0.98871	0.94169	0.85123	1.00952	1.10626	0.04922
1.15	0.98721	0.93685	0.84166	1.01093	1.11310	0.05593
1.16	0.98564	0.93196	0.83222	1.01243	1.11984	0.06298
1.17	0.98400	0.92701	0.82292	1.01403	1.12649	0.07038
1.18	0.98230	0.92200	0.81374	1.01573	1.13305	0.07812
1.19	0.98054	0.91695	0.80468	1.01752	1.13951	0.08617
1.20	0.97872	0.91185	0.79576	1.01942	1.14589	0.09453
1.21	0.97684	0.90671	0.78695	1.02140	1.15218	0.10318
1.22	0.97492	0.90153	0.77827	1.02349	1.15838	0.11213
1.23	0.97294	0.89632	0.76971	1.02567	1.16449	0.12135
1.24	0.97092	0.89108	0.76127	1.02795	1.17052	0.13085
1.25	0.96886	0.88581	0.75294	1.03033	1.17647	0.14060
1.26	0.96675	0.88052	0.74473	1.03280	1.18233	0.15061
1.27	0.96461	0.87521	0.73663	1.03537	1.18812	0.16086
1.28	0.96243	0.86988	0.72865	1.03803	1.19382	0.17135
1.29	0.96022	0.86453	0.72078	1.04080	1.19945	0.18206
1.30	0.95798	0.85917	0.71301	1.04366	1.20499	0.19299
1.31	0.95571	0.85380	0.70536	1.04662	1.21046	0.20413
1.32	0.95341	0.84843	0.69780	1.04968	1.21585	0.21548
1.33	0.95108	0.84305	0.69036	1.05283	1.22117	0.22702
1.34	0.94873	0.83766	0.68301	1.05608	1.22642	0.23876
1.35	0.94637	0.83227	0.67577	1.05943	1.23159	0.25068
1.36	0.94398	0.82689	0.66863	1.06288	1.23669	0.26277
1.37	0.94157	0.82151	0.66158	1.06642	1.24173	0.27504
1.38	0.93914	0.81613	0.65464	1.07007	1.24669	0.28747
1.39	0.93671	0.81076	0.64778	1.07381	1.25158	0.30006
1.40	0.93425	0.80539	0.64103	1.07765	1.25641	0.31281
1.41	0.93179	0.80004	0.63436	1.08159	1.26117	0.32570
1.42	0.92931	0.79469	0.62779	1.08563	1.26587	0.33874
1.43	0.92683	0.78936	0.62130	1.08977	1.27050	0.35191
1.44	0.92434	0.78405	0.61491	1.09401	1.27507	0.36522

M	T_t/T_t^*	T/T^*	p/p^*	p_t/p_t^*	V/V^*	S_{max}/R
1.45	0.92184	0.77874	0.60860	1.09835	1.27957	0.37865
1.46	0.91933	0.77346	0.60237	1.10278	1.28402	0.39221
1.47	0.91682	0.76819	0.59623	1.10732	1.28840	0.40589
1.48	0.91431	0.76294	0.59018	1.11196	1.29273	0.41968
1.49	0.91179	0.75771	0.58421	1.11670	1.29700	0.43358
1.50	0.90928	0.75250	0.57831	1.12155	1.30120	0.44758
1.51	0.90676	0.74732	0.57250	1.12649	1.30536	0.46169
1.52	0.90424	0.74215	0.56676	1.13153	1.30945	0.47589
1.53	0.90172	0.73701	0.56111	1.13668	1.31350	0.49019
1.54	0.89920	0.73189	0.55552	1.14193	1.31748	0.50458
1.55	0.89669	0.72680	0.55002	1.14729	1.32142	0.51905
1.56	0.89418	0.72173	0.54458	1.15274	1.32530	0.53361
1.57	0.89168	0.71669	0.53922	1.15830	1.32913	0.54824
1.58	0.88917	0.71168	0.53393	1.16397	1.33291	0.56295
1.59	0.88668	0.70669	0.52871	1.16974	1.33663	0.57774
1.60	0.88419	0.70174	0.52356	1.17561	1.34031	0.59259
1.61	0.88170	0.69680	0.51848	1.18159	1.34394	0.60752
1.62	0.87922	0.69190	0.51346	1.18768	1.34753	0.62250
1.63	0.87675	0.68703	0.50851	1.19387	1.35106	0.63755
1.64	0.87429	0.68219	0.50363	1.20017	1.35455	0.65265
1.65	0.87184	0.67738	0.49880	1.20657	1.35800	0.66781
1.66	0.86939	0.67259	0.49405	1.21309	1.36140	0.68303
1.67	0.86696	0.66784	0.48935	1.21971	1.36475	0.69829
1.68	0.86453	0.66312	0.48472	1.22644	1.36806	0.71360
1.69	0.86212	0.65843	0.48014	1.23328	1.37133	0.72896
1.70	0.85971	0.65377	0.47562	1.24024	1.37455	0.74436
1.71	0.85731	0.64914	0.47117	1.24730	1.37774	0.75981
1.72	0.85493	0.64455	0.46677	1.25447	1.38088	0.77529
1.73	0.85256	0.63999	0.46242	1.26175	1.38398	0.79081
1.74	0.85019	0.63545	0.45813	1.26915	1.38705	0.80636
1.75	0.84784	0.63095	0.45390	1.27666	1.39007	0.82195
1.76	0.84551	0.62649	0.44972	1.28428	1.39306	0.83757
1.77	0.84318	0.62205	0.44559	1.29202	1.39600	0.85322
1.78	0.84087	0.61765	0.44152	1.29987	1.39891	0.86889
1.79	0.83857	0.61328	0.43750	1.30784	1.40179	0.88459
1.80	0.83628	0.60894	0.43353	1.31592	1.40462	0.90031
1.81	0.83400	0.60464	0.42960	1.32413	1.40743	0.91606
1.82	0.83174	0.60036	0.42573	1.33244	1.41019	0.93183
1.83	0.82949	0.59612	0.42191	1.34088	1.41292	0.94761
1.84	0.82726	0.59191	0.41813	1.34943	1.41562	0.96342
1.85	0.82504	0.58774	0.41440	1.35811	1.41829	0.97924

(Continued)

M	T_t/T_t^*	T/T^*	p/p^*	p_t/p_t^*	V/V^*	S_{max}/R
1.86	0.82283	0.58359	0.41072	1.36690	1.42092	0.99507
1.87	0.82064	0.57948	0.40708	1.37582	1.42351	1.01092
1.88	0.81845	0.57540	0.40349	1.38486	1.42608	1.02678
1.89	0.81629	0.57136	0.39994	1.39402	1.42862	1.04265
1.90	0.81414	0.56734	0.39643	1.40330	1.43112	1.05853
1.91	0.81200	0.56336	0.39297	1.41271	1.43359	1.07441
1.92	0.80987	0.55941	0.38955	1.42224	1.43604	1.09031
1.93	0.80776	0.55549	0.38617	1.43190	1.43845	1.10621
1.94	0.80567	0.55160	0.38283	1.44168	1.44083	1.12211
1.95	0.80358	0.54774	0.37954	1.45159	1.44319	1.13802
1.96	0.80152	0.54392	0.37628	1.46164	1.44551	1.15393
1.97	0.79946	0.54012	0.37306	1.47180	1.44781	1.16984
1.98	0.79742	0.53636	0.36988	1.48210	1.45008	1.18575
1.99	0.79540	0.53263	0.36674	1.49253	1.45233	1.20167
2.00	0.79339	0.52893	0.36364	1.50310	1.45455	1.21758
2.01	0.79139	0.52525	0.36057	1.51379	1.45674	1.23348
2.02	0.78941	0.52161	0.35754	1.52462	1.45890	1.24939
2.03	0.78744	0.51800	0.35454	1.53558	1.46104	1.26529
2.04	0.78549	0.51442	0.35158	1.54668	1.46315	1.28118
2.05	0.78355	0.51087	0.34866	1.55791	1.46524	1.29707
2.06	0.78162	0.50735	0.34577	1.56928	1.46731	1.31296
2.07	0.77971	0.50386	0.34291	1.58079	1.46935	1.32883
2.08	0.77782	0.50040	0.34009	1.59244	1.47136	1.34470
2.09	0.77593	0.49696	0.33730	1.60423	1.47336	1.36056
2.10	0.77406	0.49356	0.33454	1.61616	1.47533	1.37641
2.11	0.77221	0.49018	0.33182	1.62823	1.47727	1.39225
2.12	0.77037	0.48684	0.32912	1.64045	1.47920	1.40807
2.13	0.76854	0.48352	0.32646	1.65281	1.48110	1.42389
2.14	0.76673	0.48023	0.32382	1.66531	1.48298	1.43970
2.15	0.76493	0.47696	0.32122	1.67796	1.48484	1.45549
2.16	0.76314	0.47373	0.31865	1.69076	1.48668	1.47127
2.17	0.76137	0.47052	0.31610	1.70371	1.48850	1.48703
2.18	0.75961	0.46734	0.31359	1.71680	1.49029	1.50278
2.19	0.75787	0.46418	0.31110	1.73005	1.49207	1.51852
2.20	0.75613	0.46106	0.30864	1.74345	1.49383	1.53424
2.21	0.75442	0.45796	0.30621	1.75700	1.49556	1.54994
2.22	0.75271	0.45488	0.30381	1.77070	1.49728	1.56563
2.23	0.75102	0.45184	0.30143	1.78456	1.49898	1.58130
2.24	0.74934	0.44882	0.29908	1.79858	1.50066	1.59696
2.25	0.74768	0.44582	0.29675	1.81275	1.50232	1.61259
2.26	0.74602	0.44285	0.29446	1.82708	1.50396	1.62821
2.27	0.74438	0.43990	0.29218	1.84157	1.50558	1.64381
2.28	0.74276	0.43698	0.28993	1.85623	1.50719	1.65939

M	T_t/T_t^*	T/T^*	p/p^*	p_t/p_t^*	V/V^*	S_{max}/R
2.29	0.74114	0.43409	0.28771	1.87104	1.50878	1.67496
2.30	0.73954	0.43122	0.28551	1.88602	1.51035	1.69050
2.31	0.73795	0.42838	0.28333	1.90116	1.51190	1.70602
2.32	0.73638	0.42555	0.28118	1.91647	1.51344	1.72152
2.33	0.73482	0.42276	0.27905	1.93195	1.51496	1.73700
2.34	0.73326	0.41998	0.27695	1.94759	1.51646	1.75246
2.35	0.73173	0.41723	0.27487	1.96340	1.51795	1.76790
2.36	0.73020	0.41451	0.27281	1.97939	1.51942	1.78332
2.37	0.72868	0.41181	0.27077	1.99554	1.52088	1.79872
2.38	0.72718	0.40913	0.26875	2.01187	1.52232	1.81409
2.39	0.72569	0.40647	0.26676	2.02837	1.52374	1.82944
2.40	0.72421	0.40384	0.26478	2.04505	1.52515	1.84477
2.41	0.72275	0.40122	0.26283	2.06191	1.52655	1.86008
2.42	0.72129	0.39864	0.26090	2.07895	1.52793	1.87536
2.43	0.71985	0.39607	0.25899	2.09616	1.52929	1.89062
2.44	0.71842	0.39352	0.25710	2.11356	1.53065	1.90585
2.45	0.71699	0.39100	0.25522	2.13114	1.53198	1.92106
2.46	0.71558	0.38850	0.25337	2.14891	1.53331	1.93625
2.47	0.71419	0.38602	0.25154	2.16685	1.53461	1.95141
2.48	0.71280	0.38356	0.24973	2.18499	1.53591	1.96655
2.49	0.71142	0.38112	0.24793	2.20332	1.53719	1.98167
2.50	0.71006	0.37870	0.24615	2.22183	1.53846	1.99676
2.51	0.70871	0.37630	0.24440	2.24054	1.53972	2.01182
2.52	0.70736	0.37392	0.24266	2.25944	1.54096	2.02686
2.53	0.70603	0.37157	0.24093	2.27853	1.54219	2.04187
2.54	0.70471	0.36923	0.23923	2.29782	1.54341	2.05686
2.55	0.70340	0.36691	0.23754	2.31730	1.54461	2.07183
2.56	0.70210	0.36461	0.23587	2.33699	1.54581	2.08676
2.57	0.70081	0.36233	0.23422	2.35687	1.54699	2.10167
2.58	0.69952	0.36007	0.23258	2.37696	1.54816	2.11656
2.59	0.69826	0.35783	0.23096	2.39725	1.54931	2.13142
2.60	0.69700	0.35561	0.22936	2.41774	1.55046	2.14625
2.61	0.69575	0.35341	0.22777	2.43844	1.55159	2.16106
2.62	0.69451	0.35122	0.22620	2.45935	1.55272	2.17584
2.63	0.69328	0.34906	0.22464	2.48047	1.55383	2.19059
2.64	0.69206	0.34691	0.22310	2.50179	1.55493	2.20532
2.65	0.69084	0.34478	0.22158	2.52334	1.55602	2.22002
2.66	0.68964	0.34266	0.22007	2.54509	1.55710	2.23470
2.67	0.68845	0.34057	0.21857	2.56706	1.55816	2.24934
2.68	0.68727	0.33849	0.21709	2.58925	1.55922	2.26396
2.69	0.68610	0.33643	0.21562	2.61166	1.56027	2.27856

