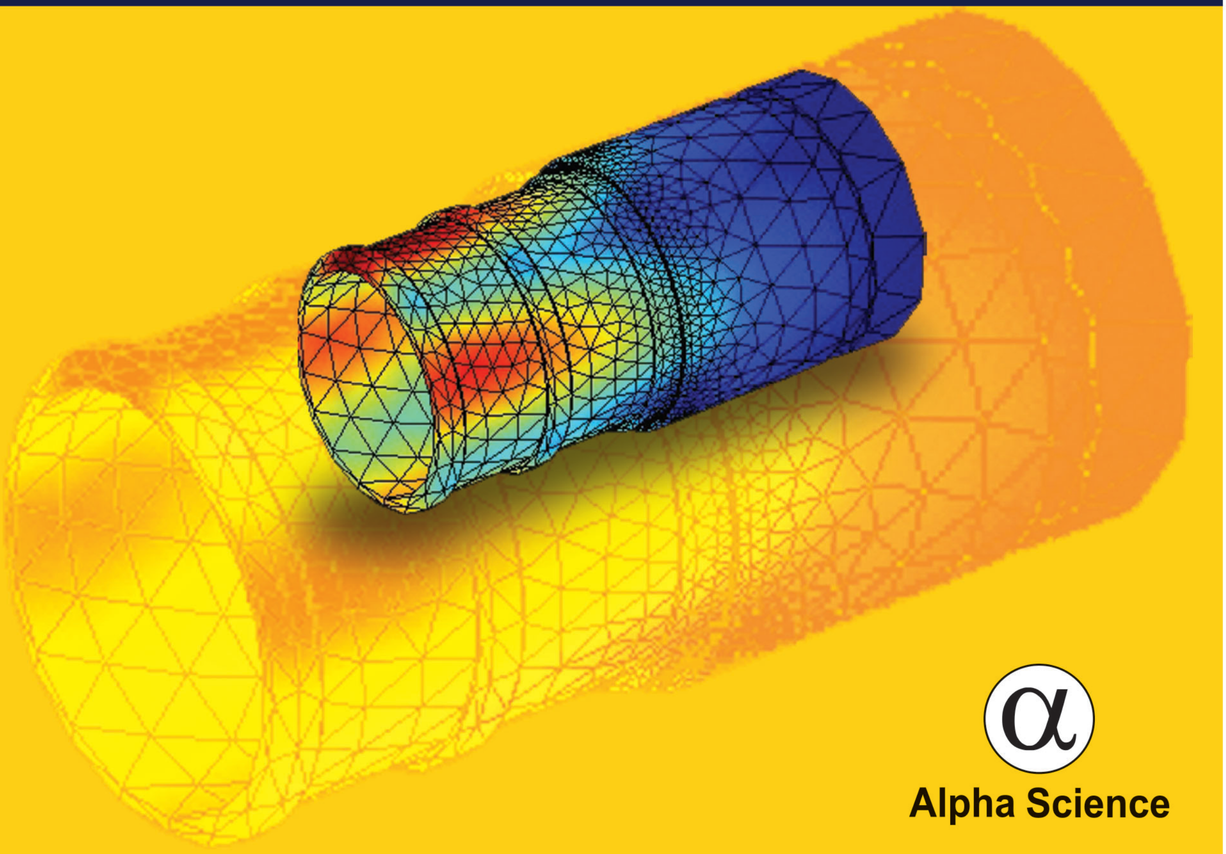
K.B.M. Nambudiripad

ADVANCED MECHANICS OF SOLIDS

A Gentle Introduction



Alpha Science

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ADVANCED MECHANICS OF SOLIDS
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Dedicated to the Memory of

My Parents and Teachers

Preface

This is a book on Advanced Mechanics of Solids, but presented at a more elementary level, and hence the qualifier *A Gentle Introduction* in spite of the obvious self-contradiction. This is written mainly for the undergraduate students of Civil, Mechanical and Aerospace branches of the relatively new APJ Abdul Kalam Technological University of Kerala, sometimes abbreviated as KTU. The look of helplessness writ large on the young innocent faces of the students is the main motivation for venturing to write this book.

This second course on the crucially important Mechanics of Solids has the reputation for being an extremely difficult, "impossible" subject. Large scale failures are commonplace. The students have a good reason to have such a hardened opinion. Engineering students nowadays are not of high academic calibre as in yesteryears. Far too many engineering colleges have mushroomed all over the place and, consequently, there is a severe shortage of experienced faculty members. Universities continue to believe that a final three-hour 'closed book examination' of the traditional kind is the only correct method of assessment in every course (subject).

I had the privilege of 'teaching' a batch of 35 young faculty members in two 'innings', first in late December 2016, and again in early January 2017. KTU had realised that the faculty members chosen to handle this course needed to be given intensive training. ICT Academy of Kerala had arranged these classes on behalf of KTU. This initiative taken by KTU is doubtless admirable; there is a real need to have such training sessions. The experience was an eye-opener for me. The young teacher-students had unanimously asked me to help them teach from 'the KTU examination point of view' and to give them suitable study material. Apparently they found the standard textbooks "too difficult to follow". I am not an admirer of 'teaching from an examination point of view', because in effect it is advising the students to resort to selective learning. That is to say, learning a set of worked out examples, avoiding all theory, and praying for deliverance. It seems to work, because this is what is happening!

But is this the right way? By doing so, students do not learn anything worthwhile. Worse still, they hate the subject and will never, never again, learn this subject. Writing yet another book will not solve all these problems. But perhaps the faculty members and the students may probably find this book helpful.

I feel that there must be greater emphasis on tutorial classes and home assignments. Challenging problems must be worked out by *all* students with partial help from teachers. They should be able to consult books. Why shouldn't they? It is not fair to expect them to commit to memory long equations with the unfriendly symbols of higher mathematics. The closed book examination, if there must be one, could only be to test their conceptual understanding of the key concepts. But I am digressing; this is not the place to articulate my views on examinations.

This book is, in a manner of speaking, a 'derivative' of two of my earlier books *Cartesian Tensors and the Equations of Solid Mechanics* and *Variational Methods in Engineering.* As

these two books are written at a slightly higher level, I have made what I consider to be appropriate changes to suit the intended audience. I do not know if my decision is in the best interest of the students. Perhaps my experienced learned colleagues will advise me. As the first book does not contain solution of stress analysis problems, new chapters are written to cover this important part.

The book opens with an introductory chapter that discusses the prerequisites expected of the students and the important fundamental concepts upon which the mechanics of solids is based. This is followed by a chapter on Special Problems in Bending. After a brief review of the theory of simple bending, various special cases like unsymmetrical bending and curved beams are discussed. The next chapter introduces the index notation as a preparatory material for the nature of stress at a point. The theory of the invariant, symmetric, stress tensor is the most important topic to be assimilated. Without a mastery of this topic, it is impossible to learn the mechanics of solids, the theory of elasticity, experimental stress analysis, and machine design. These topics are borrowed for the most part from my earlier book, though diluted to some extent. As this chapter is fairly long, some more of the important material is relegated to the next two chapters. The chapters on strains and constitutive equations that follow are kept at an elementary level so that the students can learn these without difficulty.

These are the preliminary chapters in one sense; only the governing equations are set up. Actual stress analysis problems are solved in the subsequent chapters. In an elementary book of this kind, only two-dimensional problems can be taken up; advanced problems are all left out. The chapter on energy methods is challenging more for the author than, perhaps, for the readers. The power and beauty of energy methods can be seen only when presented on a large canvas. Unfortunately, such a treatment will be abstract and mathematically demanding. Many sacrifices are made in a spirit of compromise. Some useful matter is presented at an elementary level. Torsion of non-circular prismatic bars is discussed in some detail. I hope that the students will find this chapter useful, interesting, and readable in spite of the mathematics used.

The level of the book seems to be increasing slowly but steadily as it progresses. This is, I believe, as it should be. After all, there should be a clear difference in the academic level of the students before and after taking this course.

I have to be deferential to the wishes of my teacher-students. I have, therefore, included a large chapter containing about 60 worked out examples. I hope this chapter will please them as well as the young students. In some places they supplement the material given in the theory part. A few of the problems are not worked out fully. Not all are numerical problems, and not all are of the same level of difficulty. In this second course, the emphasis is not, or ought not to be, on numerical problems.

The place of mathematics in the engineering curriculum is special. It serves two purposes, both vitally important. One is as a tool. We need to acquire the necessary tools such as, say, the method to find the maximum / minimum. The other, which is even more important, is that learning mathematics is a mental tonic. It sharpens our brain power; it helps us in the study of other subjects also.

Engineering science courses like this Advanced Mechanics of Solids are not easy; they should not be. If they are easy, it simply means that they are not intellectually or academically challenging. We need to read *good* books, spending time and asking questions. Questioning is not attacking; it is the expression of an open mind willing to explore other possibilities. All this can be accomplished *only if* we enjoy learning. There is a thrill, excitement and joy in learning. When we read a great book written by a master, it is effectively spending time in his company. Imagine spending some time every day with somebody like Albert Einstein or Subramaniam Chandrasekhar!

Imitating the saying *Child is the Father of Man*, let me state that *Student is the Teacher*. Yes, my students are / were my best teachers. I am grateful to them. But it is to my teachers that I owe the most. My early formative years at IIT, Kharagpur were beautiful; I enjoyed my stay and study there. My teachers at Kharagpur and other places have shown me that learning is a pleasure.

It was Sri K.J. Veera Raghavan of ICT Academy of Kerala who requested me to 'teach' or lead the training sessions arranged for the student-teachers. I thank him sincerely for giving me this honour as the resource person. My friends and colleagues have been helpful: "all, always, in all ways". My colleagues now at Vidya have been uniformly kind, encouraging, and highly supportive. Among them special mention must be made about Professors V.N. Krishnachandran, Sudha Balagopalan, K.V. Leela, Sooraj K. Prabha and N.K. Sudev. The seniors of the Vidya family, Ers P.K. Asokan (former Chairman), G. Mohanachandran (Executive Director), Suresh Lal (Finance Director), Dr B. Anil (Academic Director) and others have always been kind to me. Their very presence and kind words were a constant inspiration. The real guiding light was, and still is, Dr Gangan Prathap. Sri Sreerag Srinivasan drew almost all the figures promptly and cheerfully. I thank them all sincerely. Finally, I thank the publishers, M/s Narosa Publishing House, New Delhi and in particular, Shri N.K. Mehra, Publisher and Managing Director.

It is unrealistic to expect that there are no mistakes in this book. I shall sincerely appreciate if the mistakes, if any, are pointed out to me. Comments, criticisms, and suggestions are always welcome. They will certainly be gratefully acknowledged. If some of the young students and faculty members find this book useful, I shall feel amply rewarded for the labour of writing this book.

Calicut

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Contents

Chapter 1

INTRODUCTION

Let me offer with humility salutations[1] to all my teachers, most of all to the One Real Innermost Guru, the Guru of all gurus, the real source of everything beautiful and sublime.

WHAT THIS BOOK IS ABOUT

This book is a gentle introduction — an elementary treatment in spite of the obvious contradiction — to Advanced Mechanics of Solids[2]. Mechanics of Solids is arguably the most important course (subject) for Civil, Mechanical and Aerospace engineers. This is really a modern version of the traditional Strength of Materials. Our subject is of crucial importance in Design, Stress Analysis (both experimental and theoretical), and Finite Element Method.

WHY SHOULD WE LEARN ADVANCED MECHANICS OF SOLIDS?

Engineering students, particularly of the three afore-mentioned branches, invariably take a first course in the mechanics of solids. This is usually at an elementary level, and is not sufficient at all for a proper understanding. Hence a follow-up course is essential. The subject matter of this book aims at meeting two objectives: (i) a more mature understanding of the elementary concepts, and (ii) at least a peep into advanced topics. There are some technically important problems that cannot be solved by using the theory studied in the traditional elementary course. Special problems in bending, stresses (and deformations) in a rotating disc, stress concentration, contact stresses, bending stresses in shells, torsion of non-circular cross-sections, and assessment of possible failure of a crack (fracture mechanics) are some of these non-elementary problems that fall outside the scope of a first course[3].

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[1] It is an Indian tradition to remember with pleasure and gratitude one's teachers and to offer one's salutations. Let me too follow this great tradition.

[2] An elementary treatment of Advanced Mechanics of Solids?

[3] It is not necessary, nor is it possible for a practising engineer, to be able to obtain solutions for these difficult problems. It is quite sufficient if he has a *sound knowledge and understanding* of the fundamentals, and to be aware of the limitations. If these conditions are met, he will know when he has to consult experts.

We know that these are extremely important technical problems. No engineer can afford to be totally ignorant of these topics. In addition, there are some fundamental issues to be settled: are the stresses in a body independent of the material, can there be two different solutions for the same stress analysis problem, how do we deal with nonlinearity, and how can we calculate long term creep effects? There are a large number of questions that cannot be answered within the framework of an elementary first course in mechanics of solids. Even more important is the need to have greater conceptual clarity. Hence it is essential to have a second course in the mechanics of solids.

The trouble, however, is that these problems are difficult to solve. They demand a higher level of maturity and understanding for their solution. The level of mathematics needed is quite high.

WHAT ARE THE COMPLICATIONS?

Advanced Mechanics of Solids demands a higher or better level of conceptual clarity. Some of the fundamental notions like stress at a point are abstract; it takes time to absorb and digest some concepts like *invariance, stress tensor, transformations of coordinates and the induced transformations*. These are technical words pregnant with meaning. To discuss these matters it is desirable (if not essential) to use the index notation (sometimes called the tensor notation) and concepts from tensors analysis (at least cartesian tensors). A higher level of mathematics is also needed.

In addition, before we solve a stress analysis problem — stress analysis is said to be the 'centre of gravity' of the mechanics of deformable bodies — it is necessary to formulate it. Often it turns out to be a boundary value problem which is almost the same as a problem in partial differential equations. The techniques of solution are based on higher mathematics. It is necessary, when formulating a problem, to change from volume integrals to surface integrals, and surface integrals to line integrals, and vice versa. To be able to do so, we need to use Gauss' and Stokes' theorems. Although all students nowadays learn these integral theorems, they are still not comfortable with them. Part of the reason is that these theorems are generally taught by mathematicians without the support of the physics of the problem. The physical significance is practically never brought out and discussed. Maximisation / minimisation under constraints using Lagrange multipliers is another technique that is often used. Students seem to have a hardened opinion that mathematics is really useless for engineers, and that it is only of nuisance value. This seriously mistaken notion is to be corrected. One cannot get very far in engineering without the support of mathematics.

Such an attitudinal change — with a healthy respect and love for mathematics — is necessary before one undertakes the study of serious mathematical-analytical courses (subjects) like advanced mechanics of solids. We need to learn several techniques in mathematics; these tools are always very useful. Furthermore, learning mathematics enhances our ability to learn other subjects too. Learning is a joy. Enjoy learning!

This situation is not unlike the condition in the primary health centres in rural areas. It is quite sufficient if the non-specialist general physicians there can identify complications and advise the patients to consult specialist doctors.

PREREQUISITES

This being an 'advanced' course, students need to be proficient in the first course on the mechanics of solids. A superstructure cannot be built on unsound foundations. It is quite possible that because of various reasons, some students may have serious gaps and deficiencies in the first course. It is essential to remedy these weaknesses, and to have a fairly good mastery of the subject. There are several excellent books for a first course. Den Hartog [3],[4], Popov [11], Timoshenko [15], [16] and several other books by him, and Crandall & Dahl (and others) [2] are excellent[4].

Students are advised to revisit all the topics of their undergraduate syllabus, and be prepared for an advanced course. Here below are a few comments about some of the topics.

(i) **Bending moment, shearing force, axial thrust and twisting moment diagrams:** Bending moment, shearing force, axial thrust, and twisting moments and their variations along the length of a (one-dimensional) bar are traditionally taught as part of Strength of Materials or Mechanics of Solids. Actually these are part of statics (unless the problem is statically indeterminate).

The beams considered are almost always horizontal. Students tend to classify structural elements as beams and columns depending on whether they are horizontal or vertical! It is dangerous to give a large number of similar problems as exercises, because in course of time students tend to work out the problems mechanically (with the brain in the switched off mode)! It is necessary to expose the students to different looking structures. Airplane structures provide a wide variety of such elements.

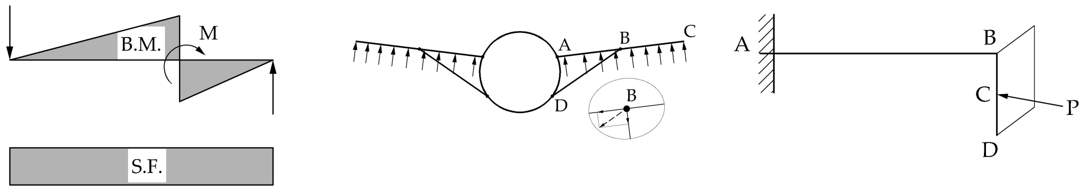
A simply supported beam with a concentrated moment, a log of wood floating on water, an L-shaped member, a balcony beam (horizontal in plan), a semi-circular beam, etc. are examples where students seem to have difficulty. When problems are assigned, there must be variety, so that every time the students are required to think and proceed with the calculations. Some examples are shown in the figures below.

Note that the shear force diagram will have a jump discontinuity at the point where a concentrated force is applied, the magnitude of the jump being exactly equal to the magnitude of the force applied. A similar remark is applicable for the other force resultants like an axial force, a bending moment, and a twisting moment also. The case of an L-shaped member should receive special attention.

(ii) **Love-Kirchhoff assumption in the theory of simple bending:** This and its consequences are important. These are discussed later in the book.

(iii) **Deflection of beams:** It is very convenient to use six simple formulae to obtain the deflection of beams in many, if not most, cases[5]. We may recall the expressions for the slope and the deflection of a simple cantilever given below.

[4] My own personal likes and prejudices may have influenced this choice of mine. There is no doubt at all that these are all highly respected authors with their affiliations to MIT, U. of C. (Berkeley), Stanford,



(a) A beam with a moment (b) A wing of an aircraft (c) An inclined load on an L-bar

Figure 1.1: Some cases to be considered. The load P on the L-bar is not perpendicular to the member BD. It is not vertical; it is not in the vertical plane either.

End slope	$\dfrac{Ml}{\mathbf{1}EI}$	$\dfrac{Pl^2}{\mathbf{2}EI}$	$\dfrac{wl^3}{\mathbf{6}EI}$
End deflection	$\dfrac{Ml^2}{\mathbf{2}EI}$	$\dfrac{Pl^3}{\mathbf{3}EI}$	$\dfrac{wl^4}{\mathbf{8}EI}$

It is easy enough to remember this: 1 2 2 3 6 8, simpler than a telephone number! Everything else can be written "in each case by dimensional reasoning". With a little practice, the final formula can be written down effortlessly as if the formula is reproduced from memory!

We shall demonstrate the procedure by just one example[6]. A cantilever loaded by two concentrated forces, P_1 at C and P_2 at B is shown in Fig. 1.2. We desire to calculate the end deflection. First we shall solve this by superposition. The total deflection at B = the deflection at B due to the load P_1 alone plus the deflection at the same point B due to the load P_2 alone. This gives us

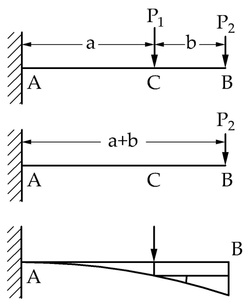
$$\delta_B = \frac{P_1\, a^3}{3\, EI} + \left(\frac{P_1\, a^2}{2\, EI}\right) b + \frac{P_2\, (a+b)^3}{3\, EI}$$

We may also arrive at the answer differently [Fig. 1.2b]. We note that (i) the shear force immediately to the left of C is $P_1 + P_2$, (ii) the bending moment at C is $P_2\, b$, and

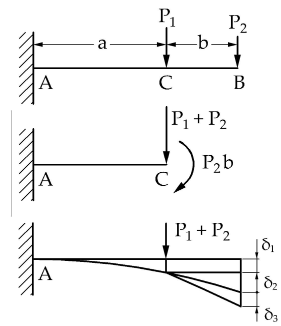
and MIT, respectively.

[5] Den Hartog [3] calls this the Myosotis method. He gives this testimony: "After the alphabet and the tables of multiplication, nothing has proved quite so useful in my professional life as these six expressions, and undoubtedly some of my students will have the same experience." See [3] for more details.

[6] This example is borrowed from Den Hartog [3].



(a) Deflection at B: superposition



(b) Deflection at B: alternative method

Figure 1.2: The deflection at the end B of a cantilever is calculated in two different ways, first by superposition [Fig. 1.2a], and later differently [Fig. 1.2b].

that (iii) the right part CB of the cantilever can be considered as fixed at an angle downward. Thus, the total deflection at B can also be calculated as:

$$\text{deflection at B} = \left[\frac{(P_1+P_2)\,a^3}{3\,EI} + \frac{(P_2\,b)\,a^2}{2\,EI}\right] + \left[\frac{(P_1+P_2)\,a^2}{2\,EI} + \frac{(P_2\,b)\,a}{EI}\right] b + \left[\frac{P_2\,b^2}{3\,EI}\right].$$

(iv) **Coulomb's theory of torsion:**
This theory is developed with the help of two assumptions: (i) cross-section do not warp when the twisting moment is applied; and (ii) straight radial lines on the cross-sections remain straight. That is, the cross-sections do not distort in their own planes[7]. Actually these are not assumptions, but facts. These can be proved by an argument of rotational symmetry. See Den Hartog [3].

INTRODUCTION

This is a revisit, and not a first time introduction to our subject of Mechanics of Solids. Still a fairly detailed introduction is given here because most students do not seem to have a solid foundation.

Opening Remarks

Our subject took shape in the first half of the nineteenth century under the influence of a strong group of French scientists, largely mathematicians, such as Poisson, Lamé and Navier[8]. Their influence is still sufficiently strong and compelling[9] that the original name

[7] Straight radial lines, on application of the twisting moment, do not curve in or curve out. Using the terminology in cricket, the radial lines are neither in-swingers nor out-swingers!

[8] Gabriel Lamé (July 1795 - May 1870) was an outstanding French mathematician and engineer. Siméon Denis Poisson (June 1781 - April 1840) was a French mathematician and physicist. Claude-Louis Navier (February 1784 - August 1836) was a French engineer, mathematician and physicist. They were all eminent as engineer-mathematician-scientists. It seems natural that they all studied in the prestigious École Polytechnique, and that their names are among the 72 great names of engineers, mathematicians and physicists written in gold letters on the Eiffel tower. See S.P. Timoshenko [14].

[9] Readers are advised to read the beautifully written book [3] by the well known author, J.P. Den Hartog.

résistance de matériaux[10] continues to be used by engineers and academicians. In recent times, however, the approach has changed, dictated mainly by the technological needs of aerospace applications. The new name that the subject has now taken on, reflecting a more modern approach, is *Solid Mechanics. Mechanics of Materials* is another popular name that is, perhaps, a compromise name accommodating a little bit of both approaches and points of view.

The traditional name of strength of materials is misleading. It does not deal with the 'strength'[11] of materials. It is the allied subject of *Mechanical Testing of Materials* that deals with this aspect. It supplements our subject by giving experimental support to it principally by providing the numerical values of the various parameters that describe the mechanical properties of the various materials used. *Theory of Elasticity* is another related subject. The approach in this is more mathematical. Nevertheless, in the practical applications of all these subjects, these are more or less the same. In every case, the problem is to find out the internal state of a body (stresses, strains and displacements, defined later) when the geometry (shape) and the composition (material or materials) of the body and the applied loads are specified. There are other problems also that are addressed by these subjects. Strength of materials is a simplified engineers' theory which is empirical in some places. Theory of elasticity discusses the same problems, but in a more mathematically thorough style, with frequent excursions into mathematical proofs and (pointless?) arguments[12]. Solid mechanics is the 'in-between' subject, concentrating primarily on the practical applications, yet partially accommodating what elasticians have to say. Thus, to gain solidity and maturity in this subject, further follow-up courses on *Structural Analysis*, *Advanced Solid Mechanics*, *Theory of Elasticity*, *Theory of Plates & Shells*, *Continuum Mechanics*, *Experimental Stress Analysis* and *Computational Mechanics*[13] are recommended. It would also be nice to be bilingual, that is, to be comfortable in the different languages used by engineers and mathematicians. To learn advanced theory, it is necessary to learn *Tensor Calculus*, *Differential Geometry*, *Abstract Algebra* and *Functional Analysis.*

Our subject is of great relevance to Mechanical, Civil and Aerospace Engineering. Next to *Mechanics (Statics and Dynamics of Rigid Bodies)*, this is arguably the engineering science subject most relevant to these branches of engineering. Investment of time and

[10] "The first mathematician to consider the nature of the resistance of solids to rupture", writes A.E.H. Love, a towering figure in the theory of elasticity, "was Galileo". The first edition of this celebrated book was published in 1892 - 1893. He goes on to write: "In the history of the theory started by the question of Galileo, undoubtedly the two great landmarks are the discovery of Hooke's law in 1660, and the formulation of the general equations by Navier in 1821". Ambitious students are advised to read the history of this subject. Love's book gives a 31- page historical introduction.

[11] 'Strength' as used here is somewhat ambiguous. It is not only the material, but perhaps even more importantly, the proper shape and design that contribute to 'strength'. Surprisingly, there are cases where the 'strength' of a machine part is actually increased if some material is removed! In any case, a deeper probe into these issues is necessary.

[12] Marsden, J.E. & Hughes, T.J.R.: *Mathematical Foundations of Elasticity*, Courier Corporation, (1994) state that, among "the three things that every beginner in elasticity theory should know", "the first is that Kirchhoff has two h's in it", $\cdots$, and "the third is that researchers in elasticity theory are very opinionated, even when they are wrong."

[13] *Fracture Mechanics*, *Composites*, *Theory of Plasticity*, etc. are also useful.

energy in this crucial subject in the form of clarity of concepts and understanding is sure to pay rich dividends in future. This subject is part of *mechanics*; in fact, it can be appropriately called the *Statics of Deformable Elastic Bodies*[14].

Mechanics and Its Various Branches

Mechanics is one branch that received attention even in ancient times. By the time of Archimedes[15], it had developed considerably. *Mechanics* is divided into *Mechanics of Fluids* and *Mechanics of Solids.* While fluid mechanics is important, we are concerned with only solid mechanics now. This, in turn, is sub-divided into *Mechanics of Rigid Bodies* and *Mechanics of Deformable Bodies*, which are both important in engineering. We shall first take up mechanics of rigid bodies before considering the latter.

Rigid bodies:

A body is called *rigid* if it does not undergo any deformation (change in shape or volume), no matter how large the applied force is. This means that the distance between *any* two points of the body remains unchanged, irrespective of how large the applied load is. A rigid body is a mathematical abstraction or an idealisation[16]. Perfectly rigid bodies do not exist; all real bodies undergo some deformation when a load is applied. Sometimes the deformation may be so small, or it may be irrelevant, that we are justified in neglecting it, and treating the body as rigid.

Particle:

Often the body can be considered as a *particle*. In this idealisation, the entire mass of the body (which is really distributed throughout the volume of the body) is regarded as a point mass located at the centre of mass. The body (which actually has finite dimensions) is treated as if it were just a particle concentrated at a single point (occupying no volume).

Deformable body:

There are situations where the deformations of a body are of prime concern to us in our analysis. In such cases, we do not get any result if the body is regarded as a particle or as a rigid body. The body then is to be treated as deformable. The question is not whether a body is truly deformable; all real bodies are deformable to varying degrees.

Which among them is correct?

A particle, a rigid body and a deformable body[17] are all various simplifications, or idealisations, to solve a problem. The scope of the problem and the method of solution decide which among these is appropriate. For example, let us say we are analysing the path of a satellite. If we are interested only in the path of the centre of mass, how long it will

[14] If the need arises, the scope can be extended to encompass *Dynamics* also. Furthermore, plastic, viscoelastic, viscoplastic, thermoelastic, and indeed a host of other special cases can also be discussed using similar methods. In such an event, one would need more advanced concepts and methods which cannot be adequately covered in a first introductory course.

[15] Archimedes of Syracuse (287 BC - 212 BC) was a renowned Greek mathematician, astronomer, physicist, engineer and inventor, all rolled into one. He was born and assassinated in Syracuse, Italy.

[16] Ideal in the sense of being hypothetical, and not in the sense of being the most desirable.

[17] There is no simplification, or idealisation, in treating a body as deformable; this is really the case. What is intended is that these are all models meant to solve a given problem for the limited purpose of obtaining the desired answer in a given context.

take to reach a certain altitude, etc., it is sufficient to treat it as a particle. The governing equations and their solution will now be relatively simple. The price that we pay is that this idealisation, or this model, gives us only partial information. We would know all about the motion of the centre of mass, but we would have no information at all of the motion of the body about (that is, the motion of other points relative to) the centre of mass. We would not be able to know anything about the orientation and the rotation of the body.

If, on the other hand, we need additional information on the motion of the body about the centre of mass, the same body, the very same body, has now to be treated as a rigid body (and not as a particle any more). If, additionally, we wish to predict the stresses inside, and estimate its safety and structural integrity, the same body has to be treated (modelled) as a deformable body. A rigid body model can give no information about the internal states of stress in exactly the same way as a particle model is unable to provide information on the motion of the body about the centre of mass.

In some problems, there is a strong interaction, or coupling, between the forces acting on a body and its shape. An example is a falling rain drop. The aerodynamic forces on the rain drop are strongly influenced by its shape; the shape, in turn, of such a highly deformable body is decisively influenced by the forces. A far better and technically important example is the flutter of aircraft wings. The aerodynamic force causes bending of the wings; this bending is accompanied by twisting of the wings. The twist changes the angle of attack of the fluid stream, thereby changing the flow pattern and, thus, the aerodynamic force. Problems with such strong cross-effects (coupling) are difficult to solve. In many problems, however, the coupling is not very strong. It then suffices to regard the body as *rigid* (in the first phase of the solution of the problem) for the determination of the forces. Thereafter (in the second phase of the solution of the problem), the same body is now considered as *deformable* to determine the stresses, strains and displacements inside. By way of abundant caution, such a two-step procedure can be iterated. The enormous computational capacity and speed of modern computers make it easy to do such iterations[18].

In general, we should choose the simplest model that gives the information that we seek, because we would like to keep the analysis as simple as possible without losing the heart of the problem.

Scope of Our Subject

Engineers are often concerned with the design, construction or manufacture, and assembly of engineering structures. The design of the various components of the structure entails estimation of the loads (often by a separate elaborate calculation that itself can be complex and challenging as to be the subject matter of one or more separate courses), choosing a

[18] Such a procedure can be tried for problems where there is strong coupling also. But then such a procedure may or may not converge. Additional investigation is necessary before one can be sure that such a scheme will converge, that it will converge to the correct answer, and that it will converge to the correct answer irrespective of the starting point. Engineers and physicists try out these procedures and see if they are successful, even before convergence is assured. This then becomes a research problem for mathematicians to investigate. The motivation for several problems in mathematics, we can see from the long history, comes from the rich and varied problems in physics and engineering!

proper material, and arriving at the proper dimensions[19]. This process of 'dimensioning' involves calculation of the stresses, strains and displacements at the interior points of the component bodies[20]. It is here that our subject is relevant and important. Thus, a large class of problems that fall within the scope of strength of materials[21] (or mechanics of materials, or solid mechanics) and the theory of elasticity, and which are of great importance to engineers is the following.

Given the shape (geometry) and composition (material or materials) of a body in static equilibrium, and the loading and the support conditions, to determine the stresses, strains and displacements at any (i.e., every) point inside the body.

This problem as stated above is so general and complex, that even the best mathematicians have not been able to solve it so far. Some of the best brains have thought and laboured on this and similar problems for a long, long time. Yet, the solution of this problem is still elusive. The demands from practising design engineers, in the meantime, made it imperative that at least partial solutions of at least simplified cases should be found. Thus, while we wait patiently (perhaps for another 100 years?) for the mathematicians to solve the problem in all its generality, it makes sense to make drastic simplifications for expediency. Thus it is, that our simplified engineers' theory of strength of materials emerged, playing a crucial role in the design calculations of engineering structures.

Relatively recently, more refined solutions have become necessary, particularly in aerospace applications. This is because the simplified theory is unable to give even an approximate answer in more and more cases. This situation has forced engineers to have a closer second look at the theory, and to revise it making it more and more rigorous. The engineering science content has, thereby, increased, with a corresponding decrease in empiricism. This new development places greater emphasis on the underlying assumptions, and has thus led to the new *avatar* in the form of solid mechanics.

What this means is that the emphasis has shifted to the fundamentals, so that the necessary changes can be made, and a revised theory built to deal with new situations. Thus, if new situations arise where one is called upon to use special materials, perhaps

[19] The loads may be surface and body forces. These act on the surface(s) and the volume, respectively, of the body. An example of a surface force is the water pressure acting on all the wetted areas of a water tank; another is the soil thrust on a retaining wall. These forces are sometimes called tractions. Additionally, there could also be body forces that act over the volume of a body. The self-weight of a structure is the most common example of body forces. Other examples are the 'inertia force' and the magnetic force of attraction. 'Inertia forces' can be of decisive importance in dynamic problems. ('Inertia forces' are imaginary forces assumed to be acting on a body so as to convert a problem in dynamics to one in statics.)

[20] If the calculated values exceed the permissible ones, the procedure may have to be iterated.

[21] To quote from the preface of Frocht, M.M.: *Photoelasticity*, Vol.1, John Wiley & Sons, Inc., New York, (1941): "There are several aspects to the subject of strength of materials. Its center of gravity may, however, be properly be said to lie in the science of stress analysis." Frocht quotes from another book: "The fact deserves emphasis that only one tiny spot need be repeatedly stressed above the endurance limit to make the whole piece fail from the crack which starts at that point. The most highly stressed spot is the Achilles heel of the whole. Any spot must fail when it has had unbearably high local stress, no matter how harmless the applied stresses have been to the rest of the piece. Hence we must focus attention on the actual local stresses and not be misled by nominal average or calculated stresses."

smart materials in *intelligent* structures, one may have to develop the theory right from the beginning. A careful review of the classical theory, *mutatis mutandis*[22], making the necessary departure dictated by the new situation, but otherwise following the classical course, may be the path to chart. After all, originality is said to be clever imitation. In this way, the gulf between the engineers' theory and the more sophisticated mathematical theory of elasticity is narrowing. These new demands[23] have also led to the development of more and more tools which are the *Finite Element Method* (FEM) and the *Boundary Element Method* (BEM) on the one hand, and Computer Algebraic Systems (CAS) such as MAPLE and MATHEMATICA, on the other.

Also of concern to engineers in solid mechanics is the analysis of stability. Long, slender rods subjected to compressive loads can buckle out of shape with disastrous, fateful consequences. Thus, plates and shells in compression are also prone to buckling. It is essential at the stage of design itself to ensure that no buckling occurs[24].

It is difficult at this early stage to define the scope of our subject and elaborate on all the aspects. We, however, hope that the 'importance of being earnest' in learning solid mechanics is already established beyond doubt.

Stresses, strains and displacements at a point:

We had stated [p. 1-9] that the general problem is to determine the components of stress, strain, and displacements at every point inside a given loaded body. We are already familiar with these terms, but we need to have greater clarity about the components of stresses and strains. This is because stresses and strains at a point are examples of (second order) tensors, and they are consequently more difficult to understand with conceptual clarity. For this reason, the concept of tensors is explained in greater detail later. Here we shall see what is meant by the stress at a point.

Stress at a point:

The pressure or stress[25] is sometimes defined and regarded as the force per unit area. This is fine as long as we consider only uniform distributions of stress, or the average stress, over a finite area. When we focus our attention to a point and zero in, and ask for the stress at a point, this definition is inadequate inasmuch as the area of a point is zero. How shall we get over this impasse?

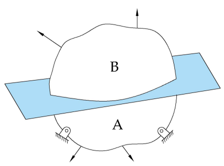
---

[22] with the necessary appropriate changes

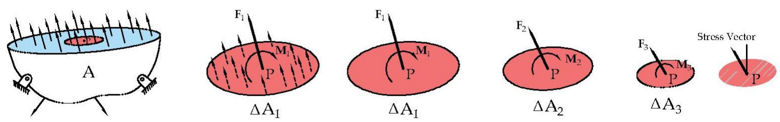
[23] Side by side with these developments is the emergence of research engineers. "Mechanical engineering is not nut and bolt engineering" any more, as the late Professor B.R. Seth was fond of saying. (Professor Bhoj Raj Seth was a noted applied mathematician, for long at Indian Institute of Technology, Kharagpur. A large number of applied mathematicians now working both in India and abroad are either his students, or his students' students.) They are called upon to play the mathematician's game, but are often inadequately trained for the job. The generally higher calibre of these people makes partial amends for this inadequate training. Thus, young teachers aspiring to go deeper into this branch will do well to invest a good part of their time and energy in mathematics, both pure and applied. It would be nice to draw inspiration from the rich academic traditions of some non-English speaking countries also.

[24] Students will do well to associate thin walled structures with the possibility of buckling. Whenever the thought of thin walled structures arises in the minds of engineers and engineering students, an alarm should ring inside warning them, caution: check to make sure that there will be no buckling!

[25] These two words are interchangeable. However, the term stress is almost universally used in our subject.



(a) A stressed body cut into two parts A and B



(b) Resultant force and moment vectors on the small areas

Figure 1.3: On the cut section, consider a small area Δ_1 enclosing the point P. There will be a resultant force vector $\boldsymbol{F}_1$ and a moment vector acting on this area ΔA_1. The area is reduced progressively to ΔA_2, ΔA_2, $\cdots$. Let us then look for the limits of $\boldsymbol{F}_n$ and $\boldsymbol{M}_n$. If a limit exists, that can be used as the definition of stress at a point on a plane. We shall assume that both limits exist, but that the limit of the ratios $\boldsymbol{M}_n/\Delta A_n = 0$. The latter assumption means that there is no locked-in moment inside the stressed body.

We note that the stress at a point depends decisively on the plane. Let us, therefore, define the stress not at a point, but at a point on a given (i.e., specified) plane passing through the point. See Fig. 1.3. The plane XX cuts the body into two parts A and B. The effect of Part B on Part A is to introduce a stress distribution on Part A; the effect of Part A on Part B, in turn, is to have an identical stress distribution (action and reaction, Newtons third law) on Part B. The stress distribution exists within the body even before the cut. This, however, is an internal state for the whole body. For any one part, say Part B, when isolated, this becomes an external distribution of stress. This is a basic concept in continuum mechanics.

We desire to define and clarify the notion of stress *at a point* on a plane (here on the plane XX). The stresses are distributed on the cut surfaces. Let us consider a small area ΔA_1 enclosing the point P. The stress distribution is equivalent to a force $\boldsymbol{F_1}$ and a moment $\boldsymbol{M_1}$. Let us calculate the ratios $\boldsymbol{F_1}/\Delta A_1$ and $\boldsymbol{M_1}/\Delta A_1$. These are the resultant force vector and the resultant moment vector *averaged* over the small area ΔA_1. We propose to use the concept of limits borrowed from the calculus. Accordingly, let us consider smaller and smaller areas ΔA_2, ΔA_3, $\cdots$, ΔA_n, and the corresponding force and moment vectors $\boldsymbol{F_2}$, $\boldsymbol{F_3}$, $\cdots$, $\boldsymbol{F_n}$ and $\boldsymbol{M_2}$, $\boldsymbol{M_3}$, $\cdots$, $\boldsymbol{M_n}$. Calculating the ratios $\boldsymbol{F_1}/\Delta A_1$, $\boldsymbol{F_2}/\Delta A_2$, $\cdots \boldsymbol{F_n}/\Delta A_n$ and $\boldsymbol{M_1}/\Delta A_1$, $\boldsymbol{M_2}/\Delta A_2, \cdots$, $\boldsymbol{M_n}/\Delta A_n$, we proceed to the limit as $n \rightarrow 0$.

At this stage, following Cauchy, we make two assumptions.

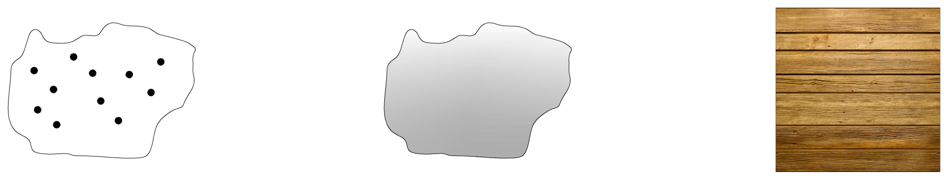
i) The limit $\boldsymbol{F_n}/\Delta A_n$ as $n \rightarrow 0$ exists. This is the resultant stress vector at the point P on the plane XX.

ii) The limit $\boldsymbol{M_n}/\Delta A_n$ as $n \rightarrow 0$ exists and is equal to 0. That is, there is no locked-in moment.

We have now defined the stress vector at a point on a plane. The nature of stress at a point is conceptually more difficult to understand. Hence this topic is discussed in greater detail later [p. 4-1]. The general problem of stress analysis is extremely difficult. Thus, it is absolutely necessary to make simplifying assumptions. Some of them are indicated in the next section below.

It was stated that several simplifying assumptions are unavoidable, especially if we are attempting to obtain analytical, closed form solutions. These simplifications are at various levels. Some of them are examined below.

SIMPLIFICATIONS



(a) Highly discontinuous matter (b) A hypothetical continuum (c) An anisotropic body

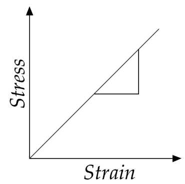
Figure 1.4: Simplification as a continuum: the actual discontinuous body is replaced by a hypothetical continuum, and calculations are made on this continuum. Such an idealisation makes calculations easy or, at least, possible. Fig. 1.4c shows an anisotropic body; the elastic properties are different in different directions.

Continuum

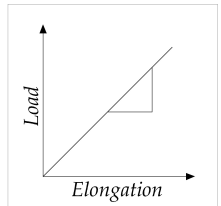
This concerns the composition of the body. We know that physicists in several situations find it convenient to conceive of matter as highly discontinuous at the micro level: particles with large volumes of empty space [Fig. 1.4a]. If our body were really of such hopelessly discontinuous composition, it would make little sense to refer to its macro properties like density. It is to get over difficulties of this kind that we assume that the body is a continuum at the macro level. (Engineers, in such applications as we have in mind here, are interested only at the macro level.) Technically, therefore, the actual body is replaced by a hypothetical continuum, and calculations are made on this continuum [Fig. 1.4b]. One hopes that the results of these calculations are valid for the real actual body. It seems reasonable to suppose that, when the smallest volume of interest at the macro level still contains a large, very large number of micro particles with voids, etc., there would be some kind of statistical averaging over these large, very large number of micro particles, etc., and that it would physically make sense to treat the body as a continuum for all practical (macro) purposes. This assumption can, therefore, be taken as reasonable and valid in all situations, except possibly when we have to deal with cases such as highly rarefied gases (when concepts and methods of analysis from statistical mechanics, molecular dynamics and other branches may have to be borrowed).

Linearity

This concerns the mathematical nature of the equations that govern the behaviour of the body. In developing a theory we are, as indicated earlier, interested in calculating (or predicting) the response of the body (stress, strain and displacement components at every point inside the body) when the stimulus or command (loads, etc.) is specified. If we regard the load as the input, and the resulting response (in the form of stress, etc.) as the output,



(a) Linear (proportional) stress-strain relationship



(b) Linear (proportional) load-elongation relationship

Figure 1.5: Linear (proportional) stress-strain and load-elongation (load-deflection) relationships: linearity implies that superposition is valid.

and the body itself as the system, then we have the classical system theory as a possible means of representation. If the governing equations are all linear, then it can be shown that the output is linearly related to the input. The relationship between the loading (input) and the response (output) can be conceived of as a *straight line passing through the origin* [Figs 1.5a, 1.5b]. This would imply that, say, twice the strain would result in twice the stress, twice the load would result in twice the deflection, etc., and that in general the effect of two or more loads is the sum, superposition, of the separate effect of each of the loads acting alone, etc. This is the principle of superposition[26].

The implications of linearity are deep and are of far reaching consequences. We do not wish to go into these details at this stage. In reality, most systems in nature are nonlinear[27]; linearity is often a simplifying assumption used to make the calculations possible or easy[28]. Linearity is usually associated with 'smallness' in some sense. In our context, it is often in the restricted case of small strains and small displacements. It can also be in the (linear, that is, proportional) relationship between stress and strain components (which is a material property)[29]. We shall briefly point out these at the appropriate places and indicate here and there the consequences of nonlinearity. Here we shall assume that our systems in this course on solid mechanics are all linear. [This is not always true. For example, consider a beam-column [Fig. 2.10]. Here the effects of (i) axial loads $P - P$ and (ii) transverse loads W_1, W_2 cannot be considered separately and then added up to obtain the total effect (such as the deflection). However, superposition is valid in a restricted sense as shown in Fig. 2.11

[26] This principle of superposition is of great fundamental importance, and has far reaching consequences. Students are advised to spend time to understand this principle and its consequences.

[27] Professor R.M. Rosenberg of U. of C., Berkeley, an acknowledged expert of *Dynamics*, is said to have remarked that dividing (differential) equations into linear and nonlinear is like dividing all the objects in the universe as bananas and non-bananas. Poor bananas (linear equations) would hopelessly lose out in any such classification!

[28] Sometimes nonlinearity is undesirable, but unavoidable. Sometimes it is avoided simply because its effect cannot be easily calculated, or calculated at all. Occasionally, nonlinearity can have beneficial effects also. This writer has heard Professor J.P. Den Hartog narrating an actual case of a German aircraft unwittingly protected by nonlinearity by limiting the resonant amplitudes to safe limits (until one fateful day immediately after it was serviced to take up clearances, etc., three of the four engines failed).

[29] The two types of nonlinearity *usually* encountered in the theory of elasticity are (i) geometric nonlinearity (in strain-displacement relations), and (ii) material nonlinearity (in stress-strain relationships).

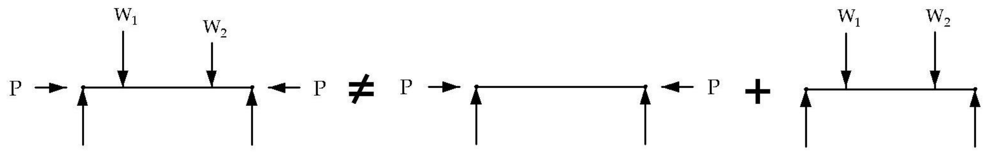


Figure 1.6: A beam-column: superposition as shown is not valid here.

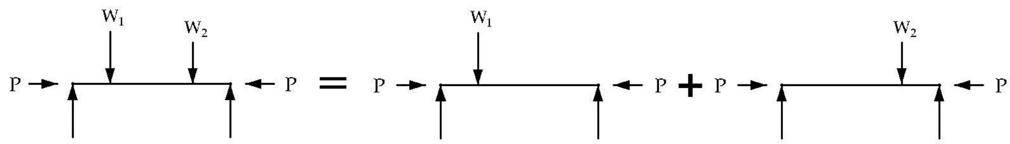
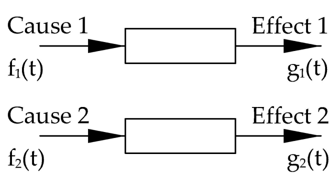


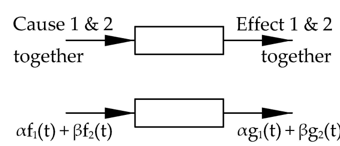
Figure 1.7: A beam-column: superposition as shown is valid here.

as long as the same axial loads $P - P$ are acting. Additionally, there are several nonlinear cases that can be considered only in advanced courses. Here we limit our considerations to simple cases within the framework of linear theory.]

The reason why we consider only linearity is primarily because this is relatively simple; nonlinear systems are far more complex. Furthermore, many technically important problems can be successfully analysed using only the elementary linear theory[30]. The linear theory is relatively well understood; there are several general results applicable to linear systems. No such general theorems exist for nonlinear systems. Theorems concerning existence and uniqueness of solution either do not exist, or have severe restrictions. Besides, there are several new phenomena in nonlinear theory that do not have any counterpart in a linear framework. Thus, no apology is needed for leaving out the much harder nonlinear theory.



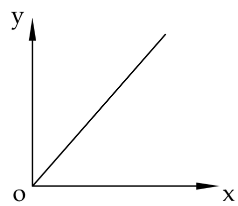
(a) Causes and effects



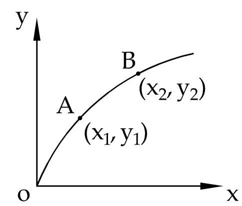
(b) Causes 1 and 2 together → effects 1 and 2.

Figure 1.8: Cause 1 alone leads to effect 1; cause 2 alone causes effect 2; causes 1 and 2 together lead to effects 1 and 2 together. More generally, ($\alpha\times$ cause 1 + $\beta\times$ cause 2) $\longrightarrow$ (α times effect 1 + $\beta\times$ effect 2).

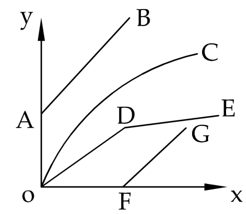
[30] Professor J.P. Den Hartog, noted not only as a great teacher and author, but also was a famous successful vibration consultant. He said more than once in his later days that most industrial vibration problems require, for their effective solution, only elementary theory. He went on to say that "otherwise, I will not be able to solve such problems at this age". This writer does not agree with him on the last sentence; it was stated only out of humility.



(a) Linear



(b) A general curve



(c) Nonlinear relationships

Figure 1.9: If $(x_1,\ y_1)$ and $(x_2,\ y_2)$ are two points on the curve [Fig. 1.9b], does the point $(x_1 + x_2, y_1 + y_2)$ fall on the curve? More generally, does the point $(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$ fall on the curve? The answer, as we can clearly see, is: (i) yes, if the curve is a straight line *passing through the origin* (0, 0); and (ii) no, otherwise.

It is necessary to understand clearly that the relationships represented by the lines AB, OC, ODE, FG in Fig. 1.9c are all *nonlinear*. The fact that these are straight line relationships does not make them *linear*; they are nonlinear, though *piece-wise linear*.

[Perhaps it is desirable to give a brief explanation of linearity in the context of differential equations. A differential equation, we know, is defined as linear if the dependent variable and its derivatives are in the first degree and there are no products of these.

First let us refer to the figure given [Fig. 1.9] and ask the following question. If $(x_1,\ y_1)$ and $(x_2,\ y_2)$ are two points on the curve, does the point (x_1+x_2, y_1+y_2) fall on the curve? More generally, do the points $(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$ (for all values of α and β) fall on the curve? The answer, as we can clearly see, is: (i) yes, if the curve is a straight line *passing through the origin* (0, 0); and (ii) no, otherwise.

Now in the light of this understanding, let us consider a differential equation

$$2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 4y = 0 \qquad (a < x < b).$$

Let us ask the question similar to the one that we asked: if $y = y_1(x)$ and $y = y_2(x)$ are two solutions (i.e., if these satisfy the differential equation), is $y_1(x)+y_2(x)$ a solution? Or more generally, are $\alpha y_1(x) + \beta y_2(x)$ also solutions (for all values of α and β)? Let us examine.

As $y = y_1(x)$ and $y = y_2(x)$ are two solutions, we have $2\dfrac{d^2y_1}{dx^2} + 3\dfrac{dy_1}{dx} + 4y_1 = 0$ and $2\dfrac{d^2y_2}{dx^2} + 3\dfrac{dy_2}{dx} + 4y_2 = 0$.

Multiplying the first equation above by α and the second one by β, and adding them, we obtain

$$\alpha 2\frac{d^2y_1}{dx^2} + \alpha 3\frac{dy_1}{dx} + \alpha 4y_1 + \beta 2\frac{d^2y_2}{dx^2} + \beta 3\frac{dy_2}{dx} + \beta 4y_2 = 0.$$

$$\text{That is,}\quad 2\frac{d^2(\alpha y_1 + \beta y_2)}{dx^2} + 3\frac{d(\alpha y_1 + \beta y_2)}{dx} + 4(\alpha y_1 + \beta y_2) = 0,$$

showing that $\alpha y_1(x) + \beta y_2(x)$ is indeed a solution for all values of α and β.

Why did, or does, this superposition hold here? This is because all the 'operators', viz.,

$$\frac{d^2}{dx^2}, \qquad \frac{d}{dx}, \qquad y \qquad \mathcal{L}(\alpha y_1 + \beta y_2) = \alpha\mathcal{L}(y_1) + \beta\mathcal{L}(y_2), \qquad \mathcal{L}(0) = 0.$$

are linear. Now if, on the other hand, we have a term like y^2, $(d/dx)^{1.5}$, $(d^2/dx^2)^{2.5}$, or $y(d/dx)$ in the differential equation, superposition will not work!

Let us also realise that equations like, say,

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

are linear variable coefficient differential equations. The independent variable does appear in the second degree, and there are products of the independent variable and the derivatives[31]. Integral operators are also linear. Note also that all straight line relationships are not linear. Fig. 1.9b shows nonlinear relationships; some of them — (How many of them, and which are they? Examine and find out.) — are nonlinear, but piece-wise linear. Linearity corresponds to proportionality; the curve must pass through the origin; recall $\mathcal{L}(0) = 0$. Note further that the linearly independent solutions of a (homogeneous) linear differential equations form a basis, and that they span the entire solution space.

The concept of linearity is profound; it has far reaching consequences. Electrical engineering students are generally better exposed to these concepts in their course on Signals and Systems. If the response of a system (linear, time invariant) due to an impulse is known, the response to any excitation can be obtained. Some key words that act as a memory trigger — no more than that — are response to a unit step function, convolution, Faltung integral, Boltzmann superposition, etc.]

Elastic Materials

We assume that all the bodies that we consider are of elastic materials[32]. We had emphasised that it is not the materials that are elastic; 'elastic material' is the model[33] that we employ in this book. Physically speaking, a large number of materials fall in this category. When a load is applied, a deformation results. When the load is removed, the deformation disappears entirely, and the body return to its original undeformed configuration. If this is the case, we say that the material is loaded within its elastic range. Most of our interest is in technical applications[34]; the physical materials (and, therefore, bodies) can be treated as entirely elastic when the load (and, therefore, the stress) is small[35]. When the loads are large, the

[31] Students are advised to spend some time, examine all the relevant aspects, understand clearly the roles of the independent and dependent variables, and form a mature understanding of linearity. It is not uncommon to see even great learned authors calling linear, variable coefficient differential equations as nonlinear. As it is not a good idea to pick on learned famous authors, the actual mistakes are not pointed out. Perhaps it also serves to illustrate that even learned famous authors do make mistakes now and then.

[32] Some exceptional cases will be considered here and there in this book.

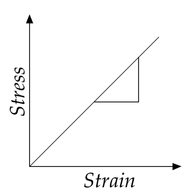
[33] We may consider in a few places thermoelastic models also. This is used when we have to consider strains because of temperature differences.

[34] in the limited scope of this book

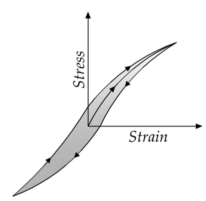
[35] The stress can be high even if the load is small. An example is the stress in the immediate neighbourhood of the point of application of a concentrated load. Another is that at a crack tip.

limit of elasticity is crossed. When the limit of elasticity is reached, some materials called ductile materials (for example, mild steel) start yielding. This is plastic flow. The state of stress in the material (and, loosely speaking, the material itself) is now said to have entered the plastic region. If the yield point is crossed, and then the load is removed, the body does not return to its original configuration. There is a residual deformation which is called the permanent set [Fig. 1.11b]. Furthermore, when the material is loaded beyond the elastic limit and unloaded, and loaded again, the elastic limit goes up. This is due to the so-called strain-hardening, sometimes called work-hardening also[36]. We can see more details when we discuss the simple tension test[37].

Associated with yielding, there are internal structural changes. The details of these are the concern of the vast area called deformation processes in metals studied in metal physics and metallurgy. We shall not go into these details, even though it is desirable to have some familiarity with them. Our study is purely phenomenological[38]; we are concerned only with the macro level descriptions and calculations. Courses in metallurgy and materials science give the details and the micro level explanation of several phenomena associated with a simple tension test to failure of a ductile material such as a mild steel specimen. When



(a) Loading and unloading



(b) Loading and unloading: hysterisis loop

Figure 1.10: With loading and unloading, a small amount of energy is dissipated. With several such cycles, quite some energy is dissipated which accounts for structural damping.

operating within the elastic range, but not outside it, the work done by the external load during the deformation is stored inside the body as internal energy. None of it is dissipated irreversibly. During unloading, this stored energy is entirely given back[39] [Figs 1.10a, 1.11a].

The elastic limit and the proportional limit are conceptually different, but they are often, though not always, close to each other[40]. Some materials even within the elastic range

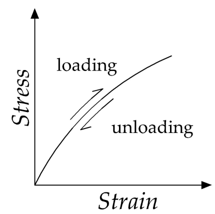
[36] This phenomenon itself is sometimes called strain-hardening or work-hardening.

[37] It is important to know about this test in detail.

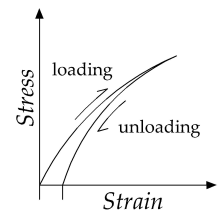
[38] phenomenological study

[39] The case of hysteresis can be of decisive influence in some situations. It is this phenomenon that is responsible for damping, or energy dissipation, when a body vibrates. In many places, however, this is not important and can be disregarded. For a hypothetical elastic material, no energy is dissipated at all. For real materials a small amount of energy is indeed dissipated as hysteresis loss. This amount is small and negligible in most cases, but it assumes significance when there are repeated loading and unloading as when a body undergoes vibration. It is these little droplets making an ocean that are responsible for the energy dissipation. When a thin bar is bent and unbent several times, we can feel an increase in temperature. This is due to the hysterisis loss manifesting as thermal energy.

[40] Hooke's law states that, within the elastic limit, the stress is proportional to the strain. This implies that



(a) Loading and unloading below the elastic limit: no permanent set



(b) Loading beyond the elastic limit and unloading: permanent set

Figure 1.11: Loading and unloading below and beyond the elastic limit. Note the permanent set on unloading. Note that the stress and the strain are not necessarily proportional for an elastic body (elastic behaviour).

may not have a proportional relationship between the load (or stress) and the resulting deformation (or strain). Concrete is one such material; copper is another. Because of the great importance of concrete as a structural material, special terms like secant modulus and tangent modulus are sometimes used to bring the calculations within the framework of linearity (proportional relationship between stress and strain).

Homogeneity, Crack-free Body in Its 'Natural' State

The body is assumed to be homogeneous, (i.e., the composition of the body is the same at all points) even though this is not usually the case in several real life situations. In reinforced concrete, for example, there are two materials, steel and concrete, steel in the form of bars surrounded by concrete. Concrete, by itself, too is inhomogeneous; it is made up of aggregates of varying sizes bonded together by cement paste. Nevertheless, this is a convenient assumption to make, particularly when considering only simple cases. Thus, we shall have only homogeneous bodies for our analysis almost everywhere.

The body is also assumed to be crack-free. The classical theory is developed for this case. A realisation has come in recent years that micro-cracks or some faults can hardly be avoided in several situations. These micro-cracks are not by themselves a cause for alarm, and should not be taken as manifestations or indications of failure or impending failure. A relatively new vigorously growing subject called *Fracture Mechanics* has techniques and methods of analysis to judge whether these micro-cracks are 'benign' (harmless) or 'malignant' (harmful) (in the sense that these cracks open up, progress further, and lead to catastrophic failure). The design philosophy is accordingly different. We, however, do not concern ourselves in this book with these relatively recent developments.

There are also cases where the body might be prestressed or prestrained. Stresses can, and do, occur inside a body even when there is no external load. Residual stresses (say, left in a body after it is machined) are a case in point. These situations present complications that we do not wish to consider now. Thus, the bodies that we consider are in their 'natural' state: when there is no external load, there is no stress inside, and when there is no stress,

the proportionality is maintained throughout the elastic region.

there is no strain. The case of thermal strain, however, does occur frequently, and is sometimes considered in some books at this level. [It is better in such situations to consider the 'material' (really model) as thermoelastic, and not as elastic. Thermoelasticity may be regarded as a different, though allied, discipline governed by a different set of constitutive equations like plasticity. We do not discuss thermoelasticity in this book.]

Isotropy

Just as materials are often considered as homogeneous for simplicity even though they are often not so in practice, these materials are also assumed to be isotropic. Isotropy means that the properties are the same is all directions. Crystals have strong directional properties: elastic, optical, piezoelectric, etc. There are cases of practical importance when this assumption of isotropy is not justified; anisotropy may have to be taken into account in the calculations. This can happen because of two different reasons or situations. One is that the material that is used may have directional properties (different properties along different directions). Wood is one example in which the properties are different along the directions parallel and perpendicular to the grains. Another is cold rolled copper where the rolling process introduces directional properties in the direction of rolling. The second situation arises because of the constructional features. One example is a concrete floor slab with parallel beams in only one direction. Another is a cast iron slab (or a steel plate) with stiffening ribs in only one direction. These are examples where the material is, or can be considered as, substantially isotropic. But the constructional features introduce anisotropy (in the examples cited, 'orthotropy', which is a simpler case of anisotropy compared to the general anisotropy) in the structure[41]. In these situations, the structure, say, the floor slab can be replaced by another slab with anisotropic (orthotropic) properties.

These are technically important cases. Nevertheless, a large number of technically important practical problems can be solved by considering only the simple case of isotropy. In recent years, technological demands have made composite materials or constructions (or 'composites' as they are called) more popular. These introduce the unavoidable complication of anisotropy. Thus, any course that aims at building technical competence cannot neglect anisotropy.

Simplifications: One-dimensional and Two-dimensional

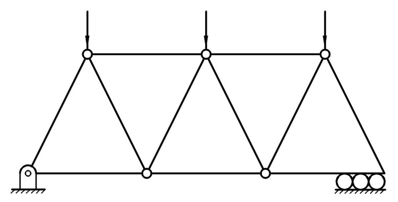
All physical bodies are really three-dimensional. However, sometimes there are strange situations. The (two-dimensional) curved surface of a sphere in a three-dimensional space has features that complicate matters. The ordinary familiar Euclidean geometry is not applicable to this case. It is desirable, especially for advanced students and teachers to be aware of the complications lurking in the background.

We often model structural elements and machine components as either one-dimensional or two-dimensional for simplicity of analysis. When two of the dimensions are small in comparison to the third, it seems appropriate to model the element as one-dimensional. As

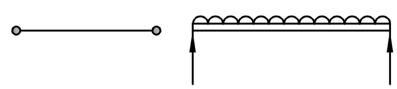
[41] Then there can be unusual materials like bio-membranes. Such membranes have to be treated as orthotropic. However, these are exceptional situations. We will not discuss such cases in this book.

an example, the two 'small' dimensions may be the cross-sectional dimensions of a beam, while the third may be its length (span). Beams, columns, shafts, members in a truss, etc. are examples of such bodies (or elements). It is mainly such bodies (or elements) that are analysed in a first course. We can, thus, obtain much simplification in our analysis.

In a similar way, we can have two-dimensional bodies also. A flat plate is, thus, a two-dimensional (and, thus, simplified) model of a really three-dimensional structural element like a roof slab. The analysis will generally be more difficult, though still much simpler than the three-dimensional analysis. We generally consider only one-dimensional bodies (or more appropriately, one-dimensional simplified approach), and thus one-dimensional problems in elementary courses.



(a) A truss



(b) One-dimensional bodies

Figure 1.12: A (plane or space) truss is made up of one-dimensional bars. These are usually considered to be in pure tension or compression (i.e., one-dimensional states of stress). A beam also is treated as a one-dimensional body.

As another example, let us consider the analysis of a hydrodynamical bearing. The heart of the problem is the analysis of the flow of the lubricant fluid as the journal rotates in the bearing. If the bearing is very short [Fig. 1.13b], the flow analysis becomes a two-dimensional problem (as explained earlier). On the other hand, if the bearing is 'infinitely long' [Fig. 1.13a], and all the cross-sections identical, we can neglect the flow in the z-axis (axial direction), and again the problem is simplified.

(a) A 'long' bearing

(b) A 'short' bearing

Figure 1.13: An 'infinitely long' (long) and an 'infinitely short' (short) bearings are shown in the figure. The analysis of both 'very long' and 'very short' bearings is greatly simplified.

Strange and paradoxical as it may seem, two-dimensional approximations are possible under either of these two extreme conditions: infinitely long (long) [Fig. 1.13a], or infinitely short (short)[Fig. 1.13b]. Let us see another situation also. Shown in Fig. 1.14a is a

(a) A cylindrical body (b) A circular cross-section

Figure 1.14: A cylindrical body (cross-section circular) is shown. Now for this geometry it is more convenient to choose a cylindrical polar (r, θ, z) coordinate system.

cylindrical body subjected to some radial loads. (These loads are such that the entire body is in static equilibrium.) The geometry of the problem makes it convenient to choose, not a rectangular cartesian coordinate system, but a cylindrical polar one (r, θ, z). Now all the quantities of interest (stresses, strains and displacements) are functions of r, θ, z. The above three-dimensional problem becomes simpler under special situations.

(i) If the cylinder (in the z-direction) is very short compared to the radius, the problem can be treated as two-dimensional with (r, θ) as the only two independent variables).

(ii) If the body is very long, and all the cross-sections identical (shape, properties, loading), the problem is again simplified as two-dimensional with (r, θ) as the two independent variables.

(iii) We can have simplifications that result from symmetry. If, for example, we have a homogeneous cylindrical body (with the same cross-section and the same loading all along its length in the z direction) with a circular cross-section loaded by a radial load that is distributed symmetrically around the circumference (Fig. 1.14a), we have a case of circular symmetry (axisymmetry, symmetry about an axis). A thick cylinder subjected to an internal fluid pressure as shown in the figure is a case in point. We can now infer that all the quantities of interest inside the body (such as stresses, strains and displacements) are independent of the coordinate θ. The problem is thus simplified: all the quantities are now dependent only on r, z.

(iv) If we have the conditions (iii) and either (ii) or (i) satisfied in a given problem, all the quantities of interest are now functions of only r. Thus, the problem has become one-dimensional.

In the above case (iv), the problem can be simplified by exploiting the inherent circular symmetry. If one fails to notice it, and chooses a rectangular cartesian x, y coordinate system, this convenience is lost, and the problem is no more one-dimensional.

Like circular symmetry, there can be spherical symmetry in some cases. An example is the problem of finding out the stresses, strains and displacements in a spherical vessel subjected to fluid pressure (internal, or external). Here the only independent variable is the radial coordinate r; thus, the problem is one-dimensional.

We may also inquire into questions of the kind: how small (and compared to what) shall the dimensions be before they can be justifiably treated as 'small'? There are theoretical issues involved in its answer. Furthermore, from a practical point of view, engineering judgement is needed in making such assumptions properly.

Perhaps it is necessary to clarify that it is really not the bodies, but the analyses, that are one-, two- or three-dimensional. A one-dimensional analysis takes into account the variation along the (only) one direction. Thus, for example, a one-dimensional analysis of fluid flow in a pipe can give information about the variation (of, say, the pressure, the velocity, etc.) along the length[42]. Variations of those variables across the cross-section cannot be obtained from this analysis[43].

It looks plausible that when the body is 'one-dimensional', a one-dimensional analysis would give good results. That is to say, the results of such a simplified analysis agree very well with those of a more detailed, rigorous three-dimensional analysis. It is perfectly possible and sensible in appropriate situations to carry out a one-dimensional (say, conduction heat transfer) analysis of a large, thick slab, even though the body is no more 'one-dimensional'.

It is interesting to note that it is also perfectly possible to carry out an analysis which is one-dimensional for one variable (say, the pressure), while it is two-dimensional, or even three-dimensional, for another variable (say, the velocity). These are all various (mathematical) models used to simplify a given problem that may be too difficult to solve with all the complications retained. Usually the motivation for effecting these simplifications is the desire to get rid of complicated (often nonlinear) mathematical terms[44]. The guiding light must still be engineering judgement[45].

[42] Some people prefer to classify only those problems as one-dimensional, in which all the field variables change only in the (only) one direction that is considered.

Consider the so-called Hagen-Poisseuille flow, which is a viscous (Newtonian) flow in a uniform, (circular) cylindrical pipe. Among the velocity components v_r, v_θ, v_z, only the axial component v_z exists; the other two are zero. This component of velocity, v_z, however, varies with r, but not with z; i.e., $v_z = v_z(r)$. The pressure p, on the other hand, varies only along the axial direction, z; i.e., $p = p(z)$. This problem is, thus, two-dimensional in this sense. Some authors call this one-dimensional. Opinion is divided on this, as indeed on everything in life!

[43] We note that, in the elementary theory of pure bending of beams, the variation of stress across the cross-section can be obtained as a result of a 'one-dimensional' analysis. This may appear to contradict the statement made above. This is only an apparent anomaly; really there is no contradiction. Readers are advised to review the development of the theory of bending, and understand the implications in the above statement.

[44] An order of magnitude estimate enables one to drop some of the terms of the governing equations, which contribute but little to the final result. Such estimates are widely employed in fluid mechanics.

[45] We may recall from our study of fluid mechanics that the governing (Navier-Stokes) equations are nonlinear and notoriously difficult to solve. Prandtl was able to get over several difficulties and come up with his boundary layer theory. He, and he alone, had the intuition and judgement to neglect all the complicating features, retaining only the essential terms. Thus, the analysis became relatively simple — boundary layer theory is still not very easy — without losing the crucial aspects (call them the heart of the problem, if you like). This was in spite of the fact that there were scientists in Göttingen at that time who had far greater mathematical prowess than Prandtl. It is often such abilities that mark out a genius from the rest.

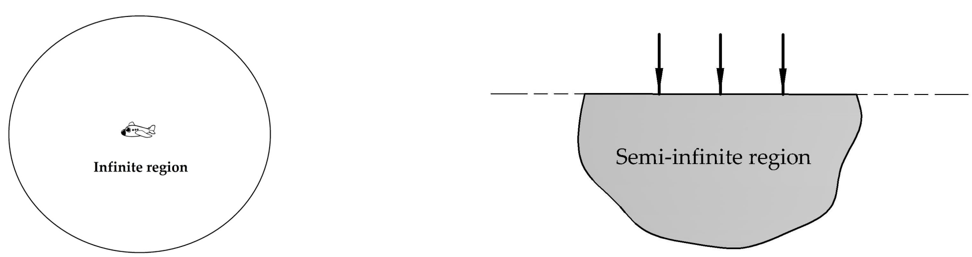
As an example, let us consider the problem of testing an airfoil in a wind tunnel that seeks to simulate the actual conditions during flight of an aircraft. If the scaled-down model[46] of the wing is small in comparison with the cross-sectional dimensions of the wind tunnel, we may assume that the conditions of flow over an object in an infinite fluid medium are met at least approximately.

Infinite and Semi-infinite Bodies / Regions

(a) An 'infinite' beam (b) A 'semi-infinite' beam

Figure 1.15: 'Infinite' and 'semi-infinite' beams: rail-road tracks extending in both left and right directions [Fig. 1.15a], and with a dead end, the track extending only to the right

When modelling physical problems we often use concepts like infinite and semi-infinite regions. 'Infinite' in mathematical modelling is only 'sufficiently large' in the physical sense. 'Sufficiently large' means that far-field conditions prevail. One example was indicated earlier. As another, we might consider a rail-road track that can properly be modelled as as an infinite beam (supported on elastic foundations) [Fig. 1.15a]. When a train moves, there is a certain distance beyond which this load has no effect. When a train moves on a rail-track in Calicut, it is obvious that there would be no consequent stress, strain or displacement in, say, Chennai. Beyond a certain distance (say, 3 times the value of a certain parameter[47]), the effects are negligibly small. If the beam is longer than this distance, it is treated as an infinite beam.



(a) An 'infinite' region (b) A 'semi-infinite' region

Figure 1.16: 'Infinite' and 'semi-infinite' regions: fluid all around a flying aircraft [Fig. 1.16a], and soil below a tall building [Fig. 1.16b]

[46] In a practical sense, an infinite medium is to be understood in this sense: when a body is introduced in a fluid flow field, there would be local disturbances. These disturbances become less and less, and eventually die down as we go farther and farther from the body. We can conceive of certain distances beyond which the undisturbed conditions (the so-called far field conditions) can be assumed to prevail. These distances are the limits. When the domain has dimensions that are larger than these distances, we may assume that the domain to be an infinite region.

[47] See the topic of beams on elastic foundations in any good book, ideally, M. Hetéyni: *Beams on Elastic Foundations*, University of Michigan Press, Ann Arbor, Mich. (1944).

At the end, the dead end, of a rail-track, the track extends only in one direction. Such cases are referred to as semi-infinite bodies [Fig. 1.15b]. Other examples are: a flying aircraft may be modelled as an object moving in an infinite region (sky) of fluid (air) [Fig. 1.16a]; a moving ship may be modelled as an object moving in a semi-infinite region (ocean) of fluid (sea water); a tall building may be modelled as a static[48] load on a semi-infinite region of elastic or viscoelastic, viscoplastic material (soil) [Fig. 1.16b]; etc.

CLOSING REMARKS

We have pointed out some simplifications. It is on the basis of these that the (simplified) engineers' theory of strength of materials or mechanics of solids is developed. We shall consider only simple cases in this book. However, there may be some exceptional cases (when one or more simplifying assumptions do not hold) that we may come across in this book. We shall call attention to these cases at the appropriate places.

In the first course, which is at an elementary level, we take a simple-minded approach. Thus, everything is simple. The geometry is regular: perfectly straight, perfectly circular, perfectly spherical, etc.; the body is homogeneous, and free from cracks, defects and inclusions; it is moreover in its 'natural' state (in the sense that there is no stress or strain when there is no external load).

After discussing simple cases in a first course, students are generally required to take a second course, where slightly more complicated cases are discussed. Special problems in bending (like curved beams, unsymmetrical bending, beam-columns and beams on elastic foundations), torsion of non-circular prismatic bars, thick cylinders and rotating discs and energy methods are of such great importance that they cannot be left out of the curriculum.

Real life problems seldom fall into these cases (elementary and slightly more complicated cases) referred to above. The geometry is often complex[49]; there can be several materials, as say, in reinforced concrete; there can be cracks. One can guess that these complex problems can usually be solved only by numerical methods like the finite element method (FEM). However, before the numerical methods are used, it is necessary to formulate the problem. To be able to do this properly, we have to learn the governing equations and the relevant boundary conditions consistent with the physical conditions that actually prevail during the operating conditions. Thus, just because there are powerful numerical methods now available, and access to powerful computers including supercomputers, it does not follow[50] that we need not learn the underlying theory properly with conceptual clarity.

[48] For studying earthquake responses, the model is to be revised; the problem is now recast as a dynamic one of a tall structure with support excitation (movement).

[49] An example might be the calculation of the thermal stresses in a cylinder block of an automobile. The geometry is admittedly far from simple. There can be strains (thermal strains) even when there is no stress. In other cases, there can be locked in 'residual stresses' as when a component is machined. We then have a case of internal stresses even when no external load is applied. There can be many more complications. These examples are cited to emphasise the fact that only classroom exercises are simple, and that real life problems are usually difficult, and fall outside the scope of a first course.

[50] Of late there is a clear shift in emphasis: rigour seems to have disappeared altogether. If ability to learn whatever is needed in later life is the objective, it is essential, absolutely essential, to learn at least six or seven courses rigorously.

New technological challenges have forced scientists and engineers to adopt more scientific approaches. Our subject which used to be called strength of materials traditionally has changed its style and methods in its new *avatar* as mechanics of solids. Structural mechanics, theory of elasticity and the more abstract continuum mechanics at a higher level are closely associated with our subject. Fracture mechanics and composites have become important. Advances in aerospace engineering have made these topics to be of great relevance . Experimental stress analysis and powerful numerical methods like the finite element method (FEM) are also important allied subjects. When these new problems and approaches are examined, it becomes necessary to learn abstract mathematics, etc. on the one hand, and experimental methods on the other. There is no end to this already long list. Ambitious readers of this book will do well to learn these topics slowly and steadily[51].

S.P. Timoshenko

One of the greatest names in the area of Strength of Materials or Mechanics of Solids is Timoshenko. It is necessary for us to be familiar with his contributions. The following few lines are meant to introduce this outstanding engineer-scientist to the young readers.

Stephen Prokofyevich Timoshenko (Dec. 22, 1878 - May 29, 1972) was a Ukranian by birth. He is the author of several highly rated books, and is known as the father of modern engineering mechanics. He was arguably the greatest ever teacher in the broad area of

The École Polytechnique has already shown us the way to proceed. It will be a folly not to take notice of such guiding lights.

[51] I presume my readers would permit me to make a few informal comments. I am reminded of a doctor friend of mine who had once told me about his professor in the medical college. The learned professor began his lecture: “There are three important things in the successful practice of medicine”. And then he would pause and wait for attention before he would resume. “Number One: diagnosis”. Again he would pause, and continue. “Number Two: diagnosis”, and pausing once again dramatically, “Number Three: diagnosis”. I so much liked this that I am persuaded to imitate him and state: there are three important things in mechanics: Number One: draw the free-body diagram; Number Two: draw the free-body diagram; and Number Three: draw the free-body diagram!

Talking about free-body diagrams, I am also persuaded to add this. Let us take the example of a simply supported beam loaded by a concentrated load. We draw the free-body diagram and find the reactions easily. However, before we spring into the detailed calculations, we would do well to spend some and examine the problem. How many unknowns do we have, and how many equations? If it is a concurrent coplanar system of forces, how many separate (linearly independent) equations do we have? If we take moments about four or five different points, will we or will we not obtain four or five linearly independent equations? It pays to have a few quiet moments, reflect on such issues, and come to a mature understanding. Such regular exercises would take us to a higher level of understanding. It is good to form the habit of examining or analysing the problem before the detailed calculations are carried out.

During our students days some of my friends and I used to think that for each problem in mechanics, there was one (and only one!) point about which the moment is to be taken. And we thought that it was important to know beforehand which point for each problem. We followed, or tried to follow, this practice until a faculty member came and told us that we could take moments about any point. That was a revelation! Yes, we can take moments about any point. But if the point is chosen cleverly, the subsequent work becomes simpler. It is not a question of correctness; it is a question of convenience in choosing the proper point about which the moment is taken. Years have rolled by, but I remember his words with much pleasure and gratitude.

engineering. There are an incredibly large number of outstanding scholars who are either his students, or his students' students. Many of them consider him to be their teacher in an extended sense.

Timoshenko graduated in 1901 from the St. Petersburg Ways and Communication Institute, and continued to teach there. He had a meteoric rise to become a Professor & Chair of Strength of Materials in 1906 and Dean (Dean of the Division of Structural Engineering) in 1909 in Kiev Polytechnic Institute. In 1922 he moved over to the US where he worked for the Westinghouse Electric Company during 1923 - 1927. He joined the University of Michigan, but in 1936 he moved to Stanford University where he taught until he was 75. He has also written *Technical Education in Russia*, *As I Remember* (his autobiography) and the famous *History of Strength of Materials*. Students are advised to read his autobiography.

In the next chapter, we shall discuss special problems in bending.

Figure 1.17: Stephen P. Timoshenko

Chapter 2

SPECIAL PROBLEMS IN BENDING

We already know the simple theory of bending of beams. Beams are one-dimensional structural elements with transverse forces as the primary loading[1]. This theory leading to the Euler-Bernoulli formula[2] is of much fundamental importance. However, there are several technically important cases that are not covered by this theory. Thus, in this closer second look at the theory of bending, we need to consider these special problems.

SPECIAL PROBLEMS IN BENDING

Some of these special problems are the following.

1. Unsymmetrical bending
2. Curved beams
3. Bending beyond the elastic limit
4. Beams of two materials
5. Beams with wide cross-sections
6. Beams on elastic foundation
7. Beam-columns

Among these special cases, we shall take up only the first two, viz., (i) unsymmetrical bending and (ii) curved beams. However, we shall make brief comments about the other

[1] Students are often tempted to call horizontal elements as beams. This mistaken idea arises because the beams that they are exposed to are all, or almost all, horizontal. It is necessary to consider a variety of situations where the beams are vertical or inclined. Aircraft structures provide several interesting cases. In fact, it is the demands of the aerospace industry that led to great advances in our subject.

[2] The Euler-Bernoulli equation governing simple bending is (with the usual notations) $\sigma/y = M/I = E/R$.

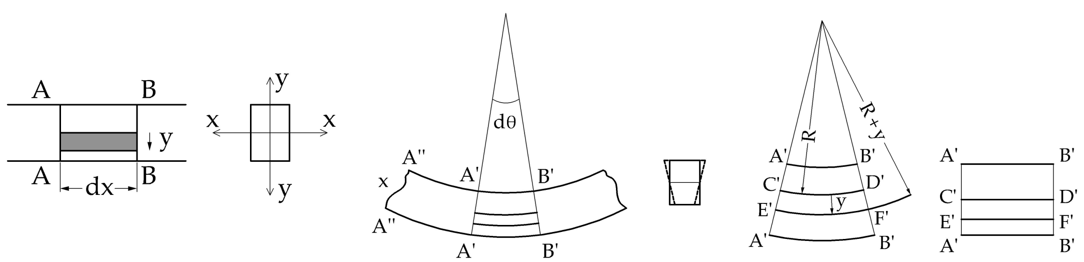
four cases also. To have a proper appreciation of these various cases, it is necessary to have a review of the elementary theory of bending which we shall take up now.

SIMPLE THEORY OF BENDING

The actual body though really three-dimensional, we recall, is treated as one-dimensional in the following analysis. When the (one-dimensional) beam is bent on application of a bending moment, there are the resulting stresses, strains and displacements. We must know these at all points when a known (or specified) bending moment is applied.

We consider the problem of pure bending, that is, bending in the absence of shearing forces. In pure bending, the bending moment M is constant. That is to say, the same bending moment M acts on both sides of the beam element. It means equivalently that there is no shear force on the cross-sections of the beam element because $dM/dx = F$.

The starting point is the Love-Kirchhoff[3] assumption: plane cross-sections continue to be plane even after bending. Let us consider a small beam element before and after deformation [Fig. 2.1]. The two cross-sections AA and BB before deformation rotate to form an angle $d\theta$ and become $A'A'$ and $B'B'$. By the Love-Kirchhoff assumption, the cross-sections do not become crooked like $A''A''$. The bottom fibres are stretched out, while the top ones are shortened. As we go from the bottom fibre to the top one, there is a fibre, somewhere in between, that is neither elongated nor compressed [Fig. 2.1c]. Thus, $CD = C'D' = R\,d\theta$. Examining the geometry of the deformed shape of the element, we can develop the theory of bending.

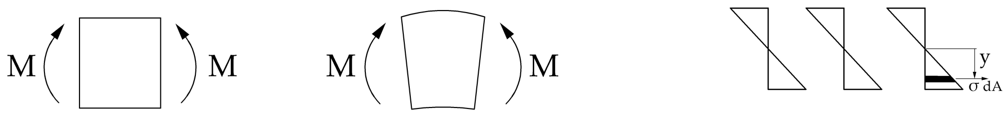


(a) Before deformation (b) After deformation (c) Geometry of deformation

Figure 2.1: The elementary theory of bending is developed on the basis of the Love-Kirchhoff assumption. The geometry of deformation is shown in the figures.

What do we mean by developing a theory of bending, or by solving the elementary bending problem? What do we want to know after solving the problem of bending? Well, we desire to obtain the stresses, the strains and the displacements when a known bending moment acts on a beam element. As we consider only pure bending (bending in the absence of cross-shear on the cross-section; this is, when there is no shear force), the bending moment M is constant. The same bending moment M acts on both sides of the element. It implies that there is no shear force on the cross-sections because $dM/dx = F$.

[3] The same assumption is made in the theories of plates and shells also.



(a) Before deformation (b) After deformation (c) Strain and stress distributions

Figure 2.2: Examining the deformation it is possible to obtain the strain distribution and, hence, the stress distribution across the cross-section.

Let us consider a typical fibre EF that, after deformation, becomes $E'F'$. The strain can now be calculated as:

$$\text{strain, } e\Big|_y = \frac{E'F' - EF}{EF} = \frac{(R+y)\,d\theta - R\,d\theta}{R\,d\theta} = \frac{y}{R}. \tag{2.1}$$

Having obtained the (expression for) strain of a fibre distant y from the neutral axis — we shall show below that the neutral axis is the same as the centroidal axis[4] — we can calculate the stress easily as:

$$\text{stress } \sigma\Big|_y = E\,e\Big|_y = \frac{E\,y}{R} \qquad \longrightarrow \qquad \frac{\sigma}{y} = \frac{E}{R}. \tag{2.2}$$

Both the strain and the stress vary linearly across the cross-section as shown in Fig. 2.2c.The consequence of this fact is that the material farthest from the neutral axis is stressed the most and that, therefore, the material is most effectively utilised when placed farthest from the neutral axis.

Let us now consider the equilibrium requirements. These are the conditions to be satisfied.

$$\int_A \sigma\,dA = 0 \qquad \text{(no net axial force)} \tag{2.3a}$$

$$\int_A (\sigma\,y)\,dA = 0 \qquad \text{(no bending moment about the } x \text{ axis)} \tag{2.3b}$$

$$\int_A (\sigma\,x)\,dA = 0 \qquad \text{(no net bending moment about the } y \text{ axis)} \tag{2.3c}$$

Using the expression for σ [Eq. (2.2] in these equilibrium equations, we are led to the following results.

$$\int_A \left(\frac{E}{R}\,y\right)\,dA = 0 \qquad \longrightarrow \qquad \int_A y\,dA = 0 \quad \text{(first moment = 0!)} \tag{2.4a}$$

[4] These two axes do not always coincide: (i) a curved beam, and (ii) even a straight beam when accompanied by an axial force are two cases in point.

$$\int_A \left(\frac{E}{R}\,y\right) y\,dA \;=\; M_x \qquad\longrightarrow\qquad \frac{E}{R}\int_A y^2\,dA = M_x \qquad\longrightarrow\qquad \frac{E}{R} = \frac{M_x}{I} \tag{2.4b}$$

$$\int_A \left(\frac{E}{R}\,y\right) x\,dA \;=\; M_y = 0 \quad\longrightarrow\quad \frac{E}{R}\int_A xy\,dA = 0 \quad (x,\, y \text{ are principal axes!}) \tag{2.4c}$$

Eq. (2.4a) shows that the neutral axis passes through the centroid!

Let us observe that we have used (i) the strain-displacement equations, (ii) the constitutive equation (material law, Hooke's law), and (iii) the equations of equilibrium. As the displacements were assumed, there is no role for the compatibility equations (or, equivalently, the compatibility equations are automatically satisfied). We shall refer to the last equation (2.4c) when we discuss unsymmetrical bending later in this chapter.

This is a recapitulation of the simple pure bending of beams. This forms the basis of the various special problems of bending that we shall discuss one by one below. As the readers would surely have learned this, we do not discuss this theory any further except to point out the amazing consequences of the Love-Kirchhoff assumption.

The Amazing Consequences of the Love-Kirchhoff Assumption

The classical theory of small deflection of thin plates is developed as a two-dimensional generalisation of the theory of bending of (one-dimensional) beams. The starting point for both is the Love-Kirchhoff assumption[5]. The consequence of this simple assumption is amazing. The theory of bending of beams is intimately related to the geometry of the bent (deflected) shape of the middle line. This means that all the physical quantities of engineering interest such as the bending moment and the shearing force; and the stress, the strain, and the displacement at every point are decided entirely by the geometry of deformation of the middle line. A similar assumption, a similar procedure, and a similar conclusion are applicable for thin plates and shells also. For a plate now we have a middle surface (instead of a middle line for a beam), and when the plate bends, the originally flat middle surface goes into a curved one. The consequence this time also (as before for a beam) is that the geometry of this curved middle surface is related to all the physical quantities of engineering interest. The Love-Kirchhoff assumption, thus, converts the engineering problem of the mechanics of solids to a mathematical one of the geometry of the curved middle surface[6]. The differential geometry of the curved middle surface now takes over; the curvature tensor[7] plays a decisive role in the theory of plates.

This time, however, the geometry is a little more complicated because we have to deal with a two-dimensional (curved) surface $w = w(x, y)$, where w is the (small) deflection of the middle surface. Now there is bending in both the directions. Thus, we now have in both

[5] Cross-sections which are plane before bending remain plane even after bending; they only rotate.

[6] In a voice, loud and clear, the Love-Kirchhoff assumption announces triumphantly: "tell me all about the geometry of the curved middle surface, and I can tell you all that you want to know about the plate such as (i) the bending moment, the shearing force, and the twisting moment; and (ii) the stresses, the strains, and the displacements."

[7] The radii of curvature and the twist, we can see, are analogous to the normal and shearing stresses, σ_{xx}, σ_{yy} and the shearing stress τ_{xy}. The analogy is based on the fact that the curvature tensor and the

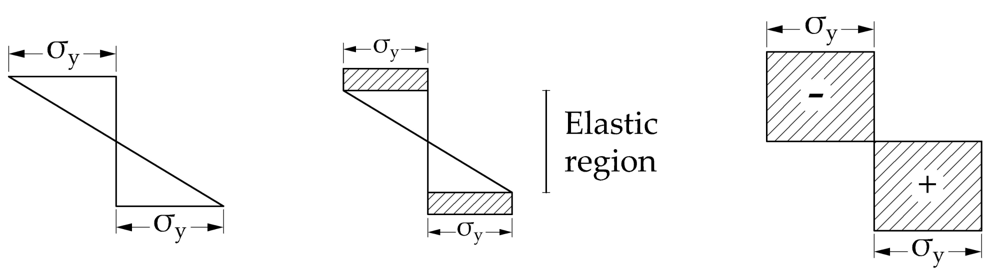
directions (i) the slopes $\partial w/\partial x, \partial w/\partial y$; the (approximate) curvatures $\partial^2 w/\partial x^2, \partial^2 w/\partial y^2$; and (iii) the twist $\partial^2 w/(\partial x \partial y) = \partial^2 w/(\partial y \partial x)$ pertaining to the geometry. On the physical side, we can expect to have (i) the bending moments M_x, M_y; the vertical shear forces Q_x, Q_y; and (iii) the twisting moments M_{xy}, M_{yx} corresponding to the two directions x, y. (For a shell, the theory is far more complicated.)

Now we are ready to take up the special problems one by one. First we shall dispose of items 3 - 7 with brief comments for each before we take up the cases of unsymmetrical bending and curved beams.

PROBLEMS IN BENDING: FURTHER EXTENSIONS

These are problems when one or more of the simplifying assumptions made in the elementary theory are dropped, only one at a time. We shall have a cursory glance at some of them with a few helpful suggestions. First we shall examine what happens if or when the elastic limit is exceeded.

Bending Beyond the Elastic Limit



(a) Fully elastic region (b) Elastic-plastic region (c) Fully plastic region

Figure 2.3: The stress distributions in the three cases: (i) fully elastic [Fig. 2.3a], (ii) elastic-plastic [Fig. 2.3b], and (iii) fully plastic [Fig. 2.3c].

We begin with the simple theory of pure bending when the entire body is in the elastic regime (that is, the stresses are below the elastic limit, which for all practical purposes can be taken to be the proportional limit). Now what happens when the applied bending moment is progressively increased? Well, the maximum bending stress (normal stress) is correspondingly increased until the elastic limit is reached. On further loading plasticity 'creeps in' and the stress distribution is modified from the linear variation [Fig. 2.3a] to a nonlinear one [Fig. 2.3b]. Now we have an elastic core sandwiched between two plastic regions shown shaded. The figure is drawn on the (simple-minded) assumption that the

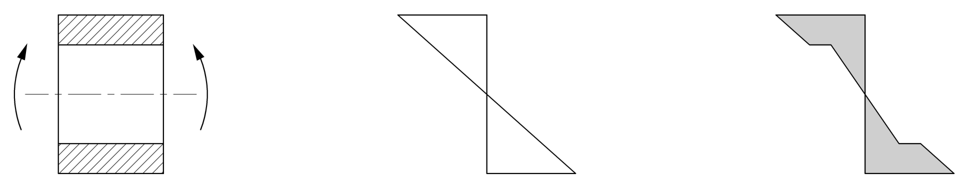
(two-dimensional) stress tensor are examples of a second order tensor.

$$\begin{bmatrix} \frac{1}{R_{xx}} & \frac{1}{R_{xy}} \\ \frac{1}{R_{yx}} & \frac{1}{R_{yy}} \end{bmatrix} \longrightarrow \begin{bmatrix} \frac{1}{R_{11}} & 0 \\ 0 & \frac{1}{R_{22}} \end{bmatrix} \text{ analogous to } \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} \longrightarrow \begin{bmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{bmatrix}.$$

yield point σ_p is the same in both tension (bottom) and compression (top)[8]. The bending moment (or resisting moment) that corresponds to this stress distribution [Fig. 2.3b] can be calculated. On further loading, 'plasticity' permeates deeper into the cross-section from both the bottom and the top ends. The maximum, the limit, is when 'plasticity' creep in and covers the entire region. This, of course, is physically impossible; the stress cannot obviously change abruptly from $+\sigma_y$ (tensile) to $-\sigma_y$ (compressive), as we cross the neutral axis from the bottom to the top of the cross-section. Now a plastic hinge is formed at that cross-section, because the section cannot develop any further resistance. The limit of resistance is reached when the entire cross-section goes into yield, the bottom half with a tensile stress σ_y, and the top half with a compressive stress σ_y.

Let us note that we still follow the Love-Kirchhoff assumption, and that the strain variation is still linear. The constitutive equation is now changed: from elastic through elastic-plastic to fully plastic.

Beams of Two Materials



(a) Two materials (b) Distribution of strain (c) 'Transformed area'

Figure 2.4: The cross-section shows two materials [Fig. 2.4a]. The strain distribution is, as in the earlier cases, linear [Fig. 2.4b]. The stress distribution, however, not linear, because the moduli of elasticity of the two materials are different [Fig. 2.4c]. It is also possible to have the concept of a 'transformed area' and to pretend that the material is the same throughout.