(*Continued*)

M	T_t/T_t^*	T/T^*	p/p^*	p_t/p_t^*	V/V^*	S_{max}/R
2.70	0.68494	0.33439	0.21417	2.63429	1.56131	2.29312
2.71	0.68378	0.33236	0.21273	2.65714	1.56233	2.30766
2.72	0.68264	0.33035	0.21131	2.68021	1.56335	2.32217
2.73	0.68150	0.32836	0.20990	2.70351	1.56436	2.33666
2.74	0.68037	0.32638	0.20850	2.72704	1.56536	2.35111
2.75	0.67926	0.32442	0.20712	2.75080	1.56634	2.36554
2.76	0.67815	0.32248	0.20575	2.77478	1.56732	2.37995
2.77	0.67705	0.32055	0.20439	2.79900	1.56829	2.39432
2.78	0.67595	0.31864	0.20305	2.82346	1.56925	2.40867
2.79	0.67487	0.31674	0.20172	2.84815	1.57020	2.42299
2.80	0.67380	0.31486	0.20040	2.87308	1.57114	2.43728
2.81	0.67273	0.31299	0.19910	2.89825	1.57207	2.45154
2.82	0.67167	0.31114	0.19780	2.92366	1.57300	2.46578
2.83	0.67062	0.30931	0.19652	2.94931	1.57391	2.47999
2.84	0.66958	0.30749	0.19525	2.97521	1.57482	2.49417
2.85	0.66855	0.30568	0.19399	3.00136	1.57572	2.50833
2.86	0.66752	0.30389	0.19275	3.02775	1.57661	2.52245
2.87	0.66651	0.30211	0.19151	3.05440	1.57749	2.53655
2.88	0.66550	0.30035	0.19029	3.08129	1.57836	2.55062
2.89	0.66450	0.29860	0.18908	3.10844	1.57923	2.56467
2.90	0.66350	0.29687	0.18788	3.13585	1.58008	2.57868
2.91	0.66252	0.29515	0.18669	3.16352	1.58093	2.59267
2.92	0.66154	0.29344	0.18551	3.19145	1.58178	2.60663
2.93	0.66057	0.29175	0.18435	3.21963	1.58261	2.62057
2.94	0.65960	0.29007	0.18319	3.24809	1.58343	2.63447
2.95	0.65865	0.28841	0.18205	3.27680	1.58425	2.64835
2.96	0.65770	0.28675	0.18091	3.30579	1.58506	2.66220
2.97	0.65676	0.28512	0.17979	3.33505	1.58587	2.67602
2.98	0.65583	0.28349	0.17867	3.36457	1.58666	2.68981
2.99	0.65490	0.28188	0.17757	3.39437	1.58745	2.70358
3.00	0.65398	0.28028	0.17647	3.42445	1.58824	2.71732
3.01	0.65307	0.27869	0.17539	3.45481	1.58901	2.73103
3.02	0.65216	0.27711	0.17431	3.48544	1.58978	2.74472
3.03	0.65126	0.27555	0.17324	3.51636	1.59054	2.75837
3.04	0.65037	0.27400	0.17219	3.54756	1.59129	2.77200
3.05	0.64949	0.27246	0.17114	3.57905	1.59204	2.78560
3.06	0.64861	0.27094	0.17010	3.61082	1.59278	2.79918
3.07	0.64774	0.26942	0.16908	3.64289	1.59352	2.81272
3.08	0.64687	0.26792	0.16806	3.67524	1.59425	2.82624
3.09	0.64601	0.26643	0.16705	3.70790	1.59497	2.83974
3.10	0.64516	0.26495	0.16604	3.74084	1.59568	2.85320
3.11	0.64432	0.26349	0.16505	3.77409	1.59639	2.86664
3.12	0.64348	0.26203	0.16407	3.80764	1.59709	2.88005

M	T_t/T_t^*	T/T^*	p/p^*	p_t/p_t^*	V/V^*	S_{max}/R
3.13	0.64265	0.26059	0.16309	3.84149	1.59779	2.89343
3.14	0.64182	0.25915	0.16212	3.87565	1.59848	2.90679
3.15	0.64100	0.25773	0.16117	3.91011	1.59917	2.92011
3.16	0.64018	0.25632	0.16022	3.94488	1.59985	2.93342
3.17	0.63938	0.25492	0.15927	3.97997	1.60052	2.94669
3.18	0.63857	0.25353	0.15834	4.01537	1.60119	2.95994
3.19	0.63778	0.25215	0.15741	4.05108	1.60185	2.97316
3.20	0.63699	0.25078	0.15649	4.08712	1.60250	2.98635
3.21	0.63621	0.24943	0.15558	4.12347	1.60315	2.99952
3.22	0.63543	0.24808	0.15468	4.16015	1.60380	3.01266
3.23	0.63465	0.24674	0.15379	4.19715	1.60444	3.02577
3.24	0.63389	0.24541	0.15290	4.23449	1.60507	3.03885
3.25	0.63313	0.24410	0.15202	4.27215	1.60570	3.05191
3.26	0.63237	0.24279	0.15115	4.31014	1.60632	3.06495
3.27	0.63162	0.24149	0.15028	4.34847	1.60694	3.07795
3.28	0.63088	0.24021	0.14942	4.38714	1.60755	3.09093
3.29	0.63014	0.23893	0.14857	4.42614	1.60816	3.10388
3.30	0.62940	0.23766	0.14773	4.46549	1.60877	3.11681
3.31	0.62868	0.23640	0.14689	4.50518	1.60936	3.12971
3.32	0.62795	0.23515	0.14606	4.54522	1.60996	3.14258
3.33	0.62724	0.23391	0.14524	4.58561	1.61054	3.15543
3.34	0.62652	0.23268	0.14442	4.62635	1.61113	3.16825
3.35	0.62582	0.23146	0.14361	4.66744	1.61170	3.18105
3.36	0.62512	0.23025	0.14281	4.70889	1.61228	3.19382
3.37	0.62442	0.22905	0.14201	4.75070	1.61285	3.20656
3.38	0.62373	0.22785	0.14122	4.79287	1.61341	3.21928
3.39	0.62304	0.22667	0.14044	4.83540	1.61397	3.23197
3.40	0.62236	0.22549	0.13966	4.87830	1.61453	3.24463
3.41	0.62168	0.22432	0.13889	4.92157	1.61508	3.25727
3.42	0.62101	0.22317	0.13813	4.96521	1.61562	3.26988
3.43	0.62034	0.22201	0.13737	5.00923	1.61616	3.28247
3.44	0.61968	0.22087	0.13662	5.05362	1.61670	3.29503
3.45	0.61902	0.21974	0.13587	5.09839	1.61723	3.30757
3.46	0.61837	0.21861	0.13513	5.14355	1.61776	3.32008
3.47	0.61772	0.21750	0.13440	5.18909	1.61829	3.33257
3.48	0.61708	0.21639	0.13367	5.23501	1.61881	3.34503
3.49	0.61644	0.21529	0.13295	5.28133	1.61932	3.35746
3.50	0.61580	0.21419	0.13223	5.32804	1.61983	3.36987
3.51	0.61517	0.21311	0.13152	5.37514	1.62034	3.38225
3.52	0.61455	0.21203	0.13081	5.42264	1.62085	3.39461
3.53	0.61393	0.21096	0.13011	5.47054	1.62135	3.40695

(*Continued*)

M	T_t/T_t^*	T/T^*	p/p^*	p_t/p_t^*	V/V^*	S_{max}/R
3.54	0.61331	0.20990	0.12942	5.51885	1.62184	3.41926
3.55	0.61270	0.20885	0.12873	5.56756	1.62233	3.43154
3.56	0.61209	0.20780	0.12805	5.61668	1.62282	3.44380
3.57	0.61149	0.20676	0.12737	5.66621	1.62331	3.45603
3.58	0.61089	0.20573	0.12670	5.71615	1.62379	3.46824
3.59	0.61029	0.20470	0.12603	5.76652	1.62427	3.48043
3.60	0.60970	0.20369	0.12537	5.81730	1.62474	3.49259
3.61	0.60911	0.20268	0.12471	5.86850	1.62521	3.50472
3.62	0.60853	0.20167	0.12406	5.92013	1.62567	3.51683
3.63	0.60795	0.20068	0.12341	5.97219	1.62614	3.52892
3.64	0.60738	0.19969	0.12277	6.02468	1.62660	3.54098
3.65	0.60681	0.19871	0.12213	6.07761	1.62705	3.55302
3.66	0.60624	0.19773	0.12150	6.13097	1.62750	3.56503
3.67	0.60568	0.19677	0.12087	6.18477	1.62795	3.57702
3.68	0.60512	0.19581	0.12024	6.23902	1.62840	3.58899
3.69	0.60456	0.19485	0.11963	6.29371	1.62884	3.60093
3.70	0.60401	0.19390	0.11901	6.34884	1.62928	3.61285
3.71	0.60346	0.19296	0.11840	6.40443	1.62971	3.62474
3.72	0.60292	0.19203	0.11780	6.46048	1.63014	3.63661
3.73	0.60238	0.19110	0.11720	6.51698	1.63057	3.64845
3.74	0.60184	0.19018	0.11660	6.57394	1.63100	3.66028
3.75	0.60131	0.18926	0.11601	6.63137	1.63142	3.67207
3.76	0.60078	0.18836	0.11543	6.68926	1.63184	3.68385
3.77	0.60025	0.18745	0.11484	6.74763	1.63225	3.69560
3.78	0.59973	0.18656	0.11427	6.80646	1.63267	3.70733
3.79	0.59921	0.18567	0.11369	6.86578	1.63308	3.71903
3.80	0.59870	0.18478	0.11312	6.92557	1.63348	3.73071
3.81	0.59819	0.18391	0.11256	6.98584	1.63389	3.74237
3.82	0.59768	0.18303	0.11200	7.04660	1.63429	3.75401
3.83	0.59717	0.18217	0.11144	7.10784	1.63469	3.76562
3.84	0.59667	0.18131	0.11089	7.16958	1.63508	3.77721
3.85	0.59617	0.18045	0.11034	7.23181	1.63547	3.78877
3.86	0.59568	0.17961	0.10979	7.29454	1.63586	3.80031
3.87	0.59519	0.17876	0.10925	7.35777	1.63625	3.81183
3.88	0.59470	0.17793	0.10871	7.42151	1.63663	3.82333
3.89	0.59421	0.17709	0.10818	7.48575	1.63701	3.83481
3.90	0.59373	0.17627	0.10765	7.55050	1.63739	3.84626
3.91	0.59325	0.17545	0.10713	7.61577	1.63777	3.85769
3.92	0.59278	0.17463	0.10661	7.68156	1.63814	3.86909
3.93	0.59231	0.17383	0.10609	7.74786	1.63851	3.88048
3.94	0.59184	0.17302	0.10557	7.81469	1.63888	3.89184
3.95	0.59137	0.17222	0.10506	7.88205	1.63924	3.90318
3.96	0.59091	0.17143	0.10456	7.94993	1.63960	3.91450

M	T_t/T_t^*	T/T^*	p/p^*	p_t/p_t^*	V/V^*	S_{max}/R
3.97	0.59045	0.17064	0.10405	8.01835	1.63996	3.92579
3.98	0.58999	0.16986	0.10355	8.08731	1.64032	3.93706
3.99	0.58954	0.16908	0.10306	8.15681	1.64067	3.94831
4.00	0.58909	0.16831	0.10256	8.22685	1.64103	3.95954
4.10	0.58473	0.16086	0.09782	8.95794	1.64441	4.07064
4.20	0.58065	0.15388	0.09340	9.74729	1.64757	4.17961
4.30	0.57682	0.14734	0.08927	10.59854	1.65052	4.28652
4.40	0.57322	0.14119	0.08540	11.51554	1.65329	4.39143
4.50	0.56982	0.13540	0.08177	12.50226	1.65588	4.49440
4.60	0.56663	0.12996	0.07837	13.56288	1.65831	4.59550
4.70	0.56362	0.12483	0.07517	14.70174	1.66059	4.69477
4.80	0.56078	0.12000	0.07217	15.92337	1.66274	4.79229
4.90	0.55809	0.11543	0.06934	17.23245	1.66476	4.88809
5.00	0.55556	0.11111	0.06667	18.63390	1.66667	4.98224
5.10	0.55315	0.10703	0.06415	20.13279	1.66847	5.07477
5.20	0.55088	0.10316	0.06177	21.73439	1.67017	5.16575
5.30	0.54872	0.09950	0.05951	23.44420	1.67178	5.25522
5.40	0.54667	0.09602	0.05738	25.26788	1.67330	5.34322
5.50	0.54473	0.09272	0.05536	27.21132	1.67474	5.42979
5.60	0.54288	0.08958	0.05345	29.28063	1.67611	5.51498
5.70	0.54112	0.08660	0.05163	31.48210	1.67741	5.59883
5.80	0.53944	0.08376	0.04990	33.82228	1.67864	5.68138
5.90	0.53785	0.08106	0.04826	36.30790	1.67982	5.76265
6.00	0.53633	0.07849	0.04669	38.94594	1.68093	5.84270
6.10	0.53488	0.07603	0.04520	41.74362	1.68200	5.92155
6.20	0.53349	0.07369	0.04378	44.70837	1.68301	5.99924
6.30	0.53217	0.07145	0.04243	47.84787	1.68398	6.07579
6.40	0.53091	0.06931	0.04114	51.17004	1.68490	6.15124
6.50	0.52970	0.06726	0.03990	54.68303	1.68579	6.22562
6.60	0.52854	0.06531	0.03872	58.39527	1.68663	6.29896
6.70	0.52743	0.06343	0.03759	62.31541	1.68744	6.37128
6.80	0.52637	0.06164	0.03651	66.45238	1.68821	6.44261
6.90	0.52535	0.05991	0.03547	70.81536	1.68895	6.51298
7.00	0.52438	0.05826	0.03448	75.41379	1.68966	6.58240
7.50	0.52004	0.05094	0.03009	102.28748	1.69279	6.91625
8.00	0.51647	0.04491	0.02649	136.62352	1.69536	7.22982
8.50	0.51349	0.03988	0.02349	179.92363	1.69750	7.52538
9.00	0.51098	0.03565	0.02098	233.88395	1.69930	7.80482
9.50	0.50885	0.03205	0.01885	300.40722	1.70082	
10.00	0.50702	0.02897	0.01702	381.61488	1.70213	
∞	0.48980	0.0	0.0	∞	1.7143	

Appendix K

Properties of Air at Low Pressure

THERMODYNAMIC PROPERTIES OF AIR AT LOW PRESSURES

This information is presented in English Engineering (EE) units

T is in °R, ϕ is in Btu/lbm-°R.
t is in °F, h and u are in Btu/lbm.

p_r and v_r are relative pressure and relative volume.

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

T	t	h	p_r	u	v_r	ϕ	T	t	h	p_r	u	v_r	ϕ
200	−259.7	47.67	0.04320	33.96	1714.9	0.36303	**600**	140.3	143.47	2.005	102.34	110.88	0.62607
210	−249.7	50.07	0.05122	35.67	1518.6	0.37470	**610**	150.3	145.88	2.124	104.06	106.38	0.63005
220	−239.7	52.46	0.06026	37.38	1352.5	0.38584	**620**	160.3	148.28	2.249	105.78	102.12	0.63395
230	−229.7	54.85	0.07037	39.08	1210.7	0.39648	**630**	170.3	150.68	2.379	107.50	98.11	0.63781
240	−219.7	57.25	0.08165	40.80	1088.8	0.40666	**640**	180.3	153.09	2.514	109.21	94.30	0.64159
250	−209.7	59.64	0.94150	42.50	983.6	0.41643	**650**	190.3	155.50	2.655	110.94	90.69	0.64533
260	−199.7	62.03	0.10797	44.21	892.0	0.42582	**660**	200.3	157.92	2.801	112.67	87.27	0.64902
270	−189.7	64.43	0.12318	45.92	812.0	0.43485	**670**	210.3	160.33	2.953	114.40	84.03	0.65263
280	−179.7	66.82	0.13986	47.63	741.6	0.44356	**680**	220.3	162.73	3.111	116.12	80.96	0.65621
290	−169.7	69.21	0.15808	49.33	679.5	0.45196	**690**	230.3	165.15	3.276	117.85	78.03	0.65973
300	−159.7	71.61	0.17795	51.04	624.5	0.46007	**700**	240.3	167.56	3.446	119.58	75.25	0.66321
310	−149.7	74.00	0.19952	52.75	575.6	0.46791	**710**	250.3	169.98	3.623	121.32	72.60	0.66664
320	−139.7	76.40	0.22290	54.46	531.8	0.47550	**720**	260.3	172.39	3.806	123.04	70.07	0.67002
330	−129.7	78.78	0.24819	56.16	492.6	0.48287	**730**	270.3	174.82	3.996	124.78	67.67	0.67335
340	−119.7	81.18	0.27545	57.87	457.2	0.49002	**740**	280.3	177.23	4.193	126.51	65.38	0.67665
350	−109.7	83.57	0.3048	59.58	425.4	0.49695	**750**	290.3	179.66	4.396	128.25	63.20	0.67991
360	−99.7	85.97	0.3363	61.29	396.6	0.50369	**760**	300.3	182.08	4.607	129.99	61.10	0.68312
370	−89.7	88.35	0.3700	62.99	370.4	0.51024	**770**	310.3	184.51	4.826	131.73	59.11	0.68629
380	−79.7	90.75	0.4061	64.70	346.6	0.51663	**780**	320.3	186.94	5.051	133.47	57.20	0.68942
390	−69.7	93.13	0.4447	66.40	324.9	0.52284	**790**	330.3	189.38	5.285	135.22	55.38	0.69251
400	−59.7	95.53	0.4858	68.11	305.0	0.52890	**800**	340.3	191.81	5.526	136.97	53.63	0.69558
410	−49.7	97.93	0.5295	69.82	286.8	0.53481	**810**	350.3	194.25	5.775	138.72	51.96	0.69860
420	−39.7	100.32	0.5760	71.52	270.1	0.54058	**820**	360.3	196.69	6.033	140.47	50.35	0.70160
430	−29.7	102.71	0.6253	73.23	254.7	0.54621	**830**	370.3	199.12	6.299	142.22	48.81	0.70455
440	−19.7	105.11	0.6776	74.93	240.6	0.55172	**840**	380.3	201.56	6.573	143.98	47.34	0.70747
450	−9.7	107.50	0.7329	76.65	227.45	0.55710	**850**	390.3	204.01	6.856	145.74	45.92	0.71037
460	0.3	109.90	0.7913	78.36	215.33	0.56235	**860**	400.3	206.46	7.149	147.50	44.57	0.71323
470	10.3	112.30	0.8531	80.07	204.08	0.56751	**870**	410.3	208.90	7.450	149.27	43.26	0.71606
480	20.3	114.69	0.9182	81.77	193.65	0.57255	**880**	420.3	211.35	7.761	151.02	42.01	0.71886
490	30.3	117.08	0.9868	83.49	183.94	0.57749	**890**	430.3	213.80	8.081	152.80	40.80	0.72163
500	40.3	119.48	1.0590	85.20	147.90	0.58233	**900**	440.3	216.26	8.411	154.57	39.64	0.72438
510	50.3	121.87	1.1349	86.92	166.46	0.58707	**910**	450.3	218.72	8.752	156.34	38.52	0.72710
520	60.3	124.27	1.2147	88.62	158.58	0.59173	**920**	460.3	221.18	9.102	158.12	37.44	0.72979
530	70.3	126.66	1.2983	90.34	151.22	0.59630	**930**	470.3	223.64	9.463	159.89	36.41	0.73245
540	80.3	129.06	1.3860	92.04	144.32	0.60078	**940**	480.3	226.11	9.834	161.68	35.41	0.73509
550	90.3	131.46	1.4779	93.76	137.85	0.60518	**950**	490.3	228.58	10.216	163.46	34.45	0.73771

560	100.3	133.86	1.5742	95.47	131.78	0.60950	**960**	500.3	231.06	11.610	165.26	33.52	0.74030
570	110.3	136.26	1.6748	97.19	126.08	0.61376	**970**	510.3	233.53	11.014	167.05	32.63	0.74287
580	120.3	138.66	1.7800	98.90	120.70	0.61793	**980**	520.3	236.02	11.430	168.83	31.76	0.74540
590	130.3	141.06	1.8899	100.62	115.65	0.62204	**990**	530.3	238.50	11.858	170.63	30.92	0.74792
1000	540.3	240.98	12.298	172.43	30.12	0.75042	**1500**	1040.3	369.17	55.86	266.34	9.948	0.85416
1010	550.3	243.48	12.751	174.24	29.34	0.75290	**1510**	1050.3	371.82	57.30	268.30	9.761	0.85592
1020	560.3	245.97	13.215	176.04	28.59	0.75536	**1520**	1060.3	374.47	58.78	270.26	9.578	0.85767
1030	570.3	248.45	13.692	177.84	27.87	0.75778	**1530**	1070.3	377.11	60.29	272.23	9.400	0.85940
1040	580.3	250.95	14.182	179.66	27.17	0.76019	**1540**	1080.3	379.77	61.83	274.20	9.226	0.86113
1050	590.3	253.45	14.686	181.47	26.48	0.76259	**1550**	1090.3	382.42	63.40	276.17	9.056	0.86285
1060	600.3	255.96	15.203	183.29	25.82	0.76496	**1560**	1100.3	385.08	65.00	278.13	8.890	0.86456
1070	610.3	258.47	15.734	185.10	25.19	0.76732	**1570**	1110.3	387.74	66.63	280.11	8.728	0.86626
1080	620.3	260.97	16.278	186.93	24.58	0.76964	**1580**	1120.3	390.40	68.30	282.09	8.569	0.86794
1090	630.3	263.48	16.838	188.75	23.98	0.77196	**1590**	1130.3	393.07	70.00	284.08	8.414	0.86962
1100	640.3	265.99	17.413	190.58	23.40	0.77426	**1600**	1140.3	395.74	71.73	286.06	8.263	0.87130
1110	650.3	268.52	18.000	192.41	22.84	0.77654	**1610**	1150.3	398.42	73.49	288.05	8.115	0.87297
1120	660.3	271.03	18.604	194.25	22.30	0.77880	**1620**	1160.3	401.09	75.29	290.04	7.971	0.87462
1130	670.3	273.56	19.223	196.09	21.78	0.78104	**1630**	1170.3	403.77	77.12	292.03	7.829	0.87627
1140	680.3	276.08	19.858	197.94	21.27	0.78326	**1640**	1180.3	406.45	78.99	294.03	7.691	0.87791
1150	690.3	278.61	20.51	199.78	20.771	0.78548	**1650**	1190.3	409.13	80.89	296.03	7.556	0.87954
1160	700.3	281.14	21.18	201.63	20.293	0.78767	**1660**	1200.3	411.82	82.83	298.02	7.424	0.88116
1170	710.3	283.68	21.86	203.49	19.828	0.78985	**1670**	1210.3	414.51	84.80	300.03	7.295	0.88278
1180	720.3	286.21	22.56	205.33	19.377	0.79201	**1680**	1220.3	417.20	86.82	302.04	7.168	0.88439
1190	730.3	288.76	23.28	207.19	18.940	0.79415	**1690**	1230.3	419.89	88.87	304.04	7.045	0.88599
1200	740.3	291.30	24.01	209.05	18.514	0.79628	**1700**	1240.3	422.59	90.95	306.06	6.924	0.88758
1210	750.3	293.86	24.76	210.92	18.102	0.79840	**1710**	1250.3	425.29	93.08	308.07	6.805	0.88916
1220	760.3	296.41	25.53	212.78	17.700	0.80050	**1720**	1260.3	428.00	95.24	310.09	6.690	0.89074
1230	770.3	298.96	26.32	214.65	17.311	0.80258	**1730**	1270.3	430.69	97.45	312.10	6.576	0.89230
1240	780.3	301.52	27.13	216.53	16.932	0.80466	**1740**	1280.3	433.41	99.69	314.13	6.465	0.89387
1250	790.3	304.08	27.96	218.40	16.563	0.80672	**1750**	1290.3	436.12	101.98	316.16	6.357	0.89542
1260	800.3	306.65	28.80	220.28	16.205	0.80876	**1760**	1300.3	438.83	104.30	318.18	6.251	0.89697
1270	810.3	309.22	29.67	222.16	15.857	0.81079	**1770**	1310.3	441.55	106.67	320.22	6.147	0.89850
1280	820.3	311.79	30.55	224.05	15.518	0.81280	**1780**	1320.3	444.26	109.08	322.24	6.045	0.90003
1290	830.3	314.36	31.46	225.93	15.189	0.81481	**1790**	1330.3	446.99	111.54	324.29	5.945	0.90155
1300	840.3	316.94	32.39	227.83	14.868	0.81680	**1800**	1340.3	449.71	114.03	326.32	5.847	0.90308
1310	850.3	319.53	33.34	229.73	14.557	0.81878	**1810**	1350.3	452.44	116.57	328.37	5.752	0.90458
1320	860.3	322.11	34.31	231.63	14.253	0.82075	**1820**	1360.3	455.17	119.16	330.40	5.658	0.90609