For this case also, the strain distribution is the same, viz., linear. The stress distribution is different, because the Young's modulus of elasticity has different numerical values, E_1 and E_2 for the two materials. Once the stress distribution is known, the expression for the bending moment can be worked out.

It is advantageous to have a large quantity of a low modulus material in the middle low-stressed area and a small quantity of a high modulus material in the farthest regions of high stress. For example, we can have wood[9] as the low modulus material in the middle, and steel as the high modulus one at the ends (shown shaded in Fig. 2.4a). Instead of considering two materials, it is possible to have the concept of a 'transformed area' and treat the beam as if it were only of one material. Thus, if E_1, E_2 (with, say, $E_2 > E_1$) are

[8] If this is not so, it is not at all difficult to deal with the case. The corresponding figure will not be skew-symmetric any more. The consequences are not difficult to explore. We, however, do not undertake this exercise here.

[9] This used to be inexpensive, but is not so any more!

the Young's moduli, respectively, the area A_2 can be transformed as $A_1 \times (E_2/E_1)$. Now the transformed A_2 will be larger than the real, or original, A_2. The modified calculations can thus be brought into the framework of the more common or convenient beams with only one material.

One of the commonest examples of a beam with two materials is a reinforced concrete beam. Steels rods are placed in the region of tensile stresses. Concrete is weak in tension, and it is common practice in design to neglect or disregard the presence of concrete in the tensile region. If the bonding between the steel rods and the surrounding concrete is proper — the bonding must be proper, so that the steel rods do not 'slip' — the stress in the steel bar is $E_{steel}/E_{concrete}$ times the stress in the concrete. The position of the neutral axis and the bending moment (or the resisting moment) can be worked out from the stress distribution across the cross-section. There are several important aspects to discuss, but we cannot do so in this book. (All this will be discussed in great detail in courses on reinforced concrete design and analysis.)

This approach if often employed in the calculations on Reinforced Cement Concrete (RCC) beams. Fig. 2.5 refers to such a case. The figures refer to an RCC beam. Steels

(a) Cross-section with two materials

(b) Distribution of strain

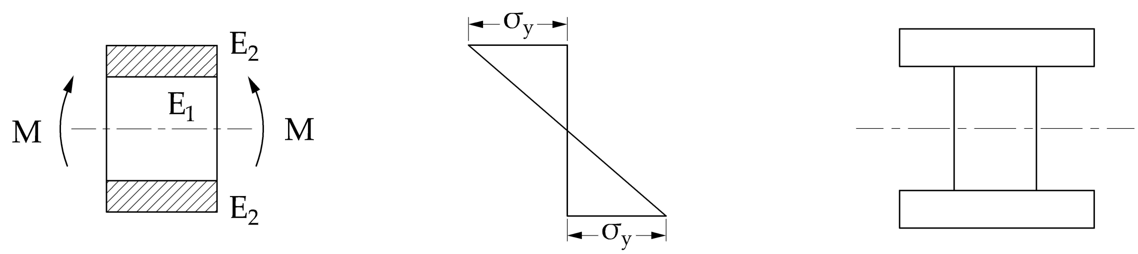
Figure 2.5: A simplified cross-section of an RCC beam and the corresponding strain distribution are shown in Fig. 2.5a. The stress distribution is shown in [Fig. 2.5b]. The tensile stress in the steel bars is much higher than that in the concrete, because E_{steel} is much higher than $E_{concrete}$. Generally the tensile stress in the concrete is neglected. The neutral axis is not at a depth of $d/2$.

rods are placed in the tensile region (at the bottom of the cross-section in this figure)[10]. The distribution of strain is as shown. The stresses are, however, not linearly distributed. The neutral axis is shifted upwards [Figs 13.4, p. 13-18].

Beams with Wide Cross-sections

In the simple bending problem that we have seen, the cross-section is as in Fig. 2.7a. If, instead, we have cross-sections as in Fig. 2.7b with the width $b >>$ than the depth d, there are some changes to be made. In the former cases as in Fig. 2.7a, the cross-sections are free to have lateral contractions (in the top layers where the bending stress is tensile), and extensions (in the bottom layers where the bending stress in compressive). However, when

[10] In other examples, the steel bars could be at the top; it depends on where tensile stresses are developed. There are many details that cannot be explained here. For students of Civil Engineering, reinforcement concrete design is a subject of great importance.



(a) Two materials (b) Distributions of strain and stress (c) 'Transformed area'

Figure 2.6: A cross-section of a beam with two materials [Fig. 2.6a] and the corresponding strain and stress distributions are shown in Fig. 2.6b. An 'equivalent cross-section' based on the concept of a 'transformed area' is shown in Fig. 2.6c.

(a) A usual cross-section (b) Beam with a wide cross-section

Figure 2.7: Fig. 2.7a shows an example of a usual cross-section for which the simple theory of bending is worked out. Fig. 2.7b shows the case of a wide cross-section. Now the lateral strain is inhibited and, consequently, the theory is to be modified.

the cross-section is very wide (with $b >> d$), the lateral strain is restricted or inhibited. The physical effect of this is that the beam will now be stiffer.

If we assume that $e_{zz} = 0$, that is, the lateral strain is completely prevented — actually the lateral strain may not be *completely* prevented, only reduced considerably —

$$e_{zz} = \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \nu\sigma_{yy})] = 0.$$

Further, we may assume that $\sigma_{yy} = 0$ — the various layers do not press on one another —

$$\sigma_{zz} = \nu\,\sigma_{xx}.$$

If this is substituted in the first of the generalised Hooke's law $e_{xx} = \dfrac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz}]$ with $\sigma_{yy} = 0$, we obtain

$$e_{xx} = \frac{1-\nu^2}{E}\,\sigma_{xx}.$$

The bending stress σ_{xx} given by the Euler-Bernoulli equation is

$$\sigma_{xx} = \frac{M}{I_{zz}}\,y.$$

Thus, we have

$$e_{xx} = \frac{1-\nu^2}{EI_{zz}}\, My. \tag{2.5}$$

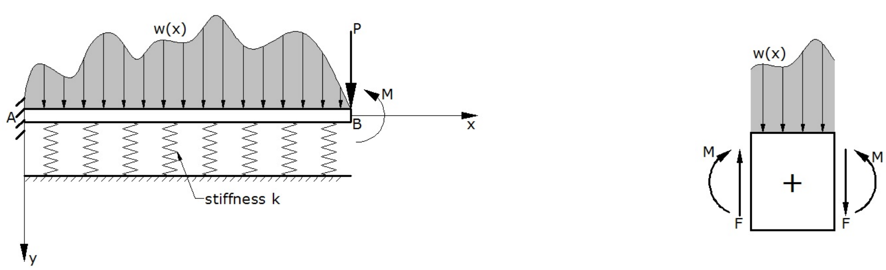
From the usual Euler-Bernoulli equation, we find that

$$\sigma_{xx} = \frac{M\,y}{I_{zz}} \qquad \longrightarrow \qquad e_{xx} = \frac{M\,y}{EI_{zz}}. \tag{2.6}$$

On comparing Eqs (2.6) (applicable for the usual simple cases) and (2.5) (applicable for the case of wide cross-sections), we note that the strain e_{xx} is reduced by the factor $(1-\nu^2)$. With this correction, the changes to be made can be understood. For example, the governing differential equation for bending becomes

$$EI_{zz}\,\frac{d^2y}{dx^2} = (1-\nu^2)\,M. \tag{2.7}$$

Beams on Elastic Foundations



(a) A beam on an elastic foundation (b) An element of the beam

Figure 2.8: A beam AB, resting on an elastic foundation with the Winkler constant (spring stiffness) k, is acted upon by an external load distribution $w(x)$. A concentrated load P and a bending moment M are applied at the ends. The left end A is fixed, while the right end B is free.

Shown in Fig. 2.8a is a beam AB on an elastic foundation (continuous elastic support). The foundation is assumed to be a Winkler foundation[11] with a Winkler constant (spring stiffness) k. In addition to an external distributed load $w(x)$, there is a concentrated load P and a bending moment M applied at the right end B. The left end A is fixed.

The governing differential equation is

$$\frac{d^2}{dx^2}\left[EI\,\frac{d^2y}{dx^2}\right] + ky = w(x) \qquad (0 < x < l). \tag{2.8}$$

[11] It is common practice to assume a Winkler foundation when beams on elastic foundations are treated. A railroad track, a beam cast integrally on a grid of other structural members, and even some problems in shells are cases where the theory of beams on elastic foundations is used. Objections can be raised on how such a foundation that is assumed to produce a reaction proportional to the deflection can realistically represent all these cases. Yet it is quite firmly established that in almost all of these applications such an assumption (as reaction $= ky$, where k is a constant called the Winkler constant) is eminently satisfactory. Hetéyni [7] contains pretty much everything that can be said on beams on continuous elastic support.

[We know from the Euler-Bernoulli theory of bending that the shear force, F; the bending moment, M; and the rate of loading, w; are related by the equations

$$\frac{dF}{dx} = -w; \qquad \frac{dM}{dx} = F; \qquad EI\frac{d^2y}{dx^2} = -M.]$$

For our problem, the beam is fixed at the left end $A\,(x = 0)$. Thus, the deflection and the slope at A are both zero. These are the prescribed boundary conditions.

$$y(x)\Big|_{x=0} \equiv y(0) = 0 \qquad \text{No deflection at } A \qquad \text{Prescribed b.c.} \tag{2.9}$$

$$\frac{dy}{dx}\Big|_{x=0} \equiv y'(0) = 0 \qquad \text{No slope at } A \qquad \text{Prescribed b.c.} \tag{2.10}$$

The differential equation, let us note, is of the fourth (4^{th}) order; it will, therefore, take four (4) boundary conditions. Two of them are the prescribed boundary conditions given above [Eqs (2.9), (2.10)]. The other two boundary conditions are the natural boundary conditions at the right end $B\,(x = l)$. Here we know neither the deflection, nor the slope. Accordingly, the natural boundary conditions prevail at B.

$$\left(EI\,\frac{d^2y}{dx^2}\right)\Big|_{x=l} = -M \qquad \text{Applied B.M. at } B \qquad \text{Natural b.c.} \tag{2.11}$$

$$\frac{d}{dx}\left(EI\,\frac{d^2y}{dx^2}\right)\Big|_{x=l} = -P \qquad \text{Applied concentrated force at } B \qquad \text{Natural b.c.} \tag{2.12}$$

Let us also note that the primary variables y and dy/dx are specified at the left end $A\,(x = 0)$. With the specification of these four boundary conditions — two prescribed boundary conditions at the left end $A\,(x = 0)$, and two natural boundary conditions at the right end $B\,(x = l)$ — the above equation (2.8) can be solved. The actual calculations, though simple and straightforward in principle, can be quite a burden.

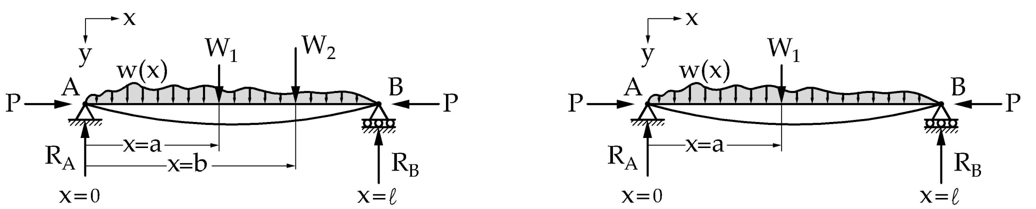
[We have referred to two kinds of boundary conditions, prescribed and natural. These must be understood in the light of the calculus of variations. Students are advised to learn at least the most elementary aspects of this extremely useful topic.]

Beam-Columns

Beam-columns are one-dimensional structural elements that carry transverse loads W_1, W_2, $\cdots$, etc., like a beam, and also axial compressive forces like $P - P$ like a column [Fig. 2.9a]. Such structural members are not uncommon, particularly in airplane structures. Beam-columns are special in the sense that superposition (of the separate effects of (i) W_1, W_2, $\cdots$ and of (ii) $P - P$) is not valid. The reason is not hard to understand. The axial forces introduce bending moments; their magnitudes depend decisively on the deflection. Thus, there is a cross effect — a coupling, as it were — between the effects of (i) the axial loads $P - P$, and of (ii) the transverse loads.

The bending moment in the two sections $(0 \leq x \leq a)$ and $(a \leq x \leq l)$ is given by

$$M = -\frac{Wb}{l}\,x - Py \qquad (0 \leq x \leq a) \tag{2.13a}$$



(a) A beam-column (b) A beam-column with a single load W_1

Figure 2.9: A beam column with the relevant details is shown in Fig. 2.9a. The same beam-column but with a single concentrated load W_1 is shown in Fig. 2.9b.

$$= -Wa + \frac{Wa}{l}\,x - Py \qquad (a \le x \le l) \tag{2.13b}$$

[From our earlier knowledge of the theory of beams, we know these equations already (with only marginal appropriate changes). M is the bending moment, EI the flexural rigidity, and P the axial compressive force. $y = y(x)$ is the vertical downward deflection. Any set of consistent units and sign conventions may be used. The main difference here is that the axial force P together with the deflection y contributes to the bending moment.]

The governing differential equation is, therefore,

$$EI\,\frac{d^2y}{dx^2} + Py = \frac{Wb}{l}\,x \qquad (0 \le x \le a) \tag{2.14a}$$

$$= Wa - \frac{Wa}{l}\,x \qquad (a \le x \le l) \tag{2.14b}$$

If we call $P/(EI) = k^2$, these equations can be rewritten as

$$\frac{d^2y}{dx^2} + k^2 y = \frac{Wb}{EI\,l}\,x \qquad (0 \le x \le a) \tag{2.15a}$$

$$= \frac{Wa}{EI} - \frac{Wa}{EI\,l}\,x \qquad (a \le x \le l) \tag{2.15b}$$

The solution of this equation defined differently in the two intervals is of the form

$$y = C_1 \sin kx + C_2 \cos kx + (\text{particular integral})_1 \tag{2.16a}$$

$$= C_3 \sin kx + C_4 \cos kx + (\text{particular integral})_2 \tag{2.16b}$$

There are four boundary conditions to determine these four arbitrary constants of integration. They are:

(a) at $x = 0$, $y = 0$;
(b) at $x = l$, $y = 0$;

(c) the slope dy/dx computed from Eqs (2.16a) and (2.16b) must be equal; and
(d) the deflection y computed from Eqs (2.16a) and (2.16b) must be equal.

The four constants can be evaluated using these boundary conditions. The procedure is simple and straightforward, though the algebra is tedious. These steps are not shown here. The deflection at mid-span for the simple special case when $a = l/2$ (that is, if W acts at mid-span) is

$$\delta \equiv y\Big|_{x=l/2} = \frac{W\,l^3}{48\,EI}\left[\frac{3(\tan u - u)}{u^3}\right], \quad \text{where} \quad u = \frac{kl}{2} = \frac{l}{2}\sqrt{\frac{P}{EI}}. \tag{2.17}$$

When $u = \pi/2$, the deflection becomes infinite!

$$\text{When } u = \frac{\pi}{2} = \frac{l}{2}\sqrt{\frac{P}{EI}}, \quad \text{i.e., when} \quad P = \frac{\pi^2\,EI}{l^2} = P_{cr},$$

the deflection becomes infinite! The parameter u is a measure of how close the axial load P is to the Euler buckling load $P_{cr} = \pi^2\,EI/l^2$. The expression $[\cdots]$ in Eq. (2.17) may be regarded as a magnification factor: the mid-span deflection of $Wl^3/(48\,EI)$ is magnified when there is an axial compressive load $P-P$!

We can next compute the deflections when (i) there is only W_1, and $P-P$; (ii) there is only W_2, and $P-P$; and (iii) there are both W_1 and W_2, and $P-P$. The procedure is similar, though more tedious. The result is insightful: superposition now holds, if the same axial load $P-P$ acts. Thus, we see the following important result.

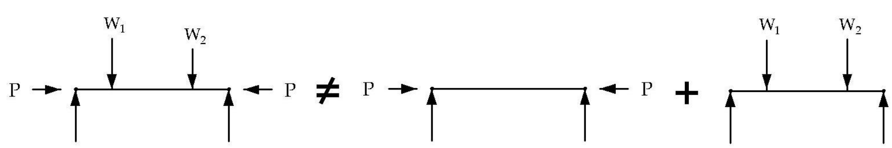


Figure 2.10: A beam-column: superposition as shown is not valid here.

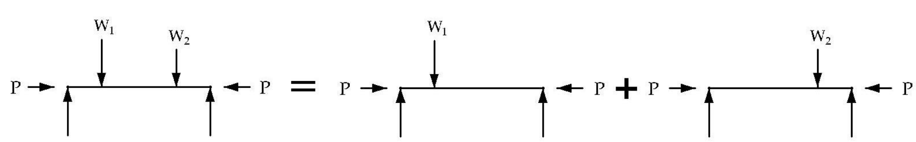
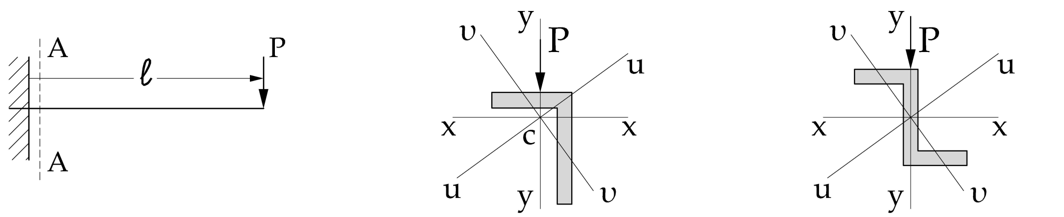


Figure 2.11: A beam-column: superposition as shown is valid here.

With this we close the discussion of beam-columns, and take up the important topic of unsymmetrical bending in the next section.

UNSYMMETRICAL BENDING

Let us consider the bending of a cantilever with two examples of cross-section: (i) an unequal angle iron, and (ii) a Z-section. Note that x and y are not principal axes (that is, $I_{xy} \neq 0$). The principal directions uu and vv are shown. The line of load — or more precisely, the



(a) A loaded cantilever (b) An angle iron cross-section (c) A Z as the cross-section

Figure 2.12: A cantilever with an end load P is shown in [Fig. 2.12a]. Two possible cross-sections, an angle iron [Fig. 2.12b] and a Z-section [Fig. 2.12c], along with the principal axes of inertia I_{uu} and I_{vv}, are shown. These are both cases of unsymmetrical bending as the line of the load — trace of the plane of the bending moment — is not along a principal axis of the second moment of area (principal axes of inertia).

trace of the plane of the applied bending moment — is along the y axis (non-principal axis), and not along the principal axis (uu or vv). The bending moment applied on a cross-section AA is $M = P\,l$. This M_x acts in the vertical plane, that is, about the axis xx. Let us try to obtain the bending stresses using the Euler-Bernoulli formula $\sigma/y = M_x/I_{xx} = E/R$. We now have the equilibrium equations as before.

Equilibrium:

$$
\begin{aligned}
\int_A \sigma\, dA &= 0 && \text{(no net axial force)}\\
\int_A (\sigma\, y)\, dA &= M_x && \text{(bending moment about the } x \text{ axis)}\\
\int_A (\sigma\, x)\, dA &= 0 && \text{(no net bending moment about the } y \text{ axis)}
\end{aligned}
$$

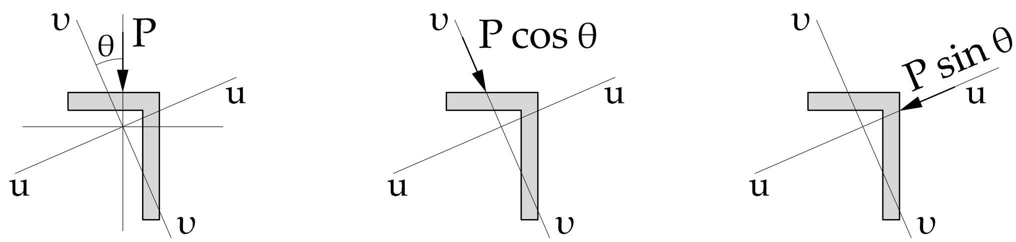
Using the expression for σ [Eq. (2.2] in these equilibrium equations, we are led to the following results.

$$
\begin{aligned}
\int_A \left(\frac{E}{R}\,y\right) dA &= 0 && \longrightarrow \quad \int_A y\, dA = 0 && \text{(first moment = 0!)}\\
\int_A \left(\frac{E}{R}\,y\right) y\, dA &= M_x && \longrightarrow \quad \frac{E}{R}\int_A y^2\, dA = M_x && \longrightarrow \quad \frac{E}{R} = \frac{M_x}{I}\\
\int_A \left(\frac{E}{R}\,y\right) y\, dA &= M_y = 0 && \longrightarrow \quad \frac{E}{R}\int_A xy\, dA = 0 && (x,\ y \text{ are principal axes!})
\end{aligned}
$$

Let us examine the last equation closely. What does it state? Physically it is an equation of equilibrium stating that there is no net bending moment M_y. But it leads to the conclusion that the (area) product of inertia $I_{xy} = 0$, which means that the x and y axes are principal! But we know that x and y axes are not principal axes. Where is the anomaly? Did we make any mistake?

Sure, we did. We applied the simple theory (of symmetrical bending) to these cases of loading. This demonstration forcibly tells us that the simple theory of bending is not valid here. This is a case of unsymmetrical bending.

What do we do? One way to deal with the problem is this. (i) To find the principal axes of inertia uu and vv, and the principal second moments of area (principal area moments of inertia) I_{uu} and I_{vv} (using Mohr's circle, or the equivalent transformation equations related to the inertia tensor); (ii) to resolve the bending moments M_u and M_v about the principal axes; and finally (iii) to change this one (relatively difficult) problem of unsymmetrical bending to two problems, each of the simple case of symmetrical bending. An important consequence is that the neutral axis is not xx, nor perpendicular to the line of loading, any more. We shall see this below. Referring to the three figures [Fig. 2.13], the (difficult)



(a) Vertical end load P (b) Load resolved along vv (c) Load resolved along uu

Figure 2.13: A (relatively difficult) problem in unsymmetrical bending [Fig. 2.13a] is converted into two (simple) problems of symmetrical bending [Figs 2.13b and 2.13c] by resolving the load P along the two principal axes of inertia uu and vv.

problem of unsymmetrical bending [Fig. 2.13a] is converted into two (simple) problems, each of symmetrical bending. The total stresses are obtained by adding up as

$$\sigma = \frac{M_u\, v}{I_{uu}} + \frac{M_v\, u}{I_{vv}} = \frac{(P\,\cos\theta)\, v}{I_{uu}} + \frac{(P\,\sin\theta)\, u}{I_{vv}}. \tag{2.20}$$

We can obtain the equation to the neutral axis by setting equal to zero the above expression for the total stress giving us

$$\frac{(P\,\cos\theta)\, v}{I_{uu}} + \frac{(P\,\sin\theta)\, u}{I_{vv}} = 0. \tag{2.21}$$

This neutral axis is not horizontal, nor is it perpendicular to the line of action of the load P! This equation (2.21) can, of course, be written in terms of x and y. An illustrative example is worked out later [p. 13-24] which will be helpful to understand the procedure.

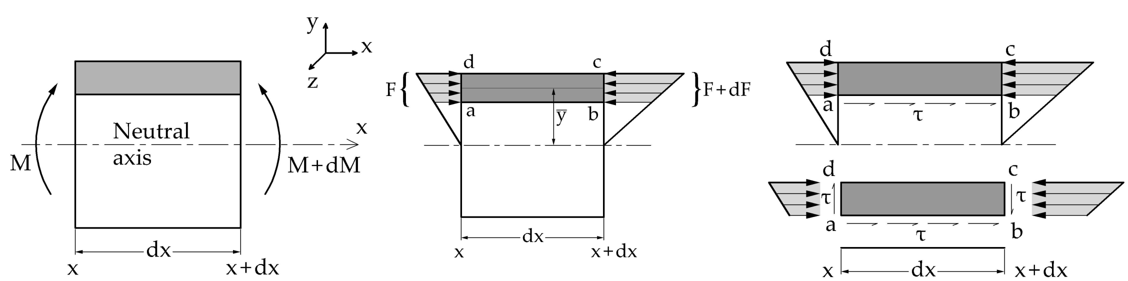
SHEAR CENTRE

Shear centre is of much importance to Civil, Mechanical and Aerospace engineers. For every cross-section used for a beam, there is a point called the shear centre, through which the external load must pass for the beam to bend without twisting. As an example, when

the wing of an aircraft bends due to the air pressure from below, it also twists[12]. It is necessary (i) to understand the theory behind this concept, (ii) to know the location of the shear centres of common cross-sections, and (iii) to be able to calculate and locate the shear centre for a given cross-section. Finally it is necessary to have a good qualitative understanding of bending and twisting[13].

To understand the concept of shear centre, we need to know the distribution of shear stresses (cross-shear) on the cross-section because of bending accompanied by a (transverse) shear force. We have studied this in our earlier first course on the mechanics of solids (or even in strength of materials), but we would still have a quick review of this, omitting several details. Students are advised to revise this topic using one or two *good* books.

Shear Stress (Cross-shear) on the Cross-section



(a) Unequal bending moments M and $M + dM$ (b) Unequal bending stresses on the two sides (c) Shear stress τ on the underside for force balance

Figure 2.14: A beam element of length dx is considered [Fig. 2.14a]. Because the bending moment is not a constant, the corresponding bending stresses are also different on the two faces of the element [Fig. 2.14b]. To keep the shaded portion in equilibrium [Fig. 2.14c], a shear stress τ — on the y plane in the x direction — must develop on the underside. The counterpart τ_{yx} — shear and complementary shear — will then act on the cross-section in the vertical direction. It is this shear stress τ_{xy} on the cross-section that we are after.

First we note that there is a (vertical) shear force on the cross-section, say, F. This means that the bending moment $M = M(x)$ is not a constant, because $dM/dx = F$. If we have a beam element [Fig. 2.14a], the bending moments are M and $M + dM$ on the two faces. It follows that the bending stresses are different on the two faces or cross-sections.

If we consider a small portion shown shaded in Figs 2.14b and 2.14c of the beam element, we can see that the bending stresses are different on the two faces: to the left at x and to

[12] This twist changes the angle of attack causing changes in the aerodynamics forces acting on the wing. The changed forces, in turn, lead to changes in the bending of the wings and the consequent twisting of the wing. There is thus a strong coupling between the bending and the associated twisting, and the associated aerodynamic forces acting on the wing. This situation can, under certain situations, lead to serious problems of flutter.

[13] Books on the theory theory of elasticity treat bending and twisting of bars together, twisting being a sub-problem of a larger general problem of bending.

the right at $x + dx$. Inasmuch as the bending moment $M + dM$ at $x + dx$ is larger than M at x, the bending stress also is correspondingly larger [Fig. 2.14b]. The shaded part must, of course, be in (horizontal) equilibrium. This requirement makes it essential that some shear stress τ is developed on the underside [Fig. 2.14c] to have equilibrium (in the horizontal direction). The counterpart of this shear stress τ_{yx} on the underside is τ_{xy} acting on the cross-section in the vertical direction. It is this vertical shear stress on the cross-section — the cross-shear on the cross-section due to bending — that we are after.

Once this argument is well understood, it is easy enough to make the calculations and obtain an explicit formula for this shear stress τ_{xy}.

Consider the horizontal equilibrium of the shaded portion *abcd*. To the right acts the bending stress on the cross-section x over the area *ad*, and to the left acts the similar, but larger, bending stress on the cross-section $x + dx$ over the area *bc*. These two are not equal. The resultant horizontal (longitudinal) force F on the face *ad* [Fig. 2.14b] acting to the right is less than the resultant horizontal (longitudinal) force $F + dF$ on the face *bc*. The difference dF can be computed by integrating the expressions for the bending stress on both faces. This difference dF is balanced by the shear stress τ (actually τ_{yx}) acting on the underside *ab* on the area $ab\times$ width $= dx \times b$. For equilibrium, $\tau \times b \times dx = dF$ where b is the width of the cross-section at the level (height) of the line *ab*. We can compute dF as

$$dF = q\,dx = \frac{dM}{I}\int_{abcd} y\,dA = \frac{dM}{I}A_{abcd} \times \bar{y} = \frac{dM}{I}\,Q$$

Q is the moment of the area *abcd* about the neutral axis. This dF acting to the left must be balanced by the shear force developed on the underside *ab*. Thus, $dF = \tau \times b \times dx$ where b is the width of the cross-section at the level (height) of the line *ab*. The shear stress τ on the cross-section — this is the sought after cross-shear — can be calculated as

$$\tau = \frac{dV}{b\,dx} = \frac{dM}{dx}\,\frac{A_{abcd}\,\bar{y}}{I\,b} = \frac{V\,Q}{I\,b} = \frac{q}{b}.$$

Q, we repeat, is the moment of the area above the line *ab* about the neutral axis, and V the vertical shear force on the cross-section.

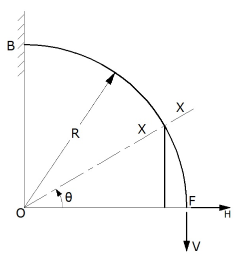
$$\tau = \frac{V\,Q}{I\,b} = \frac{q}{b} \qquad \left[\frac{F\,Q}{I\,b} = \frac{q}{b} \text{ if the vertical shear force is designated by } F\right]. \tag{2.22}$$

gives us the distribution of the shear stress on the cross-section. This we can take as the operating equation for such shear stress calculations. Although we have already studied this earlier, we shall clarify this further by two illustrative examples in a later chapter.

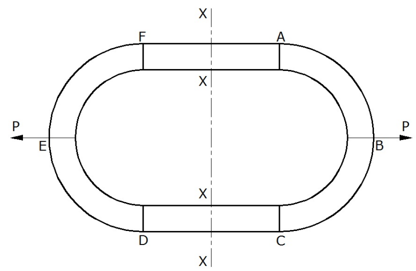
CURVED BEAMS

The theory of pure bending leading to the Euler-Bernoulli equation was developed for beams that are initially straight. When the beam is curved even before the bending moment is applied, there are some major departures in the theory. Crane hooks, chain links, C-clamps and curved frames of punching machines are some examples where the curved members are to be treated using the curved beam theory. We shall develop this theory leading to the so-called Winkler-Bach solution.

General Remarks



(a) A straight beam!



(b) A chain link

Figure 2.15: 'Straight' and curved beams. The first [Fig. 2.15a] is technically a straight beam! The second [Fig. 2.15b] is indeed a curved beam. What decides the issue is whether the depth of the cross-section is small (straight beam) or not (curved beam) compared to the radius of curvature. Thus, the Winkler-Bach theory is to be used for curved beams.

We again use the Love-Kirchhoff assumption: that cross-sections which are plane before bending continue to remain plane; they only rotate, but do not become crooked or wavy. We shall see that the curvature introduces qualitative differences. The most important of these differences is that (i) the distribution of strains and, therefore, of stresses is not linear across the cross-section, (ii) the neutral axis does not pass through the centroid of the cross-section. The distribution of the bending stresses is hyperbolic, and not linear, and the neutral axis is shifted from the centroidal axis towards the centre of curvature. We shall also see that these differences arise only if the depth of the beam is comparable to the radius of curvature. Thus, a thin circular bar (with the radius of curvature several times, say, 20 times, the depth of the cross-section) behaves like a straight beam in spite of its obvious appearance as curved.

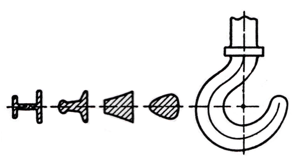


Figure 2.16: A crane hook

Fig. 2.16 shows a crane hook. The depth of the beam is not small compared to the radius of curvature. A few possible cross-sections are shown alongside.

The calculation of the property Z is a little complicated for such cross-sections.

We shall see that a certain property of the area of cross-section defined by

$$Z = -\frac{1}{A}\int_A \frac{y}{R+y}\, dA$$

is important in curved beam calculations. This is somewhat, but not quite, analogous to I, the second moment of the area of cross-section in straight beam theory. Z is dependent not only on the geometric property of the cross-section, but also on the radius of curvature R. Its numerical calculations can sometimes be tricky; unless special care is taken, the error

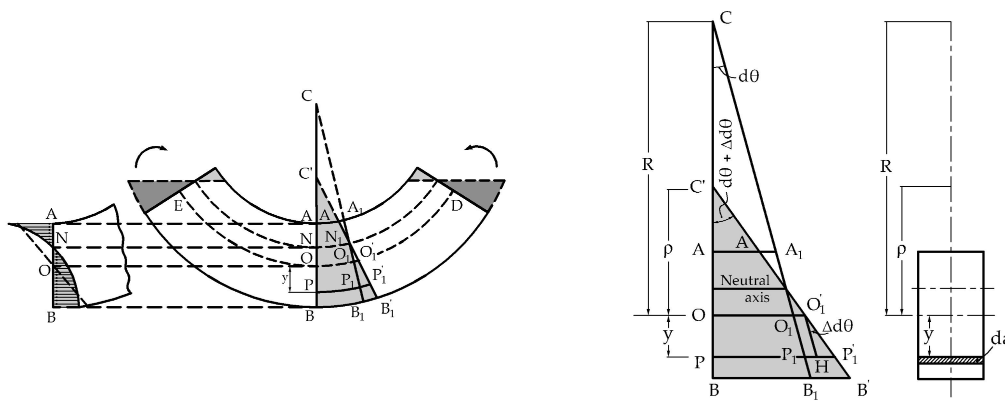
(a) A C-clamp (b) A crane hook

Figure 2.17: Two examples of a curved beam: a C-clamp and a crane hook

in the result can be serious. Additionally, in curved beam calculations the distance y is usually reckoned from the centroidal axis, and not from the neutral axis. This is because the neutral axis is not known *a priori*; it is known only after calculations. The centroidal axis, on the other hand, is known as soon as the cross-section is specified.

These are some of the departures from the straight beam theory. These facts will emerge as consequences of the theory that we are about to develop below known as the Winkler-Bach theory[14].

Winkler-Bach Theory



(a) An element before and after deformation (b) Geometry of deformation

Figure 2.18: A curved beam element is shown. Plane sections before deformation continue to remain plane even after deformation. The details of the deformation, the cross-section, and the stresses on it can be seen.

Let us consider a curved beam element. As always we assume that cross-sections which were plane before bending continue to remain plane even after bending. They do not become crooked; they only rotate. Thus, the line A_1B_1 rotates to become $A'B'$ [Fig. 2.18a]. The bottom fibres are stretched out, and the top ones shortened. As we go up from the bottom

[14]named after the German engineers E. Winkler and C. Bach

fibres to the top ones, there is one fibre that is neither elongated nor compressed. This is the neutral line NN_1. The cross-section AB, because of symmetry, stays where it is without any movement (rotation). Thus, the fibre PP_1 is stretched to become PP_1', P_1P_1' being the stretch. This fibre is at a distance of y measured from the centroidal axis (and *not from the neutral axis*) [Fig. 2.18a]. The cross-section is taken, to be specific and for convenience, as a rectangle [Fig. 2.18b]. The coordinate axes x, y, z are as marked in Fig. 2.18b passing through the centroid O of the cross-section.

Now we shall consider the deformations. As in the simple straight beam theory, the deformation (displacement, elongation or compression) is again proportional to the distance from the neutral axis, but the strains are not! This is because the fibres are not of equal lengths: strain = change in length ÷ original length; the original lengths of the various fibres are different! This is the first major departure from the straight beam theory. We shall now take a closer look at the geometry of deformation, and work out the expression for the strain [Figs 2.18].

The general procedure is natural and straightforward. Once the variation of the strain e with y (measured from the centroidal axis) [Figs 2.18] is obtained, we can use the constitutive equation $\sigma = E\,e$ (simple Hooke's law) and obtain the expression for the variation of the stress σ with y. Next we use the equilibrium requirements and proceed further.

General procedure:
Our plan is to look at the geometry of deformation clearly [Fig. 2.17a] and to work out an expression for the strain e at any fibre distant *y from the centroidal axis*. We then obtain the corresponding expression for the stress σ using the constitutive equation $\sigma = E\,e$. Next, if we use the equilibrium requirements (equilibrium equations) we can obtain the expression for the bending moment M. By processing these expressions we can obtain the Winkler-Bach formula expressing the bending stress σ in terms of the bending moment and the properties of the cross-sectional area. (This formula takes the place of the Euler-Bernoulli equation of the straight beam theory.) This formula will be derived below.

Detailed calculations:
The strain of the fibre at the centroidal axis serves as some kind of reference. Let us call it e_c. Note once again that this would be zero if we had considered the neutral axis!

$$e_c = \frac{\text{change in length}}{\text{original length}} = \frac{OO_1' - OO_1}{OO_1} = \frac{O_1O_1'}{OO_1} \quad \longrightarrow \quad O_1O_1' = e_c\,ds = e_c\,R\,d\theta$$

The strain of a fibre at a distance of y from the centroidal axis is now calculated as

$$e\Big|_y = \frac{\text{change in length}}{\text{original length}} = \frac{PP_1' - PP_1}{PP_1} = \frac{P_1P_1'}{PP_1} = \frac{P_1H + HP_1'}{PP_1} = \frac{O_1O_1' + HP_1'}{PP_1}.$$

Let us note that $O_1O_1' = e_c\,R\,d\theta$; $HP_1' = O_1'H(\Delta\,d\theta) = y\,\Delta\,d\theta$; and $PP_1 = (R+y)\,d\theta$.

The expression for the strain e at any distance y from the centroidal axis is, thus,

$$e\Big|_y = \frac{e_c\,R\,d\theta + y\,\Delta\,d\theta}{(R+y)\,d\theta} = \frac{R\,e_c + y(\Delta\,d\theta/d\theta)}{(R+y)}.$$

If we call $(\Delta\, d\theta/d\theta)$ as ω for convenience, we obtain

$$\begin{aligned} e\Big|_y &= \frac{R\,e_c + y\,\omega}{R+y} = \frac{R\,e_c + (y\,e_c - y\,e_c) + y\,\omega}{R+y} = \frac{e_c(R+y) + (\omega - e_c)y}{R+y} \\ &= e_c + (\omega - e_c)\frac{y}{R+y}. \end{aligned} \tag{2.23}$$

This is the variation of the strain with the distance y. Accordingly, the variation of the bending stress σ is obtained as

$$\sigma\Big|_y = E\,e = E\left[e_c + (\omega - e_c)\frac{y}{R+y}\right]. \tag{2.24}$$

Now let us consider the equilibrium requirements. They are

$$\int_A \sigma\,dA \quad = 0 \qquad \text{(no net axial force)} \sum F_x = 0; \tag{2.25a}$$

$$\int_A (\sigma\,y)\,dA \quad = M_z \qquad \text{(bending moment about the } z \text{ axis)} = M_z. \tag{2.25b}$$

Substituting the expression for σ [Eq. (2.24], these equilibrium requirements give us

$$\int_A \sigma\,dA = \int_A E\left[e_c + (\omega - e_c)\frac{y}{R+y}\right]dA = 0 \tag{2.26a}$$

$$M = \int_A (\sigma\,y)\,dA = \int_A E\,y\left[e_c + (\omega - e_c)\frac{y}{R+y}\right]dA = 0. \tag{2.26b}$$

As the Young's modulus E is a constant — this is so if there is only one material with the usual simplifying assumptions (i) the behaviour is linear; and (ii) the values of E in both tension and compression are the same — the above two equations [(2.26a) and (2.26b] appear as follows.

$$e_c \int_A dA = -(\omega - e_c)\int_A \frac{y}{R+y}\,dA;\ \text{and} \tag{2.27a}$$

$$M = E\left[e_c \int_A y\,dA + (\omega - e_c)\int_A \frac{y^2}{R+y}\,dA\right]. \tag{2.27b}$$

We can simplify these equations by using a notation Z and noting the following.

$$Z = -\frac{1}{A}\int_A \frac{y}{R+y}\,dA; \qquad \int_A dA = A; \qquad \int_A y\,dA = 0 \ \text{(first moment = 0.)} \tag{2.28}$$

$$\int_A \frac{y^2}{R+y}\,dA = \int_A \left[y - R\,\frac{y}{R+y}\right]dA = -R\int_A \frac{y}{R+y}\,dA = Z\,A\,R \tag{2.29}$$

With these Eqs (2.27a) and (2.27b) are simplified as

$$e_c = (\omega - e_c)Z; \tag{2.30a}$$

$$M = E(\omega - e_c)Z\,A\,R. \tag{2.30b}$$

Solving for e_c and ω — that is, expressing these in terms of M, E, A and R, we obtain

$$\omega - e_c = \frac{M}{E\,Z\,A\,R} \qquad e_c = \frac{M}{E\,A\,R} \qquad \omega = \frac{1}{E\,A}\left(\frac{M}{R} + \frac{M}{R\,Z}\right).$$

If we substitute these expressions in Eq. (2.24), we arrive at the equation

$$\sigma = \frac{M}{A\,R}\left(1 + \frac{1}{Z}\,\frac{y}{R+y}\right) \qquad \left(\text{where} \quad Z = -\frac{1}{A}\int_A \frac{y}{R+y}\,dA\right). \tag{2.31}$$

This is the Winkler-Bach formula giving the bending stress in terms of the applied bending moment and the properties of the cross-section[15]. [See the illustrative examples given in a later chapter.]

Calculation of Z:

The calculation of the numerical value of Z is tricky even for simple sections like a rectangle of a circle[16]. For trapezia, and combinations of rectangles, analytical methods are possible. For more complicated shapes, numerical integration will have to be used.

If we use the formula

$$Z = -\frac{1}{A}\int_A \frac{y}{R+y}\,dA$$

and try to evaluate this by integration — which is essentially a summation procedure — the process involves addition of a series of terms. These are positive where y is positive, and negative where y is negative. The procedure leads to finding the (small) difference between two (relatively large) numbers. This can lead to serious errors unless special efforts are taken to obviate the difficulty[17]. The powerful and efficient Gaussian quadrature is much better than the trapezoidal rule or even the Simpson's rule for numerical integration. Graphical integration used to be popular some time ago, but not any more, for a good reason.

What happens when R is very large?

When R is very large, the curvature effects fade away as it were. It is then sufficient to use the simple straight beam theory. (This is what we do in cases like the beam shown in Fig. 2.15a, where in spite of the obvious curvature of the beam, the straight beam theory can be applied!) We shall now show that this is so.

[15] The influence of [12] can be seen throughout this section on the topic of Curved Beams. This was one of the books that my teacher, the distinguished professor C.N. Lakshminarayana of IIT Kharagpur, used when we were students. The help is gratefully acknowledged. That first love for the book still remains with me. There is a later edition of the book, but I am still proud of my old copy. Fond memories!

[16] Calculation of Z for these cross-sections is shown later on pp.13-29, 13-30.

[17] One method which is successfully used is to rewrite Z as

$$Z = \frac{1}{A\,R}\int_A \frac{y^2}{R+y}\,dA$$

where, unlike the earlier case, the integrand is always positive. Now in carrying out the integration process, there is no more the difficulty indicated above. We do not propose to give further details here.

To do so, let us begin with the Winkler-Bach formula, and process it as shown below.

$$\sigma = \frac{M}{A\,R}\left(1+\frac{1}{Z}\,\frac{y}{R+y}\right) = \frac{M}{A\,R} + \frac{M}{ZAR}\,\frac{y}{R+y}$$
$$= \frac{M}{A\,R} + \frac{M}{\int_A \frac{y^2}{R+y}\,dA}\,\frac{y}{R+y} = \frac{M}{A\,R} + \frac{M\,y}{\left(1+\frac{y}{R}\right)\int_A \frac{y^2\,dA}{1+y/R}}$$

$$\text{If } R \longrightarrow \infty,\ \sigma \longrightarrow \frac{M\,y}{\int_A y^2\,dA} = \frac{M\,y}{I},$$

which is the straight beam formula.

Further beyond:
What we have obtained is the formula for bending (circumferential) stress. In addition, there are radial stresses. These are zero on the outer (convex) and the inner (concave) surfaces. They are small inside for solid cross-sections like trapezoidal, rectangular and circular ones. However, they can be large and serious in the web for cross-sections like I, H and T. These can be analysed without much difficulty, but we do not propose to do this here[18].

In the next chapter, we shall discuss the index notation.

[18] There are many details that must be understood by an engineer when he designs curved beams. But we will not discuss them here, because this is only a gentle introduction to some advanced topics in the mechanics of solids.

Chapter 3

INDEX NOTATION

In this chapter, we shall consider the index notation. The summation convention, which is extensively used, is part of it. First we shall see why we need to use the index notation.

INTRODUCTION: WHY INDEX NOTATION?

The index notation and the summation convention are so much a part of tensor analysis that they are sometimes referred to as the tensor notation. We shall see how these, together with the Kronecker delta and the permutation symbols (defined later below), help us write several important equations concisely. Using these methods we can quickly arrive at results that are otherwise long and complicated. If we follow some systematic procedures when manipulating these symbols and indices, we can work out complicated expressions correctly without straining ourselves; good notations seem to be able to guide us to correct results without much intellectual effort.

Let us now see how the index notation, together with two conventions, enables us to pack a large number of similar looking equations in a capsule.

INDEX NOTATION

Let us consider a set of linear algebraic simultaneous equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= c_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= c_3. \end{aligned}$$

These long equations (in matrix notation $\boldsymbol{Ax} = \boldsymbol{c}$) can be abbreviated as

$$\sum_{j=1}^{3} a_{1j}x_j = c_1, \quad \sum_{j=1}^{3} a_{2j}x_j = c_2, \quad \sum_{j=1}^{3} a_{3j}x_j = c_3, \quad \text{or as}$$

$$\sum_{j=1}^{3} a_{ij}x_j = c_i, \quad i = 1, 2, 3.$$

[Generally only subscripts such as a_{ij} are used when we work with cartesian tensors; superscripts such as a^{ij}, or mixed superscripts and subscripts such as a^i_j are used only when dealing with general tensors. When superscripts are used such as, for instance, P^1, P^2, P^3, we should realise that these numbers $1, 2, 3$ are superscripts, and not powers. Thus, P^2 and P^3 are not P squared and P cubed. This fact would be clear from the context. Thus, we will use only subscripts in this book.]

Such sets of equations appear often enough that it is found convenient to introduce two conventions. The same repeating pattern makes it possible to abbreviate the above equations further as

$$a_{ij}x_j = c_i, \tag{3.1}$$

where it is understood, according to the two conventions, that
(i) there is summation over the repeated index j, and that
(ii) the equation is valid for every (applicable) possible value of the index i.

In the above example, the index j is repeated in the term $a_{ij}x_j$ — let us call this a term, even though it is really the sum of three separate terms — and, thus, according to the convention (i) above, there is implied summation over the repeated index j. Furthermore, the equation (3.1) — let us call this an equation, even though it is really three separate equations[1] — holds, according to the convention (ii) above, for every possible value[2] of the index i. The repeated index j is called a dummy index, because $a_{ij}x_j$ and, say, $a_{ik}x_k$ mean the same[3]. The index i is called the free (identifying) index. These conventions are called Einstein's conventions. To avoid ambiguity, we must be sure to see that the same index does not appear more than twice in any 'term'.

Generally Latin letters are used for the indices as subscripts (and superscripts). Sometimes Greek letters are also used[4]. The letters at the end of the alphabet such as x, y, z and u, v, w are not generally used for these indices serving as subscripts (and superscripts). This is not a rule; it is a common practice just as we generally use the letters x, y, z, $\cdots$ for variables, and a, b, c, $\cdots$ for constants.

We can see readily that a matrix $\boldsymbol{A}$ can be represented conveniently by its components a_{ij}, and its transpose $\boldsymbol{A}^T$ by a_{ji}. The (row by column) multiplication of two matrices $\boldsymbol{A}_{m\times n}$ $(a_{ij},\ i = 1, 2, 3, \cdots, m;\ j = 1, 2, 3, \cdots, n)$ and $\boldsymbol{B}_{n\times p}$ $(b_{ij}, i = 1, 2, 3, \cdots, n;\ j =$

[1] Shall we call this a three-in-one, if it is not blasphemous to consider such a trinity?

[2] Applicable in the given situation. If there are three equations, $i = 1, 2, 3$.

[3] In this regard, it is analogous to the (dummy) variable of integration in a definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy.$$

[4] For example, when we discuss the theory of shells, we encounter a situation of a two-dimensional surface (the curved middle surface of the shell) embedded in the surrounding three-dimensional flat surface, R^2 embedded in E^3. In such cases, to distinguish clearly between R^2 and E^3, the Greek letters such as α and β are used to refer to the Riemannian space R^2 (curved surface of the shell) $(\alpha, \beta = 1, 2)$, while the Latin letters such as i, j, k refer to the Euclidean space E^3 $(i, j, k = 1, 2, 3)$.

$1, 2, 3, \cdots, p$) can be indicated easily and naturally as $a_{ij}\, b_{jk}$ which is convenient for implementation in a computer program.

Symmetry and skew-symmetry of (square) matrices can be defined and represented as

$$e_{ij} = e_{ji} \quad \text{(symmetric)}, \quad \text{and} \quad \omega_{ij} = -\omega_{ji} \quad \text{(skew-symmetric)}.$$

The elements on the leading diagonal (where $i = j$) of a skew-symmetric matrix like ω_{ij} must necessarily be zero.

The well known result that any (square) matrix can be written as the sum of a symmetric and a skew-symmetric one is easily demonstrated as follows.

$$a_{ij} = \frac{1}{2}\,(a_{ij} + a_{ji}) + \frac{1}{2}\,(a_{ij} - a_{ji}) \equiv e_{ij} + \omega_{ij},$$

where

$$e_{ij} = \frac{1}{2}\,(a_{ij} + a_{ji}) = \frac{1}{2}\,(a_{ji} + a_{ij}) = e_{ji} \quad \text{(symmetric); and} \tag{3.2}$$

$$\omega_{ij} = \frac{1}{2}\,(a_{ij} - a_{ji}) = \frac{1}{2}\,(-a_{ji} + a_{ij}) = -e_{ji} \;\text{(skew-symmetric).} \tag{3.3}$$

A useful result that we often need to use when manipulating symmetric and skew-symmetric matrices in index notation is $e_{ij}\omega_{ij} = 0$.

$$\begin{aligned}
e_{ij}\,\omega_{ij} &= e_{ji}\,\omega_{ij} \;(\text{symmetry}, e_{ij} = e_{ji}),\\
&= -e_{ji}\,\omega_{ji} \;(\text{skew-symmetry}, \omega_{ij} = \omega_{ji}),\\
&= -e_{mn}\,\omega_{mn} \;(i, j \text{ are both dummy indices; replace them by } m, n)\\
&= -e_{ij}\,\omega_{ij} \;(\text{for the same reason, replace them by } i, j),\\
&= 0 \;(\text{something is equal to its own negative; each has to be zero}).
\end{aligned} \tag{3.4}$$

KRONECKER DELTA

We shall see that it is very convenient to use Kronecker[5] delta defined as

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

[5] Leopold Kronecker (Dec. 1823 - Dec. 1891) was a wealthy and influential German mathematician. His remark “Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk” (“God created the integers; everything else is the work of man.”) is well known. His major contributions are in elliptic functions and the theory of algebraic numbers. He had unbelievably unusual, and sometimes seriously wrong, academic convictions, and had bitter antagonism with several famous mathematicians including Georg Cantor. He was the leader of the so-called ‘intuitionists’ (intuitionism stressing that intuition has priority over logic in mathematics) in the battle of wits against the ‘formalists’ led by David Hilbert. Hilbert referred to him as ‘Verbotsdiktator’ (forbidding dictator).

We begin to see the convenience immediately. This δ_{ij} represents a unit matrix in index notation. For a unit matrix (3×3, or 2×2, which is usually the case for our applications),

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \qquad \boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For all the elements on the leading diagonal where i = j, the value is 1; for the off-diagonal elements ($i \neq j$), the value is 0. We note in passing that a hydrostatic state of stress is represented in matrix form as

$$\begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} = -p \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -p\boldsymbol{I},$$

and in index notation as $-p\,\delta_{ij}$.

The Kronecker delta often plays the role of a 'substitution operator'. To see this, let us consider a 'term' $a_{ij}\,\delta_{jk}$. As the index j is repeated, there is running summation over j. Thus,

$$a_{ij}\,\delta_{jk} = \sum_{j=1}^{n} a_{ij}\,\delta_{jk} = a_{i1}\,\delta_{1k} + a_{i2}\,\delta_{2k} + \cdots + a_{in}\,\delta_{nk}.$$

All these terms except one vanish because $\delta = 0$ except when its subscripts are equal. As k is a free index, it can take up different values in *different* expressions. Thus, we cannot state which term (the first, second, $\cdots$, etc.) survives. However, we can be sure that only when j hits the value of k (it does not matter what value k takes), that term alone survives. We can, therefore, conclude that

$$a_{ij}\,\delta_{jk} = a_{ik}. \tag{3.5}$$

The effect of 'multiplying' a_{ij} with δ_{jk} is simply this: the dummy index j in a_{ij} is substituted by the free index k in the Kronecker delta. Hence the name substitution operator. The matrix equivalent of Eq.(3.5) is $\boldsymbol{AI} = \boldsymbol{A}$.

Let us see one more simple example which might appear as contrived and trivial at first sight. If x_1, x_2, x_3 are *independent* variables, it is obvious that $\partial x_1/\partial x_2, \partial x_2/\partial x_3, \partial x_3/\partial x_1$, etc., are all zero. It is also obvious that $\partial x_1/\partial x_1, \partial x_2/\partial x_2, \partial x_3/\partial x_3$ are all equal to 1. That is, $\partial x_i/\partial x_j$ vanishes if the two indices i and j are different, but is equal to 1 when i and j have the same value $(1, 2$ or $3)$. Thus, $\partial x_i/\partial x_j = \delta_{ij}$

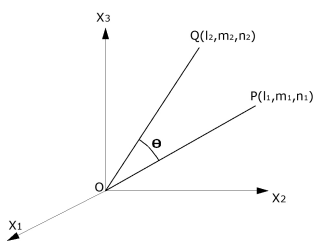
When we discuss tensors in general, transformation of the coordinates and the induced transformation of the components of a tensor in these various coordinate systems are of fundamental importance. When the axes are changed from the 'old' ones x_i to the 'new' ones x'_i, we would need to consider functional relations of the kind $x'_i = x'_i(x_1, x_2, x_3)$ and $x_i = x_i(x'_1, x'_2, x'_3)$. It would be necessary in this context to compute the partial derivatives $\partial x'_i/\partial x_j$ (and/or $\partial x_i/\partial x'_j$), and during simplifications the results in the example considered will be useful.

AN EXAMPLE: RECTANGULAR ORTHOGONALITY OF AXES

We shall see, as an example, how the index notation, along with the two conventions mentioned above, can be used to advantage in writing the conditions of orthogonality of two sets of rectangular cartesian coordinate axes[6]. This example is of importance in the study of our subject (as in the transformation of stress components). But first we need to recapitulate two simple, but important results relating to the direction cosines.

Direction Cosines: Two Simple Results

$$\cos(Ox_1, OP) = l_1, \qquad \cos(Ox_2, OP) = m_1, \qquad \cos(Ox_3, OP) = n_1 \tag{3.6a}$$
$$\cos(Ox_1, OQ) = l_2, \qquad \cos(Ox_2, OQ) = m_2, \qquad \cos(Ox_3, OQ) = n_2 \tag{3.6b}$$



The following two equations are very useful.

Result (a): $l^2 + m^2 + n^2 = 1$
Result (b): $l_1\, l_2 + m_1\, m_2 + n_1\, n_2 = \cos\theta$

Figure 3.1: The direction cosines enjoy these two useful relations.

Result (a): $l^2 + m^2 + n^2 = 1$
The components of OP along the coordinate directions are, accordingly,

$$OP \cos(Ox_1, OP) \equiv OP\ l_1 \tag{3.7a}$$
$$OP \cos(Ox_2, OP) \equiv OP\ m_1 \tag{3.7b}$$
$$OP \cos(Ox_3, OP) \equiv OP\ n_1. \tag{3.7c}$$

If these three components are resolved back again along OP and added, we must obviously obtain the same original length OP. Thus,

$$(OP\ l_1)\, l_1 + (OP\ m_1)\, m_1 + (OP\ n_1)\, n_1 = OP, \tag{3.8}$$

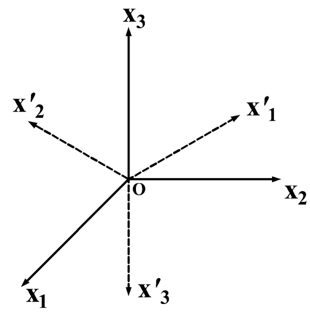
showing that $l_1^2 + m_1^2 + n_1^2 = 1$. Thus, the direction cosines (l, m, n) satisfy the relation $l^2 + m^2 + n^2 = 1$.
Result (b): $l_1\, l_2 + m_1\, m_2 + n_1\, n_2 = \cos\theta$
Next, let us resolve the components [See Eqs (3.7a), (3.7b) and (3.7c).] along the direction OQ and add up. Then we obtain

$$(OP\, l_1)\ \cos(Ox_1, OQ) + (OP\, m_1)\ \cos(Ox_2, OQ) + (OP\, n_1)\ \cos(Ox_3, OQ), \tag{3.9}$$

[6] Only such transformations from one rectangular cartesian coordinate system to another (rectangular cartesian coordinate system) are relevant when we work with only cartesian tensors, and not with general tensors.

New coordinates \ Old coordinates	x_1	x_2	x_3
x'_1	a_{11}	a_{12}	a_{13}
x'_2	a_{21}	a_{22}	a_{23}
x'_3	a_{31}	a_{32}	a_{33}

Table 3.1: Table of direction cosines



which must be the same as the projection of the length OP along OQ, which is obviously $OP\,\cos(OP, OQ)$. We, thus, obtain the result

$$(OP\,l_1)\,l_2 + (OP\,m_1)\,m_2 + (OP\,n_1)\,n_2 = \cos(OP, OQ) \equiv \cos\theta, \tag{3.10}$$

showing that

$$l_1\,l_2 + m_1\,m_2 + n_1\,n_2 = \cos\theta \equiv \cos(OP, OQ). \tag{3.11}$$

There are two important, special cases of this equation (3.11): (i) when $\theta = 0$ (OP and OQ coincide), and (ii) when $\theta = \pi/2$ (OP and OQ are perpendicular — orthogonal — to each other). In these two special cases, we have

$$l^2 + m^2 + n^2 = 1, \text{ when } l_1 = l_2 = l;\ m_1 = m_2 = m;\ n_1 = n_2 = n; \text{ and} \tag{3.12a}$$

$$l_1\,l_2 + m_1\,m_2 + n_1\,n_2 = 0, \text{ when } \theta = \pi/2. \tag{3.12b}$$

These are the two simple, but important results mentioned in the caption of this subsection.

Next we shall see the matrix of the direction cosines.

Matrix of the Direction Cosines

Let us consider two sets of rectangular cartesian coordinate systems, the 'old' x_i and the 'new' x'_i $(i = 1, 2, 3)$. (See the figure by the side of Table 3.1.) The orientation of the new coordinate system x'_i with respect to the old one x_i is given by the direction cosine matrix given in Table 3.1.

We shall now see how the index notation can be conveniently used to write down the orthogonality relations[7].

(a) x'_1, x'_2, x'_3 are mutually orthogonal.

These 'new' axes are mutually orthogonal: their direction cosines w.r.t. the 'old' axes x_1, x_2, x_3 should satisfy the orthogonality conditions. This case is examined below.

[7] These are simple to understand. However, experience shows that they are also confusing unless we are mentally alert, at any rate, when we are learning this for the first time. Therefore, we shall develop the equations slowly and patiently.

Step 1:
Let us observe that the axis x'_1, with the direction cosines (a_{11}, a_{12}, a_{13}) w.r.t. the 'old' axes (x_1, x_2, x_3), is perpendicular to the axis x'_2, with the direction cosines (a_{21}, a_{22}, a_{23}) w.r.t. the 'old' axes (x_1, x_2, x_3). Thus, we see from Eq: (3.12b) that

$$a_{1\underline{1}}\, a_{2\underline{1}} + a_{1\underline{2}}\, a_{2\underline{2}} + a_{1\underline{3}}\, a_{2\underline{3}} = 0. \tag{3.13}$$

This is the same as Eq. (3.12b) with the direction cosines suitably interpreted so as to be applicable to this context. Note that the second subscripts[8] in the three terms are $11, 22, 33$. Recalling the summation convention, this equation in longhand written as

$$a_{1\underline{1}}\, a_{2\underline{1}} + a_{1\underline{2}}\, a_{2\underline{2}} + a_{1\underline{3}}\, a_{2\underline{3}} \equiv \sum_{j=1}^{3} a_{1j}\, a_{2j} = 0$$

may be written in index notation as $a_{1j}\, a_{2j} = 0$ (repeated index j; implied running summation over all values of j; here $j = 1, 2, 3$).

In exactly the same way, we argue that the axes x'_2 with the direction cosines $(a_{21},\ a_{22},\ a_{23})$ w.r.t. the 'old' axes (x_1, x_2, x_3), and x'_3 with the direction cosines $(a_{31},\ a_{32},\ a_{33})$ w.r.t. the 'old' axes (x_1, x_2, x_3) are perpendicular to each other. This observation enables us to write

$$a_{2\underline{1}}\, a_{3\underline{1}} + a_{2\underline{2}}\, a_{3\underline{2}} + a_{3\underline{3}}\, a_{2\underline{3}} \equiv \sum_{j=1}^{3} a_{2j}\, a_{3j} = 0.$$

Thus, we note the following equations.

$$a_{1\underline{1}}\, a_{2\underline{1}} + a_{1\underline{2}}\, a_{2\underline{2}} + a_{1\underline{3}}\, a_{2\underline{3}} \equiv \sum_{j=1}^{3} a_{1j}\, a_{2j} = 0 \text{ (axes 1 and 2 perpendicular),} \tag{3.14a}$$

$$a_{2\underline{1}}\, a_{3\underline{1}} + a_{2\underline{2}}\, a_{3\underline{2}} + a_{2\underline{3}}\, a_{3\underline{3}} \equiv \sum_{j=1}^{3} a_{2j}\, a_{3j} = 0 \text{ (axes 2 and 3 perpendicular), and} \tag{3.14b}$$

$$a_{3\underline{1}}\, a_{1\underline{1}} + a_{3\underline{2}}\, a_{1\underline{2}} + a_{3\underline{3}}\, a_{1\underline{3}} \equiv \sum_{j=1}^{3} a_{3j}\, a_{1j} = 0 \text{ (axes 3 and 1 perpendicular).} \tag{3.14c}$$

Using the index notation, these three can be written in capsule form as

$$a_{ij}\, a_{kj} = 0 \qquad (i \neq k),$$

where the dummy index j runs through the entire set of values $1, 2, 3$. Furthermore, this equation is valid for each of the values of $1, 2, 3$ for each of the free indices i and k, $(i \neq k)$.

[8] Such underlining is not resorted to in actual practice. Here we have done so to attract attention, and, thus, to have greater clarity. Underlining like $a_{\underline{ii}}$ or as $a_{\underline{i}\,\underline{i}}$ is sometimes used if we wish to indicate that the summation is suspended.

They together tell us that the axes x_1', x_2', x_3' are mutually perpendicular. Notice that the running sum (repeated index) is on the second subscript. Notice further that the first subscript refers to the axis concerned, 1 for x_1', 2 for x_2', 3 for x_3'.

Step 2:

Now if we had taken the x_1' axis, and had computed the sum of the squares of its direction cosines, we would have obtained the result 1. See Eq. (3.12a).

$$a_{11}^2 + a_{12}^2 + a_{13}^2 = 1.$$

A similar result holds for the other two 'new' axes, x_2' and x_3', also.

$$a_{21}^2 + a_{22}^2 + a_{23}^2 = 1, \text{ and}$$
$$a_{31}^2 + a_{32}^2 + a_{33}^2 = 1.$$

Step 3:

Combining the results shown above in Steps 1 and 2, we obtain the following.

The first set of three equations below, Eqs (3.15a), (3.15b), (3.15c), refer to two *different* axes; the result is 0 then. The three equations in the second set, Eqs (3.16a), (3.16b), (3.16c), refer to the *same* axis; the result is 1 then.

$$a_{1\underline{1}}\, a_{2\underline{1}} + a_{1\underline{2}}\, a_{2\underline{2}} + a_{1\underline{3}}\, a_{2\underline{3}} \equiv \sum_{j=1}^{3} a_{1j}\, a_{2j} = 0 \text{ (different axes 1 and 2),} \tag{3.15a}$$

$$a_{2\underline{1}}\, a_{3\underline{1}} + a_{2\underline{2}}\, a_{3\underline{2}} + a_{2\underline{3}}\, a_{3\underline{3}} \equiv \sum_{j=1}^{3} a_{2j}\, a_{3j} = 0 \text{ (different axes 2 and 3)} \tag{3.15b}$$

$$a_{3\underline{1}}\, a_{1\underline{1}} + a_{3\underline{2}}\, a_{1\underline{2}} + a_{3\underline{3}}\, a_{1\underline{3}} \equiv \sum_{j=1}^{3} a_{3j}\, a_{1j} = 0.\text{(different axes 3 and 1),} \tag{3.15c}$$

$$a_{1\underline{1}}\, a_{1\underline{1}} + a_{1\underline{2}}\, a_{1\underline{2}} + a_{1\underline{3}}\, a_{1\underline{3}} \equiv \sum_{j=1}^{3} a_{1j}\, a_{1j} = 1 \text{ (same axis, 1 and 1),} \tag{3.16a}$$

$$a_{2\underline{1}}\, a_{2\underline{1}} + a_{2\underline{2}}\, a_{2\underline{2}} + a_{2\underline{3}}\, a_{2\underline{3}} \equiv \sum_{j=1}^{3} a_{2j}\, a_{2j} = 1 \text{ (same axis 2 and 2)} \tag{3.16b}$$

$$a_{3\underline{1}}\, a_{3\underline{1}} + a_{3\underline{2}}\, a_{3\underline{2}} + a_{3\underline{3}}\, a_{3\underline{3}} \equiv \sum_{j=1}^{3} a_{3j}\, a_{3j} = 1 \text{ (same axis 3 and 3).} \tag{3.16c}$$

Let us look at these equations carefully and understand what they mean. It is, in words, this: (i) when the direction cosines of two *different* axes such as x_1' and x_2' are multiplied together and added, we get the result 0 (as demonstrated in Step 1, Eqs (3.14a), (3.14b), (3.14c)), and (ii) when the direction cosines of the *same* axis are squared and added, we get the result 1 as demonstrated in Step 2, Eqs (3.16a), (3.16b), (3.16c)). To repeat for

further clarification, when the free indices (i and k) in $a_{\mathbf{i}j}\, a_{\mathbf{k}j}$, shown in bold here to attract attention, are different, the result is 0. When, instead, these are the same, the result is 1. We can exploit the conveniently defined Kronecker delta, and express the result together in one equation as shown below in Eq. (3.17).

Thus, we can combine all the six (6) equations and display them together in a small compact form using the index notation as

$$a_{ij}\, a_{kj} = \delta_{ik}. \tag{3.17}$$

Now we turn around and notice that the 'old' axes (x_1, x_2, x_3) are mutually orthogonal too. These orthogonality conditions are similar, except that now the summation is on the first subscript. We shall examine this case below, though not in as great detail as before.

(b) $\boldsymbol{x_1, x_2, x_3}$ are mutually orthogonal.

As the axes x_1 and x_2 axes are perpendicular, their direction cosines satisfy the equation

$$a_{\underline{1}1}\, a_{\underline{1}2} + a_{\underline{2}1}\, a_{\underline{2}2} + a_{\underline{3}1}\, a_{\underline{3}2} = a_{ij}\, a_{ik} = 0. \tag{3.18}$$

This equation is similar to Eq. (3.13), except that the running sum now is on the first subscripts, underlined as before to attract attention.

Orthogonality relations:

Thus, in conclusion, the orthogonality relations are

$$a_{ij}\, a_{kj} = \delta_{ik} \qquad (x_1', x_2', x_3' \text{ are mutually orthogonal}), \text{ and} \tag{3.19a}$$

$$a_{ij}\, a_{ik} = \delta_{jk} \qquad (x_1, x_2, x_3 \text{ are mutually orthogonal}). \tag{3.19b}$$

These are important results. We can see that, while these are not difficult, they can be confusing. Some practice is needed before the notation and its message become familiar and effortless to understand.

The index notation becomes more useful and powerful only when accompanied by Levi-Civita's permutation symbols. We shall discuss them in the next section.

PERMUTATION SYMBOLS

Permutation symbols are a system of 3-index symbols. They are also known as an e-system, Levi-Civita symbols, anti-symmetric symbols, or alternating symbols. These are part of the index notation, and are very useful when dealing with it.

Definition

This permutation symbol e_{ijk} is defined in 3 dimensions as follows.

$$e_{ijk} = \begin{cases} +1 & \text{if } i, j, k \text{ is an even permutation of } 1, 2, 3; \\ -1 & \text{if } i, j, k \text{ is an odd permutation of } 1, 2, 3; \text{ and} \\ 0 & \text{in other cases when two or three indices are the same.} \end{cases} \tag{3.20}$$

We shall clarify this further. This permutation symbol e_{ijk} is equal to

- $+1$ if these three indices are in an even permutation of the sequence $1, 2, 3$ (cyclic sequence $1, 2, 3$; or $2, 3, 1$; or $3, 1, 2$);
- -1 if these three indices are in an odd permutation of the sequence $1, 2, 3$ (anti-cyclic sequence $1, 3, 2$; or $2, 1, 3$; or $3, 2, 1$);
- 0 in other cases, that is, if any two or all the three indices are the same (equal) (not cyclic; not anti-cyclic; acyclic $1, 1, 2$; or $1, 2, 1$; or $3, 3, 3$; $3, 2, 2$; etc.). See Figs 3.2a and 3.2b.

(a) Even permutation: $+1$ (b) Odd permutation: -1

Figure 3.2: Even permutations of $1, 2, 3$ are shown on the left hand side (Fig. 3.2a), and odd ones on the right (Fig. 3.2b). Thus, for example, $e_{123} = e_{231} = e_{312} = +1$, while $e_{132} = e_{213} = e_{321} = -1$.

They are also defined in 2 dimensions as follows.

$$e_{ij} = \begin{cases} +1 & \text{if } i, j \text{ is an even permutation of } 1, 2; \\ -1 & \text{if } i, j \text{ is an odd permutation of } 1, 2; \text{ and} \\ 0 & \text{in other cases when the two indices are the same.} \end{cases} \tag{3.21}$$

This definition can be shown in matrix form as

$$\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.22}$$

[They can be defined similarly in higher dimensions also. For example, in 4 dimensions,

$$e_{ijkl} = \begin{cases} +1 & \text{if } i, j, k, l \text{ is an even permutation of } 1, 2, 3, 4; \\ -1 & \text{if } i, j, k, l \text{ is an odd permutation of } 1, 2, 3, 4; \text{ and} \\ 0 & \text{when any two, any three, or all the indices are the same.} \end{cases} \tag{3.23}$$

We hardly ever use these symbols in higher dimensions. Our interest is usually only in 2 and 3 dimensions.]

Expansion of Determinants Using e_{ijk}

The long expanded form of any determinant may be recast in abbreviated form using these symbols. We shall be concerned mostly with 3×3, and occasionally with 2×2, determinants in the context of applications to our subject.

A 3×3 determinant

$$a = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \equiv |a_{ij}| \tag{3.24}$$

may be written in the form

$$\begin{aligned} a &= a_{i1}\, a_{j2}\, a_{k3}\, e_{ijk} \\ &= a_{11} \times [\text{its cofactor}] + a_{12} \times [\text{its cofactor}] + a_{13} \times [\text{its cofactor}] \end{aligned} \tag{3.25}$$

which is expansion of the determinant by column, here by the first column (as here a_{11}, a_{12}, a_{13} are the elements in the *first* column).

It can also be written in an alternative form

$$\begin{aligned} a &= a_{3i}\, a_{2j}\, a_{1k}\, e_{ijk} \\ &= a_{31} \times [\text{its cofactor}] + a_{32} \times [\text{its cofactor}] + a_{33} \times [\text{its cofactor}] \end{aligned} \tag{3.26}$$

which is expansion of the determinant by row, here by the third row (as here a_{31}, a_{32}, a_{33} are the elements in the *third* row).

[It would be nice if this expression is patiently written out in full so that the pattern becomes clear. Another possibility is to write a similar expression for the simpler case of a 2×2 determinant, and work it out fully. In any case, it is necessary to convince ourselves that out of the $3 \times 3 \times 3 = 27$ terms ($i, j, k = 1, 2, 3$), only the expression shown above survives. We shall indicate below the kind of steps to be taken and the kind of arguments to be used. The full expansion is not completely worked out.

$$\begin{aligned} a_{3i}\, a_{2j}\, a_{1k}\, e_{ijk} &= a_{31} \times [a_{2j}\, a_{1k}\, e_{1jk}] + a_{32} \times [a_{2j}\, a_{1k}\, e_{2jk}] + a_{33} \times [a_{2j}\, a_{1k}\, e_{3jk}] \\ &= a_{31} \times [\{a_{21}\, a_{1k}\, e_{11k}\} + \{a_{22}\, a_{1k}\, e_{12k}\} + \{a_{23}\, a_{1k}\, e_{13k}\}] + \\ &\quad\; a_{32} \times [\cdots] + a_{33} \times [\cdots] \end{aligned}$$

Let us note that

(i) the first 'term' $\{a_{21}\, a_{1k}\, e_{11k}\}$ in the last equation need not be further expanded because the factor $e_{11k} = 0$ for all values of k;

(ii) the second 'term' $\{a_{22}\, a_{1k}\, e_{12k}\}$ is to be expanded over all values of $k = 1, 2, 3$. However, only e_{123} is non-zero; $e_{121} = e_{122} = 0$; and that

(iii) similarly the 'third' term $\{a_{23}\, a_{1k}\, e_{13k}\}$, when expanded, leads to the only non-zero term $\{a_{23}\, a_{12}\; e_{132}\}$, which is equal to $-\{a_{23}\, a_{12}\}$, because $e_{132} = -1$ (anti-cyclic sequence of $1, 2, 3$).]

If, in either of the two forms (of expansion), two subscripts are interchanged, the value of the determinant becomes $-a$. This is not difficult to understand; we know that the sign of a determinant is changed when two columns (rows) are interchanged. Thus,

$$a = a_{i1}\, a_{j2}\, a_{k3}\, e_{ijk}$$

$$-a = a_{i2}\, a_{j1}\, a_{k3}\, e_{ijk} \text{ (interchange of two columns, the first and the second)}$$
$$= a_{3i}\, a_{2j}\, a_{1k}\, e_{ijk} \text{ (interchange of two rows, the first and the third).}$$

We can use such expressions, manipulate them, and simplify them by using the rules to prove the results of operation on determinants. For example, the determinant of a product matrix $\mathbf{P}$ of two component square matrices $\mathbf{A}$ and $\mathbf{B}$ (that is, $\mathbf{P} = \mathbf{AB}$), is the product of the determinants. Thus, if

$$\mathbf{P} = \mathbf{AB}, \text{ then } |P| = |A|\,|B|,$$

where

$$|A| \equiv a \equiv |a_{ij}|, \quad |B| \equiv b \equiv |b_{ij}|, \text{ and } |P| \equiv p \equiv |p_{ik}| = |a_{ij}\, b_{jk}|.$$

When any two columns (rows) of a determinant are the same, the value of the determinant is zero. This can be seen easily from Eqs (3.25) and (3.26), because $e_{ijk} = 0$ whenever any two indices are the same.

Cross Product of Vectors Using e_{ijk}

Cross products of vectors can be conveniently expressed in index notation using these permutation symbols. We first begin by noting the following relationships among the unit vectors ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$) in a rectangular cartesian coordinate system.

$$\begin{array}{lll} \mathbf{i}\times\mathbf{j} = \mathbf{k}; & \mathbf{j}\times\mathbf{k} = \mathbf{i}; & \mathbf{k}\times\mathbf{i} = \mathbf{j}; \\ \mathbf{j}\times\mathbf{i} = -\mathbf{k}; & \mathbf{k}\times\mathbf{j} = -\mathbf{i}; & \mathbf{i}\times\mathbf{k} = -\mathbf{j}; \\ \mathbf{i}\times\mathbf{i} = \mathbf{0}; & \mathbf{j}\times\mathbf{j} = \mathbf{0}; & \mathbf{k}\times\mathbf{k} = \mathbf{0}; \end{array}$$

or equivalently,

$$\begin{array}{lll} \mathbf{e_1}\times\mathbf{e_2} = \mathbf{e_3}; & \mathbf{e_2}\times\mathbf{e_3} = \mathbf{e_1}; & \mathbf{e_3}\times\mathbf{e_1} = \mathbf{e_2}; \\ \mathbf{e_2}\times\mathbf{e_1} = -\mathbf{e_3}; & \mathbf{e_3}\times\mathbf{e_2} = -\mathbf{e_1}; & \mathbf{e_1}\times\mathbf{e_3} = -\mathbf{e_2}; \\ \mathbf{e_1}\times\mathbf{e_1} = \mathbf{0}; & \mathbf{e_2}\times\mathbf{e_2} = \mathbf{0}; & \mathbf{e_3}\times\mathbf{e_3} = \mathbf{0}; \end{array}$$

which are all condensed into the following 'single' equation that states that the three coordinate axes are orthogonal: $\mathbf{e_i} \times \mathbf{e_j} = e_{ijk}\, \mathbf{e_k}$.

We are now ready to consider the cross product of any two vectors $\mathbf{A}$ and $\mathbf{B}$ using the above result.

$$\mathbf{P} \equiv \mathbf{A}\times\mathbf{B} = (A_i\, \mathbf{e_i}) \times (B_j\, \mathbf{e_j})$$
$$= (A_i\, B_j)(\mathbf{e_i} \times \mathbf{e_j}) = A_i\, B_j\, e_{ijk}\, \mathbf{e_k}.$$

If $\mathbf{P}$ is written as $P_k\, \mathbf{e_k}$, we obtain $\mathbf{P} = P_k\, \mathbf{e_k} = A_i\, B_j\, e_{ijk}\, \mathbf{e_k}$, which is the same in content as

$$P_k = A_i\, B_j\, e_{ijk}. \tag{3.27}$$

Triple Scalar Product

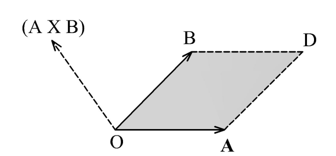
The triple scalar product $(\mathbf{A} \times \mathbf{B}) \,.\, \mathbf{C}$ of three vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ can be represented in index notation as

$$\mathbf{A} \times \mathbf{B} \,.\, \mathbf{C} = A_i \, B_j \, C_k \, e_{ijk}.$$

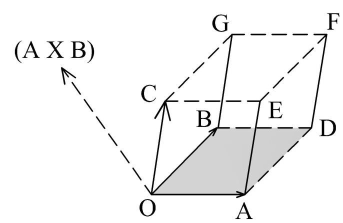
This can also be written as any one of the expressions shown below without any ambiguity. Let us note that the positions of the dot and the cross in a triple scalar product are immaterial as long as the sequence $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ is maintained as an even sequence. When the sequence is broken, the sign of the triple scalar product is reversed.

$$\mathbf{A} \times \mathbf{B} \,.\, \mathbf{C} = A_i \, B_j \, C_k \, e_{ijk} = \mathbf{A} \,.\, \mathbf{B} \times \mathbf{C} = \mathbf{B} \,.\, \mathbf{C} \times \mathbf{A} = \mathbf{C} \,.\, \mathbf{A} \times \mathbf{B}. \tag{3.28}$$

Cross Product and Triple Scalar Product: Physical Meaning



(a) Cross product of two vectors



(b) Triple scalar product

Figure 3.3: Cross product $\boldsymbol{A} \times \boldsymbol{B}$ of two vectors [Fig. 3.3a] represents the shaded area $OADB$; the cross product is along the normal to the shaded area. The triple scalar product $(\boldsymbol{A} \times \boldsymbol{B}) \,.\, \boldsymbol{C}$ represents the volume of the parallelepiped shown in dotted lines [Fig. 3.3b].

These products, the cross product and the triple scalar product, can be given physical or geometrical meanings.

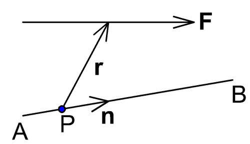
The cross product represents the area of a parallelogram defined by the two vectors $\mathbf{A}$ and $\mathbf{B}$ as shown in Fig. 3.3a. The surface area can be given a direction also, viz., the direction of the normal to the surface. This observation will be useful when we discuss the area on which a stress component (or a stress vector) acts.

The triple scalar product $(\mathbf{A} \times \mathbf{B}) \,.\, \mathbf{C}$ represents the volume of the block in Fig. 3.3b. It is clear from the figure also that the above triple scalar product may be written in any one of the forms given in Eq. (3.28), because all of them represent the same volume.

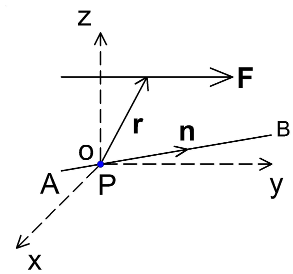
If a triple scalar product vanishes, the three vectors concerned are coplanar. That is, if $(\mathbf{A} \times \mathbf{B}) \,.\, \mathbf{C} = 0$, it means that $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are coplanar. Then, any one of the three vectors can be written as a linear combination of the other two as $\boldsymbol{A} = \alpha \boldsymbol{B} + \beta \boldsymbol{C}$.

Moment of a Force

The moment $\mathbf{M}$ of a force $\mathbf{F}$ about a point P (Fig. 3.4a), we may recall from elementary mechanics, is given by $\mathbf{M} = \mathbf{r} \times \mathbf{F}$, where $\mathbf{r}$ is the position vector. The equation may be



(a) Moment of a force $\boldsymbol{F}$ about a point $P = \boldsymbol{r} \times \boldsymbol{F}$



(b) Moment of a force $\boldsymbol{F}$ about a point P in a given direction $AB = (\boldsymbol{r} \times \boldsymbol{F}) \cdot \boldsymbol{n}$

Figure 3.4: Moment of a force $\boldsymbol{F}$ about a point $P = \boldsymbol{r} \times \boldsymbol{F}$ [Fig. 3.4a], where $\boldsymbol{r}$ is the position vector. The component along the line AB of the moment of the force $\boldsymbol{F}$ about the point P is $(\boldsymbol{r} \times \boldsymbol{F}) \cdot \boldsymbol{n}$ [Fig. 3.4b], where $\boldsymbol{n}$ is the unit vector along AB.

written in index notation as

$$M_k = r_i\, F_j\, e_{ijk}.$$

The component of this moment $\mathbf{M}$ along an axis, say, AB is a scalar, and is represented by the triple scalar product $(\mathbf{r} \times \mathbf{F}) \cdot \mathbf{n}$, where $\mathbf{n}$ is a unit vector along AB [See Fig. (3.4b).] This equation may be written as

$$M_{AB} = r_i\, F_j\, n_k\, e_{ijk}. \tag{3.29}$$

We may, if desired, set up a coordinate system, work out the moments from first principles, and convince ourselves of the correctness of these equations.

PERMUTATION IDENTITY ($e - \delta$ IDENTITY)

The permutation symbols and the Kronecker deltas are related by an identity known as the permutation identity or $e - \delta$ identity. This is useful when manipulating expressions and arriving at relationships in index notation. It is a convenient tool for simplification of expressions in index notation. It reads

$$e_{ijk}\, e_{ilm} = \delta_{jl}\, \delta_{km} - \delta_{jm}\, \delta_{kl}. \tag{3.30}$$

Two other results involving these permutation symbols are

$$e_{ilm}\, e_{jlm} = 2\delta_{ij} \text{ and} \tag{3.31}$$

$$e_{ijk}\; e_{ijk} = 6. \tag{3.32}$$

The corresponding results in 2 dimensions are

$$e_{ij}\, e_{mn} = \delta_{im}\, \delta_{jn} - \delta_{in}\, \delta_{jm}; \tag{3.33}$$

$$e_{ij}\, e_{in} = \delta_{jn}; \text{ and} \tag{3.34}$$

$$e_{ij}\; e_{ij} = 2. \tag{3.35}$$

A FEW ILLUSTRATIVE EXAMPLES

We shall discuss and work out a few illustrative examples below for further clarification. We hope these will be helpful for beginners.

Example 1

When we substitute one expression in another, we must be sure to avoid having the same index appearing more than twice in any 'term'. For example, if we have an equation $a_{ij}\,x_i\,y_j = 0$, and if we wish to replace the variables x_i's by a new set of variables z_i's linearly related by $x_i = b_{ij}\,z_j$, a straightforward substitution gives us an equation $a_{ij}\,b_{ij}\,z_j\,y_j = 0$, which is meaningless, the index j appearing too many times in the same 'term'. In such cases, the proper way to deal with this is to write the first (original) equation in terms of some other dummy indices, say k and l, as $a_{kl}\,x_k\,y_l = 0$, where $x_k = b_{kj}\,z_j$. Thus, we obtain $a_{kl}\,b_{kj}\,z_j\,y_l = 0$, which is a valid equation.

Example 2

Solve a system of linear, simultaneous, algebraic equations $a_{ij}\,x_j = c_i$.

$$a_{ij}\,x_j = c_i; \qquad \mathbf{A}\mathbf{x} = \mathbf{c}.$$

Multiply both sides by b_{kl}, the inverse of the matrix a_{ij} (and sum over the index i by setting $l = i$). Thus,

$$b_{kl}\,a_{ij}\,x_j = b_{kl}\,c_i; \qquad \mathbf{B}\mathbf{A}\,\mathbf{x} = \mathbf{B}\mathbf{c}; \qquad \mathbf{A}^{-1}\mathbf{A}\,\mathbf{x} = \mathbf{A}^{-1}\,\mathbf{c}.$$

That is, by setting $l = i$, $b_{ki}\,a_{ij} = \delta_{kj}$. Hence we have

$$\begin{aligned} \delta_{kj}\,x_j &= b_{ki}\,c_i; \qquad & \mathbf{I}\,\mathbf{x} = \mathbf{B}\,\mathbf{c} \equiv \mathbf{A}^{-1}\,\mathbf{c}; \\ x_k &= b_{ki}\,c_i; \qquad & \mathbf{x} = \mathbf{B}\,\mathbf{c} = \mathbf{A}^{-1}\,\mathbf{c}. \end{aligned}$$

Note that we used the substitution property of the Kronecker delta. We finally obtain the solution as $x_k = b_{ki}\,c_i$. We further note that the inverse of a (square) matrix exists[9] only if its determinant does not vanish[10].

[9] Given a (square) matrix (corresponding to such sets of linear, simultaneous, algebraic equations), how its inverse can be calculated, and questions of the existence and uniqueness of the solution, and related matter are discussed in great detail in books on Linear Algebra.

[10] When solving technologically important problems as, say, when using the finite element method, the size of the matrix is usually very large. In such cases, calculating the inverse to solve a set of linear, simultaneous, algebraic equations is not computationally wise; it is never done in practice. To demonstrate this in the simplest, almost trivial case, let us solve the equation $3x = 6$ in one unknown. Calculating the inverse as $1/3$ involves one operation, and solving for x involves another operation, viz., $1/3 \times 6$. On the other hand, a straightforward solution (by Gaussian elimination) gives us the answer as $x = 6/3 = 2$ in 1 operation (compared to 2 operations in the previous case).

There are several details that are of great importance when a large number of algebraic, linear, simultaneous equations are to be solved. These cannot be discussed here. Advanced books like Bathe, K.J., *Finite Element Procedures*, 2^{nd} ed., Watertown, MA, (2014), and Datta, B.N., *Numerical Linear Algebra and Applications*, 2^{nd} ed., SIAM, (2000) have to be consulted.

Example 3

Write down the following equation in full longhand. The comma (,) denoted partial differentiation[11] with respect to the space variable x_i.

$$\sigma_{ji,j} + X_i = 0.$$

This is sometimes written as $\sigma_{ij,j} + X_i = 0$, because $\sigma_{ji} = \sigma_{ij}$ (symmetry of the stress matrix).

(Summation over the repeated (dummy) index j; hence there is summation over the entire set of values for $j : (j = 1, 2, 3)$). Thus,

$$\sigma_{1i,1} + \sigma_{2i,2} + \sigma_{3i,3} + X_i = 0.$$

(Free index i; hence separate equations corresponding to all (possible) values of $i : (i = 1, 2, 3)$). Thus, we have

$$\begin{array}{ll}
i = 1;\ 1 \equiv x_1 \equiv x;\ X_1 \equiv X & \sigma_{11,1} + \sigma_{21,2} + \sigma_{31,3} + X_1 = 0;\ \text{that is,} \\
 & \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{yx}}{\partial y} + \dfrac{\partial \sigma_{zx}}{\partial z} + X = 0; \\
i = 2;\ 2 \equiv x_2 \equiv y;\ X_2 \equiv Y & \sigma_{12,1} + \sigma_{22,2} + \sigma_{32,3} + X_2 = 0;\ \text{that is,} \\
 & \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \sigma_{zy}}{\partial z} + Y = 0; \\
i = 3;\ 3 \equiv x_3 \equiv z;\ X_3 \equiv Z & \sigma_{13,1} + \sigma_{23,2} + \sigma_{33,3} + X_3 = 0;\ \text{that is,} \\
 & \dfrac{\partial \sigma_{xz}}{\partial x} + \dfrac{\partial \sigma_{yz}}{\partial y} + \dfrac{\partial \sigma_{zz}}{\partial z} + Z = 0.
\end{array}$$

We shall see later (p. 5-4) that these are the differential equations of equilibrium[12] in the mechanics of solids or the theory of elasticity.

Example 4

Write down the following equation in full[13]. The comma (,) again stands for partial differentiation with respect to the space variable indicated immediately after the comma.

$$e_{ij} = \frac{1}{2}\left(e_{i,j} + e_{j,i}\right)$$

[11] Here in this book we deal with only cartesian tensors; general tensors are not considered. For general tensors, a comma stands for what is called covariant differentiation. For the simplified case of cartesian tensors, this is greatly simplified to be just the usual partial differentiation with respect to the space variable indicated immediately after the comma.

[12] The shear stresses are written as τ_{xy} instead of as σ_{xy}, etc.. We, engineers, know that when the subscripts are different, the stress components are shear stresses.

[13] This is quite similar to the previous example. Yet we write it out in full, because these two examples 3 and 4 are of much importance to the mechanics of solids and / or the theory of elasticity.

There is no repeated index here; thus, there is no running summation. The free indices i and j take on different values $1, 2, 3$ in the nine (9) equations.

$$
\begin{array}{lll}
i=1;\, j=1; & e_{11} = \frac{1}{2}\left(u_{1,1}+u_{1,1}\right); & e_{xx} = \frac{\partial u}{\partial x} \\
i=1;\, j=2; & e_{12} = \frac{1}{2}\left(u_{1,2}+u_{2,1}\right); & e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\
i=1;\, j=3; & e_{13} = \frac{1}{2}\left(u_{1,3}+u_{3,1}\right); & e_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\
i=2;\, j=1; & e_{21} = \frac{1}{2}\left(u_{2,1}+u_{1,2}\right); & e_{yx} = \frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) \\
i=2;\, j=2; & e_{22} = \frac{1}{2}\left(u_{2,2}+u_{2,2}\right); & e_{yy} = \left(\frac{\partial v}{\partial y}\right) \\
i=2;\, j=3; & e_{23} = \frac{1}{2}\left(u_{2,3}+u_{3,2}\right); & e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\
i=3;\, j=1; & e_{31} = \frac{1}{2}\left(u_{3,1}+u_{1,3}\right); & e_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) \\
i=3;\, j=2; & e_{32} = \frac{1}{2}\left(u_{3,2}+u_{2,3}\right); & e_{zy} = \frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) \\
i=3;\, j=3; & e_{33} = \frac{1}{2}\left(u_{3,3}+u_{3,3}\right); & e_{zz} = \left(\frac{\partial w}{\partial z}\right)
\end{array}
$$

Again, we shall see later that these are the strain-displacement relations. (Although there are $3\times 3 = 9$ equations here, only six (6) of them are independent because of the symmetry, $e_{ij} = e_{ji}$).

Example 5

Carry out the partial differentiation indicated below, and obtain the result in index notation. The C's are all constants.

$$\frac{\partial}{\partial x_i}\left(C_{kl}\, x_k\, x_l\right)$$

$$
\begin{aligned}
\frac{\partial}{\partial x_i}\left(C_{kl}\, x_k\, x_l\right) &= C_{kl}\left(x_k \frac{\partial x_l}{\partial x_i} + x_l \frac{\partial x_k}{\partial x_i}\right) \\
&= C_{kl}\left(x_k\, \delta_{li} + x_l\, \delta_{ki}\right) \\
&= C_{ki}\, x_k + C_{il}\, x_l \\
&= C_{ki}\, x_k + C_{ik}\, x_k, \text{ changing the dummy index } l \text{ to } k \\
&= \left(C_{ki} + C_{ik}\right)\, x_k
\end{aligned}
$$

Note that we have used the result $\frac{\partial x_l}{\partial x_i} = \delta_{li}$ above. We have also made use of the substitution property of the Kronecker delta.

Example 6

(a) The dot product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is given by
$\mathbf{A}\,.\,\mathbf{B} = A_i\,B_i = A_i\,\delta_{ij}\,B_j = \delta_{ij}\,A_i\,B_j$.

(b) $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

(c) $\delta_{ij}\,\delta_{jk} = \delta_{i1}\,\delta_{1k} + \delta_{i2}\,\delta_{2k} + \delta_{i3}\,\delta_{3k}$,
which is equal to 1 if $i = k$, and 0 if $i \neq k$. Thus, it is equal to δ_{ik}. It does not matter whether $i = j = 1$ or 2 or 3. Only one of the three terms above survives.