(*Continued*)

T	t	h	p_r	u	v_r	ϕ	T	t	h	p_r	u	v_r	ϕ
1330	870.3	324.69	35.30	233.52	13.958	0.82270	**1830**	1370.3	457.90	121.79	332.45	5.566	0.90759
1340	880.3	327.29	36.31	235.43	13.670	0.82464	**1840**	1380.3	460.63	124.47	334.50	5.476	0.90908
1350	890.3	329.88	37.35	237.34	13.391	0.82658	**1850**	1390.3	463.37	127.18	336.55	5.388	0.91056
1360	900.3	332.48	38.41	239.25	13.118	0.82848	**1860**	1400.3	466.12	129.95	338.61	5.302	0.91203
1370	910.3	335.09	39.49	241.17	12.851	0.83039	**1870**	1410.3	468.86	132.77	340.66	5.217	0.91350
1380	920.3	337.68	40.59	243.08	12.593	0.83229	**1880**	1420.3	471.60	135.64	342.73	5.134	0.91497
1390	930.3	340.29	41.73	245.00	12.340	0.83417	**1890**	1430.3	474.35	138.55	344.78	5.053	0.91643
1400	940.3	342.90	42.88	246.93	12.095	0.83604	**1900**	1440.3	477.09	141.51	346.85	4.974	0.91788
1410	950.3	345.52	44.06	248.86	11.855	0.83790	**1910**	1450.3	479.85	144.53	348.91	4.896	0.91932
1420	960.3	348.14	45.26	250.79	11.622	0.83975	**1920**	1460.3	482.60	147.59	350.98	4.819	0.92076
1430	970.3	350.75	46.49	252.72	11.394	0.84158	**1930**	1470.3	485.36	150.70	353.05	4.744	0.92220
1440	980.3	353.37	47.75	254.66	11.172	0.84341	**1940**	1480.3	488.12	153.87	355.12	4.670	0.92362
1450	990.3	356.00	49.03	256.60	10.954	0.84523	**1950**	1490.3	490.88	157.10	357.20	4.598	0.92504
1460	1000.3	358.63	50.34	258.54	10.743	0.84704	**1960**	1500.3	493.64	160.37	359.28	4.527	0.92645
1470	1010.3	361.27	51.68	260.49	10.537	0.84884	**1970**	1510.3	496.40	163.69	361.36	4.458	0.92786
1480	1020.3	363.89	53.04	262.44	10.336	0.85062	**1980**	1520.3	499.17	167.07	363.43	4.390	0.92926
1490	1030.3	366.53	54.43	264.38	10.140	0.85239	**1990**	1530.3	501.94	170.50	365.53	4.323	0.93066
2000	1540.3	504.71	174.00	367.61	4.258	0.93205	**2500**	2040.3	645.78	435.7	474.40	2.125	0.99497
2010	1550.3	507.49	177.55	369.71	4.194	0.93343	**2510**	2050.3	648.65	443.0	476.58	2.099	0.99611
2020	1560.3	510.26	181.16	371.79	4.130	0.93481	**2520**	2060.3	651.51	450.5	478.77	2.072	0.99725
2030	1570.3	513.04	184.81	373.88	4.069	0.93618	**2530**	2070.3	654.38	458.0	480.94	2.046	0.99838
2040	1580.3	515.82	188.54	375.98	4.008	0.93756	**2540**	2080.3	657.25	465.6	483.13	2.021	0.99952
2050	1590.3	518.61	192.31	378.08	3.949	0.93891	**2550**	2090.3	660.12	473.3	485.31	1.9956	1.00064
2060	1600.3	521.39	196.16	380.18	3.890	0.94026	**2560**	2100.3	662.99	481.1	487.51	1.9709	1.00176
2070	1610.3	524.18	200.06	382.28	3.833	0.94161	**2570**	2110.3	665.86	489.1	489.69	1.9465	1.00288
2080	1620.3	526.97	204.02	384.39	3.777	0.94296	**2580**	2120.3	668.74	497.1	491.88	1.9225	1.00400
2090	1630.3	529.75	208.06	386.48	3.721	0.94430	**2590**	2130.3	671.61	505.3	494.07	1.8989	1.00511
2100	1640.3	532.55	212.1	388.60	3.667	0.94564	**2600**	2140.3	674.49	513.5	496.26	1.8756	1.00623
2110	1650.3	535.35	216.3	390.71	3.614	0.94696	**2610**	2150.3	677.37	521.8	498.46	1.8527	1.00733
2120	1660.3	538.15	220.5	392.83	3.561	0.94829	**2620**	2160.3	680.25	530.3	500.65	1.8302	1.00843
2130	1670.3	540.94	224.8	394.93	3.510	0.94960	**2630**	2170.3	683.13	538.9	502.85	1.8079	1.00953
2140	1680.3	543.74	229.1	397.05	3.460	0.95092	**2640**	2180.3	686.01	547.5	505.05	1.7861	1.01063
2150	1690.3	546.54	233.5	399.17	3.410	0.95222	**2650**	2190.3	688.90	556.3	507.25	1.7646	1.01172
2160	1700.3	549.35	238.0	401.29	3.362	0.95352	**2660**	2200.3	691.79	565.2	509.44	1.7434	1.01281
2170	1710.3	552.16	242.6	403.41	3.314	0.95482	**2670**	2210.3	694.68	574.2	511.65	1.7225	1.01389
2180	1720.3	554.97	247.2	405.53	3.267	0.95611	**2680**	2220.3	697.56	583.3	513.85	1.7019	1.01497

2190	1730.3	557.78	251.9	407.66	3.221	0.95740	**2690**	2230.3	700.45	592.5	516.05	1.6817	1.01605
2200	1740.3	560.59	256.6	409.78	3.176	0.95868	**2700**	2240.3	703.35	601.9	518.26	1.6617	1.01712
2210	1750.3	563.41	261.4	411.92	3.131	0.95996	**2710**	2250.3	706.24	611.3	520.47	1.6420	1.01819
2220	1760.3	566.23	266.3	414.05	3.088	0.96123	**2720**	2260.3	709.13	620.9	522.68	1.6226	1.01926
2230	1770.3	569.04	271.3	416.18	3.045	0.96250	**2730**	2270.3	712.03	630.7	524.88	1.6035	1.02032
2240	1780.3	571.86	276.3	418.31	3.003	0.96376	**2740**	2280.3	714.93	640.5	527.10	1.5847	1.02138
2250	1790.3	574.69	281.4	420.46	2.961	0.96501	**2750**	2290.3	717.83	650.4	529.31	1.5662	1.02244
2260	1800.3	577.51	286.6	422.59	2.921	0.96626	**2760**	2300.3	720.72	660.5	531.53	1.5480	1.02348
2270	1810.3	580.34	291.9	424.74	2.881	0.96751	**2770**	2310.3	723.62	670.7	533.74	1.5299	1.02453
2280	1820.3	583.16	297.2	426.87	2.841	0.96876	**2780**	2320.3	726.53	681.0	535.96	1.5122	1.02558
2290	1830.3	585.99	302.7	429.01	2.803	0.96999	**2790**	2330.3	729.42	691.4	538.17	1.4948	1.02662
2300	1840.3	588.82	308.1	431.16	2.765	0.97123	**2800**	2340.3	732.33	702.0	540.40	1.4775	1.02767
2310	1850.3	591.66	313.7	433.31	2.728	0.97246	**2810**	2350.3	735.24	712.7	542.62	1.4606	1.02870
2320	1860.3	594.49	319.4	435.46	2.691	0.97369	**2820**	2360.3	738.15	723.5	544.85	1.4439	1.02974
2330	1870.3	597.32	325.1	437.60	2.655	0.97489	**2830**	2370.3	741.05	734.4	547.06	1.4274	1.03076
2340	1880.3	600.16	330.9	439.76	2.619	0.97611	**2840**	2380.3	743.96	745.5	549.29	1.4112	1.03179
2350	1890.3	603.00	336.8	441.91	2.585	0.97732	**2850**	2390.3	746.88	756.7	551.52	1.3951	1.03282
2360	1900.3	605.84	342.8	444.07	2.550	0.97853	**2860**	2400.3	749.79	768.1	553.74	1.3794	1.03383
2370	1910.3	608.68	348.9	446.22	2.517	0.97973	**2870**	2410.3	752.71	779.6	555.98	1.3638	1.03484
2380	1920.3	611.53	355.0	448.38	2.483	0.98092	**2880**	2420.3	755.61	791.2	558.19	1.3485	1.03586
2390	1930.3	614.37	361.3	450.54	2.451	0.98212	**2890**	2430.3	758.53	802.9	560.43	1.3333	1.03687
2400	1940.3	617.22	367.6	452.70	2.419	0.98331	**2900**	2440.3	761.45	814.8	562.66	1.3184	1.03788
2410	1950.3	620.07	374.0	454.87	2.387	0.98449	**2910**	2450.3	764.37	826.8	564.90	1.3037	1.03889
2420	1960.3	622.92	380.5	457.02	2.356	0.98567	**2920**	2460.3	767.29	839.0	567.13	1.2892	1.03989
2430	1970.3	625.77	387.0	459.20	2.326	0.98685	**2930**	2470.3	770.21	851.3	569.37	1.2749	1.04089
2440	1980.3	628.62	393.7	461.36	2.296	0.98802	**2940**	2480.3	773.13	863.8	571.60	1.2608	1.04188
2450	1990.3	631.48	400.5	463.54	2.266	0.98919	**2950**	2490.3	776.05	876.4	573.84	1.2469	1.04288
2460	2000.3	634.34	407.3	465.70	2.237	0.99035	**2960**	2500.3	778.97	889.1	576.07	1.2332	1.04386
2470	2010.3	637.20	414.3	467.88	2.209	0.99151	**2970**	2510.3	781.90	902.0	578.32	1.2197	1.04484
2480	2020.3	640.05	421.3	470.05	2.180	0.99266	**2980**	2520.3	784.83	915.0	580.56	1.2064	1.04583
2490	2030.3	642.91	428.5	472.22	2.153	0.99381	**2990**	2530.3	787.75	928.2	582.79	1.1932	1.04681
3000	2540.3	790.68	941.4	585.04	1.1803	1.04779	**3500**	3040.3	938.40	1829.3	698.48	0.7087	1.09332
3010		793.61	955.0	587.29	1.1675	1.04877	**3510**		941.38	1852.1	700.78	0.7020	1.09417
3020		796.54	968.7	589.53	1.1549	1.04974	**3520**		944.36	1875.2	703.07	0.6954	1.09502
3030		799.47	982.5	591.78	1.1425	1.05071	**3530**		947.34	1898.6	705.36	0.6888	1.09587
3040		802.41	996.4	594.03	1.1302	1.05168	**3540**		950.32	1922.1	707.65	0.6823	1.09671
3050	2590.3	805.34	1010.5	596.28	1.1181	1.05264	**3550**	3090.3	953.30	1945.8	709.95	0.6759	1.09755

(*Continued*)

T	t	h	p_r	u	v_r	ϕ	T	t	h	p_r	u	v_r	ϕ
3060		808.28	1024.8	598.52	1.1061	1.05359	**3560**		956.28	1969.8	712.24	0.6695	1.09838
3070		811.22	1039.2	600.77	1.0943	1.05455	**3570**		959.26	1993.9	714.54	0.6632	1.09922
3080		814.15	1053.8	603.02	1.0827	1.05551	**3580**		962.25	2018.3	716.84	0.6571	1.10005
3090		817.09	1068.5	605.27	1.0713	1.05646	**3590**		965.23	2043.0	719.14	0.6510	1.10089
3100	2640.3	820.03	1083.4	607.53	1.0600	1.05741	**3600**	3140.3	968.21	2067.9	721.44	0.6449	1.10172
3110		822.97	1098.5	609.79	1.0488	1.05836	**3610**		971.20	2093.0	723.74	0.6389	1.10255
3120		825.91	1113.7	612.05	1.0378	1.05930	**3620**		974.18	2118.4	726.04	0.6330	1.10337
3130		828.86	1129.1	614.30	1.0269	1.06025	**3630**		977.17	2144.0	728.34	0.6272	1.10420
3140		831.80	1144.7	616.56	1.0162	1.06119	**3640**		980.16	2169.9	730.64	0.6214	1.10502
3150	2690.3	834.75	1160.5	618.82	1.0056	1.06212	**3650**	3190.3	983.15	2196.0	732.95	0.6157	1.10584
3160		837.69	1176.4	621.08	0.9951	1.06305	**3660**		986.14	2222.4	735.26	0.6101	1.10665
3170		840.64	1192.5	623.35	0.9848	1.06398	**3670**		989.13	2249.0	737.57	0.6045	1.10747
3180		843.59	1208.7	625.60	0.9746	1.06491	**3680**		992.12	2275.8	739.87	0.5990	1.10828
3190		846.53	1225.1	627.86	0.9646	1.06584	**3690**		995.11	2302.9	742.17	0.5936	1.10910
3200	2740.3	849.48	1241.7	630.12	0.9546	1.06676	**3700**	3240.3	998.11	2330.3	744.48	0.5882	1.10991
3210		852.43	1258.5	632.39	0.9448	1.06768	**3710**		1001.11	2358.0	746.79	0.5829	1.11071
3220		855.38	1275.5	634.65	0.9352	1.06860	**3720**		1004.10	2385.9	749.10	0.5776	1.11152
3230		858.33	1292.7	636.92	0.9256	1.06952	**3730**		1007.10	2414.0	751.41	0.5724	1.11233
3240		861.28	1310.0	639.19	0.9162	1.07043	**3740**		1010.09	2442.4	753.73	0.5672	1.11313
3250	2790.3	864.24	1327.5	641.46	0.9069	1.07134	**3750**	3290.3	1013.09	2471.1	756.04	0.5621	1.11393
3260		867.19	1345.2	643.73	0.8977	1.07224	**3760**		1016.09	2500.0	758.35	0.5571	1.11473
3270		870.15	1363.1	646.00	0.8886	1.07315	**3770**		1019.09	2529.2	760.66	0.5522	1.11553
3280		873.11	1381.2	648.27	0.8797	1.07405	**3780**		1022.09	2558.7	762.98	0.5473	1.11633
3290		876.06	1399.5	650.54	0.8708	1.07495	**3790**		1025.09	2588.4	765.29	0.5424	1.11712
3300	2840.3	879.02	1418.0	652.81	0.8621	1.07585	**3800**	3340.3	1028.09	2618.4	767.60	0.5376	1.11791
3310		881.98	1436.6	655.09	0.8535	1.07675	**3810**		1031.09	2648.9	769.92	0.5328	1.11870
3320		884.94	1455.4	657.37	0.8450	1.07764	**3820**		1034.09	2679.5	772.23	0.5281	1.11948
3330		887.90	1474.5	659.64	0.8366	1.07853	**3830**		1037.10	2710.3	774.55	0.5235	1.12027
3340		890.86	1493.7	661.92	0.8283	1.07942	**3840**		1040.10	2741.5	776.87	0.5189	1.12105
3350	2890.3	893.83	1513.0	664.20	0.8202	1.08031	**3850**	3390.3	1043.11	2772.9	779.19	0.5143	1.12183
3360		896.80	1532.6	666.48	0.8121	1.08119	**3860**		1046.11	2804.6	781.51	0.5098	1.12261
3370		899.77	1552.5	668.76	0.8041	1.08207	**3870**		1049.12	2836.6	783.83	0.5054	1.12339
3380		902.73	1572.6	671.04	0.7962	1.08295	**3880**		1052.13	2869.0	786.16	0.5010	1.12416
3390		905.69	1592.8	673.32	0.7884	1.08383	**3890**		1055.13	2901.6	788.48	0.4966	1.12494
3400	2940.3	908.66	1613.2	675.60	0.7807	1.08470	**3900**	3440.3	1058.14	2934.4	790.80	0.4923	1.12571
3410		911.64	1633.9	677.89	0.7732	1.08558	**3910**		1061.15	2967.6	793.12	0.4881	1.12648

3420		914.61	1654.8	680.17	0.7657	1.08645	**3920**		1064.16	3001.1	795.44	0.4839	1.12725
3430		917.58	1675.9	682.46	0.7582	1.08732	**3930**		1067.17	3034.9	797.77	0.4797	1.12802
3440		920.55	1697.2	684.75	0.7508	1.08818	**3940**		1070.18	3069.0	800.10	0.4756	1.12879
3450	2990.3	923.52	1718.7	687.04	0.7436	1.08904	**3950**	3490.3	1073.19	3103.4	802.43	0.4715	1.12955
3460		926.50	1740.4	689.32	0.7365	1.08990	**3960**		1076.20	3138.1	804.75	0.4675	1.13031
3470		929.48	1762.3	691.61	0.7294	1.09076	**3970**		1079.22	3173.0	807.08	0.4635	1.13107
3480		932.45	1784.5	693.90	0.7224	1.09162	**3980**		1082.23	3208.3	809.41	0.4595	1.13183
3490		935.42	1806.8	696.19	0.7155	1.09247	**3990**		1085.24	3243.8	811.73	0.4556	1.13259
4000	3540.3	1088.26	3280	814.06	0.4518	1.13334	**4500**	4040.3	1239.86	5521	931.39	0.3019	1.16905
4010		1091.28	3316	816.39	0.4480	1.13410	**4510**		1242.91	5576	933.76	0.2996	1.16972
4020		1094.30	3352	818.72	0.4442	1.13485	**4520**		1245.96	5632	936.12	0.2973	1.17040
4030		1097.32	3389	821.06	0.4404	1.13560	**4530**		1249.00	5687	938.48	0.2951	1.17107
4040		1100.34	3427	823.39	0.4367	1.13635	**4540**		1252.05	5743	940.84	0.2928	1.17174
4050	3590.3	1103.36	3464	825.72	0.4331	1.13709	**4550**	4090.3	1255.10	5800	943.21	0.2906	1.17241
4060		1106.37	3502	828.05	0.4295	1.13783	**4560**		1258.16	5857	945.58	0.2884	1.17308
4070		1109.39	3540	830.39	0.4259	1.13857	**4570**		1261.21	5914	947.94	0.2862	1.17375
4080		1112.42	3579	832.73	0.4223	1.13932	**4580**		1264.26	5972	950.30	0.2841	1.17442
4090		1115.44	3617	835.06	0.4188	1.14006	**4590**		1267.31	6030	952.67	0.2820	1.17509
4100	3640.3	1118.46	3656	837.40	0.4154	1.14079	**4600**	4140.3	1270.36	6089	955.04	0.2799	1.17575
4110		1121.49	3696	839.74	0.4119	1.14153	**4610**		1273.42	6148	957.41	0.2778	1.17642
4120		1124.51	3736	842.08	0.4085	1.14227	**4620**		1276.47	6208	959.77	0.2757	1.17708
4130		1127.54	3776	844.41	0.4052	1.14300	**4630**		1279.52	6268	962.14	0.2736	1.17774
4140		1130.56	3817	846.75	0.4018	1.14373	**4640**		1282.58	6328	964.51	0.2716	1.17840
4150	3690.3	1133.59	3858	849.09	0.3985	1.14446	**4650**	4190.3	1285.63	6389	966.88	0.2696	1.17905
4160		1136.61	3899	851.44	0.3953	1.14519	**4660**		1288.69	6451	969.25	0.2676	1.17970
4170		1139.64	3940	853.78	0.3920	1.14592	**4670**		1291.75	6513	971.62	0.2656	1.18036
4180		1142.67	3982	856.12	0.3888	1.14665	**4680**		1294.80	6575	973.99	0.2637	1.18101
4190		1145.69	4024	858.46	0.3857	1.14737	**4690**		1297.86	6638	976.36	0.2617	1.18167
4200	3740.3	1148.72	4067	860.81	0.3826	1.14809	**4700**	4240.3	1300.92	6701	978.73	0.2598	1.18232
4210		1151.75	4110	863.15	0.3795	1.14881	**4710**		1303.98	6765	981.10	0.2579	1.18297
4220		1154.78	4153	865.50	0.3764	1.14953	**4720**		1307.03	6830	983.47	0.2560	1.18362
4230		1157.81	4197	867.84	0.3734	1.15025	**4730**		1310.09	6895	985.85	0.2541	1.18427
4240		1160.84	4241	870.18	0.3704	1.15097	**4740**		1313.15	6960	988.23	0.2523	1.18491
4250	3790.3	1163.87	4285	872.53	0.3674	1.15168	**4750**	4290.3	1316.21	7026	990.60	0.2505	1.18556
4260		1166.90	4330	874.88	0.3644	1.15239	**4760**		1319.27	7092	992.97	0.2486	1.18620
4270		1169.94	4375	877.23	0.3615	1.15310	**4770**		1322.33	7159	995.35	0.2468	1.18684
4280		1172.97	4421	879.58	0.3586	1.15381	**4780**		1325.39	7226	997.73	0.2451	1.18749

(*Continued*)

T	t	h	p_r	u	v_r	ϕ	T	t	h	p_r	u	v_r	ϕ
4290		1176.00	4467	881.93	0.3558	1.15452	**4790**		1328.45	7294	1000.10	0.2433	1.18813
4300	3840.3	1179.04	4513	884.28	0.3529	1.15522	**4800**	4340.3	1331.51	7362	1002.48	0.2415	1.18876
4310		1182.08	4560	886.63	0.3501	1.15593	**4810**		1334.57	7431	1004.86	0.2398	1.18940
4320		1185.11	4607	888.98	0.3474	1.15663	**4820**		1337.64	7500	1007.24	0.2381	1.19004
4330		1188.15	4654	891.33	0.3446	1.15734	**4830**		1340.70	7570	1009.61	0.2364	1.19068
4340		1191.19	4702	893.69	0.3419	1.15804	**4840**		1343.76	7640	1011.99	0.2347	1.19131
4350	3890.3	1194.23	4750	896.04	0.3392	1.15874	**4850**	4390.3	1346.83	7711	1014.37	0.2330	1.19194
4360		1197.26	4799	898.39	0.3366	1.15943	**4860**		1349.90	7782	1016.76	0.2313	1.19257
4370		1200.30	4848	900.75	0.3339	1.16012	**4870**		1352.97	7854	1019.14	0.2297	1.19320
4380		1203.34	4897	903.10	0.3313	1.16082	**4880**		1356.03	7926	1021.52	0.2281	1.19383
4390		1206.38	4947	905.45	0.3287	1.16151	**4890**		1359.10	7999	1023.90	0.2264	1.19445
4400	3940.3	1209.42	4997	907.81	0.3262	1.16221	**4900**	4440.3	1362.17	8073	1026.28	0.2248	1.19508
4410		1212.46	5048	910.17	0.3236	1.16290	**4910**		1365.24	8147	1028.66	0.2233	1.19571
4420		1215.50	5099	912.52	0.3211	1.16359	**4920**		1368.30	8221	1031.04	0.2217	1.19633
4430		1218.55	5150	914.88	0.3186	1.16427	**4930**		1371.37	8296	1033.43	0.2201	1.19696
4440		1221.59	5202	917.24	0.3162	1.16496	**4940**		1374.44	8372	1035.81	0.2186	1.19758
4450	3990.3	1224.64	5254	919.60	0.3137	1.16565	**4950**	4490.3	1377.51	8448	1038.20	0.2170	1.19820
4460		1227.68	5307	921.95	0.3113	1.16633	**4960**		1380.58	8525	1040.58	0.2155	1.19982
4470		1230.72	5360	924.31	0.3089	1.16701	**4970**		1383.65	8602	1042.97	0.2140	1.19944
4480		1233.77	5413	926.67	0.3066	1.16769	**4980**		1386.72	8680	1045.36	0.2125	1.20006
4490		1236.81	5467	929.03	0.3042	1.16837	**4990**		1389.79	8758	1047.74	0.2111	1.20067
5000	4540.3	1392.87	8837	1050.12	0.20959	1.20129	**5500**	5040.3	1547.07	13568	1170.04	0.15016	1.23068
5010		1395.94	8917	1052.51	0.20814	1.20190	**5510**		1550.17	13680	1172.45	0.14921	1.23124
5020		1399.01	8997	1054.90	0.20670	1.20252	**5520**		1553.26	13793	1174.87	0.14826	1.23180
5030		1402.08	9078	1057.29	0.20527	1.20313	**5530**		1556.36	13906	1177.28	0.14732	1.23236
5040		1405.16	9159	1059.68	0.20385	1.20374	**5540**		1559.45	14020	1179.69	0.14638	1.23292
5050	4590.3	1408.24	9241	1062.07	0.20245	1.20435	**5550**	5090.3	1562.55	14135	1182.10	0.14545	1.23348
5060		1411.32	9323	1064.45	0.20106	1.20496	**5560**		1565.65	14250	1184.52	0.14453	1.23404
5070		1414.39	9406	1066.84	0.19968	1.20557	**5570**		1568.74	14366	1186.93	0.14362	1.23459
5080		1417.46	9489	1069.23	0.19831	1.20617	**5580**		1571.84	14483	1189.34	0.14272	1.23515
5090		1420.54	9573	1071.62	0.19696	1.20678	**5590**		1574.93	14601	1191.75	0.14182	1.23570
5100	4640.3	1423.62	9658	1074.02	0.19561	1.20738	**5600**	5140.3	1578.03	14719	1194.16	0.14093	1.23626
5110		1426.70	9743	1076.41	0.19428	1.20799	**5610**		1581.13	14838	1196.58	0.14005	1.23681
5120		1429.77	9829	1078.80	0.19296	1.20859	**5620**		1584.23	14958	1198.99	0.13918	1.23736
5130		1432.85	9916	1081.19	0.19165	1.20919	**5630**		1587.33	15079	1201.40	0.13831	1.23791
5140		1435.94	10003	1083.59	0.19035	1.20979	**5640**		1590.43	15201	1203.82	0.13745	1.23847