This result is more readily obtained by invoking the substitution property of the Kronecker delta: $\delta_{ij}\,\delta_{jk} = \delta_{ik}$.

(d)

$$\begin{aligned} \beta_{ij}\,a_i\,b_j &\neq \beta_{ij}\,b_i\,a_j \\ (\beta_{ij} + \beta_{ji})\,a_i\,b_j &\neq 2\beta_{ij}\,a_i\,b_j \\ \beta_{ij}\,(a_i + b_j) &\neq \beta_{ij}\,a_i + \beta_{ij}\,b_j \end{aligned}$$

(e) $\beta_{imn}\,(a_i + b_m)\,c_n \neq \beta_{imn}\,a_i\,c_n + \beta_{imn}\,b_m\,c_n$ (No free index on the left hand side!)

(f)

$$\begin{aligned} \beta_{ij}\,(a_j + b_j) &= \beta_{ij}\,a_j + \beta_{ij}\,b_j \\ \beta_{ij}\,a_i\,b_j &= \beta_{ij}\,b_j\,a_i \\ \beta_{ij}\,a_i\,a_j &= \beta_{ij}\,a_j\,a_i \end{aligned}$$

(g)

$$\begin{aligned} (\beta_{ij} + \beta_{ji})\,a_i\,a_j &= 2\beta_{ij}\,a_i\,a_j \\ (\beta_{ij} - \beta_{ji})\,a_i\,a_j &= 0 \end{aligned}$$

(h) $\delta_{ij}\,e_{ijk} = 0$ (Whenever $i \neq j$, $\delta_{ij} = 0$, and whenever $i = j$, $e_{ikj} = 0$, two indices of the permutation symbol being equal.)

Example 7

If $e_{ijk}\,\sigma_{jk} = 0$, show that $\sigma_{ij} = \sigma_{ji}$.

To see this, all that is needed is to write down this equation for each value of i. For example, if $i = 1$, we obtain $e_{1jk}\,\sigma_{jk} = 0$, which means

$$e_{11k}\,\sigma_{1k} + e_{12k}\,\sigma_{2k} + e_{13k}\,\sigma_{3k} = 0.$$

Recalling that the permutation symbols e_{ijk} vanish whenever two indices are equal (the same), we see from the above equation that

$$e_{12k}\,\sigma_{2k} + e_{13k}\,\sigma_{3k} = 0; \text{ that is,}$$

$$(e_{121}\,\sigma_{21} + e_{122}\,\sigma_{22} + e_{123}\,\sigma_{23}) + (e_{131}\,\sigma_{31} + e_{132}\,\sigma_{32} + e_{133}\,\sigma_{32}) = 0.$$

Again noting that only the third term in the first set of braces ( ), and the second in the second set survive, we obtain

$$e_{123}\,\sigma_{23} + e_{132}\,\sigma_{32} = 0,$$

which implies that $\sigma_{23} = \sigma_{32}$, (because $e_{123} = 1$ and $e_{132} = -1$). The other two similar sets of equations, viz., $\sigma_{13} = \sigma_{31}$ and $\sigma_{21} = \sigma_{12}$, can similarly be obtained by setting the values of i to 2 and 3. We shall see later that these are the consequences, (the shear and the complementary shear being equal), of the equations of moment equilibrium.

Example 8

The vector triple product $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \,.\, \boldsymbol{C})\,\boldsymbol{B} - (\boldsymbol{A} \,.\, \boldsymbol{B})\,\boldsymbol{C}$.

A typical component, say, the first, of the left hand side, is

$$\begin{aligned}
[\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})]_1 &= A_2\,(B_1C_2 - B_2C_1) - A_3\,(B_3C_1 - B_2C_3) \\
&= B_1\,(A_2C_2 + A_3C_3) - C_1\,(A_2B_2 + A_3B_3) \\
&= B_1\,(A_1C_1 + A_2C_2 + A_3C_3) - C_1\,(A_1B_1 + A_2B_2 + A_3B_3) \\
&= B_1\,(\boldsymbol{A} \,.\, \boldsymbol{C}) - C_1\,(\boldsymbol{A} \,.\, \boldsymbol{B})
\end{aligned}$$

If the second and the third terms are also processed similarly, and all the results combined together, we obtain the result $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \,.\, \boldsymbol{C})\,\boldsymbol{B} - (\boldsymbol{A} \,.\, \boldsymbol{B})\,\boldsymbol{C}$. Let us not fail to notice that the brackets are important here, because $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) \neq (\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C}$.

Example 9

A linear transformation $y_i = a_{ij}\,x_j$ from the variables x_i's to the variables y_i's is followed by another linear transformation $z_i = b_{ij}\,y_j$. Write down the product transformation (after the two successive transformations) from x_i's to z_i's.

If we substitute $y_i = a_{ij}\,x_j$ directly into the equation $z_i = b_{ij}\,y_j$, we would have the index j too many times in the same 'term'. This, as explained earlier, becomes quite meaningless. Thus, the equation $y_i = a_{ij}\,x_j$ is first to be rewritten as $y_j = a_{jk}\,x_k$ before the substitution is made. Then we obtain $z_i = b_{ij}\,y_j = b_{ij}\,a_{jk}\,x_k$. The equivalent matrix equations are $\boldsymbol{Y} = \boldsymbol{AX}$, $\boldsymbol{Z} = \boldsymbol{BY}$, and $\boldsymbol{Z} = \boldsymbol{BAX}$. We note that the indicated order is to be maintained, as the products $\boldsymbol{BA}$ and $\boldsymbol{AB}$ are, in general, different.

Example 10

Expand a determinant $|a_{ij}|$ in terms of these elements and their cofactors.

We know that a determinant $|a_{ij}| \equiv a$ may be expanded row-wise or column-wise in terms of its elements and their cofactors. Thus, if A_{ij} is the cofactor of the element a_{ij},

$$|a_{ij}| \equiv a = a_{ij}\,A_{ij} \quad \text{(row-wise Laplace expansion, no sum on } i\text{);}$$
$$|a_{ij}| \equiv a = a_{ij}\,A_{ij} \quad \text{(column-wise Laplace expansion, no sum on } j\text{)}$$

We further know that, if the elements of *any* row / (column) are multiplied by the cofactors of *any other* row / (column) and added up, the result is zero. Thus,

$$a_{ij}\,A_{kj} = \delta_{ik}\,a \;\; \text{(row-wise Laplace expansion), and}$$
$$a_{ij}\,A_{jk} = \delta_{jk}\,a \;\; \text{(column-wise Laplace expansion).}$$

Example 11

Use the above result to derive Cramer's rule for the solution of a system of linear, simultaneous, algebraic equations $a_{ij}\,x_j = c_i$.

Multiply both sides of the equation by A_{ik} (and sum over the index i). This gives us

$$A_{ik}\,a_{ij}\,x_j = A_{ik}\,c_k;$$
$$\text{i.e.,}\quad a\,\delta_{kj}\,x_j = A_{ik}\,c_i \quad \longrightarrow \quad a\,x_k = A_{ik}\,c_i,$$
$$\text{giving us the result,}\quad x_i = \frac{A_{ik}\,c_i}{a}.$$

Example 12

Show that $(\boldsymbol{A}\times\boldsymbol{B})\,.\,(\boldsymbol{C}\times\boldsymbol{D}) = (\boldsymbol{A}\,.\,\boldsymbol{C})(\boldsymbol{B}\,.\,\boldsymbol{D}) - (\boldsymbol{A}\,.\,\boldsymbol{D})(\boldsymbol{B}\,.\,\boldsymbol{C})$.

We recall that

$$\boldsymbol{A}\,.\,\boldsymbol{B} = (A_i\,e_i)\,.\,(B_j\,e_j);\;\; \text{and}$$
$$\boldsymbol{A}\times\boldsymbol{B} = (A_i\,e_i)\times(B_j\,e_j) = A_iB_j\,e_{ijk}\,e_k.$$
$$\text{Thus,}\quad (\boldsymbol{A}\times\boldsymbol{B})\,.\,(\boldsymbol{C}\times\boldsymbol{D}) = [(A_i\,\boldsymbol{e}_i)\times(B_j\,\boldsymbol{e}_j)]\,.\,[(C_k\,\boldsymbol{e}_k)\times(D_l\,\boldsymbol{e}_l)]$$
$$= (A_iB_j\,e_{ijk}\,\boldsymbol{e}_k)\,.\,(C_kD_l\,e_{klm}\,\boldsymbol{e}_m)$$

Now when the two $(\cdots)$ and $(\cdots)$ are taken together, there are too many k's as indices. The indices in the second $(\cdots)$ are, therefore, replaced by another set of indices, say, p, q and r. Accordingly,

$$\begin{aligned}(\boldsymbol{A}\times\boldsymbol{B})\,.\,(\boldsymbol{C}\times\boldsymbol{D}) &= [(A_i\,\boldsymbol{e}_i)\times(B_j\,\boldsymbol{e}_j)]\,.\,[(C_k\,\boldsymbol{e}_k)\times(D_l\,\boldsymbol{e}_l)]\\ &= (A_iB_j\,e_{ijk}\,\boldsymbol{e}_k)\,.\,(C_pD_q\,e_{pqr}\,\boldsymbol{e}_r)\\ &= A_iB_jC_pD_q\,e_{ijk}\,e_{pqr}\,\delta_{kr}\\ &= A_iB_jC_pD_q\,e_{ijk}\,e_{pqk} \quad \text{because}\;\; e_{pqr}\,\delta_{kr} = e_{pqk}\\ &= A_iB_jC_pD_q(\delta_{ip}\,\delta_{jq} - \delta_{iq}\,\delta_{jp}) \quad \text{using } e-\delta \text{ identities}\end{aligned}$$

A FEW PROBLEMS FOR EXERCISE

To gain mastery over the manipulations with the index notation, it is necessary to work out some problems. A few problems are given below for exercise.

1. The strain-displacement relations when the displacements are large are given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}\, u_{k,j})$$

   Write these equations in full in terms of the variables x, y, z. (Note the presence of the nonlinear terms $u_{k,i}\, u_{k,j}$. Such terms introduce geometric nonlinearity and consequent difficulties in the mathematical theory of elasticity.)

2. The stress-strain relations for an isotropic, linearly elastic material are given by the so-called generalised Hooke's law

$$\sigma_{ij} = \lambda\, \delta_{ij}\, e_{kk} + 2\, \mu\, e_{ij}.$$

   How many separate (independent) equations are contained in this? Write down these equations in full in terms of the variables x, y, z.

3. Show, using the index notation and operations, that the following scalar triple products are equal (each representing the volume of a parallelepiped defined by the three vectors, as explained earlier).

$$\boldsymbol{A} \times \boldsymbol{B}\,.\,\boldsymbol{C} = \boldsymbol{A}\,.\,\boldsymbol{B} \times \boldsymbol{C}$$

4. Evaluate $e_{ijk}\, e_{ijk}$ and e_{ijj}.

5. Expand the expression $a_{ij}\, x_i\, x_j$ for a bilinear form in full. How many terms does this have?

6. The stress components in a Newtonian fluid are given by the relation

$$\sigma_{ij} = -p\, \delta_{ij} + 2\, \mu\, e_{ij} - \frac{2}{3}\, \delta_{ij}\, e_{kk}.$$

   Write this out in full to obtain all the equations explicitly. Also express the above equation in the usual Gibbs' notation of boldface for vectors.

7. Expand in full the expressions given below (which define the 'metric properties' of a metric space).

$$(ds)^2 = g_{ij}\, dx^i\, dx^j; \qquad (ds)^2 = \delta_{ij}\, dx_i\, dx_j$$

8. Write down, in index notation, the following quadratic form:

$$Q \equiv X^T A X,$$

   where X is the row vector $[x_i \quad x_2 \quad x_3]$. Under what condition is this quadratic form positive definite? What is the matrix A that corresponds to the following quadratic form?

$$4x_i^2 + 6x_2^2 + 3x_3^2 + 6x_1\, x_2 + 5x_2\, x_3 - 2x_3\, x_1$$

   Is the matrix unique? Can we consider the matrix A as symmetric without any loss in generality? Diagonalise the matrix into its diagonal canonical form, and write the quadratic form in terms of the new variables y_1, y_2, y_3.

9. The quadratic forms

$$T = \frac{1}{2} a_{kl}\, \dot{q}_{kl}\, q_l \qquad \text{and} \qquad V = \frac{1}{2} b_{kl}\, q_k\, q_l$$

represent, respectively, the kinetic and potential energies of a certain dynamical system that has the n generalised coordinates q_k's and the n generalised velocities $\dot{q}_k$'s. The Lagrange's equation of motion for this (conservative) system are

$$a_{kl}\, \ddot{q}_l + b_{kl}\, q_l = 0.$$

The constants a_{kl} and b_{kl} are symmetric. Write down the Lagrange's equations in full explicitly.

10. Calculate

$$\frac{\partial}{\partial y_k}(b_{klm}\, y_k\, y_l\, y_m)$$

and show that it is equal to

$$(b_{mkl} + b_{kml} + b_{klm})\, y_k\, y_l,$$

if b_{klm} are all constants.

11. Recast in the index notation the equation $(\boldsymbol{A} \,.\, \boldsymbol{B})\boldsymbol{C} + \boldsymbol{A} \times \boldsymbol{D} = \boldsymbol{E}$ in Gibbs' vector notation.

12. Show, using the index notation, that $(A\,B)^T = B^T\, A^T$.

CLOSURE

We hope we have demonstrated at least some of the advantages of using the index notation. We shall follow up on these ideas by applying these techniques to the ideas and the basic equations of solid mechanics.

We shall discuss in the next chapter the important topic of the state of stress at a point.

Chapter 4

THE STATE OF STRESS AT A POINT

The state of stress at a point is a topic of fundamental importance. It is absolutely essential to understand all the aspects (both physical and mathematical) related to this clearly. This is a most crucial topic to master. We cannot learn basic subjects like the mechanics of solids, the theory of elasticity, experimental stress analysis and machine design without a sound background of this topic. We shall discuss the physical aspects first even though several concepts are abstract. It is desirable to learn the index notation, sometimes called the tensor notation. It is necessary to build the mathematical tools before we can effectively understand the contents of this chapter. Side by side with the analytical-mathematical treatment, some physical feel also is necessary. Geometrical visualisation is of great help to understand the nature of stress at a point. Some concepts are somewhat abstract, and have to be absorbed and digested slowly.

INTRODUCTION

We begin by pointing out the difference in the state of stress at a point between fluids at rest and solids. We may recall Pascal's[1] law from our earlier study[2]. This law states that for all fluids at rest, the resultant pressure on every plane passing through a given point has the same magnitude (compressive), and is always normal to the plane[3].

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[1] Blaise Pascal (June 1623 - Aug. 1662) was a French mathematician and physicist with a "multiplicity of gifts" in other fields too. Pascal's law of pressure is explained in his book *Treatise on the Equilibrium of Liquids* (1653).

[2] In fluid statics. It is valid for all fluids at rest, real (viscous) and ideal (inviscid, without any viscosity; and not in the sense of being the most desirable kind of fluid), but only for ideal fluids *in motion*.

[3] The numerical value is the same on all planes at the same point. In some books this is stated as "the same in all directions". We consider that such a statement is conceptually in error. A picture often given alongside, like rays shooting out radially outwards all around from a centrally located circular (or spherical) sun, reinforces the error. We believe that this distinction must be emphasised: the pressure is the same *on all planes* (and not "$\cdots$ the same in all directions"). We shall see a little later that pressure at a point does not have, or cannot be assigned, any direction without specifying the plane.

Pressure at a Point in a Fluid at Rest

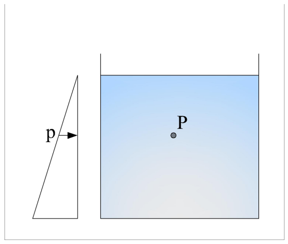
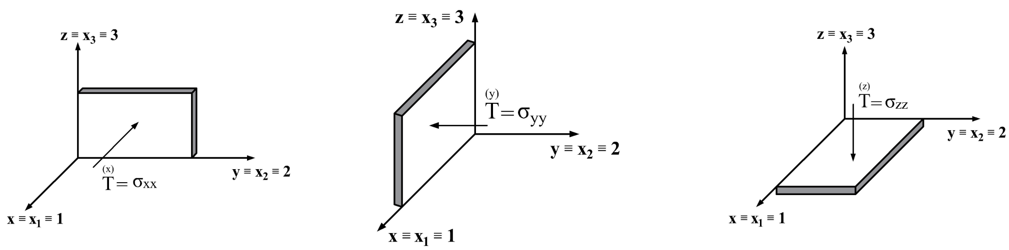


Figure 4.1: Variation of pressure in a fluid at rest

We know that the pressure in a fluid at rest varies linearly with the depth as shown. We also know that it remains constant, the same, at all points on the same horizontal line. But we are not discussing this here. Our purpose is now to understand the nature of pressure *at a point*.

We shall return to the question of how the stress components vary inside a body (typically a machine part). We will not discuss fluid statics further except to point out later that the pressure at a point is an isotropic tensor.



(a) Stress vector on the x-plane

(b) Stress vector on the y-plane

(c) Stress vector on the z-plane

Figure 4.2: In a fluid at rest, the pressure vectors on different planes passing through the same point are all normal to the plane concerned. Furthermore, the magnitudes are all the same. It is necessary to emphasise that the two parallel planes shown are one and the same. They are shown as separated in order to show clearly that the pressure is compressive.

Let us discuss the pressure at a point in a fluid at rest. We know from our earlier studies that the pressure varies linearly with the depth [Fig. 4.1], and that it has the same value at all points on the same horizontal line. But we are not discussing now *how the pressure is distributed from point to point*[4]. The pressure vector on any plane is always normal to the plane concerned. Furthermore, it has the same magnitude. This is symbolically represented in Fig. 4.2. Let us note that the two parallel planes are one and the same.

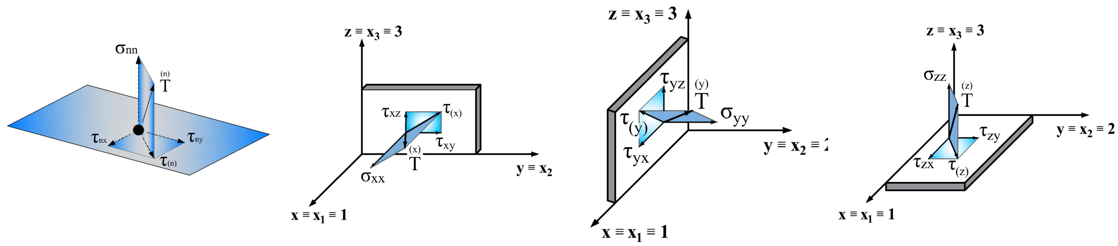
Stress at a Point in a Solid in Equilibrium

The resultant stress would vary, of course, from point to point. Often the interest in the solution of problems is to obtain this distribution, i.e., variation, from point to point. However, we are not discussing such variations here. We need to know in great detail the

---

The pressure may, in general, vary from point to point, but we are not concerned about such variations here in this chapter.

[4] Our interest does lie eventually when discussing design, etc. in how the pressure is distributed at different points, but now we are focussing on the nature of pressure *at a point*.



(a) Stress vector on a plane

(b) Stress vector on the x-plane

(c) Stress vector on the y-plane

(d) Stress vector on the z-plane

Figure 4.3: Unlike a fluid at rest, the stress vectors on different planes passing through the same point in a solid, in general, are not normal to the plane concerned. Their magnitudes also are different. There can be exceptional cases where the state of stress is similar to that of a fluid at rest. We shall call such a special state as a hydrostatic state of stress in a figurative sense even though there is no fluid.

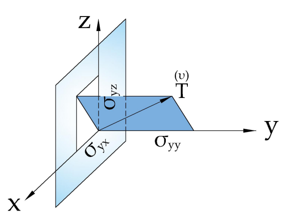


Figure 4.4: Stress vector and its components

For solids, however, the situation is different. The resultant pressure (more commonly referred to as the resultant stress) at a point, even at the same point, is different[a] on different planes in both magnitude and direction. Now the resultant stress at the same point depends decisively on the plane considered. It is, in general, no more normal to the plane in question.

[a] This is so in general. There can surely be special cases where it may be the same, but these are exceptions.

nature of stress at a point, before we can embark on the discussion of the larger problem. Thus, for the rest of this chapter, and even beyond, we shall focus our attention on the state of stress at the same point; we emphasise: at a point, at a point, at a point[5]. This statement holds until we make a further announcement waiving or superseding this.

We shall now proceed to define the stress at a point on a plane, and then to describe the state of stress at a point. We shall state here, in anticipation of what we shall see, that the stress (i) at a point is a (second order) tensor, (ii) at a point on a (given) plane is a vector, and at a point on a plane in a (given) direction is a scalar[6]. The stress at a point is to be defined using concepts from the calculus such as the limit. This has already been

[5] Repetition is said to produce wonderful end results, even though it may appear to be monotonous and pointless! Besides, Sanskrit aestheticians have emphasised that repetition is not only permissible but is even desirable in teaching and love making!

[6] It is not quite correct or precise to state that the stress *is* a tensor. Strictly speaking we should state that the state of stress *is an example* of a tensor, $\cdots$. Once the concept is unmistakably clear, we may occasionally make such imprecise statements to avoid circumlocution (long drawn out sentences).

Note the steady climbing down step by step: stress at a point; stress at a point on a plane; stress at a point on a plane in a direction.

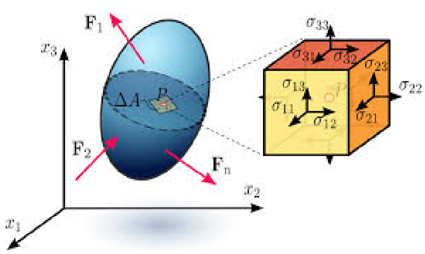
done in our earlier classes and briefly reviewed in the previous chapter [p. 1-11]. Here we shall not repeat it. We shall assume that we are all sufficiently familiar with it.

Surface and Volume Forces

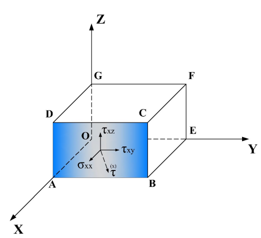
There are two kinds of forces, surface forces and volume forces. As the qualifications indicate, these act on the (relevant) surface and volume, respectively, of a body. Examples of the former are the water pressure acting on the bottom and side surfaces of a water tank, and the soil pressure on a retaining wall, while those of the latter are weight (gravitational forces), 'inertia forces', and forces of attraction. These latter ones act on the volume of a body. In this book we represent surface force components by X, Y, Z or X_i per unit area, and body force components by F_x, F_y, F_z or F_i per unit volume. The body forces may be reckoned on a 'per-unit-mass' basis also. The total body force will be $F_i\,(dV)$ or $F_i\,(\rho\,dV)$, $(\rho\,dV)$ being the mass of the volume element of the body. The external surface force applied on the (part or the entire external) surface of the body is also referred to as the (applied) surface traction. In addition, there can be couples acting on the entire volume. There may be locked-in moments. The presence of such locked-in couples (moments) leads to couple stresses and considerable complications in the resulting theory. As this theory is hard and abstract, we do not discuss couple stresses[7] at all in this book.

STRESS COMPONENTS AT A POINT

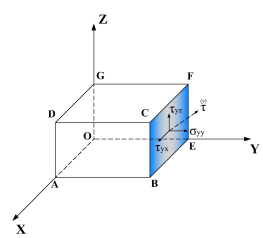
Let us discuss a stress vector at a point on a plane and its three components along the coordinate axes. Figs 4.5a and 4.5b show the stress vectors in dotted lines and the components in firm lines on the x- and y-planes, respectively[8].



(a) Stress components at a point



(b) Stress components on the x-plane $(ABCD)$



(c) Stress components on the y-plane $(EFCD)$

Figure 4.5: Stress components at a point (the subscripts $1, 2, 3$ refer, respectively, to x, y, z); stress components on the x- and y-planes $(ABCD)$ and $(EFCB)$, respectively

[7] The two brothers Eugéne-Maurice-Pierre Cosserat (Mar. 1866 - May 1931) and François Cosserat (Oct. 1852 - March 1914) had proposed a theory as far back as in 1909, and developed it to some extent. It was not taken much notice of until about 1954 when R.D. Mindlin and J.P. Nowacki revived and developed it much further. The Cosserat continuum theory is now known as the theory of micro-polar elasticity. This helps us understand certain situations like dislocation theory, a reduction in the stress concentration factor around holes and cracks, and the wave speed of plane dilatational waves. This is very hard. One of the complications is that the stress tensor (stress matrix) is not symmetric $\sigma_{ij} \neq \sigma_{ji}$ any more with all its complicating consequences.

[8] It is not proper to draw both (i) the resultant and (ii) the three components in firm lines.

Stress Vector on a Plane and Its Three Components

We may represent the state of stress at a point as $\boldsymbol{\sigma}$ (Gibbs' coordinate-free representation) or by its components w.r.to to a coordinate system. Choosing the latter method, we set up a right handed cartesian coordinate system (x, y, z) at a typical point P [Fig. 4.5]. If we choose a plane, we obtain a stress vector on this plane [Fig. 4.3a]. This stress vector on this plane is represented as shown in the figure and as explained below.

The stress vector $\overset{(\nu)}{\boldsymbol{T}}$ is resolved first into two components: (i) normal and (ii) tangential. The dependence of the stress vector on the plane is explicitly shown by (ν) on top of the letter $\boldsymbol{T}$. The tangential part (that is, the resultant shear stress at the point on the plane ν, marked as $\tau_{(\nu)}$ in Fig. 4.3a) is further resolved along the two coordinate directions, thereby obtaining the three stress components at the point on this plane. See Fig. 4.3a.

Similarly, the stress vector $\overset{(x)}{\boldsymbol{T}}$ on the x-plane — a plane is defined by its normal; the x-plane is that plane for which the x-axis is the normal — is resolved[9] into a normal and a tangential components. The normal stress is σ_{xx}. The tangential stress (resultant or total) shear stress is further resoled along the coordinate directions y- and z- directions as τ_{xy} and τ_{xz}, respectively [Fig. 4.5a]. All these three stress components have two subscripts[10] each, the first subscript (here x) denoting the plane, and the second the direction of the stress component concerned (y for the shear stress component τ_{xy} and z for the shear stress components τ_{xz}). When the subscripts are the same, the stress component is a normal stress; when there are different, it is a shear stress[11].

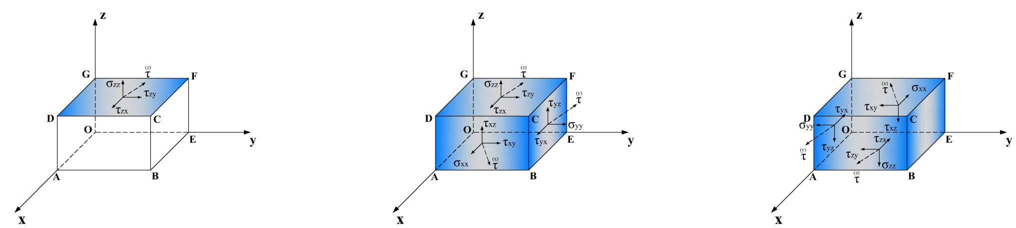
Stress Components on an Inclined Plane

In exactly the same way, the stress vectors $\overset{(y)}{\boldsymbol{T}}$ and $\overset{(z)}{\boldsymbol{T}}$ on the other two planes (the y- and the z-planes) can be represented and shown as in Figs 4.5c and 4.6a. These are all shown in Fig. 4.6b. There are thus, $3 \times 3 = 9$ stress components together describing the state of stress at the point P when referred to the chosen rectangular cartesian coordinate system (x, y, z). These nine (9) components together give a complete description of the state of stress at the point P. These components are displayed in the form of a matrix as shown below. The diagonal elements are normal stresses, and the others shear stresses. Let us note that for the elements on the leading diagonal, the subscripts are the same (normal stresses), while for the off-diagonal ones the subscripts are different (shear stresses). [For the elements on the leading diagonal, the row number and the column number are obviously the same. This is why these elements with the same subscripts represent normal stresses.]

[9] Strictly speaking, a stress cannot be resolved into components like a force. But if the reference is to *the same plane*, then resolving the stress in effect is the same as resolving the force. When different planes are involved, however, this statement takes on special importance, and should serve as a caution.

[10] Two subscripts, because these are the components of a tensor of rank 2.

[11] Mathematicians use only one letter, either σ or τ. Actually, the same letter can represent both normal and shear stresses: normal when the two subscripts are the same, and shear otherwise. Engineers generally prefer to use σ to denote normal stresses and τ for shear stresses. Whether a stress component is a normal stress or a shear stress makes much difference to engineers.



(a) Stress components on the z-plane ($GDCF$)

(b) Stress components on all the three planes

(c) Stress components marked on the other three planes

Figure 4.6: Stress components on all the three planes ($ABCD, EFCB$ and $GDCF$)

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \quad \begin{array}{ll} \longrightarrow & \text{the three components of } \overset{(x)}{\boldsymbol{T}} \text{ in the three directions} \\ \longrightarrow & \text{the three components of } \overset{(y)}{\boldsymbol{T}} \text{ in the three directions} \\ \longrightarrow & \text{the three components of } \overset{(z)}{\boldsymbol{T}} \text{ in the three directions} \end{array}$$

Caution: If is perhaps tempting to add all the three elements in the same column as, say, $\sigma_{xx} + \tau_{yx} + \tau_{zx}$ and call it the resultant stress in the x direction. But, no; a firm no. They are acting on different planes and, it is meaningless to add stress components on *different* planes even if they are acting in the same direction[12].

Let us call this as the stress matrix, that is, the matrix representation referred to the rectangular cartesian coordinate system (x, y, z) of the invariant[13] symmetric stress tensor. There are three key words here: invariant, symmetric, (stress) tensor. They are pregnant with meaning. We shall explain these terms slowly and carefully. See p. 4-38.

Figs 4.6b, 4.6c together show all the stress components[14] on all the six faces. The three pairs of planes are shown here. It is important to realise that the parallel planes are *not* physically separated. Thus, the parallel planes $ABCD$, $OEFG$ are one and the same, both being the same x-plane. Similarly the parallel planes $EFCB$, $OGDA$ (and $GDCF$, $OABE$) are also one and the same, both being the same y-plane (z-plane). They are shown as

[12] Strictly, we can add only force components and not stress components. But if the stress components are all acting on the *same plane*, adding the stress components comes to the same thing as adding the corresponding force components. But adding stress components acting on different planes is an entirely different matter.

[13] The word invariant is used with two entirely different meanings: one in the sense that it is a scalar; and the other in the sense that the tensor itself is invariant (unchanging) even when the components change. The same word with these two different meanings may create misunderstanding for some students. In addition this same word is used to refer to certain combinations of stress components like I_1, I_2, I_3 [p. 4-30].

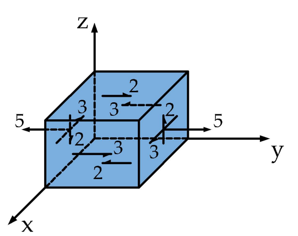
[14] Actually all the $9 \times 2 = 18$ components are to be shown in the same figure. As such a figure with too many entries will look rather cluttered, we have shown them as two sets separately.

separated in order to understand the proper directions of the various stress components[15]. [When we derive the equations of equilibrium, we should consider the changes in the stress components as we move from one point to a neighbouring point. Thus, we are required to consider two different, though neighbouring, points.]

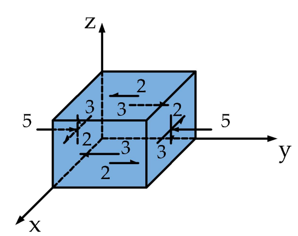
Sign Conventions for Stress Components

The stress components are reckoned as positive or negative according to the sign convention that we choose. Figs 4.6b, 4.6c show the positive directions of all the stress components. For normal stresses it is traditional to choose tension as positive and compression, therefore, as negative. For shear stresses, however, such a sign convention cannot be used, because there cannot be any such classification as tensile or compressive for shear stresses. Thus, it becomes necessary to adopt a somewhat artificial convention. This is explained below and illustrated for the cases (a) [Fig. 4.7a] and (b) [Fig. 4.7b].

To decide whether a (given) shear stress component on a (given) plane (say, the component τ_{xy}) is positive or negative, first draw a tensile stress on that plane (that is, the x-plane). Is it acting along the positive direction of the coordinate axis concerned (the x-axis)? If it is, then the *positive direction* of the shear stress (τ_{xy}) is along the *positive direction* of the coordinate axis concerned (the y-axis). This is illustrated with reference to Figs 4.7a and 4.7b.



(a) Positive normal and shear stresses

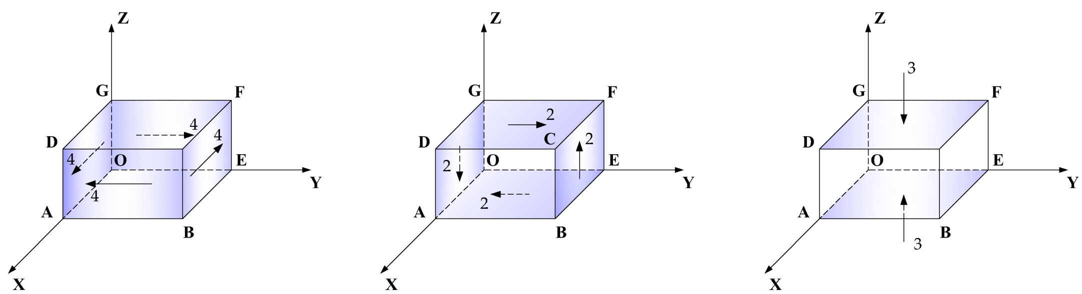


(b) Negative normal and shear stresses

Figure 4.7: Sign conventions for stress components, positive (Fig. 4.7a) and negative (Fig. 4.7b) stress components are shown. For example, $\sigma_{yy} = 5$, $\tau_{yx} = 3$ [Fig. 4.7a]; and $\sigma_{yy} = -5$, $\tau_{yx} = -3$ [Fig. 4.7b], all in MPa.

To carry the explanation further, let us refer to the figures [Figs 4.8a, 4.8b] shown. Let us see how to mark $\tau_{xy} = \tau_{yx} = -4$ MPa. The stress component τ_{xy} acts on the x-plane ($ABCD$, $OEFG$) [Fig. 4.8a]. To find the positive direction of this component on the $ABCD$ plane, consider a tensile stress on this face. This acts along the positive x-direction (coming towards us from the page). Hence, according to the sign convention for shear stresses, the positive direction of τ_{xy} on this plane $ABCD$ is along the positive

[15] We shall see a similar figure later [Fig. 5.3, p. 5-5]. There the planes are physically separated by the distances dx, dy and dz; after writing the stress components and, thus, the equations of equilibrium, we proceed to the limit as dx, dy, $dz \to 0$ to obtain the differential equations of equilibrium *at the point* P.



(a) $\tau_{xy} = \tau_{yx} = -4$ MPa (b) $\tau_{yz} = \tau_{zy} = 2$ MPa (c) $\sigma_{zz} = -3$ MPa

Figure 4.8: Sign conventions: another example for stress components: positive (Fig. 4.8a) and negative (Fig. 4.8b) stress components are shown. For example, $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0$, $\tau_{xy} = \tau_{yx} = -4$, $\tau_{yz} = \tau_{zy} = 0$, $\tau_{zx} = \tau_{xz} = 0$, all in MPa are shown in Fig. 4.8a; $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0$, $\tau_{xy} = \tau_{yx} = 0$, $\tau_{yz} = \tau_{zy} = 2$, $\tau_{zx} = \tau_{xz} = 0$, all in MPa are shown in Fig. 4.8b; and $\sigma_{xx} = \sigma_{yy} = 0$, $\sigma_{zz} = -3$, $\tau_{xy} = \tau_{yx} = -2$, $\tau_{yz} = \tau_{zy} = 0$, $\tau_{zx} = \tau_{xz} = 0$, all in MPa are shown in Fig. 4.8c. We shall see later that Fig. 4.8a and Fig. 4.8b represent cases of pure shear, and that the z-plane in Fig. 4.8c is a principal plane.

y-coordinate direction (to the right). Here $\tau_{xy} = -4$ MPa. Thus, this stress component acts to the left. Having already reversed the direction to take care of the negative sign, the magnitude is to be marked as 4 MPa (and not as -4 MPa). On the plane $OEFG$ which is also the *same* x-plane, the direction of $\tau_{xy} = -4$ MPa is in the opposite direction, i.e., to the right, as shown in Fig. 4.8a.

We know this. However, we can also arrive at it arguing out as before. On the plane $OEFG$, a tensile stress acts in the negative x-direction (going away from us). Hence the *positive direction* of the shear stress component τ_{xy} on this face $OEFG$ is along the *negative* y-direction (to the left). Here $\tau_{xy} = -4$ MPa; hence it is marked as 4 MPa (and not as -4 MPa) (to the right).

LAW OF TRANSFORMATION OF THE STRESS COMPONENTS

The nine (9) stress components, arranged in the form of a matrix following the stated pattern, constitute the stress matrix with reference to the (x, y, z) coordinate system. These are independent[16]; they together give a complete description of the state of stress at the point P. That is, the stress components at the same point, when referred to other coordinate systems, are all indirectly contained in this set of nine components.

Thus, we are naturally led to examine this question: given the nine (9) stress components at a point w.r. to a coordinate system (x, y, z), how shall we obtain the stress components at the same point w.r. to a different coordinate system (x', y', z')? The 'new' stress components in the 'new' system are obviously related to the 'old' ones in the 'old' system in a way that

[16] In all the cases that we deal with, the stress matrix turns out to be symmetric, in which case only six (6) of them are independent. The conclusion that it is symmetric follows from the moment equations of equilibrium. We shall call attention to this fact when we discuss the equations of equilibrium.

depends decisively on how the 'new' axes are oriented relative to the 'old' ones. This exercise leads to the important law of transformation of the nine stress components. We shall examine this question, but before that we need some simple results which are of the nature of prerequisites[17].

Complete Description of the State of Stress

We stated above that these nine (9) stress components describe the state of stress at a point completely. If this is so, it should be possible for us to extract from these nine components all the information that we seek, and answer questions such as the following.

(i) How can we obtain the stress components on an inclined plane?

(ii) How can we find the stress components referred to a different set of (right handed cartesian) coordinate system, say, (x', y', z')?

(iii) How can we find the maximum normal stress and locate its plane?

(iv) Are there planes on which there is no shear stress at all (even in the general case)?

(v) How can we find the maximum shear stress and identify the corresponding plane?

(vi) Is there a coordinate system referred to which there are only normal stresses? Can there be several such coordinate systems? If so, how many in general?

(vii) $\cdots$

We need to answer all these questions. We shall take them one by one. First let us obtain the stress components on an inclined plane. If this set of nine (9) stress components gives a complete description of the state of stress, then all the information pertaining to the state of stress at P is contained, directly or indirectly, in this set. In particular, it should be possible to find the normal and shear stress components on any inclined plane with the direction cosines (l, m, n) w.r.to the coordinate axes passing through the *same* point P.

Stress Components on an Inclined Plane

The general approach is this. Let us choose a plane ABC [Fig. 4.9a]. Let $\overset{(\nu)}{\boldsymbol{T}}$ be the stress vector on this inclined plane that has a normal ν with its direction cosines (l, m, n). We desire to find this stress vector (and its components) and express it (them) in terms of the known (given) nine (9) stress components σ_{ij}, $(i, j = 1, 2, 3)$. This is the problem. How can we obtain these sought after relationships?

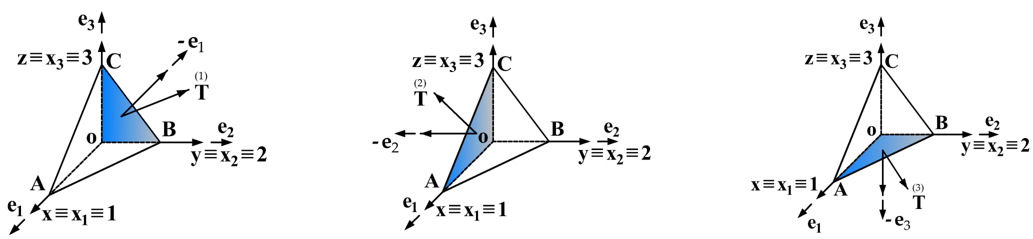
We consider the tetrahedron $OABC$ and mark all the stress components acting on all its faces. The stress components acting on the flat surfaces are known (given). We

[17] Similar observations are equally valid in other coordinate systems also. For example, in a (2-dimensional) polar coordinate system, the stress components are $\sigma_{rr}, \sigma_{\theta\theta}, \tau_{r\theta} = \tau_{\theta r}$. As we are concerned only about cartesian tensors, we consider only rectangular cartesian coordinate systems.

(a) $\overset{(\nu)}{\boldsymbol{T}}$ on ABC (normal ν) (b) $\overset{(\nu)}{\boldsymbol{T}}$ on ABC resolved along the axes

Figure 4.9: To calculate the stress components on an inclined plane ABC

desire to know the normal and shear stresses on the inclined surface ABC. The body is in equilibrium, and hence there must be force balance in each of three coordinate directions x, y and z. The stress components multiplied by the respective areas give us the forces; it is the equilibrium of this system of forces that we consider. The unknown stress components on the inclined plane are thus written in terms of the known nine (9) stress components σ_{ij} $(i, j = 1, 2, 3)$. After obtaining these relationships, let us move the inclined plane ABC parallel to itself making the tetrahedron smaller and smaller, until in the limit the inclined plane passes through the point P. This is the general plan. Now we shall carry out the various steps in accordance with this plan.



(a) $\overset{(x)}{\boldsymbol{T}}$ on OBC (normal x) (b) $\overset{(y)}{\boldsymbol{T}}$ on OCA (normal y) (c) $\overset{(z)}{\boldsymbol{T}}$ on OAB (normal z)

Figure 4.10: To calculate the stress components on an inclined plane ABC

$\overset{(\nu)}{\boldsymbol{T}}$ (with its components $\overset{(\nu)}{T}_i$, $(i = x, y, z)$ along the coordinate directions) is the resultant stress vector on the plane ABC shown [Fig. 4.9a] shaded for clarity. This is, in general, not along the normal. The components along the coordinate directions, are shown in Fig. 4.9b. The normal to this inclined plane has the direction cosines (l, m, n). Let us resolve each of the three stress vectors in terms of the base vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$.

$$\text{stress vector on } OBC \qquad \overset{(x)}{\boldsymbol{T}} = -\sigma_{xx}\,\boldsymbol{i} - \tau_{xy}\,\boldsymbol{j} - \tau_{xz}\,\boldsymbol{k}; \tag{4.1a}$$

$$\text{stress vector on } OAC \qquad \overset{(y)}{\boldsymbol{T}} = -\sigma_{yx}\,\boldsymbol{i} - \sigma_{yy}\,\boldsymbol{j} - \tau_{yz}\,\boldsymbol{k}; \text{ and} \tag{4.1b}$$

$$\text{stress vector on } OAB \qquad \overset{(x)}{\boldsymbol{T}} = -\sigma_{xx}\,\boldsymbol{i} - \tau_{xy}\,\boldsymbol{j} - \tau_{xz}\,\boldsymbol{k}. \tag{4.1c}$$

The tetrahedron $OABC$ is in equilibrium. Hence, there must be force balance in every direction. Let us consider the equation of equilibrium in, say, the y-direction. Let Δ be the area of the inclined face ABC. Then, the area OBC is its projection on the x-plane. Its magnitude is, therefore, $l\,\Delta$. Similarly, the areas OAC and OAB are, respectively, $m\,\Delta$ and $n\,\Delta$. The force components that appear in the equation of equilibrium are the following.

Face ABC:	stress component	$\overset{(\nu)}{T}_y$;	area Δ;	force $\overset{(\nu)}{T}_y\,\Delta$
Face OAC:	stress component	σ_{yy};	area $m\,\Delta$;	force $\sigma_{yy}\,m\,\Delta$
Face OBC:	stress component	τ_{xy};	area $l\,\Delta$;	force $\tau_{xy}\,l\,\Delta$
Face OAB:	stress component	τ_{zy};	area $n\,\Delta$;	force $\tau_{zy}\,n\,\Delta$
Volume $OABC$:	body force	F_y;	volume *small*;	force $F_y \times$ *small*

$$\text{Force balance in the } y\text{-direction:} \quad \overset{(\nu)}{T}_y\,\Delta + \text{body force} = \tau_{xy}\,l\,\Delta + \sigma_{yy}\,m\,\Delta + \tau_{zy}\,n\,\Delta.$$

Now let us make the tetrahedron smaller and smaller, and watch how this equation of equilibrium behaves as $\Delta \to 0$. During this process the plane ABC must retain its orientation. Thus, it stays parallel to the original ABC with the same direction cosines (l, m, n). Let us note that, as Δ becomes *small*, the surface forces become *smaller and smaller*, while the body force become *smaller and smaller and smaller*. Thus, the body force term $\to 0$ faster and, consequently does not come into our reckoning. The inclined plane in the limit passes through the point P, and the equation of equilibrium in the y-direction appears as

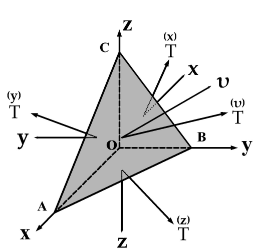
$$\overset{(\nu)}{T}_y \equiv p_{\nu y} = \tau_{xy}\,l + \sigma_{yy}\,m + \tau_{zy}\,n.$$

Along with its companion equations in the x- and z- directions, the three equations of equilibrium are:

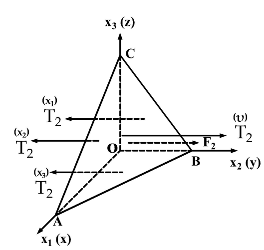
$$\overset{(\nu)}{T}_x \equiv p_{\nu x} = \sigma_{xx}\,l + \sigma_{yx}\,m + \tau_{zx}\,n \tag{4.2a}$$

$$\overset{(\nu)}{T}_y \equiv p_{\nu y} = \tau_{xy}\,l + \sigma_{yy}\,m + \tau_{zy}\,n \tag{4.2b}$$

$$\overset{(\nu)}{T}_z \equiv p_{\nu z} = \tau_{xz}\,l + \tau_{yz}\,m + \sigma_{zz}\,n \tag{4.2c}$$



(a) Stress vectors on all the four faces



(b) All the stress components in the y-direction

Figure 4.11: To write down the equation of equilibrium (in the y-direction)

This set of equations appears in index notation as

$$\overset{(\nu)}{T_i} = \sigma_{ji}\, n_j \quad \text{or} \quad T_i = \sigma_{ji}\, n_j \qquad (i, j = 1, 2, 3). \tag{4.3}$$

This is known as Cauchy's formula or Cauchy's[18] result. We shall see this again. In view of the importance of this result, let us honour this by enclosing it in a box.

$$\boxed{\text{Cauchy's result:} \ \ \overset{(\nu)}{T_i} = \sigma_{ji}\, n_j \quad (i, j = 1, 2, 3)}$$

Now how do we calculate the normal and shear stress on this inclined plane? That is not difficult. If we resolve $p_{\nu x}$, $p_{\nu y}$, $p_{\nu z}$ back again on the normal ν to the plane, and add them up, we would obtain the normal stress. Thus,

$$\begin{aligned}
\sigma_{\nu\nu} &= \overset{(\nu)}{\boldsymbol{T}} \,.\, \boldsymbol{\nu} = p_{\nu x}\, l + p_{\nu y}\, m + p_{\nu z}\, n \\
&= [l\sigma_{xx} + m\tau_{yx} + n\tau_{zx}]\ l + [l\tau_{xy} + m\sigma_{yy} + n\tau_{zy}]\ m + [l\tau_{xz} + m\tau_{yz} + n\sigma_{zz}]\ n \\
&= l^2\sigma_{xx} + m^2\sigma_{yy} + n^2\sigma_{zz} + 2lm\tau_{xy} + 2mn\tau_{yz} + 2nl\tau_{zx},
\end{aligned} \tag{4.4}$$

where we have assumed the symmetry of the stress matrix ($\tau_{ij} = \tau_{ji}$).

Having obtained the normal stress, we can calculate the (total, resultant) shear stress on the inclined plane as follows.

$$\begin{aligned}
\tau_{(\nu)} &= \sqrt{\left|\overset{(\nu)}{\boldsymbol{T}}\right|^2 - \left|\sigma_{\nu\nu}\right|^2} = \sqrt{[p_{\nu x}^2 + p_{\nu y}^2 + p_{\nu z}^2] - [\sigma_{\nu\nu}]^2} \\
\tau_{(\nu)}^2 &= [l\sigma_{xx} + m\tau_{yx} + n\tau_{zx}]^2 + [l\tau_{xy} + m\sigma_{yy} + n\tau_{zy}]^2 + [l\tau_{xz} + m\tau_{yz} + n\sigma_{zz}]^2 \\
&\quad - \left[l^2\sigma_{xx} + m^2\sigma_{yy} + n^2\sigma_{zz} + 2lm\tau_{xy} + 2mn\tau_{yz} + 2nl\tau_{zx}\right]^2
\end{aligned} \tag{4.5}$$

We shall work out a small numerical example to illustrate this.

A Numerical Example:

The state of stress at a point is given by the matrix given on p. 13-2, Sec. 2 w.r.to the axes (x, y, z). We desire to calculate the normal and shear stresses on a plane defined by the direction cosines $(0.42, 0.50, 0.76)$.

Note first of all that the direction cosines (l, m, n) satisfy the relation

$$l^2 + m^2 + n^2 = 0.42^2 + 0.50^2 + 0.76^2 = 1.$$

[18] Augustine-Louis Cauchy (Aug. 1789 - May 1857) was an outstanding French mathematician. Students of engineering would be interested to know that Cauchy graduated in Civil Engineering and that he was a civil engineer in the earlier part of his career. His fame is, however, as a great mathematician. He had established several important results in the theory of elasticity.

Referring to the stress matrix given on p. 13-2, Sec. 2 w.r.to the axes (x, y, z), we recognise that the entries in the same (horizontal) row are the stress components acting in the same direction. Thus,

$$\text{the stress vector on the } x\text{-plane, } \overset{(x)}{\boldsymbol{T}} = 10\,\boldsymbol{i} + 12\,\boldsymbol{j} + 8\,\boldsymbol{k};$$

$$\text{the stress vector on the } y\text{-plane, } \overset{(y)}{\boldsymbol{T}} = 12\,\boldsymbol{i} + 15\,\boldsymbol{j} - 5\,\boldsymbol{k}; \text{ and}$$

$$\text{the stress vector on the } z\text{-plane, } \overset{(z)}{\boldsymbol{T}} = 8\,\boldsymbol{i} - 5\,\boldsymbol{j} - 7\,\boldsymbol{k}.$$

The components of the stress vector $\overset{(\nu)}{\boldsymbol{T}}$ are

$$\begin{aligned}
\overset{(\nu)}{T_x} \equiv p_{\nu x} &= l\sigma_{xx} + m\tau_{yx} + n\tau_{zx} \\
&= 0.42 \times 10 + 0.50 \times 12 + 0.76 \times 8 = 16.28 \text{ MPa}; \\
\overset{(\nu)}{T_y} \equiv p_{\nu y} &= l\tau_{xy} + m\sigma_{yy} + n\tau_{zy} \\
&= 0.42 \times 12 + 0.50 \times 15 - 0.76 \times 5 = 8.74 \text{ MPa}; \text{ and} \\
\overset{(\nu)}{T_y} \equiv p_{\nu z} &= l\tau_{xz} + m\tau_{yz} + n\sigma_{zz} \\
&= 0.42 \times 8 + 0.50 \times (-5) + 0.76 \times (-7) = -4.46 \text{ MPa}.
\end{aligned}$$

The magnitude of the stress vector on the inclined plane is, therefore,

$$\begin{aligned}
\left|\overset{(\nu)}{\boldsymbol{T}}\right| &= \sqrt{p_{\nu x}^2 + p_{\nu y}^2 + p_{\nu z}^2} \equiv \sqrt{[\overset{(\nu)}{T_x}]^2 + [\overset{(\nu)}{T_y}]^2 + [\overset{(\nu)}{T_y}]^2} \\
&= \sqrt{16.28^2 + 8.74^2 + 4.46^2} = 19.01 \text{ MPa}.
\end{aligned}$$

The normal stress on the inclined (ν) plane $= lp_{\nu x} + mp_{\nu y} + np_{\nu z}$

$$= 0.42 \times 16.28 + 0.50 \times 8.74 - 0.76 \times 4.46 = 7.82 \text{ MPa},$$

and the shear stress on the same inclined (ν) plane $= \sqrt{19.01^2 - 7.82^2} = 17.33$ MPa.

Having learned how to calculate the stresses on an inclined plane, we are now ready to tackle the problem of stress transformation.

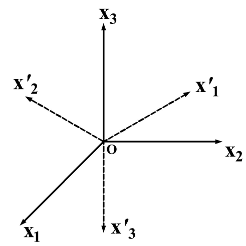
Transformation of Stress Components

Let us now revert to the question that we raised earlier: given the nine (9) stress components at a point P with reference to a rectangular orthogonal coordinate system, how can we obtain the new stress components when the axes are changed from the 'old' ones (x, y, z), that is, (x_1, x_2, x_3), to the 'new' ones (x_1', x_2', x_3')? As $(x_1, x_2, x_3) \quad \longrightarrow \quad (x_1', x_2', x_3')$,

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}_{(x_1,x_2,x_3)} \longrightarrow \begin{bmatrix} \sigma'_{x'x'} & \tau'_{x'y'} & \tau'_{x'z'} \\ \tau'_{y'x'} & \sigma'_{y'y'} & \tau'_{y'z'} \\ \tau'_{z'x'} & \tau'_{z'y'} & \sigma'_{z'z'} \end{bmatrix}_{(x_1',x_2',x_3')} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}_{(x_1',x_2',x_3')}$$

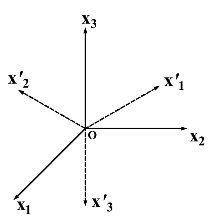
New coordinates	Old coordinates x_1	x_2	x_3
x_1'	a_{11}	a_{12}	a_{13}
x_2'	a_{21}	a_{22}	a_{23}
x_3'	a_{31}	a_{32}	a_{33}

Table 4.1: Table of direction cosines

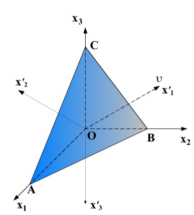


First we note that the nine (9) 'new' stress components $\sigma'_{i'j'}$ depend on the 'old' ones in a way that depends decisively on how the 'new' coordinates are disposed (oriented) relative to the 'old' ones. Thus, we must know or specify the various direction cosines. Table 4.1 and the figure alongside specify the orientations of the 'new' axes relative to the 'old' ones.

Shown here is a table of direction cosines that defines or specifies the orientation of the 'new' coordinate system relative to the 'old' one. Thus, for example, $a_{12} \equiv a_{1'2} \equiv \cos(x_1', x_2)$. Note further that $a_{12} \equiv a_{1'2} \neq a_{12'} \equiv a_{21}$; that is, this matrix of direction cosines is not symmetric in general. We know that the direction cosines satisfies the two sets of orthogonal conditions. The procedure is explained below.



(a) Two coordinate systems, 'old' and 'new'



(b) Choose an inclined plane ABC

Figure 4.12: 'Old' (x_1, x_2, x_3) and 'new' (x_1', x_2', x_3') coordinate systems: an inclined plane ABC is chosen with its normal along the x_1' axis.

(i) First calculate the stress components on the x_1' plane, viz., $(\sigma'_{x_1'x_1'},\ \tau'_{x_1'x_2'},\ \tau'_{x_1'x_3'})$[19]. For this, construct a triangle ABC with its normal along the (x_1') axis [Fig. 4.12].

(ii) Next calculate the stress components $p_{\nu x} \equiv \overset{(\nu)}{T}_1,\ \ p_{\nu y} \equiv \overset{(\nu)}{T}_2,\ \ p_{\nu z} \equiv \overset{(\nu)}{T}_3$. These are given by Eqs (4.2a, 4.2b, 4.2c) (Cauchy's result) as

$$\overset{(\nu)}{T}_x \equiv p_{\nu x} = \sigma_{xx}\, l + \sigma_{yx}\, m + \tau_{zx}\, n$$
$$\overset{(\nu)}{T}_y \equiv p_{\nu y} = \tau_{xy}\, l + \sigma_{yy}\, m + \tau_{zy}\, n$$

[19]Hereafter we will drop the prime marks on the subscripts and write only as σ'_{11} for $\sigma'_{x'x'}$. The primes are dropped only for the number subscripts, but not for the letter subscripts.

$$\overset{(\nu)}{T_z} \equiv p_{\nu z} = \tau_{xz}\, l + \tau_{yz}\, m + \sigma_{zz}\, n$$

(iii) We now note that x_1' is normal to the plane ABC and, therefore, x_2' and x_3' being perpendicular to x_1' lie along the plane ABC. Thus, the stress components on the plane ABC along these directions are shear stresses.

(iv) We have already calculated $\overset{(\nu)}{T_x} \equiv p_{\nu x}$; $\overset{(\nu)}{T_y} \equiv p_{\nu y}$; $\overset{(\nu)}{T_z} \equiv p_{\nu z}$ in terms of the given stress components w.r.to the 'old' coordinate system (x, y, z).

To obtain the 'new' stress components, all we have to do is to resolve the above components in the direction concerned and add up. Thus,

$$\begin{aligned}
\sigma'_{x'x'} =& p_{\nu x}\, \cos(x', x) + p_{\nu y}\, \cos(x', y) + p_{\nu z}\, \cos(x', z) \\
=& l\,[l\sigma_{xx} + m\tau_{yx} + n\tau_{zx}] + m\,[l\tau_{xy} + m\sigma_{yy} + n\tau_{zy}] + n\,[l\tau_{xz} + m\tau_{yz} + n\sigma_{zz}] \\
=& \cos(x', x)\,\big[\cos(x', x)\,\sigma_{xx} + \cos(x', y)\,\tau_{yx} + \cos(x', z)\,\tau_{zx}\big] \\
&+ \cos(x', y)\,\big[\cos(x', x)\,\tau_{xy} + \cos(x', y)\,\sigma_{yy} + \cos(x', z)\,\tau_{zy}\big] \\
&+ \cos(x', z)\,\big[\cos(x', x)\,\tau_{xz} + \cos(x', y)\,\tau_{yz} + \cos(x', z)\,\sigma_{zz}\big] \\
=& a_{11}\sigma_{xx}a_{11} + a_{11}\tau_{yx}a_{12} + a_{11}\tau_{zx}a_{13} \\
&+ a_{12}\tau_{xy}a_{11} + a_{12}\sigma_{yy}a_{12} + a_{12}\tau_{zy}a_{13} \\
&+ a_{13}\tau_{xz}a_{11} + a_{13}\tau_{yz}a_{12} + a_{13}\sigma_{zz}a_{13}
\end{aligned} \tag{4.7}$$

In the same way, we can project $p_{\nu x},\ p_{\nu y},\ p_{\nu z}$ on (i) on the y' axis and add them together to obtain $\tau_{x'y'}$; and (ii) on the z' axis to obtain $\tau_{x'z'}$. These are not difficult at all, but they can be confusing. The pattern, once recognised, would help us.

$$\begin{aligned}
\tau'_{x'y'} =& p_{\nu x}\, \cos(y', x) + p_{\nu y}\, \cos(y', y) + p_{\nu z}\, \cos(y', z) \\
=& a_{21}\,[a_{11}\sigma_{xx} + a_{12}\tau_{yx} + a_{13}\tau_{zx}] \\
&+ a_{22}\,[a_{11}\tau_{xy} + a_{12}\sigma_{yy} + a_{13}\tau_{zy}] \\
&+ a_{23}\,[a_{11}\tau_{xz} + a_{12}\tau_{yz} + a_{13}\sigma_{zz}] \\
=& a_{21}\sigma_{xx}a_{11} + a_{21}\tau_{yx}a_{12} + a_{21}\tau_{zx}a_{13} \\
&+ a_{22}\tau_{xy}a_{11} + a_{22}\sigma_{yy}a_{12} + a_{22}\tau_{zy}a_{13} \\
&+ a_{23}\tau_{xz}a_{11} + a_{23}\tau_{yz}a_{12} + a_{23}\sigma_{zz}a_{13}
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
\tau'_{x'z'} =& p_{\nu x}\, \cos(z', x) + p_{\nu y}\, \cos(z', y) + p_{\nu z}\, \cos(z', z) \\
=& a_{31}\,[a_{11}\sigma_{xx} + a_{12}\tau_{yx} + a_{13}\tau_{zx}] \\
&+ a_{32}\,[a_{11}\tau_{xy} + a_{12}\sigma_{yy} + a_{13}\tau_{zy}] \\
&+ a_{33}\,[a_{11}\tau_{xz} + a_{12}\tau_{yz} + a_{13}\sigma_{zz}] \\
=& a_{31}\sigma_{xx}a_{11} + a_{31}\tau_{yx}a_{12} + a_{31}\tau_{zx}a_{13} \\
&+ a_{32}\tau_{xy}a_{11} + a_{32}\sigma_{yy}a_{12} + a_{32}\tau_{zy}a_{13} \\
&+ a_{33}\tau_{xz}a_{11} + a_{33}\tau_{yz}a_{12} + a_{33}\sigma_{zz}a_{13}
\end{aligned} \tag{4.9}$$

(v) We have thus far obtained the three stress components on the x' plane. To obtain the stress components of the y', we can proceed similarly by considering an inclined plane with y' as its normal. After obtaining the three stress components on this y' plane, start afresh and consider an inclined plane with z' as its normal. The procedure is entirely similar. In this way all the nine 'new' stress components can be obtained. All these nine 'new' stress components — really only six because of the symmetry of the stress matrix — are now expressed in terms of the given 'old' stress components and the set of nine direction cosines a_{ij}, $(i, j = 1, 2, 3)$. These are the stress transformation equations.

STRESS TRANSFORMATION EQUATIONS

The final results of the above exercise are displayed below in full.

$$\begin{aligned}
\sigma'_{x'x'} \equiv \sigma'_{11} = & a_{11}\sigma_{xx}a_{11} + a_{11}\tau_{yx}a_{12} + a_{11}\tau_{zx}a_{13} \\
& + a_{12}\tau_{xy}a_{11} + a_{12}\sigma_{yy}a_{12} + a_{12}\tau_{zy}a_{13} \\
& + a_{13}\tau_{xz}a_{11} + a_{13}\tau_{yz}a_{12} + a_{13}\sigma_{zz}a_{13}
\end{aligned} \tag{4.10a}$$

$$\begin{aligned}
\tau'_{x'y'} \equiv \sigma'_{12} = & a_{21}\sigma_{xx}a_{11} + a_{21}\tau_{yx}a_{12} + a_{21}\tau_{zx}a_{13} \\
& + a_{22}\tau_{xy}a_{11} + a_{22}\sigma_{yy}a_{12} + a_{22}\tau_{zy}a_{13} \\
& + a_{23}\tau_{xz}a_{11} + a_{23}\tau_{yz}a_{12} + a_{23}\sigma_{zz}a_{13}
\end{aligned} \tag{4.10b}$$

$$\begin{aligned}
\tau'_{x'z'} \equiv \sigma'_{13} = & a_{31}\sigma_{xx}a_{11} + a_{31}\tau_{yx}a_{12} + a_{31}\tau_{zx}a_{13} \\
& + a_{32}\tau_{xy}a_{11} + a_{32}\sigma_{yy}a_{12} + a_{32}\tau_{zy}a_{13} \\
& + a_{33}\tau_{xz}a_{11} + a_{33}\tau_{yz}a_{12} + a_{33}\sigma_{zz}a_{13}
\end{aligned} \tag{4.10c}$$

$$\begin{aligned}
\tau'_{y'x'} \equiv \sigma'_{21} = & a_{11}\sigma_{xx}a_{21} + a_{11}\tau_{yx}a_{22} + a_{11}\tau_{zx}a_{23} \\
& + a_{12}\tau_{xy}a_{21} + a_{12}\sigma_{yy}a_{22} + a_{12}\tau_{zy}a_{23} \\
& + a_{13}\tau_{xz}a_{21} + a_{13}\tau_{yz}a_{22} + a_{13}\sigma_{zz}a_{23}
\end{aligned} \tag{4.10d}$$

$$\begin{aligned}
\sigma'_{y'y'} \equiv \sigma'_{22} = & a_{21}\sigma_{xx}a_{21} + a_{21}\tau_{yx}a_{22} + a_{21}\tau_{zx}a_{23} \\
& + a_{22}\tau_{xy}a_{21} + a_{22}\sigma_{yy}a_{22} + a_{22}\tau_{zy}a_{23} \\
& + a_{23}\tau_{xz}a_{21} + a_{23}\tau_{yz}a_{22} + a_{23}\sigma_{zz}a_{23}
\end{aligned} \tag{4.10e}$$

$$\begin{aligned}
\tau'_{y'z'} \equiv \sigma'_{23} = & a_{31}\sigma_{xx}a_{21} + a_{31}\tau_{yx}a_{22} + a_{31}\tau_{zx}a_{23} \\
& + a_{32}\tau_{xy}a_{21} + a_{32}\sigma_{yy}a_{22} + a_{32}\tau_{zy}a_{23} \\
& + a_{33}\tau_{xz}a_{21} + a_{33}\tau_{yz}a_{22} + a_{33}\sigma_{zz}a_{23}
\end{aligned} \tag{4.10f}$$

$$\begin{aligned}
\tau'_{z'x'} \equiv \sigma'_{31} = & a_{11}\sigma_{xx}a_{31} + a_{11}\tau_{yx}a_{32} + a_{11}\tau_{zx}a_{33} \\
& + a_{12}\tau_{xy}a_{31} + a_{12}\sigma_{yy}a_{32} + a_{12}\tau_{zy}a_{33} \\
& + a_{13}\tau_{xz}a_{31} + a_{13}\tau_{yz}a_{32} + a_{13}\sigma_{zz}a_{33}
\end{aligned} \tag{4.10g}$$

$$\begin{aligned}
\tau'_{z'y'} \equiv \sigma'_{32} = & a_{21}\sigma_{xx}a_{31} + a_{21}\tau_{yx}a_{32} + a_{21}\tau_{zx}a_{33} \\
& + a_{22}\tau_{xy}a_{31} + a_{22}\sigma_{yy}a_{32} + a_{22}\tau_{zy}a_{33} \\
& + a_{23}\tau_{xz}a_{31} + a_{23}\tau_{yz}a_{32} + a_{23}\sigma_{zz}a_{33}
\end{aligned} \tag{4.10h}$$

$$\begin{aligned}\sigma'_{z'z'} \equiv \sigma'_{33} = & a_{31}\sigma_{xx}a_{31} + a_{31}\tau_{yx}a_{32} + a_{31}\tau_{zx}a_{33} \\ & + a_{32}\tau_{xy}a_{31} + a_{32}\sigma_{yy}a_{32} + a_{32}\tau_{zy}a_{33} \\ & + a_{33}\tau_{xz}a_{31} + a_{33}\tau_{yz}a_{32} + a_{33}\sigma_{zz}a_{33}\end{aligned} \tag{4.10i}$$

Let us also note that $\tau'_{x'y'} = \tau'_{y'x'} \qquad \tau'_{y'z'} = \tau'_{z'y'} \qquad \tau'_{x'z'} = \tau'_{z'x'}$.

(that is, $\sigma'_{ij} = \sigma'_{ji}$ symmetry of the stress matrix).

These formidable equations [Eqs 4.10a - 4.10i] — nine (9) long equations each of which has nine (9) terms on the right hand side! — can be encapsulated in one small equation using the index notation as

$$\sigma'_{i'j'} = a_{i'k}\, a_{j'l}\, \sigma_{kl} \qquad \text{more commonly written as} \qquad \sigma'_{ij} = a_{ik}\, a_{jl}\, \sigma_{kl}. \tag{4.11}$$

If we take this equation (4.11) and carry out the implied summations, we can obtain the full length formulae shown above [Eqs (4.10a - 4.10i]. We shall demonstrate this for one special case, say, for $i = 1,\ j = 2$.

$$\begin{aligned} i = 1,\ j = 2 \qquad \sigma'_{12} \equiv \tau_{xy} &= \sum_{k=1}^{3} [a_{11}\, a_{2l}\, \sigma_{1l} + a_{12}\, a_{2l}\, \sigma_{2l} + a_{13}\, a_{2l}\, \sigma_{3l}] \\ &= a_{11}\, [a_{21}\, \sigma_{11} + a_{22}\, \sigma_{12} + a_{23}\, \sigma_{13}] \\ &+ a_{12}\, [a_{21}\, \sigma_{21} + a_{22}\, \sigma_{22} + a_{23}\, \sigma_{23}] \\ &+ a_{13}\, [a_{21}\, \sigma_{31} + a_{22}\, \sigma_{32} + a_{23}\, \sigma_{33}] \\ &= a_{11}\, \sigma_{xx}\, a_{21} + a_{11}\, \tau_{xy}\, a_{22} + a_{11}\, \tau_{xz}\, a_{23} \\ &+ a_{12}\, \tau_{yx}\, a_{21} + a_{12}\, \sigma_{yy}\, a_{22} + a_{12}\, \tau_{yz}\, a_{23} \\ &+ a_{13}\, \tau_{zx}\, a_{21} + a_{13}\, \tau_{zy}\, a_{22} + a_{13}\, \sigma_{zz}\, a_{23} \end{aligned}$$

which agrees with Eq. (4.10b and 4.10d).

We may also write Eq. (4.11), which is the same as the set of equations (4.10a - 4.10i, in the matrix form as

$$[\sigma'] = [a][\sigma][a]^T \tag{4.12}$$

where $[a]$ is the matrix of the direction cosines [Table: 4.1, p. 4-14]. More explicitly, the transformation law is given by the matrix equation

$$\begin{bmatrix} \sigma'_{x'x'} & \tau'_{x'y'} & \tau'_{x'z'} \\ \tau'_{y'x'} & \sigma'_{y'y'} & \tau'_{y'z'} \\ \tau'_{z'x'} & \tau'_{z'y'} & \sigma'_{z'z'} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}. \tag{4.13}$$

PRINCIPAL PLANES AND PRINCIPAL STRESSES

Principal planes and principal (normal) stresses are of fundamental importance in the science of stress analysis. What are principal planes? Well, they are those planes on which

there is no shear stress[20]. That is, the stress vector on a principal plane is entirely normal to it. There are a number of associated questions: (i) are there such planes at every point for the general state of stress; (ii) how many such planes can we have in general; etc. We shall answer all such questions in due course. But first let us formulate the problem.

Formulation of the Problem

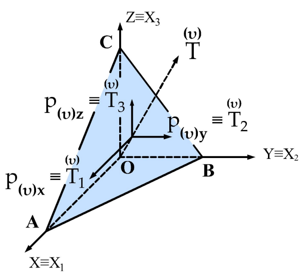


Figure 4.13: ABC: a principal plane

If ABC is a principal plane, then the stress vector $\overset{(\nu)}{\boldsymbol{T}}$ on this plane is perfectly normal to the plane. Let us call this magnitude by σ. This means that there are no shear stresses on this plane. It also means that $p_{\nu x}$, $p_{\nu y}$, $p_{\nu z}$ are the components of σ. Thus,

$$p_{\nu x} = l\,\sigma; \qquad p_{\nu y} = m\,\sigma; \qquad p_{\nu z} = n\,\sigma;$$

where σ is the magnitude of the principal stress (normal stress on the principal plane ABC) vector.

We have already obtained the expressions for $p_{\nu x}$, $p_{\nu y}$, $p_{\nu z}$ [Eqs (4.2a, 4.2b, 4.2c), p. 4-12]. Substituting these in the above equation, we obtain

$$\begin{aligned} l\,\sigma_{xx} + m\,\tau_{yx} + n\,\tau_{zx} &= l\,\sigma; \\ l\,\tau_{xy} + m\,\sigma_{yy} + n\,\tau_{zy} &= m\,\sigma; \\ l\,\tau_{xz} + m\,\tau_{yz} + n\,\sigma_{zz} &= n\,\sigma. \end{aligned}$$

This set of equations may be written in matrix form as

$$\begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \sigma \begin{Bmatrix} l \\ m \\ n \end{Bmatrix}; \quad \text{i.e.,} \quad \boldsymbol{\sigma}\,\boldsymbol{x} = \sigma\,\boldsymbol{x}. \tag{4.14}$$

What are known, and what are unknown, in this equation? The stress components are known (or given); σ, a parameter, is unknown at this stage. What are to be determined? The non-trivial solutions for l, m, n and the values of the parameter σ for which these non-trivial solutions exist.

We recognise this immediately as an eigenvalue problem (often written as $\boldsymbol{A}\boldsymbol{x} = \lambda\,\boldsymbol{x}$, where λ is the eigenvalue, and $\boldsymbol{x}$ the eigenvector). We have thus formulated the physical

[20] We shall see that the normal stress reaches its stationary values on the principal planes. We shall explain this clearly a little later. We will see that we can have two equally acceptable possibilities of defining principal planes: (i) as those planes on which the shear stress is zero; and then we can show that on these special planes, the normal stress reaches its stationary values; or (ii) as those planes on which the normal stress reaches its stationary values; and then we can show that on these special planes the shear stress is zero. Both are equally tenable. However, stationary values are a little more difficult conceptually to understand. Hence we shall define principal planes as those on which the shear stress is zero.

problem of determining the principal planes and the corresponding principal stresses as an eigenvalue problem. Here for our physical problem, the matrix $\boldsymbol{A}$ is the (given) stress matrix $\boldsymbol{\sigma}$, the eigenvalue λ is the principal stress σ, and the eigenvector $\boldsymbol{x}$ is the direction cosines. The eigenvector, we recall, is the non-trivial solution which exists when the parameter σ takes on one of the special values called the eigenvalues. The physical meanings here are: (i) the matrix $\boldsymbol{A}$ referred to above is here the (given) stress matrix $\boldsymbol{\sigma}$; (ii) σ, the eigenvalue, stands for the principal stresses; and (iii) $\boldsymbol{x}$, the eigenvector for the direction cosines (direction ratios)[21].

When written as a system of linear, algebraic, simultaneous, homogeneous equations, this appears as

$$\begin{bmatrix} (\sigma_{xx}-\sigma) & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & (\sigma_{yy}-\sigma) & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & (\sigma_{zz}-\sigma) \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}; \tag{4.15}$$

that is, as

$$(\boldsymbol{\sigma} - \sigma \boldsymbol{I})\,\boldsymbol{x} = \mathbf{0}. \tag{4.16}$$

We shall now continue with further analysis and discussion of this problem.

Further Analysis and Discussion

We notice that $l = m = n = 0$ is always[22] a solution of Eq. (4.14)). This is the trivial solution. But from a physical point of view, these are the direction cosines and, therefore, these must satisfy the condition $l^2+m^2+n^2 = 1$. Thus, the trivial solution is not physically acceptable. The question that we now ask is: is there a non-trivial solution and, if so, under what conditions?

We know from our earlier study[23] of linear algebra that, for a system of linear, algebraic, simultaneous, *homogeneous*[24] equations, a trivial solution always exists, and that for a non-trivial solution to exist the determinant of the coefficients must vanish.

Non-trivial solution, characteristic equation, eigenvalues:

Physically, if ABC is indeed a principal plane, then there must be at least one set of physically possible direction cosines. The trivial solution $l = m = n = 0$ will not do; this (trivial solution) cannot represent a physically meaningful set of direction cosines, because the direction cosines must satisfy the condition $l^2 + m^2 + n^2 = 1$. Thus, we can conclude that there must be a non-trivial solution. The condition for the existence of a non-trivial solution is that the determinant of the coefficients of the homogeneous set of equations shall

[21] It is clear from Eq. (4.14) that we can obtain only the direction ratios. The direction cosines may be obtained from these direction ratios by normalisation using the result $l^2 + m^2 + n^2 = 1$, that is, $x_1^2 + x_2^2 + x_3^2 = 1$.

[22] By 'always' we mean 'for all values of σ'.

[23] Readers are advised to refresh their knowledge of linear algebra. Such questions arise often enough in applied mathematics or mathematical engineering that it is wise to invest some time and energy to learn linear algebra fairly well.

[24] the 'right hand side' equal to zero

vanish. Thus, for ABC to be a principal plane,

$$\begin{vmatrix} \sigma_{xx}-\sigma & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy}-\sigma & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz}-\sigma \end{vmatrix} = 0. \tag{4.17}$$

This, we note, is the characteristic equation of the stress matrix σ_{ij} $(\boldsymbol{\sigma})$. When this is written out in full in terms of the unknown parameter σ, we obtain

$$\begin{aligned} \sigma^3 - \left[\sigma_{xx}+\sigma_{yyu}+\sigma_{zz}\right]\sigma^2 + \left[\sigma_{xx}\sigma_{yy}+\sigma_{yy}\sigma_{zz}+\sigma_{zz}\sigma_{xx}-\tau_{xy}^2-\tau_{yz}^2-\tau_{zx}^2\right]\sigma \\ - \left[\sigma_{xx}\sigma_{yy}\sigma_{zz}-\sigma_{xx}\tau_{yz}^2-\sigma_{yy}\tau_{zx}^2-\sigma_{zz}\tau_{xy}^2+2\tau_{xy}\tau_{yz}\tau_{zx}\right] = 0. \end{aligned} \tag{4.18}$$

This is conveniently written as

$$\text{characteristic equation:} \qquad \sigma^3 - [I_1]\sigma^2 + [I_2]\sigma - [I_3] = 0, \tag{4.19}$$

where I_1, I_2, I_3 are the coefficients as defined by Eq. (4.19). This, we repeat, is the characteristic equation of the stress matrix. Its roots are the eigenvalues. Physically they represent the principal stresses.

$$I_1 = \left[\sigma_{xx}+\sigma_{yy}+\sigma_{zz}\right] \tag{4.20a}$$

$$I_2 = \left[\sigma_{xx}\sigma_{yy}+\sigma_{yy}\sigma_{zz}+\sigma_{zz}\sigma_{xx}-\tau_{xy}^2-\tau_{yz}^2-\tau_{zx}^2\right] \tag{4.20b}$$

$$I_3 = \left[\sigma_{xx}\sigma_{yy}\sigma_{zz}-\sigma_{xx}\tau_{yz}^2-\sigma_{yy}\tau_{zx}^2-\sigma_{zz}\tau_{xy}^2+2\tau_{xy}\tau_{yz}\tau_{zx}\right] \tag{4.20c}$$

The roots of the characteristic equation [Eq. (11.18) which is the same as Eq. (4.19)] are the eigenvalues. Being a cubic equation in σ, this characteristic equation has exactly three roots. There are all necessarily real; a theorem in linear algebra guarantees this[25]. Let us call them σ_{11}, σ_{22}, σ_{33}. These are the eigenvalues representing the three principal stresses. Let us emphasise that some of them may be repeated roots. For example, we may have (i) $\sigma_{11} \neq \sigma_{22} \neq \sigma_{33}$ (all distinct); (ii) $\sigma_{11} = \sigma_{22} \neq \sigma_{33}$ (two of them are equal); and (iii) $\sigma_{11} = \sigma_{22} = \sigma_{33}$ (all the three of them are equal). Usually the algebraically largest is designated by σ_{11}, and the algebraically smallest by σ_{33}. That is, if the principal stresses are, say, -20 (that is, 20 compressive), 10 and 5 (all in MPa), we call them as $\sigma_{11} = 10$, $\sigma_{22} = 5$, $\sigma_{33} = -20$ (all in MPa).

Eigenvectors:

The eigenvectors corresponding to these eigenvalues represent the planes of the principal stresses. They can be found as explained below.

At this stage, the three eigenvalues are known. Let us first determine the eigenvector corresponding to the first eigenvalue $\sigma = \sigma_{11}$ which is now known. Substitute this known value in the set of equations [Eq. (11.16)] to yield

$$\begin{bmatrix} (\sigma_{xx}-\sigma_{11}) & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & (\sigma_{yy}-\sigma_{11}) & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & (\sigma_{zz}-\sigma_{11}) \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{4.21}$$

[25] The eigenvalues of a real symmetric matrix — and of a Hermitian symmetric matrix — are always real. We do not encounter Hermitian symmetric matrices in our discussions here.

Let us examine this set of equations. If $(l_1,\, m_1,\, n_1)$ is a solution, then $(\alpha l_1,\, \alpha m_1,\, \alpha n_1)$ is also a solution for all values of α. What does this mean? Well, this means that there are several solutions; there is no unique solution. We will be able to obtain only the ratios, and not the absolute values of $(l_1,\, m_1,\, n_1)$. This realisation — that we can obtain only the ratios, and not the absolute values of $(l_1,\, m_1,\, n_1)$ — leads us to conclude that there is no harm in giving any value to any one of $(l_1,\, m_1,\, n_1)$. Let us, then, put $l_1^* = 1$, and work out the corresponding values of m_1^* and n_1^*. These starred quantities stand for the direction ratios.

Eq. (4.21) is the same as

$$\begin{aligned}
(\sigma_{xx} - \sigma_{11})\, l_1^* + \tau_{yx}\, m_1^* + \tau_{zx}\, n_1^* &= 0 \\
\tau_{xy} l_1^* + (\sigma_{yy} - \sigma_{11})\, m_1^* + \tau_{zy}\, n_1^* &= 0 \\
\tau_{xz}\, l_1^* + \tau_{yz}\, m_1^* + (\sigma_{zz} - \sigma_{11})\, n_1^* &= 0.
\end{aligned}$$

On putting $l_1^* = 1$ — we can obtain only the ratios as argued out above — these equations appear as

$$\tau_{yx}\, m_1^* + \tau_{zx}\, n_1^* = \sigma_{11} - \sigma_{xx} \tag{4.22a}$$

$$(\sigma_{yy} - \sigma_{11})\, m_1^* + \tau_{zy}\, n_1^* = -\tau_{xy} \tag{4.22b}$$

$$\tau_{yz}\, m_1^* + (\sigma_{zz} - \sigma_{11})\, n_1^* = -\tau_{xz}. \tag{4.22c}$$

Here are now three equations in only two unknowns, viz., m_1^* and n_1^*! But there arises the question: are these equations consistent? The condition for the consistency of these equations — three equations in only two unknowns — is that the rank of the coefficient matrix is equal to the rank of the augmented matrix. The coefficient matrix is 3×2, which can have at the most a rank of 2. The augmented matrix is 3×3, but we know that its determinant is zero. Thus, the augmented matrix cannot have a rank 3!

Thus, any two equations from Eqs (4.22a, 4.22b, 4.22c) may be used to solve for m_1^* and n_1^*. We can be sure that the third equation (which was not used) will surely be satisfied. How can we be so sure? Well, we have discussed the consistency of these equations[26].

Having thus obtained the direction ratios $(l_1^*,\, m_1^*,\, n_1^*)$, we can get the direction cosines by normalising them as

$$l_1 = \frac{l_1^*}{\sqrt{(l_1^*)^2 + (m_1^*)^2 + (n_1^*)^2}};\;\; m_1 = \frac{m_1^*}{\sqrt{(l_1^*)^2 + (m_1^*)^2 + (n_1^*)^2}};\;\; n_1 = \frac{n_1^*}{\sqrt{(l_1^*)^2 + (m_1^*)^2 + (n_1^*)^2}}.$$

We have now calculated the eigenvector $(l_1\; m_1\; n_1)^T$ corresponding to the first eigenvalue σ_{11}. The physical significance is, let us repeat, that the eigenvalue represents the (first) principal stress, and the eigenvector the direction cosines of the normal to this principal plane.

[26] A special case arises when the set of three equations is of rank 1. Now we can set, say, $m_1^* = 2$ and solve for n_1^*. Recall an important result in linear algebra: if the rank of a set of $n \times n$ is r, there are exactly $(n - r)$ linearly independent solutions. This means that we can choose at will the values of two of the unknowns, say, l_1^* *and* m_1^* and calculate the value of the third unknown, here n_1^*.

Other eigenvectors:
The other eigenvectors also are calculated in the same way. Consider again the equation (11.16) in which we substitute σ_{22} for σ, and solve for the unknown direction ratios. Let us put $l_2^* = 1$ and work out the direction ratios (l_2^*, m_2^*, n_2^*), and calculate the direction cosines (l_2, m_2, n_2) by normalising them. In exactly the same way, to calculate the third eigenvector, let us go back to Eq. (11.16) and substitute σ_{33}, and then calculate the corresponding direction ratios. As before, let us put $l_3^* = 1$, calculate the direction ratios, and finally obtain the direction cosines by normalising them.

$$l_2 = \frac{l_2^*}{\sqrt{(l_2^*)^2 + (m_2^*)^2 + (n_2^*)^2}}; \quad m_2 = \frac{m_2^*}{\sqrt{(l_2^*)^2 + (m_2^*)^2 + (n_2^*)^2}}; \quad n_2 = \frac{n_2^*}{\sqrt{(l_2^*)^2 + (m_2^*)^2 + (n_2^*)^2}};$$
$$l_3 = \frac{l_3^*}{\sqrt{(l_3^*)^2 + (m_3^*)^2 + (n_3^*)^2}}; \quad m_3 = \frac{m_3^*}{\sqrt{(l_3^*)^2 + (m_3^*)^2 + (n_3^*)^2}}; \quad n_3 = \frac{l_3^*}{\sqrt{(l_3^*)^2 + (m_3^*)^2 + (n_3^*)^2}}.$$

Orthogonality of Eigenvectors

These eigenvectors have the important property of orthogonality. If the eigenvectors are distinct, the corresponding eigenvectors are orthogonal.

$$l_1\, l_2 + m_1\, m_2 + n_1\, n_2 = 0; \qquad l_2\, l_3 + m_2\, m_3 + n_2\, n_3 = 0; \qquad l_3\, l_1 + m_3\, m_1 + n_3\, n_1 = 0. \quad (4.23)$$

Physically this means that, if the principal stresses are distinct, the corresponding principal planes are mutually orthogonal (perpendicular). This result is an example of an important result in linear algebra: the eigenvectors corresponding to distinct eigenvalues are mutually orthogonal.

If perchance the principal stresses are not distinct, the corresponding principal planes may not be orthogonal. However, it is still possible always to find a set of three mutually orthogonal (perpendicular) principal planes. This is achieved by the so-called Gram-Schmidt procedure.

How many principal planes are there in general? Three in most cases when the principal stresses are all distinct. When two or three principal planes are equal, there are infinite number of principal planes. In fact, when all the principal stresses are equal, that is a case of hydrostatic stress represented by an isotropic tensor. Now every plane is a principal plane. It is obvious that all of them cannot be mutually orthogonal. We can, as we stated above, find a set of three mutually orthogonal (perpendicular) principal planes by the Gram-Schmidt procedure[27]. The Lamé's ellipsoid of stresses helps us to understand these in a way that we will never, never forget these.

STATE OF STRESS IN A PRINCIPAL COORDINATE SYSTEM

All expressions involving stress components are simplified in a principal coordinate system. This is obvious; the shear stresses are all zero on the principal planes.

[27] It is essential to understand all this clearly. However, students generally do not have a sound background of linear algebra making it difficult to absorb all this readily. Perhaps we should be realistic; these ideas must be digested slowly. We shall call attention to these facts again and again. These ideas would become clearer when we discuss the geometrical representation of stress at a point and Lamé's stress ellipsoid.

Stress Matrix Referred to the Principal Coordinate System

How does the stress matrix appear in the principal coordinate system $(1,2,3)$?

$$I_1 = [\sigma_{xx} + \sigma_{yy} + \sigma_{zz}] \tag{4.24a}$$
$$I_2 = \left[\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2\right] \tag{4.24b}$$
$$I_3 = \left[\sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx}\right] \tag{4.24c}$$

As $(x_1, x_2, x_3) \quad \longrightarrow \quad (1,2,3)$,

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}_{(x_1,x_2,x_3)} \longrightarrow \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}_{(1,2,3)}$$

When referred to the principal coordinate system $(1,2,3)$, there are entries only on the diagonal; there are no shear stresses. (On the principal planes there cannot be any shear stresses.) Thus, the stress matrix is now diagonalised into its diagonal canonical form! Given a stress matrix, the problem of finding the principal stresses is exactly the same as that of diagonalising the matrix.

The characteristic equation corresponding to the above diagonal matrix is

$$(\sigma - \sigma_{11})(\sigma - \sigma_{22})(\sigma - \sigma_{33}) = 0 \tag{4.25}$$

so that the roots, the eigenvalues which we have seen are the principal stresses, are $(\sigma_{11},\ \sigma_{22},\ \sigma_{33})$. The coefficients I_1, I_2, I_3 are [compare with Eqs (4.24a), (4.24b), (4.24c)]:

$$I_1 = [\sigma_{11} + \sigma_{22} + \sigma_{33}]\,; \tag{4.26a}$$
$$I_2 = [\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}]\,; \tag{4.26b}$$
$$I_3 = [\sigma_{11}\sigma_{22}\sigma_{33}]\,. \tag{4.26c}$$

These coefficients $(I_1,\ I_2,\ I_3)$ are called the first, the second and the third invariants, respectively, of the stress matrix. [See later p. 4-30, Sec. 4.6.6 for more explanation.]

Stress Components on an Inclined Plane

The calculation of the stress components becomes simpler when we work with the principal planes. The reason is obvious; there are no shear stresses on the principal planes.

We shall now work out the expressions for the stress components on an inclined plane. The general approach is the same as in 4.3.1, p. 4-5. Fig. 4.14 shows a tetrahedron. We shall mark all the forces acting on the various faces, write down the equation(s) of equilibrium, proceed to the limit as the tetrahedron becomes smaller and smaller, and thereby obtain the required expressions. From the figure it is clear that

$$\overset{(1)}{\boldsymbol{T}} = -\sigma_{11}\,\boldsymbol{i}; \quad \overset{(2)}{\boldsymbol{T}} = -\sigma_{22}\,\boldsymbol{j}; \quad \overset{(3)}{\boldsymbol{T}} = -\sigma_{33}\,\boldsymbol{k}. \tag{4.27}$$

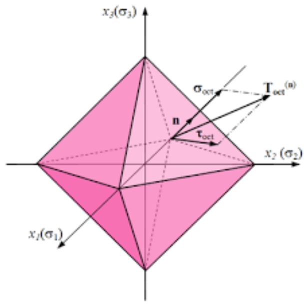


Figure 4.14: Octahedral plane

Shown in the figure 4.14 is a small tetrahedron $OABC$. The coordinate axes are along the principal axes $1, 2, 3$. The stress vectors on all the planes are shown. The normal to the inclined plane is ν with its direction cosines l, m, n referred to the principal axes $1, 2, 3$. We can see that

$$\overset{(1)}{\boldsymbol{T}} = -\sigma_{11}\,\boldsymbol{i}; \quad \overset{(2)}{\boldsymbol{T}} = -\sigma_{22}\,\boldsymbol{j}; \quad \overset{(3)}{\boldsymbol{T}} = -\sigma_{33}\,\boldsymbol{k}. \tag{4.28}$$

Obviously there are no shear stresses on the flat faces, because they are principal planes.

The body, the tetrahedron, is in equilibrium. Hence

$$\Delta_1 \overset{(1)}{\boldsymbol{T}} + \Delta_2 \overset{(2)}{\boldsymbol{T}} + \Delta_3 \overset{(3)}{\boldsymbol{T}} + \Delta \overset{(\nu)}{\boldsymbol{T}} = \boldsymbol{0}. \tag{4.29}$$

We have omitted the body force[28]. The areas $\Delta_1, \Delta_2, \Delta_3$ are the projections of the area Δ on the $1, 2, 3$ planes, respectively. Accordingly, $\Delta_1 = l\,\Delta$; $\Delta_2 = m\,\Delta$; $\Delta_3 = n\,\Delta$. When these and the results of Eq. (4.28) are substituted in Eq. (4.29), we obtain

$$\begin{aligned} \overset{(\nu)}{\boldsymbol{T}} &= -l\,\overset{(1)}{\boldsymbol{T}} - -m\,\overset{(2)}{\boldsymbol{T}} - n\,\overset{(3)}{\boldsymbol{T}} \\ &= l^2\sigma_{11}\,\boldsymbol{i} + m^2\sigma_{22}\,\boldsymbol{j} + n^2\sigma_{11}\,\boldsymbol{k}. \end{aligned}$$

From this we can find the expressions for the normal and shearing stresses as

$$\sigma_{\nu\nu} = \overset{(\nu)}{\boldsymbol{T}}\,\boldsymbol{.}\,\boldsymbol{\nu} = l^2\sigma_{11} + m^2\sigma_{22} + n^2\sigma_{33};\ \text{and} \tag{4.30}$$

$$\tau_{(\nu)} = \left[\left|\overset{(\nu)}{\boldsymbol{T}}\right| - \sigma_{\nu\nu}\right]^{\frac{1}{2}} = \left[\left(l^2\sigma_{11} + m^2\sigma_{22} + n^2\sigma_{33}\right) - \left(l^2\sigma_{11} + m^2\sigma_{22} + n^2\sigma_{33}\right)^2\right]^{\frac{1}{2}} \tag{4.31}$$

These ideas may appear abstract and difficult. Perhaps it would help if a few numerical problems are worked out as illustrative examples.

Numerical Example 1:

Let the stress matrix referred to the principal coordinates $(1, 2, 3)$ be

$$\begin{bmatrix} 40 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & -10 \end{bmatrix} \text{(all in MPa)}. \tag{4.32}$$

[28] The reason is that the body force term goes to zero faster as the tetrahedron is made smaller and smaller. We had seen this earlier.

We shall work out two different problems. First, let us refer this matrix to a non-principal set of axes. After obtaining this new stress matrix, let us pretend that we do not know the principal stresses and the corresponding principal planes. We shall, we must, obtain the principal stresses (the elements on the diagonal of the original diagonal matrix), viz., $\sigma_{11} = 40$; $\sigma_{22} = 20$; $\sigma_{33} = -10$, all in MPa.

Part I:

Let the matrix of the direction cosines be:

New coordinates	Old coordinates: $x_1 \equiv 1$	$x_2 \equiv 2$	$x_3 \equiv 3$
x'_1	0.2666	−0.3928	0.8801
x'_2	−0.5653	0.6759	0.4729
x'_3	0.7806	0.6236	0.04187

Table 4.2: Table of direction cosines

Stress transformation - matrix referred to a non-principal set of axes:

Let us transform this matrix [Eq. (4.32)] to a non-principal set of axes, viz., (x'_1, x'_2, x'_3) whose orientation is defined by Table 4.2.

(a) Stress components on the x'_1 plane:

Following the procedure and terminology used earlier, let us consider an inclined plane (a triangle ABC, normal along the x'_1-axis), and calculate the stress components $p_{x'_1}$.

$$\begin{aligned} p_{x'_1 1} &= \quad \sigma_{11} \times \cos(x'_1,\, 1) = 40 \times 0.2666 = 10.664 \text{ MPa};\\ p_{x'_1 2} &= \sigma_{22} \times \cos(x'_1,\, 2) = 20 \times (-0.3928) = -7.856 \text{ MPa};\\ p_{x'_1 3} &= \sigma_{33} \times \cos(x'_1,\, 3) = (-10) \times 0.8801 = -8.801 \text{ MPa}. \end{aligned}$$

The three stress components on this x'_1 plane are calculated as shown below.

$$\begin{aligned} \sigma_{x'_1 x'_1} &= p_{x'_1 1} \times \cos(x'_1,\, 1) + p_{x'_1 2} \times \cos(x'_1,\, 2) + p_{x'_1 3} \times \cos(x'_1,\, 3)\\ &= 10.666 \times 0.2666 + (-7.856) \times (-0.3928) + (-8.801) \times 0.8801 = -1.817 \text{ MPa};\\ \tau_{x'_1 x'_2} &= p_{x'_1 1} \times \cos(x'_2,\, 1) + p_{x'_1 2} \times \cos(x'_2,\, 2) + p_{x'_1 3} \times \cos(x'_2,\, 3)\\ &= 10.664 \times (-0.5653) + (-7.856) \times 0.6759 + (-8.801) \times 0.4729 = -15.500 \text{ MPa};\\ \tau_{x'_1 x'_3} &= p_{x'_1 1} \times \cos(x'_3,\, 1) + p_{x'_1 2} \times \cos(x'_3,\, 2) + p_{x'_1 3} \times \cos(x'_3,\, 3)\\ &= 10.664 \times 0.7806 + (-7.856) \times 0.6326 + (-8.801) \times 0.4729 = 3.055 \text{ MPa}. \end{aligned}$$

Let us now proceed to the next plane x'_2.

(b) Stress components on the x'_2 plane:

Let us now choose a different inclined plane and a different triangle such that x'_2 is its normal. We proceed exactly as before.

$$p_{x'_2 1} = \sigma_{11} \times \cos(x'_2,\, 1) = 40 \times (-0.5653) = -22.612 \text{ MPa};$$

$$p_{x'_2 2} = \sigma_{22} \times \cos(x'_2, 2) = 20 \times 0.6759 = 13.518 \text{ MPa};$$
$$p_{x'_2 3} = \sigma_{33} \times \cos(x'_2, 3) = (-10) \times 0.4729 = -4.729 \text{ MPa}.$$

Resolving these along the relevant directions and adding up the components we obtain the required stress components on this plane.

$$\begin{aligned}
\sigma_{x'_2 x'_2} &= p_{x'_2 1} \times \cos(x'_2, 1) + p_{x'_2 2} \times \cos(x'_2, 2) + p_{x'_2 3} \times \cos(x'_2, 3) \\
&= -22.612 \times (-0.5653) + 13.515 \times 0.6759 + (-4.729) \times 0.4729 = 19.683 \text{ MPa}; \\
\tau_{x'_2 x'_1} &= p_{x'_2 1} \times \cos(x'_1, 1) + p_{x'_2 2} \times \cos(x'_1, 2) + p_{x'_2 3} \times \cos(x'_1, 3) \\
&= -22.612 \times 0.2666 + 13.518 \times (-0.3928) + (-4.729) \times 0.8801 = -15.500 \text{ MPa}; \\
\tau_{x'_2 x'_3} &= p_{x'_2 1} \times \cos(x'_3, 1) + p_{x'_2 2} \times \cos(x'_3, 2) + p_{x'_2 3} \times \cos(x'_3, 3) \\
&= -22.612 \times 0.7806 + 13.518 \times 0.6236 + (-4.729) \times 0.04187 = -9.419 \text{ MPa}.
\end{aligned}$$

Now we calculate the stress components on the x'_3 plane.

(c) Stress components on the x'_3 plane:

$$p_{x'_3 1} = \sigma_{11} \times \cos(x'_3, 1) = 40 \times 0.7806 = 31.224 \text{ MPa};$$
$$p_{x'_3 2} = \sigma_{22} \times \cos(x'_3, 2) = 20 \times 0.6236 = 12.472 \text{ MPa};$$
$$p_{x'_3 3} = \sigma_{33} \times \cos(x'_3, 3) = (-10) \times 0.04187 = -0.4187 \text{ MPa}.$$

Again we resolve these along the relevant directions and add up the components to obtain the required stress components on this plane.

$$\begin{aligned}
\sigma_{x'_3 x'_3} &= p_{x'_3 1} \times \cos(x'_3, 1) + p_{x'_3 2} \times \cos(x'_3, 2) + p_{x'_3 3} \times \cos(x'_3, 3) \\
&= 31.224 \times 0.7806 + 12.472 \times 0.6236 + (-0.4187) \times 0.04187 = 32.133 \text{ MPa}; \\
\tau_{x'_3 x'_1} &= p_{x'_3 1} \times \cos(x'_1, 1) + p_{x'_3 2} \times \cos(x'_1, 2) + p_{x'_3 3} \times \cos(x'_1, 3) \\
&= 31.224 \times 0.2666 + 12.472 \times (-0.3928) + (-0.4187) \times 0.8801 = 3.055 \text{ MPa}; \\
\tau_{x'_3 x'_2} &= p_{x'_3 1} \times \cos(x'_2, 1) + p_{x'_3 2} \times \cos(x'_2, 2) + p_{x'_3 3} \times \cos(x'_2, 3) \\
&= 31.224 \times (-0.5653) + 12.472 \times 0.6759 + (-0.4187) \times 0.4729 = -9.419 \text{ MPa}.
\end{aligned}$$

Now the task is completed. However, it is a good habit to check the correctness to see if (i) $\tau_{x'_1 x'_2} = \tau_{x'_2 x'_1}$; $\tau_{x'_2 x'_3} = \tau_{x'_3 x'_2}$; $\tau_{x'_3 x'_1} = \tau_{x'_1 x'_3}$ and (ii) the numerical values of the three invariants. Here we have

$$\tau_{x'_1 x'_2} = \tau_{x'_2 x'_1} = -15.500; \;\; \tau_{x'_2 x'_3} = \tau_{x'_3 x'_2} = -9.419; \;\; \tau_{x'_3 x'_1} = \tau_{x'_1 x'_3} = 3.055; \text{ all in MPa.}$$

The first invariant is $I_1 = -1.817 + 19.683 + 32.133 = 40 + 20 - 10 = 50$ MPa. The second and third invariants are not quite as easy to compute, and so they are not checked here.

The stress matrix in the new coordinate system appears as

$$\left[\sigma'_{i'j'}\right] = \begin{bmatrix} -1.817 & -15.500 & 3.055 \\ -15.500 & 19.683 & -9.419 \\ 3.055 & -9.419 & 32.133 \end{bmatrix}, \text{ all values in MPa.} \tag{4.33}$$

We have completed the first part of the example. Now for the second part.
Part II:
Our starting point is the given stress matrix [Eq. (4.33)], and we shall ask for the eigenvalues and eigenvectors of this stress matrix. We know, of course, that the eigenvalues are 40, 20, −10, all in MPa. We shall pretend that we do not know this, and we shall obtain the principal stresses and the principal planes as if nothing else is known about this problem.

Numerical Example 2:

Let us take the stress matrix [13-2, Sec. 2] and compute the principal stresses and the corresponding principal planes. All the numerical values are in MPa.

$$\begin{bmatrix} 10 & 12 & 8 \\ 12 & 15 & -5 \\ 8 & -5 & -7 \end{bmatrix}$$

Characteristic equation and eigenvalues:
The characteristic equation of this stress matrix is [Eq. (4.34)]

$$\begin{vmatrix} 10-\sigma & 12 & 8 \\ 12 & 15-\sigma & -5 \\ 8 & -5 & -7-\sigma \end{vmatrix} = 0. \tag{4.34}$$

$$I_1 = [\sigma_{xx} + \sigma_{yy} + \sigma_{zz}] = 10 + 15 - 7 = 18 \text{ MPa}; \tag{4.35a}$$

$$\begin{aligned} I_2 &= \left[\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2\right] \\ &= 10 \times 15 + 15 \times (-7) + (-7) \times 10 - 12^2 - (-5)^2 - 8^2 = -258 \text{ (MPa)}^2; \end{aligned} \tag{4.35b}$$

$$\begin{aligned} I_3 &= \left[\sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx}\right] \\ &= 10 \times 15 \times (-7) - 10 \times (-5)^2 - 15 \times 8^2 - (-7) \times 12^2 + 2 \times 12 \times (-5) \times 8 = -2212 \\ &= -2212 \text{ (MPa)}^3. \end{aligned} \tag{4.35c}$$

Thus, the characteristic equation is

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \quad \longrightarrow \quad \sigma^3 - 18\sigma^2 - 258\sigma + 2212 = 0. \tag{4.36}$$

The (three) roots of this characteristic equation are the eigenvalues, which are the principal stresses. The three roots[29] are 24.806, 6.635, −13.441 (all in MPa). We can thus, or in

[29] Nowadays there are easy methods of obtaining the roots. In the earlier days of hand computation, solving a cubic equation like $f(x) = x^3 + ax^2 + bx + c = 0$ was not as easy. Here this problem is relatively simple, because we have an assurance beforehand that all the roots are real. How do we know this? A result in linear algebra guarantees this: the eigenvalues of real symmetric matrices are always real. One method is to guess the value of the largest root. Another method is to plot the value of the function $f(x)$ for various of x and see where the curve cuts the x-axis. That gives us an approximate value of one root. Here we can invoke the physical meaning of the largest root: it is the largest principal stress. It is the largest value

some other way, obtain the roots. We shall call them as $\sigma_{11} = 24.806$ (algebraically largest), $\sigma_{22} = 6.635$, and $\sigma_{33} = -13.441$ (algebraically smallest), all in MPa.

Having calculated the eigenvalues, let us proceed to calculate the eigenvectors. We shall discuss the calculation of the first eigenvector (the eigenvector corresponding to the first — largest — eigenvalue).

Calculation of the first eigenvector:
Let us now determine the eigenvector[30] corresponding to $\sigma = \sigma_{11}$. Let us now substitute the known value of $\sigma_{11} = 24.806$ in Eq. (4.21), p. 4-20, to yield

$$\begin{bmatrix} (\sigma_{xx} - 24.806) & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & (\sigma_{yy} - 24.806) & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & (\sigma_{zz} - 24.806) \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{4.37}$$

This is the same as

$$\begin{aligned} (\sigma_{xx} - 24.806)\, l + \tau_{yx}\, m + \tau_{zx}\, n &= 0; \\ \tau_{xy}\, l + (\sigma_{yy} - 24.806)\, m + \tau_{zy}\, n &= 0; \\ \tau_{xz}\, l + \tau_{yz} + (\sigma_{zz} - 24.806)\, n &= 0. \end{aligned}$$

An examination of any of these sets of equations reveals that the solution is not unique. If, for example, (l_1, m_1, n_1) is a solution, then $(\alpha l_1, \alpha m_1, \alpha n_1)$ is also a solution for all values of α. What does this mean? Well, it means that we cannot obtain the absolute values of (l_1, m_1, n_1); we can obtain only the ratios. This observation leads us to conclude that there is no harm in specifying the value of any one unknown. Thus, we may set, say, $l_1 = 1$ and compute the corresponding values of m_1 and n_1. We would then obtain not the *direction cosines*, but only the *direction ratios*.

In view of this realisation, shall we put[31] $l_1^* = 1$? Substituting the known numerical

of the normal stress; thus, it cannot obviously be 15 MPa or less. Thus, a probable value is, say, 22 MPa. If this is an approximate value, we can obtain the 'correction' h by using the Newton-Raphson method. This involves expansion in a Taylor series.

$$\text{To solve } f(x) = 0; \quad f(x_0) \neq 0; \quad f(x_0 + h) = 0;$$
$$f(x_0) + hf'(x_0) + \cdots = 0 \quad \longrightarrow \quad h_0 = -\frac{f(x_0}{f'(x_0)}.$$

If the assumed value x_0 is a good approximation, the value of h_0 calculated thus will be small. If it is not, revise the assumed approximate value as $x_1 - x_0 + h_o$ and go through the procedure again. Once one root is obtained, we can divide the original cubic equation and obtain a quadratic equation, etc.

[30] Is this *the* eigenvector (meaning that the eigenvector is unique), or only *an* eigenvector (implying that there can be several — certainly more than one — eigenvectors)? We can see that the eigenvector can be calculated only to within an arbitrary factor. In other words we can obtain only the ratios m_1/l_1 and n_1/l_1. These are not the *direction cosines*, but only the *direction ratios*. The direction cosines can be calculated from the direction ratios by normalising them. We have stated above how this is done, but we shall show this below again.

[31] We shall use the starred letters to represent the direction ratios. Accordingly, the triplet (l_1^*, m_1^*, n_1^*) stands for the direction ratios.

values of the stress components, we obtain

$$12m_1^* + 8n_1^* = 14.806; \tag{4.38a}$$
$$9.806m_1^* + 5n_1^* = 12; \tag{4.38b}$$
$$5m_1^* + 31.806n_1^* = 8. \tag{4.38c}$$

Here are now a set of *three* equations in *only two* unknowns, viz., m_1^* and n_1^*! There arises the question of consistency. Are these equations consistent? Sure, they are. The condition for the consistency, we recall from linear algebra, is that the rank of the coefficient matrix is equal to the rank of the augmented matrix. Here the augmented matrix cannot have the rank 3, because the eigenvalues were found by setting the determinant of the coefficient matrix equal to zero. We may, therefore, take any two of these three equations, solve for m_1^* and n_1^*, and check that the third is satisfied. There is really no need to check, but this is an illustrative example. Therefore, we shall check and convince ourselves that the equations are indeed consistent, and that all is well.

From the first two equations [Eqs (4.38a), (4.38b)] we obtain $m_1^* = 1.1909152$, $n_1^* = 0.0643771$. These satisfy the third equation (4.38c). If instead, we choose the second and the third equations, we again obtain the same set of values. Having obtained the direction ratios, we can readily work out the direction cosines as $l_1 = -0.642, \quad m_1 = -0.765, \quad n_1 = -0.041$.

Other eigenvectors:

The other two eigenvectors can also be determined by the same procedure. To calculate the eigenvector corresponding to the second eigenvalue (of 6.635 MPa), we substitute $\sigma = 6.635$ in Eq. (4.21), p. 4-20 and follow the same procedure. It is long; there are no new lessons to learn. Therefore this part of the example is omitted. The result is displayed as below.

Eigenvectors:

The corresponding eigenvectors are

$$\begin{array}{lll}
\text{eigenvalue 1: } \sigma_{11} = 24.806 & \text{eigenvector:} & (l_1,\ m_1,\ n_1) = (-0.642,\ -0.765,\ -0.041); \\
\text{eigenvalue 2: } \sigma_{22} = 6.635 & \text{eigenvector:} & (l_2,\ m_2,\ n_2) = (0.617,\ -0.549,\ 0.564); \\
\text{eigenvalue 3: } \sigma_{33} = -13.441 & \text{eigenvector:} & (l_3,\ m_3,\ n_3) = (0.454,\ -0.337,\ -0.825).
\end{array}$$

Verification - orthogonality:

We can verify — it is always a good habit to check for the numerical correctness and our own certitude — that these eigenvectors are orthogonal. The eigenvalues are distinct and, hence, the eigenvectors will be orthogonal.

$$\begin{array}{lll}
\text{1 and 2:} & (-0.642)\times(0.617) + (-0.765)\times(-0.549) + (-0.041)\times(0.564) & = 0; \\
\text{2 and 3:} & (0.617)\times(0.454) + (-0.549)\times(-0.337) + (0.564)\times(-0.825) & = 0; \\
\text{3 and 1:} & (0.454)\times(-0.642) + (-0.337)\times(-0.765) + (-0.825)\times(-0.041) & = 0.
\end{array}$$

Verification - invariants:

We can also verify that [Compare with Eq. (4.36).]

$$\sigma_{11} + \sigma_{22} + \sigma_{33} = 24.806 + 6.635 + 13.441 = 18 \text{ MPa}$$

$$
\begin{aligned}
&= \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 10 + 15 - 7 = 18 \text{ MPa} = I_1; \\
\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} &= 258 \text{ (MPa)}^2 = I_2; \\
\sigma_{11}\,\sigma_{22}\,\sigma_{33} &= 24.806 \times 6.635 \times (-13.441) = 2212 \text{ (MPa)}^3 = I_3.
\end{aligned}
$$

The Case of Repeated Eigenvalues

When two or even three eigenvalues are equal, the corresponding eigenvectors may not be orthogonal. Now the calculations can be a little tricky unless we have the support of clear understanding[32]. There are no difficulties in the calculation of the eigenvalues (principal stresses). However, the calculation of the eigenvectors can pose some difficulties.

Now that the eigenvalues are not all distinct, the eigenvectors may not be orthogonal. Now there are infinite principal planes. However, it is still possible to find a set of three mutually perpendicular principal planes by using a scheme called the Gram-Schmidt orthogonalisation procedure. These facts can be seen readily and effortlessly from Lamé's stress ellipsoid. Seeing is believing!

These calculations have to be performed as part of our learning process. Once the concepts are clearly understood, any software tool like MAPLE or MATLAB may be used to relieve the tedium of the calculations.

Invariants of the Stress Matrix

Let us realise that both the stress matrices $[\sigma_{ij}]$ and $[\sigma'_{i'j'}]$ represent the state of stress *at the same point*, though the coordinate systems are different. Both are matrix representations of the *same* invariant, symmetric stress tensor *at the same point*. Now the principal stresses and the corresponding principal planes at the same point cannot depend upon the choice of the coordinate system. Thus, whether we work with the stress matrix $[\sigma_{ij}]$ or with $[\sigma'_{i'j'}]$, we must necessarily obtain the same principal stresses. This means that the characteristic equations shown below must necessarily have the same coefficients.

$$
\begin{aligned}
\sigma^3 - [\cdots]\sigma^2 + [\cdots]\sigma - [\cdots] = 0; &\quad \text{in the } x_i \text{ system} \\
\sigma^3 - [\cdots]\sigma^2 + [\cdots]\sigma - [\cdots] = 0; &\quad \text{in the } x'_i \text{ system}
\end{aligned}
$$

This observation leads us immediately to the conclusion that the coefficients must be the same. That is,

$$
\begin{array}{lll}
I_1 \ \ [\text{in the } x_i \text{ system}] = I_1 & [\text{in the } x'_i \text{ system}] & \text{first invariant } I_1 \text{ of the stress matrix;} \\
I_2 \ \ [\text{in the } x_i \text{ system}] = I_2 & [\text{in the } x'_i \text{ system}] & \text{second invariant } I_2 \text{ of the stress matrix;} \\
I_3 \ \ [\text{in the } x_i \text{ system}] = I_3 & [\text{in the } x'_i \text{ system}] & \text{third invariant } I_3 \text{ of the stress matrix.}
\end{array}
$$

On transformation of the coordinate systems, the stress components $[\sigma_{ij}]$ *do* change to $[\sigma'_{i'j'}]$, but these three combinations do not change; these are the three invariants of the

[32]Routine calculations with the brain in the switched-off mode may not be possible. On the mathematical side, the rank of the matrix, the fact there are $n - r$ linearly independent solutions, etc. have to be understood. A geometrical picture like Lamé's stress ellipsoid would greatly help. We shall revisit this topic later after our discussion of the geometrical representation of the state of stress at a point.

stress matrix.

$$
\begin{aligned}
I_1 &= [\sigma_{xx} + \sigma_{yyu} + \sigma_{zz}] = \left[\sigma'_{x'x'} + \sigma'_{y'y'} + \sigma'_{z'z'}\right] \\
&= [\sigma_{11} + \sigma_{22} + \sigma_{33}] && (4.39a) \\
I_2 &= \left[\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2\right] \\
&= \left[\sigma'_{x'x'}\sigma'_{y'y'} + \sigma'_{y'y'}\sigma'_{z'z'} + \sigma'_{z'z'}\sigma'_{x'x'} - \tau'^2_{x'y'} - \tau'^2_{y'z'} - \tau'^2_{z'x'}\right] \\
&= [\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}] && (4.39b) \\
I_3 &= \left[\sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx}\right] \\
&= \left[\sigma'_{x'x'}\sigma'_{y'y'}\sigma'_{z'z'} - \sigma'_{x'x'}\tau'^2_{y'z'} - \sigma'_{y'y'}\tau'^2_{z'x'} - \sigma'_{z'z'}\tau'^2_{x'y'} + 2\tau'_{x'/y'}\tau'_{y'z'}\tau'_{z'x'}\right] \\
&= [\sigma_{11}\sigma_{22}\sigma_{33}] && (4.39c)
\end{aligned}
$$

These three invariants play important roles in theoretical developments. When numerical problems are worked out, it is helpful to check the values of these invariants.

Comments:

The stress matrices $[\sigma_{ij}]$ and $[\sigma'_{i'j'}]$ are similar[33] matrices. Similar matrices have the same eigenvalues, but not the same eigenvectors. How do we understand this statement in the context of the state of stress at a point? Well, the principal stresses are the same, and so are the principal planes. However, the same principal planes will be represented by two *different* sets of direction cosines, because the coordinate systems are different. Thus, the eigenvalues are the same, but the eigenvectors are — will have to be — different such that the *same* principal planes are referred to.

Principal Stresses as Stationary Values

Let us recall that we defined the principal stress as the (normal) stress on a principal plane which, in turn, was defined as that on which there is no shear stress. We may also seek the stationary values of the expression for the normal stress and, thereby, obtain the principal stress. We shall presently undertake this exercise.

The expression for the normal stress, we have seen, is given by Eq. (4.4) which is reproduced below.

$$f(l,m,n) \equiv \sigma_{\nu\nu} = l^2\sigma_{xx} + m^2\sigma_{yy} + n^2\sigma_{zz} + 2lm\tau_{xy} + 2mn\tau_{yz} + 2nl\tau_{zx}. \tag{4.40}$$

Let us try to find the stationary values of this expression. This is a function of three variables (l, $m\,n$). Being direction cosines, they must satisfy the condition

$$g(l,m,n) \equiv l^2 + m^2 + n^2 - 1 = 0. \tag{4.41}$$

Stationary values with a constraint: Lagrange multiplier method:

When we are required to find the maximum / minimum / stationary values of a function, say, $f(x,y,z)$ subject to a constraint condition, say, $g(x,y,z) = 0$, there is a beautiful

[33] Similar is a technical word with a specific technical meaning; it should not taken as an ordinary word.

method[34] called the Lagrange multiplier method[35]. We define a new function $h(x, y, z) = f(x, y, z) - \lambda g(x, y, z)$ and find the stationary values of $h(x, y, z)$ *subject to no constraint.*

Extremise[36] $f(l, m, n)$ subject to the constraint $g(l, m, n) = 0$. To do this, define a new function $h(l, m, n) = f(l, m, n) - \lambda g(l, m, n)$ and extremise this function subject to no constraint. Thus, we have

$$\frac{\partial h}{\partial l} = 0; \quad \frac{\partial h}{\partial l} = 0; \quad \frac{\partial h}{\partial l} = 0. \tag{4.42}$$

The function $h(l, m, n)$ here is [See Eqs (4.40), (4.41).]

$$h(l, m, n,) \equiv l^2\sigma_{xx} + m^2\sigma_{yy} + n^2\sigma_{zz} + 2lm\tau_{xy} + 2mn\tau_{yz} + 2nl\tau_{zx} - \lambda l^2 + m^2 + n^2 - 1. \tag{4.43}$$

Applying the (necessary) conditions [Eq. (4.42)] to this function $h(l, m, n)$, we obtain

$$2\left[l\sigma_{xx} + m\tau_{yx} + n\tau_{zx}\right] - 2\lambda l = 0; \tag{4.44a}$$
$$2\left[l\tau_{xy} + m\sigma_{yy} + n\tau_{zy}\right] - 2\lambda m = 0; \tag{4.44b}$$
$$2\left[l\tau_{xz} + m\sigma_{yz} + n\sigma_{zz}\right] - 2\lambda n = 0; \tag{4.44c}$$

which are the same as

$$\begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \lambda \begin{Bmatrix} l \\ m \\ n \end{Bmatrix}; \quad \text{i.e.,} \quad \boldsymbol{\sigma x} = \lambda \boldsymbol{x}. \tag{4.45}$$

If we compare this equation with Eq. (4.14), we find that λ introduced as a Lagrange parameter now appears as the eigenvalue! Note particularly the dual role[37] played by the Lagrange parameter!

The statement that we made earlier that the principal planes may be defined differently (viz., (i) the planes on which the shear stress is zero, and / or (ii) the planes on which the normal stress becomes stationary) stands vindicated.

Maximum Shear Stresses as Stationary Values

The maximum shear stresses also may be obtained as the stationary values of the expression for the shear stress on an inclined plane. The latter, expressed in terms of the principal stresses (σ_{11}, σ_{22}, σ_{33}), is given by

$$\tau_{(\nu)} = \left[l^2m^2\left(\sigma_{11} - \sigma_{22}\right)^2 + m^2n^2\left(\sigma_{22} - \sigma_{33}\right)^2 + n^2l^2\left(\sigma_{33} - \sigma_{11}\right)^2\right]^{\frac{1}{2}}. \tag{4.46}$$

Let us find the stationary values of this expression subject to the constraint $l^2 + m^2 + n^2 = 1$. A similar problem was discussed just now. We have seen that to extremise $f(l, m, n)$

[34] Named after the famous French mathematician Joseph-Louis Lagrange (Jan. 1736 - April 1813)

[35] Readers who are not familiar with this method are advised to read this from some good books and understand it clearly. This is a common technique used extensively in applied mathematics.

[36] Find the maximum / minimise / stationary value(s).

[37] We should not fail to note and appreciate the beauty of applied mathematics in action.

subject to the constraint $g(l,m,n)=0$, it is sufficient to extremise $h(l,m,n) \equiv f(l,m,n) + \lambda\, g(l,m,n)$ subject to no constraint, where λ is a parameter, unknown to begin with, called as the Lagrange parameter. Thus, this problem is formulated[38] as:

$$\text{extremise} \quad f(l,m,n) \equiv \left[l^2m^2\left(\sigma_{11}-\sigma_{22}\right)^2 + m^2n^2\left(\sigma_{22}-\sigma_{33}\right)^2 + n^2l^2\left(\sigma_{33}-\sigma_{11}\right)^2\right]$$

$$\text{subject to the constraint} \quad g(l,m,n) \equiv l^2+m^2+n^2-1=0.$$

We proceed by defining $h(l,m,n) \equiv f(l,m,n)+\lambda g(l.m.n)$ and extremising $h(l,m,n)$ subject to no constraint. This procedure gives us

$$\frac{\partial h}{\partial l}=0 \quad \longrightarrow \quad lm^2\left(\sigma_{11}-\sigma_{22}\right)^2 + ln^2\left(\sigma_{33}-\sigma_{11}\right)^2 - \lambda l = 0; \tag{4.47a}$$

$$\frac{\partial h}{\partial m}=0 \quad \longrightarrow \quad l^2m\left(\sigma_{11}-\sigma_{22}\right)^2 + mn^2\left(\sigma_{22}-\sigma_{33}\right)^2 - \lambda m = 0; \tag{4.47b}$$

$$\frac{\partial h}{\partial n}=0 \quad \longrightarrow \quad m^2n\left(\sigma_{22}-\sigma_{33}\right)^2 + l^2n\left(\sigma_{33}-\sigma_{11}\right)^2 - \lambda n = 0. \tag{4.47c}$$

These are accompanied by the constraint condition $l^2+m^2+n^2-1=0$.

Case (a) Minimum values:

We know from our understanding of the physical problem even without any calculation that the shear stress on the principal planes is zero. The inclined plane with its normal (ν) becomes a principal plane when

(i) $(l=\pm 1,\ m=n=0)$; **(ii)** $(m=\pm 1,\ l=n=0)$; **(iii)** $(n=\pm 1,\ l=n=0)$.

Case (b) Maximum values:

The maximum values are reached corresponding to three different sets of the direction cosines. These are the following.

(i) $(l=\pm\frac{1}{\sqrt{2}},\ m=n=0)$; corresponding $\tau_{(max)_1} = \pm\frac{1}{2}\left(\sigma_{22}-\sigma_{33}\right);$ (4.48a)

(ii) $(m=\pm\frac{1}{\sqrt{2}},\ l=n=0)$; corresponding $\tau_{(max)_2} = \pm\frac{1}{2}\left(\sigma_{33}-\sigma_{11}\right);$ (4.48b)

(iii) $(n=\pm\frac{1}{\sqrt{2}},\ l=m=0)$; corresponding $\tau_{(max)_3} = \pm\frac{1}{2}\left(\sigma_{11}-\sigma_{22}\right).$ (4.48c)

The maximum shear at the point is the grand maximum among these three maxima.

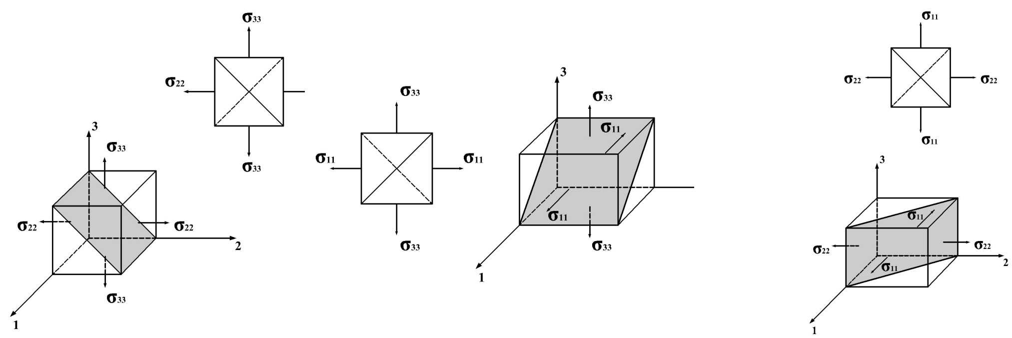
$$\begin{aligned}\tau_{grandmax} &= \text{maximum of} \quad \left[\tau_{(max)_1};\ \ \tau_{(max)_2};\ \ \tau_{(max)_3}\right] \\ &= \text{maximum of} \quad \left[\frac{1}{2}\left(\sigma_{22}-\sigma_{33}\right);\ \ \frac{1}{2}\left(\sigma_{33}-\sigma_{11}\right);\ \ \frac{1}{2}\left(\sigma_{11}-\sigma_{22}\right)\right].\end{aligned}$$

What are these maxima? How can there be three maxima? Fig. 4.15 shows three planes on which the shear stress reaches its stationary values. These three planes are shown shaded

[38] Extremising $\tau_{(\nu)}$ is the same as extremising $\tau^2_{(\nu)}$. This is done merely to simplify the algebra.

[Figs 4.15a, 4.15b, 4.15c]. These planes, as we can see from these figures, perpendicular to the 12, 23, and 31 planes. The maxima on these planes are, respectively,

$$\frac{1}{2}\left(\sigma_{11}-\sigma_{22}\right); \qquad \frac{1}{2}\left(\sigma_{22}-\sigma_{33}\right); \qquad \frac{1}{2}\left(\sigma_{33}-\sigma_{11}\right).$$



(a) Maximum shear stress on the shaded plane

(b) Maximum shear stress on the shaded plane

(c) Maximum shear stress on the shaded plane

Figure 4.15: Planes of maximum shear stress (shaded): perpendicular to the planes 12, 23, 31. The maxima are $(\sigma_{11}-\sigma_{22})/2$, $(\sigma_{22}-\sigma_{33})/2$, $(\sigma_{33}-\sigma_{11})/2$.

Octahedral Planes and Stresses on These Planes

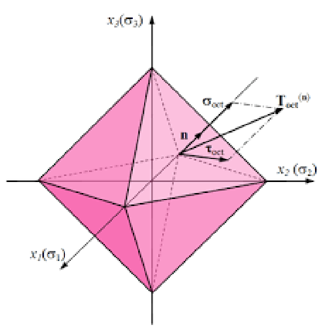


Figure 4.16: Octahedral plane

Planes equally inclined to the three principal planes are called octahedral planes. There are eight (8) such planes. Their direction cosines referred to the principal axes 1, 2, 3 are

$$\pm\frac{1}{\sqrt{3}}; \quad \pm\frac{1}{\sqrt{3}}; \quad \pm\frac{1}{\sqrt{3}}.$$

The stresses on such planes are important because they are related to the theories of failure.

Planes which are equally inclined to the principal planes are called octahedral planes [Fig. 4.16]. Being equally inclined, their direction cosines must satisfy the condition

$$l^2+m^2+n^2=1; \quad l=m=n; \quad \text{thus,} \quad \pm\frac{1}{\sqrt{3}}; \quad \pm\frac{1}{\sqrt{3}}; \quad \pm\frac{1}{\sqrt{3}}.$$

There are thus eight (8) such planes. The stresses, or the expressions for the stresses, on these octahedral planes have special significance because they are related to the theories of failure of ductile materials.

If $(\sigma_{11}, \sigma_{22}, \sigma_{33})$ are the principal stresses, the stress components, both normal and shear, can be obtained by stress transformation. We can see that the expressions for the stress components, resultant p_{oct}, normal σ_{oct} and shear τ_{oct}, are given by

$$p_{oct} = \sqrt{\frac{\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2}{3}}; \tag{4.49a}$$

$$\sigma_{oct} = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}; \tag{4.49b}$$

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2} \tag{4.49c}$$

$$= \frac{1}{3}\sqrt{2\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)^2 - 6\left(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}\right)}. \tag{4.49d}$$

These expressions are in terms of the principal stresses. These will, naturally enough, look differently when expressed in terms of the (non-principal) stress components. There will now be some shear terms also which will be absent when the expressions are written in principal coordinate systems. They are [See Example 2, p. 13-3.]

$$\sigma_{oct} = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} \tag{4.50a}$$

$$\tau_{oct} = \frac{1}{3}\sqrt{2\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)^2 - 6\left(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{xy}^2 - \tau_{yz}^2 + \tau_{zx}^2\right)} \tag{4.50b}$$

$$= \frac{1}{3}\sqrt{\left(\sigma_{xx} - \sigma_{yy}\right)^2 + \left(\sigma_{yy} - \sigma_{zz}\right)^2 + \left(\sigma_{zz} - \sigma_{xx}\right)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)} \tag{4.50c}$$

TENSOR CHARACTER OF THE STRESS AT A POINT

We have seen some aspects of the stress tensor. We shall now see the number of (independent) components for scalars, vectors and tensors, and gain further insights into the tensor character.

We know that the stress at a point can be specified by four (4) components, viz., σ_{xx}, τ_{xy}; τ_{yx}, σ_{yy} in two dimensions, and σ_{xx}, τ_{xy}, τ_{xz}; τ_{yx}, σ_{yy}, τ_{yz}; τ_{zx}, τ_{zy}, σ_{yy} in three dimensions. A little reflection would convince us that, in one dimension, there can be only one component, viz., σ_{xx} for the stress at a point[39].

Next let us consider a vector such as, say, the velocity (of flow of a fluid) at a point. There are now one (1) component, viz., V_x in one dimension; two (2) components, viz., V_x, V_y in two dimensions; and three (3) components, viz., V_x, V_y, V_z in three dimensions. A scalar such as, say, the temperature at a point has just one component T in one, two and three dimensions. We can see that T_x, T_y do not make sense.

Let us display the results of these observations systematically in the form of a table [Table 4.3] below. The pattern is unmistakably clear. A scalar, a vector, and a tensor (of

[39] These are with respect to a rectangular cartesian coordinate system. In other coordinate systems also, the number of (independent) stress components still remains the same, although they are now represented differently like σ_{rr}, $\tau_{r\theta}$, etc. The fact that $\tau_{xy} = \tau_{yx}$ (to satisfy the equations of equilibrium in the absence of body couples) does not make any difference in principle like $4 - 1 = 3$, or $9 - 3 = 6$.

rank two) have n^0, n^1, n^2 components in n dimensions. Thus, they have, respectively

$$\begin{aligned} \text{scalar: } & 1^0 = 1, \quad 2^0 = 1, \quad 3^0 = 1; \\ \text{vector: } & 1^1 = 1, \quad 2^1 = 2, \quad 3^1 = 3; \text{ and} \\ \text{tensor: } & 1^2 = 1, \quad 2^2 = 4, \quad 3^2 = 9 \end{aligned}$$

components in one, two, and three dimensions. We may generalise further and state that in an n-dimensional space, a tensor of rank r has n^r (independent) components. This observation also may lead us to regard a scalar as a tensor of rank 0, and a vector as a tensor of rank 1. Judged in this light, a scalar is one step (generation?) lower than a vector which, in turn, is one step (generation) lower than a tensor of rank two[40].

Table 4.3: Number of components of a scalar, a vector, and a tensor

n (dimensions)	scalar (e.g., temperature)	vector (e.g., velocity)	tensor (e.g., stress)
One-dimensional: $n = 1$	$1\ (= n^0)$	$1\ (= n^1)$	$1\ (= n^2)$
Two-dimensional: $n = 2$	$1\ (= n^0)$	$2\ (= n^1)$	$4\ (= n^2)$
Three-dimensional: $n = 3$	$1\ (= n^0)$	$3\ (= n^1)$	$9\ (= n^2)$
...	...	...	...
n-dimensional: n	$1\ (= n^0)$	$n\ (= n^1)$	$n^2\ (= n^2)$

Scalars, Vectors and Tensors

By now we have had some familiarity with scalars, vectors and (second order) tensors. Let us have a few quiet moments to look at them, reflect upon them, and gain an appreciation of their relationships to one another. We shall take the nature of stress at a point as an example for this exercise. Let us note:

(i) the stress *at a point* is (an example of) a second order tensor with 1, 4 and 9 components in 1, 2 and 3 dimensions;

(ii) the stress *at a point on a (given) plane* is a vector with 1, 2 and 3 components in 1, 2 and 3 dimensions; and

(iii) the stress *at a point on a (given) plane in a (given) direction* is a scalar with 1 component only in all the three cases of 1, 2 and 3 dimensions.

[40] Shall we then state, if we will not be taken too literally and seriously, that the relationship between a tensor (of rank 2) is exactly the same as that between a vector and a scalar? Perhaps we might point out that the relationship between a (paternal) grandfather and a father is the same as that between the father and his son!

Thus, a scalar can be considered as a tensor of order zero, and a vector as a tensor of order one. Furthermore,

(i) a second order tensor associates a vector with every direction — a given direction specifies a plane which has a stress vector acting on it — just as

(ii) a vector associates a scalar with every direction — a given direction defines or specifies the stress component in that direction on that plane, the component being a scalar.

The order and the rank of a tensor are the same; these two words may be used interchangeably.

Tensors: Transformation Law

We have pointed out a few aspects of tensors, but the crucial qualifying aspect is not any of the above-mentioned facts. It is the transformation property (of the components of vectors and tensors) that is of decisive importance[41]. We do not discuss this any further, because this aspect is dealt with in much greater detail elsewhere in the book.

Some Comments

We may also note that, just as a vector $\boldsymbol{P}$ can be written in terms of the base[42] vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ as

$$\boldsymbol{P} = P_1\,\boldsymbol{e}_1 + P_2\,\boldsymbol{e}_2 + P_3\,\boldsymbol{e}_3,$$

a second order tensor also may be written as

$$\begin{aligned}\boldsymbol{\sigma} =& \sigma_{11}\,\boldsymbol{e}_1\,\boldsymbol{e}_1 + \sigma_{12}\,\boldsymbol{e}_1\,\boldsymbol{e}_2 + \sigma_{13}\,\boldsymbol{e}_1\,\boldsymbol{e}_3 + \sigma_{21}\,\boldsymbol{e}_2\,\boldsymbol{e}_1 + \sigma_{22}\,\boldsymbol{e}_2\,\boldsymbol{e}_2 + \sigma_{23}\,\boldsymbol{e}_2\,\boldsymbol{e}_3 + \\ & \sigma_{31}\,\boldsymbol{e}_3\,\boldsymbol{e}_1 + \sigma_{32}\,\boldsymbol{e}_3\,\boldsymbol{e}_2 + \sigma_{33}\,\boldsymbol{e}_3\,\boldsymbol{e}_3 \\ =& \sigma_{ij}\,\boldsymbol{e}_i\,\boldsymbol{e}_j,\end{aligned}$$

where $\boldsymbol{e}_1\,\boldsymbol{e}_1$, $\boldsymbol{e}_1\,\boldsymbol{e}_1$, etc. are referred to as dyads. A unit second order tensor (a 2×2 matrix $\boldsymbol{I}$ with transformation properties), in particular, may be written as

$$\boldsymbol{I} = \delta_{ij}\,\boldsymbol{e}_i\,\boldsymbol{e}_j.$$

[To digress a little bit, let us return to the state of pressure at a point in a fluid at rest. The pressure, we had seen, is the same on all planes passing through the point, and is entirely normal to the plane.

$$\begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}_{(xyz)} \longrightarrow \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}_{(x'y'z')} \longrightarrow \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}_{(x''y''z'')}$$

[41] It is when judged from this correct point of view that 'three-index entities' like the Christoffel symbols are not tensors, even though they too have the relevant number of components.

[42] These are also the unit vectors in our context of only rectangular cartesian coordinates. When we use the general curvilinear coordinates, we shall see that the base vectors are not unit vectors! It is only in some special cases that the base vectors are unit vectors.

There is no tangential component; that is, a fluid at rest cannot sustain any shear stress. Let us note that this matrix has the same appearance *in all* coordinate systems! This is a very special case of the general state of stress at a point in a solid, and is an example of an isotropic tensor.

We repeat for emphasis: the pressure at a point in a fluid at rest is an example of a second order tensor, but being an isotropic tensor it can be regarded as a scalar. Just one number (say, $-p = -4$ kPa) will do to specify the pressure at a point, which is a feature of a scalar.]

We shall close this section with the following statements.

$$\begin{aligned}
\text{The stress } \textit{at a point} &\quad\longrightarrow\quad \text{a second order tensor;}\\
\text{the stress } \textit{at a point on a plane} &\quad\longrightarrow\quad \text{a vector;}\\
\text{the stress } \textit{at a point on a plane in a direction} &\quad\longrightarrow\quad \text{a scalar;}\\
\text{the stress at a point} &\quad\longrightarrow\quad \text{invariant, symmetric.}
\end{aligned}$$

STATES OF STRESS: HYDROSTATIC AND PURE SHEAR

We have seen earlier the two special states of (i) hydrostatic stress and (ii) pure shear. They are also referred to as the spherical and deviatoric parts of the stress tensor. These qualifiers, spherical and deviatoric, are appropriate. For the case of a hydrostatic stress, the three principal stresses are equal, and that the Lamé's ellipsoid of stresses becomes a sphere. Furthermore, for a (two-dimensional) curvature tensor, when the two eigenvalues — the principal curvatures — are equal, the surface there is indeed spherical!

Hydrostatic and Deviatoric Parts in Stress Space

It is also insightful to look at these special states of stress in a Westergaard stress space [Fig. 4.17]. The line OA being equally inclined to the principal stress axes σ_{11}, σ_{22}, σ_{33} represents all hydrostatic states of stress. AB is a measure of how far the state of stress A deviates from a state of hydrostatic stress — that is, how far it deviates from the line OP. OB and BA represent, respectively, the spherical and the deviatoric parts of the state of stress A. The hydrostatic part corresponds to the spherical part, and the deviatoric part to the pure shear part of the stress tensor (state of stress represented by the point A in the Westergaard stress space).

Hydrostatic State of Stress (Spherical Part of the Stress Tensor)

We have seen that a hydrostatic state of stress is represented by the stress matrix

$$\begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} \quad\longrightarrow\quad p\boldsymbol{I} \quad\longrightarrow p\,\delta_{ij}.$$

This matrix has the same appearance in every coordinate system. The corresponding Lamé's ellipsoid of stresses becomes a sphere.

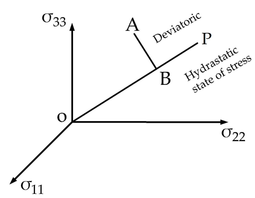


Figure 4.17: A point A in the Westergaard stress space

A point A is marked in the Westergaard stress space. The line OB is equally inclined to the three principal axes σ_{11}, σ_{22}, σ_{33}. Accordingly, every point on this line (and on its extension) represents a state of hydrostatic state of stress. The distance AB is a measure of how far the state of stress represented by the point A deviates from the hydrostatic state of stress. Thus, OB and BA represent, respectively, the spherical and the deviatoric parts of the state of stress A.

A State of Pure Shear (Deviatoric Part of the Stress Tensor)

It seems natural to define a state of pure shear as explained below. If the stress matrix appears in the form

$$\begin{bmatrix} 0 & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & 0 & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & 0 \end{bmatrix}$$

with no normal stress components — that is, only zeros on the leading diagonal — it can be regarded as representing a state of pure shear. However, referred to the principal axes, even this matrix representing a state of pure shear will appear as a diagonal matrix:

$$\begin{bmatrix} 0 & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & 0 & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & 0 \end{bmatrix}_{xyz} \longrightarrow \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}_{123}.$$

If the first matrix (the one on the left) represents a state of pure shear, the second matrix (the one on the right) too must represent the same state of pure shear. After all, they both represent the same invariant stress tensor! Hence it becomes necessary for us to revise our definition of the state of pure shear. Just because there are non-zero elements on the leading diagonal, the stress matrix need not necessarily cease to represent a state of pure shear. We know that for the two matrices shown, the first invariant[43] must be the same. Thus, we are led to the conclusion that as long as the trace of the matrix (the first invariant) $I_1 = 0$, that matrix will represent a state of pure shear. It is possible to show that a *necessary* and *sufficient* condition for a matrix to represent a state of pure shear is that

$$\text{the first invariant, } I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_{11} + \sigma_{22} + \sigma_{33} = 0. \tag{4.51}$$

The first part — the necessity — is obvious. The second part — the sufficiency — is not quite as easy to prove. That is, we are required to prove[44] that, if the first invariant vanishes, there exists a *rectangular cartesian* coordinate system referred to which the stress matrix appears with only zero elements on its leading diagonal.

[43] Not only the first invariant, but the other two invariants too

[44] Such proofs are important for mathematicians. They are not so important for engineers. A plausibility argument is quite sufficient for us. It is always nice to have the support of a rigorous proof, though.

Any (general) state of stress can always be broken up into two special states of stress, viz., (a) a hydrostatic state of stress, and (b) a state of pure shear. But why do we need to do this? When we learn theories of failure (also called theories of strength), we come across a famous criterion of failure[45] called the von Mises' criterion. This criterion is associated with the strain energy of the pure shear component. Thus, it is necessary for us to split the general stress matrix (at the most vulnerable point; that is, wherever we desire to check the safety of the machine part) into the two matrices, and then compute the strain energy of the pure shear part. This part of the strain energy is called the distortion strain energy.

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix} + \begin{bmatrix} \sigma_{xx}-\sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy}-\sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz}-\sigma \end{bmatrix}. \tag{4.52}$$

We desire that the first matrix in Eq. (4.52) should represent a hydrostatic state of stress, and that the second one a state of pure shear. The first matrix is now in good shape to represent a state of hydrostatic stress; any value of σ will do. But if the second matrix should represent a state of pure shear, only one value of σ can be used. The condition of pure shear, we recall, is that the first invariant should be zero. Thus, we require that

$$1^{\text{st}} \text{ invariant (of the second matrix):} \quad [\sigma_{xx}-\sigma]+[\sigma_{yy}-\sigma]+[\sigma_{zz}-\sigma]=0.$$

This gives us the required value of σ as $\sigma = \frac{1}{3}[\sigma_{xx}+\sigma_{yy}+\sigma_{zz}]$. With this choice for the value of σ, the desired decomposition is obtained. It is of theoretical and practical importance to do this. Theory of plasticity and theories of strength are two areas of application among others where such a decomposition is necessary.

THEORIES OF FAILURE: TRESCA'S AND VON MISES' CRITERIA

Theories of failure (also called theories of strength) are suggested to predict in advance when a material (say, a machine part) subjected to a complex state of stress fails. This is to be done with the limited information obtained from a simple tension test.

When does a material or a machine part fail? We discuss failure only from the point of stress analysis (and not from other considerations like bad appearance and unsuitability to serve the very purpose of the machine part). Perhaps it is natural to postulate that a material fails when the maximum stress exceeds the safe limit. The safe limit is, of course, to be obtained experimentally in a material testing laboratory. The experiment is the simple tension test usually performed on a universal testing machine. This is for ductile materials. (For a material like concrete, a compression test is used.)

[45] Named after the Austrian-German-American applied mathematician Richard Edler von Mises (April 1983 - July 1953) — not to be confused with Ludwig von Mises, the well known Austrian theoretical economist. He had made a mark as an applied mathematician. It was he who founded the prestigious journal ZAMM (*Zeitschrift für Angewande Mathematik und Mechanik*) (Journal of Applied Mathematics and Mechanics), the first of its kind.

Simple Tension Test

The usual tension test is the most common and fundamentally important test. This test is under a uniaxial state of stress[46] [Fig. 4.18][47]. With this (limited) information, we are required to judge when a (vulnerable, critically 'stressed') point in a machine part subjected to complex three dimensional stresses fails. This is what a theory of failure or a theory of strength is concerned about[48].

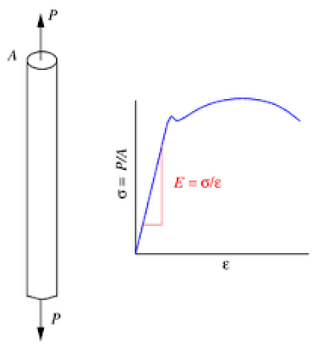


Figure 4.18: A simple tension test

One-dimensional state of stress: only $\sigma_{11} = \sigma_y$ exists.
The stress matrix corresponding to the simple tension test is

$$\begin{bmatrix} \sigma_y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where σ_y, the yield stress — the stress at which the specimen is deemed to have failed — is obtained experimentally. With this meagre information, we are required to judge when a machine part will fail in a complex — in general, a 3-dimensional — state of stress. This is the relevance of a theory of failure, also called a theory of strength.

What Causes Failure?

How are we sure that it is the high stress that is responsible for the failure? It could very well be the high strain that accompanies the high stress that causes failure? How do we know for sure? To settle this issue, we have to be innovative. We should devise a loading so that the stress exceeds the 'safe' limit, but the strain does not, or vice versa, if that would be possible. But there are other situations that force us to abandon both these theories. For example, we know that at the bottom of the sea, there are rocks that are perfectly intact. If we bring a specimen of the same rock and test it in the laboratory, it gets crushed at a much, much lower stress! How can we explain this? Well, we have to come up with some plausibility arguments[49]. Usually there are extra supporting experimental facts, or established theories, that lend credence to such hypotheses.

[46] The state of stress is complex and no more uniaxial in the later part of the test after the phenomenon of 'necking', but we disregard all these complexities and consider the 'simple' tension as a uniaxial test producing a uniaxial state of stress in the test specimen.

[47] There are many details pertaining to this important test. The specimen is not of uniform cross-section as shown here. The figure is in this sense somewhat misleading.

[48] Readers not exposed to the details of this topic should consult two or three good books. Some books on machine design also give some aspects of this topic fairly well.

[49] To come up with some credible stories is given the more honorable name of proposing a new theory. This is said to be science, but it is no more than suggesting a possibility that has no inherent internal contradictions and that is able to explain the observed phenomenon or phenomena and the experimental evidence. A hypothesis in science is really a bed time story except that there are pompous technical words and several unfriendly symbols of higher mathematics! No irreverence is meant by this statement.

(a) Tresca's criterion (b) Von Mises' criterion

Figure 4.19: The figures represent in a 2-dimensional stress space the 'safe' regions (shaded). Tresca's criterion is slightly more conservative than the von Mises' one.

On the physical side we know — it is our experience — that brittle and ductile materials behave differently. Thus, we may have to understand the internal mechanism[50] that is responsible for failure before we can propose a theory of strength convincingly[51]. These considerations take us away into the realm of materials science (deformation processes in metals, dislocation theory, etc.), a vast area in its own right. We cannot discuss these here. Nevertheless, we may perhaps point out that yielding (in ductile materials) is the result of slippage of certain crystal planes. This happens along the surface of maximum shear stress. Thus, there is some physical ground to assume that the maximum shear stress is of greater relevance.

Shall we, then, postulate that it is the maximum shear stress or some entity dependent on this that causes failure? Are there further arguments supporting this point of view?

Is Shear Stress the Villain?

A brilliant idea is to suppose that it is the shear stress, or some entity dependent on the shear stress, that causes failure. This would explain why rocks deep down at the bottom of the sea remain safe; the state of stress that the rocks are subjected to is one of hydrostatic stress. No matter how high the water pressure is, there is no shear stress at all on any plane!

[50] Siavouche (Sia) Nemat-Nasser, the leading expert in mechanics at (U. of California at) San Diego, states in the preface of his book, *Plasticity: A Treatise on the Finite Deformation of Heterogeneous Inelastic Materials*, Cambridge University Press, New York, (2004): "··· It became clear to me that true advances in the basic understanding of the mechanics of materials, and particularly the inelastic deformation of materials and geomaterials, can be achieved only by moving beyond the traditional phenomenological approach to plasticity models ···" combining "··· the mathematical rigor of solid mechanics with the physics-based micro-structural understanding of the materials science ···".

[51] To see how hard it is to understand the phenomenon of failure, we suggest that we try to define when a person can be said to have died. Even with our perfect ignorance in matters related to life and death, we might be persuaded to propose this in different ways. When can a person be said to have died? When his brain fails, heart fails, kidney fails, all the vital organs fail? We shall try to avoid further complications by restricting ourselves to clinical death. (Has not Shakespeare said that fools die several times?) We could even state that a person is dead when his wife becomes a widow! This is fine, except that we must also address issues like the following. (i) Is there an independent way of finding out if and when a (married) woman becomes a widow? (ii) What about death of women, unmarried men, children, etc.? (iii) Yogis are said to be able to enter a state of *samādhi* when all breathing is suspended. They look, it is said, lifeless for all untrained people.

This is why, this theory postulates, rocks can stand very, very high[52] hydrostatic stresses while they crumble at much, much lower non-hydrostatic states of stress. In this way two important theories emerged: (i) the maximum shear stress theory, and (ii) the maximum shear energy (related to the shear stress part only) theory. The first is called the Tresca's[53] criterion of failure, while the second is known as the von Mises' criterion. These are the most commonly used theories of failure to check whether or not a machine part is safe. We shall discuss these two theories briefly.

Tresca's Criterion

Tresca's criterion of failure, known also known as Guest's criterion, states that the (ductile) material fails when the maximum shear stress (at the most vulnerable, critically stressed point) in a complex 3-dimensional state of stress reaches the critical value of the maximum shear stress in a simple 1-dimensional tensile test[54]. If the material fails — yielding is considered to be failure — at a yield stress of $\sigma_{11} = \sigma_y$ which corresponds to the maximum shear stress $\tau_{max} = (\sigma_{11} - 0)/2 = (\sigma_y - 0)/2$ in a simple tension test, the limit of safety is, as this theory suggests, when the maximum shear stress in a complex 3-dimensional state of stress reaches this value of $(\sigma_y - 0)/2$. We have seen that the maximum shear stress is half the value of the difference between any two principal stresses. Thus, the onset of yielding is predicted by this theory as: the maximum shear in the complex state of stress = the maximum shear stress at failure in a simple tension test, i.e., $\sigma_y/2$. The limit of failure is thus given by Eq. (4.53) below.

$$\begin{aligned}
\text{simple tension test:} \quad & \sigma_{11} = \sigma_y; \quad \tau_{max} = \frac{\sigma_y}{2}; \\
\text{complex stress:} \quad \tau_{max} = \quad & \text{max. of} \quad \left[\frac{|\sigma_{11} - \sigma_{22}|}{2}; \ \frac{|\sigma_{22} - \sigma_{33}|}{2}; \ \frac{|\sigma_{33} - \sigma_{11}|}{2}\right]; \\
\text{onset of yielding:} \quad & \text{max. of} \quad \left[\frac{|\sigma_{11} - \sigma_{22}|}{2}; \ \frac{|\sigma_{22} - \sigma_{33}|}{2}; \ \frac{|\sigma_{33} - \sigma_{11}|}{2}\right] = \frac{\sigma_y}{2}; \\
\text{i.e.,} \quad & \text{max. of} \quad [|\sigma_{11} - \sigma_{22}|; \ |\sigma_{22} - \sigma_{33}|; \ |\sigma_{33} - \sigma_{11}|] = \sigma_y.
\end{aligned} \tag{4.53}$$

If $\sigma_{33} = 0$ (2-d), the above criterion or prediction of the onset of yielding becomes[55]

$$\text{max. of} \quad [|\sigma_{11} - \sigma_{22}|; \ |\sigma_{22} - 0|; \ |0 - \sigma_{11}|] = \sigma_y. \tag{4.54}$$

Tresca's criterion is popular, but there is another criterion, called von Mises' criterion, that is also just as popular, if not more popular. We shall see this below.

[52] If we recognise that the sea is in some places as deep as 10 km or even deeper, and that the density of salt water is slightly more than that of water, we can see how high the water pressure is at the bottom of the sea. We suggest that the young readers do carry out this numerical calculation. The value is, we venture to state, much larger than what we might imagine.

[53] Henri Édourd Tresca (Oct. 1814 - June 1885), French, professor of mechanical engineering

[54] Named after H.É. Tresca. Also called Guest's criterion.

[55] Let us note clearly that in this case of $\sigma_{33} = 0$ if, say, $\sigma_{11} = 30$ and $\sigma_{22} = 20$ (both in MPa), the maximum shear stress at the point is *not* $(30 - 20)/2 = 5$ MPa, but $(30 - 0)/2 = 15$ MPa!

Von Mises' Criterion

This theory[56] postulates that it is the strain energy associated with the pure shear part — the deviatoric part — that decides failure. When this component of the strain energy, called the distortion energy reaches its limit equal to the distortion energy at the point of failure (yielding) in a simple tension test, the material fails.

To develop the equation that predicts failure according to this theory, we again start from the result of the simple tension test. The stress matrix for this case of one-dimensional loading is broken or decomposed into the hydrostatic (spherical component) and the pure shear (deviatoric component) parts. We then calculate the strain energy associated with the pure shear part. This is the limiting value at failure of the distortion energy.

For the simple tension test:

We need to split up the stress tensor into (i) the hydrostatic part and (ii) the pure shear part as shown below. We can readily recognise that the two matrices on the right hand side below represent, respectively, (i) the hydrostatic part (the three principal stresses are all equal, spherical part, isotropic tensor) and (ii) the pure shear part (sum of the three elements on the diagonal = 0, deviatoric part).

$$\begin{bmatrix} \sigma_y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sigma_y}{3} & 0 & 0 \\ 0 & \frac{\sigma_y}{3} & 0 \\ 0 & 0 & \frac{\sigma_y}{3} \end{bmatrix} + \begin{bmatrix} \frac{2\sigma_y}{3} & 0 & 0 \\ 0 & -\frac{\sigma_y}{3} & 0 \\ 0 & 0 & -\frac{\sigma_y}{3} \end{bmatrix} \tag{4.55}$$

The distortion energy (the strain energy corresponding to the second matrix, the pure shear part) per unit volume is calculated[57]. It is equal to

$$\text{distortion energy (simple tension test):} \quad \frac{1+\nu}{3E}\,\sigma_y^2. \tag{4.56}$$

For the complex 3- dimensional loading:

For this case, we need to split up the matrix representing the state of stress into two parts as before. The distortion energy corresponding to the second matrix — the pure shear part — is to be calculated again as before.

$$\begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} + \begin{bmatrix} \sigma_{11}-p & 0 & 0 \\ 0 & \sigma_{22}-p & 0 \\ 0 & 0 & \sigma_{33}-p \end{bmatrix}, \tag{4.57}$$

[56] Also known as the Maxwell-Huber-Hencky-von Mises theory. It seems to have been formulated by James Clark Maxwell as early as 1865, but is usually attributed to Richard Edler von Mises (1913). Tytus Maksymilian Huber is said to have anticipated this (1904).

[57] The calculation of strain energy presupposes some knowledge of the strain components related to the stress components through the constitutive equations. Thus, to understand this properly, we need to wait until we discuss the strain components and the constitutive constants (the elastic constants), ν (Poisson's ratio) and E (Young's modulus of elasticity). As the readers have already had some familiarity with these ideas, we let this sentence stay here as it is.

where

$$p = \frac{[\sigma_{11} + \sigma_{22} + \sigma_{33}]}{3}.$$

The distortion energy (the strain energy corresponding to the second matrix, the pure shear part) per unit volume is calculated. It is equal to

$$\text{distortion energy (general case):} \quad \frac{1+\nu}{6E}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2\right]. \tag{4.58}$$

Equating these two expressions [Eqs 4.56, 4.58], we find that the criterion for yielding (failure) is

$$\frac{1+\nu}{6E}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2\right] = \frac{1+\nu}{3E}\sigma_y^2;$$
$$\text{i.e.,} \quad \frac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2\right] = \sigma_y^2. \tag{4.59}$$

Onset of yielding:

The onset of failure (yielding) is thus predicted by this theory by the equation

$$\frac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2\right] = \sigma_y^2. \tag{4.60}$$

In two dimensions with $\sigma_{33} = 0$ (the case of plane stress), the corresponding equation — the von Mises' criterion — is

$$|\sigma_{11}^2 - \sigma_{11}\sigma_{22} + \sigma_{22}^2| = \sigma_y^2. \tag{4.61}$$

It should be obvious that for the material to be safe, Eqs (4.60, 4.61) will appear slightly modified, respectively, as

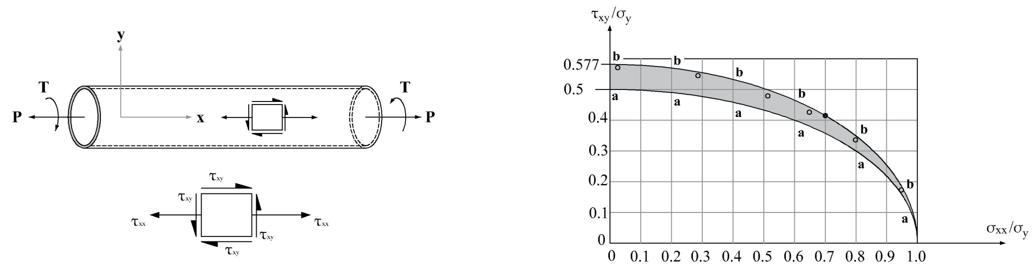
$$\frac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2\right] < \sigma_y^2; \text{ and} \tag{4.62}$$
$$|\sigma_{11}^2 - \sigma_{11}\sigma_{22} + \sigma_{22}^2| < \sigma_y^2. \tag{4.63}$$

Tresca's and von Mises' Criteria: A Comparison

Both these criteria are widely used. Tresca's criterion is slightly more conservative than the von Mises' one. We can see this from the fact the Tresca's yield surface is circumscribed by the von Mises' one.

Many scientists have performed experiments to check the validity of these criteria. The experiments by Taylor & Quinney[58] are very famous. They applied various combinations of

[58] G.I. Taylor & H. Quinney (1932). Geoffrey Ingram Taylor (Mar. 1886 - June 1975) was an outstanding British mathematician-scientist-engineer who was for the most part at the prestigious Cavendish Laboratory. The three great engineer-mathematician-scientists, the other two being S.P. Timoshenko (Stanford) and Th. von Kármán (CalTech) dominated the scene of mathematical engineering. They all had long and active professional lives and lived at about the same time. Taylor's student, G.K. Batchelor, a leading light in Fluid Mechanics in his own right, writes glowingly about his teacher. See his book, G.K. Batchelor: *The Life and Legacy of G.I. Taylor*, Cambridge University Press, (2008).



(a) A thin (copper) tube subjected to axial and torsional loading

(b) Experimental results and the predictions as per the two criteria

Figure 4.20: Taylor & Quinney: Experiments on thin tubes subjected to combined axial tension and torsion. Fig. 4.20a shows the specimen, a thin tube of circular cross-section subjected to (a pair of) axial forces and to (a pair of) twisting moments. Their results are shown in Fig. 4.20b. It can be seen that the experimental points (marked by heavy dots) lie between the curves *aaaa* (the lower one) and *bbbb* which are the predictions as per the Tresca's and von Mises' criteria, respectively.

axial tension and torque on thin-walled tubes and, thus, obtained several cases of combined stresses. Their results[59] show that the experimental points fall between the curves predicted by the two criteria. The von Mises' prediction is slightly more accurate (because the experimental points are closer), but it must be conceded that it is harder to apply compared to the Tresca's criterion. The Tresca's criterion is about 15% more conservative than the von Mises' one. These two are of importance in the theory of plasticity also.

CLOSING REMARKS

This topic of stress at a point is of great importance. This serves as a foundation upon which are built the mechanics of solids, the theory of elasticity, continuum mechanics, experimental and theoretical stress analysis and finite element methods. To be able to solve stress analysis problems we need to discuss all the governing equations also. We shall see them in the subsequent chapters.

In the next chapter, we shall have some more discussion of stresses.

[59] Thin-walled tubes may be subjected to various combinations of internal pressure and axial tension. This is another possibility. Such experiments were carried out and experimental results obtained by Lode (1928) and confirmed by other scientists.

Chapter 5

MORE ABOUT STRESSES

So far we confined ourselves to examining the nature of stress at a point. In a stress analysis problem, our interest is not limited to a single point; we are concerned with how the various stress components change (vary) from point to point. For example, if we look for the most critically stressed point, we have in our minds the entire stress field, and we locate the critical (vulnerable) point where the normal stress (or shear stress, or the distortional energy per unit volume) is the largest. Thus, the topics treated in the remaining part of this chapter still refer *to a point*, but have reference to several points in the stress field.

We need to discuss all the governing equations before we can solve stress analysis problems. These governing equations will be discussed one by one later in this book. The most important among these topics is the differential equations of equilibrium.

DIFFERENTIAL EQUATIONS OF EQUILIBRIUM

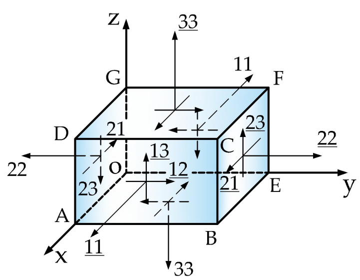
We have so far been discussing the state of stress *at a point*. We shall now start thinking about how the stress components can change from point to point. Such variations are governed by some fundamental equations known as the field equations. Among them the ones that are of the greatest importance are the (differential) equations of equilibrium. These differential equations also are *local* equations, in the sense that they must hold at each point inside the domain. In this sense, it is not inappropriate to discuss the differential equations of equilibrium as part of the nature of stress at a point.

The physical requirement that these equations demand is that every part of the body must be in equilibrium. Fair enough! If we consider an arbitrary block, small or large, of (stressed) material in the domain, there are stress components acting at all points of the block. The components of stress may change from point to point, but they may change only in such a way as to keep the block, small or large, in equilibrium. This basic requirement places some restrictions on how the stress components may change from point to point. This requirement is finally expressed as a set of differential equations to be satisfied at all points in the domain (inside the body). We shall see the details below.

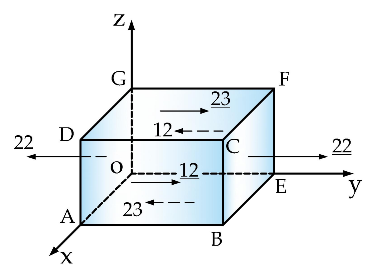
Introduction

We shall now consider the differential equations of equilibrium. The basic physical idea behind these equations, as stated above, is simple: the entire body, adequately supported, is in equilibrium. Furthermore, every tiny little bit of the body also is in equilibrium. We shall invoke the (necessary and sufficient) conditions of equilibrium and arrive at these important equations. These equations can be modified by including the so-called 'inertia forces and inertia moments' so that they are now applicable for dynamical problems also (using D'Alembert's principle).

Stress Components on a Block



(a) The stress components acting on a block



(b) Stress components in the y-direction

Figure 5.1: The stress components acting on the various faces of the block are shown in Fig. 5.1a. Only the stress components acting along the y-direction are marked in Fig. 5.1b. These are used to derive the differential equation of equilibrium in the y-direction.

As the forces are the externally applied forces and those associated with the stress components, let us consider the stress components acting on a block. To reduce the writings in the figure so as not to clutter them as far as possible, the following notations are used.

$$
\begin{array}{lll}
11 : \sigma_{xx} & 22 : \sigma_{yy} & 33 : \sigma_{zz} \\
\underline{11} : \sigma_{xx} + \dfrac{\partial \sigma_{xx}}{\partial x}\, dx & \underline{22} : \sigma_{yy} + \dfrac{\partial \sigma_{yy}}{\partial y}\, dy & \underline{33} : \sigma_{zz} + \dfrac{\partial \sigma_{zz}}{\partial z}\, dz \\
12 : \tau_{xy} & 23 : \tau_{yz} & 31 : \tau_{zx} \\
\underline{12} : \tau_{xy} + \dfrac{\partial \tau_{xy}}{\partial x}\, dx & \underline{23} : \tau_{yz} + \dfrac{\partial \tau_{yz}}{\partial y}\, dy & \underline{31} : \tau_{zx} + \dfrac{\partial \sigma_{zx}}{\partial z}\, dz \\
23 : \tau_{yz} & 31 : \tau_{zx} & 12 : \tau_{xy} \\
\underline{23} : \tau_{yz} + \dfrac{\partial \tau_{yz}}{\partial y}\, dy & \underline{31} : \tau_{zx} + \dfrac{\partial \tau_{zx}}{\partial z}\, dz & \underline{12} : \tau_{xy} + \dfrac{\partial \sigma_{xy}}{\partial x}\, dx \\
31 : \tau_{zx} & 12 : \tau_{xy} & 23 : \tau_{yz} \\
\underline{31} : \tau_{zx} + \dfrac{\partial \tau_{zx}}{\partial z}\, dz & \underline{12} : \tau_{xy} + \dfrac{\partial \tau_{xy}}{\partial x}\, dx & \underline{23} : \tau_{yz} + \dfrac{\partial \sigma_{yz}}{\partial y}\, dy
\end{array}
$$

The entire body is in equilibrium under the influence of surface and body forces. Every part of it, small or large, is also in equilibrium. Let us consider a block and mark all the stress components on the various faces of the block [Fig. 5.1a]. The various faces, their areas, and the stress components concerned are shown below.

$OEFG$	$dy\,dz$	σ_{xx}	$ABCD$	$dy\,dz$	$\sigma_{xx} + \dfrac{\partial \sigma_{xx}}{\partial x}\,dx$
$OEFG$	$dy\,dz$	τ_{xy}	$ABCD$	$dy\,dz$	$\tau_{xy} + \dfrac{\partial \tau_{xy}}{\partial x}\,dx$
$OEFG$	$dy\,dz$	τ_{xz}	$ABCD$	$dy\,dz$	$\tau_{xz} + \dfrac{\partial \tau_{xz}}{\partial x}\,dx$
$OGDA$	$dx\,dz$	τ_{yx}	$EFCB$	$dx\,dz$	$\tau_{yx} + \dfrac{\partial \tau_{yx}}{\partial y}\,dy$
$OGDA$	$dx\,dz$	σ_{yy}	$EFCB$	$dx\,dz$	$\sigma_{yy} + \dfrac{\partial \sigma_{yy}}{\partial y}\,dy$
$OGDA$	$dx\,dz$	τ_{yz}	$EFCB$	$dx\,dz$	$\tau_{yz} + \dfrac{\partial \tau_{yz}}{\partial y}\,dy$
$OABE$	$dx\,dy$	τ_{zx}	$EFCB$	$dx\,dy$	$\tau_{zx} + \dfrac{\partial \tau_{zx}}{\partial z}\,dz$
$OABE$	$dx\,dy$	σ_{zz}	$GDCF$	$dx\,dy$	$\sigma_{zz} + \dfrac{\partial \sigma_{zz}}{\partial z}\,dz$
$OABE$	$dx\,dy$	τ_{zy}	$GDCF$	$dx\,dy$	$\tau_{zy} + \dfrac{\partial \tau_{zy}}{\partial z}\,dz$

These are the various stress components acting on the various faces of the block. These are all acting on surfaces and they contribute to surface forces. In addition, there can be body forces acting on the volume of the block. If F_x, F_y, F_z are the body forces per unit volume, they lead to the forces $F_x\,dV = F_x\,dx\,dy\,dz$; $F_y\,dV = F_y\,dx\,dy\,dz$; $F_z\,dV = F_z\,dx\,dy\,dz$ in the x, y, z directions, respectively. These too have to be taken into account.

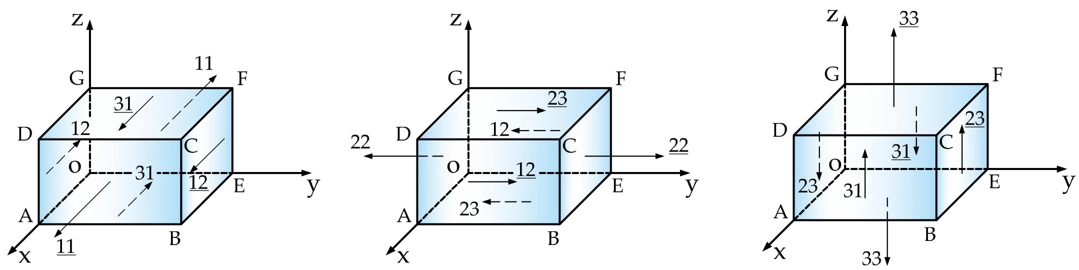
Fig. 5.1a is too cluttered. Hence for greater clarity, the stress components are shown separately in three figures [5.2a, 5.2b, 5.2c]. These are the stress components in the x-, y-, and z-directions relevant when the equations of equilibrium are written.

Equations of Equilibrium: Forces

Having considered all the forces on an elemental block, we can now write down the equations of motion. Let us refer to Fig. 5.1b where the relevant stresses that contribute to forces in the relevant direction — here the y-direction — are shown. To these we must add the body force $F_y\,dx\,dy\,dz$ also. Thus, we have

$$\left[-\tau_{xy}\,dy\,dz + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\,dx\right)\,dy\,dz\right] + \left[-\sigma_{yy}\,dx\,dz + \left(\sigma_{yy} + \frac{\partial \sigma_{yy}}{\partial y}\,dy\right)\,dx\,dz\right] +$$
$$\left[-\tau_{zy}\,dx\,dy + \left(\tau_{zy} + \frac{\partial \tau_{zy}}{\partial z}\,dz\right)\,dx\,dy\right] + F_y\,dx\,dy\,dz = 0.$$

On cleaning this up, we obtain the differential equation of equilibrium in the y-direction. Its companion equations in the x- and y-directions can be obtained similarly (by changing



(a) Stress components in the x-direction (b) Stress components in the y-direction (c) Stress components in the z-direction

Figure 5.2: The stress components acting on the various faces of the block, *but all in the same direction.* are shown in the three figures, the first [Fig. 5.2a - x direction] and so on. These are convenient when we derive the differential equations of equilibrium in the three (x, y, z) directions.

x to y, y to z, and z to x). These three equations are displayed below, and enclosed in a box because of their importance.

Differential equations of equilibrium:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = 0; \tag{5.1a}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + F_y = 0; \tag{5.1b}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z = 0. \tag{5.1c}$$

In index notation these are written as

$$\sigma_{ji,j} + F_i = 0 \quad \text{which is the same as} \quad \sigma_{ij,j} + F_i = 0, \qquad (i = 1, 2, 3). \tag{5.2}$$

The differential equations of equilibrium may be presented in matrix form as

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} + \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{5.3}$$

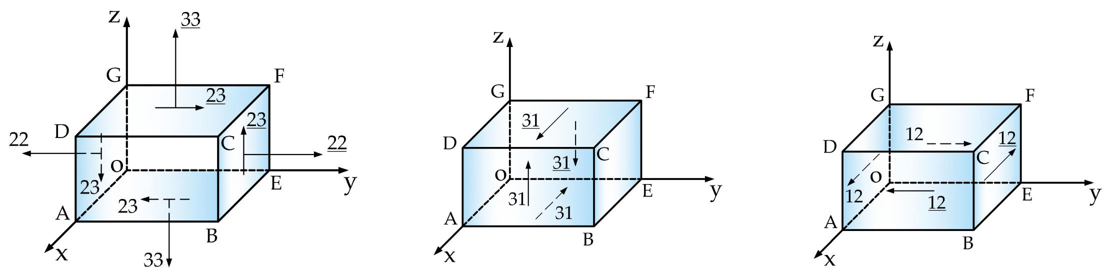
We may write this in the form

$$L^T \boldsymbol{\tau} + \boldsymbol{F} = \boldsymbol{0}, \tag{5.4}$$

where L is a linear (differential) operator operating here on the stress matrix $\boldsymbol{\tau}$ written[1] slightly differently as

$$\{\boldsymbol{\tau}\} = [\sigma_{xx} \;\; \sigma_{yy} \;\; \sigma_{zz} \;\; \tau_{xy} \;\; \tau_{yz} \;\; \tau_{zx}]^T \tag{5.5}$$

We shall see this operator L again later in a different context[2]. We shall obtain this result again later following a different procedure.



(a) Moment about the x-axis: $\tau_{yz} = \tau_{zy}$ (b) Moment about the y-axis: $\tau_{xz} = \tau_{zx}$ (c) Moment about the z-axis: $\tau_{xy} = \tau_{yx}$

Figure 5.3: To show that $\tau_{ij} = \tau_{ji}$: the three moment equations of equilibrium lead to the result that the stress matrix is symmetric. If there is a locked-in moment, the stress matrix will not be symmetric, which leads to a lot of complications as in micro-polar elasticity.

Equations of Equilibrium: Moments

We have seen the equations of equilibrium of forces. We shall now examine the equations of moment equilibrium. We shall wee that the moment equations of equilibrium lead to the conclusion that the stress matrix is symmetric, i.e., $\tau_{ij} = \tau_{ji}$.

We shall first consider the moment equation of equilibrium about the x-axis. Referring to, say, Fig. 5.1b, the relevant shear stresses that contribute to this moment are:

$$\begin{array}{lrl}
\text{face OGDA:} & \tau_{yz} & \text{area: } dx\,dz; \\
\text{face EFCB:} & \tau_{yz} + \dfrac{\partial \tau_{yz}}{\partial y}\,dy & \text{area: } dx\,dz; \\
\text{face OABE:} & \tau_{zy} & \text{area: } dx\,dy; \\
\text{face GDCF:} & \tau_{zy} + \dfrac{\partial \tau_{yz}}{\partial z}\,dz & \text{area: } dx\,dy;
\end{array}$$

[1] The stress matrix $\boldsymbol{\tau}$ is written here as a column matrix. It is displayed here as the transpose of a row matrix to save space. A notation $\{\boldsymbol{\tau}\}$ different from $[\boldsymbol{\sigma}]$ (which is a 3×3 matrix) is used here.

[2] Strain-displacement relations connecting the strain components and the associated displacement ones. It looks surprising that the same operator should appear in these two contexts which are entirely different from the physical point of view: (i) equilibrium of forces associated with stress components (no geometrical considerations here), and (ii) the strain components related to the associated displacements based on geometry (no equilibrium considerations here)! Nature seems to have interconnections at a deeper level. Is everything connected to everything else, as Buddhists are said to believe?

Personal discussions with Dr Gangan Prathap are gratefully acknowledged. How delightful and insightful!

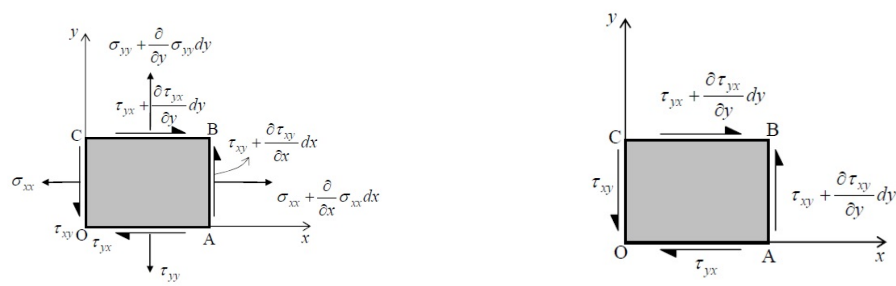
If we take moments of these associated shear forces about the x-axis, we find that

$$\left(\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y}\, dy\right) \times (dx\, dz) \times dy - \left(\tau_{zy} + \frac{\partial \tau_{zy}}{\partial z}\, dz\right) \times (dx\, dy) \times dz = 0,$$

from which we conclude that $\tau_{yz} = \tau_{zy}$. Quantities of higher order of smallness can be, and are, neglected. We may also mention that the contribution to this moment (about the x-axis) of (i) the normal stresses on the faces $OGDA$, $EFCB$, $OABE$, $GDCF$ and of (ii) the shear stresses on the faces $OEFG$, $ABCD$ is only negligibly small (higher order of smallness). During the limiting process, the quantities of higher order of smallness $\to 0$ much faster and, therefore, drop out of the resulting equation.

In the same way we can show that[3] (i) $\tau_{zx} = \tau_{xz}$, and (ii) $\tau_{xy} = \tau_{yx}$. We shall obtain this result again later following a different procedure[4].

Two-dimensional Case



(a) Stress components on a block

(b) To show that $\tau_{xy} = \tau_{yx}$

Figure 5.4: The stress components acting on a rectangular block are shown in Fig. 5.4a. In Fig. 5.4b are shown the shear stresses acting on the various faces. These are meant to be of help in deriving the equations of equilibrium, and in showing that $\tau_{xy} = \tau_{yx}$.

Two-dimensional cases are of special importance. The general three-dimensional problem is so difficult to solve that often simplifications are made. Among them perhaps the most important one is the two-dimensional simplification. In experimental stress analysis it is essential to know the two-dimensional case thoroughly.

A rectangular block with the stress components is shown [Fig. 5.4]. We shall take the help of Fig. 5.4a to derive the two differential equations of equilibrium, and of Fig. 5.4b to show that $\tau_{xy} = \tau_{yx}$ (which is really the result obtained from the moment equation of equilibrium). As in earlier cases, we consider the stress components and the associated areas, and write down the equations of equilibrium in the x- and y- directions. The body forces are F_x and F_y per *unit area*. The various faces, their areas and the stress components acting on them are identified below. The thickness of the block is taken as 1.

face CO: area: $dy \times 1$ σ_{xx}; τ_{xy};

[3] Sometimes this result $\tau_{ij} = \tau_{ji}$ is stated as: 'shear and complementary shear are equal'.

[4] Physically it is the same procedure of computing the moments; mathematically the method used is different.

$$
\begin{array}{llll}
\text{face OA:} & \text{area: } dx\times 1 & \sigma_{yy}; & \tau_{yx}; \\
\text{face AB:} & \text{area: } dy\times 1 & \sigma_{xx}+\dfrac{\partial\sigma_{xx}}{\partial x}\,dx; & \tau_{xy}+\dfrac{\partial\tau_{xy}}{\partial x}\,dx; \\
\text{face BC:} & \text{area: } dx\times 1 & \sigma_{yy}+\dfrac{\partial\sigma_{yy}}{\partial y}\,dy; & \tau_{yx}+\dfrac{\partial\tau_{yx}}{\partial y}\,dy.
\end{array}
$$

These are all surface forces. Additionally there are, in general, body forces. If these are F_x and F_y per unit area[5] in the x- and y- directions, respectively. With this information, we can write down the equations of equilibrium in the x- and y-directions. They are:

$$
\begin{aligned}
x\text{-direction:}\quad & \left[\left(\sigma_{xx}+\frac{\partial\sigma_{xx}}{\partial x}\,dx\right)(dy\times 1)-\sigma_{xx}(dy\times 1)\right]+ \\
& \left[\left(\sigma_{yx}+\frac{\partial\sigma_{yx}}{\partial y}\,dy\right)(dx\times 1)-\tau_{yx}(dx\times 1)\right]+F_x(dx\times dy\times 1)=0; \\
y\text{-direction:}\quad & \left[\left(\sigma_{yy}+\frac{\partial\sigma_{yy}}{\partial y}\,dy\right)(dx\times 1)-\sigma_{yy}(dx\times 1)\right]+ \\
& \left[\left(\tau_{xy}+\frac{\partial\tau_{xy}}{\partial x}\,dx\right)(dy\times 1)-\tau_{xy}(dy\times 1)\right]+F_y(dx\times dy\times 1)=0.
\end{aligned}
$$

Cleaning these up, we obtain the differential equations of equilibrium. They are honoured by placing them below in an enclosure.

Differential equations of equilibrium:

$$
\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+F_x=0; \tag{5.6a}
$$

$$
\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+F_y=0. \tag{5.6b}
$$

In index notation these are written as

$$
\sigma_{ji,j}+F_i=0 \quad \text{which is the same as} \quad \sigma_{ij,j}+F_i=0, \qquad (i=1,2). \tag{5.7}
$$

The differential equations of equilibrium may be presented in matrix form as

$$
\begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}
\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}
+\begin{Bmatrix} F_x \\ F_y \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{5.8}
$$

We may write this in the form

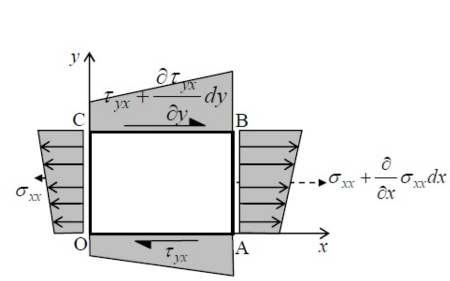
$$
L^T\boldsymbol{\tau}+\boldsymbol{F}=\boldsymbol{0}, \tag{5.9}
$$

[5] per unit area is the same as per unit volume because the thickness is 1.

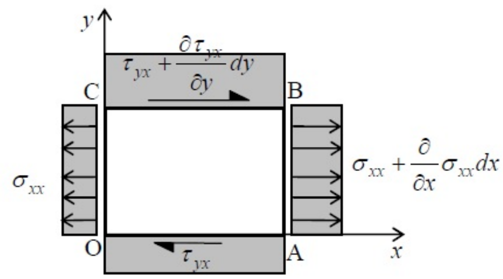
where L is a linear (differential) operator operating here on the stress matrix $\boldsymbol{\tau}$ written[6] slightly differently as

$$\{\boldsymbol{\tau}\} = [\sigma_{xx} \;\; \sigma_{yy} \;\; \tau_{xy}]^T \tag{5.10}$$

A Little More Rigorous Derivation



(a) A more rigorous analysis



(b) A simplified analysis

Figure 5.5: The stress components acting on a rectangular block are shown. The variations along both $x-$ and $y-$directions are taken into account [Fig. 5.5a].

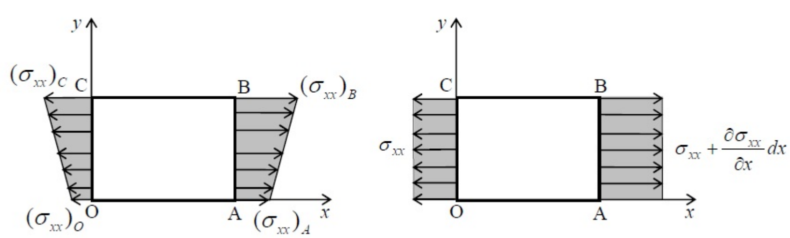


Figure 5.6: To show that a simplified analysis is sufficient to obtain the correct equations

Here we shall derive the same equations of equilibrium a little more rigorously. If $(\sigma_{xx})_O$ is the stress component σ_{xx} at the point O, there is a variation of this component along OA and also along OC [Fig. 5.5a]. Thus,

$$(\sigma_{xx})_A = (\sigma_{xx})_O + \frac{\partial (\sigma_{xx})_O}{\partial x} \, dx;$$
$$(\sigma_{xx})_C = (\sigma_{xx})_O + \left(\frac{\partial \sigma_{xx}}{\partial y} \right)_O dy.$$

[6] The stress matrix $\boldsymbol{\tau}$ is written here as a column matrix. It is displayed here as the transpose of a row matrix to save space. A notation $\{\boldsymbol{\tau}\}$ different from $[\boldsymbol{\sigma}]$ (which is a 3×3 matrix) is used here.

As the distances OA, OC, etc. are small — we are considering an elemental rectangle which would shrink to a point in the limit — the variations may be assumed to be linear as shown in Fig. 5.5a. Similarly, considering the variation of σ_{xx} along AB,

$$\begin{aligned}(\sigma_{xx})_B &= (\sigma_{xx})_A + \frac{\partial(\sigma_{xx})_A}{\partial y}\,dy \\ &= (\sigma_{xx})_O + \frac{\partial(\sigma_{xx})_O}{\partial x}\,dx + \frac{\partial}{\partial y}\left[(\sigma_{xx})_O + \frac{\partial(\sigma_{xx})_O}{\partial x}\,dx\right]dy \\ &= (\sigma_{xx})_O + \frac{\partial(\sigma_{xx})_O}{\partial x}\,dx + \frac{\partial(\sigma_{xx})_O}{\partial y}\,dy + \frac{\partial^2(\sigma_{xx})_O}{\partial x\,\partial y}\,dx\,dy. \end{aligned} \tag{5.11}$$

The net force in the x-direction corresponding to this stress component σ_{xx} is

$$\begin{aligned}\text{net force} &= \frac{1}{2}\left[(\sigma_{xx})_B + (\sigma_{xx})_A\right] \times dy \times 1 - \frac{1}{2}\left[(\sigma_{xx})_C + (\sigma_{xx})_O\right] \times dy \times 1 \\ &= \frac{1}{2}\left[\left\{(\sigma_{xx})_O + \frac{\partial(\sigma_{xx})_O}{\partial x}\,dx + \frac{\partial(\sigma_{xx})_O}{\partial y}\,dy + \frac{\partial^2(\sigma_{xx})_O}{\partial x\,\partial y}\,dx\,dy\right\}\right] + \\ &\quad \frac{1}{2}\left[\left\{(\sigma_{xx})_O + \frac{\partial(\sigma_{xx})_O}{\partial x}\,dx\right\}\right]dy - \frac{1}{2}\left[\left\{(\sigma_{xx})_O + \frac{\partial(\sigma_{xx})_O}{\partial y}\,dy\right\} + (\sigma_{xx})_O\right]dy \\ &= \frac{\partial(\sigma_{xx})_O}{\partial x}\,dx\,dy + \frac{\partial^2(\sigma_{xx})_O}{\partial x\,\partial y}\,dx\,dy. \end{aligned} \tag{5.12}$$

For the simplified case [Fig. 5.5b] we have

$$\begin{aligned}\text{net force} &= \left[\left\{\sigma_{xx} + \frac{\partial\sigma_{xx}}{\partial x}\,dx\right\}\Big|_{AB} - \sigma_{xx}\Big|_{OC}\right]dy \\ &= \frac{\partial\sigma_{xx}}{\partial x}\,dx\,dy. \end{aligned} \tag{5.13}$$

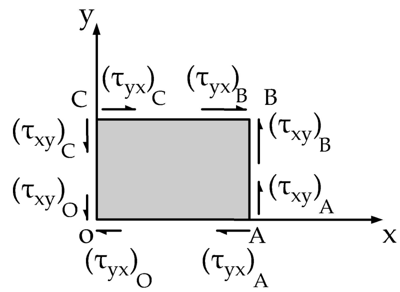
A comparison of Eqs (5.12) and (5.13) shows that the result is the same except for a term of a higher order of smallness which can be neglected. We are looking for the limiting values as the rectangle is made smaller and smaller until it degenerates to a point. The conclusion is this: it is quite sufficient to consider the simplified case represented by Fig. 5.5a. The same conclusion will be arrived at when we consider the net force in the x-direction because of the shear stresses. We have here as before two cases, the more rigorous one allowing for the variation of the shear stress components τ_{xy} and τ_{yx} [Fig. 5.7a], and the simplified case represented by Fig. 5.7b. The conclusion is the same.

The net force in the x-direction associated with the shear stress τ_{yx} [Fig. 5.7a] is

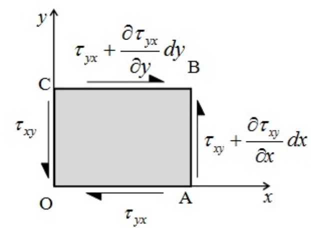
$$\frac{1}{2}\left[(\tau_{yx})_B + (\tau_{yx})_C\right]dy \times 1 - \frac{1}{2}\left[(\tau_{yx})_B + (\tau_{yx})_C\right]dy \times 1,$$

where we write as before

$$(\tau_{yx})_A = (\tau_{yx})_O + \frac{\partial(\tau_{yx})_O}{\partial x}\,dx;$$



(a) More rigorous analysis



(b) A simplified analysis

Figure 5.7: The shear stress components acting on a rectangular block are shown in Fig. 5.7a where the variations along both $x-$ and $y-$directions are taken into account. In Fig. 5.7b a simplified case is shown. By comparing the two cases we can conclude that the simplified case is sufficient to obtain the equations of equilibrium correctly.

$$(\tau_{yx})_B = (\tau_{yx})_C + \frac{\partial (\tau_{yx})_C}{\partial x}\, dx,$$

etc. We assume, again as before, that the variations are linear. The net force in the x-direction associated with the shear stress τ_{yx} using the simplified stress distribution [Fig. 5.7b] is

$$\left[\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\, dy\right] dx \times 1 - [\tau_{yx}]\, dx \times 1. \tag{5.14}$$

A detailed calculation and comparison with Eq. (5.14) lead us to the same conclusion: a simplified calculation is sufficient! Obviously, similar conclusions will be arrived at in the case of the net force in the y-direction also.