5150	4690.3	1439.02	10091	1085.98	0.18906	1.21038	**5650**	5190.3	1593.53	15323	1206.24	0.13659	1.23902
5160		1442.09	10179	1088.37	0.18778	1.21097	**5660**		1596.63	15446	1208.65	0.13574	1.23956
5170		1445.17	10268	1090.77	0.18651	1.21157	**5670**		1599.74	15569	1211.07	0.13491	1.24010
5180		1448.26	10358	1093.17	0.18525	1.21217	**5680**		1602.84	15694	1213.48	0.13407	1.24065
5190		1451.33	10448	1095.56	0.18401	1.21276	**5690**		1605.94	15820	1215.89	0.13324	1.24120
5200	4740.3	1454.41	10539	1097.96	0.18279	1.21336	**5700**	5240.3	1609.04	15946	1218.31	0.13242	1.24174
5210		1457.50	10630	1100.36	0.18156	1.21395	**5710**		1612.15	16072	1220.73	0.13161	1.24229
5220		1460.58	10722	1102.76	0.1834	1.21454	**5720**		1615.25	16200	1223.15	0.13080	1.24283
5230		1463.66	10815	1105.15	0.17914	1.21513	**5730**		1618.35	16329	1225.57	0.12999	1.24337
5240		1466.75	10908	1107.55	0.17795	1.21572	**5740**		1621.46	16458	1227.99	0.12919	1.24391
5250	4790.3	1469.83	11002	1109.95	0.17677	1.21631	**5750**	5290.3	1624.57	16588	1230.41	0.12840	1.24445
5260		1472.92	11097	1112.35	0.17560	1.21689	**5760**		1627.67	16720	1232.82	0.12762	1.24498
5270		1476.01	11192	1114.75	0.17443	1.21747	**5770**		1630.77	16852	1235.24	0.12684	1.24552
5280		1479.09	11288	1117.15	0.17328	1.21806	**5780**		1633.88	16984	1237.67	0.12607	1.24606
5290		1482.17	11384	1119.55	0.17214	1.21864	**5790**		1636.98	17117	1240.08	0.12530	1.24660
5300	4840.3	1485.26	11481	1121.95	0.17101	1.21923	**5800**	5340.3	1640.09	17252	1242.50	0.12454	1.24714
5310		1488.35	11579	1124.35	0.16988	1.21981	**5810**		1643.20	17388	1244.93	0.12378	1.24767
5320		1491.43	11678	1126.75	0.16876	1.22039	**5820**		1646.30	17524	1247.35	0.12303	1.24821
5330		1494.52	11777	1129.15	0.16765	1.22097	**5830**		1649.41	17661	1249.77	0.12229	1.24874
5340		1497.61	11877	1131.56	0.16655	1.22155	**5840**		1652.52	17799	1252.19	0.12155	1.24927
5350	4890.3	1500.70	11978	1133.96	0.16547	1.22213	**5850**	5390.3	1655.63	17937	1254.62	0.12082	1.24981
5360		1503.79	12079	1136.36	0.16439	1.22270	**5860**		1658.73	18076	1257.04	0.12009	1.25034
5370		1506.88	12181	1138.77	0.16332	1.22327	**5870**		1661.84	18216	1259.46	0.11937	1.25087
5380		1509.97	12283	1141.17	0.16226	1.22385	**5880**		1664.95	18357	1261.88	0.11865	1.25140
5390		1513.05	12386	1143.57	0.16120	1.22442	**5890**		1668.06	18500	1264.30	0.11794	1.25193
5400	4940.3	1516.14	12490	1145.98	0.16015	1.22500	**5900**	5440.3	1671.17	18643	1266.73	0.11723	1.25246
5410		1519.24	12595	1148.38	0.15911	1.22557	**5910**		1674.28	18787	1269.15	0.11653	1.25298
5420		1522.33	12700	1150.78	0.15809	1.22614	**5920**		1677.39	18931	1271.58	0.11584	1.25351
5430		1525.42	12806	1153.19	0.15707	1.22671	**5930**		1680.50	19078	1274.00	0.11515	1.25403
5440		1528.51	12913	1155.60	0.15606	1.22728	**5940**		1683.61	19224	1276.43	0.11447	1.25456
5450	4990.3	1531.60	13021	1158.01	0.15506	1.22785	**5950**	5490.3	1686.73	19371	1278.86	0.11379	1.25508
5460		1534.70	13129	1160.41	0.15407	1.22841	**5960**		1689.84	19519	1281.29	0.11312	1.25560
5470		1537.79	13238	1162.82	0.15308	1.22898	**5970**		1692.96	19668	1283.72	0.11244	1.25613
5480		1540.88	13348	1165.23	0.15209	1.22954	**5980**		1696.07	19818	1286.14	0.11178	1.25665
5490		1543.98	13458	1167.63	0.15112	1.23011	**5990**		1699.18	19968	1288.57	0.11112	1.25717
6000	5540.3	1702.29	20120	1291.00	0.11047	1.25769	**6300**	5840.3	1795.88	25123	1364.02	0.09289	1.27291
6010		1705.41	20274	1293.43	0.10981	1.25821	**6310**		1799.01	25306	1366.46	0.09237	1.27341
6020		1708.52	20427	1295.86	0.10917	1.25872	**6320**		1802.13	25489	1368.90	0.09185	1.27390
6030		1711.64	20582	1298.29	0.10853	1.25924	**6330**		1805.26	25674	1371.35	0.09133	1.27440
6040		1714.76	20738	1300.72	0.10789	1.25976	**6340**		1808.39	25860	1373.79	0.09082	1.27489
6050	5590.3	1717.88	20894	1303.15	0.10726	1.26028	**6350**	5890.3	1811.51	26046	1376.23	0.09031	1.27538

(*Continued*)

T	t	h	p_r	u	v_r	ϕ	T	t	h	p_r	u	v_r	ϕ
6060		1720.99	21051	1305.58	0.10664	1.26079	**6360**		1814.63	26233	1378.66	0.08981	1.27587
6070		1724.10	21210	1308.01	0.10602	1.26130	**6370**		1817.76	26422	1381.10	0.08931	1.27636
6080		1727.22	21369	1310.44	0.10540	1.26182	**6380**		1820.89	26611	1383.54	0.08881	1.27685
6090		1730.33	21529	1312.87	0.10479	1.26233	**6390**		1824.01	26802	1385.98	0.08832	1.27734
6100	5640.3	1733.45	21691	1315.30	0.10418	1.26284	**6400**	5940.3	1827.14	26994	1388.43	0.08783	1.27783
6110		1736.57	21853	1317.73	0.10357	1.26335	**6410**		1830.27	27187	1390.88	0.08734	1.27832
6120		1739.69	22016	1320.16	0.10297	1.26386	**6420**		1833.40	27381	1393.32	0.08685	1.27881
6130		1742.81	22180	1322.60	0.10238	1.26437	**6430**		1836.53	27577	1395.76	0.08637	1.27929
6140		1745.93	22345	1325.04	0.10179	1.26488	**6440**		1839.66	27773	1398.21	0.08590	1.27978
6150	5690.3	1749.05	22512	1327.47	0.10120	1.26539	**6450**	5990.3	1842.79	27970	1400.65	0.08542	1.28026
6160		1752.17	22678	1329.90	0.10062	1.26589	**6460**		1845.92	28169	1403.09	0.08495	1.28074
6170		1755.29	22846	1332.34	0.10004	1.26639	**6470**		1849.05	28369	1405.53	0.08448	1.28123
6180		1758.41	23016	1334.77	0.09946	1.26690	**6480**		1852.18	28569	1407.98	0.08402	1.28171
6190		1761.53	23186	1337.20	0.09889	1.26741	**6490**		1855.31	28772	1410.42	0.08356	1.28219
6200	5740.3	1764.65	23357	1339.64	0.09833	1.26791	**6500**	6040.3	1858.44	28974	1412.87	0.08310	1.28268
6210		1767.77	23529	1342.08	0.09777	1.26841							
6220		1770.89	23703	1344.52	0.09721	1.26892							
6230		1774.02	23877	1346.95	0.09666	1.26942							
6240		1777.14	24052	1349.39	0.09611	1.26992							
6250	5790.3	1780.27	24228	1351.83	0.09556	1.27042							
6260		1783.39	24405	1354.27	0.09502	1.27092							
6270		1786.51	24583	1356.71	0.09448	1.27142							
6280		1789.63	24762	1359.14	0.09395	1.27192							
6290		1792.75	24942	1361.58	0.09342	1.27241							

Source: Condensed with permission from Table 1 of J. H. Keenan and J. Kaye, *Gas Tables*, copyright 1948, John Wiley & Sons, New York.

Appendix L

Specific Heats of Air at Low Pressures

SPECIFIC HEATS OF AIR AT LOW PRESSURES

This information is presented in English Engineering (EE) units.

T is in °R,	c_p is in Btu/lbm-°R.
t is in °F,	c_v is in Btu/lbm-°R.
a is in ft/sec,	$\gamma = c_p/c_v$.

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

T	t	c_p	c_v	γ	a	***T***	t	c_p	c_v	γ	a
100	−359.7	0.2392	0.1707	1.402	490.5	**1900**	1440.3	0.2750	0.2064	1.332	2084
150	−309.7	0.2392	0.1707	1.402	600.7	**2000**	1540.3	0.2773	0.2088	1.328	2135
200	−259.7	0.2392	0.1707	1.402	693.6	**2100**	1640.3	0.2794	0.2109	1.325	2185
250	−209.7	0.2392	0.1707	1.402	775.4	**2200**	1740.3	0.2813	0.2128	1.322	2234
300	−159.7	0.2392	0.1707	1.402	849.4	**2300**	1840.3	0.2831	0.2146	1.319	2282
350	−109.7	0.2393	0.1707	1.402	917.5	**2400**	1940.3	0.2848	0.2162	1.317	2329
400	−59.7	0.2393	0.1707	1.402	980.9	**2600**	2140.3	0.2878	0.2192	1.313	2420
450	−9.7	0.2394	0.1708	1.401	1040.3	**2800**	2340.3	0.2905	0.2219	1.309	2508
500	40.3	0.2396	0.1710	1.401	1096.4	**3000**	2540.3	0.2929	0.2243	1.306	2593
550	90.3	0.2399	0.1713	1.400	1149.6	**3200**	2740.3	0.2950	0.2264	1.303	2675
600	140.3	0.2403	0.1718	1.399	1200.3	**3400**	2940.3	0.2969	0.2283	1.300	2755
650	190.3	0.2409	0.1723	1.398	1248.7	**3600**	3140.3	0.2986	0.2300	1.298	2832
700	240.3	0.2416	0.1730	1.396	1295.1	**3800**	3340.3	0.3001	0.2316	1.296	2907
750	290.3	0.2424	0.1739	1.394	1339.6	**4000**	3540.3	0.3015	0.2329	1.294	2981
800	340.3	0.2434	0.1748	1.392	1382.5	**4200**	3740.3	0.3029	0.2343	1.292	3052
900	440.3	0.2458	0.1772	1.387	1463.6	**4400**	3940.3	0.3041	0.2355	1.291	3122
1000	540.3	0.2486	0.1800	1.381	1539.4	**4600**	4140.3	0.3052	0.2367	1.290	3191
1100	640.3	0.2516	0.1830	1.374	1610.8	**4800**	4340.3	0.3063	0.2377	1.288	3258
1200	740.3	0.2547	0.1862	1.368	1678.6	**5000**	4540.3	0.3072	0.2387	1.287	3323
1300	840.3	0.2579	0.1894	1.362	1743.2	**5200**	4740.3	0.3081	0.2396	1.286	3388
1400	940.3	0.2611	0.1926	1.356	1805.0	**5400**	4940.3	0.3090	0.2405	1.285	3451
1500	1040.3	0.2642	0.1956	1.350	1864.5	**5600**	5140.3	0.3098	0.2413	1.284	3513
1600	1140.3	0.2671	0.1985	1.345	1922.0	**5800**	5340.3	0.3106	0.2420	1.283	3574
1700	1240.3	0.2698	0.2013	1.340	1977.6	**6000**	5540.3	0.3114	0.2428	1.282	3634
1800	1340.3	0.2725	0.2039	1.336	2032	**6200**	5740.3	0.3121	0.2435	1.282	3693
						6400	5940.3	0.3128	0.2442	1.281	3751

Source: Adapted with permission from Table 2 of J. H. Keenan and J. Kaye, *Gas Tables*, copyright 1948, John Wiley & Sons, New York.