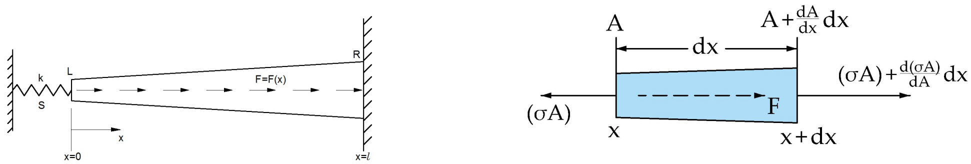
When we calculate the moments, the normal stresses also contribute to the moments. However, if we make detailed calculations, we can convince ourselves that they really do not have to be reckoned for the same reason: quantities of a higher order of smallness can be left out. Fig. 5.6 also leads us to the same conclusion.

One-dimensional Case: An Axial Bar

We shall now take up the simplest case, perhaps, of a one-dimensional body. A (one-dimensional) bar [Fig. 5.8a] is subjected to an axially distributed load F. We desire to obtain the differential equation of equilibrium.

Let us consider the various forces on the elemental block. The stress components are σ on the left end at $x = 0$, and $\sigma + (d\sigma/dx)\, dx$. The area of cross-section A is variable: $A = A(x)$. The force components acting on the end cross-sections are marked in Fig. 5.8b. In addition there is an applied force F per unit length which is variable and distributed along the length. The equation of equilibrium is, thus,

$$-(\sigma\, A) + \left[(\sigma\, A) + \frac{d(\sigma\, A)}{dx}\, dx\right] + F\, dx = 0;$$



(a) An axial bar with a distributed load (b) Force components on an element

Figure 5.8: To derive the equations of equilibrium for an axial bar

$$\text{i.e.,} \quad \frac{d}{dx}\,(\sigma\, A) + F = 0, \qquad (0 < x < l). \tag{5.15}$$

[This differential equation of equilibrium is in terms of the stress components. This is the natural approach. However, there are occasions when it is desirable to recast this equation in terms of the displacements. We shall explain this later. (See the section on displacement formulation on p. 12-13.) We shall see this problem again on p. 12-22 where this differential equation is rewritten in terms of the (unknown) displacement $u = u(x)$. At that time our focus will be on writing the boundary conditions in terms of the displacement u and its derivatives. We shall see that this is a mixed boundary value problem in the sense that one boundary condition is on the displacement u, while the other is on du/dx. This is to be taken notice of; the techniques used may be different for such situations.]

Equations of Equilibrium in Other Coordinate Systems

It is often convenient, and sometimes even necessary, to have the equations of equilibrium[7] written in other coordinate systems. The reasons are that (i) sometimes the (boundary value) problems are easier to formulate, and (ii) sometimes the boundary conditions are far easier to write in one coordinate system than in another mainly because of the geometry of the problem. Besides, even more importantly, some problems when formulated in one coordinate system can be easily solved, while they can be incredibly difficult, or even impossible, in another[8]. An example is, say, the problem of determining the stresses in a thick cylinder subjected to internal fluid pressure. The geometry makes it convenient — well, almost essential — to choose polar coordinates in order to exploit the inherent circular symmetry (axisymmetry) of the problem. Thus, there is a strong case to use coordinate systems other than the simple rectangular coordinate system that we have used so far.

In addition to the above factors, there is the natural academic curiosity of examining the possibility of formulation and solution of problems in a more general setting. For these reasons, it is extremely important to consider other coordinate systems also. However, there are several complications when a general (curvilinear) coordinate system is used. In

[7] and, indeed, the other governing equations such as (i) the strain-displacement relations, (ii) the constitutive equations

[8] Readers familiar with the dynamics of rigid bodies may recall how easy the problem of torque-free motion of a gyroscope is to solve when Euler angles are used, and how difficult it is otherwise.

spite of the complications, it seems appropriate to give here an elementary derivation of the differential equations of equilibrium in cylindrical and polar coordinates. However, before we undertake this task, we need to have a glimpse of some of the complications in coordinate systems other than the rectangular cartesian coordinates.

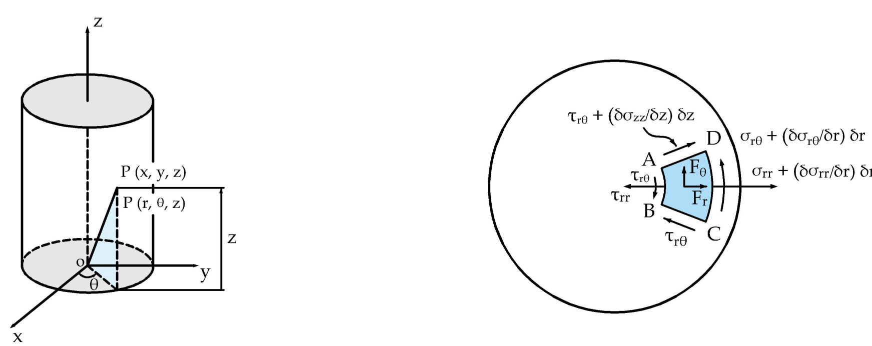
Governing Equations in Other Coordinate Systems

Let us say we desire to obtain the differential equations of equilibrium in a cylindrical coordinate system. One way is to draw an elemental block, mark all the stress components acting on the various faces and the components of the body force, and consider the equilibrium of the block. The necessary condition is that the net resultant force acting in any direction — more specifically, in the r, θ, z (radial, tangential, and axial directions, respectively) directions — should vanish. After writing these equations, we proceed to the limit as the element is made smaller and smaller, and we obtain the required equation.

A second method is to change the equation from rectangular cartesian to cylindrical polar coordinates by a mathematical transformation changing the independent variables from (x, y, z) to (r, θ, z). This is not difficult, but we would not have the physical feel of the various stress components and their directions, etc. However, this method is quite effective in spite of the various mathematical manipulations involved.

A third method is by using general tensors, but it is too difficult for us now.

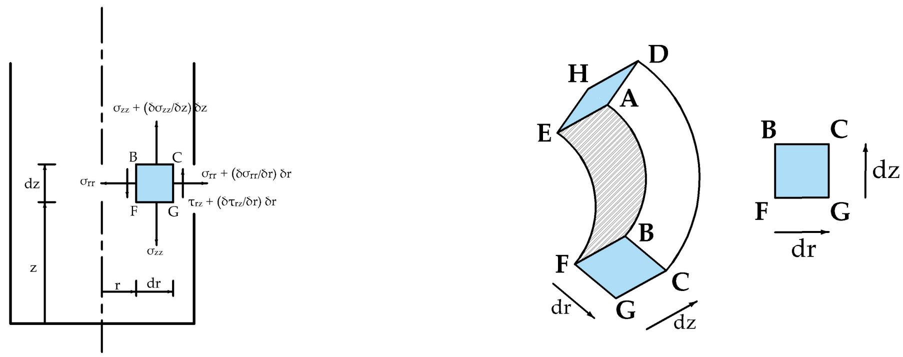
Equations of Equilibrium in Cylindrical Coordinate Systems



(a) A cylindrical polar coordinate system (b) An elemental block with stress components

Figure 5.9: To derive the equations of equilibrium in cylindrical coordinates

Figs 5.9, 5.10 show an elemental block in a cylindrical polar coordinate system. The various stress components and the corresponding areas are listed below. As usual, we calculate the net force acting in each of the r, θ, z directions, and set it equal to zero to obtain the equations of equilibrium in the radial (r), tangential (θ) and axial (z) directions. The curved surfaces ($ABFE$, $DCGH$) are in the vertical plane. (They do not appear to be so in the figure shown.) The various faces, their areas, and the stress components concerned are shown below. We can calculate the forces concerned and write down the



(a) An elemental block (b) An elemental block with its faces

Figure 5.10: The faces of an elemental block and the stress components are shown. The edges (BF, CG, AE, DH) are vertical, and BC, FG, AD, EH) horizontal.

(three) equations of equilibrium in the radial (r), tangential θ and axial (z) directions.

$$
\begin{array}{llllll}
AEFB & r\,d\theta\,dz & \sigma_{rr} & DHGC & (r+dr)\,d\theta\,dz & \sigma_{rr} + \dfrac{\partial \sigma_{rr}}{\partial r}\,dr \\
AEFB & r\,d\theta\,dz & \tau_{r\theta} & DHGC & (r+dr)\,d\theta\,dz & \tau_{r\theta} + \dfrac{\partial \tau_{r\theta}}{\partial r}\,dr \\
AEFB & r\,d\theta\,dz & \tau_{rz} & DHGC & (r+dr)\,d\theta\,dz & \tau_{rz} + \dfrac{\partial \tau_{rz}}{\partial r}\,dr \\
BFGC & dr\,dz & \sigma_{\theta\theta} & AEHD & dr\,dz & \sigma_{\theta\theta} + \dfrac{\partial \sigma_{\theta\theta}}{\partial \theta}\,d\theta \\
BFGC & dr\,dz & \tau_{\theta r} & AEHD & dr\,dz & \tau_{\theta r} + \dfrac{\partial \tau_{\theta r}}{\partial \theta}\,d\theta \\
BFGC & dr\,dz & \tau_{\theta z} & AEHD & dr\,dz & \tau_{\theta z} + \dfrac{\partial \tau_{\theta z}}{\partial \theta}\,d\theta \\
EFGH & r\,dr\,d\theta & \sigma_{zz} & ABCD & r\,dr\,d\theta & \sigma_{zz} + \dfrac{\partial \sigma_{zz}}{\partial z}\,dz \\
EFGH & r\,dr\,d\theta & \tau_{zr} & ABCD & r\,dr\,d\theta & \tau_{zr} + \dfrac{\partial \tau_{zr}}{\partial z}\,dz \\
EFGH & r\,dr\,d\theta & \tau_{z\theta} & ABCD & r\,dr\,d\theta & \tau_{z\theta} + \dfrac{\partial \tau_{z\theta}}{\partial z}\,dz.
\end{array}
$$

These are the various stress components acting on the various faces of the block. These are all acting on surfaces and they contribute to surface forces. In addition, there can be body forces acting on the volume of the block. If F_r, F_θ, F_z are the body forces per unit volume in the radial (r), tangential (θ) and axial (z) directions, they lead to the forces $F_r\,dV = F_r\,r\,dr\,d\theta\,dz$; $F_\theta\;dV = F_\theta\,r\,dr\,d\theta\,dz$; $F_z\,dV = F_z\,r\,dr\,d\theta\,dz$ in the r, θ, z directions, respectively. These too have to be reckoned.

Let us note that the tangential stresses $\sigma_{\theta\theta}$ and $\sigma_{\theta\theta} + \dfrac{\partial \sigma_{\theta\theta}}{\partial \theta}\,d\theta$ are not parallel; there is a small angle $d\theta$ between them. This means that there is a contribution from these stress components *in the radial direction* (radially inwards, towards the centre).

If these equations are written (noting the correct direction of each stress component) and cleaned up which involves some algebraic manipulations, we obtain the (linear differential) equations

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + F_r = 0; \tag{5.16a}$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} + \frac{2\tau_{r\theta}}{r} + F_\theta = 0; \tag{5.16b}$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\tau_{rz}}{r} + F_z = 0. \tag{5.16c}$$

Polar Coordinates

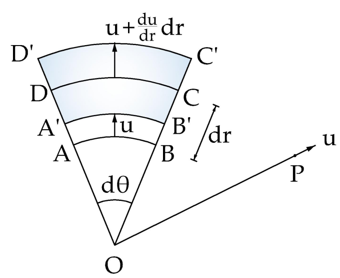
These equations are simplified when (two-dimensional) polar coordinates (r, θ) are used. Obviously, we will have two equations of equilibrium. The derivation is along the same lines; there is much simplification. The final equations are shown below.

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\,\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + F_r = 0; \tag{5.17a}$$

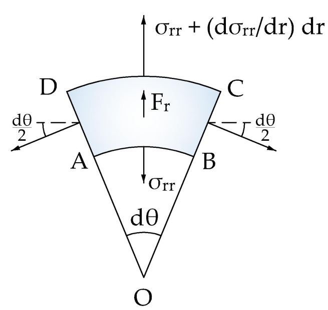
$$\frac{1}{r}\,\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\,\tau_{r\theta}}{r} + F_\theta = 0. \tag{5.17b}$$

The Simplified Axisymmetric Case

There are important cases where there is much simplification because of axisymmetry (also called rotational symmetry). A cylinder, thin or thick, subjected to a fluid pressure[9] and a rotating disc are two examples. All the stress components acting on the element are shown



(a) Deformations



(b) Stress components

Figure 5.11: To explain the simplified case of an axisymmetric problem

[9] This problem is relevant even when there is no fluid at all! If we wish to make interference fit calculations, the stresses are calculated using the well known Lamé's equations for the stresses in a thick cylinder subjected to fluid pressure. Some of the equations of fluid mechanics become relevant even when there is no fluid. The techniques used in fluid mechanics are applicable in several other situations quite unrelated to fluids. This is so in the case of most engineering science courses. This is the reason why a rigorous study of engineering science courses is emphasised.

here. Because of the axisymmetry[10], there is no equation of equilibrium in the tangential direction[11]. The only equation of equilibrium is in the radial direction. The relevant stress components in the r-direction are (the thickness t is taken to be 1):

AB	$r\,d\theta \times 1$	σ_{rr}	CD	$(r+dr)\,d\theta \times 1$	$\sigma_{rr} + \dfrac{d\sigma_{rr}}{dr}\,dr$
BC	$dr \times 1$	$\sigma_{\theta\theta}$	AD	$dr \times 1$	$\sigma_{\theta\theta}$

Because of axisymmetry, BC and AD and, hence, AB and CD are principal planes; there are no shear stresses on them. Furthermore, all the variables of interest are independent of θ; the derivatives are ordinary, and not partial.

The tangential stress components are not quite tangential; they have a component in the radial direction, because of the small angle $d\theta$ between the edges BC and AD. The radial force component is

$$2 \times \sigma_{\theta\theta} \times dr \times 1 \times \frac{d\theta}{2} = \sigma_{\theta\theta}\,dr\,d\theta \quad \text{radially inwards.}$$

Additionally, there can be a body force component F_r per unit volume[12] in the r-direction. Collecting all these terms, the (only) equation of equilibrium (in the r-direction) is

$$\left(\sigma_{rr} + \frac{d\sigma_{rr}}{dr}\,dr\right)(r+dr)\,d\theta - \sigma_{rr}\,r\,d\theta - \sigma_{\theta\theta}\,dr\,d\theta + F_r\,r\,dr\,d\theta \times 1 = 0;$$

$$\text{i.e.,}\quad \frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + F_r = 0. \tag{5.18}$$

Other Cases

There are several more cases of interest and practical use in stress analysis problems; we cannot obviously discuss all of them here.

Concluding Remarks

We have presented some information about one prime requirement, viz., the (differential) equations of equilibrium shall be satisfied. The same requirement may be met by demanding that the principle of virtual work shall hold. We can show that the principle of virtual work is both a necessary and sufficient condition for equilibrium.

[10] rotational symmetry about the axis passing through O perpendicular to the plane of the paper. Axisymmetry leads to considerable simplifications, because all the variables will now be independent of the independent variable θ.

[11] What is meant is not that there is no equation of equilibrium, but that it is a trivial one. This equation is a correct, but useless equation being of the form: [something] = [the same something].

[12] Is this per unit volume, or is it per unit area? The thickness being 1, do these two come to the same thing? Let us examine and settle the issue, and not accept this statement. Let us learn to question. Questioning is not attacking or challenging. It is an expression of an open mind willing to explore other possibilities. Gullibility is not a virtue!

The differential equations of equilibrium are, traditionally and almost always, written in terms of the stress components. However, it is possible to write these in terms of strain components (using the stress-strain relations), and / or in terms of the displacement components using the strain-displacement relations. One of the motivations for doing so is to express all the unknowns in terms of, say, the strain components or, better still, in terms of displacements. There may be some convenience. Some problems can be solved by this approach. However, we must realise that the equations, though expressed in terms of displacements, are still equilibrium equations. Furthermore, inasmuch as we have used the constitutive equations, the resulting equations of equilibrium are now restricted to elastic materials only!

We shall see that the governing equations are generally written in terms of different (unknown, dependent) variables: (i) the equations of equilibrium in terms of the stress components $\sigma_{ij},\ (i,j=x,y,z)$; (ii) the strain-displacement relations in terms of the strain and displacement components $e_{ij},\ \ (i,j\ =\ x,y,z)$; $u_i\ \ (i\ =\ 1,2,3)$ (i.e., u,v,w); and the constitutive equation in terms of σ_{ij} and $e_{ij}\ \ (i,j=x,y,z)$. This situation is admittedly inconvenient. Can we recast them all in terms of the same unknowns, the displacement components u,v,w? Sure enough, this can be done. We shall do this later [p. 12-13].

SOME CONCEPTS RELATED TO 2-D STRESS FIELDS

There are some important terms and concepts related to two-dimensional *stress fields* — not just at a point — that have special relevance and significance in photoelasticity. Experimental stress analysis is a separate discipline in own right[13]. It is not possible to deal with it here. Nevertheless we shall discuss briefly a few terms. There are many aspects to be discussed in detail before we can appreciate, or even understand, the significance of these technical terms. We shall use p and q to refer to the two principal stresses just for this section only.

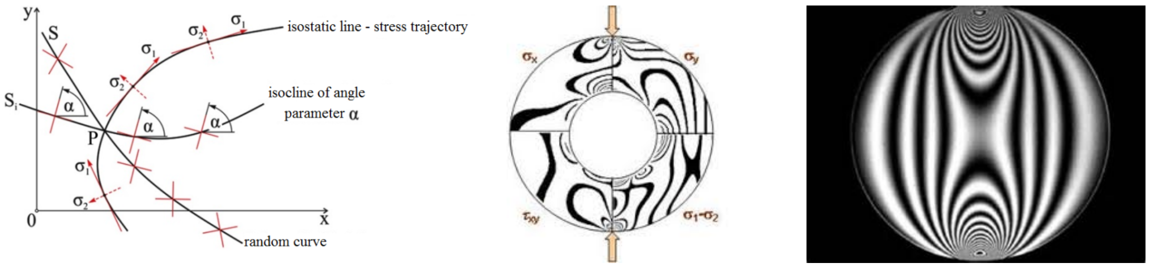
Experimental Stress Analysis: Photoelasticity

Most of the methods of experimental stress analysis are based on measurement of strain. Photoelasticity is an exception where stress, and not strain, is directly measured (or calculated from experiments). The following has special relevance for photoelasticity.

Dependence of the Stresses on the Material Properties

We use photoelasticity to determine the stresses in actual structures like a dam and machine parts. This is essentially a two-dimensional analysis (although three-dimensional photoelastic analysis also is possible). Behind this procedure, there lies a fundamental question: are

[13] Photoelasticity is an important part of experimental stress analysis. The method is based on the phenomenon of double refraction. When polarised light passes through a stressed material, a phase angle is introduced between the two rays, called the ordinary and the extraordinary. Thus, fringes are formed. These fringes are obtained experimentally using a photoelastic bench and polarised light. Fig. 5.12 shows two fringe patters and two special curves of special importance in photoelasticity. Students are urged to be familiar with this extremely useful topic. Most colleges have facilities at least as demonstration experiments in photoelasticity.



(a) Some important curves (b) A fringe pattern (c) Another fringe pattern

Figure 5.12: Photoelastic fringes and curves related to photoelasticity

the stresses in an actual structure or a machine part the same as those in the photoelastic model? This question leads us to the examination of an even more fundamental question: are the stresses in a body independent of the material? Let us examine.

Let us consider a few specific cases and examine if the stresses are dependent only on the applied loads but independent of the elastic properties of the material.

(i) A uniform bar loaded by a pair of (tensile) loads $P - P$:

The (tensile) stress is given by $\sigma = P/A$ which is clearly independent of the material (constitutive) properties.

(ii) A uniform circular shaft loaded by a pair of twisting moments $T - T$:

The (shear) stress is given by $\tau = T\,r/J$ which is again independent of the material.

(iii) A uniform beam in pure bending loaded by $M - M$:

The bending stress is given by $\sigma = M\,y/I$ which is independent of the material.

(iv) Lamé's thick cylinder problem: a thick cylinder subjected to internal pressure

The stresses are given by $\sigma_{rr} = A + B/r^2$ and $\sigma_{\theta\theta} = A - B/r^2$ (independent of the material).

With these examples in mind, we may be tempted to conclude that the stresses depend only on the load applied and never on the elastic (constitutive) properties. But wait a minute. Let us not jump into hasty conclusions. Let us consider two more problems.

(v) Rotating discs: the stresses now depend on the elastic properties!

$$\sigma_{rr} = A + \frac{B}{r^2} - \frac{3+\nu}{8}\,\rho\omega^2 r^2,$$
$$\sigma_{\theta\theta} = A - \frac{B}{r^2} - \frac{1+3\nu}{8}\,\rho\omega^2 r^2,$$

(vi) The free end of a uniform (horizontal) cantilever lifted by a specified amount of displacement, δ

Now the load P needed to bring about the end deflection $= \dfrac{3\,EI}{l^3}\,\delta$. The stresses are dependent on the material properties!

From the examination of these few cases, what can we understand? In all the problems (i) - (iv), the displacements do depend on the elastic properties. In (v), even the stresses depend on the constitutive properties. Problem (v) is the only one where a body force is present. What does all this lead us to?

Well, the general problem is very difficult to tackle. We will not attempt to attack the problem in all its generality. Instead we shall merely obtain a partial answer applicable to two-dimensional (plane stress and plane strain) problems.

The differential equations of equilibrium and the traction boundary conditions are:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + F_x = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y = 0; \tag{5.19}$$

$$l\,\sigma_{xx} + m\,\tau_{yx} = p_{\nu x} \qquad l\,\tau_{xy} + m\,\sigma_{yy} = p_{\nu y}. \tag{5.20}$$

Now there are three unknowns (the three stress components), but only two equations. We need an additional equation which is the compatibility equation

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x\,\partial y}. \tag{5.21}$$

The constitutive equations for the two cases of plane stress and plane strain are:

$$\left\{\begin{matrix} e_{xx} \\ e_{yy} \\ e_{xy} \end{matrix}\right\} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & (1+\nu) \end{bmatrix}\left\{\begin{matrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{matrix}\right\} \qquad \text{(plane stress);} \tag{5.22}$$

$$\left\{\begin{matrix} e_{xx} \\ e_{yy} \\ e_{xy} \end{matrix}\right\} = \frac{1+\nu}{E}\begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 1 \end{bmatrix}\left\{\begin{matrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{matrix}\right\} \qquad \text{(plane strain);} \tag{5.23}$$

Substituting for the stress components in Eq. (5.21) using the constitutive equations Eq. (5.22, plane stress), we obtain

$$\frac{1}{E}\left[\frac{\partial^2 \sigma_{xx}}{\partial y^2} - \nu\frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{\partial^2 \sigma_{yy}}{\partial x^2} - \nu\frac{\partial^2 \sigma_{xx}}{\partial x^2}\right] = \frac{2(1+\nu)}{E}\frac{\partial^2 \tau_{xy}}{\partial x\,\partial y}. \tag{5.24}$$

We now desire to get rid of τ_{xy} in this equation. Towards this end, let us differentiate Eq. (5.19) w.r.to x and Eq. (5.20) w.r.to y and add to yield

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = -2\,\frac{\partial^2 \tau_{xy}}{\partial x\,\partial y}.$$

From here if we substitute for τ_{xy} on the right hand side of Eq. (5.24), we obtain on simplification

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = -(1+\nu)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right) \qquad \text{(plane stress).} \tag{5.25}$$

In the same way we can obtain for the case of plane strain

$$\nabla^2(\sigma_{xx}+\sigma_{yy}) = -\frac{1}{(1-\nu)}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right) \qquad \text{(plane strain).} \tag{5.26}$$

If the body forces are absent or constant, the right hand sides of Eqs (5.25, 5.26) vanish and we obtain an important result: the sum of the principal stresses $(\sigma_{11}+\sigma_{22}) = (\sigma_{xx}+\sigma_{yy})$ satisfy the Laplace's equation

$$\nabla^2(\sigma_{xx}+\sigma_{yy}) = 0.$$

Now the equations of equilibrium, the traction boundary conditions, and the compatibility conditions are all independent of the elastic properties. Thus, in conclusion we note that under the afore-mentioned conditions, the stresses inside the body depend only on the loading and not on the elastic properties of the material.

A similar result holds for the three-dimensional case also. However, the general problem is far more complex; we do not discuss this at all.

Isochromatics

These are the lines along which the maximum shear stress has the same value. This maximum value will change from fringe to fringe. These isochromatics are obtained experimentally. [There are several details that must be understood before a photoelastic experimental analysis can be successfully completed. For example, along with the isochromatics, there will be isoclinics also. How to distinguish between these different curves is an important aspect that must be addressed. Would it help if the magnitude of the applied load is slightly increased or decreased without changing the pattern of loading? Would one set of lines, but not the other, change? Would this technique help us to distinguish between the two sets? We regret our inability to discuss all these aspects here.]

Isotropic Point

This is a special point where both the principal stresses are equal: $p = q$. The Mohr's circle degenerates to a point; every plane is a principal plane and every stress a principal stress. There is no shear stress[14] on any plane[15] passing through the isotropic point.

Principal Stresses and Principal Directions

The principal stresses and their planes (directions) are given by

$$\sigma_{11} \equiv p = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}; \tag{5.27a}$$

[14] In photoelasticity, there are fringes which are formed depending on the maximum shear stress reaching some specified values. At an isotropic point, the maximum shear stress is always zero. Consequently, in a standard photoelastic image, there is a dark spot at an isotropic point. Readers not familiar with photoelasticity are advised to read about it from some good books on Experimental Stress Analysis.

[15] It must be understood clearly that on planes that are inclined to the xy-plane, there will be shear stresses.

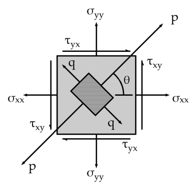


Figure 5.13: Stress components at a point

Fig. 5.13 is a pictorial representation of the (2-dimensional) state of stress at a point. The principal stresses $p \equiv \sigma_{11}$ and $q \equiv \sigma_{22}$ and their directions are marked in the figure. The angle θ defines the direction of one of the principal stresses with respect to the x-axis.

Traditionally the angle θ is measured anticlockwise from the x-axis to the nearer principal direction. The value of θ is never negative, and never larger than $90\,^\circ$.

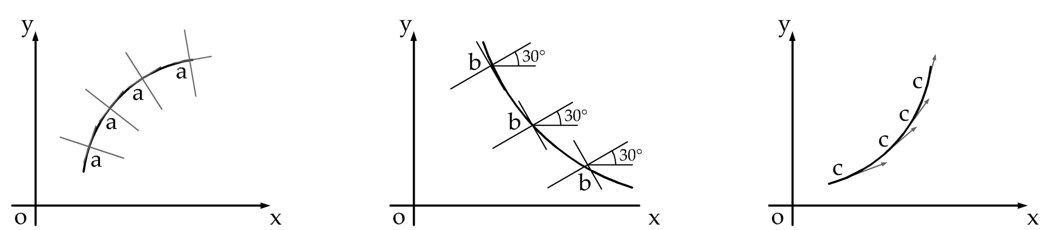
Figure 5.14: Stress components at a point

$$\sigma_{22} \equiv q = \frac{\sigma_{xx} - \sigma_{yy}}{2} + \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}; \tag{5.27b}$$

$$\tan\theta = \frac{2\tau_{xy}}{(\sigma_{xx} - \sigma_{yy})}. \tag{5.27c}$$

The third principal stress is zero, because we are considering only the 2-d case now. The angle θ often serves as a parameter which, in general, varies from point to point.

Isoclinics



(a) An arbitrary curve aaaa (b) An isoclinic bbbb ($\theta = 30\,^\circ$) (c) A stress trajectory cccc

Figure 5.15: The first curve aaaa [Fig. 5.15a] is an arbitrary curve; naturally enough, the principal directions will be different at different points. The second curve bbbb [Fig. 5.15b] is an isoclinic of parameter $\theta = 30\,^\circ$; one of the principal directions makes an angle of $30\,^\circ$ with the x-axis. The third curve cccc [Fig. 5.15c] is a stress trajectory; its tangent is along a principal direction.

Let us consider an arbitrary curve aaaa in the stress field [Fig. 5.15a]. The value of θ will, naturally enough, vary from point to point. Now there arises the question: is there a (continuous) curve along which θ has the same value, say, $30\,^\circ$? The answer is yes. This curve bbbb in the stress field [Fig. 5.15b] is called an *isoclinic* of parameter $\theta = 30\,^\circ$. In other words, an isoclinic of parameter θ is a (continuous) curve in the stress field such that at every point on it, one of the principal directions is always in the same direction as shown

in Fig. 5.15b. It is obvious that, if θ is the parameter of an isoclinic, then $\theta + 90\,^\circ$ also is a parameter of the same isoclinic. Traditionally, the parameter is measured anticlockwise from the x-axis to the nearer principal direction (of p or of q). The parameter is never negative, nor is it ever larger than $90\,^\circ$. [There are many details that must be discussed before we can understand the significance of these terms.]

Through each point in the stress field, there always exists a curve (like bbbb) along which the angle θ (which means that both θ_p and θ_q are constant) remains constant. Thus,

$$\frac{\partial \theta_p}{\partial s} = \frac{\partial \theta_p}{\partial s} = 0 \text{ along an isoclinic like bbbb.}$$

Furthermore, in general[16] there is only set of principal directions at each point. Thus, there can be only one isoclinic passing through a given point (as long as it is not an isotropic point). At an isotropic point, however, the situation is different, because

$$\sigma_{xx} = \sigma_{yy} = p = q;\ \tau_{xy} = 0. \text{ Thus, from Eq. (5.27c), } \ \theta_{p,\,q} = \frac{1}{2}\tan^{-1}\frac{0}{0} \ \text{(indeterminate).}$$

We, therefore, conclude that there can be isoclinics of different parameters passing through an isotropic point. Isoclinics can be obtained experimentally by photoelasticity techniques. At a free boundary, the parameter of the isoclinic must match the known direction of the principal stresses at a boundary point. There are several details which must be understood by an experimental stress analyst if he wishes to conduct serious studies using photoelasticity. White light is generally used to improve the quality of the isoclinics obtained experimentally.

Isoclinics are not easy to obtain analytically, but they can be obtained experimentally quite easily from photoelastic measurements. They are usually drawn (sketched or obtained) at regular intervals of $10\,^\circ$ or $5\,^\circ$. Isoclinics of different parameters can converge at the point of application of a concentrated load as at an isotropic point. [There are several details that must be understood. Furthermore, in recent years the face of photoelasticity has changed greatly, almost out of recognition, from the days of Coker and Filon[17].]

Stress Trajectory

Stress trajectories are (continuously differentiable) curves like cccc [Fig. 5.15c] such that the tangent at any point to the curve is along one of the principal directions. They are also known as the principal stress trajectories and isostatics. Written along such stress trajectories are the Lamé-Maxwell equations of equilibrium. We do not discuss these here.

We shall now discuss how a better understanding of the nature of stress at a point can be had by geometric visualisation.

---

[16] As an isotropic point there can be several; in fact, every direction (in the xy-plane) is a principal direction.

[17] The early investigators on photoelasticity, Ernest George Coker (April 1869 - April 1946), British mathematician and engineer (a noted expert of stress analysis and photoelasticity) and Louis Napolean George Filon (Nov. 1875 - Dec. 1937), a French-born English applied mathematician

STRESS AT A POINT: GEOMETRIC VISUALISATION

The nature of stress at a point can be abstract and difficult to understand. It pays to have visual support. Mohr's circle, Cauchy's stress quadric, and Lamé's ellipsoid of stress[18] are widely used[19]. Their value is not as computational tools; nobody uses graphical methods to obtain numerical values. Their primary use is to help us understand the qualitative features clearly. Seeing is believing. A picture is far more appealing and effective than elaborate explanations. With this introduction we shall first see Lamé's stress ellipsoid.

LAMÉ'S ELLIPSOID OF STRESSES

(a) The stress resultant $\overset{(\nu)}{\boldsymbol{T}}$ on a plane (b) The tips of the stress resultants

Figure 5.16: The tips of the stress resultants on the various planes fall on an ellipsoid.

We know that the stress vector on different planes passing through a point are, in general, different in both magnitude and direction. If the tips of these various stress vectors on various planes are joined together, we obtain a closed surface. This, in general, will be the surface of an ellipsoid known as Lamé's ellipsoid of stresses [See Fig. 5.16b.]. Let us first orient the x, y, z axes along the principal axes $1, 2, 3$. Then the equations expressing the normal stresses on an inclined plane (whose normal ν has the direction cosines l, m, n) [Eqs (4.2a), (4.2b), (4.2c), p. 4-12 reproduced here] are simplified as shown below.

$$\overset{(\nu)}{T}_x \equiv p_{\nu x} = \sigma_{xx}\, l + \sigma_{yx}\, m + \tau_{zx}\, n \qquad \longrightarrow \qquad \overset{(\nu)}{T}_x \equiv p_{\nu x} = l\, \sigma_{11} \tag{5.28a}$$

$$\overset{(\nu)}{T}_y \equiv p_{\nu y} = \tau_{xy}\, l + \sigma_{yy}\, m + \tau_{zy}\, n \qquad \longrightarrow \qquad \overset{(\nu)}{T}_y \equiv p_{\nu y} = m\, \sigma_{22} \tag{5.28b}$$

$$\overset{(\nu)}{T}_z \equiv p_{\nu z} = \tau_{xz}\, l + \tau_{yz}\, m + \sigma_{zz}\, n \qquad \longrightarrow \qquad \overset{(\nu)}{T}_z \equiv p_{\nu z} = n\, \sigma_{zz} \tag{5.28c}$$

But we know that the direction cosines (l, m, n) [Eqs (5.28a), (5.28b), (5.28c)] should satisfy the condition

$$l^2 + m^2 + n^2 = 1 \qquad \longrightarrow \qquad \left(\frac{p_{\nu x}}{\sigma_{11}}\right)^2 + \left(\frac{p_{\nu y}}{\sigma_{22}}\right)^2 + \left(\frac{p_{\nu z}}{\sigma_{33}}\right)^2 = 1. \tag{5.29}$$

[18] Named after the German engineer, C.O. Mohr; the French mathematician (also a civil engineer), A.L. Cauchy; and the French mathematician and elastician, G. Lamé

[19] The influence of the book [5] is gratefully acknowledged.

This, we can see, is an ellipsoid with $\sigma_{11}, \sigma_{22}, \sigma_{33}$ as its semi-axes. If we consider various planes passing through the point O, and mark off the lengths $OP_1, OP_2, \cdots, OP_n$, etc. the tips of these lengths fall on this ellipsoid. Let OP be one such line. The locus of P for various planes is the stress ellipsoid.

If we desire, the equation to the ellipsoid may be written in the more familiar form

$$\frac{x^2}{\sigma_{11}^2} + \frac{y^2}{\sigma_{22}^2} + \frac{z^2}{\sigma_{33}^2} = 1.$$

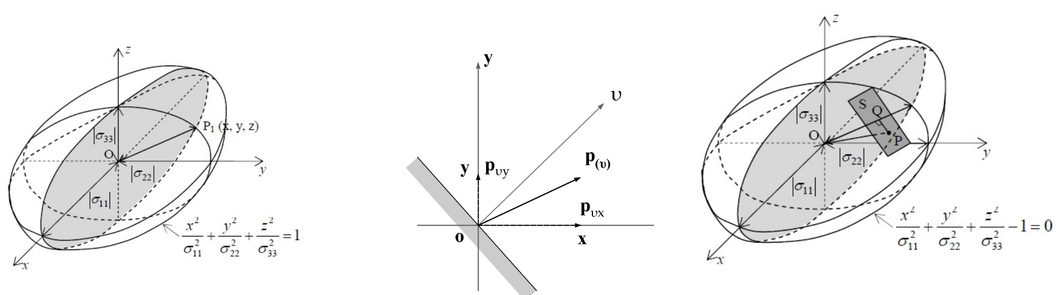
If the coordinates of the tip P of a typical line OP are (x, y, z), the lengths Ox, Oy, Oz are the projections of the length OP which represents the magnitude of the resultant stress vector on some plane. [This plane is not known just at present; some auxiliary construction is needed to locate it. We shall see this later.] The projections of the resultant stress vector (on this unknown plane) are the stress components in the coordinate directions. Thus,

$$x = \overset{(\nu)}{T}_x = p_{\nu x}; \qquad y = \overset{(\nu)}{T}_y = p_{\nu y}; \qquad z = \overset{(\nu)}{T}_z = p_{\nu z}.$$

Now Eq. (5.29) appears in the form

$$\frac{x^2}{\sigma_{11}^2} + \frac{y^2}{\sigma_{22}^2} + \frac{z^2}{\sigma_{33}^2} = 1. \tag{5.30}$$

We do not have to know this plane ν on which this resultant stress vector $\overset{(\nu)}{\boldsymbol{T}}$ acts. But if we so desire, we can locate it by an auxiliary construction.



(a) The stress ellipsoid and a radius vector P_1

(b) A typical plane

(c) The stress ellipsoid and a tangent plane

Figure 5.17: In Fig. 5.17a the radius vector OP_1 represents the resultant stress vector $p_{(\nu)}$ on an unidentified plane. In Fig. 5.17c it is OQ (and not OP) that represents the stress resultant. The tangent plane at P is shown. Fig. 5.17b shows a typical plane, its normal, a typical radius vector, and the stress resultant on this plane.

Features of Lamé's Stress Ellipsoid

In connection with the Lamé's stress ellipsoid, let us note the following.

(i) The stress ellipsoid is visualised and drawn in a Westergaard stress space. [See the various figures of Lamé's stress ellipsoid in this subsection with the axes along the principal axes σ_{11}, σ_{22}, σ_{33}.]

(ii) The tips of the resultant stress vectors on the various planes passing through the point fall on the surface of this ellipsoid [Fig. 5.16b]. However, as indicated earlier, we cannot quickly know the plane on which the resultant stress vector, say, OP acts.

(iii) The resultant stress vector acts along the normal of the plane — that is, the normal component of the stress resultant is all we have; there are no shear components — only when the plane is a principal plane. [The point Q defined above in the auxiliary construction falls on the surface of the ellipsoid only at six points in general. In general, there are only three principal planes and principal stresses.]

(iv) The first invariant I_1 is proportional to the sum of the principal radii of the ellipsoid. This is easy to understand; the semi-axes are the three principal stresses σ_{11}, σ_{11}, σ_{11}.

(v) The second invariant I_2 is proportional to the three principal areas of the ellipsoid. (On each plane the projection of the ellipsoid gives an ellipse; the areas of these three ellipses are called the principal areas of the ellipsoid.)

(vi) The third invariant I_3 is proportional to the volume of the ellipsoid.

(vii) The stress ellipsoid remains the same no matter what the coordinate system is. If the coordinate system is chosen along the principal directions, the equation to the ellipsoid contains no cross-terms like xy, yz, zx, but if the coordinate system is chosen along non-principal axes [Fig. 5.18], the equation will contain some cross-term like xy.

To understand this clearly, let us consider a quadratic form $4u^2 + 9v^2$. The equation $4u^2 + 9v^2 = 36$ clearly represents an ellipse with its semi-axes 3 and 2.

$$4u^2 + 9v^2 = 36 \qquad \longrightarrow \qquad \left(\frac{u}{3}\right)^2 + \left(\frac{v}{2}\right)^2 = 1.$$

If the variables u, v are changed to x, y by a linear transformation, we obtain the equation of the same ellipse in terms of the new variables x, y. If $u = c_{11}x + c_{12}y$; $v = c_{21}x + c_{22}y$, the above equation to the ellipse will now have the appearance

$$4\left(c_{11}x + c_{12}y\right)^2 + 9\left(c_{21}x + c_{22}y\right)^2 = 36. \tag{5.31}$$

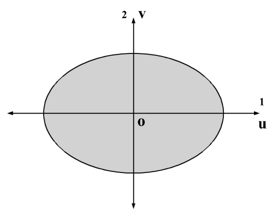
This equation contains a term in xy. What this example demonstrates is the following. As a special case, when the new axes x, y are oriented at 45° to the old u, v axes, the associated rotation matrix is

$$\begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

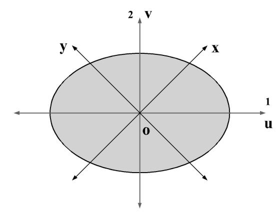
We know that this rotation matrix is an orthogonal matrix; that is, (i) its determinant is equal to 1, and (ii) its inverse is the same as its transpose. Thus, the equation (5.31) to the ellipse now appears in terms of x, y as

$$\frac{13}{2}x^2 + \frac{13}{2}y^2 - 5xy = 36. \tag{5.32}$$

The ellipse referred to is the same, but now it is referred to a set of non-principal axes x, y. The equation has a cross-term $-5xy$. [See Figs 5.18a, 5.18b.]



(a) Lamé's stress ellipse referred to the principal axes u, v



(b) The same stress ellipse referred to a set of non-principal axes x, y

Figure 5.18: Lamé's stress ellipse referred to principal and non-principal axes. With respect to a principal set of axes u, v [Fig. 5.18a], there is no cross-term uv in its equation but w.r.to a set of non-principal axes like x, y [Fig. 5.18b], the equation contains a cross-term xy.

(viii) The quadratic forms and the matrices with and without the cross-term, are

$$\text{cross-term } -5xy, \quad 6.5x^2 + 6.5y^2 - 5xy = \begin{Bmatrix} x & y \end{Bmatrix} \begin{bmatrix} 6.5 & -2.5 \\ -2.5 & 6.5 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} \tag{5.33a}$$

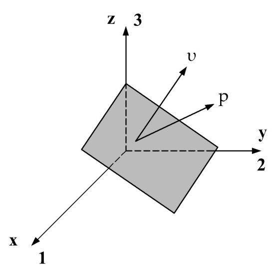
$$\text{no cross-term } uv, \quad 4u^2 + 9v^2 = \begin{Bmatrix} u & v \end{Bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \tag{5.33b}$$

When there is a cross-term as in Eq. (5.33a), the matrix is non-diagonal. If there is no cross-term as in Eq. (5.33b), the corresponding matrix is a diagonal one. Thus, to remove the cross-term(s) in the appearance of a quadratic form, the (square) matrix associated with the quadratic form is to be diagonalised!

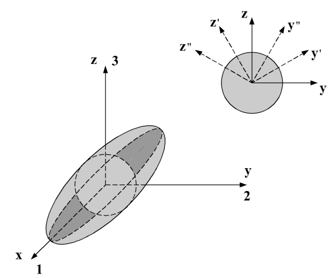
[Why should we remove the cross-term(s) in a quadratic form? Well, there are several reasons. One of them is to check, or know for sure, if the quadratic form is positive definite or not, so necessary in the discussion of stability of dynamical systems. It is the presence of the cross-terms that makes it difficult to decide this. If the associated (square) matrix is rendered into its diagonal canonical form, the issue is settled. The diagonal elements are the eigenvalues, and we can conclude that a quadratic form is positive definite if all the eigenvalues of its associated matrix are positive.]

(ix) If the three principal stresses are all equal, i.e., if $\sigma_{11} = \sigma_{22} = \sigma_{33}$, it is obvious that the ellipsoid becomes a sphere [Fig. 5.20c]. Now every plane is a principal plane; there is no shear stress on any plane. This represents a state of hydrostatic

stress (appropriately called a case of spherical stress tensor). The stress matrix now represents an isotropic state of stress.



(a) Lamé's stress ellipsoid when $\sigma_{11} \neq \sigma_{22} \neq \sigma_{33}$



(b) Lamé's stress ellipse of revolution when $\sigma_{11} \neq \sigma_{22} = \sigma_{33}$

Figure 5.19: When two principal stresses are equal, Lamé's stress ellipsoid degenerates into an ellipse of revolution (in this case about the 11 axis) [Fig. 5.19b].

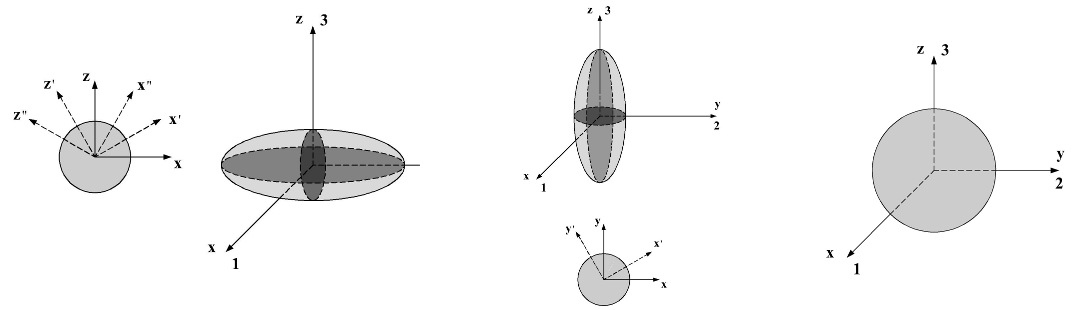
(x) If two principal stresses are equal, say, $\sigma_{11} \neq \sigma_{22} = \sigma_{33}$, the stress ellipsoid degenerates into an ellipse of revolution about the principal axis 11 [Fig. 5.19b]. All normal cross-sections (normal to the 11 axis) are, thus, circles. One such circle projected on the yz, i.e., 23 plane, is shown in this figure. Let us note that y, z; y', z'; y'', z'' are all principal planes. There are just three (3), no more and no less, principal planes for the first case when the three principal stresses are distinct [Fig. 5.19a], infinite principal planes in the second case [Fig. 5.19a]. Note that even though y and y' are both principal planes, they are not normal (perpendicular) to each other. We recall from linear algebra that the eigenvectors are guaranteed to be orthogonal to one another only if the eigenvalues (principal stresses) are distinct. These facts must be understood clearly.

We hope that the figures [Figs 5.20a, 5.20b, 5.20c] throw further light on what happens when two of the principal stresses are equal. Among all the planes normal to the 23 plane, for example, the shaded plane [Fig. 5.21a] is the one on which the shear stress reaches a maximum; its value is $(\sigma_{11} - \sigma_{22})/2$.

(xi) Referring to the two figures 5.17a, 5.17b, we note that the point Q is quite different from the point P, in general. The points Q and P coincide only in very special cases. We know that the resultant stress vector on a plane ν is normal to it only when ν is a principal plane.

Thus, Q touches the surface of the ellipsoid only in six places in general where the surface cuts the coordinate axes. These, we know, correspond to the principal planes.

All these aids for geometric visualisation have their counterparts in a two-dimensional setting when there are considerable simplifications. We shall review them briefly below.



(a) Ellipse of revolution when $\sigma_{11} = \sigma_{33} \neq \sigma_{22}$

(b) Ellipse of revolution when $\sigma_{11} = \sigma_{22} \neq \sigma_{33}$

(c) Sphere when $\sigma_{11} = \sigma_{22} = \sigma_{33}$

Figure 5.20: As before when two principal stresses are equal, Lamé's stress ellipsoid degenerates into an ellipse of revolution (in this case about the 22 axis) when $\sigma_{11} = \sigma_{33} \neq \sigma_{22}$ [Fig. 5.20a] and about the 33 axis when $\sigma_{11} = \sigma_{22} \neq \sigma_{33}$ [Fig. 5.20b]. When $\sigma_{11} = \sigma_{22} = \sigma_{33}$ [Fig. 5.20c], the ellipsoid degenerates into a sphere.

Two-dimensional Case: Lamé's Ellipse of Stresses

There are obviously simplifications when we consider the above topics in the more limited context of two dimensions. Now among the nine (9) / six (6) stress components, there are only four (4) / three (3), viz., σ_{xx}, $\sigma_{xy} = \sigma_{yx}$, σ_{yy}. In other words, $\sigma_{yz} = \sigma_{zy} = 0$, $\sigma_{zx} = \sigma_{xz} = 0$, $\sigma_{zz} = 0$. This corresponds to the so-called plane stress simplification[20]. Readers are urged to refer to the earlier (more general) three-dimensional case repeatedly, and realise that all the results and explanations given earlier are relevant here too.

Let us consider a general two-dimensional state of stress represented by the two principal stresses σ_{11} and σ_{22} [Fig. 5.22a]. If the coordinate axes x, y are oriented along these principal axes, we have

$$\sigma_{xx} = \sigma_{11}; \quad \sigma_{yy} = \sigma_{22}; \quad \sigma_{xy} = 0.$$

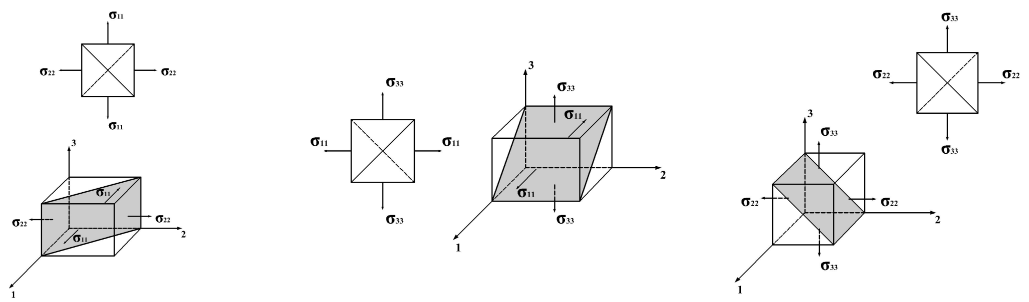
The normal and shearing stresses and the resultant stress on an inclined plane ν [Fig. 5.22b] are given by

$$p_{\nu x} = \sigma_{11} \cos(\nu,\, x) = l\, \sigma_{11}; \tag{5.34a}$$

[20] Readers who are not exposed to these concepts are advised to consult any good book on the Theory of Elasticity, say, Timoshenko & Goodier [16]. There are other simplifications such as plane strain and generalised plane stress. If the plane stress problem is solved, the plane strain solution also (and vice versa) can be obtained by making a few changes:

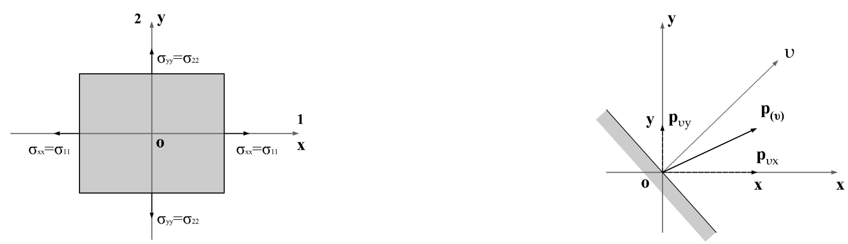
$$\text{plane stress solution} \longrightarrow \text{plane strain solution:} \quad E \longrightarrow \frac{E}{1-\nu^2}; \quad \nu \longrightarrow \frac{\nu}{1-\nu};$$

$$\text{plane strain solution} \longrightarrow \text{plane stress solution:} \quad E \longrightarrow \frac{E(1+2\nu)}{(1+\nu)^2}; \quad \nu \longrightarrow \frac{\nu}{1+\nu}.$$



(a) Plane(s) normal to the 23 plane (b) Plane(s) normal to the 13 plane (c) Plane(s) normal to the 12 plane

Figure 5.21: Planes of maximum shear stress: the figures show the planes on which the maximum shear stress occurs. The maximum values, we know, are $(\sigma_{11} - \sigma_{22})/2$, $(\sigma_{22} - \sigma_{33})/2$, $(\sigma_{33} - \sigma_{11})/2$. The maximum shear stress at the point is the grand maximum among these three values.



(a) A two-dimensional state of stress (b) Resultant stress on an inclined plane

Figure 5.22: A two-dimensional state of stress is shown in Fig. 5.22a. The resultant stress vector on an inclined plane ν is shown in Fig. 5.22b.

$$p_{\nu y} = \sigma_{22} \cos(\nu,\, y) = m\,\sigma_{22}; \tag{5.34b}$$

$$p_{(\nu)} = \sqrt{p_{\nu x}^2 + p_{\nu y}^2} = \sqrt{l^2\,\sigma_{11}^2 + m^2\,\sigma_{22}^2}; \tag{5.34c}$$

$$\tau_{(\nu)} = -\sigma_{11}\,\sin\theta\,\cos\theta = \frac{-(\sigma_{11} - \sigma_{22})}{2}\,\sin 2\theta. \tag{5.34d}$$

[The last equation is a special case of Eq. (4.10b), p. 4-16, simplified for this context with the direction cosines appropriately interpreted. It can also be obtained by considering the equilibrium of a small rectangular block acted upon by the relevant stress components on all its faces. When working in two dimensions it is often more convenient to use $\sin\theta$ and $\cos\theta$ instead of the direction cosines.]

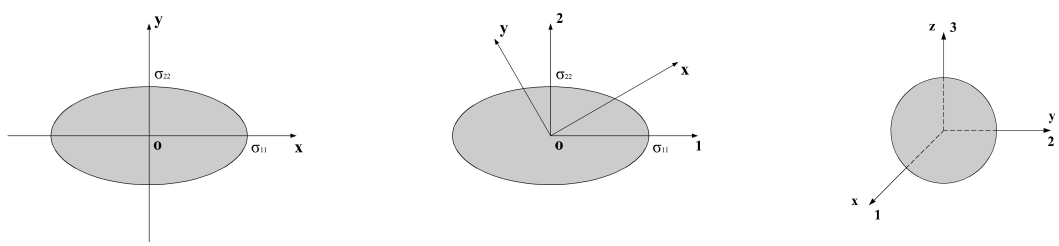
Let us consider the resultant stress vectors on all the inclined planes passing through the point under consideration. The tips of all these resultant stress vectors will lie on an ellipse. This is the Lamé's stress ellipse [Fig. 5.23a]. [Compare with the explanation of Lamé's ellipsoid of stresses on p. 5-22.]

To obtain the equation to the ellipse, we calculate l^2, m^2 (that is, $\cos^2(\nu, x), cos^2(\nu, y)$) from Eqs (5.34a, 5.34b) and use the result that $l^2 + m^2 = 1$ (that is, $\sin^2\theta + \cos^2\theta = 1$). This gives us

$$\frac{p_{\nu x}^2}{\sigma_{11}^2} + \frac{p_{\nu y}^2}{\sigma_{22}^2} = 1. \tag{5.35}$$

Noting from Fig. 5.22b that the tip of the resultant stress vector $p_{(\nu)}$ has the coordinates (x, y) (that is, $p_{\nu x} = x$ and $p_{\nu y} = y$), we obtain the equation to the ellipse [Fig. 5.23a] as

$$\frac{x^2}{\sigma_{11}^2} + \frac{y^2}{\sigma_{22}^2} = 1. \tag{5.36}$$



(a) Lamé's ellipse of stresses (b) Ellipse: non-principal coordinates (c) Special case: circle

Figure 5.23: Lamé's ellipse of stresses [Fig. 5.23a] referred to principal axes; there is no 'cross-term'. The same ellipse when referred to non-principal axes appears as in Fig. 5.23b. The ellipse is the same; it remains unchanged (invariant). Its equation will now contain a 'cross-term'. The ellipse becomes a circle when both the principal stresses are equal: $\sigma_{11} = \sigma_{22}$ (a two-dimensional state of hydrostatic stress).

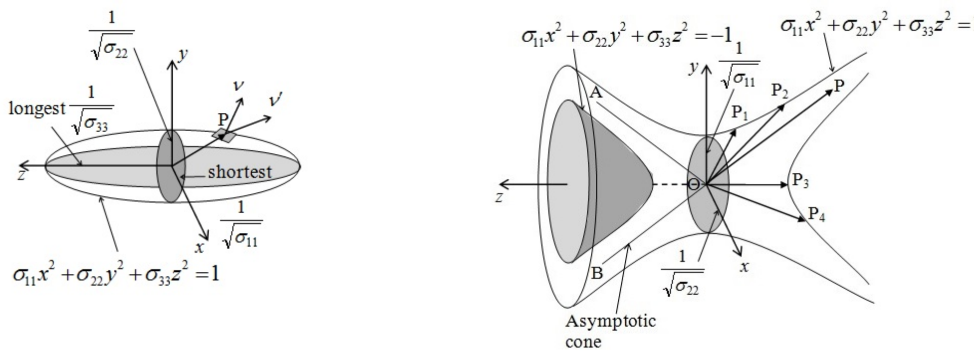
There are two more special cases of the ellipse of stresses. One is when $\sigma_{11} = -\sigma_{22}$ which is a state of pure shear. The first invariant $I_1 = \sigma_{11} + \sigma_{22} = 0$. As $|\sigma_{11}| = |\sigma_{22}|$, the stress ellipse is a circle. The magnitude of the resultant stress vector $|p_{(\nu)}|$ is the same on every plane (perpendicular to the xy plane). Let us not fail to note that we are now discussing only two-dimensional states of stresses. The other special case is when σ_{11} alone is present. Now $\sigma_{22} = 0$; the stress ellipse degenerates into a straight line of magnitude $2|\sigma_{11}|$.

CAUCHY'S STRESS QUADRIC

Next, we shall discuss Cauchy's stress quadric. We repeat that these geometrical constructions are never used for actual computation. Referring to a set of principal axes $(x, y, z) = (1, 2, 3)$ for convenience, the quadric surface

$$\sigma_{11}x^2 + \sigma_{22}y^2 + \sigma_{33}z^2 = \pm 1 \tag{5.37}$$

represents, in general, either an ellipsoid or a hyperboloid [Fig. 5.24]. The hyperboloid may have one or two sheets. [If the reference axes are non-principal, the equation will have



(a) An ellipsoid

(b) A hyperboloid of one or two sheets

Figure 5.24: Cauchy's stress quadric: this is either an ellipsoid [Fig. 5.24a] or a hyperboloid of one or two sheets [Fig. 5.24b] depending on whether the principal stresses are of like or unlike signs. This is explained further in the text.

'cross-terms' like xy, yz and zx.] OP any radius vector from the origin O touching the quadric surface at the point $P(x, y, z)$ has the direction cosines

$$l = \cos(\nu,\, x); \quad m = \cos(\nu,\, y); \quad n = \cos(\nu,\, z).$$

Then the length OP is inversely proportional to the square root of the absolute value of the normal stress $|\sigma_{\nu\nu}|$. This acts on a plane ν through O; that is, the normal to this plane is along OP [direction cosines: $l = \cos(\nu,\, x)$; $m = \cos(\nu,\, y)$; $n = \cos(\nu,\, z)$]. We shall prove this for a general point P. But let us first verify this for the special cases where P is A, B, $C, \cdots$ where the principal axes cuts the coordinate axes $(x, y, z) = (1, 2, 3)$.

$$OA = \frac{1}{\sqrt{\sigma_{11}}} = OB; \quad OC = \frac{1}{\sqrt{\sigma_{22}}} = OD; \quad OE = \frac{1}{\sqrt{\sigma_{33}}} = OF.$$

If the coordinates of such a typical point P are (x, y, z), we can see that

$$x = l\,OP = OP\,\cos(\nu,\, x); \quad y = m\,OP = OP\,\cos(\nu,\, y); \quad z = n\,OP = OP\,\cos(\nu,\, z).$$

The equation (5.37) to the quadric surface now takes the form

$$\sigma_{11}\,\left[(OP)^2\,\cos^2(\nu,\, x)\right] + \sigma_{22}\,\left[(OP)^2\,\cos^2(\nu,\, y)\right] + \sigma_{33}\,\left[(OP)^2\,\cos^2(\nu,\, z)\right] = \pm 1;$$

$$\text{that is,}\quad l^2\,\sigma_{11} + m^2\,\sigma_{22} + n^2\,\sigma_{33} = \pm\frac{1}{(OP)^2}; \tag{5.38}$$

$$\text{or,}\quad \sigma_{11}\,\cos^2(\nu,\, x) + \sigma_{22}\,\cos^2(\nu,\, y) + \sigma\,\cos^2(\nu,\, z) = \pm\frac{1}{(OP)^2}. \tag{5.39}$$

But we know — we have seen this — that

$$\sigma_{\nu\nu} = l^2\,\sigma_{11}^2 + m^2\,\sigma_{22}^2 + n^2\,\sigma_{33}^2 = \sigma_{11}\,\cos^2(\nu, x) + \sigma_{22}\,\cos^2(\nu, y) + \sigma_{33}\,\cos^2(\nu, z). \tag{5.40}$$

A comparison of Eqs (5.39), (5.40) shows that

$$\sigma_{\nu\nu} = \pm \frac{1}{(OP)^2}; \quad \text{i.e., } OP = \frac{1}{\sqrt{|\sigma_{\nu\nu}|}}. \tag{5.41}$$

Fig. 5.24a is an ellipsoid somewhat similar to the Lamé's ellipsoid of stresses. But there are important differences. In Lamé's ellipsoid, a radius vector OP represents the magnitude of the resultant stress vector. Its plane is not immediately apparent; it is to be found by an auxiliary construction. Here, however, a radius vector OP represents the reciprocal of the magnitude of the normal stress $\sigma_{\nu\nu}$; its normal ν is also known. But here an auxiliary construction is needed to determine the plane on which the resultant stress vector acts. The other difference is that here the ellipsoid is the shortest along the x-axis (1-axis) and the longest along the z-axis (3-axis), because $|\sigma_{11}| > |\sigma_{33}|$ and, therefore, $1/\sqrt{|\sigma_{11}|} < 1/\sqrt{|\sigma_{33}|}$.

Fig. 5.24b shows the hyperboloid of (i) one sheet, and (ii) of two sheets. These correspond, respectively, to

(i) one sheet: $\sigma_{11}x^2 + \sigma_{22}y^2 + \sigma_{33}z^2 = +1$ shaped like a cooling tower;
(ii) two sheets: $\sigma_{11}x^2 + \sigma_{22}y^2 + \sigma_{33}z^2 = -1$ shaped like two bowls.

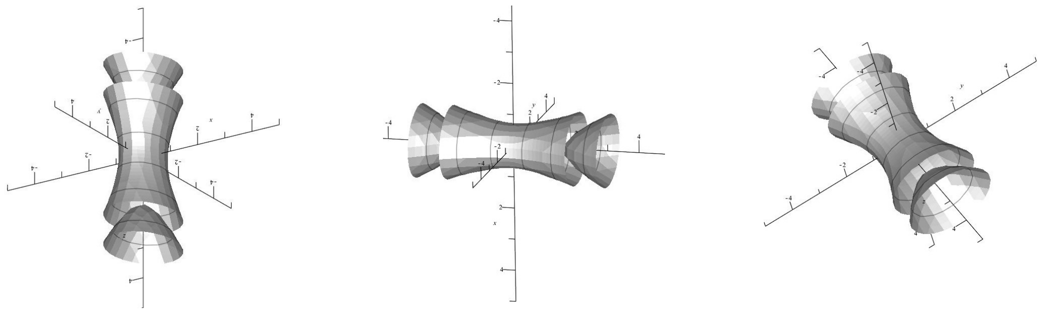
The radius vectors OP_1, OP_2, $\cdots$ with their tips on the hyperboloid of one sheet represents positive (tensile) normal stresses. On the other hand, OP_3, OP_4, $\cdots$ with their tips on the hyperboloid of two sheets represent negative (compressive) stresses. If P falls on the asymptotic cone, it is the in-between case. [One of these two asymptotic cones is shown in Fig. 5.24b.] As $P_n \longrightarrow \infty$ (asymptotically), the normal stress $\longrightarrow 0$. As there are infinite number of points on the asymptotic cone, P may fall on any one of the infinite generators of this cone corresponding to the planes on which the normal stress is zero. [We recall that the length of the radius vector OP is the reciprocal of the square root of the magnitude of the normal stress: $OP = 1/\sqrt{|\sigma_{\nu\nu}|}$.]

The cross-section, shown shaded in Fig. 5.24b, of the hyperboloid of one sheet is an ellipse whose equation — Eq. (5.37) in which z is set equal to 0 corresponding to the xy plane — is

$$\sigma_{11}x^2 + \sigma_{22}y^2 = +1.$$

[To have a better idea of the shape of these surfaces, let us take the help of MAPLE. For the numerical values chosen, the hyperboloid of one and two sheets are drawn in the same figure and displayed in Figs 5.25a, 5.25b, 5.25c. It is the same figure looked at from three different 'perspectives'. The governing equations to draw these figures are

$$50x^2 + 30y^2 - 10z^2 = 25 \qquad (-5 < x < 5;\ -5 < x < 5;\ -2 < x < 2);$$
$$50x^2 + 30y^2 - 10z^2 = -25 \qquad (-5 < x < 5;\ -5 < x < 5;\ -3 < x < 3).]$$



(a) Stress quadric: one view (b) Stress quadric: a second view (c) Stress quadric: a third view

Figure 5.25: Cauchy's stress quadric drawn by MAPLE is shown here.

MOHR'S CIRCLE

Mohr's circle is the most popular and best known pictorial representation of the state of stress at a point. Readers of this book are expected to have had a good understanding of this topic for the two-dimensional case[21]. They would also have seen that such a visualisation is possible for the state of strain at point also. Indeed, it is equally valid for all examples of a second order tensor (inertia tensor, curvature tensor, etc.).

We shall not discuss Mohr's circle in two dimensions as it is elementary. Here we shall consider Mohr's circles in three dimensions. Now there are three (or rather, three pairs of) Mohr's circles. As in the two-dimensional case, these are also drawn on the normal and shear stress axes, $\sigma_{(\nu)}$ and $\tau_{(\nu)}$, respectively. If the principal stresses (σ_{11}, σ_{22}, σ_{33}) are known or given, these circles tell us quickly, though not as quickly as in two-dimensions, the normal and shear stress components — $\sigma_{(\nu)}$ and $\tau_{(\nu)}$ — respectively of the resultant stress vector on any plane ν passing through the point.

We assume $\sigma_{11} > \sigma_{22} > \sigma_{33}$. The centres and the radii of the three pairs of circles are

$$\text{(i) } \sigma_{11} \text{ and } \sigma_{22} \quad \text{centre } E\text{: } \frac{\sigma_{11}+\sigma_{22}}{2}; \quad \text{radii: } \frac{\sigma_{11}-\sigma_{22}}{2}; \quad \sigma_{33}-\frac{\sigma_{11}+\sigma_{22}}{2}; \quad \text{Fig. 5.26a;}$$

$$\text{(i) } \sigma_{22} \text{ and } \sigma_{33} \quad \text{centre } F\text{: } \frac{\sigma_{22}+\sigma_{33}}{2}; \quad \text{radii: } \frac{\sigma_{22}-\sigma_{33}}{2}; \quad \sigma_{11}-\frac{\sigma_{22}+\sigma_{33}}{2}; \quad \text{Fig. 5.26b;}$$

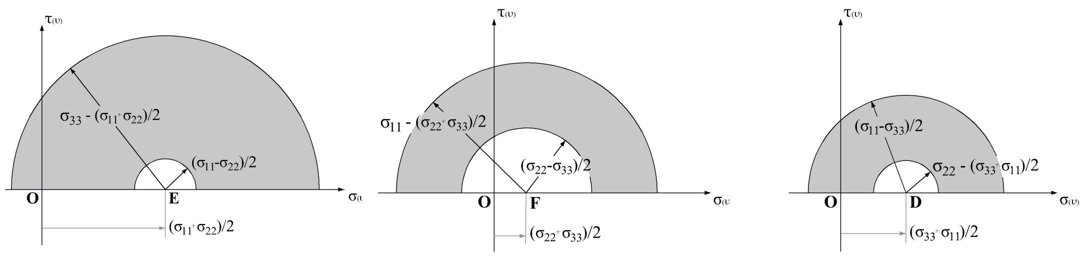
$$\text{(i) } \sigma_{33} \text{ and } \sigma_{11} \quad \text{centre } D\text{: } \frac{\sigma_{33}+\sigma_{11}}{2}; \quad \text{radii: } \frac{\sigma_{11}-\sigma_{33}}{2}; \quad \sigma_{22}-\frac{\sigma_{33}+\sigma_{11}}{2}; \quad \text{Fig. 5.26c.}$$

These pairs of circles are shown, one pair at a time, in these figures 5.26a, 5.26b, 5.26c. Here we have displayed the circles[22], but we have not carried out the calculations that lead us to these. We shall now undertake this exercise.

Let us begin from the following three equations that we already know. The first one follows from Eqs (5.28a, 5.28b, 5.28c), while the second one expresses the normal stress on an

[21] This construction, Mohr's circle, is treated quite well in almost all good books on the Mechanics of Solids and its previous *avatar* as Strength of Materials.

[22] The circles are drawn for a general case where all the three principal stresses are distinct and positive: $\sigma_{11} > \sigma_{22} > \sigma_{33} > 0$. A set of numerical values $\sigma_{11} = 50$, $\sigma_{22} = 30$, $\sigma_{33} = -10$, all in MPa is considered in the example shown later [p. 13-34].



(a) The first pair of circles (b) The second pair of circles (c) The third pair of circles

Figure 5.26: Shown here are three pairs of circles. These are the limiting circles corresponding to the limiting values of 0 and 1 that each direction cosine can have. When all of these pairs of circles are considered together, only the common area (shown shaded in Fig. 5.29b) is of interest to us.

inclined plane ν in terms of the direction cosines $l \equiv \cos(\nu,\, x)$, $m \equiv \cos(\nu,\, y)$, $n \equiv \cos(\nu,\, z)$ and the (given) principal stresses. The third one is obvious.

$$\sigma_{11}^2\, l^2 + \sigma_{22}^2\, m^2 + \sigma_{33}^2\, n^2 = p_{(\nu)}^2 \equiv |\overset{(\nu)}{\boldsymbol{T}}|^2; \tag{5.42a}$$

$$\sigma_{11}\, l^2 + \sigma_{22}\, m^2 + \sigma_{33}\, n^2 = \sigma_{\nu\nu}; \tag{5.42b}$$

$$l^2 + m^2 + n^2 = 1. \tag{5.42c}$$

The solution is, using Cramer's rule,

$$l^2 = \frac{\begin{vmatrix} (\sigma_{\nu\nu}^2 + \tau_{(\nu)}^2) & \sigma_{22}^2 & \sigma_{33}^2 \\ \sigma_{\nu\nu} & \sigma_{22} & \sigma_{33} \\ 1 & 1 & 1 \end{vmatrix}}{|D|}; \quad m^2 = \frac{\begin{vmatrix} \sigma_{11}^2 & (\sigma_{\nu\nu}^2 + \tau_{(\nu)}^2) & \sigma_{33}^2 \\ \sigma_{11} & \sigma_{\nu\nu} & \sigma_{33} \\ 1 & 1 & 1 \end{vmatrix}}{|D|}; \quad n^2 = \frac{\begin{vmatrix} \sigma_{11}^2 & \sigma_{22}^2 & (\sigma_{\nu\nu}^2 + \tau_{(\nu)}^2) \\ \sigma_{11} & \sigma_{22} & \sigma_{\nu\nu} \\ 1 & 1 & 1 \end{vmatrix}}{|D|}$$

where the denominator determinant $|D|$ is given by

$$|D| = \begin{vmatrix} \sigma_{11}^2 & \sigma_{22}^2 & \sigma_{33}^2 \\ \sigma_{11} & \sigma_{22} & \sigma_{33} \\ 1 & 1 & 1 \end{vmatrix}.$$

At this stage, we have to branch off to three different cases, because there are qualitative differences among them.

(a) The principal stresses are distinct: now the denominator determinant $D \neq 0$. A unique non-trivial solution exists.

(b) Two principal stresses are equal: now the denominator determinant $D = 0$. For a non-trivial solution to exist, all the three numerator determinants have to be separately zero. We have two cases to consider:
(i) $\sigma_{11} = \sigma_{22} > \sigma_{33}$; and (ii) $\sigma_{11} > \sigma_{22} = \sigma_{33}$.

(c) All the principal stresses are equal: $\sigma_{11} = \sigma_{22} = \sigma_{33}$.

(i) All the principal stresses are positive: $\sigma_{11} > \sigma_{22} > \sigma_{33} > 0$. Now the positive (+) sign is to be taken in Eq. (5.37). The quadric surface is an ellipsoid [Fig. 5.24a]. We can see, on comparing Eq. (5.37) with the equation to an ellipsoid in the standard form [Eq. (5.29)], that the semi-axes are $1/\sqrt{\sigma_{11}}$, $1/\sqrt{\sigma_{11}}$, $1/\sqrt{\sigma_{11}}$. For this case, the normal stress on every plane is positive, i.e., tensile.

(ii) All the principal stresses are negative: $\sigma_{33} < \sigma_{22} < \sigma_{11} < 0$. Now the negative ($-$) sign is to be taken in Eq. (5.37). The quadric surface is again an ellipsoid [Fig. 5.24a]. The normal stress on every plane will obviously be negative, i.e., compressive. For these two cases [items (i), (ii)] there is no plane on which there is only shear stress.

(iii) As remarked earlier, when all the principal stresses are equal, the quadric surface becomes a sphere, representing a state of hydrostatic stress (spherical state of stress). Every plane is a principal stress; there is no shear stress on any plane.

(iv) If two of the principal stresses are equal, the quadric degenerates into an ellipse of revolution. The conclusions of this paragraph and of the one just above this, are similar to the ones mentioned for Lamé's stress ellipsoid.

(v) If one principal stress is negative and the others are positive, that is, if $\sigma_{11} \geq \sigma_{22} > 0 > \sigma_{33}$, both the signs in Eq. (5.37) are to be taken. Corresponding to the positive (+) sign, we have a hyperboloid of one sheet (shaped in the form of a cooling tower). Corresponding to the negative ($-$) sign, we have a hyperboloid of two sheets. These are shown in Figs 5.24b, 5.25.

(vi) If two principal stresses are equal, that is, if $\sigma_{11} = \sigma_{22} > 0 > \sigma_{33}$, the quadric surfaces are hyperbolae of revolution.

(vii) If one principal stress, say σ_{zz} is zero ($\sigma_{11} \geq \sigma_{22} > \sigma_{33} = 0$), Eqs (5.37) tell us what happens. Now this equation takes the form

$$\sigma_{11}x^2 + \sigma_{22}y^2 = +1 \quad (\sigma_{11},\, \sigma_{22} \text{ both positive, only } +1),$$

which is independent of z. This obviously represents a cylinder — prismatic, every cross-section is the same, independent of z — which is an ellipse if $\sigma_{11} > \sigma_{22}$, and a circle if $\sigma_{11} = \sigma_{22}$.

If $\sigma_{11} > \sigma_{22} = 0 > \sigma_{33}$, the two principal stresses are unlike, and both the signs (+) and ($-$) are to be taken. Eq. (5.37) now takes the form

$$\sigma_{11}x^2 + \sigma_{33}z^2 = \pm 1 \quad (\sigma_{11} > 0,\, \sigma_{33} < 0,\text{ both sign, both } +1 \text{ and } -),$$

which represents two hyperbolic cylinders [Fig. 5.27a drawn for the numerical values (units irrelevant) of $\sigma_{11} = 3$; $\sigma_{33} = -1$]. These are parallel to the y-axis and asymptotic to the plane

$$x = \pm\sqrt{\frac{|\sigma_{33}|}{\sigma_{11}}}\, z.$$

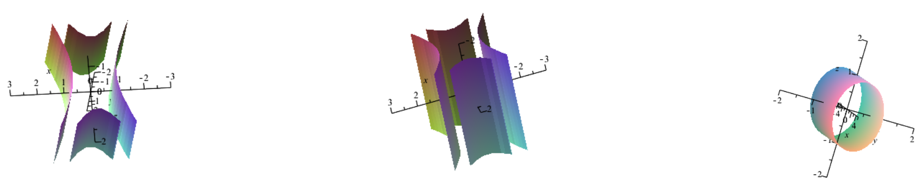
When the tip P of the radius vector OP falls on any one of these asymptotic planes, the corresponding normal stress is zero. Why? Because

$$\text{as} \quad OP \quad \longrightarrow \quad \infty, \quad \sigma_{(\nu\nu)} = \frac{1}{\sqrt{OP}} \quad \longrightarrow \quad 0.$$

If $\sigma_{11} = 0 > \sigma_{22} \geq \sigma_{33}$, this is similar to the case discussed above. The only difference is that the changed version of Eq. (5.37)

$$\sigma_{22} y^2 + \sigma_{33} z^2 = -1 \quad (\sigma_{22},\, \sigma_{33} < 0,\ \text{both negative, only } -),$$

is now independent of x. All the normal stresses are negative (compressive). The cylinder — prismatic, everywhere the same — is independent of x. The cross-section is an ellipse if $\sigma_{22} > \sigma_{33}$, and a circle if $\sigma_{22} = \sigma_{33}$. This elliptic / circular cylinder is parallel to the x-axis [Fig. 5.27b drawn for the numerical values (units irrelevant) of $\sigma_{22} = -2$; $\sigma_{33} = -1$].



(a) Graph of $\sigma_{11}\, x^2 + \sigma_{33}\, z^2 = \pm 1; \sigma_{11} = 3;\ \sigma_{33} = -1.$

(b) Graph of $\sigma_{22}\, y^2 + \sigma_{33}\, z^2 = \pm 1$; another view

(c) Graph of $\sigma_{22}\, y^2 + \sigma_{33}\, z^2 = -1; \sigma_{22} = -2;\ \sigma_{33} = -1.$

Figure 5.27: Two special cases of Cauchy's stress quadric drawn by MAPLE

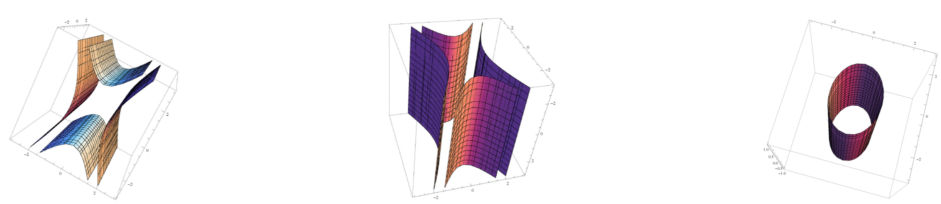


Figure 5.28: The same figures drawn by MATHEMATICA

(viii) What happens when two principal stresses are zero? Well, now we have a one-dimensional state of stress, the cases of uniaxial tension ($\sigma_{11} > \sigma_{22} = \sigma_{33} = 0$), and uniaxial compression (uniaxial) ($\sigma_{11} = \sigma_{22} = 0 > \sigma_{33}$). For these cases, Eq. (5.37) become, respectively,

$$\sigma_{11}\, x^2 = +1 \quad (\sigma_{11} > 0,\ \text{only } +1) \quad \longrightarrow \quad x = \pm\sqrt{\frac{1}{\sigma_{11}}};$$

$$\sigma_{33}\, z^2 = -1 \quad (\sigma_{33} < 0,\ \text{only } -1) \quad \longrightarrow \quad x = \pm\sqrt{\frac{1}{|\sigma_{33}|}}.$$

These represent two infinite planes parallel, respectively, to the yz and xy planes.

This list pretty much covers all situations. The plane ν of the normal stress $\sigma_{\nu\nu}$ and its direction are readily defined by the radius vector OP. But what is the direction of the resultant stress vector on this plane ν? This is to be found out by an auxiliary construction.

Auxiliary construction:

We have seen that the radius vector OP [Fig. 5.24a] touches the quadric surface at the point P. Let us consider the normal ν' to the quadric surface at the point P and calculate its direction ratios. This calculation is similar to what we have done earlier. The equation to the Cauchy's stress quadric is $f(x, y, z) = \sigma_{11}\, x^2 + \sigma_{22}\, y^2 + \sigma_{33}\, z^2 \mp 1 = 0$, and the direction ratios, given by the partial derivatives, are

$$\frac{\partial f}{\partial x} = 2\sigma_{11}\, x = 2\sigma_{11}\, OP\, \cos(\nu,\, x);$$
$$\frac{\partial f}{\partial y} = 2\sigma_{22}\, y = 2\sigma_{22}\, OP\, \cos(\nu,\, y);$$
$$\frac{\partial f}{\partial y} = 2\sigma_{33}\, z = 2\sigma_{33}\, OP\, \cos(\nu,\, z).$$

Using the equations (5.28a, 5.28b, 5.28c), we obtain

$$\frac{\partial f}{\partial x} = 2\sigma_{11}\, OP\, \cos(\nu,\, x) = 2\, OP\, \overset{(\nu)}{T_x};$$
$$\frac{\partial f}{\partial y} = 2\sigma_{22}\, OP\, \cos(\nu,\, y) = 2\, OP\, \overset{(\nu)}{T_y};$$
$$\frac{\partial f}{\partial z} = 2\sigma_{33}\, OP\, \cos(\nu,\, z) = 2\, OP\, \overset{(\nu)}{T_z}.$$

We, thus, find that the components $\overset{(\nu)}{T_x}, \overset{(\nu)}{T_y}, \overset{(\nu)}{T_z}$ of the resultant stress vector $\overset{(\nu)}{\boldsymbol{T}}$ are proportional to the direction ratios of the normal ν'. This fact leads to the conclusion that the resultant stress vector is along the normal ν' to the stress quadric at P.

Three different cases are identified here; we have already seen and discussed these cases earlier. We repeat them here emphasising this geometric picture when the normal ν' is along the radius vector OP.

(i) The principal stresses are distinct ($\sigma_{11} \neq \sigma_{22}$; $\sigma_{22} \neq \sigma_{33}$; $\sigma_{33} \neq \sigma_{11}$): ellipsoid.
The radius vector OP is along the direction ν' to the stress quadric at P only when OP is along a coordinate direction. This is understandable; the resultant stress vector on a plane can be entirely normal to it, only if there are no tangential (shear) stresses. This can happen only for the principal stresses (on the principal planes, when the principal directions are along the coordinate directions). There are three and only three principal directions now.

(ii) Two of the principal stresses are equal ($\sigma_{11} \neq \sigma_{22} = \sigma_{33}$): ellipse of revolution.
The surface now becomes an ellipse of revolution about the x-axis (1-axis). The normal ν' is along OP if, and only if, OP is perpendicular to the x-axis (in addition, of course, to the case when OP is along the x-axis). Now the x-direction and all directions perpendicular to the x-direction are principal directions. The cases when two of the other principal stresses are equal ($\sigma_{22} \neq \sigma_{33} = \sigma_{11}$, ellipse of revolution about the y-axis) and ($\sigma_{33} \neq \sigma_{11} = \sigma_{22}$, ellipse of revolution about the z-axis) are also similar.

(iii) All the principal stresses are equal ($\sigma_{11} = \sigma_{22} = \sigma_{33}$): sphere.
The quadric surface for this case is a sphere. Now the normal ν' is along the radius vector OP for all points P. This means that for this case, every direction is principal; the resultant stress vector on any plane is always normal to this plane.

(iv) If (x, y, z) are non-principal axes, there will be 'cross-terms' (xy, yz, zx), and the equation to the stress quadric appears in the more complex but equivalent form

$$\sigma_{xx}\, x^2 + \sigma_{yy}\, y^2 + \sigma_{zz}\, z^2 + 2\tau_{xy}\, xy + 2\tau_{yz}\, yz + 2\tau_{zx}\, zx = \pm k^2,$$

$$\text{corresponding to} \quad \frac{x^2}{\left(\frac{k^2}{\sigma_{11}}\right)} + \frac{y^2}{\left(\frac{k^2}{\sigma_{22}}\right)} + \frac{z^2}{\left(\frac{k^2}{\sigma_{33}}\right)} = \pm 1.$$

There is nothing special to discuss for the simplified case of a two-dimensional state of stress (plane stress, $\sigma_{33} = 0$).

We shall take up these cases (a), (b), and (c) separately.

Distinct Principal Stresses: $\boldsymbol{\sigma_{11} > \sigma_{22} > \sigma_{33}}$

If the principal stresses are distinct, that is, if $\sigma_{11} > \sigma_{22} > \sigma_{33}$, the denominator determinant $|D| \neq 0$. The solution for l^2, m^2, n^2 is now unique, and is given by

$$l^2 \equiv \cos(\nu,\, x) = \frac{\tau_{(\nu)}^2 + (\sigma_{(\nu)} - \sigma_{22})(\sigma_{(\nu)} - \sigma_{33})}{(\sigma_{11} - \sigma_{22})(\sigma_{11} - \sigma_{33})}; \tag{5.43a}$$

$$m^2 \equiv \cos(\nu,\, y) = \frac{\tau_{(\nu)}^2 + (\sigma_{(\nu)} - \sigma_{33})(\sigma_{(\nu)} - \sigma_{11})}{(\sigma_{22} - \sigma_{33})(\sigma_{33} - \sigma_{11})}; \tag{5.43b}$$

$$n^2 \equiv \cos(\nu,\, z) = \frac{\tau_{(\nu)}^2 + (\sigma_{(\nu)} - \sigma_{11})(\sigma_{(\nu)} - \sigma_{22})}{(\sigma_{33} - \sigma_{11})(\sigma_{33} - \sigma_{22})}. \tag{5.43c}$$

Equation for l^2 (5.43a):
The first equation (5.43a) is recast as

$$\tau_{(\nu)}^2 + \sigma_{(\nu)}^2 - \sigma_{(\nu)}(\sigma_{22} + \sigma_{33}) + \sigma_{22}\,\sigma_{33} - \cos^2(\nu\, x)(\sigma_{11} - \sigma_{22})(\sigma_{11} - \sigma_{22}) = 0;$$

$$\text{i.e.,} \quad \tau_{(\nu)}^2 + \left(\sigma_{(\nu)} - \frac{\sigma_{22} + \sigma_{33}}{2}\right)^2 = \left(\frac{\sigma_{22} - \sigma_{33}}{2}\right)^2 + \cos^2(\nu,\, x)(\sigma_{11} - \sigma_{22})(\sigma_{11} - \sigma_{33}).$$

This equation, after some algebraic manipulations, takes the form

$$\tau_{(\nu)}^2 + \left(\sigma_{(\nu)} - \frac{\sigma_{22}+\sigma_{33}}{2}\right)^2 = \left(\sigma_{11} - \frac{\sigma_{22}+\sigma_{33}}{2}\right)^2 \cos^2(\nu,\, x) + \left(\frac{\sigma_{22}-\sigma_{33}}{2}\right)^2 \left[1-\cos^2(\nu,\, x)\right]. \tag{5.44}$$

This equation represents a family of circles in the $\sigma_{(\nu)}\,\tau_{(\nu)}$ plane. The principal stresses $\sigma_{11} > \sigma_{22} > \sigma_{33}$ are all known (given). The direction cosines may be regarded as a parameter that can take all values between [0, 1].

Family of circles, all with the same centre: $\left[\dfrac{\sigma_{22}+\sigma_{33}}{2},\ \right]$, $\quad 0 \le \cos^2(\nu,\, x) \le 1;$

$$\text{for } \cos(\nu,\, x) = 0 \qquad \tau_{(\nu)}^2 + \left(\sigma_{(\nu)} - \frac{\sigma_{22}+\sigma_{33}}{2}\right)^2 = \left(\frac{\sigma_{22}-\sigma_{33}}{2}\right)^2; \tag{5.45a}$$

$$\text{for } \cos(\nu,\, x) = 1 \qquad \tau_{(\nu)}^2 + \left(\sigma_{(\nu)} - \frac{\sigma_{22}+\sigma_{33}}{2}\right)^2 = \left(\sigma_{11} - \frac{\sigma_{22}+\sigma_{33}}{2}\right)^2. \tag{5.45b}$$

These are the smallest and the largest limiting circles [Fig. 5.26a]. The operating area between these limiting circles is shown shaded.

Equations for m^2 (5.43b) and n^2 (5.43c):

These two equations, on exactly similar treatment, lead to the following.

Family of circles, all with the same centre: $\left[\dfrac{\sigma_{33}+\sigma_{11}}{2},\ 0\right]$, $\quad 0 \le \cos^2(\nu,\, y) \le 1;$

$$\text{for } \cos(\nu,\, y) = 0 \qquad \tau_{(\nu)}^2 + \left(\sigma_{(\nu)} - \frac{\sigma_{33}+\sigma_{11}}{2}\right)^2 = \left(\frac{\sigma_{33}-\sigma_{11}}{2}\right)^2; \tag{5.46a}$$

$$\text{for } \cos(\nu,\, y) = 1 \qquad \tau_{(\nu)}^2 + \left(\sigma_{(\nu)} - \frac{\sigma_{33}+\sigma_{11}}{2}\right)^2 = \left(\sigma_{22} - \frac{\sigma_{33}+\sigma_{11}}{2}\right)^2. \tag{5.46b}$$

These are the smallest and the largest limiting circles [Fig. 5.26b]. The operating area between these limiting circles is shown shaded.

Family of circles, all with the same centre: $\left[\dfrac{\sigma_{11}+\sigma_{22}}{2},\ 0\right]$, $\quad 0 \le \cos^2(\nu,\, z) \le 1;$

$$\text{for } \cos(\nu,\, z) = 0 \qquad \tau_{(\nu)}^2 + \left(\sigma_{(\nu)} - \frac{\sigma_{11}+\sigma_{22}}{2}\right)^2 = \left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2; \tag{5.47a}$$

$$\text{for } \cos(\nu,\, z) = 1 \qquad \tau_{(\nu)}^2 + \left(\sigma_{(\nu)} - \frac{\sigma_{11}+\sigma_{22}}{2}\right)^2 = \left(\sigma_{33} - \frac{\sigma_{11}+\sigma_{22}}{2}\right)^2. \tag{5.47b}$$

These are the smallest and the largest limiting circles [Fig. 5.26c]. The operating area between these limiting circles is shown shaded.

(a) All the three pairs of circles (b) Shaded part: common (operative) part

Figure 5.29: All the pairs of Mohr's semi-circles are shown in Fig. 5.29a. The shaded part, which alone is the operative region, is shown in Fig. 5.29b. Some of the important or special points are also marked here.

Further Discussion

We have seen that only the shaded regions in the three pairs of circles is operative. Now when all the principal stresses are acting, a typical point P with its coordinates ($\sigma_{(\nu)}$, $\tau_{(\nu)}$) in the $\sigma_{(\nu)}$, $\tau_{(\nu)}$ plane can lie only the shaded region common to all the shaded regions shown earlier [Fig. 5.29b].

What are the maximum and minimum normal stresses at the point P? What are the maximum and minimum shear stresses at P? Will there be a normal stress on these planes? These and a number of similar questions can be answered readily from the Mohr's circle construction. From Figs 5.29b, 5.30c we can note the following. [If the circles are drawn to scale, the (approximate) numerical values can be measured from these Mohr's circles.]

Point A:	$\sigma_{max} = \sigma_{11}$	Point B:	$\sigma_{min} = \sigma_{33}$
Point C:	$\tau_{max} = \dfrac{\sigma_{11} - \sigma_{33}}{2}$	Point E:	$\sigma_{(max\,\tau)} = \dfrac{\sigma_{11} + \sigma_{33}}{2}$
Points A, B, D:	$\tau_{min} = 0$		

On which plane(s) do these maximum and minimum shear stresses act? To answer this, all we have to do is to substitute these values in Eqs (5.43a, 5.43b, 5.43c). In this way, the direction cosines are obtained which define the plane that we seek.

We can now make the following statements about the nature of stress at a typical point P [coordinates $\sigma_{(\nu)}$, $\tau_{(\nu)}$] in the shaded region.

(i) The state of stress at P is defined completely by the three principal stresses (and their directions).

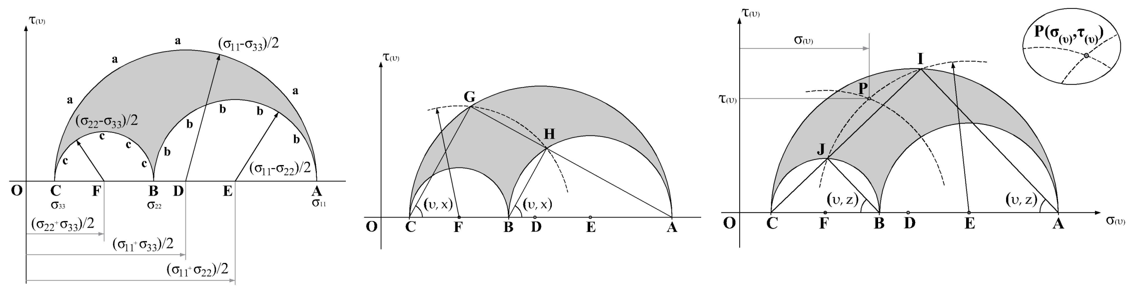
(ii) The maximum and minimum normal stresses are, respectively, σ_{11} and σ_{33}.

(iii) The minimum shear stress, $\tau_{min} = 0$, acts on the principal planes.

(iv) The maximum shear stress, $\tau_{max} = (\sigma_{11} - \sigma_{33})/2$ acts on two planes. These two planes are perpendicular to the σ_{33} principal direction, and inclined at $45°$ to the σ_{11} and σ_{22} principal directions.

(v) The normal stress $\sigma_{(max\ \tau)}$ on these two planes is $(\sigma_{11} + \sigma_{33})/2$.

Mohr Circle Construction



(a) Mohr's circle: stage 1 (b) Mohr's circle: stage 2 (c) Mohr's circles: stage 3

Figure 5.30: The three circles *aaaa*, *bbbb*, *cccc* are shown. The procedure of completing the construction is explained in the text.

The Mohr's circles are constructed as described below. We shall assume that the principal stresses σ_{11}, σ_{22}, σ_{33} are known. We shall see how the normal stress $\sigma_{(\nu)}$ and the shearing stress $\tau_{(\nu)}$ on a plane ν of given direction cosines $l \equiv \cos(\nu, x)$, $m \equiv \cos(\nu, y)$, $n \equiv \cos(\nu, z)$ can be obtained from this graphical construction.

(i) Draw (to a convenient scale) the horizontal $\sigma_{(\nu)}$ and the vertical $\tau_{(\nu)}$ axes, and locate the points $A\,(\sigma_{11})$, $B\,(\sigma_{22})$, $C\,(\sigma_{33})$ [Fig. 5.30a].

(ii) Choose these values pairwise, and draw the three circles *aaaa*, *bbbb*, *cccc* [Fig. 5.30a].

$$
\begin{array}{llll}
\text{circle aaaa:} & \text{centre } D\text{:} \ \dfrac{\sigma_{11} + \sigma_{33}}{2} & \text{radius} & \left|\dfrac{\sigma_{11} - \sigma_{33}}{2}\right|; \\
\text{circle bbbb:} & \text{centre } E\text{:} \ \dfrac{\sigma_{22} + \sigma_{11}}{2} & \text{radius} & \left|\dfrac{\sigma_{22} - \sigma_{11}}{2}\right|; \\
\text{circle cccc:} & \text{centre } F\text{:} \ \dfrac{\sigma_{33} + \sigma_{22}}{2} & \text{radius} & \left|\dfrac{\sigma_{33} - \sigma_{22}}{2}\right|.
\end{array}
$$

(iii) Calculate the angles corresponding to the given direction cosines $l \equiv \cos(\nu, x)$, $m \equiv \cos(\nu, y)$, $n \equiv \cos(\nu, z)$. [Actually only two angles are sufficient, just like only two of the direction cosines need be given, the third one being automatically specified or known because of the relation $l^2 + m^2 + n^2 = 1$.]

$$\theta_l \equiv (\nu,\, x) \equiv \cos^{-1}(\nu,\, x); \quad \theta_m \equiv (\nu,\, y) \equiv \cos^{-1}(\nu,\, y); \quad \theta_n \equiv (\nu,\, z) \equiv \cos^{-1}(\nu,\, z).$$

(iv) Draw two lines CG, at the calculate angle $\theta_l \equiv (\nu,\, x) \equiv \cos^{-1}(\nu,\, x)$, and AG through the points C and A to touch the circle aaaa at the point G [Fig. 5.30b].

(v) With D as centre, draw a circular arc connecting A and G.

(vi) In the same way, draw two lines AI, at the calculated angle $\theta_n \equiv (\nu,\, z) \equiv \cos^{-1}(\nu,\, z)$, and CI through the points A and C to touch the circle aaaa at the point I [Fig. 5.30c].

(vii) With E as centre, draw a circular arc through the points I and J. These two circular arcs intersect at the point P.

(viii) Now the point P represents the point at which we are considering the state of stress. Thus, its two coordinates $\sigma_{(\nu)}$ and $\tau_{(\nu)}$ give us the desired normal stress and the shear stress on the plane ν.

The construction can be carried out in the reverse direction also; that is, if the normal and shearing stresses are given, and we are required to identify the plane on which these act.

We have described a method to draw the two circular arc (shown in dotted lines in Fig. 5.30c). The radii of these circular arcs can be obtained analytically also[23] from Eq. (5.44) and its companion equations (which are not explicitly shown). The three circular arcs — we have shown only two of them — will necessarily pass through the unique point P (with its coordinates $\sigma_{(\nu)}$, $\tau_{(\nu)}$).

We are now required to prove that the angles that we have set out and marked as (ν, x), $(\nu,\, y)$ and $(\nu,\, z)$ are indeed the given direction cosines. We shall carry out this exercise presently as shown below.

To show that the angles marked (ν, x), $(\nu,\, y)$, $(\nu,\, z)$ are the direction cosines
Referring to Fig. 5.30c, let us call for convenience the angle EAI (marked $(\nu,\, z) = \phi$ in the figure) ϕ. We need to prove that $\phi = (\nu,\, z)$. From the triangle EAI (not fully drawn in the figure), we have

$$(EI)^2 = (AE)^2 + (AI)^2 - 2(AE)(AI)\cos\phi.$$

EI is the radius R_{12} that we have calculated. From the following equations,

$$(EI)^2 = R_{12}^2 = \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \cos^2(\nu,\, z)(\sigma_{33} - \sigma_{11})(\sigma_{33} - \sigma_{22});$$

[23] These can be calculated readily for the given values of the direction cosines. These equations, we can see, are of the form $x^2 + y^2 = r^2$. Thus, the radii of these three arcs can be seen to be:

$$R_{23} = \sqrt{\left(\frac{\sigma_{22} - \sigma_{33}}{2}\right)^2 + \cos^2(\nu,\, x)(\sigma_{11} - \sigma_{22})(\sigma_{11} - \sigma_{33})};$$
$$R_{31} = \sqrt{\left(\frac{\sigma_{33} - \sigma_{11}}{2}\right)^2 + \cos^2(\nu,\, y)(\sigma_{22} - \sigma_{33})(\sigma_{22} - \sigma_{11})};$$
$$R_{12} = \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \cos^2(\nu,\, z)(\sigma_{33} - \sigma_{11})(\sigma_{33} - \sigma_{22})}.$$

$$(AE)^2 = \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2;$$
$$(AI) = (CA)\cos\phi = (\sigma_{11} - \sigma_{22})\cos\phi.$$

Solving for $\cos\phi$, we find that $\cos\phi = \cos(\nu,\, z)$.

Two Equal Principal Stresses: $\boldsymbol{\sigma_{11} = \sigma_{22} > \sigma_{33}}$

Let us return to Eqs (5.42a,5.42b,5.42c). We recall that the denominator determinant $|D|$ is given by

$$|D| = \begin{vmatrix} \sigma_{11}^2 & \sigma_{22}^2 & \sigma_{33}^2 \\ \sigma_{11} & \sigma_{22} & \sigma_{33} \\ 1 & 1 & 1 \end{vmatrix},$$

which now vanishes because two of the principal stresses are equal. For this case, a non-trivial solution for $(l^2,\, m^2,\, n^2)$ [Eqs (5.42a, 5.42b, 5.42c)] can exist only if all the numerator determinants are zero. This condition gives us

$$\begin{vmatrix} (\sigma_{\nu\nu}^2 + \tau_{(\nu)}^2) & \sigma_{22}^2 & \sigma_{33}^2 \\ \sigma_{\nu\nu} & \sigma_{22} & \sigma_{33} \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \sigma_{11}^2 & (\sigma_{\nu\nu}^2 + \tau_{(\nu)}^2) & \sigma_{33}^2 \\ \sigma_{11} & \sigma_{\nu\nu} & \sigma_{33} \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \sigma_{11}^2 & \sigma_{11}^2 & (\sigma_{\nu\nu}^2 + \tau_{(\nu)}^2) \\ \sigma_{11} & \sigma_{11} & \sigma_{\nu\nu} \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

The third determinant is zero here, because its first two columns are the same ($\sigma_{11} = \sigma_{22}$). The first and the second ones when equated to zero give us the same equation

$$\tau_{(\nu)}^2 + \left(\sigma_{(\nu)} - \frac{\sigma_{11} + \sigma_{22}}{2}\right)^2 = \left(\frac{\sigma_{11} + \sigma_{22}}{2}\right)^2,$$

which represents a circle. As $\sigma_{11} = \sigma_{22}$, the two circles bbbb and cccc coalesce; the shaded area degenerates to the semi-circle connecting the points C and A. This now represents a simpler state of stress (axi-symmetry about the $3\,(z)$-axis)[24]. The normal $\sigma_{(\nu)}$ and the shear stresses $\tau_{(\nu)}$ are the same for all (consistent, physically possible) values of the two direction cosines $l \equiv \cos(\nu,\, x)$ and $m \equiv \cos(\nu,\, y)$.

The other case of (two principal stresses being equal) $\sigma_{11} > \sigma_{22} = \sigma_{33}$ is similar to the one considered above and it is, therefore, left out.

All Principal Stresses Equal: $\boldsymbol{\sigma_{11} = \sigma_{22} = \sigma_{33}}$

When ($\sigma_{11} = \sigma_{22} = \sigma_{33}$), all the circles degenerate to the single point $[\sigma_{(\nu)},\, 0]$. Now every plane is a principal plane and that every stress a principal one. There is no shear stress on any plane. This represents a case of hydrostatic state of stress.

$$\sigma_{(\nu)} = \sigma_{11} = \sigma_{22} = \sigma_{33}; \qquad \tau_{(\nu)} = 0 \text{ on all planes } \nu.$$

We shall discuss in the next chapter the analysis of strain at a point.

[24] Lamé's ellipsoid becomes an ellipse of revolution. The consequences or the conclusions should be obvious by now.

Chapter 6

ANALYSIS OF STRAIN AT A POINT

We have seen the analysis of stress at a point. Next, we have to carry out a similar exercise on the nature of strain at a point. To do this, we have to examine the details of the deformation in the neighbourhood of a typical point. A few remarks are made to understand the broad issues involved. Later only a simple-minded analysis is used.

GENERAL INTRODUCTORY REMARKS

There is a great deal of common ground between the analyses of stress and strain at a point; they are both very similar mathematically. They are examples of a second order tensor. The mathematical equations are almost similar. All the topics in the analysis of stress carry over to the analysis of strain also with very minor changes. However, the physical basis of the analysis of strain is different: the geometry of deformation instead of the equilibrium of the body under the influence of the forces[1]. This is a major difference.

TO TRACK THE DEFORMATION

The body, to begin with, is unloaded and stress-free. When an external load is applied, the body is deformed. We need to track this and examine this in detail. The body may undergo rigid body motion; it may also be deformed or strained. The rigid body motion may be a translation and / or a rotation. We need to identify and filter out the rigid body motion. It is only the remaining part, the deformation part, that concerns us. Only this part is associated with stresses and strains.

To explain this, let us take an example. When a number of policemen are deployed to provide security support, the Chief of Police may wish to know where each policeman is at every instant. Here the policemen have identity tags, their names or identity numbers. For a deformable solid / fluid, the situation is similar but harder; the material particles have no such identification tags. How shall we address this difficulty? This is our first concern.

[1] I am grateful to Dr Gangan Prathap for his suggestion to refer to these two as kinematics vs kinetics.

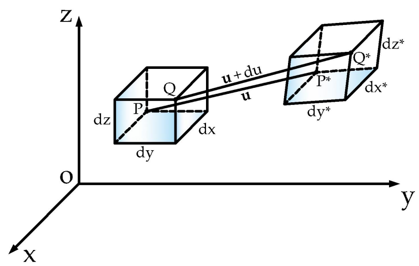


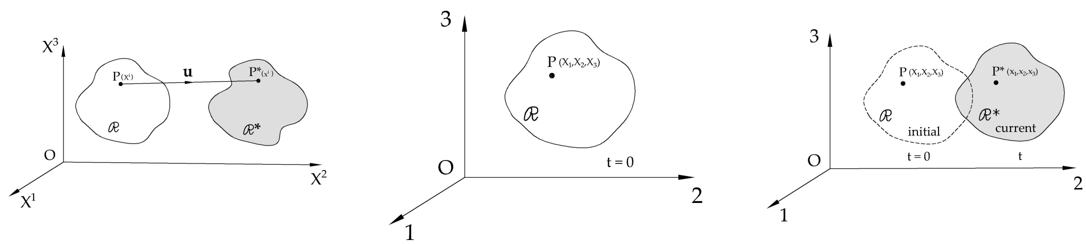
Figure 6.1: Deformation in the neigbourhood of a point

When the undeformed body is loaded, it is deformed. Fig. 6.1 shows the deformation in the neigbourhood of a point. We can establish a mapping that relates every point in the undeformed body to a corresponding point in the deformed body. In particular, a brick element before and after deformation is shown.

We need to consider its deformation, and examine the changes in its lengths and angles as the body deforms. We shall carry out this important exercise a little later.

A material particle cannot possibly occupy two (or more) places (points), nor can two (or more) material particles occupy the same place (point) at the same instant of time.

Labelling of Material Particles



(a) Deformation (b) Initial configuration (c) Current configuration

Figure 6.2: To assign an identification label to each particle: the coordinates (X_1, X_2, X_3) of a particle at a reference time, say $t = t_0$, serve as the identification label of this particle.

Let us try to label the various material particles so that their movements can be conveniently tracked. Fig. 6.2 shows a typical material particle $\mathcal{P}$ in its initial (time $t = t_0$) and current configurations, $P(X_1, X_2, X_3, 0)$ and $P^*(x_1, x_2, x_3, t)$ at a general instant of time t, respectively. The coordinates of P at time $t = t_0$ are (X_1, X_2, X_3). These numbers (X_1, X_2, X_3) uniquely specify a material particle. They are, in a manner of speaking, the 'place of birth' of the particle! When the particle moves, its position is changed from $P(X_1, X_2, X_3)$ to $P^*(x_1, x_2, x_3)$ at time t. Thus,

$$x_i = x_i(X_1, X_2, X_3, t) \quad (i = 1, 2, 3) \tag{6.1}$$

specifies the position (place) x_i (that is, x_1, x_2, x_3) at time t of the particle (X_1, X_2, X_3). (X_1, X_2, X_3) serves as the particle identification label, and (x_1, x_2, x_3) the place that it occupies at time t. This is the same as stating that the Mumbai man — 'place of birth':

Mumbai (X_1, X_2, X_3) — is now at time t at Paris (x_1, x_2, x_3), etc. No matter where he goes, his identity label (X_1, X_2, X_3) remains the same, the Mumbai man!

This one-to-one mapping given by Eq. (6.1) can be inverted to yield

$$X_i = X_i(x_1, x_2, x_3, t) \quad (i = 1, 2, 3). \tag{6.2}$$

Let us note that the independent variables are (X_1, X_2, X_3) in Eq. (6.1) and (x_1, x_2, x_3) in Eq. (6.4). These form the basis of the two approaches, the Lagrangian and the Eulerian.

Lagrangian and Eulerian Approaches

We note that a quantity of importance, say, the velocity may be expressed either in terms of (X_1, X_2, X_3) as the independent coordinates as

$$V_i = V_i(X_1, X_2, X_3, t) \quad (i = 1, 2, 3), \tag{6.3}$$

or in terms of (x_1, x_2, x_3) as the independent variable as

$$v_i = v_i(x_1, x_2, x_3, t) \qquad (i = 1, 2, 3). \tag{6.4}$$

It is not a good idea to use the same dependent variable V_i (or v_i) for both these representations. Why is this so? Because the functional forms are different[2].

We have seen that (X_1, X_2, X_3) serve as the particle identification labels. If these are the independent variables, we refer to this as the Lagrangian approach (also called the material or substantial approach). On other hand, if the position (or place) variables (x_1, x_2, x_3) are the independent variables, this is called the Eulerian approach[3] (also called the local approach). (X_1, X_2, X_3) are called the Lagrange's coordinates, and (x_1, x_2, x_3) the Euler's coordinates[4].

The above explanation is from the mathematical point of view. Physically what is the basic difference between the two? Let us examine.

In the Lagrangian approach, we focus on a material particle, and watch the different places that it visits at different times. This is the physical meaning of the equation (6.1). Obviously all the material particles are covered. This equation is a satisfactory method of describing the motion by a mathematical / analytical equation. The actual functional form

---

[2] To explain this point clearly, let us take an example.

Let $y = y(x) = x^2$ and let $x = x(t) = 1 + t$. If we substitute $x = 1 + t$ in the first equation, we obtain $y = y(x) = x^2 = (1 + t)^2 = 1 + 2t + t^2$. This is not the same as $y(t)$ because $y(t)$ means t^2. Thus, the correct way to write is $y = y(x) = y(x(t)) = (1 + t)^2 = 1 + 2t + t^2 = y_1(t)$. Note that we now have a different functional form $y_1(t)$, which must be distinguished carefully from $y(t)$.

$v = v(x_1, x_2, x_3, t) = v(x_1(X_i), x_2(X_i), x_3(X_i), t) \neq v(X_1, X_2, X_3, t)$!

[3] Actually both approaches are due to the great Leonard Euler (April 1707 - Sept. 1783). Euler is known for his extraordinary magnanimity in letting others take the credit for his original work.

[4] Not to be confused with the Euler angles ϕ, θ, ψ used in rigid body dynamics when discussing the motion of a gyroscope

cannot be known at this time of formulating the problem; it can be known only after the (boundary value) problem is solved.

In the Eulerian approach, we focus on a place, and watch the various material particles that visit this chosen place. Again, all the places are covered. Either of these two approaches gives a complete description of the motion. In both cases, we need to be able to derive the mathematical expressions for the velocity, momentum, kinetic energy, etc. This is part of the development of the theoretical foundation of continuum mechanics that seeks to encompass both solid and fluid mechanics in its scope[5].

Representation of a Function and Its Differentiation

We shall now see how functions are represented, and differentiation carried out, in the two approaches. Let a function, to be specific, the velocity V_i be represented in the Lagrangian coordinate system as

$$V_i = V_i(X_1, X_2, X_3, t) \quad (i = 1, 2, 3). \tag{6.5}$$

If we differentiate this w.r.to time, we obtain

$$\frac{dV_i}{dt} = \left.\frac{\partial V_i}{\partial t}\right|_{\mathcal{L}}, \tag{6.6}$$

where the special notation is used to indicate the Lagrangian point of view. The displacement vector is

$$u_i = x_i(X_1, X_2, X_3, t) - X_i \qquad (i = 1, 2, 3), \tag{6.7}$$

so that

$$\frac{\partial u_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j} - \frac{\partial X_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j} - \delta_{ij} \qquad (i, j = 1, 2, 3). \tag{6.8}$$

In the Eulerian approach,

$$v_i = v_i(x_1, x_2, x_3, t) \qquad (i = 1, 2, 3), \tag{6.9}$$

which represents the velocity of an unidentified material particle now at time t occupying the place (x_1, x_2, x_3). On differentiation, we obtain

$$\frac{dv_i}{dt} = \frac{\partial v_i}{\partial x_j}\dot{x}_j + \frac{\partial v_i}{\partial t}, \quad \text{(summation on the repeated index } j\text{)}. \tag{6.10}$$

[5] Perhaps an analogy might help clarify this difference.

In a coed college, let us assume that one youngster follows a particular girl and notes down the various places she visits at various instants of time. For example, he follows a girl, say, Usha, and notes down that at 8:50 a.m. she reaches the college gate, that at 8:55 a.m. she visits the girls' common room, that at 8:58 a.m. she enters her classroom, etc. Another boy similarly follows another girl, say, Lakshmi, and similarly notes down the places along with the instants of time when she visits these places. If all such records are collected together, we get a complete picture of where every girl was at every instant of time.

Let us now turn around and take a different way to do this. One boy stations himself at a given fixed point, say, the canteen and notes down the various girls who reach there: Usha at 10:15 a.m., Lakshmi at 11:20 a.m., and so on. If every girl is covered in the first approach, and every place in the second, we have a complete picture of the movement of every girl at every instant of time.

"taruṇastāvad taruṇīsaktah"!

From our earlier study of fluid mechanics, we recognise the left hand side to be the substantial, or material, derivative[6]. The two terms on the right hand side are the convective derivative and the local derivative, respectively. Thus, as v_i represents the velocity of an unidentified material particle, and as the acceleration is to be tied down to a material particle — its identity is not revealed directly, but is indicated indirectly by the place (x_1, x_2, x_3) that it occupies at time t — we recognise them as the convective and local accelerations, respectively.

The displacement vector

$$u_i = x_i - X_i(x_1, x_2, x_3, t) \quad (i = 1, 2, 3) \tag{6.11}$$

may be differentiated w.r.to the space variable x_j as

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial x_i}{\partial x_j} - \frac{\partial X_i}{\partial x_j} = \delta_{ij} - \frac{\partial X_i}{\partial x_j}. \tag{6.12}$$

Simplified Case for Solids Only

These discussions are equally valid for solids and fluids. Such a treatment is necessary in continuum mechanics where we try to build a theory equally applicable for both solids and fluids. As we do not consider flow, time is no more a factor. Thus, we disregard the variable time t in all further discussions and rewrite Eqs (6.1) and (6.7) as

$$x_i = x_i(X_1, X_2, X_3) \qquad (i = 1, 2, 3) \tag{6.13}$$

$$u_i = x_i(X_1, X_2, X_3) - X_i \quad (i = 1, 2, 3), \tag{6.14}$$

respectively. Similarly, Eq. (6.11) is also recast as

$$u_i = x_i - X_i(x_1, x_2, x_3) \qquad (i = 1, 2, 3), \tag{6.15}$$

A SIMPLIFIED APPROACH FOR THE LINEAR THEORY

There several cases where the analysis can be complicated: a general (curvilinear) coordinate system, convected coordinates, large deformation theory, etc. We shall drop all such complications and discuss the analysis of deformation sufficient for the linear infinitesimal theory of elasticity staying in a rectangular coordinate system.

Deformations in the Neighbourhood of a Typical Point

We have only one rectangular cartesian coordinate system $Oxyz$ now; one is sufficient. The deformations are so small (unlike the general case of large deformations) that it is no more necessary to distinguish between the undeformed $\mathcal{R}$ and the deformed $\mathcal{R}^*$ configurations. As before, our interest is in the analysis of deformation in a neighbourhood of a typical point P. By doing so, we obtain the relevant information. Equilibrium of forces is not considered here; only geometry is relevant.

[6] often written as $\frac{Dv_i}{Dt}$

(a) 'Small' deformation of a body (b) The details near a typical point

Figure 6.3: A rectangular cartesian coordinate system $Oxyz$ is chosen to discuss the nature of deformation in the neighbourhood of a typical point $P(x_0,\, y_0,\, z_0)$. The figure on the right up is a blow up of the neighbourhood of (i) a point $P(x_0,\, y_0,\, z_0)$ and its map $P^*(x_0^*,\, y_0^*,\, z_0^*)$, and (ii) a point $Q(x,\, y,\, z)$ and its map $Q^*(x^*,\, y^*,\, z^*)$.

A typical point $P(x_0,\, y_0,\, z_0)$ and its neighbour $Q(x,\, y,\, z)$ are carried to $P^*(x_0^*,\, y_0^*,\, z_0^*)$ and $Q^*(x^*,\, y^*,\, z^*)$, respectively. Comparing the coordinates of (i) $P(x_0,\, y_0,\, z_0)$ and its map $P^*(x_0^*,\, y_0^*,\, z_0^*)$, and (ii) $Q(x,\, y,\, z)$ and its map $Q^*(x^*,\, y^*,\, z^*)$ [Fig. 6.3a], we see that

$$dx = x - x_0; \qquad dy = y - y_0; \qquad dz = z - z_0; \tag{6.16a}$$

$$dx^* = x^* - x_0^*; \qquad dy^* = y^* - y_0^*; \qquad dz^* = z^* - z_0^*; \tag{6.16b}$$

$$x^* = x + u(x, y, z) = x + u(x_0 + dx,\, y_0 + dy,\, z_0 + dz); \tag{6.16c}$$

$$y^* = y + v(x, y, z) = y + v(x_0 + dx,\, y_0 + dy,\, z_0 + dz); \tag{6.16d}$$

$$z^* = z + w(x, y, z) = z + w(x_0 + dx,\, y_0 + dy,\, z_0 + dz). \tag{6.16e}$$

Expanding these equations in a Taylor series about the base station $P(x_0,\, y_0,\, z_0)$, we have

$$x^* = x + u(x_0,\, y_0,\, z_0) + \left(\frac{\partial u}{\partial x}\right)_P dx + \left(\frac{\partial u}{\partial y}\right)_P dy + \left(\frac{\partial u}{\partial z}\right)_P dz; \tag{6.17a}$$

$$y^* = y + v(x_0,\, y_0,\, z_0) + \left(\frac{\partial v}{\partial x}\right)_P dx + \left(\frac{\partial v}{\partial y}\right)_P dy + \left(\frac{\partial v}{\partial z}\right)_P dz; \tag{6.17b}$$

$$z^* = z + w(x_0,\, y_0,\, z_0) + \left(\frac{\partial w}{\partial x}\right)_P dx + \left(\frac{\partial w}{\partial y}\right)_P dy + \left(\frac{\partial w}{\partial z}\right)_P dz. \tag{6.17c}$$

The series may be terminated with the terms explicitly shown for two separate cases: (i) if the investigation is restricted to the immediate neighbourhood of the point P, so that dx, dy, dz are small, and (ii) if the higher derivatives of the displacements components like $\partial^2 u/\partial x^2$ and $\partial^2 v/\partial y^2$ are zero, or at least very small. The latter condition means that the derivatives like $\partial u/\partial x$ and $\partial v/\partial y$ — these are strain components — are either constants or nearly so. If the strain components are nearly constant, it means that all parts of the body are subjected to nearly the same strain, which corresponds to the special case of homogeneous deformation.

It is sometimes more convenient to write these equations (6.17a - 6.17c) differently so that the position (that is, the coordinates) of Q^* depends only on the positions (that is, the

coordinates) of Q and P. This can be done by using Eq. (6.16a) to eliminate dx, dy, dz. On doing so, the above equations (6.17a - 6.17c) appear as

$$\begin{aligned}
x^* &= u(x_0,\, y_0,\, z_0) - \left(\frac{\partial u}{\partial x}\right)_P x_0 - \left(\frac{\partial u}{\partial y}\right)_P y_0 - \left(\frac{\partial u}{\partial z}\right)_P z_0 \\
&\quad + \left[1 + \left(\frac{\partial u}{\partial x}\right)_P\right] x + \left(\frac{\partial u}{\partial y}\right)_P y + \left(\frac{\partial u}{\partial z}\right)_P z; \qquad (6.18a)\\
y^* &= v(x_0,\, y_0,\, z_0) - \left(\frac{\partial v}{\partial x}\right)_P x_0 - \left(\frac{\partial v}{\partial y}\right)_P y_0 - \left(\frac{\partial v}{\partial z}\right)_P z_0 \\
&\quad + \left[1 + \left(\frac{\partial v}{\partial y}\right)_P\right] y + \left(\frac{\partial v}{\partial x}\right)_P x + \left(\frac{\partial v}{\partial z}\right)_P z; \qquad (6.18b)\\
z^* &= w(x_0,\, y_0,\, z_0) - \left(\frac{\partial w}{\partial x}\right)_P x_0 - \left(\frac{\partial w}{\partial y}\right)_P y_0 - \left(\frac{\partial w}{\partial z}\right)_P z_0 \\
&\quad + \left[1 + \left(\frac{\partial w}{\partial z}\right)_P\right] z + \left(\frac{\partial w}{\partial x}\right)_P x + \left(\frac{\partial w}{\partial y}\right)_P y. \qquad (6.18c)
\end{aligned}$$

Let us examine these equations. For a given base station $P(x_0,\, y_0,\, z_0)$, the first four terms are constants; they depend only on the values at the base station P. Thus, for these equations that define the mapping from Q to Q^*, these four terms being constant represent a translation. They do not affect the stretching and rotation of the line element. Calling them $(C_u)_P$, $(C_v)_P$, $(C_w)_P$, let us write the above three equations (6.18a - 6.18c) as

$$\begin{aligned}
x^* &= (C_u)_P + \left[1 + \left(\frac{\partial u}{\partial x}\right)_P\right] x + \left(\frac{\partial u}{\partial y}\right)_P y + \left(\frac{\partial u}{\partial z}\right)_P z; \qquad (6.19a)\\
y^* &= (C_v)_P + \left(\frac{\partial v}{\partial x}\right)_P x + \left[1 + \left(\frac{\partial v}{\partial y}\right)_P\right] y + \left(\frac{\partial v}{\partial z}\right)_P z; \qquad (6.19b)\\
z^* &= (C_w)_P + \left(\frac{\partial w}{\partial x}\right)_P x + \left(\frac{\partial w}{\partial y}\right)_P y + \left[1 + \left(\frac{\partial w}{\partial z}\right)_P\right] z. \qquad (6.19c)
\end{aligned}$$

These equations have only linear terms; there are no quadratic or higher degree terms. Such transformation equations are known as affine transformations or linear transformations. Let us emphasise that the linearity[7] arises because we confined ourselves to the behaviour in the immediate neighbourhood of the base station.

These equations can also be recast into a form so that we can see how the elemental lengths dx, dy, dz are mapped to dx^*, dy^*, dz^*. We may use Eqs (6.16c - 6.16e) to make the necessary changes.

$$\begin{gathered}
dx^* = x^* - x_0^*; \qquad dy^* = y^* - y_0^*; \qquad dz^* = z^* - z_0^*; \\
x^* = x_0 + u(x_0,\, y_0,\, z_0); \quad y^* = y_0 + v(x_0,\, y_0,\, z_0); \quad z^* = z_0 + w(x_0,\, y_0,\, z_0).
\end{gathered}$$

[7] Even in the case of a general (nonlinear) transformation, the transformation equations are *locally* linear. Hence linear transformations have great importance, even though nearly all transformations may be nonlinear.

Thus, we have

$$dx^* = \left[1 + \left(\frac{\partial u}{\partial x}\right)_P\right] dx + \left(\frac{\partial u}{\partial y}\right)_P dy + \left(\frac{\partial u}{\partial z}\right)_P dz; \tag{6.20a}$$

$$dy^* = \left(\frac{\partial v}{\partial x}\right)_P dx + \left[1 + \left(\frac{\partial v}{\partial y}\right)_P\right] dy + \left(\frac{\partial v}{\partial z}\right)_P dz; \tag{6.20b}$$

$$dz^* = \left(\frac{\partial w}{\partial x}\right)_P dx + \left(\frac{\partial w}{\partial y}\right)_P dy + \left[1 + \left(\frac{\partial w}{\partial z}\right)_P\right] dz. \tag{6.20c}$$

These also represent an affine transformation; these may be inverted to give dx, dy, dz in terms of dx^*, dy^*, dz^* which will also represent an affine transformation. Because of their importance, we shall briefly discuss below affine transformations.

Affine Transformations

Eqs (6.19a, 6.19b, 6.19c) and (6.20a, 6.20b, 6.20c) are examples of affine transformations. They have some important properties. We shall see some of them below.

Let us define[8] an affine transformation (also known as an affine mapping, a homogeneous mapping, and a linear mapping) from the variables x_i $(i = 1, 2, 3)$ to the variable y_i $(i = 1, 2, 3)$ defined by the equation

$$y_i = c_{i0} + (\delta_{ij} + c_{ij})\, x_j \quad (i, j = 1, 2, 3), \tag{6.21}$$

where c_{i0} and c_{ij} are constants. For this to be a one-to-one mapping, the determinant of the coefficients shall not vanish: det $(\delta_{ij} + c_{ij}) \neq 0$. Then the equation (6.21) may be inverted to yield another affine mapping, this time from the variables y_i $(i = 1, 2, 3)$ to the original variables x_i $(i = 1, 2, 3)$, as

$$x_i = b_{i0} + (\delta_{ij} + b_{ij})\, y_j \quad (i, j = 1, 2, 3). \tag{6.22}$$

Such an affine transformation has several — important and useful in applications — properties. Some of them relevant for us are discussed below.

Properties of Affine Mappings

(i) *Planes are mapped onto planes.*

(ii) *Straight lines are mapped onto straight lines.*

(iii) *Equal vectors — directed line segments — are mapped onto equal vectors.*

(iv) *Parallel planes are mapped onto parallel planes.*

(v) *Identical polygons are mapped onto identical polygons.*

[8] A general definition acceptable to mathematicians is not attempted here. The definition is limited to the context of deformation of a body and to a discussion of strains and deformations of a solid.

(vi) *Two successive affine transformations:* This is the product transformation, the result of two transformations, one followed by the other. Let us now restrict the coefficients c_{i0}, c_{ij}; d_{k0}, d_{ki} to small values. The mappings are now *infinitesimal affine transformations.*

(vii) *Infinitesimal affine transformations:* The coefficients are small; their products are neglected. This is the case in most engineering structures and machine parts.

Thus, such product terms can be dropped in the product transformation (the result of successive transformations). Then the coefficients in the product transformation are simply the sum of the corresponding coefficients in the two separate transformations.

(viii) *The order of the two separate transformation is unimportant.* Note particularly that we obtain the same product mapping irrespective of the order of the two separate component transformations.

This has consequences in the linear, infinitesimal, theory of elasticity. The final result of combined loading on a body is independent of the order of application of the two separate component loadings. This property is an important consequence of the linearity of the transformation and the associated principle of superposition.

(ix) *Several successive infinitesimal affine transformations.* The same result is true for more number of successive, infinitesimal, affine transformations also.

Having discussed affine transformations, let us return to Eqs (6.20a)-(6.20c).

STRAINS AND STRAIN TRANSFORMATIONS

(a) A rectangular parallelepiped (b) Shape after deformation

Figure 6.4: A brick shaped block [Fig. 6.4a] is deformed to the shape shown in Fig. 6.4b

Fig. 6.4 shows a small block before and after deformations: AB, AP, $AD, \cdots \rightarrow A^*B^*$, A^*P^*, $A^*D^*, \cdots$. It is clear that parallel planes such as AB, CD are carried, on deformation, to the parallel planes such as A^*B^*, C^*D^*. Small parallelepipeds — why small? Otherwise the transformation may not be affine (linear) — are deformed as parallelepipeds. The sides of the undeformed parallelepiped [Fig. 6.4a] are chosen along the coordinate axes of lengths $dx \equiv \Delta x$, $dy \equiv \Delta y$, $dz \equiv \Delta z$. These lengths, on deformation, become dx^*, dy^*, dz^*, respectively.

Linear Strain

We now calculate the square of the deformed length $(P^*Q^*)^2 = (dx^*)^2 + (dy^*)^2 + (dz^*)^2$ using the equations (6.20a - 6.20c) and dropping quantities of higher order of small (because of the assumed 'small' deformations). This gives us

$$\begin{aligned}(P^*Q^*)^2 = &\left[1+2\left(\frac{\partial u}{\partial x}\right)_P\right](dx)^2 + \left[1+2\left(\frac{\partial v}{\partial y}\right)_P\right](dy)^2\left[1+2\left(\frac{\partial w}{\partial z}\right)_P\right](dz)^2 + \\ &2\left[\left(\frac{\partial u}{\partial y}\right)_P + \left(\frac{\partial v}{\partial x}\right)_P\right](dx)(dy) + 2\left[\left(\frac{\partial v}{\partial z}\right)_P + \left(\frac{\partial w}{\partial y}\right)_P\right](dy)(dz) + \\ &2\left[\left(\frac{\partial w}{\partial x}\right)_P + \left(\frac{\partial u}{\partial z}\right)_P\right](dz)(dx) \end{aligned} \tag{6.23}$$

If l, m, n are the direction cosines of the undeformed line element PQ,

$$l = \frac{dx}{PQ}; \quad m = \frac{dy}{PQ}; \quad n = \frac{dx}{PQ}; \qquad \longrightarrow (PQ)^2 = (dx)^2 + (dy)^2 + (dz)^2,$$

the above equation (6.23) can be recast as

$$\begin{aligned}(1+e_{PQ})^2 = 1 &+ 2\left(\frac{\partial u}{\partial x}\right)_P l^2 + 2\left(\frac{\partial v}{\partial y}\right)_P m^2 + 2\left(\frac{\partial w}{\partial z}\right)_P n^2 + 2\left[\left(\frac{\partial u}{\partial y}\right)_P + \left(\frac{\partial v}{\partial x}\right)_P\right] lm \\ &+2\left[\left(\frac{\partial v}{\partial z}\right)_P + \left(\frac{\partial w}{\partial y}\right)_P\right] mn + 2\left[\left(\frac{\partial w}{\partial x}\right)_P + \left(\frac{\partial u}{\partial z}\right)_P\right] nl.\end{aligned}$$

Expanding the left hand side, leaving out the square of the strain e_{PQ}, and simplifying, we obtain the expression for the strain in a given direction specified by the direction cosines (l, m, n) in terms of the strain components w.r.to the x, y, z coordinates.

$$\begin{aligned}e_{PQ} = &\left(\frac{\partial u}{\partial x}\right)_P l^2 + \left(\frac{\partial v}{\partial y}\right)_P m^2 + \left(\frac{\partial w}{\partial z}\right)_P n^2 + \left[\left(\frac{\partial u}{\partial y}\right)_P + \left(\frac{\partial v}{\partial x}\right)_P\right] lm \\ &+\left[\left(\frac{\partial v}{\partial z}\right)_P + \left(\frac{\partial w}{\partial y}\right)_P\right] mn + \left[\left(\frac{\partial w}{\partial x}\right)_P + \left(\frac{\partial u}{\partial z}\right)_P\right] nl.\end{aligned} \tag{6.24}$$

Shearing Strain

To obtain the shearing strain, or rather the expression for the shearing strain, we need to calculate the change is the angle between (i) the lines $PQ\,(l_1, m_1, n_1)$ and $PR\,(l_2, m_2, n_2)$, and (ii) their images $P^*Q^*\,(l_1^*, m_1^*, n_1^*)$ and $P^*R^*\,(l_2^*, m_2^*, n_2^*)$. The direction cosines are, as we can readily see, the following.

$$\begin{aligned} l_1 &= \frac{dx_1}{PQ} & m_1 &= \frac{dy_1}{PQ} & n_1 &= \frac{dz_1}{PQ} \\ l_2 &= \frac{dx_2}{PR} & m_2 &= \frac{dy_2}{PR} & n_2 &= \frac{dz_2}{PR} \\ l_1^* &= \frac{dx_1^*}{P^*Q^*} & m_1^* &= \frac{dy_1^*}{P^*Q^*} & n_1^* &= \frac{dz_1^*}{P^*Q^*} \end{aligned}$$

$$l_2^* = \frac{dx_2^*}{P^*R^*} \qquad m_2^* = \frac{dy_2^*}{P^*R^*} \qquad n_2^* = \frac{dz_2^*}{P^*R^*}$$

The first angle is easy to calculate; it is given by

$$\cos(PQ,\, PR) = l_1\, l_2 + m_1\, m_2 + n_1\, n_2.$$

The second angle is similarly given by

$$\cos(P^*Q^*,\, P^*R^*) = l_1^*\, l_2^* + m_1^*\, m_2^* + n_1^*\, n_2^* = \frac{dx_1^*\, dx_2^* + dy_1^*\, dy_2^* + dz_1^*\, dz_2^*}{(P^*Q^*)(P^*R^*)}$$

We have already worked out the expressions for dx^*, dy^*, dz^* [Eqs (6.20a) - (6.20c)]. It is merely a question of substituting these (rather long) expressions, neglecting quantities of a higher order of smallness, and cleaning up the expressions. When these steps are carried out, we obtain the expression for the second angle $(P^*Q^*,\, P^*R^*)$.

$$\begin{aligned}\gamma_{(PQ,\,PR)} &= (1 - e_{PQ} - e_{PR})\,[(l_1\, l_2 + m_1\, m_2 +\; n_1\, n_2) + 2(e_{xx}\, l_1\, l_2 + e_{yy}\, m_1\, m_2 + e_{zz}\, n_1\, n_2)] \\ &\quad + (1 - e_{PQ} - e_{PR})\,[\gamma_{xy}(l_1\, m_2 + l_2\, m_1) + \gamma_{yz}(m_1\, n_2 + m_2\, n_1) + \gamma_{zx}(n_1\, l_2 + n_2\, l_1)]\end{aligned}$$

But this analysis is too general. The procedure outlined above tells us the change in angle between two elemental line elements in two *arbitrary directions*. We do not need results with such generality. It is sufficient if we can compute the change in angle when the elemental lines $(PQ,\, PR)$ are at right angles. For this case, there is considerable simplification. Now in the last equation $(l_1\, l_2 + m_1\, m_2 +\; n_1\, n_2) = 0$, because PQ and PR are at right angles.

Thus, collecting all the information that we have gathered, the strain transformation equations are obtained as follows. We would do well to change over from γ_{xy}, etc. to $2e_{xy}$, etc. (to preserve the tensor character in the transformation equations).

If $(l_1,\, m_1,\, n_1)$; $(l_2,\, m_2,\, n_2)$; $(l_3,\, m_3,\, n_3)$ are the direction cosines of the orthogonal $x',\, y',\, z'$ axes w.r.to the $(x,\, y,\, z)$ axes, the strain transformation laws are:

$$e'_{x'x'} = l_1^2\, e_{xx} + m_1^2\, e_{yy} + n_1^2\, e_{zz} + 2l_1\, m_1\, \frac{\gamma_{xy}}{2} + 2m_1\, n_1\, \frac{\gamma_{yz}}{2} + 2n_1\, l_1\, \frac{\gamma_{zx}}{2} \tag{6.25}$$

$$\frac{\gamma'_{x'y'}}{2} = l_1\, l_2\, e_{xx} + m_1\, m_2\, e_{yy} + n_1\, n_2\, e_{zz} + (l_1\, m_2 + l_2\, m_1)\, \frac{\gamma_{xy}}{2} \tag{6.26}$$

$$+ (m_1\, n_2 + m_2\, n_1)\, \frac{\gamma_{yz}}{2} + (n_1\, l_2 + n_2\, l_1)\, \frac{\gamma_{zx}}{2} \tag{6.27}$$

The companion equations can be written down using cyclic changes: $x \rightarrow y$; $y \rightarrow z$; $z \rightarrow x$. They may be written compactly as

$$e'_{ij} = a_{ik}\, a_{jl}\, e_{kl} \qquad \text{and / or} \qquad [e'] = [a][e][a]^T.$$

We can see that all this is analogous to the stress transformation equations. Let us repeat for emphasis: use e_{xy}, etc. instead of γ_{xy}, etc. In the equations shown above, $\gamma_{xy}/2$, etc. are used so that the equations are in good shape to be rewritten in terms of e_{xy}, etc. We may remark here that all the transformation equations valid for stresses are equally valid for strains also with this one difference indicated about the factor of 1/2.

FURTHER BEYOND

Having established that the strain is another example of a second order tensor, we can borrow all the results from the analysis of stress at a point. Thus, for example, principal stresses and principal planes will now, in the context of strains, be referred to as principal strains and the corresponding principal directions[9]. A few minor changes may have to be made to suit the context and the physical meanings of the terms. It is the geometry of deformations that is considered here, whereas it was the equilibrium of forces that was the basis for deriving the equations in the theory of stresses. An example is given below.

Spherical and Deviatoric Parts of the Strain Tensor

Recall that the stress tensor represented as a matrix may be written as the sum of two matrices, the first one representing a hydrostatic stress (all the eigenvalues are equal, spherical part) and the other a state of pure shear (the first invariant vanishes, deviatoric part). Here they correspond to the two components of the deformation: (i) only volume change, no change in shape — spherical part, there are no off-diagonal terms representing (half) the shearing strains — and (ii) no volume change, only change in shape or distortion (deviatoric part, the first invariant representing volumetric strain is zero.)[10]

$$\text{volumetric strain } \; e_v = e_{ii} = e_{11} + e_{22} + e_{33} = 0.$$

Some minor changes may have to be made to suit the context. The decomposition into the pure shear (deviatoric part) and the hydrostatic (spherical part) is, in index notation

$$e_{ij} = \left[e_{ij} - \frac{1}{3}\,\delta_{ij}\,e_{kk} \right] + \left[\frac{1}{3}\,\delta_{ij}\,e_{kk} \right].$$

Principal Strains and Principal Directions

Eq. (6.25) tells us the linear strain in a direction defined by the direction cosines l, m, n. Now we might enquire what its maximum / minimum (in general, stationary values) value is. This leads us to the principal strains and the corresponding directions to the associated principal directions. We can begin with Eq. (6.25) and find the maximum / minimum under the constraint $(l^2 + m^2 + n^2 - 1 = 0)$. This problem is exactly similar to what we discussed earlier on p. 4-31, and for exactly the same reason, we do not proceed further.

There are two important topics that follow, or ought to follow, the analysis of strain at a point. These are the strain-displacement relations and the compatibility conditions. These are discussed later in the book.

We shall discuss constitutive equations in the next chapter.

[9] Recall that we had stated as an aside in a lighter vein that it is the same story — the same events, the same ending, everything is the same — except the names of the characters. This is particularly true here.

[10] Incompressibility is an important consideration in several places. Plastic flow is associated with no volume change.

Chapter 7

CONSTITUTIVE EQUATIONS

This chapter refers to the constitutive equations, known also as the material laws. These are the equations that refer to the properties of the *material*[1]. Elaborate theories have been developed. Our scope of investigation is limited to elastic materials. Several approaches are possible, but we shall first see the most natural and simplest of them. The appropriate equations relevant to our context are Hooke's law[2], and its extension to three dimensions called the generalised Hooke's law.

HOOKE'S LAW AND THE GENERALISED HOOKE'S LAW

It seems to be easiest to begin with Hooke's law which when applied to our context of one-dimensional stresses and strains is $e = \sigma/E$, where E is the Young's modulus[3] of elasticity. When this is extended to three dimensions, we begin with the equations

$$e_{xx} = \frac{\sigma_{xx}}{E}; \qquad e_{yy} = \frac{\sigma_{yy}}{E}; \qquad e_{zz} = \frac{\sigma_{zz}}{E}.$$

But then there is the Poisson's effect and the associated Poisson's ratio. Thus, the strains would be modified as

$$e_{xx} = \frac{\sigma_{xx}}{E} - \nu\left(\frac{\sigma_{yy}}{E} + \frac{\sigma_{zz}}{E}\right); \quad e_{yy} = \frac{\sigma_{yy}}{E} - \nu\left(\frac{\sigma_{zz}}{E} + \frac{\sigma_{xx}}{E}\right); \quad e_{zz} = \frac{\sigma_{zz}}{E} - \nu\left(\frac{\sigma_{xx}}{E} + \frac{\sigma_{yy}}{E}\right).$$

---

1 strictly, not so much to the material, but to the model that we choose

2 Named after Robert Hooke (July 1635 - March 1703), British scientist and natural philosopher. To quote from Den Hartog [3]: "$\cdots$ This law was first enunciated by Robert Hooke (1635 - 1703) in Elizabethan England in connection with his invention of applying a hairspring to a watch or clock". $\cdots$ "Hooke, after the manner of his age, published his invention in the shape of a riddle, or "anagram" *ceiiinosssttuv*, which, when unscrambled, was supposed to read *Ut tensio sic vis*, Latin for " as the extension, so is the force".

Hooke had three entirely different stages in life: (i) the early years when he had no money, (ii) the middle years when he had lots and lots of money and fame, and (iii) the later years when he had indifferent health.

3 Named after Thomas Young (June 1773 - May 1829), English physicist and physician. It appears that the concept was used 25 years earlier by G. Riccati, and that the idea can be traced to one of Euler's papers 80 years before Young's publication in 1807. Some people considered, or perhaps still consider, Young to be the last person "who knew everything" although that honour is usually given to Leibnitz.

Just as the linear stress and linear strain are related by the Young's modulus of elasticity (E), the shear stress and the shear strain are related by the modulus of rigidity (G). Thus,

$$\gamma_{xy} = 2\, e_{xy} = \frac{\tau_{xy}}{G}; \qquad \gamma_{yz} = 2\, e_{yz} = \frac{\tau_{yz}}{G}; \qquad \gamma_{zx} = 2\, e_{zx} = \frac{\tau_{zx}}{G}.$$

There is no Poisson's effect[4] here. Thus, combining the above equations, we obtain the generalised Hooke's law as

$$e_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right]; \tag{7.1a}$$
$$e_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{zz} + \sigma_{xx}\right)\right]; \tag{7.1b}$$
$$e_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right]; \tag{7.1c}$$
$$e_{xy} = \frac{1}{2}\,\gamma_{xy} = \frac{1}{2}\,\frac{\tau_{xy}}{G} = \frac{1}{2}\,\gamma_{yx} = e_{yx}; \tag{7.1d}$$
$$e_{yz} = \frac{1}{2}\,\gamma_{yz} = \frac{1}{2}\,\frac{\tau_{yz}}{G} = \frac{1}{2}\,\gamma_{zy} = e_{zy}; \tag{7.1e}$$
$$e_{zx} = \frac{1}{2}\,\gamma_{zx} = \frac{1}{2}\,\frac{\tau_{zx}}{G} = \frac{1}{2}\,\gamma_{xz} = e_{xz}. \tag{7.1f}$$

The numerical values of these (elastic) constants are to be obtained experimentally in a material testing laboratory[5]. This is one approach to obtain the generalised Hooke's law. But this is not entirely satisfactory. Other approaches may shed more light on the topic on hand. We shall examine an alternative approach.

AN ALTERNATIVE APPROACH

We shall start by stating that the stress is a function of strain. This function obviously must be continuous. Thus we may write $\sigma = \sigma(e)$. Now we know that a continuous function may be written in the form

$$\sigma = \sigma(e) = a_o + a_1\, e + a_2\, e^2 + a_3\, e^3 + \cdots + a_n\, e^n,$$

where the larger the number of terms, the better will be the approximation. We can have any degree of approximation. This fact is intuitively clear, but there is a formal proof.

[4] Named after Siméon Denis Poisson (June 1781 - April 1840), the famous French mathematician and physicist. He obtained (1829) the value of Poisson's ratio, not by experiments in the lab, but a mathematical / physical argument.

Mild steel has a Poisson's ratio of about 0.3. Concrete has a Poisson's ratio between 0.1 and 0.2 and aluminium about 0.32. Rubber has a value nearly equal to 0.5, while cork has a value that is almost zero. Poisson's ratio cannot be less than -1.0 or greater than 0.5. There are some materials that have negative values of Poisson's ratio. Such materials are called auxetic materials. Recently strange cases have been reported.

[5] There are several details concerning such experiments. We cannot discuss them here. Most of the readers, we hope, would have performed such experiments, at least the straightforward simple ones, as part of their earlier studies.

(There is a theorem called the Stone-Weierstrass theorem[6] that justifies this.) Next we argue that the body is in a 'natural state'. That is, when there is no strain, there is no stress[7]. This means that the constant a_0 vanishes ($a_0 = 0$). Thus, the above equation is simplified to the form

$$\sigma = \sigma(e) = a_1\, e + a_2\, e^2 + a_3\, e^3 + \cdots + a_n\, e^n,$$

where n is usually limited to one or two. (The departure from linearity is usually only slight; there is generally no need to take higher powers.) The consequence is that the stress strain curve now passes through the origin.

Here we are discussing only the one-dimensional case, when both σ and e are but numbers. Such a material law can be extended to the general three-dimensional case by simply regarding the above as a matrix equation, when it reads as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{e}) = \boldsymbol{a_1}\, \boldsymbol{e} + \boldsymbol{a_2}\, \boldsymbol{e}^2 + \boldsymbol{a_3}\, \boldsymbol{e}^3 + \cdots + \boldsymbol{a_n}\, \boldsymbol{e}^n.$$

Here $\boldsymbol{\sigma}$ and $\boldsymbol{e}$ are, in a three-dimensional setting, 3×3 matrices. There is a well known theorem called the Cayley-Hamilton theorem[8] that permits the higher degree terms to be expressed in terms of $\boldsymbol{e}$ and $\boldsymbol{e}^2$. When this is done, the above equation appears as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{e}) = \boldsymbol{c_1}\, \boldsymbol{e} + \boldsymbol{c_2}\, \boldsymbol{e}^2.$$

Thus, we do not need to consider higher orders of $\boldsymbol{e}$ even in the general case. This constitutive equation is nonlinear. This kind of nonlinearity is called material nonlinearity in contrast to the other kind called geometric nonlinearity. The latter refers to the nonlinear relationship in the strain-displacement equations.

Let us decide to consider only the simple case of linear theory[9], when the above equation simplifies further as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{e}) = \boldsymbol{D}\, \boldsymbol{e}, \tag{7.2}$$

where $\boldsymbol{\sigma}$, $\boldsymbol{e}$, and $\boldsymbol{D}$ are matrices. We shall take Eq. (7.2) as the starting point for further discussions.

[6] Named after Marshall H. Stone (April 1903 - Jan. 1989), American mathematician and Karl Weierstrass (Oct. 1816 - Feb. 1897), a German mathematician regarded as the father of modern rigour in mathematics. Weierstrass' approximation theorem — every continuous function defined on a closed interval $[a,\, b]$ can be uniformly approximated as closely as desired by a polynomial function — was generalised by Stone. He also simplified the proof.

[7] There can be situations where (i) there is stress when there is no strain, and (ii) there is no stress, there is strain. These are, shall we say, exceptional cases that we need not take into account in our context.

[8] Named after the British lawyer-mathematician Arthur Cayley (Aug. 1821 - Jan. 1895) and the Irish mathematician and astronomer William Rowan Hamilton (Aug. 1805 - Sept. 1865). The theorem states that every square matrix satisfies its own characteristic equation in the matrix sense. Thus, if $\boldsymbol{A}$ is a 3×3 matrix,

$$\boldsymbol{A}^3 - (...)\, \boldsymbol{A}^2 + (...)\, \boldsymbol{A} - (...)\, \boldsymbol{I} = \boldsymbol{0},$$

making it possible to express $\boldsymbol{A}^3$ in terms of $\boldsymbol{A}^2$, $\boldsymbol{A}$ and $\boldsymbol{I}$.

[9] Neither material nonlinearity nor geometric nonlinearity is permitted in this simpler theory.

Further Discussion

We begin by taking the above equation (7.2) in matrix form as

$$\{\sigma\} = [D]\,\{e\}, \qquad \text{i.e.,} \qquad \sigma_i = \sum_{j=1}^{6} D_{ij}\, e_j \quad (j = 1, \cdots, 6), \tag{7.3}$$

where $[D]$ is called the stiffness matrix. This may be inverted to yield

$$\{e\} = [C]\,\{\sigma\}, \qquad \text{i.e.,} \qquad e_i = \sum_{j=1}^{6} C_{ij}\, e_j \quad (e = 1, \cdots, 6), \tag{7.4}$$

where $[C]$ is known as the compliance matrix. The matrices $[C]$ and $[D]$ are related to each other as

$$[C] = [D]^{-1} \qquad \text{and / or} \qquad [D] = [C]^{-1},$$

which makes it appropriate to refer to $[D]$ as the stiffness matrix. Both D and C may be called constitutive matrices. [Some authors use other notations also, which is but natural.] There are a number of 'constants' (or coefficients, or moduli) in these matrices. How many, that is the question. We shall examine.

How Many Elastic Constants?

In the light of what we have seen so far, let us begin by assuming that each stress component is *linearly* related to each of the strain components. Then we will have

$$\left\{\begin{matrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \cdots \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xz} \end{matrix}\right\} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & \cdots & D_{18} & D_{19} \\ D_{21} & D_{22} & D_{23} & \cdots & D_{28} & D_{29} \\ D_{31} & D_{32} & D_{33} & \cdots & D_{38} & D_{39} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ D_{71} & D_{72} & D_{73} & \cdots & D_{78} & D_{79} \\ D_{81} & D_{82} & D_{83} & \cdots & D_{88} & D_{89} \\ D_{91} & D_{92} & D_{93} & \cdots & D_{98} & D_{99} \end{bmatrix} \left\{\begin{matrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \cdots \\ e_{yz} \\ e_{zx} \\ e_{xz} \end{matrix}\right\}, \tag{7.5}$$

where there are $9 \times 9 = 81$ elastic constants. These are far too many; we cannot deal with so many in the solution of any real problem. Furthermore, how do we evaluate the numerical values of these elastic constants by experiments? We must, thus, look for reasons, arguments and procedures to cut them down to a manageably small number. How shall we proceed?

Let us note that both the stress and strain matrices are symmetric. Then the number of stress and strain components will be reduced from 9 to 6. Then we will have $6 \times 6 = 36$ constants.

$$\left\{\begin{matrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{matrix}\right\} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix} \left\{\begin{matrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{xy} \\ e_{yz} \\ e_{zx} \end{matrix}\right\}, \tag{7.6}$$

But this also is too large a number. Can we reduce this further? Yes, this is possible, but only if we make further assumptions.

Further Reduction

To have further reduction in the number of elastic constants, we need to make use of the concept of a strain energy density function, $\mathcal{U} = \mathcal{U}(e_{ij})$. This relates the stress components and the strain components (constitutive equation)[10] by

$$\sigma_{ij} = \frac{\partial \mathcal{U}}{\partial e_{ij}} \quad (i,\, j = 1,\, 2,\, 3).$$

The strain energy U is the integral of the strain energy density function $\mathcal{U}$ integrated over the entire volume $\mathcal{V}$ concerned:

$$U = U(e_{ij}) = \iiint_{\mathcal{V}} \mathcal{U}(e_{ij})\, d\mathcal{V}.$$

This strain energy function U is expressed as

$$U = \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} D_{ijkl}\, e_{ik}\, e_{kl} = \frac{1}{2} \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} D_{\alpha\beta}\, e_\alpha\, e_\beta = \frac{1}{2}\, \{e\}^T\, [D]\, \{e\},$$

from which we note that

$$\frac{\partial^2 U}{\partial e_\alpha\, \partial e_\beta} = D_{\alpha\beta} = D_{\beta\alpha}, \quad \text{since} \quad \frac{\partial^2 U}{\partial e_\alpha\, \partial e_\beta} = \frac{\partial^2 U}{\partial e_\beta\, \partial e_\alpha}.$$

Thus, we note that the stiffness matrix (constitutive matrix) D is symmetric. The number of elastic constants is, thus, reduced from 36 to 21. We have so far made no assumption at all about isotropy. Hence, we conclude that the most general anisotropic material — anisotropic model — has 21 elastic constants[11].

We have so far reduced the number of elastic constants in stages as

$$81 \quad \longrightarrow \quad 36 \quad \longrightarrow \quad 21.$$

[10] This can be considered as a special case of the more general thermoelastic constitutive equation. Thermodynamics also enters here. For two special cases, (i) isothermal and (ii) reversible adiabatic (isentropic) processes, this function can be identified or interpreted as (i) Gibbs' free energy, and (ii) the internal energy function. To understand these we need to consult advanced books.

This strain energy function U enjoys the property of positive definiteness which has important consequences. Some of them are the uniqueness theorem of linear theory of elasticity, and the principles of minimum total potential (energy) and minimum complementary energy.

[11] Lekhnitskii, S.G. *Theory of Elasticity of an Anisotropic Body*, Mir Publishers, Moscow, translated from Russian, (1981), states "In (the) general case of anisotropy the number of elastic constants is 21, but among them the *independent* constants are fewer." $\cdots$ "Consequently, even in the most general case, the number of independent elastic constants is not 21, but fewer, viz., 18." He has more to say on this.

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{12} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{13} & D_{23} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{14} & D_{24} & D_{34} & D_{44} & D_{45} & D_{46} \\ D_{15} & D_{25} & D_{35} & D_{45} & D_{55} & D_{56} \\ D_{16} & D_{26} & D_{36} & D_{46} & D_{56} & D_{66} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{xy} \\ e_{yz} \\ e_{zx} \end{Bmatrix}, \tag{7.7}$$

Crystals are the most important real bodies that exhibit anisotropy[12]. Composite, plywood, wood and rock have decidedly anisotropic elastic properties. Besides, metallic components can have cylindrical anisotropy because of the technological processes of producing them like extrusion and wire drawing. But we will not discuss such matters here.

We have seen rectilinear anisotropy. Here all parallel directions are elastically equivalent. But this is not the case always. One case of wood is mentioned later when we discuss transtropy. Some bodies may have curvilinear anisotropy. Among various possibilities, Saint-Venant long, long ago had studied (i) cylindrical and (ii) spherical anisotropy. These are of great practical interest.

The generalised Hooke's law for a material exhibiting curvilinear anisotropy can be written as Eq. (7.7) with reference to a rectangular cartesian coordinate system (x, y, z). But the trouble is that the constitutive coefficients (constitutive moduli) are now not constants. Their values change from point to point *even if the body is homogeneous* due to changes in the coordinate directions. In a general curvilinear coordinate system (ξ, η, ζ), the constitutive equation can be written in the form

$$\begin{Bmatrix} \sigma_{\xi\xi} \\ \sigma_{\eta\eta} \\ \sigma_{\zeta\zeta} \\ \tau_{\xi\eta} \\ \tau_{\eta\zeta} \\ \tau_{\zeta\xi} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{12} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{13} & D_{23} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{14} & D_{24} & D_{34} & D_{44} & D_{45} & D_{46} \\ D_{15} & D_{25} & D_{35} & D_{45} & D_{55} & D_{56} \\ D_{16} & D_{26} & D_{36} & D_{46} & D_{56} & D_{66} \end{bmatrix} \begin{Bmatrix} e_{\xi\xi} \\ e_{\eta\eta} \\ e_{\zeta\zeta} \\ e_{\xi\eta} \\ e_{\eta\zeta} \\ e_{\zeta\xi} \end{Bmatrix},$$

as before. There are 21 elastic constants[13]. The above constitutive equation can be simplified exactly as in the case of rectilinear anisotropy.

One Plane of Elastic Symmetry

Further, if we assume that there is one plane of elastic symmetry, the number of elastic constants is further reduced to 13. We shall see this reduction below.

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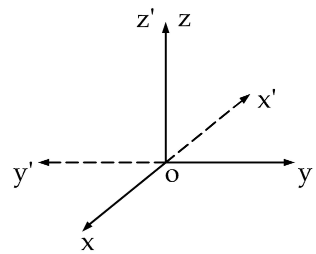
[12] It appears that all natural crystals have one or the other of 32 kinds of symmetry. Their elastic properties can be only in one of the nine types. But we are not interested in crystals.

[13] See the footnote on the previous page.



New coordinates		Old coordinates		
		x_1	x_2	x_3
	x_1'	a_{11}	a_{12}	a_{13}
	x_2'	a_{21}	a_{22}	a_{23}
	x_3'	a_{31}	a_{32}	a_{33}

The figure shows a set of ('old') axes $(x,\, y,\, z)$. These axes are changed to a new set of axes $(x',\, y',\, z')$ by rotating $(x,\, y,\, z)$ through 180° about the vertical z (also z') axis. Elastic symmetry about one plane implies that the constitutive matrix $[D]$ will still remain the same. Referred to the new axes, the stress and strain coordinates will obviously change. These changes can be calculated; we have already seen such calculations when the axes are changed from the 'old' $(x,\, y,\, z)$ system to the 'new' $(x',\, y',\, z')$ one. Here these results can be obtained merely by inspection. Alternatively, these can be calculated using the stress and strain transformation equations. In either case, the result is

$$(x,y,z) \;\longrightarrow\; (x',\, y',\, z'): \qquad [\sigma'] = [a]\,[\sigma]\,[a]^T; \qquad [e'] = [a]\,[e]\,[a]^T;$$
$$\sigma'_{x'x'} = \sigma_{xx}; \quad \sigma'_{y'y'} = \sigma_{yy}; \quad \sigma'_{z'z'} = \sigma_{zz}; \quad \tau'_{x'y'} = \tau_{xy}; \quad \tau'_{y'z'} = -\tau_{yz}; \quad \tau'_{z'x'} = -\tau_{zx};$$
$$e'_{x'x'} = e_{xx}; \quad e'_{y'y'} = e_{yy}; \quad e'_{z'z'} = e_{zz}; \quad e'_{x'y'} = e_{xy}; \quad e'_{y'z'} = -e_{yz}; \quad e'_{z'x'} = -e_{zx}.$$

Let us realise that the constitutive matrix relating (i) σ_{ij} and e_{ij}, and (ii) $\sigma'_{i'j'}$ and $e'_{i'j'}$ is the same; this is what elastic symmetry (about one plane) implies. Thus, we have

$$\begin{Bmatrix} \sigma'_{x'x'} \\ \sigma'_{y'y'} \\ \sigma'_{z'z'} \\ \tau'_{x'y'} \\ \tau'_{y'z'} \\ \tau'_{z'x'} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{12} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{13} & D_{23} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{14} & D_{24} & D_{34} & D_{44} & D_{45} & D_{46} \\ D_{15} & D_{25} & D_{35} & D_{45} & D_{55} & D_{56} \\ D_{16} & D_{26} & D_{36} & D_{46} & D_{56} & D_{66} \end{bmatrix} \begin{Bmatrix} e'_{x'x'} \\ e'_{y'y'} \\ e_{z'z'} \\ e'_{x'y'} \\ e'_{y'z'} \\ e_{z'x'} \end{Bmatrix}; \tag{7.8}$$

$$\text{i.e.,} \quad \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ -\tau_{yz} \\ -\tau_{zx} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{12} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{13} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{14} & D_{24} & D_{34} & D_{44} & D_{45} & D_{46} \\ D_{15} & D_{25} & D_{35} & D_{45} & D_{55} & D_{56} \\ D_{16} & D_{26} & D_{36} & D_{46} & D_{56} & D_{66} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ -e_{xy} \\ -e_{yz} \\ e_{zx} \end{Bmatrix}; \tag{7.9}$$

$$\text{i.e.,} \quad \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & -D_{15} & -D_{16} \\ D_{12} & D_{22} & D_{23} & D_{24} & -D_{25} & -D_{26} \\ D_{13} & D_{23} & D_{33} & D_{34} & -D_{35} & -D_{36} \\ D_{14} & D_{24} & D_{34} & D_{44} & -D_{45} & -D_{46} \\ -D_{15} & -D_{25} & -D_{35} & -D_{45} & D_{55} & D_{56} \\ -D_{16} & -D_{26} & -D_{36} & -D_{46} & D_{56} & D_{66} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{xy} \\ e_{yz} \\ e_{zx} \end{Bmatrix}; \tag{7.10}$$

Comparing Eqs (7.7) and (7.10), we conclude that

$$D_{15} = D_{25} = D_{35} = D_{46} = D_{16} = D_{26} = D_{36} = D_{46} = 0.$$

The constitutive matrix for a material — really a model — with one plane of symmetry is thereby simplified, and the constitutive equation appears as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & 0 & 0 \\ D_{12} & D_{22} & D_{23} & D_{24} & 0 & 0 \\ D_{13} & D_{23} & D_{33} & D_{34} & 0 & 0 \\ D_{14} & D_{24} & D_{34} & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & D_{56} \\ 0 & 0 & 0 & 0 & D_{56} & D_{66} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{xy} \\ e_{yz} \\ e_{zx} \end{Bmatrix}, \tag{7.11}$$

There are now only 13 elastic constants.

Two Planes of Elastic Symmetry

By an argument similar to the one used above, it is possible to show that there are now nine (9) elastic constants. The constitutive equation looks as shown below.

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ D_{12} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{13} & D_{23} & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{xy} \\ e_{yz} \\ e_{zx} \end{Bmatrix}, \tag{7.12}$$

Three Planes of Symmetry: Orthotropy

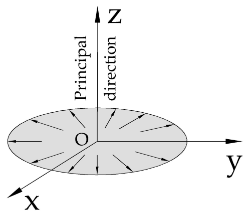
Two planes of symmetry implies three planes of symmetry! A little reflection would perhaps help us to have some kind of (intuitive?) understanding. There is a formal proof[14]. This is the case of orthotropic elasticity. The constitutive equation has the same appearance as above; there are now nine (9) elastic constants. Orthotropy has taken on special significance in relatively recent years when composites are used more and more.

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ D_{12} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{13} & D_{23} & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{xy} \\ e_{yz} \\ e_{zx} \end{Bmatrix}, \tag{7.13}$$

Transverse Isotropy (Transtropy)

Consider a material at each point of which there is a principal direction of elastic symmetry (along the z axis in the figure). In a plane (xy) normal to the principal direction (z-axis)

[14] Symmetry in a general sense is a deep and rather abstract concept. These concepts are all exploited in studying crystal symmetry. Group theory enters into these discussions. We cannot discuss all this here.



(a) Transtropy

(b) Wood

Figure 7.1: To discuss transtropy: The principal direction is the z axis. On the plane normal to this principal direction, that is, on the xy plane (shown shaded), there is isotropy. That is, the elastic properties are the same in all directions in this xy plane. Wood [Fig. 7.1b] is sometimes modelled as a transtropic material. The cross-section of wood shown corresponds to the xy plane where the elastic properties are the same in all directions. [Cylindrical polar coordinates can be used with advantage in such cases.]

— here xy is the plane normal to the principal direction (z axis) — there is isotropy. Such a special simplified case of anisotropy is referred to as transverse isotropy, or transtropy, and the material is said to be transversely isotropic or transtropic.

Wood is an example of a transtropic material. It is sometimes modelled as transtropic. The direction of the grains is the one principal direction. There are an infinite number of principal directions in the plane normal to the direction of the grains. The material has the same elastic properties, and hence isotropy, in all directions on the cross-section shown in Fig. 7.1b and on the shaded cross-section xy in Fig. 7.1a. [It is advantageous to exploit this property by using cylindrical polar coordinates in such cases.]

Now the constitutive equation can be seen to be

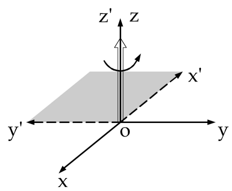
$$\begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{xy} \\ e_{yz} \\ e_{zx} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & (C_{11}-C_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}.$$

There are now only five (5) independent elastic 'constants'.

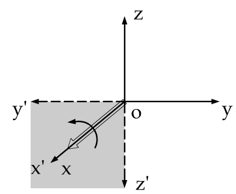
This is as far as we can go unless we assume isotropy. Isotropy means that the elastic properties are the same in all directions. We shall now demonstrate how the number of elastic constants can be further reduced if we assume that the material is isotropic. We shall see that there are three elastic constants now, but that even these three are not independent. There are only two independent elastic constants if the material is isotropic.

$$81 \longrightarrow 36 \longrightarrow 21 \longrightarrow 13 \longrightarrow 9 \longrightarrow 5 \longrightarrow 2$$

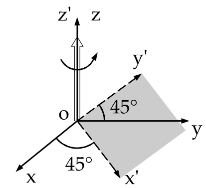
In the next section, we shall discuss this most important case of isotropic materials.



(a) A coordinate transformation



(b) Another coordinate transformation



(c) Yet another coordinate transformation

Figure 7.2: To discuss isotropy: three pairs of coordinate systems, and the corresponding transformations, are considered.

ISOTROPIC MATERIALS

What does isotropy imply? Well, it means that the elastic properties are the same *in all directions* at the same point. We shall discuss this in detail.

First Possibility

We may write the constitutive equation in tensor form as

$$\sigma_{ij} = D_{ijkl}\, e_{kl} \tag{7.14}$$

where D_{ijkl} is a fourth order tensor. [We may write this, as some authors do, as

$$\boldsymbol{D} = D_{ijkl}\, \boldsymbol{e}_i\, \boldsymbol{e}_j\, \boldsymbol{e}_k\, \boldsymbol{e}_l$$

emphasising the difference between the tensor $\boldsymbol{D}$ and its components D_{ijkl}.] Accordingly, these components transform, on change of coordinates as

$$D'_{ijkl} = a_{ip}\, a_{jq}\, a_{kr}\, a_{ls}\, D_{pqrs}. \tag{7.15}$$

When the material is isotropic, $D'_{ijkl} = D_{ijkl}$. We shall not proceed with this any further.

Second Possibility

We shall consider five (5) sets of coordinate transformations. The basic idea is what we have employed earlier.
Step 1:
Let us change the coordinate axes (x, y, z) to (x', y', z'), and see how (i) the stress components, (ii) the strain components, and (iii) the constitutive relationships change [Fig. 7.2a]. We can see just by inspection or, if we are too dependent on transformation equations, the following.

$$\begin{array}{lll}
\sigma'_{x'x'} = \sigma_{xx} & \sigma'_{y'y'} = \sigma_{yy} & \sigma'_{z'z'} = \sigma_{zz} \\
\tau'_{x'y'} = \tau_{xy} & \tau'_{y'z'} = -\tau_{yz} & \tau'_{z'x'} = -\tau_{zx} \\
e'_{x'x'} = e_{xx} & e'_{y'y'} = e_{yy} & e'_{z'z'} = e_{zz} \\
e'_{x'y'} = e_{xy} & e'_{y'z'} = -e_{yz} & e'_{z'x'} = -e_{zz}
\end{array}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{14} & D_{24} & D_{34} & D_{44} & D_{45} & D_{46} \\ D_{15} & D_{25} & D_{35} & D_{45} & D_{55} & D_{56} \\ D_{16} & D_{26} & D_{36} & D_{46} & D_{56} & D_{66} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{xy} \\ e_{yz} \\ e_{zx} \end{Bmatrix} \tag{7.16}$$

$$\begin{Bmatrix} \sigma'_{x'x'} \\ \sigma'_{y'y'} \\ \sigma'_{z'z'} \\ \tau'_{x'y'} \\ \tau'_{y'z'} \\ \tau'_{z'x'} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{14} & D_{24} & D_{34} & D_{44} & D_{45} & D_{46} \\ D_{15} & D_{25} & D_{35} & D_{45} & D_{55} & D_{56} \\ D_{16} & D_{26} & D_{36} & D_{46} & D_{56} & D_{66} \end{bmatrix} \begin{Bmatrix} e'_{x'x'} \\ e'_{y'y'} \\ e'_{z'z'} \\ e'_{x'y'} \\ e'_{y'z'} \\ e'_{z'x'} \end{Bmatrix} \tag{7.17}$$

The same stiffness matrix appears in both equations (7.16 and 7.17) because the material is isotropic. As

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{Bmatrix} \sigma'_{x'x'} \\ \sigma'_{y'y'} \\ \sigma'_{z'z'} \\ \tau'_{x'y'} \\ -\tau'_{y'z'} \\ -\tau'_{z'x'} \end{Bmatrix} \qquad \text{and} \qquad \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{xy} \\ e_{yz} \\ e_{zx} \end{Bmatrix} = \begin{Bmatrix} e'_{x'x'} \\ e'_{y'y'} \\ e'_{z'z'} \\ e'_{x'y'} \\ -e'_{y'x'} \\ -e'_{z'x'} \end{Bmatrix},$$

it follows, on comparing the two stiffness matrices after making the necessary changes, that

$$D_{15} = D_{16} = D_{25} = D_{26} = D_{35} = D_{36} = D_{45} = D_{46} = 0;$$
$$D_{51} = D_{52} = D_{53} = D_{54} = D_{61} = D_{62} = D_{63} = D_{64} = 0.$$

With these simplifications, the constitutive equation becomes

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & 0 & 0 \\ D_{12} & D_{22} & D_{23} & D_{24} & 0 & 0 \\ D_{13} & D_{23} & D_{33} & D_{34} & 0 & 0 \\ D_{14} & D_{24} & D_{34} & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & D_{56} \\ 0 & 0 & 0 & 0 & D_{56} & D_{66} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{xy} \\ e_{yz} \\ e_{zx} \end{Bmatrix}, \tag{7.18}$$

Step 2:
We shall now consider another coordinate transformation. The coordinate system is rotated about the Ox axis by 180°. Proceeding as before, we can conclude that

$$D_{14} = D_{24} = D_{34} = D_{41} = D_{42} = D_{43} = D_{56} = 0.$$

Step 3:
A rotation of the coordinate system about the Ox axis by 90° leads, by the same argument, to the result

$$D_{12} = D_{13}; \qquad D_{21} = D_{31}; \qquad D_{22} = D_{33}.$$

Step 4:
A similar rotation of the coordinate system by 90°, but this time about the Oz axis gives us the following simplification:

$$D_{13} = D_{23}; \qquad D_{11} = D_{22}; \qquad D_{44} = D_{55}.$$

The generalised Hooke's law has now become simpler. They now read as

$$\sigma_{xx} = D_{11}\, e_{xx} + D_{12}\,(e_{yy} + e_{zz}) \tag{7.19a}$$
$$\sigma_{yy} = D_{11}\, e_{yy} + D_{12}\,(e_{zz} + e_{xx}) \tag{7.19b}$$
$$\sigma_{zz} = D_{11}\, e_{zz} + D_{12}\,(e_{xx} + e_{yy}) \tag{7.19c}$$
$$\tau_{xy} = D_{44}\, e_{xy} \tag{7.19d}$$
$$\tau_{yz} = D_{44}\, e_{yz} \tag{7.19e}$$
$$\tau_{zx} = D_{44}\, e_{zx} \tag{7.19f}$$

Step 5:
As the fifth and last step, let us rotate the coordinate system about the Oz axis through 45° [Fig. 7.2c]. From the stress / strain transformation laws we can work out how the 'new' stress / strain components are related to the 'old' ones. The results are:

$$\begin{Bmatrix} \sigma'_{x'x'} \\ \sigma'_{y'y'} \\ \sigma'_{z'z'} \\ \tau'_{x'y'} \\ \tau'_{y'z'} \\ \tau'_{z'x'} \end{Bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}. \tag{7.20}$$

This means that D_{ijkl} is a fourth order isotropic (elasticity) tensor. We have discussed elsewhere some features of isotropic tensors. We shall take up such an approach, and proceed further later [p. 12-11].

$$\begin{Bmatrix} e'_{x'x'} \\ e'_{y'y'} \\ e'_{z'z'} \\ e'_{x'y'} \\ e'_{y'x'} \\ e'_{z'x'} \end{Bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{xy} \\ e_{yz} \\ e_{zx} \end{Bmatrix}. \tag{7.21}$$

As we have isotropy, the constitutive equations written above in the 'old' coordinate system will appear in the 'new' also in exactly the same form. Thus, we have

$$\sigma'_{x'x'} = D_{11}\, e'_{x'x'} + D_{12}\,(e'_{y'y'} + e'_{z'z'}) \tag{7.22a}$$
$$\sigma'_{y'y'} = D_{11}\, e'_{y'y'} + D_{12}\,(e'_{z'z'} + e'_{x'x'}) \tag{7.22b}$$
$$\sigma'_{z'z'} = D_{11}\, e'_{z'z'} + D_{12}\,(e'_{x'x'} + e'_{y'y'}) \tag{7.22c}$$

$$\tau'_{x'y'} = D_{44}\, e'_{x'y'} \tag{7.22d}$$
$$\tau'_{y'z'} = D_{44}\, e'_{y'z'} \tag{7.22e}$$
$$\tau'_{z'x'} = D_{44}\, e'_{z'x'}. \tag{7.22f}$$

If Eqs (7.20, 7.21) are substituted into Eq. (7.22a), we obtain

$$\frac{1}{2}\left(\sigma_{xx}+\sigma_{yy}\right)+\tau_{xy}=\sigma'_{x'x'}=D_{11}\left(\frac{1}{2}e_{xx}+\frac{1}{2}e_{yy}+e_{xy}\right)+D_{12}\left(\frac{1}{2}e_{xx}+\frac{1}{2}e_{yy}-e_{xy}+e_{zz}\right).$$

The stress components σ_{xx} and σ_{yy} are given by Eqs (7.19a, 7.19b). When these are substituted in the above equation, we obtain

$$\tau_{xy} = \left(D_{11}-D_{12}\right) e_{xy}.$$

Comparison of this expression for τ_{xy} with Eq. (7.19d) yields

$$D_{44} = (D_{11} - D_{12}).$$

Replacing the constitutive (stiffness) coefficients by the physically meaningful Lamé's constants λ and G, we arrive at the generalised Hooke's law as given below.

$$\sigma_{xx} = (2G+\lambda)\, e_{xx} + \lambda(e_{yy}+e_{zz}) \tag{7.23a}$$
$$\sigma_{yy} = (2G+\lambda)\, e_{yy} + \lambda(e_{zz}+e_{xx}) \tag{7.23b}$$
$$\sigma_{zz} = (2G+\lambda)\, e_{zz} + \lambda(e_{xx}+e_{yy}) \tag{7.23c}$$
$$\tau_{xy} = 2G\, e_{xy} \tag{7.23d}$$
$$\tau_{yz} = 2G\, e_{yz} \tag{7.23e}$$
$$\tau_{zx} = 2G\, e_{zx} \tag{7.23f}$$

These equations may be inverted to express the strain components in terms of the stress components as

$$e_{xx} = \frac{(\lambda+G)}{G(3\lambda+2G)}\,\sigma_{xx} - \frac{\lambda}{2G(3\lambda+2G)}(\sigma_{yy}+\sigma_{zz}) \tag{7.24a}$$
$$e_{yy} = \frac{(\lambda+G)}{G(3\lambda+2G)}\,\sigma_{yy} - \frac{\lambda}{2G(3\lambda+2G)}(\sigma_{zz}+\sigma_{xx}) \tag{7.24b}$$
$$e_{zz} = \frac{(\lambda+G)}{G(3\lambda+2G)}\,\sigma_{zz} - \frac{\lambda}{2G(3\lambda+2G)}(\sigma_{xx}+\sigma_{yy}) \tag{7.24c}$$
$$e_{xy} = \frac{1}{2G}\,\tau_{xy} \tag{7.24d}$$
$$e_{yz} = \frac{1}{2G}\,\tau_{yz} \tag{7.24e}$$
$$e_{zx} = \frac{1}{2G}\,\tau_{zx} \tag{7.24f}$$

We thus conclude that for isotropic materials, there are two and only two independent elastic constants. Mathematicians sometimes prefer to use these Lamé's constants G and λ, while engineers generally work with the Young's modulus E, Poisson's ratio ν and modulus of rigidity G. It is repeated for emphasis that other elastic constants such as bulk modulus can be defined, but only two of them are independent.

Physical Significance of These Elastic Constants

The elastic constants (stiffness coefficients, compliance coefficients, together referred to as constitutive coefficients) were introduced as some numbers. But they stand for physically meaningful material properties. To see this, we shall apply simple loadings like (i) a simple uniaxial stress and (ii) a simple shearing stress, and work out the corresponding strain components. First let us apply a uniaxial stress, say, $\sigma_{xx} = p$, all the other stress components being zero. The resulting strain components can be found out from the above equations. We know what to expect: a strain in the x-direction only ($e_{xx} = p/E$ and $e_{yy} = e_{zz} = -\nu p/E$ representing the Poisson's effect). Next we shall apply $\tau_{xy} = \tau_{yx}$, a simple pair of 'shear and complementary shear' and see that the only resulting shearing strain is $e_{xy} = e_{yx} = \tau_{xy}/(2G) = \tau_{yx}/(2G)$, showing that this G is nothing other than the familiar modulus of rigidity.

Constitutive Equations

These are also known as the material law, Hooke's law, or the generalised Hooke's law in the restricted case of the classical linear theory of elasticity.

The nine (9) stress components (9 reduced to 6 because of symmetry) are related to the nine (9) strain components (9 reduced to 6 because of symmetry again). In the simplest case of infinitesimal, linear, isotropic, elastic materials (which alone we consider for the most part here), these are given by the following equations.

Generalised Hooke's Law

The generalised Hooke's law is

$$e_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right], \tag{7.25a}$$

$$e_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{zz} + \sigma_{xx}\right)\right], \tag{7.25b}$$

$$e_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right], \tag{7.25c}$$

$$\gamma_{xy} \equiv 2e_{xy} = 2e_{yx} \equiv \gamma_{yx} = \frac{\tau_{xy}}{G}, \tag{7.25d}$$

$$\gamma_{yz} \equiv 2e_{yz} = 2e_{zy} \equiv \gamma_{zy} = \frac{\tau_{yz}}{G}, \tag{7.25e}$$

$$\gamma_{zx} \equiv 2e_{zx} = 2e_{xz} \equiv \gamma_{xz} = \frac{\tau_{zx}}{G}. \tag{7.25f}$$

The above equations may be inverted to give the stress components in terms of the strain components as

$$\sigma_{xx} = \frac{\nu E}{(1+\nu)(1-2\nu)}\, e + \frac{E}{1+\nu}\, e_{xx}, \tag{7.26a}$$

$$\sigma_{yy} = \frac{\nu E}{(1+\nu)(1-2\nu)}\, e + \frac{E}{1+\nu}\, e_{yy}, \tag{7.26b}$$

$$\sigma_{zz} = \frac{\nu E}{(1+\nu)(1-2\nu)}\, e + \frac{E}{1+\nu}\, e_{zz}, \tag{7.26c}$$

$$\tau_{xy} = G\gamma_{xy}, \tag{7.26d}$$

$$\tau_{yz} = G\gamma_{yz}, \tag{7.26e}$$

$$\tau_{zx} = G\gamma_{zx}, \tag{7.26f}$$

where $e \equiv e_{xx} + e_{yy} + e_{zz} = e_{ii}$ is the volumetric strain (the first invariant of $\boldsymbol{e}$).

In Terms of Lamé's Constants

It is sometimes convenient to write these in the form

$$\sigma_{xx} = \lambda e + 2Ge_{xx}, \tag{7.27a}$$

$$\sigma_{yy} = \lambda e + 2Ge_{yy}, \tag{7.27b}$$

$$\sigma_{zz} = \lambda e + 2Ge_{zz}, \tag{7.27c}$$

$$\tau_{xy} = G\gamma_{xy} = 2Ge_{xy}, \tag{7.27d}$$

$$\tau_{yz} = G\gamma_{yz} = 2Ge_{yz}, \tag{7.27e}$$

$$\tau_{zx} = G\gamma_{zx} = 2Ge_{zx}, \tag{7.27f}$$

in terms of the Lamé's constants λ and G, which are related to the Young's modulus of elasticity, E and the Poisson's ratio, ν by the equations

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad \text{and} \qquad G = \frac{E}{2(1+\nu)}.$$

We note that there are only two (2) independent elastic constants for a linear, elastic, isotropic material. These are usually taken as E and ν by engineers. Applied mathematicians and elasticians sometimes prefer to work in terms of the Lamé's constants λ and G[15]. The constitutive equations, that is, the generalised Hooke's law in this case, may be written in index notation as

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2Ge_{ij}, \tag{7.28}$$

where δ_{ij} is the Kronecker delta[16].

From Eq. (7.28) we obtain, on setting $i = j$

$$\sigma_{ii} = (3\lambda + 2G)\, e_{ii}. \tag{7.29}$$

Substituting for e_{kk} in Eq. (7.28) from Eq. (7.29) — ($e_{ii} \equiv e_{kk}$ and $\sigma_{ii} \equiv \sigma_{kk}$) — and solving for e_{ij} we have

$$e_{ij} = \frac{1}{2G}\left[\sigma_{ij} - \frac{\lambda}{3\lambda + 2G}\, \delta_{ij}\, \sigma_{kk}\right]. \tag{7.30}$$

[15] Some people use λ and μ for the Lamé's constants. The constant, μ, of course, is the same as G, the modulus of rigidity.

[16] Cauchy had proposed this form of the generalised Hooke's law (not in index notation, of course!) in 1822.

In direct notation, these equations (7.28) and (7.30) appear as

$$\boldsymbol{\sigma} = \lambda\,(\mathrm{tr}\,\boldsymbol{e})\,\boldsymbol{I} + 2G\,\boldsymbol{e}$$
$$\boldsymbol{e} = \frac{1}{2G}\left[\boldsymbol{\sigma} - \frac{\lambda}{3\lambda+2G}\,(\mathrm{tr}\,\boldsymbol{\sigma})\,\boldsymbol{I}\right].$$

These may also be written as

$$\sigma_{ij} = \frac{E}{1+\nu}\left[e_{ij} + \frac{\nu}{1-2\nu}\,\delta_{ij}\,e_{kk}\right] \qquad \boldsymbol{\sigma} = \frac{E}{1+\nu}\left[\boldsymbol{e} + \frac{\nu}{1-2\nu}\,(\mathrm{tr}\,\boldsymbol{e})\,\boldsymbol{I}\right]$$
$$e_{ij} = \frac{1+\nu}{E}\,\sigma_{ij} - \frac{\nu}{E}\,\delta_{ij}\,\sigma_{kk} \qquad \boldsymbol{e} = \frac{1+\nu}{E}\,\boldsymbol{\sigma} - \frac{\nu}{E}\,(\mathrm{tr}\,\boldsymbol{\sigma})\,\boldsymbol{I}.$$

The constitutive equations in index notation for an isotropic (linearly elastic) material are displayed below in a box.

Constitutive equations:

$$\sigma_{ij} = \lambda\,\delta_{ij}\,e_{kk} + 2G\,e_{ij};\ \text{ and / or} \tag{7.31a}$$
$$e_{ij} = \frac{1}{2G}\,\sigma_{ij} - \frac{\lambda}{2G}\,e_{kk}\,\delta_{ij}. \tag{7.31b}$$

See p. 12-10 for more details.

Principal Stresses and Principal Strains

We have already discussed the topic of principal stresses and the associated principal planes. We have also seen principal strains and the corresponding principal directions. Now the question arises: are the principal axes of stress coincident with the principal axes of strain? The answer is: yes, they are coincident for isotropic materials.

We have established the following relationship between the components of the stress and the strain matrices.

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{xy} \\ e_{yz} \\ e_{zx} \end{Bmatrix},$$

This is written in a general rectangular cartesian coordinate system (x, y, z). In particular, if the axes are taken along the principal axes of strain, we will have $e_{xx} \equiv e_{11}$, $e_{yy} \equiv e_{22}$, $e_{zz} \equiv e_{11}$. We will then obtain the corrsponding (above) stress matrix in the diagonalised form

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}.$$

We know that if the stress matrix is diagonalised (into its diagonal canonical form), the components on the leading diagonal are, indeed, the principal stresses. Thus, we note that the principal axes of strain are automatically the principal axes of stress also.

If we carry out a similar exercise expressing the strain components in terms of the stress components, and if we choose the axes (x, y, z) along the principal axes of stress, we can see that the strain matrix is diagonalised too. Thus, we note that the principal axes of stress are automatically the principal axes of strain also.

These two sets of principal axes are always the same (coincident) as long as the material is isotropic. This may not always be the case of more general constitutive relationships. [As a relatively simple example where these two sets of principal axes are not the same (coincident), we can consider a material with one plane of elastic symmetry. If we go through a similar exercise starting from a matrix equation relating the stress components and the strain components, and if we choose the axes along the principal axes of strain, we will find that the corresponding stress matrix is not in the diagonal canonical form. What does this show? Well, it shows that principal axes of strain are not always the principal axes of stress!]

NUMERICAL VALUES OF SOME IMPORTANT PROPERTIES

As engineers it is desirable for us to have an idea of the approximate values of some useful properties of common materials. This cannot be discussed in a few lines because there is so much to explain. But this is here for a limited purpose. With this caution and warning, some representative values are given. For serious purposes such as for design, and for solving real problems, these values have to be taken from reliable sources.

Here the table 7.1 gives the approximate numerical value of (a) Young's modulus of elasticity, E; (b) Poisson's ratio, ν; (c) modulus of rigidity, G; (d) yield stress, $\sigma_{y.p.}$; and (e) ultimate stress, σ_{ult}, for four common materials, (i) structural steel, (ii) medium carbon steel, (iii) aluminium, (iv) brass, and (v) concrete (M30).

Table 7.1: Numerical values of some properties for a few common engineering materials

Material	**E (GPa)**	**ν**	**G (GPa)**	**$\sigma_{y.p.}$ (MPa)**	**σ_{ult} (MPa)**
Structural steel	200	0.30	77	275	410
Medium carbon steel	200	0.30	77	310	560
Aluminium	70	0.33	26	270	310
Brass	97	0.31	37	220	400
Concrete (M20, M30)	25-30	0.20	10.5-12.5	-	20-40

ANISOTROPY AND ORTHOTROPY

Materials cannot always be treated as isotropic. Some materials have intrinsically different properties along different directions. Cold rolled copper and wood are two cases in point. The properties are different along the directions of rolling. Similarly, the properties along the grains are distinctly different from those perpendicular to this direction. Sometimes the method of construction introduces anisotropy in a structure. If ribs are provided in one direction, but not in the other, a slab or a plate, made up of essentially an isotropic material, will exhibit anisotropy. More significantly, composites, which have in recent times become an important structural material, are decidedly anisotropic. To deal with such materials, we need to consider anisotropic elasticity[17] also.

If the nine (9) stress components are related to the nine (9) strain components by the linear equations $e_{ij} = C_{ijkl}\,\sigma_{kl}$, we can see that there are $9 \times 9 = 81$ elastic constants C_{ijkl} $(i, j, k, l = 1, 2, 3;\;\; 3^4 = 81)$. If the symmetry conditions $\sigma_{ij} = \sigma_{ji}$ and $e_{ij} = e_{ji}$ are invoked, this number 81 reduces to 36 $(6 \times 6 = 36)$. If, furthermore, the existence of a strain energy density function is assumed, this number 36 reduces to 21. Thus, in the general case of anisotropy, there are 21 elastic constants.

To continue with such simplifications or reductions, if there is one plane of elastic symmetry, this number will be reduced further to 13. If the material has three planes of elastic symmetry, there will be only nine (9) elastic constants. These are called orthotropic materials. (Many types of bio-membranes such as cell walls should be modelled as orthotropic.) For transtropic materials there are five (5) elastic constants. And finally, when the material is isotropic, this number five (5) reduces further to three (3). Even these three (3) are not independent. Thus, we may conclude that there are just two (2) independent elastic constants for a linear, elastic, isotropic material. These, as indicated earlier, are usually taken as E and ν, or as the Lamé's constants λ and G.

In this chapter on the constitutive equations, we have restricted ourselves for the most part to isotropic materials. Most of the materials for the traditional engineering structures could be considered as isotropic (although materials like wood, concrete slabs with stiffeners in one, but not in both, directions, and plates with ribs running all along the length in one direction were used in the past). We have discussed only aspect of the constitutive (material) properties. We have not considered the physical aspects at all. These are important too for engineers. These, we hope, will be discussed in other courses like deformation processes in metals.

In the next chapter we shall see some important general considerations.

[17] The classical book that deals with this topic is Lekhnitskii, S.G.: *Theory of Elasticity of an Anisotropic Body*, Holden-Day (1963), translated from Russian.

Chapter 8

GENERAL CONSIDERATIONS

In this chapter we shall consider some useful concepts, results and theorems. They are applicable for nearly all the topics that we discuss in this book. So far we have discussed only the fundamentals; we have not considered the actual solution of problems in stress analysis. We are building the necessary tools for the solution of problems.

COORDINATE SYSTEMS

It is obvious that we need a suitable coordinate system to work with. There are several possibilities. Which system shall we use? There are some considerations that decide the most suitable system in a given context.

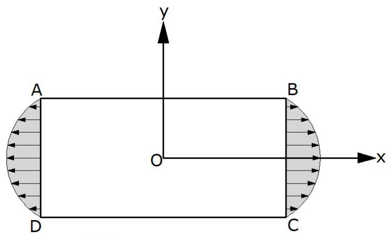
Coordinate Systems

The easiest is the familiar rectangular cartesian system. This has the great advantage of being the simplest among all the choices. We can appreciate the convenience and simplicity of this only when we are exposed to the complications in other coordinate systems.