Selected References

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

Reference numbers referred to in the text correspond to those listed below:

Dimensional Analysis

1. Lemons, D. S., *A Student's Guide to Dimensional Analysis*, Cambridge University Press, Cambridge, UK, 2017.
2. Anderson, J. D., *Fundamentals of Aerodynamics*, 3rd ed., McGraw-Hill, New York, 2001.

Thermodynamics

3. Moran, M. J., and Shapiro, H. N., *Fundamentals of Engineering Thermodynamics*, John Wiley & Sons, New York, 1999.
4. Mooney, D. A., *Mechanical Engineering Thermodynamics*, Prentice Hall, Englewood Cliffs, NJ, 1953.
5. Reynolds, W. C., and Perkins, H. C., *Engineering Thermodynamics*, 2nd ed., McGraw-Hill, New York, 1977.
6. Obert, E. F., *Concepts of Thermodynamics*, McGraw-Hill, New York, 1960.
7. Sonntag, R. E., Borgnakke, C., and Van Wylen, C. J., *Fundamentals of Thermodynamics*, 5th ed., John Wiley & Sons, New York, 1997.
8. Dittman, R. H., and Zemansky, M. W., *Heat and Thermodynamics*, 7th ed., McGraw-Hill, New York, 1996.

Fluid Mechanics

9. Pao, R. H. F., *Fluid Mechanics*, John Wiley & Sons, New York, 1961.
10. Shames, I. H., *Mechanics of Fluids*, 3rd ed., McGraw-Hill, New York, 1992.
11. Streeter, V. L., and Wylie, E. B., *Fluid Mechanics*, 8th ed., McGraw-Hill, New York, 1985.
12. Street, R. L., Walters, G. Z., and Vennard, J. K., *Elementary Fluid Mechanics*, 7th ed., John Wiley & Sons, New York, 1995.

Gas Dynamics

13. Cambel, A. B., and Jennings, B. H., *Gas Dynamics*, McGraw-Hill, New York, 1958.
14. Anderson, J. D., *Modern Compressible Flow*, 3rd ed., McGraw-Hill, New York, 2003.
15. Hall, N. A., *Thermodynamics of Fluid Flow*, Prentice Hall, Englewood Cliffs, NJ, 1951.
16. John, J. E. A., and Keith, T. G., *Gas Dynamics*, 3rd ed., Prentice Hall, Englewood Cliffs, NJ, 2006.

17. Liepmann, H. W., and Roshko, A., *Elements of Gasdynamics*, John Wiley & Sons, New York, 1957.
18. Saad, M. A., *Compressible Fluid Flow*, 2nd ed., Prentice Hall, Englewood Cliffs, NJ, 1993.
19. Shapiro, A. H., *The Dynamics and Thermodynamics of Compressible Fluid Flow*, Vol. I, John Wiley & Sons, New York, 1953.
20. Zucrow, M. J., and Hoffman J. D., *Gas Dynamics*, Vol. I, John Wiley & Sons, New York, 1976.

Propulsion

21. Archer, R. D., and Saarlas, M., *An Introduction to Aerospace Propulsion*, Prentice Hall, Upper Saddle River, NJ, 1996.
22. (a) Mattingly, J. D., *Elements of Gas Turbine Propulsion*, Tata McGraw-Hill Edition, New Delhi, 2005; (b) Oates, G. C., *Aerothermodynamics of Gas Turbine and Rocket Propulsion*, 3rd ed., AIAA Education Series, Reston, VA, 1997.
23. Hill, P. G., and Peterson C. R., *Mechanics and Thermodynamics of Propulsion*, 2nd ed., Addison-Wesley, Reading, MA, 1992.
24. Sutton, G. P., and Biblarz, O., *Rocket Propulsion Elements*, John Wiley & Sons, Hoboken, NJ, 8th ed., 2010; and 9th ed., 2017.
25. Zucrow, M. J., *Aircraft and Missile Propulsion*, Vols. I and II, John Wiley & Sons, New York, 1958.

Real Gases

26. Pierce, F. J., *Microscopic Thermodynamics*, International Textbook Co., Scranton, PA, 1968.
27. Incropera, F. P., *Molecular Structure and Thermodynamics*, John Wiley & Sons, New York, 1974.
28. Thompson, P. A., *Compressible Fluid Dynamics*, McGraw-Hill, New York, 1972.
29. Anderson, J. D., *Hypersonic and High Temperature Gas Dynamics*, McGraw-Hill, New York, 1989 (presently available as an AIAA textbook).
30. Owczarek, J. A., *Fundamentals of Gas Dynamics*, International Textbook Co., Scranton, PA, 1964.

Tables and Charts

31. (a) Keenan, J. H., and Kaye, J., *Gas Tables*, John Wiley & Sons, New York, 1948. (b) Keenan, J. H., Chao, J., and Kaye, J., *Gas Tables International Version*, 2nd ed., John Wiley & Sons, New York, 1983.

32. Ames Research Staff, *Equations, Tables, and Charts for Compressible Flow*, NACA Report 1135, 1953.
33. Sims, J. L., *Tables for Supersonic Flow around Right Circular Cones at Zero-Angle-of-Attack*, NASA Report SP-3004, 1964.

Answers to Problems

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas

Most answers have been computed by interpolation from tabular entries and have been rounded off to three significant figures at the end (except for answers beginning with 1, where four significant figures have been retained). This procedure yields values consistent with standard engineering practice.

Chapter 1

1.1. Pretty close.

1.2. (a) Yes; (b) vertical lines.

1.3. (a) 2; (b) −52.0 Btu/lbm, −52.0 Btu/lbm.

1.4. 0, 0.24×10^6 N · m, 0, 0.24×10^6 N · m, 0.

1.5. (a) 393 ΔT J/kg; (b) no.

Chapter 2

2.2. (a) $U_m/2$; (b) $U_m/3$; (c) $2U_m/3$.

2.3. 13/2.

2.4. $(2.0)V^2/2g_c$ in laminar flow.

2.5. (a) 38.9 ft/sec.; (b) $1400/D^2$ ft/sec.

2.6. 44.4 ft/sec.

2.7. 19,010 hp.

2.8. 111.2 hp

2.9. (a) 1906 m/s; (b) 5.07 kg/s.

2.10. −0.0147 Btu/lbm.

2.11. (a) 78.1 m/s; (b) 4.18.

2.12. (a) 2880 ft/sec, (b) 1.15.

2.13. (a) 661 m/s; (b) 0.0625 bar abs.

2.14. (a) 382 Btu/sec; (b) 0.03%.

2.15. 4.34×10^5 J/kg.

Check Test:

2.3. $7\rho AB_mU_m/30$.

2.5. $\dot{m}_2\beta_2 + \dot{m}_3\beta_3 - \dot{m}_1\beta_1$.

Chapter 3

3.4. 246 ft/sec.

3.5. (a) −450 J/kg; (b) 0.11 K.

3.6. (a) 2260 ft/sec; (b) 732°F; (c) 103.1 psia.

3.7. Shaft work input.

3.9. (a) 7.51 ft-lbf/lbm; (b) 2.87 psig.

3.10. 54.4 m.

3.11. (a) 46.6 ft-lbf/lbm; (b) flow from 2 to 1.

3.12. 14.82 cm.

3.13. (b) 35 ft.

3.14. Case B.

3.16. (a) 7200A lbf; (b) 1.50 lbf/ft^2.

3.17. (a) 1.50 bar abs; (b) 7810 N; (c) −56,800 J/kg.

3.18. (a) 80 ft/sec, 6.37 psig; (b) 3600 lbf.

3.19. (a) 32.1 ft/sec; (b) 174.9 lbm/sec; (c) 151 lbf.

3.20. 5000 N.

3.21. 4.36 ft^2.

3.22. 180°.

3.23. (4/3)**V** in laminar flow.

Check Test:

3.4. 2.

3.5. (a) $q = w_s = 0$, yes; (b) no losses.

3.6. (a) s.

Chapter 4

4.1. 16,836 ft/sec (steel), 5100 ft/sec (water), 1128.5 (air at 70°F)

4.2. 278 K, 189 K, 33.3 K.

4.4. (a) 295 ft/sec; (b) 298 ft/sec; (c) 1291 ft/sec, 1492 ft/sec; (d) at low Mach numbers.

4.5. 0.564.

4.6. (a) 286 m/s, 0.700; (b) 2.8 kg/m^3.

4.7. 2.1, 402 psia.

4.8. 1266 m/s.

4.9. 524°R, 1779 psfa.

4.10. 1.28×10^5 N/m^2, 330 K, 491 m/s.

4.11. $M = \infty$.

4.12. Flows toward 50 psia, 0.0204 Btu/lbm-°R.

4.13. (a) 457 K, 448 m/s; (b) 9.65 bar abs.; (c) 0.370.

4.14. (a) 451°R, 20.95 psia; (b) 0.0254 Btu/lbm-°R; (c) 1571 lbf.

4.15. (a) 156.8 m/s; (b) 32.5 J/kg · K; (c) 0.763.

4.16. (a) 85.8 lbm/sec; (b) 1.91, 578°R, 2140 ft/sec, 0.0758 lbm/ft^3, 0.528 ft^2; (c) −6960 lbf.

Check Test:

4.2. (a) Into; (b) $M_2 < M_1$.

4.3. (a) True; (b) false; (c) false; (d) true; (e) true.

Chapter 5

5.1. (a) 0.18, 94.9 psia; (b) 2.94, 320°R.

5.2. 2.20, 1.64.

5.3. (a) 0.50, 35.6 psia, 788°R; (b) nozzle; (c) 0.67, 26.3 psia, 723°R.

5.4. 239 K.

5.5. (a) 0.607, 685 ft/sec, 23.1 psia; (b) 0.342, 395 ft/sec, 30.4 psia; (c) 0.855.

5.7. (a) 0.00797 Btu/lbm-°R; (b) 0.1502.

5.8. (a) 52.3 J/kg · K; (b) 16.43 cm.

5.10. (a) 26.5 lbm/sec; (b) no change; (c) 53.0 lbm/sec.

5.11. (a) 320 m/s; (b) 0.808 kg/s; (c) 0.844 kg/s.

5.12. 671°R, 0.768, 975 ft/sec.

5.13. (a) 77.9 psia; (b) 3.77 psia; (c) 0.0406 lbm/ft^3, 2050 ft/sec.

5.14. (a) 38.6 cm^2; (b) 9.14 kg/s.

5.15. 430 ft/sec.

5.16. (a) 140.4 lbm/sec; (b) 0.491 ft^2; (c) 0.787 ft^2.

5.17. (b) 3.53 cm^2; (c) 4.09 cm^2.

5.18. (a) 1.71; (b) 91.9%; (c) 0.01152 Btu/lbm-°R.

5.19. (a) 163.9 K, 1.10 bar abs., 8.61 bar abs.; (b) 2.10; (c) 0.1276 m^2; (d) 300 kg/s.

5.20. (a) 23.7 psia; (b) 97.4%; (c) 4.14.

5.23. (a) 3.5, 436 lbm/sec-ft^2; (b) $p_{rec} \le 6.63$ psia; (c) same.

Check Test:

5.3. $T_2^* > T_1^*$.

5.6. (a) 132.1 psia; (b) 0.514 lbm/ft^3, 1001 ft/sec; (c) 0.43.

Chapter 6

6.1. (b) 0.01421 Btu/lbm-°R; (c) 0.0646 Btu/lbm-°R, 0.1237 Btu/lbm-°R.

6.2. 84.0 psia.

6.3. (a) $[(\gamma - 1)/2\gamma]^{1/2}$; (b) $\rho_2/\rho_1 = (\gamma + 1)/(\gamma - 1)$.

6.4. 2.47, 3.35.

6.5. (a) 2.88; (b) 1.529.

6.6. 0.69, 2.45.

6.7. (a) 0.965, 0.417, 0.0585; (b) 144.8 psia, 62.6 psia, 8.78 psia; (c) 15.54 psia, 36.0 psia, 256 psia.

6.8. (a) 19.30 cm^2; (b) 10.52×10^5 N/m^2; (c) 18.65×10^5 N/m^2.

6.9. 1.30 ft^2.

6.10. (a) 0.119, 0.623; (b) 0.0287 Btu/lbm-°R.

6.11. 0.498.

6.12. (a) 4.6 in^2; (b) 5.35 in^2; (c) 79 psia; (d) 6.58 in^2; (e) 1.79.

6.13. (a) 3.56; (b) 0.475.

6.14. 0.67 or 1.405.

6.15. (a) 0.973, 0.375, 0.0471; (b) 0.43; (c) 2.64, 2.50.

6.16. (a) 0.271; (b) 0.0455 Btu/lbm-°R; (c) 2.48; (d) 0.281.