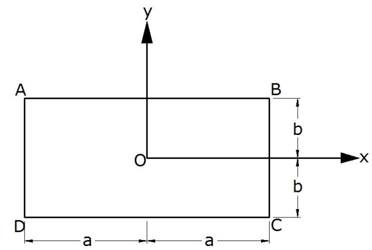
A suitable coordinate system for a specific problem:
The choice of a coordinate system suitable in a given context is relatively easy. In a two-dimensional (plane) boundary value problem (b.v.p.) [Fig. 8.1a], a rectangular coordinate system as shown in Fig. 8.1b is convenient. The axes (x, y) should be as shown to exploit the symmetry inherent in the problem. The problem here is to find the stresses inside for the given boundary conditions.

Let us consider two more problems that have circular symmetry: (i) a circular disc in diametral compression [Fig. 8.2a] and a thick cylinder subjected to an internal fluid pressure. [Fig. 8.2b]. It is obvious that a rectangular coordinate system, or a polar coordinate system with the origin elsewhere would be a poor choice. (However, when a general theory is set up, or a software developed, it is versatility that is of prime concern.)

Choice of a suitable coordinate system - two choices:
We have two choices for a suitable coordinate system. In one we use a coordinate system fixed in space. In the other, we shall have a coordinate system fixed in the body and,

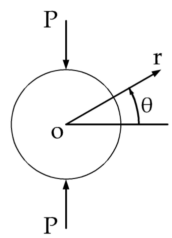


(a) A b.v.p. that has symmetry

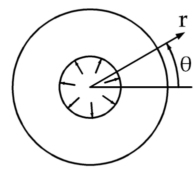


(b) A coordinate system (x, y) for the problem

Figure 8.1: A boundary value problem (b.v.p.) and a convenient coordinate system to exploit the symmetry. Choosing the origin elsewhere with inclined axes would be inconvenient. So is a polar coordinate system here.



(a) A circular disc in compression



(b) A thick cylinder subjected to a fluid pressure

Figure 8.2: Both these b.v.p. have circular symmetry. It is natural to choose a polar (r, θ) system for both these problems with circular symmetry.

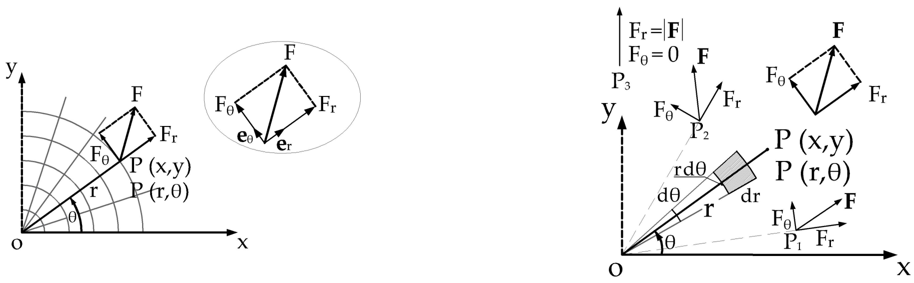
therefore, moving and deforming with the body. This second choice is called a convected coordinate system. When we take up advanced studies, it will be necessary for us to understand all this and use general tensor analysis. But we shall be simple minded; we will not use convected coordinate systems at all in this book.

Coordinate systems other than the rectangular cartesian one:
We have seen that the geometry of the problem makes it necessary, or at least convenient, to use coordinate systems other the simple rectangular cartesian system. These are more difficult to handle because of various complicating features. A sound knowledge of general tensor analysis is necessary to understand and deal with the complicating features.

A polar coordinate system:
Fig. 8.3 shows a polar coordinate system. The relevant base vectors are shown. Note also the metric: it is not of the form $(ds)^2 = (dr)^2 + (d\theta)^2$; it is $(ds)^2 = (dr)^2 + (r\,d\theta)^2$.

Some features of curvilinear coordinate systems:
Some of the features of a curvilinear system are indicated below. Readers who may have worked only with the simple rectangular coordinate system should note the following.

(i) A (two-dimensional) polar (r, θ) coordinate system is shown in Fig. 8.3. The concentric circular arcs r = constant, and the radial lines θ = constant are shown. These are the curvilinear coordinate lines. A typical point P has the coordinates (x, y) and



(a) A polar coordinate system (b) The same polar coordinate system

Figure 8.3: A polar (r, θ) coordinate system with the relevant base vectors is shown. The elemental area $dA = J\,dr\,d\theta = r\,dr\,d\theta$, not the more familiar $dA = dx\ dy$ valid in a rectangular cartesian system. Note also the metric; it is $(ds)^2 = (dr)^2 + (r\,d\theta)^2$.

(r, θ) referred to the rectangular cartesian and polar coordinates, respectively. The familiar relationships connecting the two sets of coordinates are shown later [p. 9-14]. We can also see that the elemental area dA is given by the expressions $dx\,dy$ (rectangular cartesian), and $J\,dr\,d\theta$ (polar), respectively, where J is the Jacobian of the transformation. A vector $\boldsymbol{F}$ — to be specific, a force vector — is shown; its physical components F_r, F_θ along the radial and tangential directions along with the base vectors concerned $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$ are also shown. The elemental distance ds, we can see, is given by $(ds)^2 = (dr)^2 + (r\,d\theta)^2$. The base vector $\boldsymbol{e}_\theta$ here is not a unit vector!

(ii) The force vector $\boldsymbol{F}$ may be written as $\boldsymbol{F} = F_r\,\boldsymbol{e}_r + F_\theta\,\boldsymbol{e}_\theta$ where F_r, F_θ are the components of $\boldsymbol{F}$, and $\boldsymbol{e}_r$ (radial) and $\boldsymbol{e}_\theta$ the base vectors. What is the unit of F_θ? What is the length of the base vector $\boldsymbol{e}_\theta$? Examine and find out.

(iii) If this expression is to be differentiated w.r.to a space variable, say, r, we should realise that the base vectors themselves are not constants; they change with position. Accordingly, this force vector is to be differentiated as indicated below:

$$\frac{\partial \boldsymbol{F}}{\partial r} = \frac{\partial F_r}{\partial r}\,\boldsymbol{e}_r + F_r\,\frac{\partial \boldsymbol{e}_r}{\partial r} + \frac{\partial F_\theta}{\partial r}\,\boldsymbol{e}_\theta + F_\theta\,\frac{\partial \boldsymbol{e}_\theta}{\partial r}; \ \text{and}$$
$$\frac{\partial \boldsymbol{F}}{\partial \theta} = \frac{\partial F_r}{\partial \theta}\,\boldsymbol{e}_r + F_r\,\frac{\partial \boldsymbol{e}_r}{\partial \theta} + \frac{\partial F_\theta}{\partial \theta}\,\boldsymbol{e}_\theta + F_\theta\,\frac{\partial \boldsymbol{e}_\theta}{\partial \theta}.$$

(iv) Note that the base vectors are all unit vectors, and are constants in both magnitudes and directions in rectangular cartesian coordinates. Here the situation is different; the base vectors have different magnitudes and different directions at different points (spatial locations).

(v) In Fig. 8.3b are shown some force vectors and some force components at three points P_1, P_2, P_3 in addition to the point P in Fig. 8.3a. If the force is the same, its components, F_r, F_θ are different. When a vector is a constant (at different points), its components are not constant! Conversely, when the (radial and tangential) components are constants, the vector is not a constant!

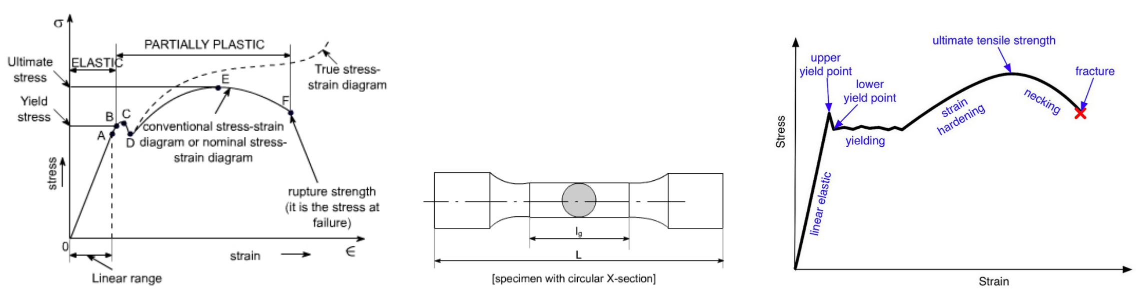
(vi) We further note that at the origin $r = 0$ where the Jacobian $J = 0$ ($dx\,dy \rightarrow J\,dr\,d\theta = r\,dr\,d\theta$; $J = r = 0$), the mapping is not one-to-one! The origin represented by ($x = 0$, $y = 0$) in rectangular coordinates is mapped into (i) ($r = 0$, $\theta = 0$), (ii) ($r = 0$, $\theta = \pi/4$), (iii) ($r = 0$, $\theta = \pi/2$), etc. in polar coordinates!

(vii) Let us write $F_r\,dr + F_\theta\,d\theta$ in imitation of the expression $F_x\,dx + F_y\,dy$. Physically this latter expression may represent a virtual work. The units of the various elements are: (i) (F_x, F_y) (forces) N, and (ii) (dx, dy) (length) m. Now $d\theta$ is dimensionless in the expression $F_r\,dr + F_\theta\,d\theta$, forcing F_θ, as it were, to have the unit Nm. The component F_θ has to have the unit Nm, making it impossible for F_θ to be a physical component of a force!

The moral of this demonstration is this: the vector component may not be the physical component in a general curvilinear coordinate system[1]! The elemental area dA and the elemental length ds are given by $dA = r\,dr\,d\theta$ and $(ds)^2 = (dr)^2 + (r\,d\theta)^2$.

These are some of the complicating features that must be considered when we work with coordinate systems other than the simple rectangular cartesian one.

SIMPLE TENSION TEST



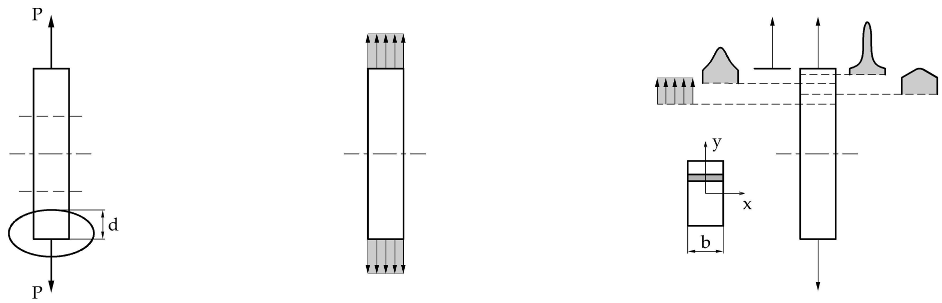
(a) A stress-strain curve (b) A typical specimen (c) A stress-strain curve again

Figure 8.4: Shown in the figure are two stress-strain curves and a typical specimen. These curves are obtained by performing a (uniaxial) tension test on a universal testing machine.

To solve practical engineering problems we need to obtain the solutions in terms of numbers. For example, when we design a beam, or a column, or a shaft, we need to know the dimensions. We depend upon experiments to know the properties such as the ultimate stress, the yield stress and the Poisson's ratio. Although there are several useful and even indispensable experiments, perhaps the most fundamental or basic among them is the simple tension test. There are several details to be understood. All we do here is to show the strain-strain curve of a ductile material like mild steel. Further details are not given here. Students will do well to learn at least some of the details.

[1] To take this discussion further, let us rewrite the expression as $F_r\,dr + F_\theta\,d\theta = (F_r)\,dr + (F_\theta/r)\,(r\,d\theta)$ and establish the relationship between the vector components and the physical components of a vector. But we will stop at this, and hope that some ambitious students would inquire further into such issues.

SAINT-VENANT'S PRINCIPLE



(a) Loaded by $P-P$ (b) Loaded by u.d.l. (c) Stress distribution at various sections

Figure 8.5: A bar is loaded by $P-P$ [Fig. 8.5a]. In Fig. 8.5b is shown the same bar, but this time loaded by a uniformly distributed load at the two ends. If these loadings are statically equivalent, the stresses will be the same at all remote points. After a certain distance (say, the linear dimension of the bar) the stress distribution is almost the same. Fig. 8.5c shows how the stress distribution tends to the value $\sigma = P/A$ (uniform) as we move farther and farther from the ends.

Barre de Saint-Venant proposed in 1855 an important principle. If, in an elastic body in equilibrium, the actual forces and moments acting on a small part of the surface are replaced by their static equivalents, the replacement causes changes only in the immediate vicinity of where the replacement is made (that is, there will be changes in the stress field[2] only locally); there is no great difference in the stress field at any point remote from where the replacement is made. After a short 'Saint-Venant distance' from each end, the stress distribution is almost the same. This distance is, say, the linear dimension of the cross-section in the case of the bar referred to for practical purposes.

This principle is applicable in a number of situations[3]. We shall see two or three examples to clarify the statement.

Example (i)
If in the bar AB loaded by two concentrated forces $P-P$ [Fig. 8.5a], the forces are replaced by a statically equivalent force system (which may include moments also) — uniformly distributed forces at the ends [Fig. 8.5b], this replacement caused differences in the stress distribution near the ends where the replacement is made. However, in all parts of the body (bar) remote from where the replacement is made, that is, away from the ends, there is no great difference in the stress field. A practical guide may be to leave a 'Saint-Venant

[2] and strain and displacement fields also (except the rigid body displacements)

[3] This Saint-Venant's principle is very useful and easy to apply in practical situations. However, there are deeper issues. Several investigators have called attention to some of them. Fung [6] points out several of them. He makes a general statement that "the justification of the principle is largely empirical and, as such, its interpretation is not entirely clear." Following some well known investigators, he points out how Saint-Venant's principle may be formulated with mathematical precision. Timoshenko [16] states that "$\cdots$ vanishing of the resultant is not an adequate criterion for the degree of localisation".

distance' equal to the linear dimension (shown shaded in the figures 8.5a and 8.5b).

(a) A body: loads (a) (b) The same body: loads (b)

Figure 8.6: The applied loads on one small part (shown shaded) is replaced by a statically equivalent system of forces. There will be some difference in the stress, strain and displacement fields on a nearby cross-section like $X - X$, but practically no difference on remote sections like $X' - X'$ and $X'' - X''$.

Den Hartog [3] states: "$\cdots$ Kelvin made use of a common-sense proposition known as Saint-Venant's principle, which states that: If the loading on a small part of the boundary of an elastic system is replaced by a different loading, which is statically equivalent to the original loading, then the stress distribution in the system will be sensibly changed only in the neighborhood of the change; the stresses at a distance from the disturbance equal to the size of the disturbance itself will be changed by a few per cent only."

Consider a cantilever with an end load P. The simple strength of materials solution gives us a normal stress and a shear stress distributions at the fixed end. Actually, however, there is some uncertainty. It is difficult to achieve these 'idealised distributions'. The practical implication of this principle is that the stress distribution is unaltered because of these uncertainties.

PRINCIPLE OF SUPERPOSITION

We had mentioned this earlier [p. 1-15]. Here we shall discuss this in a general setting. The validity of superposition, as we had pointed out earlier, is based on the linearity of all the governing equations. Thus, this theorem is valid only in the linear theory of elasticity. When the more general nonlinear theory (large deformation theory, also called finite elasticity as distinct from infinitesimal elasticity) is taken up, superposition is no more valid because some of the governing equations are no more linear. (There can be different kinds of nonlinearity and difficulties, but we do not discuss them in this book.)

To be specific, let us consider the all stress formulation (all the governing equations being expressed in terms of the stress components). The compatibility equations (the Beltrami-Michell equations) play a decisive role. The governing equations are, therefore: (i) equations of equilibrium; (ii) equations of compatibility; and (iii) the boundary conditions. [The constitutive equations are also to be included, but they are already used in (ii).]

Let us consider an elastic body in equilibrium and two sets of solutions. Each of them must, and does, satisfy (i) the equations of equilibrium, (ii) the equations of compatibility

expressed in terms of the stresses, and (iii) the boundary conditions. That is, each of these two sets satisfies all the governing equations and, thus, qualifies to be a solution. We need to prove that the sum (and / or the difference) also satisfies (satisfy) all these governing equations and, therefore, qualifies (qualify) to be a solution (solutions). We proceed to demonstrate this.

(i) Let $[(\sigma_{xx})_1,\, (\sigma_{yy})_1,\, \cdots,\, (\tau_{zx})_1]$ be the stress components obtained as one solution. The surface forces and the body forces are, respectively,

$$[(\overset{\nu}{T}_x)_1,\, (\overset{\nu}{T}_y)_1,\, (\overset{\nu}{T}_y)_1] \qquad \text{and} \qquad [(F_x)_1,\, (F_y)_2, (F_z)_3].$$

(ii) Let $[(\sigma_{xx})_2,\, (\sigma_{yy})_2,\, \cdots,\, (\tau_{zx})_2]$ also be the stress components obtained as another solution. The surface forces and the body forces are, respectively, the

$$[(\overset{\nu}{T}_x)_2,\, (\overset{\nu}{T}_y)_2,\, (\overset{\nu}{T}_y)_2] \qquad \text{and} \qquad [(F_x)_2,\, (F_y)_2, (F_z)_2].$$

Here are two solutions. We need to show that the sum / difference also is a solution. This is the idea of superposition. We shall show this.

As $[(\sigma_{xx})_1,\, (\sigma_{yy})_1,\, \cdots,\, (\tau_{zx})_1]$ are the stress components obtained when the problem is solved, these must satisfy the differential equations of equilibrium. Thus,

$$\begin{aligned}
\frac{(\partial\sigma_{xx})_1}{\partial x} + \frac{(\partial\tau_{yx})_1}{\partial y} + \frac{(\partial\tau_{zx})_1}{\partial z} + (F_x)_1 = 0;\\
\frac{(\partial\sigma_{yy})_1}{\partial y} + \frac{(\partial\tau_{zy})_1}{\partial z} + \frac{(\partial\tau_{xy})_1}{\partial x} + (F_y)_1 = 0;\\
\frac{(\partial\sigma_{zz})_1}{\partial z} + \frac{(\partial\tau_{xz})_1}{\partial x} + \frac{(\partial\tau_{yz})_1}{\partial y} + (F_z)_1 = 0.
\end{aligned}$$

For the same reason, $[(\sigma_{xx})_2,\, (\sigma_{yy})_2,\, \cdots,\, (\tau_{zx})_2]$ also must satisfy these equations. Thus,

$$\begin{aligned}
\frac{(\partial\sigma_{xx})_2}{\partial x} + \frac{(\partial\tau_{yx})_2}{\partial y} + \frac{(\partial\tau_{zx})_2}{\partial z} + (F_x)_2 = 0;\\
\frac{(\partial\sigma_{yy})_2}{\partial y} + \frac{(\partial\tau_{zy})_2}{\partial z} + \frac{(\partial\tau_{xy})_2}{\partial x} + (F_y)_2 = 0;\\
\frac{(\partial\sigma_{zz})_2}{\partial z} + \frac{(\partial\tau_{xz})_2}{\partial x} + \frac{(\partial\tau_{yz})_2}{\partial y} + (F_z)_1 = 0.
\end{aligned}$$

It is clear that the sum / difference also satisfies the equations of equilibrium.

$$\begin{aligned}
\frac{[(\partial\sigma_{xx})_1 + (\partial\sigma_{xx})_2]}{\partial x} + \frac{[(\partial\tau_{yx})_1 + (\partial\tau_{yx})_2]}{\partial y} + \frac{[(\partial\tau_{zx})_1 + (\partial\tau_{zx})_2]}{\partial z} + [(F_x)_1 + (F_x)_2] = 0;\\
\frac{[(\partial\sigma_{yy})_1 + (\partial\sigma_{yy})_2]}{\partial y} + \frac{[(\partial\tau_{zy})_1 + (\partial\tau_{zy})_2]}{\partial z} + \frac{[(\partial\tau_{xy})_1 + (\partial\tau_{xy})_2]}{\partial x} + [(F_y)_1 + (F_y)_2] = 0;\\
\frac{[(\partial\sigma_{zz})_1 + (\partial\sigma_{zz})_2]}{\partial z} + \frac{[(\partial\tau_{xz})_1 + (\partial\tau_{xz})_2]}{\partial x} + \frac{[(\partial\tau_{yz})_1 + (\partial\tau_{yz})_2]}{\partial y} + [(F_z)_1 + (F_z)_2] = 0.
\end{aligned}$$

This is because of the linearity of the equations. Similarly, the boundary conditions also can be written, and a similar conclusion can be arrived at. (i) Let $l\,(\sigma_{xx})_1 + m\,(\tau_{yx})_1 + n\,(\tau_{zx})_1 = (p_x)_1$ and (ii) $l\,(\sigma_{xx})_2 + m\,(\tau_{yx})_2 + n\,(\tau_{zx})_2 = (p_x)_2$ be the two boundary conditions corresponding to the two solutions. Then it follows, again from the linearity of the equations, by adding these two equations, that the sum (iii) $l[(\sigma_{xx})_1 + (\sigma_{xx})_2] + m[(\tau_{yx})_1 + (\tau_{yx})_2] + n[(\tau_{zx})_1 + (\tau_{zx})_2] = [(p_x)_1 + (p_x)_2]$ satisfies the boundary condition for the 'superposed' sum problem. We see that (i) and (ii) lead to the conclusion (on adding up / subtraction) (iii).

The third set of equations is the Beltrami-Michell equations which are the compatibility equations (six in number) in terms of the stress components. We note that in the all stress formulation, these compatibility equations are indeed the third set of governing equations. These [Eqs (12.22), (12.23), p. 12-17, and their companion equations] are also linear equations and, therefore, superposition does hold. The same argument can be used for all such linear governing equations. The conclusion is that the sum (as well as the difference) is also a solution.

There is, however, one major restriction. The result is true only if the deformations are small, so that the geometry is not affected when the second set of loads is applied. This is often, but not always, the case. An important exception arises in the case of beam-columns. This fact is pointed out at the appropriate place.

UNIQUENESS THEOREM

We shall now examine if a stress analysis problem can have more than one solution. We shall that this is impossible; the solution is unique. This was established by Kirchhoff in 1859, and is known as the uniqueness theorem. For a linear elastic solid with a positive definite strain energy function, there is always a one-to-one correspondence between the forces acting on the body and the resulting elastic deformation.

If possible[4], let there be *two* solutions[5] represented by

$$(i) \quad (\sigma_{xx})_1,\ (\tau_{yx})_1,\ \cdots,(\sigma_{zz})_1); \quad (e_{xx})_1,\ (eyx)_1,\ \cdots,(e_{zz})_1); \quad (u)_1,\ (v)_1,\ (w)_1; \quad \text{and}$$

$$(ii) \quad (\sigma_{xx})_2,\ (\tau_{yx})_2,\ \cdots,(\sigma_{zz})_2); \quad (e_{xx})_2,\ (eyx)_2,\ \cdots,(e_{zz})_2); \quad (u)_2,\ (v)_2,\ (w)_2$$

for the same problem with the same (a) fixity conditions, (b) surface tractions, and (c) displacements specified on the boundary

$$F_x,\ F_y,\ F_z \qquad \overset{(\nu)}{T}_x,\ \overset{(\nu)}{T}_y,\ \overset{(\nu)}{T}_z \qquad u,\ v,\ w.$$

(i) (a) The differential equations of equilibrium, (b) the stress-strain relations, (c) the strain-displacement relations, (d) the compatibility conditions, and (e) the boundary conditions will be satisfied by the first set (i). Thus,

$$(a) \qquad \frac{\partial(\sigma_{xx})_1}{\partial x} + \frac{\partial(\tau_{yx})_1}{\partial y} + \frac{\partial(\tau_{zx})_1}{\partial z} + F_x = 0;$$

[4] We are trying to prove that this is not possible.

[5] These two solutions can possibly differ by rigid body displacements that do not affect the deformations and the stresses.

$$\begin{array}{lllll} & \cdots & \cdots & \cdots & \cdots\,; \\ & \cdots & \cdots & \cdots & \cdots\,; \\ & \cdots & \cdots & \cdots & \cdots\,; \\ (e) & \multicolumn{4}{l}{l(\sigma_{xx})_1 + m(\tau_{yx})_1 + n(\tau_{zx})_1 = \overset{(\nu)}{T}_x;} \\ & \cdots & \cdots & \cdots & \cdots\,. \end{array}$$

(ii) All these equations are satisfied by the stress, strain and displacement components of the second solution also. Thus,

$$\begin{array}{lllll} (a) & \multicolumn{4}{l}{\dfrac{\partial(\sigma_{xx})_2}{\partial x} + \dfrac{\partial(\tau_{yx})_2}{\partial y} + \dfrac{\partial(\tau_{zx})_2}{\partial z} + F_x = 0;} \\ & \cdots & \cdots & \cdots & \cdots\,; \\ & \cdots & \cdots & \cdots & \cdots\,; \\ & \cdots & \cdots & \cdots & \cdots\,; \\ (e) & \multicolumn{4}{l}{l(\sigma_{xx})_2 + m(\tau_{yx})_2 + n(\tau_{zx})_2 = \overset{(\nu)}{T}_x;} \\ & \cdots & \cdots & \cdots & \cdots\,. \end{array}$$

Using superposition, by subtraction, we find that

$$F_x = F_y = F_z = 0 \qquad \overset{(\nu)}{T}_x = \overset{(\nu)}{T}_y = \overset{(\nu)}{T}_z = 0 \qquad (u_1 - u_2),\ (v_1 - v_2),\ (w_1 - w_2)$$

also satisfy all the governing equations. Thus,

$$\begin{array}{lllll} (a) & \multicolumn{4}{l}{\dfrac{\partial[(\sigma_{xx})_1 - (\sigma_{xx})_2]}{\partial x} + \dfrac{\partial[(\tau_{yx})_1 - (\tau_{yx})_2]}{\partial y} + \dfrac{\partial[(\tau_{zx})_1 - (\tau_{zx})_2]}{\partial z} = [F_x - F_x] = 0;} \\ & \cdots & \cdots & \cdots & \cdots\,; \\ & \cdots & \cdots & \cdots & \cdots\,; \\ & \cdots & \cdots & \cdots & \cdots\,; \\ (e) & \multicolumn{4}{l}{l[(\sigma_{xx})_1 - (\sigma_{xx})_2] + m[(\tau_{yx})_1 - (\tau_{yx})_2] + n[(\tau_{zx})_1 - (\tau_{zx})_2] = [\overset{(\nu)}{T}_x - \overset{(\nu)}{T}_x] = 0;} \\ & \cdots & \cdots & \cdots & \cdots\,. \end{array}$$

Here is a new stress distribution — the difference between the two solutions — that correspond to zero body forces and zero surface forces (tractions)! The work done by these zero body forces and zero surface forces is, therefore, zero. This fact leads to the conclusion

$$\iiint U\,dx\,dy\,dz = 0 \quad \longrightarrow \quad [(e_{xx})_1 - (e_{xx})_2] = \cdots = [(e_{zz})_1 - (e_{zz})_2] = 0.$$

This proves that the two strain distributions and, consequently, the two stress distributions are the same, proving the uniqueness theorem.

An Alternative Proof

An alternative proof, which is mathematically more appealing, is indicated below. We know — we have seen this when we discussed constitutive equations — that the stress components σ_{ij} are related to the strain energy density function $\mathcal{U} = \mathcal{U}(e_{ij})$ by

$$\sigma_{ij} = \frac{\partial \mathcal{U}}{\partial e_{ij}} \qquad (i, j = 1, 2, 3),$$

and that the strain energy function $U = U(e_{ij})$

$$U = U(e_{ij}) = \iiint_{\mathcal{V}} \mathcal{U}(e_{ij})\, d\mathcal{V}.$$

This function enjoys the property of positive definiteness which has important, very important, consequences. We use this property here to prove the uniqueness theorem.

The differential equations of equilibrium $\sigma_{ji,j} + F_i = 0$ in terms of the stress components can be recast in terms of $\mathcal{U}$ as

$$\left(\frac{\partial \mathcal{U}}{\partial e_{ij}}\right)_{,j} + F_i = 0.$$

The surface traction $\overset{(\nu)}{T}_i$ is prescribed on part of the boundary (S_1). The displacement u_i is prescribed on the other part (S_2) of the boundary. We know, of course, that that both the displacement and the surface traction cannot be specified at the same point on the boundary.

The boundary conditions on the surface $S = S_1 + S_2$ are

(i) the traction $\overset{(\nu)}{T}_i = \dfrac{\partial \mathcal{U}}{\partial e_{ij}}\, \nu_J$ on S_1; and

(ii) the displacements u_i are prescribed on S_2.

We shall consider, as before, two strain fields $(u_i)_1$ and $(u_i)_2$ that, being two solutions if that would be possible[6], satisfy the above equations. Let us call this 'difference solution' u_i as $[(u_i)_1 - (u_i)_2] \equiv u_i$. Now for this 'difference solution' the equations of equilibrium

$$\left(\frac{\partial \mathcal{U}}{\partial e_{ij}}\right)_{,j} = 0; \tag{8.1}$$

and the boundary conditions

$$\overset{(\nu)}{T}_i = \frac{\partial \mathcal{U}}{\partial e_{ij}}\, \nu_j = 0 \text{ on } S_1; \qquad u_i = 0 \text{ on } S_2; \tag{8.2}$$

[6] We are going to prove that it is impossible to have two different solutions. We must realise, however, that two solutions can possibly differ by rigid body displacements, because rigid body displacements are related neither to strains nor to stresses.

are, must be, satisfied. From Eq. (8.1) we obtain

$$\iiint_{\mathcal{V}} u_i \left(\frac{\partial \mathcal{U}}{\partial e_{ij}}\right)_{,j} dV = 0.$$

This, on integration by parts and changing one volume integral to a surface integral, yields

$$\iint_{\mathcal{S}} u_i \frac{\partial \mathcal{U}}{\partial e_{ij}} \nu_j \, dS - \iiint_{\mathcal{V}} \frac{\partial \mathcal{U}}{\partial e_{ij}} u_{i,j} \, dV = 0.$$

The first, the surface integral, vanishes because of the boundary conditions. The second, the volume integral, is

$$\iiint_{\mathcal{V}} \frac{\partial \mathcal{U}}{\partial e_{ij}} u_{i,j} \, dV = \iiint_{\mathcal{V}} 2 \iiint_{\mathcal{V}} \frac{\partial \mathcal{U}}{\partial e_{ij}} e_{ij} \, dV = \iiint_{\mathcal{V}} 2\mathcal{U} \, dV = 0.$$

As U is positive definite, this cannot vanish unless $U = 0$ which leads to the conclusion that $(u_i)_1 = (u_i)_2$. This means that the two solutions are really the same. The uniqueness theorem is thus proved.

We can see that it is awkward to do these manipulations without using the index notation. Readers are advised to be convinced of this fact and to learn the index notation.

Comments

There are deeper issues involved in this. This theorem is proved only in the neighbourhood of the natural state. If the strain energy function is not positive definite, several solutions or a multi-valued solution may be possible. The uniqueness theorem may not hold if (i) the strain energy function $U = U(e_{ij})$ fails to be positive definite (which can happen when the material becomes unstable when flow or yielding takes place) and (ii) when there are finite (in contrast to infinitesimal) deformation, or when the forces are non-conservative and / or when the forces are dependent on — functionals of — the deformation history.

There are some of these complicating features that make the theory not easy and straightforward. In these cases, uniqueness may not necessarily be violated. Further careful examination is needed.

When we state that the solution is unique, there can still be some limitation. Even when the stresses are uniquely known, the displacements may not still be unique. When the body has rigid body movements — translations along the three axes, and rigid body rotations about the three axes — the stresses and the strains are unchanged. Thus, when the stresses and strains are known, and we seek the displacements, we have to identify the rigid body movements and filter them out. How do we do that? Well, first we need to understand the problem. With this understanding, we can arrest the rigid body displacements. For example, in a two-dimensional problem, we can specify that one point of the body is fixed; that is, $u(0,0) = v(0,0) = 0$. Now the translation is arrested; the body cannot move in the x and y directions. It can still rotate. This too is to be arrested. There are several ways. One method is to specify that an element along the x direction does not rotate. Another is

to arrest the rotation about the y axis. In two dimensions, this is sufficient. If the physical problem is well understood, there will be no difficulty.

On the mathematical side, this situation can manifest in this way. Imagine that we are trying to determine the displacements in a finite element formulation. Now the determinant may vanish; the (linear) algebraic equations may not be linearly independent. How do we deal with this situation? It is, first, by understanding the physical problem and, then, by deleting the linearly dependent rows (and columns).

SOME MISCELLANEOUS TOPICS

We shall include here some important topics that could not be included anywhere else. These topics are somewhat jumbled; there is often no logical order. We nevertheless hope that these remarks will enrich our readers' understanding.

Role of Thermodynamics in the Mechanics of Solids

A discussion on the mechanics of solids will be incomplete if no reference at all is made to the role of thermodynamics. Classical thermodynamics imposes certain restrictions on the mechanics of solids. Some information relevant to our subject can be deduced from thermodynamical restrictions[7].

Unfortunately, not much can be gained if the discussions are held at such an elementary level as can be permitted in a first level course. Thermodynamics is a major branch of science in its own right. In relatively recent years, axiomatic thermodynamics and irreversible thermodynamics have appeared. These have proved to be of great help and a great step forward. However, these cannot be discussed here.

The existence of a strain energy density function can be established by thermodynamic arguments. For two simple processes, isothermal and adiabatic, this is easier. What this means is that the stress-strain equation (constitutive equation, or material law), can be written with no reference to the temperature. Physically speaking, sacrificing some technical correctness for the sake of intuitive understanding, if the loading (straining) process is very slow, there would be sufficient time for heat transfer to occur, and for thermal equilibrium to be established; thus, nearly isothermal conditions would prevail. On the other hand, if the loading (straining) process is very fast, there would be little time for heat transfer to occur, and we would then have nearly adiabatic conditions. For the in-between cases between these two extremes of extremely slow and extremely rapid loading, non-isothermal, non-adiabatic conditions would prevail. For such realistic cases of loading as occur in practice, the situation is more complex. It is also possible to obtain relationships among the specific heats, the modulus of elasticity, latent heats of change of stress and strain at constant temperature, etc. However, for the cases of practical engineering situations, the thermal effects can

[7] Perhaps it is well to point out that classical thermodynamics refers to equilibrium conditions. Thus, thermostatics would be a technically more correct name for this branch of science. Somehow, thermostatics continues to be called erroneously as thermodynamics. We may recall that statics deals with equilibrium conditions, while non-equilibrium conditions fall in the domain of dynamics. We may consider calling this equilibrium thermodynamics in spite of the obvious self-contradiction, if we wish to continue to use the traditional time-honoured name without sacrificing technical correctness.

be almost always disregarded, and not much will be lost if we disregard thermodynamics altogether. This is why thermodynamics is hardly ever mentioned in books of mechanics of solids (except, of course, in the advanced books).

However, when dealing with plasticity, viscoelasticity, etc., dissipation of energy cannot be neglected at all. Thus, thermodynamics and even irreversible thermodynamics would now enter with greater relevance.

Even in the case of elasticity, thermodynamics is important in theoretical considerations. Thermodynamic arguments can be used to establish the inequalities $\lambda > 0$; $G > 0$; (λ, G are the Lamé's constants; these differ slightly, but not sufficiently to be significant in practical cases, under isothermal and adiabatic conditions); $E > 0$ and $-1 < \nu < 0.5$. Real materials with negative values of ν are not of concern to us here in this book[8]. Some materials have such low values of ν that they are sometimes considered as zero for simplifying calculations.

More useful for theoretical considerations is the observation that the strain energy function is positive definite. From this observation follow important conclusions such as (i) the Saint-Venant's principle (with some restrictions), (ii) the uniqueness theorem (meaning that there is only one solution), (iii) the minimum total potential (energy) theorem, and (iv) the minimum complementary energy theorem.

Each of the topics mentioned above requires considerable amplification and explanation. However, these can hardly be attempted here. The limited purpose of this small section is to draw attention to the fact that the role of thermodynamics cannot be neglected in advanced studies in the mechanics of solids.

SOME ELEMENTARY CONCEPTS REVISITED

In this section we shall consider a couple of elementary concepts in the nature of a revisit. It is always a good idea to have a few quiet moments to reflect on ideas and concepts that we may have unwittingly taken for granted. Sometimes there are tricky situations lurking in the background. A closer second look sometimes reveals to us some of them. The reward is in the form of fresh insights and conceptual clarity of fundamentals, and is certainly worth the effort.

We shall examine if the simple formula $\sigma = P/A$, perhaps the first formula (that some, perhaps most, of us may have studied in our first course in strength of materials or mechanics of solids) has aspects that lie hidden.

Stress $\sigma = P/A$?

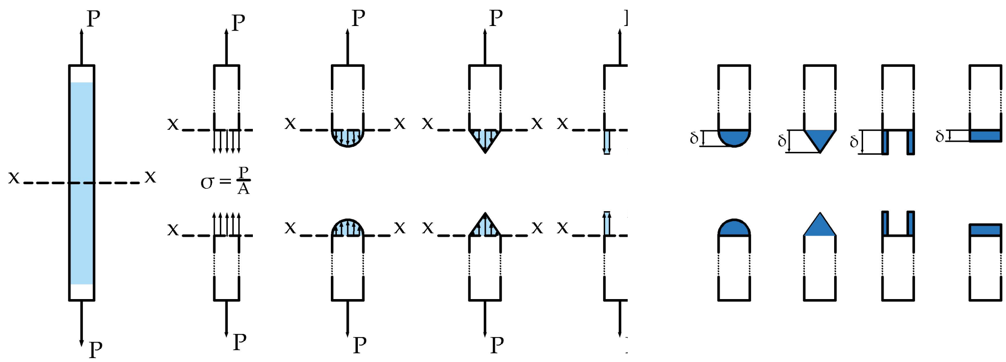
Let us review a simple problem, perhaps the simplest problem that we have studied in the spirit of a revisit.

Definition of the problem:

Consider a uniform bar loaded axially by two tensile loads $P - P$ [Fig. 8.7a]. We know, we have learned this in our first course in the mechanics of solids, that the stress is uniform[9]

[8] In relatively recent years, however, there have been references to strange situations.

[9] at all sections sufficiently far from the ends. Near the ends the stress will not be uniform.



(a) A uniform bar in tension (b) Several stress distributions (c) Deformed shapes

Figure 8.7: The stress can be seen to be uniform given by $\sigma = P/A$, but this conclusion can be arrived at only after considering the compatibility of displacements also.

and is given by $\sigma = P/A$. This answer is correct, but there is a conceptual error in its derivation as given in many books. Apparently we can arrive at the correct result using only the equation of equilibrium, viz., $\sigma \times A = P \quad \longrightarrow \quad P = \sigma/A$. One gets the impression that this is a statically determinate problem, as the correct answer was obtained by invoking *only the equation of equilibrium.*

Is this a statically determinate problem?

But is this true? No, of course, not. *All problems* in the mechanics of solids / the theory of elasticity are *statically indeterminate* internally.

A few stress distributions are shown in Fig. 8.7b. All of them satisfy the equation of equilibrium as long as the resultant of each of the stress distributions is equal to P. Among them only one can be correct. Why? Because the uniqueness theorem permits only one solution. Which is that one correct stress distribution?

Compatibility of displacements:

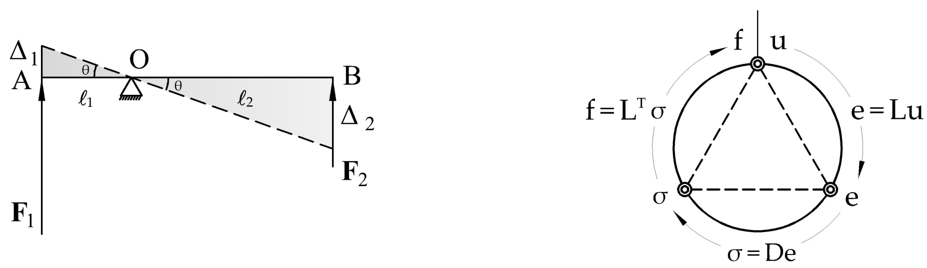
To answer this question, we have to consider the deformed shapes and examine which corresponds to a compatible displacement field. We can see almost immediately that *only the uniform stress distribution* corresponds to a strain distribution that corresponds to a displacement field that is compatible. Thus, among these various possibilities shown in Fig. 8.7b — there can be several other possibilities also — which can be regarded as eligible or qualified candidates, the only successful one is that shown in Fig. 8.7a (uniform stress distribution)! We conclude that the answer is indeed correct. But we are trying to learn with emphasis on strong conceptual understanding of the fundamentals. Thus, we must be aware of the requirement of compatibility of displacements also.

An assumption is sometimes effective:

Engineers often employ an assumption to simplify the problem. In this case, if we assume that the stress distribution is uniform across the section, we can obtain the correct answer using only the equation of equilibrium. Other examples are the calculation of the tangential (hoop) stresses in a thin cylinder subjected to an internal pressure, or more generally, in the calculation of membrane stresses in shells.

A Simple Lever

Next, we shall consider a simple lever and look at the equation of equilibrium. This may also give us fresh insights. Let us examine. Let us consider a simple lever AOB supported



(a) A simple lever AOB (b) Some interesting relationships

Figure 8.8: A kinematically admissible virtual displacement field is imposed [Fig. 8.8a]. The principle of virtual work may be written. In Fig. 8.8b some interesting relationships are shown.

at the fulcrum O, and acted upon by two vertical forces F_1 and F_2 at the two ends. We know the condition for static equilibrium: the net moment at the point $O = 0$.

Equilibrium: net moment at the point $O = F_1\, l_1 - F_2\, l_2 = 0$.

If a virtual displacement $d\theta$, consistent with the constraint conditions of the problem (technically stated as a kinematically admissible displacement field) is given, the system moves to the configuration (position) shown by the dotted line. From the geometry, we note that $\theta = \Delta_1/l_1 = \Delta_2/l_2$.

kinematics / geometry: $\theta = \dfrac{\Delta_1}{l_1} = \dfrac{\Delta_2}{l_2}$.

The bar AOB is assumed to be rigid. That is, the possible bending of the bar is not admissible. By the principle of virtual work

principle of virtual work: $-F_1\, \Delta_1 + F_2\, \Delta_2 = 0$.

These three equations are displayed below. Let us not fail to notice the role of each one of these three equations.

$$\text{equilibrium:} \qquad F_1\, l_1 - F_2\, l_2 = 0; \tag{8.3a}$$

$$\text{kinematics:} \qquad \theta = \frac{\Delta_1}{l_1} = \frac{\Delta_2}{l_2}; \tag{8.3b}$$

$$\text{virtual work:} \quad -F_1\, \Delta_1 + F_2\, \Delta_2 = 0. \tag{8.3c}$$

We can see that (8.3a) + (8.3b) $\longrightarrow$ (8.3c), and that (8.3c) + (8.3b) $\longrightarrow$ (8.3a). What this means is that the principle of virtual work is a *necessary* and *sufficient* condition for the equilibrium of a system. It is emphasised that the virtual displacement field shall be kinematically admissible. [The result of this simple demonstration — that the principle of virtual work is a necessary and sufficient condition for equilibrium — carries over to the more important and relevant case of deformable solids also.]

Let us observe the following relationships.

$$\left\{ \begin{array}{c} \Delta_1 \\ \Delta_2 \end{array} \right\} = \left\{ \begin{array}{c} -l_1 \\ l_2 \end{array} \right\} \theta; \quad \text{i.e.,} \quad \{\Delta\} = \mathcal{L}\,\theta \qquad \text{(kinematics); and} \tag{8.4}$$

$$M = \left\{ \begin{array}{cc} -l_1 & l_2 \end{array} \right\} \left\{ \begin{array}{c} -F_1 \\ F_2 \end{array} \right\}; \quad \text{i.e.,} \quad \{F\} = \mathcal{L}^T M \quad \text{(kinetics / equilibrium).} \tag{8.5}$$

Referring to Fig. 8.8b, let us note:

$$e = \mathcal{L}\,u \;\;\text{(strain-displacement);} \quad \sigma = D\,e \;\;\text{(material law);} \quad \mathcal{L}^T\,\sigma = f \;\;\text{(equilibrium).}$$

The same operator in two guises $\mathcal{L}$ and $\mathcal{L}^T$ appear in two apparently unrelated roles. At first it may appear that kinematics and kinetics / equilibrium are two entirely independent domains. But no; the same operator appears in both[10]. Nature seems to have some undercurrents or interconnections that are revealed only on close examination!

FROM THEORY TO PRACTICE

We have by now been exposed to certain aspects of the mechanics of solids. These should serve to give us an enlightened view of the practical aspects of design and construction. If the science of stress analysis is digested and absorbed, we would become better engineers. A couple of examples are given below to illustrate how a sound theoretical training is helpful in the practice of engineering profession.

Thin Walled Structures: Buckling

Whenever we deal with a thin structural member an alarm must ring inside: beware of the possibility of buckling! Thin walled members and buckling are associated ideas[11]. There are examples in industries, particularly in chemical engineering, where large cylinders containing fluids are required to boil at low temperatures. Boiling at low temperatures is possible only if the pressure is low. When, therefore, there is (partial) vacuum inside the cylinder, the atmospheric pressure can cause buckling, particularly if the wall thickness becomes effectively less as a result of prolonged exposure to harmful chemicals[12].

Stress Concentration

Stress concentration is the reason for failure in several practical problems. After learning the mathematical theory, an engineer can safely forget all the mathematics and even the formulae. He must, however, have a qualitative understanding of where stress concentration plays a decisive role. In this context, engineers will do well to distinguish between harmful and harmless corners. Shall we call them malignant and benign corners? Figs 8.9a and 8.9b show these. A practical example is shown in Fig. 8.10 where the corners C can be the sources of cracks.

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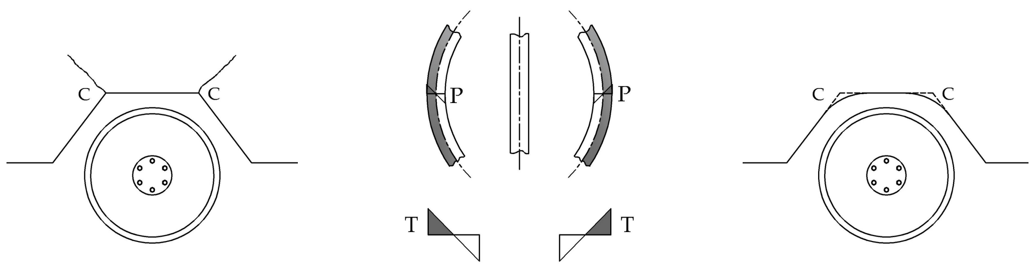
[10] It was Dr Gangan Prathap who gave me these insights. This help is gratefully acknowledged.

[11] like the problem of ragging in the season of admission of new students every year!

[12] This author, having been the consultant, is personally aware of one such massive collapse of a large cylindrical vessel. On measurement it was seen that the actual wall thickness had become 2.8 mm compared to the design value of 8.00 mm! When the cylinder buckled, it appeared that all hell had broken loose. Further details are not given here to protect the privacy.

(a) A harmless corner (b) A harmful corner

Figure 8.9: Reentrant corners are harmful; provide generous fillets at such corners.



(a) An unsatisfactory design (b) Stress reversal cycles (c) An improved design

Figure 8.10: Vibration leads to stress reversal cycles. A point like C, where there is a high stress concentration, can be a potential source of trouble. Cracks are possible. To solve this problem all we have to do is to understand the problem clearly and to reduce the stress concentration at the reentrant corners by providing generous fillets.

The figure shown [Fig. 8.10] refers to a bus. When the bus runs, particularly on rough roads, the body vibrates. This vibration causes stress reversals: tension to compression and back several times. This situation can lead to fatigue. A reentrant corner like C where there is stress concentration can be the source of cracks [Fig. 8.10a]. The solution is very simple: understand the problem and reduce the stress concentration at the reentrant corners by providing generous fillets [Fig. 8.10c]!

Crack in an R.C.C. Roof Slab

When a crack is noticed, or even suspected, the intuitive solution that occurs to untrained minds may be to provide a prop. But we engineers know that if this is attempted, a moment of the opposite sign would occur at the point of support. If such a measure is adopted, a crack which was only a suspicion will now be a confirmed reality! We engineers should not make such mistakes.

After a (RCC) shell is constructed, the supports must be removed. When doing so, several moments are called into play. This must be done strictly as directed by the designer, because several moments are called into play during this removal process[13].

[13] There was such a failure years ago in Chennai. A detailed enquiry revealed that the supports were removed without complying with the designer's clearly written directions.

Why Should We Calculate the (Small) Displacements?

Why do we attach so much importance to the calculation of displacements? For example, what does it matter whether the deflection at the end of a long structural member is two (2) mm or three (3) mm? Well, in some cases, it does not matter at all, but there are cases when it does matter. For example, in the analysis of statically indeterminate cases, the ('small') displacements lead to 'large' changes in the forces (and stresses).

Fracture Mechanics

(a) Stress near a crack tip (b) A progressing crack

Figure 8.11: The stress at a point P near a crack tip C is proportional to $1/\sqrt{r}$. Fig. 8.11b shows a crack progressing. The tip is progressing from C to C_1 and further to C_2.

Fracture mechanics is a relatively new branch that has come of age, and is now represented by a large body of literature. This has introduced new notions of the basis of design. The question that we were used to asking was this: what is the maximum safe load that can be applied so that the stress does not cross its safe limit of so much? We know that there are always micro-cracks in all real bodies. Fracture mechanics has shown that the stress near a crack tip is proportional to $1/\sqrt{r}$ where r is the distance of the point from the crack tip [Fig. 8.11a]. Thus, there is a singularity at the crack tip. The consequence is that no matter how small the applied load is, the stress will be infinitely high! In practice, however, infinitely high stresses cannot be developed, because the material would yield. Thus, near a crack tip there is a (small) region of plastic zone. The region in the immediate neighbourhood of a crack tip has assumed importance.

This situation makes it necessary for us to revise our notion of a safe design, and to ask a different question. Concepts like stress intensity factors (SIF) and crack opening displacement (COD) have assumed importance. We are also persuaded to classify cracks as harmless (benign?) and harmful (malignant?). If the crack opens by a small distance, there is some surface energy released. If this is more than the energy necessary to drive the crack further, then there could be a progressive catastrophic failure. On the other hand, if the energy released is less than that necessary to drive the crack forward, then there is perhaps no great harm if the existing crack sleeps there harmlessly! The so-called J-integral is very important in fracture mechanics, but we cannot discuss these advanced concepts here.

In the next chapter, we shall take up two-dimensional problems.

Chapter 9

TWO-DIMENSIONAL PROBLEMS

The general three-dimensional problems are generally very difficult to solve. Hence simplifications are made so that the problems may be treated as two-dimensional. These are approximations, but if the physical facts of the problems justify the simplifying assumptions, the error will be small[1]. The two main two-dimensional simplifications are (a) plane stress and (b) plane strain conditions (or assumptions)[2].

Plane Stress Condition

If the body is very thin, and if the loading is only in the plane of the thin body [Fig. 9.1a], plane stress conditions prevail. It is clear that on the two flat (stress-free) end surfaces (xy plane) $\sigma_{zz} = \tau_{zx} = \tau_{zy} = 0$. Now if the thickness is small, we may assume that these stress components are zero not only on the free surfaces at the ends, but also everywhere inside. Actually they may not be strictly zero, but they will be small. Why? Because on the left flat surface they are zero, and they can build up only slowly and gradually as we

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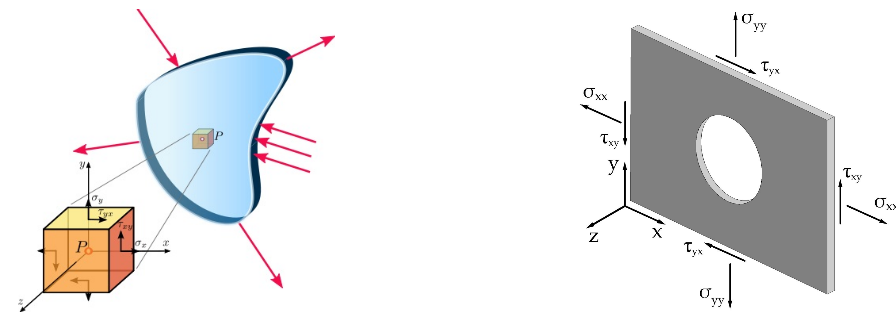
[1] The error is estimated when the approximate answer so obtained is compared to (i) the exact solution, if it is known, (ii) the experimental results, and (iii) the numerical solutions obtained by the finite element method. Theoretical methods of analysis and approximation theory also help us in assessing the goodness of approximations. We should realise that the solution is not just one number. Thus, comparison with the exact solution is not as simple as we might imagine. These considerations can be quite abstract.

Caution: Do not give an inferior status to approximate solutions. These may be approximate in the sense that the governing mathematical equations are not satisfied exactly at all points. However, the so-called exact solutions are almost always of an approximate problem where all kinds of simplifying assumptions are made. Approximate solutions are, on the other hand, (from the physical point of view) of a more realistically formulated problem. Thus, these may often be superior to the exact ones.

[2] Some people consider generalised plane stress as another approximation. We do not discuss this. Great advances have been made in the relatively recent past for two-dimensional problems. Several mathematical techniques that are thus used cannot be extended to three-dimensions. Complex variable theory, including conformal mapping, so effectively used by the great Russian elastician N.I. Mushkhelshvili — *Some Basic Problems in the Mathematical Theory of Elasticity*, P. Noordhoff, Groningen (1953); this book is a marvel — and his group is obviously not applicable for three-dimensional problems.



(a) A thin plate with only in-plane loading (b) $\sigma_{zz} = \tau_{zx} = \tau_{zy}$ on the two flat end surfaces

Figure 9.1: Fig. 9.1a shows a thin plate with only in-plane loading. The stress components $\sigma_{zz} = \tau_{zx} = \tau_{zy}$ are all zero on the two flat (stress-free) end surfaces. They may be regarded as very small inside the body also. Thus, plane stress conditions prevail for this case.

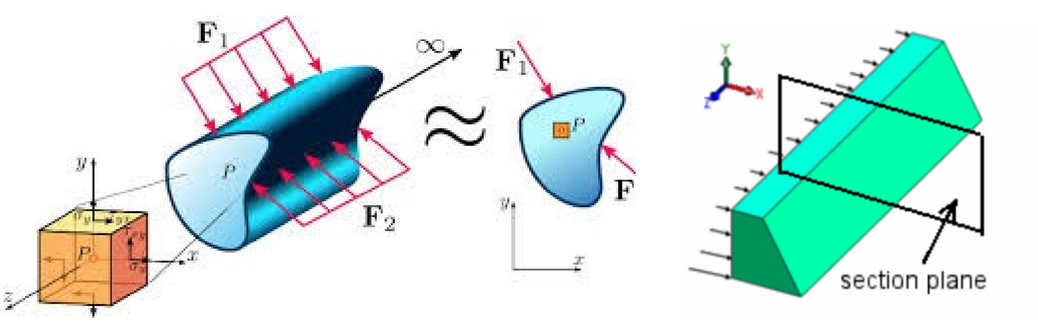
move inside (as demanded by continuity requirements), and then they would have to be zero again as we move to the right and reach the right flat free surface. Thus, only three stress components need be considered, because the others are all negligibly small.

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \qquad \longrightarrow \qquad \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that the stress matrix has effectively become of size 2×2. Note further that $e_{zz} \neq 0$.

$$e_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) \neq 0.$$

Plane Strain Condition



(a) An infinitely long body (b) An infinitely long body, a dam

Figure 9.2: Two examples where plane strain conditions prevail. The bodies are ‘infinitely’ long — sufficiently long — and all cross-sections identical and identically loaded.

Let us now go to the other extreme, as it were: the body is infinitely long, and the geometry, the material, and the loadings are identical on all cross-sections. Figs 9.2a, 9.2b

are two such examples of a long body with identical cross-sections and loadings all along the length which is infinite (sufficiently long). The cross-sections (except possibly at the ends) being identical, the strain e_{zz} can be regarded as zero. The strain matrix now becomes

$$\begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{bmatrix} \longrightarrow \begin{bmatrix} e_{xx} & e_{xy} & 0 \\ e_{yx} & e_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that the strain matrix has effectively become of size 2×2. Note further that $\sigma_{zz} \neq 0$. Why is this so? Because

$$e_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] = 0 \qquad \longrightarrow \qquad \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \neq 0.$$

For both these simplifications, it is sufficient to determine the stress components σ_{xx}, σ_{yy}, and $\tau_{xy} = \tau_{yx}$. We can show that, if one of these cases is solved, then the other too is solved by making a few changes as shown below.

$$\text{plane stress solution} \longrightarrow \text{plane strain solution:} \quad E \longrightarrow \frac{E}{1-\nu^2}; \quad \nu \longrightarrow \frac{\nu}{1-\nu};$$

$$\text{plane strain solution} \longrightarrow \text{plane stress solution:} \quad E \longrightarrow \frac{E(1+2\nu)}{(1+\nu)^2}; \quad \nu \longrightarrow \frac{\nu}{1+\nu}.$$

The best brains have laboured over our subject for 300 years or more. Thus, it is but natural that great advances have already been made. Several advanced concepts, methods and techniques have been developed and used over the years. We shall see only a few of the simpler methods. First we shall see the use of Airy's stress function.

AIRY'S STRESS FUNCTION IN RECTANGULAR COORDINATES

We know that there are three fundamental governing equations — the three pillars of the theory of elasticity or the mechanics of solids. Among them the equations of equilibrium are usually written in terms of the stress components. As *all* problems in the mechanics of solids are statically indeterminate internally, these equations of equilibrium alone cannot give us the solution. They have to be supplemented by the constitutive equations and the strain-displacement relations. The former are in terms of the stress and strain components, the latter in terms of the strain and displacement components. This situation makes it difficult to handle them effectively. If we try to have an all stress formulation, the constitutive equations can be used to write the strain components in terms of the stress components. Now the compatibility equations step in; they prevent stress distributions (satisfying the equations of equilibrium) that correspond to strain distributions which, in turn, lead to incompatible (impossible) displacement distributions. Thus, when we begin with a stress field, the compatibility condition is indeed a genuine governing equation[3].

[3] See the contrasting situation: if the displacements are assumed as, for example, in the Saint-Venant's theory of torsion, the compatibility equations have only a passive role. That is, the compatibility conditions are automatically satisfied, because impossible (incompatible) displacements will not be assumed.

Equations of Equilibrium

For the two-dimensional case, the two differential equations of equilibrium are:

$$\text{with body forces:}\quad \frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+F_x=0 \qquad \frac{\partial \sigma_{yy}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+F_y\;=0 \tag{9.1}$$

$$\text{only self-weight:}\quad \frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0 \qquad \frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}+\rho\, g\;=0 \tag{9.2}$$

$$\text{no body force:}\quad \frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0 \qquad \frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}=0 \tag{9.3}$$

There are three unknowns and two equations. In such situations, it is a usual mathematical technique to use an auxiliary function, say, $\phi=\phi(x,y)$ defined suitably so that both these equations (of equilibrium) are automatically satisfied.

$$\sigma_{xx}=\frac{\partial^2\phi}{\partial y^2}-\rho\, g\, y; \qquad \sigma_{yy}=\frac{\partial^2\phi}{\partial x^2}-\rho\, g\, y; \qquad \tau_{xy}=-\frac{\partial^2\phi}{\partial x\partial y}. \tag{9.4}$$

We can see by direct verification that Eqs (9.2) are satisfied. The advantage is that we need not consider the equations of equilibrium at all as long as we use such a cleverly defined auxiliary function $\phi=\phi(x,y)$. This auxiliary function is called Airy's stress function[4].

Compatibility Equation

The compatibility equation connecting the three strain components e_{xx}, e_{yy}, $e_{xy}=e_{yx}$ is

$$\frac{\partial^2 e_{xx}}{\partial y^2}+\frac{\partial^2 e_{yy}}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y} \qquad \left(=2\frac{\partial^2 e_{xy}}{\partial x\partial y}\right). \tag{9.5}$$

We desire to express this (restrictive[5]) equation in terms of the stress components and, thereby, in terms of ϕ. Such an equation — essentially the compatibility condition — in terms of ϕ is the governing equation to be satisfied by the Airy's stress function ϕ. We shall now derive this equation, first in terms of the stress components and later in terms of ϕ. We shall do this for the two cases of plane stress and plane strain.

(a) For the plane stress case:

For the plane stress case,

$$e_{xx}=\frac{1}{E}\left[\sigma_{xx}-\nu\sigma_{yy}\right]; \qquad e_{yy}=\frac{1}{E}\left[\sigma_{yy}-\nu\sigma_{xx}\right]; \qquad e_{xy}=\frac{1}{2\,G}\tau_{xy}=\frac{(1+\nu)}{E}\,\tau_{xy}.$$

If we substitute these expressions (in terms of the stresses) in Eq. (9.5)[6] we obtain

$$\frac{\partial^2}{\partial y^2}(\sigma_{xx}-\nu\sigma_{yy})+\frac{\partial^2}{\partial x^2}(\sigma_{yy}-\nu\sigma_{xx})=2(1+\nu)\frac{\partial^2\tau_{xy}}{\partial x\partial y}. \tag{9.6}$$

---

[4] Named after the English mathematician and astronomer Sir George Biddel Airy (July 1801 - Jan. 1892).

[5] This equation puts restrictions on possible stress fields; hence the word restrictive.

[6] For plane stress problems, there are other compatibility conditions that are violated by the assumptions made. However, it can be shown that the methods presented here will still give good results. The theory of elasticity is full of such tricky situations. It is not possible to discuss all this here.

For the case when the self-weight is the only body force, we have (on differentiating the first of Eqs (9.2) w.r.to x, and the second w.r.to y, and adding up)

$$2\,\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{\partial^2 \sigma_{xx}}{\partial x^2} - \frac{\partial^2 \sigma_{yy}}{\partial y^2}.$$

When the right hand side of Eq. (9.6) is replaced using this equation, we obtain

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_{xx} + \sigma_{yy}) = 0.$$

For the general case of body forces [Eqs (9.5)], the same procedure yields

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_{xx} + \sigma_{yy}) = -(1+\nu)\left[\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right]. \tag{9.7}$$

This is an important result: the sum of the stresses, the first stress invariant ($\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_{11} + \sigma_{22} + \sigma_{33}$) satisfies the Laplace's / Poisson's equation!

Here we reach the conclusion: the compatibility equation expressed in terms of the stress components is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_{xx} + \sigma_{yy}) = -(1+\nu)\left[\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right]. \tag{9.8}$$

(b) For the plane strain case:

For the plane strain case $\sigma_{zz} = \nu(\sigma_{xx}+\sigma_{yy})$ because $e_{zz} = \dfrac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy}\right] = 0$. Using Hooke's law

$$\begin{aligned}
e_{xx} &= \frac{1}{E}\left[(1-\nu^2)\sigma_{xx} - \nu(1+\nu)\sigma_{yy}\right],\\
e_{yy} &= \frac{1}{E}\left[(1-\nu^2)\sigma_{yy} - \nu(1+\nu)\sigma_{xx}\right],\\
e_{xy} &= \frac{(1+\nu)}{E}\,\tau_{xy},
\end{aligned}$$

we find that the compatibility equation expressed in terms of the stress components is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_{xx} + \sigma_{yy}) = -\frac{1}{(1-\nu)}\left[\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right]. \tag{9.9}$$

Plane stress and plane strain cases:

On comparing Eqs (9.7) and (9.9), we find that the plane stress and plane strain problems are similar with only minor changes. (See p. 9-3.)

For the case when the self-weight is the only body force the governing equations are the following.

equilibrium, x direction: $\dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} = 0;$

$$\text{equilibrium, } y \text{ direction:} \quad \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \rho\, g \; = 0;$$

$$\text{compatibility:} \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_{xx} + \sigma_{yy}) = 0.$$

$$\text{boundary conditions:} \quad \overset{(\nu)}{T}_x \equiv p_x = l\, \sigma_{xx} + m\, \tau_{yx}; \qquad \overset{(\nu)}{T}_y \equiv p_y = l\, \tau_{xy} + m\, \sigma_{yy}.$$

Compatibility equation in terms of the Airy's stress function:

We shall now see the compatibility equation in terms of the Airy's stress function $\phi = \phi(x, y)$.

(a) When there is no body force, the stress components are related to the Airy's stress function by

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}; \qquad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}; \qquad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}. \tag{9.10}$$

(b) When the self-weight is the only body force, the stress components are related to the Airy's stress function by Eq. (9.4) which is reproduced below.

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} - \rho\, g\, y; \qquad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} - \rho\, g\, y; \qquad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}.$$

The compatibility condition is

$$\bigtriangledown^4 \phi \equiv \bigtriangledown^2 (\bigtriangledown^2 \phi) \equiv \frac{\partial^4 \phi}{\partial x^4} + 2\, \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0, \tag{9.11}$$

which is the well known biharmonic equation! [This equation plays a major role in the theory of plates and other areas of the theory of elasticity. The Laplace's and the biharmonic (double Laplacian) equations and the consequences cover a lot of ground in applied mathematics.]

(c) More generally when the body forces are derivable from a potential, say $V = V(x, y)$,

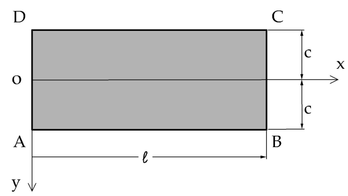
$$\sigma_{xx} - V = \frac{\partial^2 \phi}{\partial y^2} - \rho\, g\, y; \qquad \sigma_{yy} - V = \frac{\partial^2 \phi}{\partial x^2} - \rho\, g\, y; \qquad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}.$$

The compatibility condition is

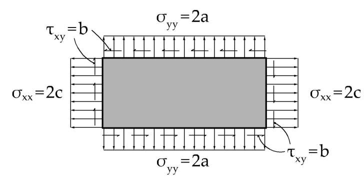
$$\bigtriangledown^4 \phi \equiv \bigtriangledown^2 (\bigtriangledown^2 \phi) \equiv \frac{\partial^4 \phi}{\partial x^4} + 2\, \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = -(1 - \nu) \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right). \tag{9.12}$$

An analogous equation holds for the case of plane strain.

In conclusion the biharmonic equation (with appropriate changes depending on the context) is the compatibility condition. It is the governing equation when we use the Airy's stress function to solve two-dimensional problems. The boundary conditions also will have to be written in terms of the partial derivatives of ϕ. We shall solve some simple problems using the Airy's stress function.



(a) A rectangular block with its dimensions



(b) Stress components on the boundary

Figure 9.3: Fig. 9.3a shows a rectangular block and Fig. 9.3b the boundary conditions. Note that the stress components $\sigma_{xx} = 2c$, $\sigma_{yy} = 2a$, $\tau_{xy} = \tau_{yx} = -b$ are all constant everywhere. Note further that the shear stress is marked in the negative direction and that, consequently, it is not marked as $-b$.

Solution of Problems Using Airy's Stress Function (Inverse Method)

Some simple problems can be solved using the Airy's stress function. However, this is an inverse method. That is, we choose a function, typically a polynomial in x and y, and find out what problem it solves. This is called an inverse method because we go backward as it were, from the answer to the problem, instead of going forward from the problem to its solution (answer). We shall demonstrate the method with a simple example.

A second degree polynomial:
Let us choose a second degree polynomial $\phi = \phi(x, y) = ax^2 + bxy + cy^2$. We know that for every choice of ϕ, the equations of equilibrium are automatically satisfied; there is no need to check even. Every choice of ϕ is fine as far as the equations of equilibrium are concerned. But the choice of ϕ may not correspond to compatible displacements. The compatibility equation is the master of the situation. Not every function ϕ will do; only those satisfying the biharmonic equation (which is the compatibility equation expressed in terms of ϕ) are acceptable. Here the governing equation is $\bigtriangledown^4\phi = 0$. Here, however, this equation is obviously satisfied. The stress components are

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = 2c; \qquad \sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = 2a; \qquad \tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -b.$$

These are all constant. If the boundary conditions are satisfied, this is *a solution.* The uniqueness theorem guarantees that this is *the only solution.* A rectangular block with the boundary conditions shown [Fig. 9.3] is the problem that this chosen polynomial solves.

Airy's Stress Function — Solution by a Polynomial

Let us choose a stress function ϕ in the form

$$\phi = \phi(x, y) = \frac{a}{4\times 3}\,x^4 + \frac{b}{3\times 2}\,x^3y + \frac{c}{2}\,x^2y^2 + \frac{d}{3\times 2}\,xy^3 + \frac{e}{4\times 3}\,y^4. \tag{9.13}$$

This ϕ must satisfy the compatibility condition $\bigtriangledown^4 = 0$. Substituting $\phi = \phi(x, y)$ in the compatibility condition $\bigtriangledown^4 = 0$, we obtain $e = -(2c + a)$.

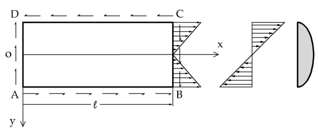


Figure 9.4: A block with stresses applied

We desire to solve this problem by using the Airy's stress function. However, this is an inverse method. We shall examine some simple cases like this, and see how a given problem can be solved by superposition.

We shall assume a polynomial with unknown coefficients, and try to determine these coefficients so that the relevant conditions are satisfied.

The stress components are, thus,

$$\begin{aligned}
\sigma_{xx} &= \frac{\partial^2 \phi}{\partial y^2} &&= c\,x^2 + d\,xy - (2c + a)y^2; \\
\sigma_{xx} &= \frac{\partial^2 \phi}{\partial y^2} &&= c\,x^2 + d\,xy - (2c + a)y^2; \\
\tau_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y} &&= -\frac{b}{2}x^2 - 2c\,xy - \frac{d}{2}y^2.
\end{aligned}$$

We can choose these constants as $a = b = c = e = 0$; $d \neq 0$. Now we find — this corresponds to the problem shown in Fig. 9.5a — that

$$\sigma_{xx} = d\,xy; \qquad \sigma_{yy} = 0; \qquad \tau_{xy} = -\frac{d}{2}. \tag{9.14}$$

Bending of a cantilever? Comments:

Now we may wonder if this corresponds to the problem of bending of a cantilever. If a cantilever of length l is fixed at the end $x = l$ and loaded by an end load P [Fig. 9.5b], the stresses are as follows.

(a) The bending stresses (i) are linearly distributed across the cross-section, and (ii) grow linearly with the distance x starting from zero at the free end $x = 0$. This indeed is what happens: the bending moment diagram shown confirms this fact.

(b) The shear stresses (cross-shear in bending) across the cross-section is parabolic with zero at the extreme fibres and a maximum at mid-section. The distribution is the same on all cross-sections. This is understandable; the shear force is constant along the length of the cantilever.

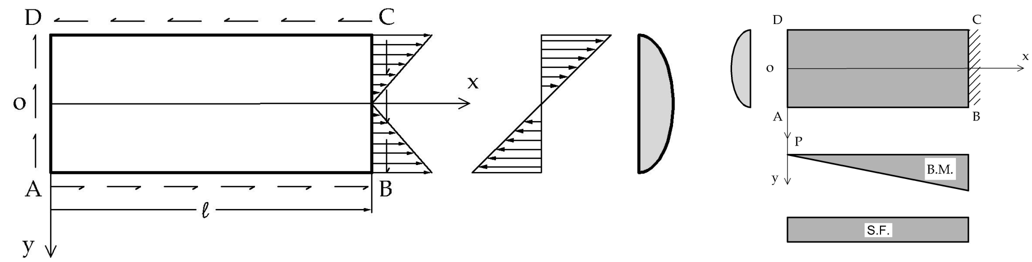
(c) However, this solution [Eq. (9.14)] indicates that there is a shear stress $\tau_{xy} = \tau_{yx}$ on the edges AB and DC. There are no such shear stresses in a cantilever in bending.

Thus, the solution [Eq. (9.14)] corresponds to the problem of a cantilever except for the shear stresses τ_{yx} on the edges AB and DC. No such shear stresses can occur because these are free edges (free surfaces).

Now the question arises: is it possible to have another solution that gives us such a shear stress distribution? Would it be possible to superpose this stress distribution so that the problem of bending of a cantilever can be solved? After all, these are all linear equations,

and superposition should be possible. The answer is: yes, this is possible. This technique of superposition is often employed. We shall demonstrate this below.

Bending of a Cantilever



(a) A body with stress components (b) A cantilever with an end load P

Figure 9.5: A rectangular block subjected to some stress components. Fig. 9.5b shows a cantilever loaded by an end load. From the solution of the problem defined by Fig. 9.5a and superposing a shear stress, the cantilever problem can be solved.

Let us try to solve the problem of bending of a uniform cantilever $ABCD$ loaded by an end load P [Fig. 9.5b]. It is clear that the edges AB and DC (representing free surfaces) are free of normal stresses σ_{yy} and shear stresses $\tau_{yx}(=\tau_{xy})$, and that the shear stresses τ_{xy} distributed on the end (AD) add up to the applied load P. Accordingly, to the stresses given by Eq. (9.14), we superpose $\tau_{xy}=-b$ and choose the various constants so that all the required conditions are satisfied. Thus,

$$\sigma_{xx}=d\,xy; \qquad \sigma_{yy}=0; \qquad \tau_{xy}=-b-\frac{d}{2}. \tag{9.15}$$

To ensure that the edges AB and DC are free of shear stresses, we should have

$$\tau_{xy}\Big|_{y=+c}=\tau_{xy}\Big|_{y=-c}=-b-\frac{d}{2}\,c^2 \qquad\longrightarrow\qquad d=-\frac{2\,b}{c^2}.$$

Similarly, to ensure that the shear stress τ_{xy} on every cross-section adds up to the applied load (shear force) P, we should have

$$-\int_{-c}^{+c}\tau_{xy}\,dy=\int_{-c}^{+c}\left(b-\frac{b}{c^2}\,y^2\right)dy=P \qquad\longrightarrow\qquad b=-\frac{3}{4}\frac{P}{c}.$$

With these values of b and d now determined, the stress components [Eq. (9.15)] now appear as

$$\sigma_{xx}=-\frac{3}{2}\frac{P}{c^2}\,xy; \qquad \sigma_{yy}=0; \qquad \tau_{xy}=-\frac{3\,P}{4\,c}\left(1-\frac{y^2}{c^2}\right). \tag{9.16}$$

As the second moment of area $I=[1\times(2c)^3]/12=2\,c^3/3$, these may be rewritten as

$$\sigma_{xx}=-\frac{P\,xy}{I}; \qquad \sigma_{yy}=0; \qquad \tau_{xy}=-\frac{P}{I}\frac{1}{2}\left(c^2-y^2\right).$$

This solution is in perfect agreement — $P\,x$ is the bending moment at any cross-section — with the result of the Euler-Bernoulli theory of bending. The shear stress τ_{xy} at the fixed end $x = l$ should be distributed as demanded by this equation. In practice, it is seldom possible to ensure this, but there is no cause for alarm. Saint-Venant's principle can be used, and we can regard that the stresses are still given by Eq. (9.16) at all cross-sections a little away from the fixed end. (See also the examples worked out in a later chapter 13.)

Having obtained the stress components [Eq. (9.16)], it is now necessary to determine the displacements. We shall illustrate the procedure for this problem.

Determination of Displacements

From the stress components [Eq. (9.16)] we can obtain the strain components, and then the displacements by direct integration.

$$e_{xx} = \frac{\partial u}{\partial x} = \frac{\sigma_{xx}}{E};\;\; e_{yy} = \frac{\partial v}{\partial y} = -\nu\frac{\sigma_{xx}}{E} = \frac{\nu\,P\,xy}{EI};\;\; e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) = \frac{\tau_{xy}}{2G} = -\frac{P}{4\,IG}\,(c^2 - y^2).$$

On integration of the first two of these equations, we obtain

$$u = -\frac{P\,xy}{2\,EI} + f(y); \quad v = \frac{\nu\,P\,xy^2}{2\,EI} + g(x) \quad [f(y),\, g(x) \text{ arbitrary (unknown) functions}]. \tag{9.17}$$

If we substitute these in the third of the equations, we have

$$-\frac{P\,x^2}{2\,EI} + f'(y) + \frac{\nu\,P\,y^2}{2\,EI} + g'(x) = -\frac{P}{2\,IG}\,(c^2 - y^2);$$

that is,

$$\left[-\frac{P\,x^2}{2\,EI} + g'(x)\right] + \left[\frac{\nu\,P\,y^2}{2\,EI} - \frac{P}{2\,IG}\,y^2 + f'(y)\right] = \left[-\frac{P}{2\,IG}\,c^2\right].$$

The expressions within the three square brackets $[\cdots]$ are, respectively, (i) a function of x only, (ii) a function of y only, and (iii) a constant. What does this equation imply? Well, each of them must be a constant, the equation being of the form $A + B = C$. Thus,

$$A + B = C = -\frac{Pc^2}{2\,IG}; \qquad g'(x) = \frac{P\,x^2}{2\,EI} + B; \qquad f'(y) = -\frac{\nu\,Py^2}{2\,EI} + \frac{Py^2}{2\,IG} + A. \tag{9.18}$$

Direct integration of the last two equations yields

$$\begin{aligned} f(y) &= -\frac{\nu\,Py^3}{6\,EI} + \frac{Py^3}{6\,EI} + Ay + D; \\ g(x) &= \frac{Px^3}{6\,EI} + Bx + E. \end{aligned}$$

If we substitute the expressions for $f(y)$ and $g(x)$ in Eq. (9.17), we obtain

$$u = -\frac{Px^2y}{2\,EI} + f(y) = -\frac{Px^2y}{2\,EI} - \frac{\nu\,Py^3}{6\,EI} + \frac{Py^3}{6\,IG} + Ay + D; \tag{9.19}$$

$$v = \frac{\nu\, Pxy^2}{2\,EI} + g(x) = \frac{\nu\, Pxy^2}{2\,EI} + \frac{Px^3}{6\,EI} + Bx + E. \tag{9.20}$$

There are four (4) constants; their values are still unknown. At this stage we should realise that the displacements can be determined only to within a rigid body movement. Why is this so? This is because rigid body movements are not associated with strains or stresses. Only if the rigid body movement is arrested (prevented) can the displacements be determined uniquely. The rigid body motions are (i) translation along, say, the x direction, (ii) translation along the y direction, and (iii) rotation in the xy plane. The three conditions arresting these three rigid body motions and the first of Eqs (9.18) give us the necessary four equations to determine the four (4) constants A, D, B and E. Now we shall proceed to determine these constants.

Determination of the four constants $\boldsymbol{A}, \boldsymbol{D}, \boldsymbol{B}$ and $\boldsymbol{E}$:

We can arrest the translation in both $x-$ and $y-$directions by fixing any point, say, the point O. Then,

$$u\Big|_{x=l,y=0} = 0 \qquad \longrightarrow \qquad D = 0;$$

$$v\Big|_{x=l,y=0} = 0 \qquad \longrightarrow \qquad E = -\frac{P\,l^3}{6\,EI} - Bl.$$

With these values, the v displacement (which is really the deflection of the central line of the beam) [Eq. (9.20) with $y = 0$], is given by

$$v\Big|_{y=0} = \frac{P\,x^3}{6\,EI} - \frac{P\,l^3}{6\,EI} - B(l - x). \tag{9.21}$$

Again we can arrest the rotation of the beam in the xy plane by demanding either (i) the horizontal element is fixed (at $x = l, y = 0$), or (ii) the vertical element is fixed (at $x = l, y = 0$). These two physical conditions correspond to

(i) $\quad \dfrac{\partial v}{\partial x}\Big|_{x=l,y=0} = 0 \quad$ (no rotation of a horizontal element at the fixed end);

(ii) $\quad \dfrac{\partial u}{\partial x}\Big|_{x=l,y=0} = 0 \quad$ (no rotation of a vertical element at the fixed end).

If we use the first condition (i), we have from Eq. (9.21) and the first of Eq. (9.18),

$$B = -\frac{P\,l^2}{2\,EI}; \qquad A = \frac{P\,l^2}{2\,EI} - \frac{P\,c^2}{2\,IG}.$$

Thus, we obtain the displacement field as

$$u = -\frac{P\,x^2 y}{2\,EI} - \frac{\nu\,P\,y^3}{6\,EI} + \frac{P\,y^3}{6\,IG} + \left(\frac{P\,l^2}{2\,EI} - \frac{P\,c^2}{2\,IG}\right) y;$$

$$v = \frac{\nu\,P\,xy^2}{2\,EI} + \frac{P\,x^3}{6\,EI} - \frac{P\,l^2 x}{2\,EI} + \frac{P\,l^3}{3\,EI}.$$

The deflection is obtained from this expression for v by setting $y = 0$ (because we are looking for the deflection of the middle line $y = 0$) as

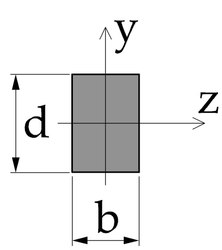
$$\text{deflection:} \quad v\Big|_{y=0} = \frac{P\,x^3}{6\,EI} - \frac{P\,l^2 x}{2\,EI} + \frac{P\,l^3}{3\,EI} \quad \longrightarrow \quad \text{end deflection:} \; v\Big|_{x=0,y=0} = \frac{P\,l^3}{3\,EI}!$$

This is the answer that we are all familiar with: the deflection at the end of a cantilever loaded by an end load P! A similar result can be obtained by using the condition (ii).

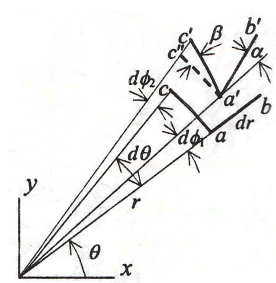
AIRY'S STRESS FUNCTION IN POLAR COORDINATES

Several problems of technical importance can be solved by using the polar coordinate (r, θ) coordinate system. The first job is to obtain the governing equations in this system. These equations are (i) the equations of equilibrium, (ii) the strain-displacement relations, and (iii) the constitutive equations — the three pillars of elasticity theory — are these equations. Additionally, (iv) the compatibility equation also plays a vital role[7]. We shall consider them one by one.

Equations of Equilibrium



(a) To derive the equations of equilibrium



(b) To derive the strain-displacement relations

Figure 9.6: To derive (i) the equations of equilibrium, and (ii) the strain-displacement relations

We have already derived the equations of equilibrium in the cylinder polar (r, θ, z) system. The equations that we seek here are the two-dimensional simplification and can be obtained from Eqs (5.16a, 5.16b, 5.16c, p. 5-14). Here, however, we shall derive the two equations afresh.

The various stress components on an elemental block are shown [Fig. 9.6a]. We can now write down the equations of equilibrium in both (i) the radial, and (ii) the tangential (circumferential) directions. Summing up the forces in the radial (r) direction, we have

$$\left[\sigma_{rr} + \frac{\partial \sigma_{rr}}{\partial r} dr\right](r + dr)d\theta + \left[\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial \theta}\, d\theta\right] dr\, \cos\left(\frac{d\theta}{2}\right) - \tau_{r\theta}\, dr\, \cos\left(\frac{d\theta}{2}\right) - \sigma_{rr}\, r\, d\theta -$$

[7] We have seen that when we work with Airy's stress functions, the equations of equilibrium are automatically satisfied. Now we try to obtain some solutions by the inverse method. When we use an Airy's stress function, we begin from all possible stress fields. All these are not the correct ones; only one of them is. Only this correct one corresponds to compatible displacements. Thus, the compatibility equation plays the real role of a governing equation.

$$\left[\sigma_{\theta\theta} + \frac{\partial\sigma_{\theta\theta}}{\partial\theta}\,d\theta\right] dr\,\sin\left(\frac{d\theta}{2}\right) - \sigma_{\theta\theta}\,dr\,\sin\left(\frac{d\theta}{2}\right) + F_r(r\,dr\,d\theta) = 0.$$

The tangential stresses $\sigma_{\theta\theta}$ do contribute a little bit to the radial forces, because the former are not quite perpendicular to the radial direction; a small angle $d\theta$ is included. Cleaning up this, we arrive at the differential equation of equilibrium in the radial direction.

By following a similar procedure, we can obtain the companion equation in the tangential direction also. These two equations are shown below[8].

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} + F_r = 0; \tag{9.22a}$$

$$\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\tau_{r\theta}}{\partial r} + \frac{2\,\tau_{r\theta}}{r} + F_\theta = 0. \tag{9.22b}$$

Next, we shall see the strain displacement relations.

Strain-displacement Relations

We are trying to establish the relations connecting the radial (u) and tangential (v) displacements, and the corresponding strains (e_{rr}) (radial) and ($e_{\theta\theta}$) (tangential). We shall do this by referring to Fig. 9.6b and working out the following expressions using geometry.

A typical line ab before deformation moves to ($a'b'$) after deformation. The length ($a'b'$) can be worked out in terms of both (i) u (radial) and v (tangential) of a; and (ii) $u + (\partial u/\partial r)\,dr$ (radial) and $v + (\partial v/\partial r)\,dr$ (tangential). Thus,

$$(a'b')^2 = \left(dr + \frac{\partial u}{\partial r}\,dr\right)^2 + \left(\frac{\partial v}{\partial r}\,dr\right)^2$$

The radial and tangential strains are expressed in terms of the displacements. This is what we desire to obtain.

$$e_{rr} = \frac{a'b' - ab}{ab} = \frac{\sqrt{(\cdots)^2 + (\cdots)^2} - dr}{dr} \approx \frac{\partial u}{\partial r}$$

$$v_a = v = r\,d\phi_1; \qquad v_c = v + \frac{\partial v}{\partial\theta}\,d\theta = r\,d\phi_2$$

$$e_{\theta\theta} = \frac{a'c'' - ac}{ac} = \frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{u}{r}; \qquad 2\,e_{r\theta} \equiv \gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial\theta} - \frac{v}{r}.$$

In this way we arrive at the following strain-displacement relations.

$$e_{rr} = \frac{\partial u}{\partial r} \tag{9.23a}$$

$$e_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{u}{r} \tag{9.23b}$$

$$2\,e_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial\theta} - \frac{v}{r} \tag{9.23c}$$

[8] See p. 13-31 also where these equations are obtained by coordinate transformation and stress transformation equations.

Constitutive Equations

We shall now consider the constitutive equations (generalised Hooke's law, suitably simplified for our context of two-dimensional stress and strain fields). There is nothing to discuss here. These are

$$e_{rr} = \frac{\sigma_{rr}}{E} - \nu\frac{\sigma_{\theta\theta}}{E} = \frac{1}{E}(\sigma_{rr} - \nu\sigma_{\theta\theta}) \tag{9.24a}$$

$$e_{\theta\theta} = \frac{\sigma_{\theta\theta}}{E} - \nu\frac{\sigma_{rr}}{E} = \frac{1}{E}(\sigma_{\theta\theta} - \nu\sigma_{rr}) \tag{9.24b}$$

$$2\,e_{r\theta} \equiv \gamma_{r\theta} = \frac{\tau_{r\theta}}{G} = \frac{2(1+\nu)}{E}\,\tau_{r\theta}. \tag{9.24c}$$

These relations are more conveniently written, (i) strains in terms of stresses, and (ii) stresses in terms of strains, in matrix form as

$$\begin{Bmatrix} e_{rr} \\ e_{\theta\theta} \\ \gamma_{r\theta} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \tau_{r\theta} \end{Bmatrix}; \text{ and } \begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \tau_{r\theta} \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{Bmatrix} e_{rr} \\ e_{\theta\theta} \\ \gamma_{r\theta} \end{Bmatrix}.$$

We need the compatibility equation also in terms of the polar coordinates (r, θ). We shall obtain this as shown below.

Compatibility Equation: Biharmonic Equation in Polar Coordinates

We have seen that the compatibility equation in the context of using Airy's stress function is the biharmonic equation [Eq. (9.11), p. 9-6]. It is

$$\bigtriangledown^4\phi \equiv \frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\,\partial y^2} + \frac{\partial^4\phi}{\partial y^4} \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) \equiv (\bigtriangledown^2)(\bigtriangledown^2\phi) = 0.$$

All we have to do is to express this equation in polar coordinates. This is done by recasting the equation with (r, θ) as the independent variables. The usual coordinate transformation using the chain rule of partial differentiation is the tool to be employed for this exercise.

The two systems of coordinates (x, y) and (r, θ) are related by

$$x = r\,\cos\theta; \qquad r = \sqrt{x^2 + y^2};$$
$$y = r\,\sin\theta; \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

The derivatives are calculated as

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta; \qquad \frac{\partial\theta}{\partial x} = -\frac{y}{r^2} = -\frac{\sin\theta}{r};$$
$$\frac{\partial r}{\partial y} = \frac{x}{r} = \sin\theta; \qquad \frac{\partial\theta}{\partial y} = \frac{x}{r^2} = \frac{\cos\theta}{r}.$$

The terms are calculated one by one as shown below.

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial^2}{\partial x^2} = \cos\theta\,\frac{\partial}{\partial r}\left(\frac{\partial}{\partial x}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial x}\right)$$

$$= \cos^2\theta\,\frac{\partial^2}{\partial r^2} - \frac{\sin 2\theta}{r}\,\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\,\frac{\partial^2}{\partial\theta^2} + \frac{\sin^2\theta}{r}\,\frac{\partial}{\partial r} + \frac{\sin 2\theta}{r^2}\,\frac{\partial}{\partial\theta} \tag{9.25a}$$

$$\frac{\partial^2}{\partial y^2} = \sin^2\theta\,\frac{\partial^2}{\partial r^2} + \frac{\sin 2\theta}{r}\,\frac{\partial^2}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\,\frac{\partial^2}{\partial\theta^2} + \frac{\cos^2\theta}{r}\,\frac{\partial}{\partial r} - \frac{\sin 2\theta}{r^2}\,\frac{\partial}{\partial\theta} \tag{9.25b}$$

$$\frac{\partial^2}{\partial x\partial y} = \sin\theta\,\cos\theta\,\frac{\partial^2}{\partial r^2} + \frac{\cos 2\theta}{r}\,\frac{\partial^2}{\partial r\partial\theta} - \frac{\sin\theta\,\cos\theta}{r^2}\,\frac{\partial^2}{\partial\theta^2} - \frac{\sin\theta\,\cos\theta}{r}\,\frac{\partial}{\partial r} - \frac{\cos 2\theta}{r^2}\,\frac{\partial}{\partial\theta}$$

If $\theta = 0$, we realise that $\sigma_{rr} = \sigma_{xx}$, $\sigma_{\theta\theta} = \sigma_{yy}$, etc., we can find how the stress components are related to the Airy's stress function in polar coordinates. See Eqs (9.29a, 9.29b, 9.29c) below. The equations above become simplified to read From these equations we find, on adding Eqs (9.25a) and (9.25b), that

$$\bigtriangledown^2 \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \qquad \longrightarrow \qquad \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2}{\partial\theta^2}\right) \tag{9.26}$$

The Laplacian operator appears, in polar coordinates (r, θ), as

$$\bigtriangledown^2 \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2}{\partial\theta^2}\right). \tag{9.27}$$

[We should note carefully that it is not quite correct to use the same ϕ in both representations, because $\phi(x, y)$ and $\phi(r, \theta)$ have different functional forms. Strictly, we must write $\phi(r, \theta) = \phi(r(x, y), \theta(x, y))$ which cannot be the same as $\phi(x, y)$. However, this distinction is important only if we use both (x, y) and (r, θ) as the independent variables at the same time. That does not happen, so that in a practical sense it does not hurt to use ϕ as the notation for Airy's stress function in both sections of this chapter.]

The compatibility equation, which is the biharmonic equation when Airy's stress function is used, appears in polar coordinates (r, θ) as

$$\bigtriangledown^4\phi \equiv \bigtriangledown^2\,\bigtriangledown^2\,\phi \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\,\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2\phi}{\partial\theta^2}\right) = 0. \tag{9.28}$$

Airy's Stress Function in Polar Coordinates

We know how the stress components are related to the Airy's stress function [Eqs 9.10]. We also know how the operators $\partial^2/\partial x^2$, $\partial^2/\partial y^2$ and $\partial^2/\partial x\,\partial y$ appear in the polar coordinate system. From these we can see that the stress components are related to the Airy's stress function as

$$\sigma_{rr} = \frac{1}{r}\,\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2\phi}{\partial\theta^2}; \tag{9.29a}$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}; \tag{9.29b}$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}. \tag{9.29c}$$

We have now set up all the equations that we need at the operational level to solve (two-dimensional) problems using Airy's stress function. These are Eqs (9.29a, 9.29b, 9.29c) (stress components in terms of ϕ), and Eq. (9.28) (compatibility equation — biharmonic equation). Next we shall see how problems are solved. We shall examine a few of the more technically important problems.

Simpler Problems: Axisymmetric Cases

Problems become much simpler if there is rotational symmetry (axisymmetry). We shall first take up these simpler cases. Now all the variables are independent of the angle θ. On the mathematical side, the variable θ drops out, and the partial derivatives become ordinary derivatives. The Airy's stress function ϕ, now dependent only on r, is the solution of

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr}\right) \equiv \frac{d^4\phi}{dr^4} + \frac{2}{r}\frac{d^3\phi}{dr^3} - \frac{1}{r^2}\frac{d^2\phi}{dr^2} + \frac{1}{r^3}\frac{d\phi}{dr} = 0. \tag{9.30}$$

This is a linear, variable coefficient, differential equation of order four (4). This is one of the easier equations to solve. (See the footnote on p. 9-26.) This can first be converted, if desired, into a linear differential equation with constant coefficients by changing the independent variable from r to, say, s by the equation $r = \exp s$. This is not necessary. In any case, the general solution of Eq. (9.30) involving four arbitrary constants is

$$\phi = A\,\log r + B\,r^2\,\log r + C\,r^2 + D. \tag{9.31}$$

This general form of solution is applicable to all (two-dimensional) problems with rotational symmetry with no body forces.