6.17. (a) $0.985p_1$, $0.296p_1$, $0.0298p_1$; (b) (i) no flow, (ii) subsonic throughout, (iii) shock in diverging portion, (iv) almost design.

6.19. (a) 54.6 in^2; (b) 18.39 lbm/sec; (c) 109.4 in^2; (d) 7.34 psia; (e) 9.24 psia; (f) 742 hp.

Check Test:

6.2. (a) Increases; (b) decreases; (c) decreases; (d) increases.

6.3. 0.973, 0.376, 0.0473.

6.5. (a) 1.625; (b) from 2 to 1.

6.6. (a) 0.380, 450 ft/sec; (b) 0.0282 Btu/lbm-°R.

Chapter 7

7.1. (a) 725°R, 42.0 psia, 922 ft/sec; (b) 0.00787 Btu/lbm-°R.

7.2. 1.024×10^6 K, 1.756×10^6 K, 20,500 bar, 135,000 bar.

7.3. 531°R, 19.75 psia, 348 ft/sec.

7.4. (a) 957 ft/sec; (b) 658°R, 34.5 psia.

7.5. (a) 310 K, 1.219×10^4 N/m^2, 50.3 m/s; (b) 328 K, 1.48×10^4 N/m^2, 340 m/s.

7.6. (a) 1453 ft/sec, 2520 ft/sec, 959 ft/sec, 2520 ft/sec; (b) 619°R, 18.05 psia; (c) 9.1°.

7.7. (a) 1.68, 25.6°; (b) 560 K, 6.10 bar; (c) weak.

7.8. (a) 52°, 77°; (b) 1013°R, 32.7 psia, 1198°R, 51.3 psia.

7.9. (a) 2.06; (b) all $M > 2.06$ cause attached shock.

7.10. (a) 1.8; (b) for $M > 1.57$.

7.11. (a) 1928 ft/sec; (b) 1045 ft/sec.

7.13. (a) 821°R, 2340 psfa, 0.0220 Btu/lbm-°R; (c) 826°R, 2470 psfa, 0.0200 Btu/lbm-°R.

7.14. (a) 2.27, 166.3 K, 5.6°; (b) 5.6°; (c) 2.01, 184.5 K, 1.43 bar.

7.15. (a) 1.453, 696°R, 24.8 psia; (b) oblique shock with $\delta = 10°$; (c) 1.031, 816°R, 42.7 psia; (d) 0.704, 906°R, 52.3 psia.

7.16. (a) 0.783, 58°; (b) 6.72, 0.837.

7.17. 1.032, 15.92, 2.61, 40°.

7.18. (a) 949 m/s; (b) 706 K; (c) 48°.

7.19. 2340 psfa, 0.0250 Btu/lbm-°R (on surface).

7.21. (a) 17,740; (b) $V_S = 1373$ m/s, $V_C = 1073$ m/s (contact), $V_R = 515$ m/s (reflected).

Check Test:

7.1. (a) $p_1 = p'_1$; (b) $T'_{t1} < T'_{t2}$; (c) none; (d) $u'_2 > u'_1$, $u'_2 = u_2$.

7.2. (a) Greater than; (b) (i) decreases, (ii) decreases.

7.6. 1667 ft/sec.

7.7. (a) 53.1°, 20°; (b) 625°R, 14.1 psia, 1.23.

Chapter 8

8.1. 2.60, 398°R, 936°R, 5.78 psia, 115 psia.

8.2. (a) 1.65, 3.04; (b) 34.2°, 52.3°.

8.3. (a) 174.5 K, 8.76×10^3 N/m^2.

8.4. 1.39.

8.5. 12.1°.

8.6. (a) 2.36, 1.986, 11.03; (b) 1.813, 2.51, 9.33; (d) no.

8.7. (a) 6.00 psia, 16.59 psia; (b) 12,020 lbf, 2120 lbf.

8.8. (c) 6.851 psia, 19.09 psia, 3.35 psia, 10.483 psia, L = 8.15×10^3 lbf/ft of span, D = 1.996×10^3 lbf/ft of span.

8.10. (a) 2.44, 392°R; (b) $\Delta\nu = 14.2°$.

8.11. (b) 241 K, 1.0 bar, 609 m/s.

8.12. (c) 1.86, 20°, 2.67, 40.5° from centerline.

8.13. (a) 15.05°; (b) 1.691, 4.14 p_{amb}; (c) expansion; (d) 2.61, p_{amb}, $0.865T_1$, 39.1° from original flow.

8.14. (a) 1.0 bar, 1.766, 6.55°, 1.4 bar, 1.536, 0°, 1.0 bar, 1.761, 6.6°.

8.15. (b) ∞; (c) 130.5°, 104.1°, 53.5°, 28.1°; (d) 3600 ft/sec.

8.16. (a) $\dfrac{L_2}{L_1} = \dfrac{1}{M_2}\left(\dfrac{\gamma+1}{2}\right)^{(\gamma+1)/2(1-\gamma)}\left(1+\dfrac{\gamma-1}{2}M_2^2\right)^{(\gamma+1)/2(\gamma-1)}$; (b) 1.343.

8.17. (a) 8.67°; (b) −10.03°; (c) no.

8.18. (a) 27.2°; (b) 1.95.

8.19. $p_b = 11.7$ kPa or about 15.24 km (50,000 ft) altitude.

Check Test:

8.4. 5.74°.

8.5. 845 lbf/ft^2.

Chapter 9

9.1. 2.22×10^5 N/m^2, 0.386.

9.2. 76.1 psia, 138.6 lbm/ft^2-sec.

9.3. (a) 21.7*D*; (b) 55.6%, 87.1%, 20.3%; (c) 0.0630 Btu/lbm-°R; (d) −0.59%, −5.9%, −5.4%, 0.00279 Btu/lbm-°R.

9.4. (a) 22.1 ft; (b) 528°R, 24.6 psia, 1072 ft/sec.

9.5. (a) 0.0313; (b) 2730 N/m^2.

9.6. (a) 551°R, 0.60; (b) from 2 to 1; (c) 0.423.

9.7. (a) 157.8 K, 2.98×10^4 N/m^2, 442 K, 10.95×10^5 N/m^2; (b) 0.0157.

9.8. (a) 556°R, 30.4 psia, 284 ft/sec; (b) 15.06 psia.

9.9. (a) 453°R, 8.79 psia; (b) 77.3 ft.

9.11. (a) 0.690, 0.877, 1128 ft/sec, 876°R, 38.0 psia; (b) 0.0205, 0.0012 ft.

9.12. (a) 324 K, 1.792 bar, 347 K, 2.27 bar; 121.8 K, 0.214 bar, 347 K, 8.33 bar; (b) 1959 hp, 4260 hp.

9.13. (a) 0.216; (b) 495°R, 10.65 psia; (c) 17.82 ft.

9.14. 229 K, 5.33×10^4 N/m^2.

9.15. (b) 0.513, 0.699; (c) 0.758.

9.16. (a) (i) 144.4 psia, (ii) 51.7 psia, (iii) 40.8 psia; (b) 15.2 psia.

9.17. (b) 0.0133; (c) 289.4 J/kg · K.

9.18. (b) $M = 0.50$; (c) 26.87 bar; (d) 0.407, 0.825.

9.19. (a) 26.0 psia; (b) 39.5 psia.

9.22. 24 psia with 2-in. tubing; choked with 1-in. tubing.

Check Test:

9.3. 43.5 psia.

9.4. 94.3 to 31.4 psia.

Chapter 10

10.1. (a) 1217°R, 1839°R; (b) 112.6 Btu/lbm added.

10.2. 1.792×10^5 J/kg removed.

10.3. 0.848, 2.83, 0.223.

10.4. (a) 3.37, 2.43×10^4 N/m^2, 126.3 K; (b) −890 J/kg · K.

10.5. (a) 767°R, 114.7 psia, 1112°R, 421 psia; (b) 68.1 Btu/lbm added.

10.8. (a) 6.39×10^5 J/kg; (b) 892 K, 0.567 atm.

10.9. (b) 2.00, 600°R, 59.8 psia; (c) 630°R, 21.0 psia, 756°R, 39.8 psia; (d) 38.7 Btu/lbm.

10.10. (a) 2180°R, 172.5 psia.

10.11. (a) 1.57×10^4 J/kg added; (b) 6.97×10^4 J/kg removed; (c) no.

10.13. 36.5 Btu/lbm removed.

10.14. (b) 0.686; (c) 1.628×10^5 J/kg.

10.15. (a) 47.4 psia; (b) 66.4 Btu/lbm added; (c) less than 1, 279 Btu/lbm for $M_2' = 0.3$.

10.17. (a) (i) True, (ii) false.

10.18. (a) $A_3 > A_4$; (b) $V_3 < V_4$, $A_3 > A_4$.

10.20. (a) $A_3 > A_2$.

Check Test:

10.4. (a) 746°R; (b) 53.1 Btu/lbm added.

Chapter 11

11.1. 128.8 Btu, 340 Btu, 469 Btu, 0.511 Btu/°R.

11.2. 36.3 Btu/lbm, 339 Btu/lbm, 0.352 Btu/lbm-°R.

11.3. 0.278 Btu/lbm-°R, 0.207 Btu/lbm-°R, 505 Btu/lbm, 367 Btu/lbm.

11.4. 1515°R, −273 Btu/lbm.

11.5. 0.1190 Btu/lbm-°R, 93.9 Btu/lbm.

11.6. 1.413.

11.7. (a) False; (b) true; (c) false; (d) false; (e) false.

11.8. 8.63 lbm/ft^3.

11.9. 0.0118 ft^3/lbm, 0.0342 ft^3/lbm by perfect gas law.

11.10. 0.0638 ft^3/lbm, 0.150 ft^3/lbm by perfect gas law.

11.11. 2.90 psia, area ratio = 11.53.

11.12. 3.06 psia, 650°R.

11.13. 3.15 psia, 656°R.

11.14. 0.02 MPa, 201 K, $M_3 = 4.39$, area ratio = 7.03.

11.15. 7.09×10^4 N/m^2, 1970 K.

Check Test:

11.3. 681 Btu/lbm, 610 Btu/lbm perfect gas.

11.4. False.

11.5. 1.018 lbm/ft^3, 0.875 lbm/ft^3 perfect gas.

11.6. at $M = 1.0$ (air) 240 psia and 2000°R, (argon) 221 psia and 1800°R (carbon dioxide) 249 psia and 2100°R.

Chapter 12

12.1. (a) 293 Btu/lbm, 129 Btu/lbm, 163.8 Btu/lbm, 322 Btu/lbm, 50.8%; (b) 21.6 lbm/sec.

12.2. (a) 269 Btu/lbm, 145 Btu/lbm, 124.4 Btu/lbm, 306 Btu/lbm, 40.6%; (b) 28.4 lbm/sec.

12.3. 37.4%, 38.5 kg/s.

12.4. (a) 24.9%; (c) 64.9%.

12.6. 4600 lbf.

12.7. 564 m/s.

12.8. 1419 ft/sec.

12.9. (a) 7820 lbf; (b) 57.1%; (c) 438 ft-lbf/lbm.

12.10. (a) 18.34 kg/s; (b) 0.257 m^2; (c) 3.12×10^5 W; (d) 28.6%; (e) 10.24×10^5 J/kg.

12.11. (a) 2880 lbf; (b) 20,800 hp.

12.12. 3290 lbf, 1.046 lbm of fuel/lbf-hr.

12.13. 6.34 ft^2, $M = 0.382$, 1309 psfa, 3400°R; 742 psfa, 2920°R, 3.96 ft^2; 6550 lbf, 1.41 lbm of fuel/lbf-hr.

12.14. 4240 lbf/ft^2, 2.20 lbm of fuel/lbf-hr.

12.15. (a) 83.3 lbm/sec; (b) 7730 ft/sec.

12.16. (a) 203 sec; (b) $(p_o - 872)$ N/m^2.

12.17. (a) 0.0402 ft^2; (b) 6060 ft/sec, 6490 ft/sec, 201 sec.

12.18. (a) 7.46, 1904 m/s; (b) 194.1 sec.

12.19. 0.924.

12.20. Need to know p_1, p_3, A_2, and γ.

12.21. (a) 0.725; (b) 0.747.

12.23. (b) $M_0 = 1.83$; (c) cannot be started.

12.24. 3.5 to 1.36.

Check Test:

12.3. 871 K, 1.184 bar.

12.5. (a) False; (b) false; (c) false; (d) false.

12.6. (a) 311 lbm/sec, 64,500 lbf; (b) 6670 ft/sec; (c) 207 sec, 5.28×10^5 hp.

12.7. $M_0 = 2.36$.

Index

Fundamentals of Gas Dynamics, Third Edition. Robert D. Zucker and Oscar Biblarz.

Companion website: www.wiley.com/go/zucker/gas