[We can realise immediately that at $r = 0$, that is, at the centre there is going to be some kind of difficulty. We could have anticipated this; the differential equation has a singular point at $r = 0$. A detailed analysis is needed before further definite conclusions can be reached. Students are advised to pay attention to such theoretical matters. It is desirable to be aware of the qualitative theory of differential equations.] The stress components corresponding to this general solution are given by

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} = \frac{A}{r^2} + B(1 + 2\,\log r) + 2C; \tag{9.32a}$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = -\frac{A}{r^2} + B(3 + 2\,\log r) + 2C; \tag{9.32b}$$

$$\tau_{r\theta} = 0. \tag{9.32c}$$

Let us choose $B = 0$[9]. Thus, we arrive at Lamé's equations! We shall discuss how the other constants can be determined from the boundary conditions of the problem. This is explained in detail later (thick cylinders).

[9] When we calculate the corresponding displacements, it turns out that the displacements will be multi-valued if $B \neq 0$. While multi-valued displacements can be accepted when discussing dislocations (dislocation theory), generally displacements have to be single-valued. Thus, B must necessarily vanish.

Axisymmetric Case: Pure Bending of Curved Bars

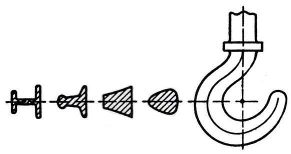


Figure 9.7: A curved beam

Fig. 9.7 shows a curved beam subjected to pure bending. The rectangular cross-section is shown alongside.

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} = \frac{A}{r^2} + B(1 + 2\,\log r) + 2C;$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = -\frac{A}{r^2} + B(3 + 2\,\log r) + 2C;$$

$$\tau_{r\theta} = 0.$$

Fig. 9.7 shows a curved beam subjected to pure bending. From our previous experience and examination of the problem, we realise that all the cross-sections are identical and identically loaded. Thus, this is an axisymmetric problem. Consequently, the solution given by Eqs (9.32a, 9.32b, 9.32c) is applicable here also. All we need to do here is to determine the constants A, B and C from the conditions of the problem. We shall show below how this is done.

The conditions that we know are the following.
(a) The curved surfaces AB and CD are traction free; there is no normal stress on them.
(b) There is no net axial (circumferential) force on the cross-sections like AD and BC.
(c) The net (bending) moment applied on the cross-sections AD and BC is M.
(d) There is no shear stress $\tau_{r\theta}$ on the boundary; that is, no tangential forces are applied on the boundary.

These facts give us the equations needed to determine the constants. They are:

$$\text{at } r = r_i, \quad \sigma_{rr} = 0; \qquad \text{at } r = r_o, \quad \sigma_{rr} = 0; \tag{9.33}$$

$$\int_{r_i}^{r_o} \sigma_{\theta\theta}\, dr = 0; \qquad \int_{r_i}^{r_o} \sigma_{\theta\theta}\, r\, dr = -M; \tag{9.34}$$

$$\text{at } r = r_i, \quad \tau_{r\theta} = 0; \qquad \text{at } r = r_o, \quad \tau_{r\theta} = 0. \tag{9.35}$$

There is no radial stress σ_{rr} on the curved boundaries. This condition leads to

$$\frac{A}{r_i^2} + B(1 + 2\,\log r_i) + 2C = 0; \tag{9.36a}$$

$$\frac{A}{r_o^2} + B(1 + 2\,\log r_o) + 2C = 0. \tag{9.36b}$$

Whenever we use Airy's stress functions, the condition of equilibrium is guaranteed. If there is any net resultant axial (circumferential) force on the cross-sections AD and BC, equilibrium will be upset. So we are sure that the condition (b) is satisfied. The moment condition gives us the third equation. This is obtained as shown below.

$$\int_{r_i}^{r_o} \sigma_{\theta\theta}\, r\, dr = \int_{r_i}^{r_o} \frac{\partial^2\phi}{\partial r^2}\, r\, dr = -M.$$

Integrating this by parts, we obtain

$$\int_{r_i}^{r_o} \frac{\partial^2 \phi}{\partial r^2}\, r\, dr = \left|\frac{\partial \phi}{\partial r}\, r\right|_{r_i}^{r_o} - \int_{r_i}^{r_o} \frac{\partial \phi}{\partial r}\, dr = \left|\frac{\partial \phi}{\partial r}\, r\right|_{r_i}^{r_o} - \left|\phi\right|_{r_i}^{r_o}.$$

From this equation we conclude that[10]

$$\left|\phi\right|_{r_i}^{r_o} = M, \quad \left(\text{because } \left|\frac{\partial \phi}{\partial r}\, r\right|_{r_i}^{r_o} = 0.\right)$$

Substituting the expression for ϕ here, we obtain the third equation for the determination of the constants A, B and C as

$$A \,\log \frac{r_o}{r_i} + B(r_o^2 \,\log r_o - r_i^2 \,\log r_i) + C(r_o^2 - r_i^2) = M. \tag{9.37}$$

The three equations (9.36a), (9.36b), (9.37) are used to solve for A, B, C as

$$A = -\frac{4\,M}{N}\, r_i^2 r_o^2 \,\log \frac{r_o}{r_i}; \qquad B = -\frac{2\,M}{N}(r_o^2 - r_i^2);$$

$$C = \frac{M}{N}[r_o^2 - r_i^2 + 2(r_o^2 \,\log r_o - r_i^2 \,\log r_i)], \qquad \text{where } N = (r_o^2 - r_i^2)^2 - 4r_o^2 r_i^2 \left(\log \frac{r_o}{r_i}\right)^2.$$

We can now obtain the stresses [Eqs (9.32a), (9.32b), (9.32c)] as:

$$\begin{aligned}
\sigma_{rr} &= -\frac{4\,M}{N}\left[\frac{r_o^2 r_i^2}{r^2} \,\log \frac{r_o}{r_i} + r_o^2 \,\log \frac{r}{r_o} + r_i^2 \,\log \frac{r_i}{r}\right]; \\
\sigma_{\theta\theta} &= -\frac{4\,M}{N}\left[-\frac{r_o^2 r_i^2}{r^2} \,\log \frac{r_o}{r_i} + r_o^2 \,\log \frac{r}{r_o} + r_i^2 \,\log \frac{r_i}{r} + r_o^2 - r_i^2\right]; \\
\tau_{r\theta} &= 0.
\end{aligned}$$

We do not discuss the consequences of the solution, plot the distribution of the stresses, work out the corresponding displacements, etc. We shall merely point out that, if the stresses are not distributed as given above at the ends, this will not be the exact solution. However, for practical purposes, we can use Saint-Venant's principle and find that a little away from the ends, the stresses are indeed as given above.

Stress Concentration: Kirsch's Solution

The problem of stress concentration is of the utmost technical importance in machine design[11]. This is one important problem that cannot be solved by the methods used in the

[10] Note that the expressions in Eqs (9.36a) and (9.36b) represent the radial stress σ_{rr} on the curved boundaries (surfaces) AD and BC. Multiplying the first of these by r_i^2 and the second by r_o^2, and subtracting one from the other we obtain

$$\sigma_{rr}\, r_o^2 - \sigma_{rr}\, r_i^2 = \sigma_{rr}\Big|_{r_i}^{r_o} = \left|\frac{\partial \phi}{\partial r}\, r\right|_{r_i}^{r_o} = 0.$$

[11] This was solved by Ernst Gustav Kirsch (Sept. 1841 - Jan. 1901), German engineer and professor in 1898, and is sometimes referred to as the Kirsch problem.

(a) A plate with a small hole in uniaxial tension

(b) Distribution of stress

Figure 9.8: Stress concentration around a circular hole. A large plate with a small hole in uniaxial tension [Fig. 9.8a]. The stress distribution obtained is shown in Fig. 9.8b
.

traditional Strength of Materials approach. Solving this problem and obtaining the stress concentration factor is of great interest to mechanical and aerospace engineers. We shall solve this problem here using the Airy's stress function, but we will not discuss the practical implications of the solution further.

Consider a small circular hole in a large plate[12] in uniaxial tension (uniform tensile stress S) [Fig. 9.8a]. We know that the stress is very high in the immediate vicinity of the hole [Fig. 9.8b]. Our interest is to solve this problem using Airy's stress function.

We know that at locations far from the (small) hole, the stress will be uniform (Saint-Venant's principle). At a typical point P sufficiently far away from the hole, the stress components are $\sigma_{xx} = S$, $\sigma_{yy} = 0$, $\tau_{r\theta} = 0$. This state of stress, when referred to the polar coordinates (r, θ), is given by the stress transformation matrix.

$$
\begin{aligned}
\begin{bmatrix} \sigma_{rr} & \tau_{r\theta} \\ \tau_{\theta r} & \sigma_{\theta\theta} \end{bmatrix} &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} S\,\cos^2\theta & -S\,\cos\theta\,\sin\theta \\ -S\,\cos\theta\,\sin\theta & S\,\sin^2\theta \end{bmatrix}.
\end{aligned}
$$

The stress components at P (at $r = r_O$, large compared to the radius of the hole r_i) are

$$
\begin{aligned}
\sigma_{rr} &= S\,\cos^2\theta = \frac{1}{2}(S + S\,\cos\theta); \\
\sigma_{\theta\theta} &= S\,\sin^2\theta = \frac{1}{2}(S - S\,\cos\theta); \\
\tau_{r\theta} &= -\frac{1}{2}S\,\sin 2\theta.
\end{aligned}
$$

Let us break this up into two parts (a) and (b). The total solution (c) is the superposition of the two parts: (c)=(a)+(b).

(a) $\sigma_{rr} = \dfrac{1}{2}\,S$ **(b)** $\sigma_{rr} = \dfrac{1}{2}\,S\,\cos 2\theta$ **(c)** $\sigma_{rr} = \dfrac{1}{2}\,(S + S\,\cos 2\theta)$;

[12] This is sometimes differently worded as 'a plate with a pin hole' or as 'a hole in an infinitely wide plate'.

$$\sigma_{\theta\theta} = \frac{1}{2}\,S \qquad \sigma_{\theta\theta} = \frac{1}{2}\,S\,\cos 2\theta \qquad \sigma_{\theta\theta} = \frac{1}{2}\,(S + S\,\cos 2\theta);$$
$$\tau_{r\theta} = \frac{1}{2}\,S \qquad \tau_{r\theta} = \frac{1}{2}\,S\,\cos 2\theta \qquad \tau_{r\theta} = \frac{1}{2}\,(S + S\,\cos 2\theta).$$

Part (a) is the Lamé's thick cylinder problem. We shall return to this after we solve the Part (b) problem.

Part (b) problem:

Assume an Airy's stress function in the form $\phi = \phi(r,\theta) = f(r)\cos 2\theta$, where $f(r,\theta)$ is unknown. We can determine it by first substituting this function in the biharmonic equation, thereby obtaining an ordinary differential equation. On solving this, we obtain the general solution. As usual, the boundary conditions are used to determine the four arbitrary constants . The Airy's stress function is now known, and the expressions for the stress components are worked out to obtain the solution. This is the procedure; we shall follow this.

Substituting $\phi = f(r)\cos 2\theta$ in the biharmonic equation, we obtain a fourth order, linear, variable coefficient, ordinary differential equation.

$$\bigtriangledown^2\bigtriangledown^2\phi = 0 \quad\longrightarrow\quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}\right)\left(\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{4f}{r^2}\right) = 0.$$

This equation may be solved directly, or by first converting this into a differential equation with constant coefficients. Either way, the general solution is seen to be

$$f(r) = A\,r^2 + B\,r^4 + \frac{C}{r^2} + D \quad\longrightarrow\quad \phi(r,\theta) = \left(A\,r^2 + B\,r^4 + \frac{C}{r^2} + D\right)\cos 2\theta.$$

The stress components are readily worked out as

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = -\left(2A + \frac{6C}{r^4} + \frac{4D}{r^2}\right)\cos 2\theta;$$
$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = -\left(2A + 12Br^2 + \frac{6C}{r^4}\right)\cos 2\theta;$$
$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = \left(2A + 6Br^2 - \frac{6C}{r^4} - \frac{2D}{r^2}\right)\sin 2\theta.$$

The boundary conditions used to determine the constants A, B, C, D are:
(a) at $r = r_i$ ('inner' radius r_i, radius of the hole), the radial stress, $\sigma_{rr} = 0$;
(b) at $r = r_i$ ('inner' radius r_i, radius of the hole), the shear stress, $\tau_{r\theta} = 0$;
(c) at $r = r_o$ (at a sufficiently large distance), the radial stress, $\sigma_{rr} = (1/2)\,S\,\cos 2\theta$; and
(d) at $r = r_0$ (at a sufficiently large distance), the shear stress, $\tau_{r\theta} = -(1/2)\,S\,\sin 2\theta$.

These conditions lead to the following four equations.

$$\text{(a)} \qquad 2A + \frac{6C}{r_i^4} + \frac{4D}{r_i^2} = 0;$$
$$\text{(b)} \qquad 2A + 6Br_i^2 - \frac{6C}{r_i^4} - \frac{2D}{r_i^2} = 0;$$

$$\text{(c)} \qquad 2A + \frac{6C}{r_o^2} + \frac{4D}{r_o^2} = -\frac{S}{2};$$
$$\text{(d)} \qquad 2A + 6Br_o^2 - \frac{6C}{r_o^4} - \frac{2D}{r_i^2} = -\frac{S}{2}.$$

These equations enable us to solve for the four constants under the condition that the hole is small (or, equivalently, that the plate is very large), i.e., $r_i/r_o \rightarrow 0$.

Multiplying the third equation (c) by r_i^2 using the condition $r_i/r_o \rightarrow 0$, we obtain $A = -S/4$. Similarly multiplying the last equation (d) by r_i^2/r_o^2 and letting $r_i/r_o \rightarrow 0$, we conclude that $B = 0$. Now we obtain from the first two equations (a) and (b) the results for the constants C and D as $C = -(r_i^4\,S)/4$ and $D = (r_i^2\,S)/2$.

With these values of the constants A, B, C, D, the stress components corresponding to case (a) are:

$$\sigma_{rr} = \frac{S}{2}\left[1 + \frac{3r_i^4}{r^4} - \frac{4r_i^2}{r^2}\right]\cos 2\theta;$$
$$\sigma_{\theta\theta} = -\frac{S}{2}\left[1 + \frac{3r_i^4}{r^4}\right]\cos 2\theta;$$
$$\tau_{r\theta} = -\frac{S}{2}\left[1 - \frac{3r_i^4}{r^4} + \frac{2r_i^2}{r^2}\right]\sin 2\theta.$$

This completes the solution corresponding to case (b). Now we shall take up the case (a).

Part (a) problem:
This is the Lamé's problem discussed elsewhere in detail. The present problem here corresponds to a thick cylinder of radii r_o (outer radius) and r_i (inner radius) subjected to an external hydrostatic pressure p_o and no internal pressure $p_i = 0$. This solution is worked out in detail (p. 9-31). Borrowing the result from there, we obtain the expressions for the stress components as

$$\sigma_{rr} = -\frac{S\,r_i^2 r_o^2}{2(r_o^2 - r_i^2)r^2} + \frac{S\,r_o^2}{2(r_o^2 - r_i^2)};$$
$$\sigma_{\theta\theta} = \frac{S\,r_i^2 r_o^2}{2(r_o^2 - r_i^2)r^2} + \frac{S\,r_o^2}{2(r_o^2 - r_i^2)};$$
$$\tau_{r\theta} = 0.$$

In the limit as $r_i/r_o \rightarrow 0$, these become

$$\sigma_{rr} = -\frac{S\,r_i^2}{2r^2\left(1 - \frac{r_i^2}{r_o^2}\right)} + \frac{S}{2\left(1 - \frac{r_i^2}{r_o^2}\right)} = \frac{S}{2}\left(1 - \frac{r_i^2}{r^2}\right);$$
$$\sigma_{\theta\theta} = \frac{S\,r_i^2}{2r^2\left(1 - \frac{r_i^2}{r_o^2}\right)} + \frac{S}{2\left(1 - \frac{r_i^2}{r_o^2}\right)} = \frac{S}{2}\left(1 + \frac{r_i^2}{r^2}\right);$$
$$\tau_{r\theta} = 0.$$

Having solved the two sub-problems (a) and (b), we can superpose these two and obtain the solution of the stress concentration problem. The strain components thus obtained are

$$\begin{aligned}
\sigma_{rr} &= \frac{S}{2}\left[1-\frac{r_i^2}{r^2}\right]+\frac{S}{2}\left[1+3\,\frac{r_i^4}{r^4}-4\,\frac{r_i^2}{r^2}\right]\,\cos 2\theta;\\
\sigma_{\theta\theta} &= \frac{S}{2}\left[1+\frac{r_i^2}{r^2}\right]-\frac{S}{2}\left[1+3\,\frac{r_i^4}{r^4}\right]\,\cos 2\theta;\\
\tau_{r\theta} &= -\frac{S}{2}\left[1-3\,\frac{r_i^4}{r^4}+2\,\frac{r_i^2}{r^2}\right]\,\sin 2\theta.
\end{aligned}$$

The interesting part is in the immediate neighbourhood of the (small) hole. At $r = r_i$, the stress components are

$$\text{at }\; r = r_i, \quad \sigma_{rr} = 0, \qquad \tau_{r\theta} = 0; \qquad \text{and} \qquad \sigma_{\theta\theta} = S - 2S\,\cos 2\theta.$$

There is nothing new in the first two; we already know that this is so. But the last one is of vital importance. It shows that the vulnerable points are A and B, where there is severe stress concentration. The peak value of $\sigma_{\theta\theta}$ is $3\times$ the nominal stress S. The stress concentration factor[13] is thus 3! At the points C and D, $\sigma_{\theta\theta} = -S$. We can also see that the magnitude of $\sigma_{\theta\theta}$ quickly drops to smaller values when we move away from A and / or B, showing that this stress concentration is a local phenomenon. Stress concentration is a very important topic having several practical applications. We, however, do not discuss this topic any further.

A Concentrated Force on a Semi-infinite Half-space

(a) A force P on an elastic half-space (b) 'A simple radial solution'

Figure 9.9: A concentrated force P acting on an elastic half-space [Fig. 9.9a] produces a stress distribution, sometimes known as 'a simple radial solution' [Fig. 9.9b].

[13] Books on machine design give more details about stress concentration factors. Elaborate charts are available for designers to help them. Books like Savin, G.N., *Stress Concentration Around Holes*, Pergamon Press, (1961) give the solution for several cases. The case of an elliptical hole is of much importance. Timoshenko [16] states: "a very slender hole (a/b large) perpendicular to the direction of the tension causes a very high stress concentration. This explains why cracks transverse to applied loads tend to spread. The spreading can be stopped by drilling holes at the ends of the crack to eliminate the sharp curvature responsible for the high stress concentration."

We shall now take up the problem of determining the stresses in a semi-infinite elastic half-space due to a concentrated (vertical) force [Fig. 9.9].

Consider a stress function $\phi = \phi(r, \theta) = Ar\,\theta\,\cos\theta$, where A is a constant. The corresponding stress components are

$$\begin{aligned}
\sigma_{rr} &= \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{A\theta\,\cos\theta}{r} - \frac{Ar(-2\,\sin\theta - \theta\,\cos\theta)}{r^2} = \frac{-2A\,\cos\theta}{r};\\
\sigma_{\theta\theta} &= \frac{\partial^2\phi}{\partial r^2} = 0;\\
\tau_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = 0.
\end{aligned}$$

When we examine the solution, we find that σ_{rr} is the only non-zero stress component and, what is more, that it is independent of θ! It is, thus, a constant at any point on any semi-circular line as shown in Fig. 9.9b. For this reason, it is sometimes known as 'the simple radial solution'. The constant A in the stress function can be evaluated in terms of the concentrated applied load — really a line load over a unit thickness — by considering the equilibrium of semi-circular element (like the shaded part in Fig. 9.9b. Thus, we have from equilibrium requirement

$$\int_0^{\frac{\pi}{2}} \sigma_{rr}\,r\,\sin\theta\,d\theta + P = 0 \qquad \longrightarrow \qquad A = \frac{P}{\pi}.$$

The radial stress is, thus,

$$\sigma_{rr} = -\frac{2P\,\sin\theta}{\pi\,r} = -\frac{2P}{\pi d},$$

the negative sign implying that it is compressive.

Geotechnical engineers may desire to calculate the stresses at a certain depth, marked h in the figure. Using the given solution and the stress transformation equations, we can work them out as shown below.

$$\begin{aligned}
\sigma_{xx} &= \sigma_{rr}\,\cos^2\theta && = -\frac{2P}{\pi r}\,\sin\theta\,\cos^2\theta = -\frac{2P}{\pi h}\,\sin^2\theta\,\cos^2\theta;\\
\sigma_{xx} &= \sigma_{rr}\,\sin^2\theta && = -\frac{2P}{\pi r}\,\sin^3\theta \qquad\;\;\; = -\frac{2P}{\pi h}\,\sin^4\theta;\\
\sigma_{xx} &= \sigma_{rr}\,\sin\theta\,\cos\theta && = -\frac{2P}{\pi r}\,\sin^2\theta\,\cos\theta = -\frac{2P}{\pi h}\,\sin^3\theta\,\cos\theta.
\end{aligned}$$

Using this solution, the stresses due to several loads, concentrated and uniformly distributed, can be worked out by superposition. It is also of interest to work out the corresponding displacements. Engineers would like to know the displacements, particularly the (vertical) settlement. But these displacement calculations, though not difficult in principle, are harder. We do not propose to discuss them here.

ROTATING DISCS

Let us consider the problem of determining the stresses in a rotating disc[14]. This is of great technical importance because rotating discs are a crucial component of steam and gas turbines. The discs are not generally flat[15], but the theory is simpler.

The disc is shrunk on the shaft with an interference fit. What is the speed at which the disc runs loose on the shaft making it impossible to transmit power? If we provide too much interference, the stresses at standstill will be very high. We need to calculate these. The boundary conditions are in terms of the radial displacement u (related to the interference) and the radial stress (related to the interference pressure between the shaft and the disc). Thus, we need to obtain the analytical expressions and the numerical values of σ_{rr}, $\sigma_{\theta\theta}$ and u. As this is not a book on machine design, technically important details are left out. The shrink allowance is of the order of 1×10^{-3}.

This is an axisymmetric problem. The equation of equilibrium was derived earlier [Eq. (5.18), p. 5-15]. We borrow it and present it here as

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + F_r = 0.$$

The body force here is the 'centrifugal force' ('inertia force' introduced to convert the problem from dynamics to statics using D'Alembert's principle). It is

$$F_r = \rho\,\omega^2\,r \;\text{ per unit volume;}$$
$$F_r \times r\,dr\,d\theta \times 1 = \rho\,\omega^2\,r^2\,dr\,d\theta \quad \text{(total radial body force).}$$

Thus, the differential equation of motion is obtained after a little manipulation as

$$\frac{d}{dr}\,(r\,\sigma_{rr}) - \sigma_{\theta\theta} + \rho\,\omega^2\,r^2 = 0. \tag{9.38}$$

Fig. 9.10a shows an element $ABCD$. Its deformed position is $A'B'C'D'$. Each point moves out only radially. A typical point P moves by u in the radial direction; there is no tangential displacement, v. Even so, a tangential strain is called into play[16].

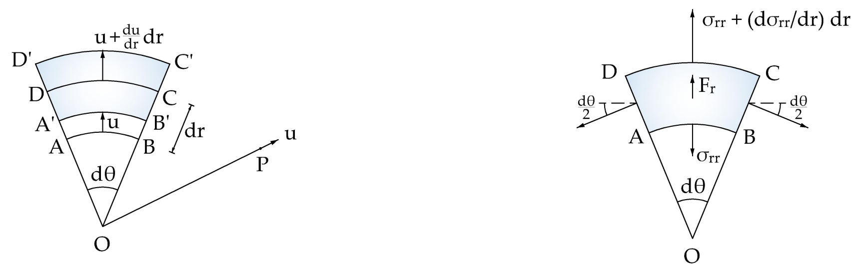
The line AB moves out radially to be the line $A'B'$. This radial displacement is $u = u(r)$. The radial displacement of the line CD (separated from the line AB by the distance dr) to be the line $C'D'$ is, therefore, $u + (\partial u/\partial dr)\,dr$. Hence, the radial strain e_r is

$$e_r = \frac{\text{change in length}}{\text{original length}} = \frac{B'C' - BC}{BC} = \frac{u + \frac{du}{dr}\,dr - u}{dr} = \frac{du}{dr}.$$

[14] This topic, like all others, is treated beautifully in Den Hartog [4]. Readers are advised to read this if they are not sufficiently familiar with this topic. The most comprehensive, thorough coverage, of nearly all topics in Engineering Dynamics is in C.B. Biezeno & R. Grammel: *Engineering Dynamics*, translation of *Technische Dynamik* in German; nothing can match this book.

[15] Flat discs, that is, discs of uniform thickness are also used.

[16] Den Hartog [4] remarks, after quoting these two strain-displacement equations: "$\cdots$ the first one is fairly obvious, and the second one refers to the feelings of a middle-aged gentleman who lets out one notch of his belt after his daily good dinner." The belly expands radially outwards; there is no tangential movement. Yet there is a tangential strain making it necessary to loosen the belt!



(a) Deformations (b) Stress components

Figure 9.10: To explain the simplified case of an axisymmetric problem

The line AB of length $r\,d\theta$, after deformation, becomes $A'B'$ of length $(r+u)\,d\theta$. Hence,

$$\text{tangential strain } e_\theta = \frac{\text{change in length}}{\text{original length}} = \frac{(r+u)\,d\theta}{r\,d\theta} = \frac{u}{r}.$$

These are the strain-displacement relations for the special case of axisymmetry.

$$\text{radial displacement,}\quad u = u(r) \qquad\qquad \text{radial strain,}\quad e_r = \frac{du}{dr}; \tag{9.39a}$$

$$\text{tangential displacement,}\quad v = 0 \qquad\qquad \text{tangential strain,}\quad e_\theta = \frac{u}{r}. \tag{9.39b}$$

We have here only the two stress components σ_{rr} and $\sigma_{\theta\theta}$; the others are all zero. Out of the three displacement components, we have here only the radial displacement, u. Let us review the situation and note the unknowns and the available equations to determine them.

The unknowns are (i) the stress components σ_{rr}, $\sigma_{\theta\theta}$ (2), (ii) the strain components e_{rr}; $e_{\theta\theta}$ (2); and (iii) the radial displacement component u (1). The available equations are (i) the equation of equilibrium [Eq. (7.10)] (1); (ii) the strain-displacement relations [Eqs (9.39a), (9.39b)] (2); and (iii) the stress-strain equations [Eqs (9.40a), (9.40b)] (2). There are, thus, $5\,(2+2+1=5)$ unknowns, and $5\,(1+2+2=5)$ equations available.

$$e_{rr} = \frac{1}{E}[\sigma_{rr} - \nu\sigma_{\theta\theta}]; \quad \text{and} \tag{9.40a}$$

$$e_{\theta\theta} = \frac{1}{E}[\sigma_{\theta\theta} - \nu\sigma_{rr}]. \tag{9.40b}$$

We can eliminate any four of them and obtain an equation in terms of the remaining one. As the boundary conditions are usually expressed in terms of σ_{rr} or u, one of these two is preferred. Choosing σ_{rr} here, the steps of the elimination procedure are shown below.

i) The strain components are eliminated by combining the strain-displacement relations [Eqs (9.39a), (9.39b)] and the constitutive equations [Eqs (9.40a), (9.40b)] as

$$\frac{du}{dr} = \frac{1}{E}[\sigma_{rr} - \nu\sigma_{\theta\theta}]; \quad \text{and} \tag{9.41a}$$

$$\frac{u}{r} = \frac{1}{E}[\sigma_{\theta\theta} - \nu\sigma_{rr}]. \tag{9.41b}$$

ii) Differentiate Eq. (9.41b) and substitute into the result Eqs (9.41a, 9.41b). The variable u is thus eliminated. We thereupon obtain the equation

$$\frac{d}{dr}(\sigma_{\theta\theta}) - \nu\frac{d}{dr}(\sigma_{rr}) + \frac{1+\nu}{r}(\sigma_{\theta\theta} - \sigma_{rr}) = 0. \tag{9.42}$$

iii) Differentiate $\sigma_{\theta\theta}$ [Eq. (9.38)] w.r.to r, and substitute the result in Eq. (9.42).

We would, thus, obtain the final governing equation in terms of $(r\,\sigma_{rr})$ as

$$r^2\,\frac{d^2}{dr^2}(r\,\sigma_{rr}) + r\,\frac{d}{dr}(r\,\sigma_{rr}) - (r\,\sigma_{rr}) + (3+\nu)\,\rho\,\omega^2\,r^3 = 0; \tag{9.43}$$

$$\text{i.e., } r^2\,\frac{d^2}{dr^2}(r\,\sigma_{rr}) + r\,\frac{d}{dr}(r\,\sigma_{rr}) - (r\,\sigma_{rr}) = -(3+\nu)\,\rho\,\omega^2\,r^3. \tag{9.44}$$

This is a linear, but variable coefficient, differential equation with a 'right hand side'. We know that the solution consists of a complementary solution and a particular integral[17]. The complementary function is of the form $(r\,\sigma_{rr}) = r^n$, as argued out in the footnote[18]. We obtain the auxiliary equation by substituting this in the ('left hand side' of the) differential equation.

$$[n(n-1) + n - 1]\,r^n = 0, \qquad \longrightarrow \qquad n = \pm 1.$$

When a particular integral is added to the complementary function, we obtain the general solution as

$$r\,\sigma_{rr} = Ar + \frac{B}{r} - \frac{3-\nu}{8}\,\rho\,\omega^2\,r^3.$$

The required expressions (for σ_{rr}, $\sigma_{\theta\theta}$, u) are obtained as

$$\sigma_{rr} = A + \frac{B}{r^2} - \frac{3+\nu}{8}\,\rho\,\omega^2\,r^2; \tag{9.45a}$$

$$\sigma_{rr} = A - \frac{B}{r^2} - \frac{1+3\nu}{8}\,\rho\,\omega^2\,r^2;\ \text{ and} \tag{9.45b}$$

$$u = \frac{r}{E}\left[(1-\nu)\,A - (1+\nu)\,\frac{B}{r^2} - \frac{1-\nu^2}{8}\,\rho\,\omega^2\,r^2\right]. \tag{9.45c}$$

[17] Some authors write rather carelessly *the* particular integral. There can be several particular integrals. Two particular integrals can differ by a complementary function.

[18] Generally variable coefficient differential equations, even if they are linear, are hard to solve. A closed form solution is more an exception than the rule. Several special functions, which have been investigated extensively, are the solutions of such linear, variable coefficient differential equations. Many of them have similar interesting and useful properties like orthogonality. Bessel functions, Legendre polynomials, Hermite polynomial, etc. are some of the well known, well explored special functions.

Our equation here is quite easy to solve. To see this, let us assume that the unknown dependent variable, here $(r\,\sigma_{rr})$, is equal to r^n. If we differentiate it once, it goes one step down, and becomes $n\,r^{(n-1)}$. If it is multiplied by r, it goes one step higher to become $n\,r^n$. Two differentiations, two steps down, but multiplication by r^2, and thus two steps up, and we are back again to the same level r^n. This is true with every term if the equation is 'homogeneous' in this sense. (The word 'homogeneous' is used in the context of linear differential equations also in the sense that the right hand side is zero.) Thus, we can understand that the solution is of the form r^n. This equation is known variously as Cauchy's equation, Euler's equation, etc.

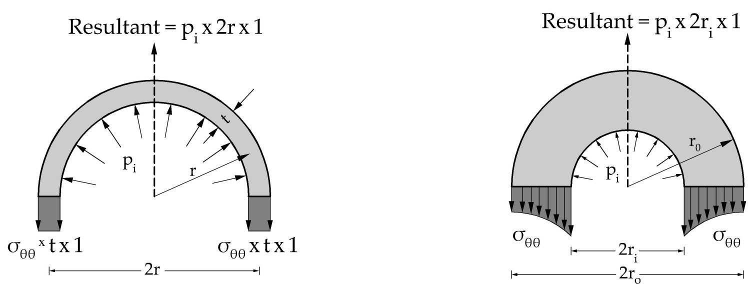
The two arbitrary constants A and B are to be determined from the usually known (sometimes unknown also) boundary conditions. [For example, when we write down the boundary condition to determine the interference pressure, it is unknown. It can be obtained later.] Some more calculations and explanations are needed for a complete solution of the problem. We do not propose to show them here.

The thick cylinder problem (the famous Lamé's problem) — to determine the stresses in a thick cylinder subjected to fluid pressure — is the same as this, except that there is no body force now. The two stresses, radial and tangential (or hoop) stresses are given by

$$\text{radial stress: } \sigma_{rr} = A + \frac{B}{r^2}; \qquad \text{hoop stress: } \sigma_{\theta\theta} = A - \frac{B}{r^2}.$$

THICK CYLINDERS: LAMÉ'S PROBLEM

We have seen this above as a special case of rotating disc when there is no rotation and, consequently, no body force. [On the mathematical side, the consequence is that there is no particular integral[19].] This problem is of much technical importance. We shall discuss this in greater detail.



(a) A thin cylinder (b) A thick cylinder

Figure 9.11: The figures show a thin [Fig. 9.17a] and a thick [Fig. 9.17b] cylinders.

Thick Cylinders

When the wall thickness is large, or when the inner pressure is very high, thick cylinder calculations are necessary. Some examples are in the design of pipes carrying high pressure fluids, gun barrels for tanks and warships, high pressure steam machinery, and hydraulic press. Another case in point is the (high pressure) fuel injection systems for diesel engines. In all these calculations calculations based on thick cylinder theory are to be used. Interference fits are another important area of application. In short, no engineer can afford not to learn thick cylinder theory.

[19] Is it better to discuss this simpler problem first before taking up the more difficult case of rotating discs following the general principle of 'from simple to complex'? Or, is it better to discuss the general problem first with rigorous details, and later obtain the simpler solutions as special cases of the general theory? Opinion is divided on this matter as in most other things in life. There are advantages and disadvantages in both approaches. It seems best to be exposed to both approaches so that each person can choose as he likes.

Thin and Thick Cylinders

Fig. 9.17 shows a thin and a thick cylinders subjected to an internal pressure p_i. Both are axisymmetric problems; that is there is circular symmetry. All the variables are independent of θ. We shall compare and contrast the stresses in the two cases.

Thin cylinders:
We know — we have studied this topic earlier — that the tangential (also called hoop), axial, and radial stresses ($\sigma_{\theta\theta}$, $\sigma_{zz} \equiv \sigma_{axial}$, σ_{rr}, respectively) in the case of a thin cylinder are given by $\sigma_{\theta\theta} = p_i\, r/t$; $\sigma_{zz} = p_i\, r/(2t)$. The radial stress σ_{rr} is small in comparison and is neglected. Because of the circular symmetry (axisymmetry) these are the principal stresses; the tangential, axial, and radial directions are the principal directions.

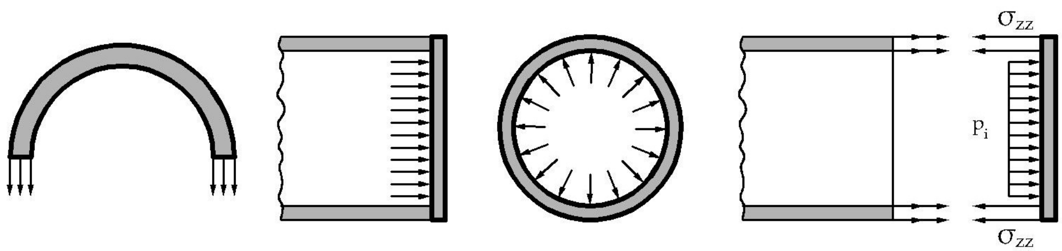


Figure 9.12: A thin cylinder subjected to an internal pressure p_i is shown. As the thickness t is small, the stress $\sigma_{\theta\theta}$ may be regarded as substantially constant across the small thickness.

For a thin[20] cylinder $t << r$ and, therefore, the tangential (or hoop) stress $\sigma_{\theta\theta}$, and the axial stress σ_{zz} (if the cylinder is closed at the ends) are several times the applied internal pressure. The radial stress σ_{rr}, being very small, is neglected. Thus,

$$\begin{aligned} \sigma_{\theta\theta} &= \frac{p_i\, r}{t} \quad \text{tensile :} \\ \sigma_{axial} \equiv \sigma_{zz} &= \frac{p_i\, r}{2\,t} \quad \text{if the cylinder is closed, tensile;} \\ &= 0 \quad \text{if the cylinder is open;} \\ \sigma_{rr} &= \text{small, between } -p \text{ and } 0 \text{ at the boundaries.} \end{aligned}$$

From the equation of equilibrium in the radial direction [Fig. 9.17a], we have

$$p_i \times 2r \times 1 = 2 \times \sigma_{\theta\theta} \times t \times 1 \qquad \longrightarrow \qquad \sigma_{\theta\theta} = \frac{p\, r}{t}.$$

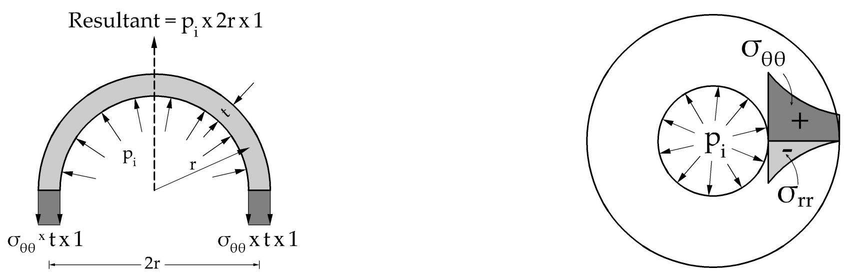
Similarly, from the equation of equilibrium in the axial direction [Fig. 9.12], we have

$$p_i \times \pi\, r^2 = \sigma_{zz} \times 2\,\pi\, r\, t \qquad \longrightarrow \qquad \sigma_{zz} = \frac{p_i\, r}{2\,t}.$$

[20] As an example let us note that the diameter of the fuselage of an aircraft is very, very large compared to the thickness — typically about 1.4 mm — of the outer skin. Here $t << r$. We can surely treat the skin as a thin cylinder! The internal pressure p_i arises because the outside pressure when the aircraft flies is much less than the atmospheric pressure maintained in the cabin. This is perhaps an extreme case.

As the thickness t increases, the tangential stress $\sigma_{\theta\theta}$ becomes less and less. In the case of a thick cylinder, this becomes comparable in magnitude to σ_{rr}. Thus, we cannot neglect σ_{rr} now. This is the difference from the point of view of stress analysis.

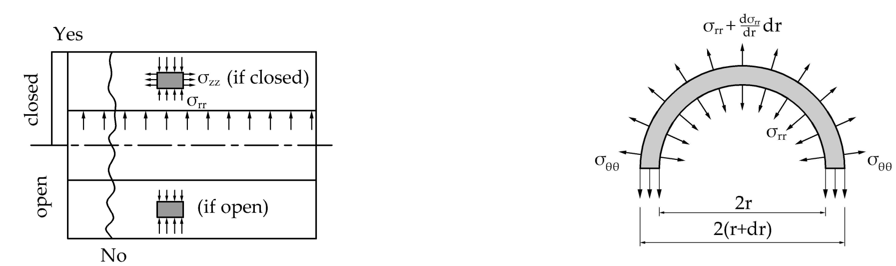
[Here arises a question: how could we obtain the stresses using only the equations of equilibrium? We know that *all* problems in stress analysis are statically indeterminate; the equations of equilibrium alone cannot give us the stress components. Where is the anomaly? Let us realise that we made an assumption that the stress component $\sigma_{\theta\theta}$ is uniformly distributed across the thickness. This is why we were able to obtain the stress components without using the strain-displacement relations and the constitutive relations.]



(a) A thin layer of a thick cylinder (b) Stress distribution in a thick cylinder

Figure 9.13: The figure show a thin layer of a thick cylinder. This is used to derive the stress distribution [Fig. 9.13b] when an internal pressure is applied in a thick cylinder.

Thick cylinders:



(a) Stress components and the stress resultants (b) A part of the thick cylinder

Figure 9.14: The figure show a thin layer of a thick cylinder. The stress components and the resultants are shown. Plane cross-sections continue to remain plane even after the (fluid) pressure is applied. From this assumption, we can derive the equation of equilibrium.

We desire to determine the stress distribution in a thick cylinder. This is an axisymmetric problem and, therefore, all the stress components (and strain and displacement components too) are functions of only the variable r. Everything is independent of the variable θ. The derivatives are all ordinary derivatives, not partial ones.

A thin half ring of thickness dr with the stress components is shown 9.14a. The resultants are shown in dotted lines. Considering the equilibrium of the half ring, we have

$$\sigma_{rr} \times 2\,r \times 1 = \left(\sigma_{rr} + \frac{d\sigma_{rr}}{dr}\,dr\right) \times 2(r + dr) \times 1 + (\sigma_{\theta\theta} \times dr \times 1) \times 2,$$

which, on dropping quantities of a higher order of smallness and cleaning up, reads as

$$\sigma_{\theta\theta} = -\sigma_{rr} - r\,\frac{d\sigma_{rr}}{dr}. \tag{9.46}$$

There are two unknowns, but only one equation of equilibrium. We, thus, need to use the other governing equations also to solve for the stress components. Recall the statement that we made that *all* problems in stress analysis are statically indeterminate internally. How shall we proceed? Where will the other equation come from?

Plane strain assumption

At this stage, let us make the assumption that cross-sections that are plane before the internal (fluid) pressure is applied, continue to remain plane [Fig. 9.14b]. [In this figure 'yes' shows that plane cross-sections remain plane, and 'no' shows that the cross-sections do not go crooked.] This implies that $e_{axial} \equiv e_{zz} = 0$. Thus,

$$\begin{aligned} e_{axial} \equiv e_{zz} = 0 &= \frac{1}{E}[0 - \nu\sigma_{\theta\theta} + \nu\sigma_{rr}] \\ &= \frac{\nu}{E}[\sigma_{rr} - \sigma_{\theta\theta}]. \end{aligned}$$

From this last equation we can note that $\sigma_{\theta\theta} - \sigma_{rr}$ = a constant, say, $2A$. Thus, the two equations that we have are

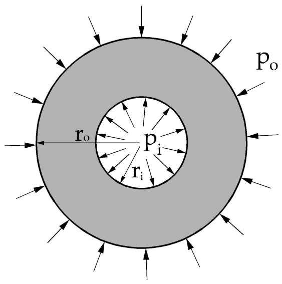
$$\begin{aligned} \sigma_{\theta\theta} - \sigma_{rr} &= \text{a constant, say, } 2A; \\ \sigma_{\theta\theta} &= -\sigma_{rr} - r\,\frac{d\sigma_{rr}}{dr}. \end{aligned}$$

Multiplying the first of these two equations by r and using the second equation, we obtain

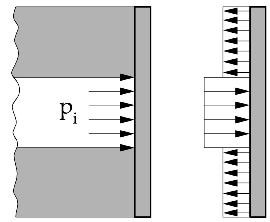
$$\begin{aligned} \sigma_{\theta\theta} - \sigma_{rr} &= -2\sigma_{rr} - r\,\frac{d\sigma_{rr}}{dr} \\ -\,2A &= -2\sigma_{rr} - r\,\frac{d\sigma_{rr}}{dr} \\ -\,2Ar &= 2(r\sigma_{rr}) + r^2\,\frac{d\sigma_{rr}}{dr} = \frac{d}{dr}(r^2\sigma_{rr}). \end{aligned}$$

On integration, we obtain the expression / formula for σ_{rr} as $-A + B/(r^2)$. Now the expression / formula for $\sigma_{\theta\theta}$ also can be readily worked out. They are (with a slight change in the definition of the constants)

$$\sigma_{rr} = A + \frac{B}{r^2}; \qquad \text{and} \qquad \sigma_{\theta\theta} = A - \frac{B}{r^2}.$$



(a) A thick cylinder subjected to internal (p_i) and external fluid pressure (p_o)



(b) Axial stresses in the thick cylinder (if, and only if, the cylinder is closed)

Figure 9.15: The figure [Fig. 9.15a] shows a thick cylinder subjected to internal (p_i) and external (p_o) pressures. [Fig. 9.15b] shows the axial stresses due to the fluid pressure if, and only if, the cylinder is closed.

Thick Cylinders Subjected to Internal and External Pressure

An important technical problem is to determine the stresses in a thick cylinder subjected to (i) and internal pressure p_i, and / or (ii) an external pressure p_o. (Sometimes these are fluid pressures. However, there are problems when there is no fluid at all as in interference fit calculations[21]. The inner and outer radii are r_i and r_o respectively.

We know that the radial and tangential stresses are given by[22]

$$\sigma_{rr} = A + \frac{B}{r^2}; \qquad \sigma_{\theta\theta} = A - \frac{B}{r^2}.$$

The constants A and B are determined using the known boundary conditions.

$$\begin{array}{llll} \text{(i)} & \text{at } r = r_i, & \sigma_{rr} = -p_i & \longrightarrow & A + \dfrac{B}{r_i^2} = -p_i \\ \text{(ii)} & \text{at } r = r_o, & \sigma_{rr} = -p_o & \longrightarrow & A + \dfrac{B}{r_o^2} = -p_o. \end{array}$$

Solving for A and B, we obtain — the slightly 'dirty' algebra is left out —

$$A = \frac{p_i\, r_i^2 - p_o\, r_o^2}{r_o^2 - r_i^2}, \qquad B = \frac{(r_o\, r_i)^2 (p_o - p_i)}{r_o^2 - r_i^2}.$$

[21] The pressure inside a gun barrel, and the pressure difference between the atmospheric pressure inside an aircraft and the low pressure outside when the aircraft flies to high altitudes are other examples.

[22] We may also write these differently, but equivalently, as

$$\sigma_{rr} = C - \frac{D}{r^2}; \qquad \sigma_{\theta\theta} = C + \frac{D}{r^2}.$$

There is no real difference; the constants are now different ($A = C$; $B = -D$). However, it is desirable to stick to one set of formulae. There is also a case to break this habit occasionally, if not for anything else, at least to remind ourselves that both sets of formulae are equally correct.

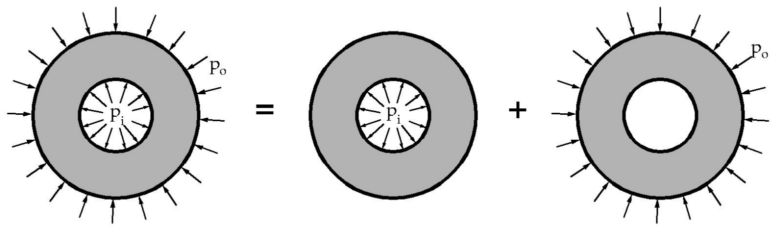


Figure 9.16: Superposition

The expressions for the stress components are

$$\sigma_{rr} = \frac{p_i\, r_i^2 - p_o\, r_o^2 + \left(\frac{r_i\, r_o}{r}\right)^2 (p_o - p_i)}{r_o^2 - r_i^2}; \qquad \sigma_{\theta\theta} = \frac{p_i\, r_i^2 - p_o\, r_o^2 - \left(\frac{r_i\, r_o}{r}\right)^2 (p_o - p_i)}{r_o^2 - r_i^2}.$$

We can solve this problem by superposition also as indicated below. Fig. 9.16 represents the problem that we solved just now. It can also be solved by superposition. The two cases shown on the right hand side of Fig. 9.16 — internal pressure p_i alone applied, and external pressure p_o alone applied — can be superposed to be the case shown on the left hand side of Fig. 9.16.

Case (a) A thick cylinder subjected to an internal pressure p_i only:
Let us consider case (a). The governing equations are, as always, in the form

$$\sigma_{rr} = A + \frac{B}{r^2}; \qquad \sigma_{\theta\theta} = A - \frac{B}{r^2}.$$

The constants A and B are determined using the known boundary conditions:

$$\begin{aligned}
&(a)\ \text{at } r = r_i, \quad \sigma_{rr} = -p_i \qquad \longrightarrow \qquad A + \frac{B}{r_i^2} = -p_i;\\
&(a)\ \text{at } r = r_o, \quad \sigma_{rr} = 0 \qquad \longrightarrow \qquad A + \frac{B}{r_o^2} = 0.
\end{aligned}$$

Solving for A and B we obtain

$$A = \frac{p_i\, r_i^2}{r_o^2 - r_i^2}; \qquad B = \frac{-p_i\, r_o^2\, r_i^2}{r_o^2 - r_i^2}.$$

The radial and tangential stresses are

$$\sigma_{rr} = A + \frac{B}{r^2} = \frac{p_i\, r_i^2}{r_o^2 - r_i^2} - \frac{p_i\, r_o^2\, r_i^2}{r^2(r_o^2 - r_i^2)} = \frac{p_i\left[r_i^2 - \left(\frac{r_o\, r_i}{r}\right)^2\right]}{r_o^2 - r_i^2};$$

$$\sigma_{\theta\theta} = A - \frac{B}{r^2} = \frac{p_i\, r_i^2}{r_o^2 - r_i^2} + \frac{p_i\, r_o^2\, r_i^2}{r^2(r_o^2 - r_i^2)} = \frac{p_i\left[r_i^2 + \left(\frac{r_o\, r_i}{r}\right)^2\right]}{r_o^2 - r_i^2}.$$

Additionally, if the cylinder is closed at the ends,

$$\sigma_{zz} = \frac{p_i\, r_i^2}{r_o^2 - r_i^2}; \quad (\sigma_{zz} = 0 \text{ if the cylinder is open at the ends}).$$

Case (b) A thick cylinder subjected to an external pressure p_o only:
Now we shall consider case (b). It is always better to begin from the fundamental equations

$$\sigma_{rr} = A + \frac{B}{r^2}; \qquad \sigma_{\theta\theta} = A - \frac{B}{r^2}.$$

The boundary conditions this time are

$$(a)\ \text{ at } r = r_i, \quad \sigma_{rr} = 0 \qquad \longrightarrow \qquad A + \frac{B}{r_i^2} = 0;$$

$$(a)\ \text{ at } r = r_o, \quad \sigma_{rr} = -p_o \qquad \longrightarrow \qquad A + \frac{B}{r_o^2} = -p_o.$$

Solving for A and B from these equations, we obtain the stress components.

$$A = \frac{-p_o\, r_o^2}{(r_o^2 - r_i^2)}; \qquad B = \frac{p_o\, r_o^2\, r_i^2}{(r_o^2 - r_i^2)}.$$

$$\sigma_{rr} = A + \frac{B}{r_i^2} = \frac{p_o\left(\frac{r_o^2\, r_i^2}{r^2} - r_o^2\right)}{(r_o^2 - r_i^2)}; \qquad \sigma_{\theta\theta} = A - \frac{B}{r_i^2} = \frac{-p_o\left(\frac{r_o^2\, r_i^2}{r^2} + r_o^2\right)}{(r_o^2 - r_i^2)}.$$

Closed cylinder:
In addition, there will be an axial stress $\sigma_{axial} \equiv \sigma_{zz}$ which is uniform on the cross-section if, and only if, the cylinder is closed [Fig. 9.15b]. The cross-sectional area is $\pi(r_o^2 - r_i^2)$. The force due to the (fluid) pressure acting on the end plate is $p_i\, \pi r_i^2$. Hence, if the axial stress is uniformly distributed[23], it is

$$\sigma_{axial} \equiv \sigma_{zz} = \frac{p_i\, \pi r_i^2}{\pi(r_o^2 - r_i^2)} = p_i\, \frac{r_i^2}{r_o^2 - r_i^2}.$$

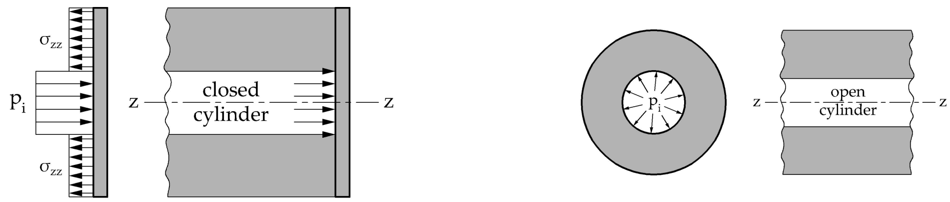
We may also note that the axial strain $e_{axial} \equiv e_{zz} =$ constant. In other words, plane cross-sections continue to be plane cross-sections even after the fluid pressure is applied. This fact can be seen readily.

$$e_{axial} \equiv e_{zz} = \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta})] = \frac{1}{E}\left[\sigma_{zz} - \nu\left(A + \frac{B}{r^2} + A - \frac{B}{r^2}\right)\right].$$

This shows that, if $e_{zz} =$ constant, then $\sigma_{zz} =$ constant, and vice versa.

Open cylinder:
If the cylinder is open, there is no end plate, and there is no axial stress. Thus, $\sigma_{zz} = 0$.

[23] It appears that Lamé had first derived these formulae assuming that the axial stress is uniformly distributed. Objections were raised when he presented his work. A sad Lamé went home and reworked the problem assuming this time that plane cross-sections remain plane (that is, assuming that $e_{zz} =$ constant). Lo, and behold! The result was the same! A beaming Lamé presented his work again to the complete satisfaction of everybody. All is well that ends well!



(a) A closed cylinder

(b) An open cylinder

Figure 9.17: The figures show a closed and an open cylinders. There will be an axial stress $\sigma_{axial} \equiv \sigma_{zz}$ in a closed cylinder, but not in an open one.

Strains and maximum strain:
The circumferential strain $e_{\theta\theta}$ is given by (the generalised Hooke's law)

$$e_{\theta\theta} = \frac{1}{E}[\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz}].$$

The maximum strain — it is this that we are concerned with — is at the inner fibre ($r = r_i$). Thus,

$$e_{max} = e_{\theta\theta}\Big|_{max} = \frac{1}{E}[\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz}]_{r=r_i}.$$

Substituting the expressions for $\sigma_{\theta\theta}$, σ_{rr}, $\sigma_{\theta\theta}$ at the inner fibre ($r = r_i$), we can obtain the maximum (circumferential or hoop) strain. Note that we have already worked out the expression for the axial stress σ_{zz}, and that it is zero ($\sigma_{zz} = 0$) for an open cylinder. When these are used, we obtain the expression for the maximum strain[24].

$$e_{max} = e_{\theta\theta}\Big|_{max} = \frac{1}{E}[\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz}]_{r=r_i}.$$

The simpler case of only an internal pressure p_i is of special interest. Now we have

$$e_{max} = e_{\theta\theta}\Big|_{max} = \begin{cases} \frac{p_i}{E}\left[\frac{r_o^2+(1-\nu)r_i^2}{r_o^2-r_i^2} + \nu\right] & \text{(closed cylinders - Clavarino's equation)} \\ \frac{p_i}{E}\left[\frac{r_o^2+r_i^2}{r_o^2-r_i^2} + \nu\right] & \text{(open cylinders - Birnie's equation)} \end{cases} \quad (9.47)$$

These are sometimes called the Clavarino's and Birnie's equations.

Displacements:
We often need for interference calculations and in other places the expressions for the displacements (especially radial displacements). The tangential displacement, of course, is zero because of the circular symmetry — axisymmetry — in our problem. For this we use the strain-displacement relations that we have already seen. In the context of axisymmetry,

[24] The numerical value, if desired, can be worked out for the specific case of interest. This expression is useful, among other places of analysis and applications, in the maximum strain theory of failure.

there is only the u displacement; $v = 0$ because of this circular symmetry; $w = 0$ because this is a two-dimensional problem. We need the following result, so crucial in interference fit calculations[25].

$$e_{rr} = \frac{du}{dr}; \qquad e_{\theta\theta} = \frac{u}{r}.$$

The Case of a Solid Disc Subjected to an (External) Pressure p_o

Whatever we have discussed so far carries over to the case of a solid disc (or shaft) also. Now, of course, it can be subjected only to an external pressure (say, p_o), and not to an internal pressure! Now also, as before we have

$$\sigma_{rr} = A + \frac{B}{r^2}; \qquad \sigma_{\theta\theta} = A - \frac{B}{r^2}.$$

The boundary conditions this time are different. One condition that is obvious is that

$$\text{at } r = r_o, \quad \sigma_{rr} = -p_o \qquad \longrightarrow \qquad A + \frac{B}{r_o^2} = -p_o.$$

What is the other boundary condition? Where does it come from? Well, we note that, if B survives (and $\neq 0$), then the stress at the centre becomes infinite. This cannot be permitted. (In other words, the singularity at $r = 0$ cannot be allowed to exist because the stresses would be infinite.) Hence $B = 0$. The stresses are now $\sigma_{rr} = \sigma_{\theta\theta} = 0$.

Both the stresses are equal and, what is more, they are both constant everywhere and equal to $-p_o$! There is no shear stress *at any point on any plane*[26]! Now the two-dimensional stress tensor is isotropic! *Every plane — every plane normal to the cross-section — at every point is a principal plane!*

We can arrive at the same conclusion by arguing differently. Let us write down the expression for the tangential strain $e_{\theta\theta}$. This gives us

$$e_{\theta\theta} = \frac{u}{r} = \frac{1}{E}[\sigma_{\theta\theta} - \nu\, \sigma_{rr}].$$

Now we can argue that the centre $r = 0$ must remain as the centre even after deformation: u must be zero at the centre $r = 0$. We arrive at the same result: $B = 0$. Thus, in conclusion, the stresses inside the solid disc (or solid shaft) are $\sigma_{rr} = \sigma_{\theta\theta} = -p_o$.

A Numerical Example

A cylinder has an internal diameter of 380 mm and an external diameter of 240 mm. Calculate the maximum pressure that can be applied so that the maximum stress does not

[25] After the stresses are determined using Lamé's equations, we can calculate the strains e_{rr} and $e_{\theta\theta}$ from the generalised Hooke's law. From $e_{\theta\theta}$, we can obtain the radial displacement. The principle of consistent deformation is in terms of the radial displacements.

[26] By any plane, we mean any plane normal to the cross-section. There is no shear stress on any plane. There can be shear stresses like τ_{xz} and τ_{yz}. Every point in the cross-section is a (two-dimensional) isotropic point!

exceed 400 MPa for the two cases: (a) no external pressure; $p_o = 0$; $p_i =?$; and (b) no internal pressure; $p_i = 0$; $p_o =?$

The dimensions tell us clearly that this is a thick cylinder.

Case (a)

We know that the maximum stress (which is $\sigma_{\theta\theta}$ occurs at the inner boundary and that its value is:

$$\sigma_{\theta\theta}\Big|_{max} = p_i \, \frac{r_o^2 + r_i^2}{r_o^{-} r_i^2} = 400 \text{ MPa}.$$

$$p_i = 400 \times \frac{190^2 - 120^2}{190^2 + 120^2} = 171.88 \text{ MPa}.$$

Thus, the maximum internal pressure that can be applied, $p_i = 171.88$ MPa.

Case (b)

When only an external pressure is applied,

$$\sigma_{\theta\theta} = -p_o \, \frac{r_o^2}{r_o^2 - r_i^2} \left(1 + \frac{r_i^2}{r^2}\right).$$

The maximum of $\sigma_{\theta\theta}$ occurs at the inner boundary where $r = r_i$. Hence,

$$\begin{aligned} \sigma_{\theta\theta}\Big|_{max} &= -2p_o \, \frac{r_o^2}{r_o^{-} r_i^2}; \\ &= -2p_o \, \frac{190^2}{190^2 - 120^2} = 400\,\text{MPa}; \\ p_o &= \frac{400 \times 21700}{2 \times 36100} = 120.22\,\text{MPa}. \end{aligned}$$

Thus, the maximum external pressure that can be applied = 120.22 MPa.

[As a rough guide to decide if a given cylinder is to be considered as thin or thick, we may perhaps say this: if the ratio of the outer and inner radii (r_o/r_i) is greater than 1.1, it is necessary to treat the cylinder as thick. This is not an absolute command. With experience and engineering judgement, such prescriptions can be ignored, overruled, or modified.]

Interference Fit Calculations

One of the important applications of theory discussed so far is for interference fit calculations. If a sleeve is to be mounted on a (solid) shaft by an interference fit, the inner diameter of the sleeve is made a little *less* or *smaller* than the diameter of the shaft. The sleeve is forced on the shaft, and for this reason, such a fit is also referred to as a force fit or a shrink fit [Figs 9.18, 9.19]. In practice, a slightly oversize part is slowly pushed through by a hydraulic ram. See p. 13-57 also.

Let us take the example of a brass sleeve force-fitted on a steel shaft [Fig. 9.19]. The outer brass sleeve has its outer and inner radii of r_o and r_i, respectively. The nominal radius of the steel shaft is the same as r_i. If the interference on the radius δ is specified or given, we

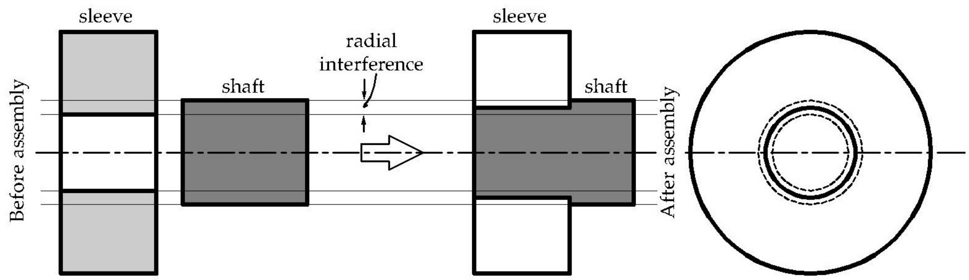
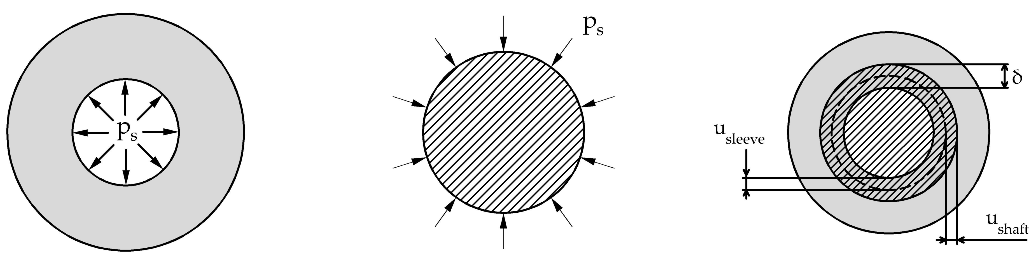


Figure 9.18: Interference fit: a sleeve mounted on a shaft. A sleeve is force-fitted on a shaft. The inner diameter of the outer sleeve is made a little less than the diameter of the shaft on which the sleeve is assembled.



(a) An outer brass sleeve (b) The inner solid steel shaft (c) The assembly

Figure 9.19: An interference fit of a sleeve on a shaft. The outer diameter of the shaft is actually a little larger, though nominally the same. When the sleeve is fitted on the shaft, the inner boundary of the sleeve moves up slightly, and the diameter of the shaft correspondingly reduced. The figure shows the diameters, before and after the assembly.

can find out the interference pressure. This pressure (which may also be called the shrinkage pressure p_s) can be calculated, and once it is known the stresses in the shaft and in the sleeve can be calculated using the standard thick cylinder formulae (Lamé's equations).

Case (a): Outer (brass) sleeve

The (unknown) interference / shrinkage pressure p_s is an external pressure acting radially inwards on the (outer) brass sleeve. The circumferential (also called hoop) and radial stresses at the interface $r = r_i$ are given by:

$$\sigma_{\theta\theta}(r_i)\Big|_{sleeve} = p_s\,\frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} \qquad \text{(tensile);}$$
$$\sigma_{rr}(r_i)\Big|_{sleeve} = -p_s \qquad \text{(compressive).}$$

The inner radius moves radially outwards. The amount of radial displacement, u can be calculated as shown:

$$\text{tangential strain,}\;\; e_{\theta\theta} = \frac{u}{r} = \frac{1}{E}[\sigma_{\theta\theta} - \nu\sigma_{rr}] \quad \longrightarrow \quad u = \frac{r}{E}[\sigma_{\theta\theta} - \nu\sigma_{rr}].$$

Thus,

$$u(r_i)\Big|_{sleeve} \equiv u_{sleeve} = \frac{r_i}{E_{brass}}[\sigma_{\theta\theta} - \nu_{brass}\sigma_{rr}] = \frac{r_i}{E_{brass}}\, p_s \left[\frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} + \nu_{brass}\right]. \qquad (9.48)$$

We have, thus, calculated δ_{sleeve}. In the same way, we can calculate δ_{shaft} also.

Case (b): Inner (steel) shaft

The inner steel shaft is subjected to the same interference pressure p_s, but radially inwards. This produces circumferential and radial stresses, $\sigma_{\theta\theta}$ and σ_{rr} of the same magnitude, both compressive. The values at the interface are:

$$\sigma_{\theta\theta}(r_i)\Big|_{shaft} = -p_s\,\frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} \qquad \text{(compressive);}$$

$$\sigma_{rr}(r_i)\Big|_{shaft} = -p_s \qquad \text{(compressive);}$$

$$u(r_i)\Big|_{shaft} \equiv u_{shaft} = \frac{r_i}{E_{steel}}[-p_s + \nu_{steel} p_s] = -\frac{r_i}{E_{steel}}\, p_s[1 - \nu_{steel}]. \qquad (9.49)$$

The radius moves radially inwards. The radial interference, $\delta = u_{sleeve} - u_{shaft} = |u_{sleeve}| + |u_{shaft}|$ [Fig. 9.19c]. Thus, the principle of consistent deformations gives us

$$\begin{aligned}\delta &= u_{sleeve} - u_{shaft} = |u_{sleeve}| + |u_{shaft}|;\\ &= p_s\, r_i \left[\frac{1}{E_{brass}}\left(\frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} + \nu_{brass}\right) + \frac{1}{E_{steel}}\,(1 - \nu_{steel})\right].\end{aligned}$$

giving us

$$p_s = \frac{\delta}{r_i\left[\frac{1}{E_{brass}}\left(\frac{r_o^2+r_i^2}{r_o^2-r_i^2} + \nu_{brass}\right) + \frac{1}{E_{steel}}\,(1 - \nu_{steel})\right]}. \qquad (9.50)$$

This is the required expression for the interference pressure. Once the interference pressure p_s is known, the stresses inside the outer sleeve and the inner shaft can be readily worked out using the thick cylinder equations (Lamé's formulae). If the numerical values are given, we can always work out the numerical values of the stresses. See p. 13-57 later.

Methods to Enhance the Elastic Strength of Thick Cylinders

In some industrial applications, the internal pressure is very large; then the cylinder would be too thick. This is not cost effective. The outer layers are not stressed much, and so the material is not utilised effectively. Can we enhance the elastic strength with a view to reducing the thickness, or to utilising all parts of the material more effectively?

Yes, we can. One way is to shrink a hollow cylinder (called a jacket, or a hoop, or a shell) on the cylinder. This will help reduce the maximum circumferential stress, and we can have a better distribution of circumferential stress $\sigma_{\theta\theta}$. The problem, we have seen, is that much of the material is under-utilised. This suggested measure will improve the distribution of stress [Figs. 9.20a, 9.20b].

Another way is to have several layers, one wrapped around the other, to have a laminated cylinder. This method will produce a more uniform stress distribution [Fig. 9.21b].

(a) Stresses due to the initial shrinking process (b) Total stresses after p_i is applied

Figure 9.20: The stress distributions (i) when a jacket is shrunk on the cylinder, and the total stress distribution when an internal pressure p_i is applied. The stress distribution is now more favourable.

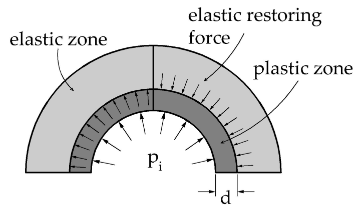
(a) Distribution of hoop prestress, $\sigma_{\theta\theta}$ (b) Total stresses after p_i is applied

Figure 9.21: The figures show the stresses in a multi-layer (laminated) cylinder. Fig. 9.21a] shows the distribution of the circumferential prestress. Fig. 9.21b shows the distribution of the total circumferential stress $\sigma_{\theta\theta}$, ie., after the internal pressure is applied to the prestressed cylinder. We can see that the distribution of $\sigma_{\theta\theta}$ is now vary favourable; it is now almost uniform across the thickness.

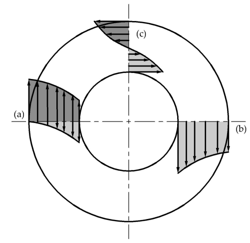
A third method is autofrettage. A very high pressure is applied initially leading to yielding at the inner fibre. With still higher pressure, the region of yielding penetrates from the inner fibre even deeper into the cylinder. When this applied internal pressure is released, there would be beneficial residual stresses inside the 'loaded and unloaded' cylinder. Now such a prestressed cylinder is capable of withstanding much larger internal pressures compared to the virgin cylinder.

Closing Comments

Two-dimensional problems are an important part of the theory of elasticity. It, therefore, deserves a far more comprehensive treatment than what is given here. There are several topics and methods of solution which are not even mentioned. We regret our inability to give this topic the honourable and well deserved important place that this deserves. We hope that the ambitious students will recognise that what is presented here is to be regarded as no more than an introduction to two-dimensional problems. With these words of apology or explanation, we close this chapter here.



(a) Autofrettage: a schematic figure



(b) Autofrettage: stresses

Figure 9.22: A schematic figure of autofrettage. An enormous pressure is applied so that plasticity creeps into the cylinder to a depth of d (autofrettage depth). When the pressure is released there are internal compressive residual (hoop) stresses. When an internal pressure is applied to such a cylinder, the total stresses are reduced because of the residual compressive stresses left in the cylinder. Fig. 9.22b shows the stress distributions.

In the next chapter, we shall discuss the important topic of energy methods.

Chapter 10

ENERGY METHODS

Energy methods are extremely important in the mechanics, particularly advanced mechanics, of solids. Several problems can be solved effectively by these methods, that are not otherwise easy to tackle. Even more importantly, these methods help us to formulate important physical problems. Thus, energy methods may be regarded as the starting point, first of setting up a general framework of concepts and methodology; next of formulating key notions, concepts and methods; and finally of solving actual physical problems of much interest to engineers at the operational level. These are intimately linked to variational methods based on the calculus of variations. In fact, energy methods and variational methods applied to deformable bodies are almost synonymous. Finite Element Method (FEM), which has become so powerful and indispensable in the analysis and design of engineering structures and machine components, is heavily dependent on these variational methods.

The starting point is Johann Bernoulli's[1] principle of virtual work. This is to be written for the case of a deformable body. We shall do this later in this chapter. We are immediately led to the more special principle of stationary potential energy. A mathematical technique called the Legendre transformation takes us to the complementary energy and the principle of complementary energy. Castigliano's theorems follow.

Unfortunately energy methods are not always easy. Beginners[2] find them difficult; they are often confused. We shall, therefore, be simple-minded and present only the relatively easier part. We hope that the ambitious students will learn the advanced material later[3].

---

[1] Johann (also called Jean I or John I) Bernoulli (July 1667 - Jan. 1748) was a famous mathematician of the illustrious Bernoulli family. He was regarded as the greatest mathematician in Europe at that time.

[2] And who are beginners? Nearly everybody except a few, very few, real *ustads* or grand masters.

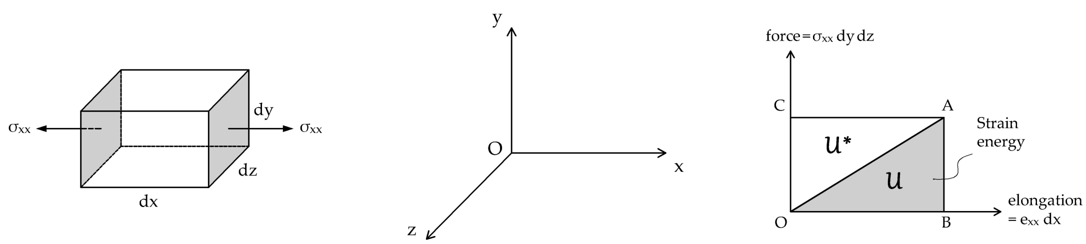
[3] What makes these difficult? Well, these principles are formulated in terms of triple integrals often using the index notation. The minimum principles make it necessary to minimise such triple integrals. The technique of minimisation uses the calculus of variations. These are frightening for those who may not have studied the calculus of variations. Actually triple integrals are nothing to be afraid of. In many problems the evaluation of these triple integral is very simple. For example, when applied to the members of a truss, the stresses and the dimensions are often constants, and the triple integral crumbles down to the evaluation of the volume of the truss member. But the real reason is that students of engineering hardly ever see mathematics used in their engineering subjects.

The concepts of strain energy and total potential energy[4] play a crucial role in these methods. The reason is that there are powerful theorems like the two Castigliano theorems and the theorem of stationary total potential energy.

As energy plays a central role, we shall begin by discussing strain energy. We shall then use it to obtain the deflection of beams, etc.

STRAIN ENERGY

We have studied and used the concept of strain energy[5] in our earlier studies. Here we shall discuss this a little more in detail.



(a) An element: only σ_{xx} (b) A coordinate system (c) $\mathcal{U}$ and $\mathcal{U}^*$

Figure 10.1: An element subjected only to uniaxial stress σ_{xx} is shown. The area OAB under the stress-strain curve (straight line) OA is the strain energy density $\mathcal{U}$. The area OAC (which 'completes the area') is the complementary energy density $\mathcal{U}^*$.

We shall now approach the concept of strain energy differently. Let us first consider a simple block [Fig. 10.1] acted upon by a uniaxial stress σ_{xx}. The force $\sigma_{xx}\, dy\, dx$ produces an elongation $e_{xx}\, dx$ in the $x-$direction. The work done, therefore, is

$$dW = \frac{1}{2}\,\text{force} \times \text{elongation} = \frac{1}{2}(\sigma_{xx}\, dy\, dz)(e_{xx}\, dx) = \frac{1}{2}\,\sigma_{xx}\, e_{xx}\, dx\, dy\, dz = \frac{1}{2}\,\sigma_{xx}\, e_{xx}\, dV,$$

where dV is the elemental volume. This is the area under the line OA [Fig. 10.1c]. This work done is stored in the body as internal energy called the strain energy dU. This is related to the strain energy density function $\mathcal{U}$ by the relation

$$dU = \mathcal{U}\, dV = \mathcal{U}\, dx\, dy\, dz.$$

The strain energy density function, we recall, is the strain energy per unit volume. Now if all the stress components are simultaneously present,

$$dU = \mathcal{U}\, dV = \frac{1}{2}(\sigma_{xx}\, e_{xx} + \sigma_{yy}\, e_{yy} + \sigma_{zz}\, e_{zz} + 2\tau_{xy}\, e_{xy} + 2\tau_{yz}\, e_{yz} + 2\tau_{zx}\, e_{zx} + 2\tau_{xy}\, e_{xy})\, dV,$$

[4] It is better to use the term *potential* instead of the more popular one potential energy. Thus, we shall prefer to use the terms *total potential* instead of total potential energy, and the *theorem of stationary total potential* instead of the theorem of stationary total potential energy, respectively.

[5] We may recall that the existence of a strain energy density function was assumed when we reduced the number of elastic constants (elastic moduli) from 36 to 21.

where $2\,e_{xy} = \gamma_{xy},\ 2\,e_{yz} = \gamma_{yz},\ 2\,e_{zx} = \gamma_{zx}$ are the shear strains.

This expression for $\mathcal{U}$ may be written either (i) in terms of the stress components only, or (ii) in terms of the strain components only using the constitutive equations (stress-strain relations, the generalised Hooke's law). Expressed in terms of the stress components, it appears as

$$\begin{aligned}\mathcal{U} &= \frac{1}{2E}(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2) - \frac{\nu}{E}(\sigma_{xx}\,\sigma_{yy} + \sigma_{yy}\,\sigma_{zz} + \sigma_{zz}\,\sigma_{xx}) + \frac{1}{2G}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \\ &= \frac{1}{2E}[I_1^2 - 2(1+\nu)I_2], \end{aligned} \tag{10.1}$$

where I_1 and I_2 are, respectively, the first and the second invariants. If, instead, $\mathcal{U}$ is expressed in terms of the strain components, the expression appears as

$$\mathcal{U} = \frac{1}{2}\lambda\,e^2 + G(e_{xx}^2 + e_{yy}^2 + e_{zz}^2) + 2G(e_{xy}^2 + e_y^2 + e_{zx}^2), \tag{10.2}$$

where

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \text{and} \qquad e = e_{xx} + e_{yy} + e_{zz}.$$

This expression for $\mathcal{U}$ tells us that the strain energy U is always positive. This property of being positive definite is of theoretical importance. Thus, $U = 0$ only if all the strain components (and hence all the stress components) are zero. This fact has important, very important, consequences. Some of them are the uniqueness theorem of linear elasticity, and the principles of minimum total potential (energy) and minimum complementary energy.

We emphasise that

$$\frac{\partial \mathcal{U}}{\partial e_{ij}} = \sigma_{ij} \qquad (i, j = 1, 2, 3).$$

Here we can verify this result using Eq. (10.2). For example,

$$\frac{\partial \mathcal{U}}{\partial e_{xx}} = \lambda\,e + 2Ge_{xx} = \sigma_{xx}.$$

Strain energy for the case of plane stress

Sometimes it is useful to have the (simplified) expression for the (simplified) case of plane stress. Now the three stress components σ_{zz}, τ_{xz}, and τ_{yz} vanish. With $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$, we can obtain the expression for $\mathcal{U}$ in terms of the stress components as

$$\mathcal{U} = \frac{1}{2E}(\sigma_{xx}^2 + \sigma_{yy}^2) - \frac{\nu}{E}\sigma_{xx}\,\sigma_{yy} + \frac{1}{2G}\tau_{xy}^2 \tag{10.3}$$

and in terms of the strain components as

$$\mathcal{U} = \frac{E}{2(1-\nu^2)}(e_{xx}^2 + e_{yy}^2 + 2\nu\,e_{xx}\,e_{yy}) + 2G\,e_{xy}^2. \tag{10.4}$$

The expression for the total strain energy U stored in the body is $U = \int \mathcal{U}\,dV$. This is a crucial concept in the mechanics of solids. The energy theorems are based on the strain energy and its dual (say, its extension in a sense), the complementary energy U^*.

(a) U and U^*: linear case (b) U and U^*: nonlinear case

Figure 10.2: The strain energy and the complementary energy for the linear [Fig. 10.2a] and nonlinear [Fig. 10.2a] cases. Note that $U = U(\delta)$, while $U^* = U^*(P)$.

Complementary Energy

The upper triangle OAC Figs 10.2 completes the rectangle $OBAC$; hence the area enclosed by OAC is a 'complementary' quantity. It has the dimension of energy. It is, therefore, called the complementary energy, even though it is not an energy[6]. In this simple case, the complementary energy U^* is related to the strain energy U by the relation

$$U^* = -U + \sigma_1 \, e_1. \tag{10.5}$$

The complementary energy U^* is obtained from the strain energy U by what is known as a Legendre transformation[7]. In the simple case that we consider here, the relationship [Eq.(10.5)] is given in this simple form. There are such dual transformations. These are both interesting and useful. However, we cannot discuss them here. Note that in the linear case [Fig. 10.2a] (but not in the nonlinear one [Fig. 10.2b]), $U = U^*$ numerically. But it is not quite correct to write $U = U^*$, because $U = U(\delta)$ while $U^* = U^*(P)$.

STRAIN ENERGY: SOME COMMON STRUCTURAL ELEMENTS

We shall define strain energy U in a general form, with the usual notations, as

$$U = \frac{1}{2} \iiint_{\mathcal{V}} (\sigma_{xx} e_{xx} + \sigma_{yy} e_{yy} + \sigma_{zz} e_{zz} + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}) \, dx \, dy \, dz. \tag{10.6}$$

Let us simplify this expression to suit the special case of a 'one-dimensional body' such as an axially loaded bar, a circular shaft in torsion and a beam.

Strain Energy for 'One-Dimensional' Bodies

For such a case of a 'one-dimensional' body (such as an axially loaded bar, a circular shaft in torsion, or a cantilever loaded by an end load) [Fig. 10.3], there are only two components of the stresses, σ_{xx} and $\tau_{xy} = \tau_{yx}$. For this special simplified case, the expression [Eq. (10.6)]

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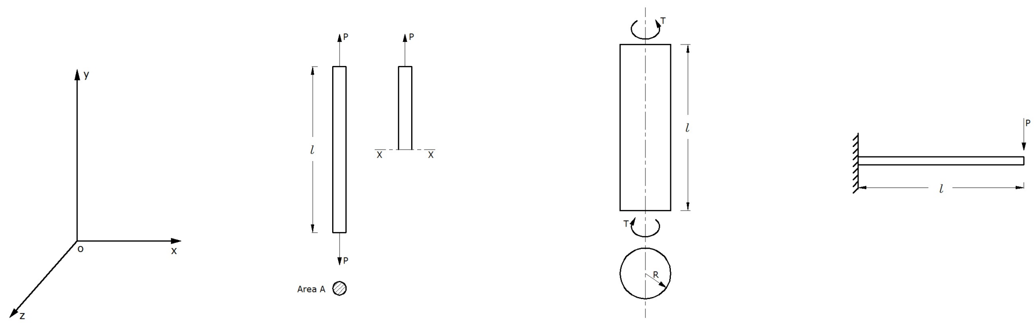
[6] Named after Harold Malcom Westergaard (Oct. 1888 - June 1950), born in Copenhagen and educated in Germany and other places in Europe, well known as a professor, first at the University of Illinois and later at Harvard.

[7] See [9] or, better still, Langhaar [8] for more details.



(a) xyz system (b) An axial bar (c) A circular shaft (d) A cantilever

Figure 10.3: Three cases of 'one-dimensional' bodies are shown: an axially loaded uniform bar [Fig. 10.3b], a uniform circular shaft subjected to a pair of torques $T-T$ [Fig. 10.3c], and a cantilever loaded by an end load, P [Fig. 10.3d].

for the strain energy gets simplified as

$$U = \frac{1}{2} \iiint_{\mathcal{V}} (\sigma_{xx} e_{xx} + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz})\, dx\, dy\, dz. \tag{10.7}$$

There is really no need at all to integrate w.r.to the variable z, because there is no variation (change) along the z-axis (perpendicular to the plane of the paper). The integration is essentially only along one axis, say the x-axis; hence the qualification as 'one-dimensional bodies'. We can replace the strain components in Eq. (10.7) in terms of the stress components using the constitutive equation (Hooke's law) and obtain

$$U = \frac{1}{2} \iiint_{\mathcal{V}} \left(\frac{\sigma_{xx}^2}{E} + \frac{\tau_{xy}^2}{G} \right) dx\, dy\, dz. \tag{10.8}$$

This equation is now in good shape for use for all 'one-dimensional' bodies such as the ones shown in Fig. 10.3. First we shall demonstrate how Eq. (10.8) may be used to obtain the elongation of the bar shown in Fig. 10.3b.

An axially loaded bar:
For this case, there is only one component of stress, viz., $\sigma_{xx} = P/A$. Accordingly, the expression for the total strain energy, U stored in the bar expressed as a function of the axial load P in a form convenient for differentiation w.r.to P (to obtain the 'work absorbing component of the displacement', which is the elongation of the bar) is

$$\begin{aligned} U = U(P) &= \frac{1}{2} \iiint_{\mathcal{V}} \frac{\sigma_{xx}^2}{E}\, dx\, dy\, dz \\ &= \frac{1}{2} \int_0^l \frac{1}{E} \left(\frac{P}{A} \right)^2 dx \times A, \qquad \text{because} \quad \iint dy\, dz = A \\ &= \int_0^l \frac{P^2}{2AE}\, dx = \frac{P^2 l}{2AE}. \end{aligned} \tag{10.9}$$

The elongation is obtained by differentiation of this expression w.r.to the load P to yield

$$\delta = \frac{dU}{dP} = \frac{Pl}{AE}, \tag{10.10}$$

which we know is correct. [Some students may find this confusing. Which P are we differentiating with respect to? And which elongation, the full elongation, or the elongation of the half bar, is this δ? It is perhaps better to consider only one half (say, the top half) of the bar. Because of symmetry, the middle cross-section $X-X$ does not move; it stays where it was with no movement. It can, therefore, be considered as fixed as shown in Fig. 10.3b. Now the strain energy is only half of what is given by Eq. (10.9). Differentiating this w.r.to. we obtain $U_{half\ bar} \equiv U_{1/2} = \frac{P^2 l}{4AE}$. Elongation of the top half of the bar $= \dfrac{dU_{1/2}}{dP} = \dfrac{Pl}{2AE}$. As there are two halves, the total elongation is $\delta = \dfrac{Pl}{AE}$. This explanation, we hope, is clearer.] Next we shall consider a uniform circular shaft subjected to a torque.

A circular shaft in torsion:
Shown in Fig. 10.3c is a uniform circular shaft — really a cylindrical shaft; by a circular shaft is meant a shaft with a circular cross-section — subjected to a pair of torques $T - T$. We shall now derive the strain energy for this member.

We know from our earlier study of Coulomb's theory of torsion (applicable only to circular shafts) that the torsional shear stress τ on the cross-section is given by[8] $\tau = Tr/J_p$. With this, the expression for the strain energy, U works out to be

$$\begin{aligned} U &= \frac{1}{2} \iiint_{\mathcal{V}} \left(\frac{\tau_{xy}^2}{G} \right) dx\, dy\, dz \\ &= \int_0^l \frac{T^2}{2GJ_p}\, dx, \quad \text{because} \quad dy\, dz = dA, \;\; \text{and} \;\; J_p = \int_{\mathcal{A}} r^2\, dA. \end{aligned} \tag{10.11}$$

The angle of twist, which is the 'work absorbing component of the displacement', is obtained by differentiating this expression given in Eq. (10.11) w.r.to the 'load' T to yield

$$\theta = \frac{dU}{dT} = \frac{d}{dT} \int_0^l \frac{T^2}{2GJ_p}\, dx = \int_0^l \frac{d}{dT} \left(\frac{T^2}{2GJ_p} \right) dx = \frac{Tl}{GJ_p}, \tag{10.12}$$

as T, J_p and G are all constants here. We know that this is the correct formula.

The third and last structural member that we mentioned above is a cantilever. We shall presently obtain the expression for the strain energy for this case.

A cantilever with an end load:
Consider a cantilever loaded by an end load P. We desire to obtain the expression for the strain energy U and, thereby, to find the deflection at the free end.

[8] Students are advised to review the theories of pure bending of beams, and of torsion of circular shafts, and to see the similarity between the Euler-Bernoulli formula and the torsion formula in Coulomb's theory of torsion (with the usual notation), $\sigma/y = M/I = E/R$ and $\sigma/y = M/I = E/R$ and $\tau/r = T/J_p = G\theta/l$.

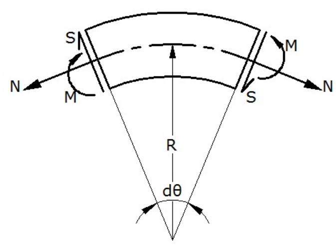
The expression for the strain energy [Eq. (10.8)] is reproduced here for convenience as

$$U = \frac{1}{2}\iiint_{\mathcal{V}}\left(\frac{\sigma_{xx}^2}{E}+\frac{\tau_{xy}^2}{G}\right)dx\,dy\,dz.$$

First, let us calculate the strain energy associated with the bending moment. The bending stress is given by the Euler-Bernoulli equation as $\sigma_{xx} = My/I$. From this expression the strain energy due to the (normal) bending stress is calculated as

$$\begin{aligned} U &= \frac{1}{2}\iiint_{\mathcal{V}}\left(\frac{\sigma_{xx}^2}{E}+\frac{\tau_{xy}^2}{G}\right)dx\,dy\,dz = \iiint_{\mathcal{V}}\frac{\sigma_{xx}^2}{2E}\,dx\,dy\,dz \\ &= \iiint_{\mathcal{V}}\frac{1}{2E}\left(\frac{My}{I}\right)^2 dx\,dy\,dz = \iiint_{\mathcal{V}}\frac{M^2y^2}{2EI^2}\,dx\,dy\,dz, \\ &= \int_0^l \frac{M^2}{2EI}\,dx, \quad \text{because} \;\; dy\,dz = dA, \;\; \text{and} \quad \int y^2\,dy\,dz = \int y^2\,dA = I. \end{aligned}$$

$$\text{Thus,} \quad U = \int_0^l \frac{M^2}{2EI}\,dx. \tag{10.13}$$



An element of the beam. Two neighbouring cross-sections, which were parallel before deformation, have now rotated (still remaining as plane), through an angle $d\theta$. The distance between the two cross-sections along the neutral axis is dx. This small length dx is shown as fairly large for clarity.

In general, on the cross-sections, there are not only bending moments $M - M$[9], but also a direct thrust N and a shear force S. Let us note further that $dx = R \times d\theta$.

Eq. (10.13) may be obtained in a different way also. We know that the strain energy stored is the work done by the bending moment on the 'work absorbing component of the displacement', which is the rotation of the cross-section. The figure above shows two neighbouring cross-sections of a bent beam. These, which were parallel before the deformation, now after the deformation subtend an angle $d\theta$. Thus, from the geometry, we have

$$d\theta = \frac{dx}{R} = \frac{dx}{\frac{EI}{M}}, \quad \text{as} \quad \frac{M}{I} = \frac{E}{R} \quad \longrightarrow \quad R = \frac{EI}{M}.$$

[9] Actually the bending moments cannot be the same if there are shear forces, because the shear force is equal to the rate of change (w.r.to the variable x) of the bending moment. However, here this introduces no error. (Why?)

$$\text{strain energy:} \quad dU = \frac{1}{2} \times \text{work done by B.M.} = \frac{1}{2} \times M \times \text{angle of rotation}$$
$$= \frac{1}{2}\,M\,d\theta = \frac{1}{2}\,M\,\frac{dx}{R} = \frac{M^2}{2EI}\,dx.$$
$$\text{Thus, the strain energy:} \quad U = \int_o^l \frac{M^2}{2EI}\,dx.$$

Next, let us calculate the strain energy associated with the shear force. The shear stress (cross shear) distribution across the cross-section (here rectangular for this example), we know, is given by the equation as

$$\tau_{xy} = \frac{S}{Ib}\int_y^{\frac{d}{2}} yb\,dy = \frac{S}{Ib}\left[\left(\frac{d}{2}\right)^2 - y^2\right]$$

with the usual notation. Using this expression we can calculate the strain energy due to the shearing stress (cross shear) as

$$\text{strain energy:} \quad U = \frac{1}{2}\iiint_{\mathcal{V}} \frac{S^2\left[\left(\frac{d}{2}\right)^2 - y^2\right]^2}{4I^2G}\,dx\,dy\,dz = \frac{3}{5}\int_0^l \frac{S^2}{AG}\,dx. \tag{10.14}$$

With these expressions for the strain energy, the deflection of the cantilever can be readily calculated. We shall calculate separately the deflection at the free end due to (i) the bending moment alone, (ii) the shear alone, and (iii) both the bending moment and the shear.

(i) *Deflection due to bending moment alone*
The bending moment along the beam [Fig. 10.3d] is given by

$$M = M(x) = Px \quad (0 \leq x \leq l).$$

The strain energy U_M stored in the beam due to the bending moment alone is

$$U_M = U_M(P) = \int_0^l \frac{M^2}{2AE}\,dx = \int_o^l \frac{(Px)^2}{2AE}\,dx.$$

The deflection at the free end due to the bending moment alone is

$$\delta_M = \frac{dU_M}{dP} = \frac{d}{dP}\int_0^l \frac{(Px)^2}{2EI}\,dx = \int_0^l \frac{\partial}{\partial P}\left(\frac{(Px)^2}{2EI}\right)dx = \frac{Pl^3}{3EI}, \tag{10.15}$$

which we know to be correct.

(ii) *Deflection due to the shear force alone*
We shall now calculate the deflection due to the shear force alone. We have seen that the shearing force along the beam [Fig. 10.3d] is given by

$$S = S(x) = P \quad (0 < x < l).$$

The strain energy U_S stored in the beam due to the shearing force alone is

$$U_S = U_S(P) = \frac{3}{5}\frac{P^2 l}{AG}.$$

The deflection at the free end due to the shearing force alone is

$$\delta_S = \frac{dU_S}{dP} = \frac{d}{dP}\int_0^l \frac{3}{5}\frac{P^2 l}{AG}\,dx = \int_0^l \frac{\partial}{\partial P}\left(\frac{3}{5}\frac{P^2 l}{AG}\right)dx = \frac{6}{5}\frac{Pl^2}{AG}. \tag{10.16}$$

(iii) *Deflection due to both the bending moment and the shear force*
If both the effects (due to the bending moment and the shear force) are considered together, the total deflection is

$$\delta = \delta_M + \delta_S = \frac{Pl^3}{3EI} + \frac{6}{5}\frac{Pl}{AG} = \frac{Pl^3}{3EI}\left[1 + \left\{0.6(1+\mu)\frac{d^2}{l^2}\right\}\right]. \tag{10.17}$$

Comments
A few comments seem to be appropriate here. In Eq. (10.17), the first term is the deflection due to the bending moment alone. The term within the double brackets $\{\cdots\}$ is the 'correction' to be added to account for the effects of shear. If we take some realistic figures and calculate the numerical values, we can see that this second term is very small indeed in almost all of the usual cases when the length (span) l of the beam is much larger than its depth d.

A close look at this equation (10.17) rewards us with invaluable insights. In the case of a *long, slender* beam, the deflection due to the bending moment is all that matters; the contribution of the shear is very, very small. On the other hand, if we go to the other extreme of a *short, stubby* beam, the contribution of the bending moment towards the total deflection (which is admittedly much smaller than in the earlier case) is very, very small; the deflection due to the shear force is all that matters. For the in-between case, both the bending moment and the shear force do contribute to the total deflection. In other words, while for most beams that we see in practice, the contribution of the bending moment to the total deflection may still be the lion's share, the effect of the shear force may not be negligible if the dimensions of the beam are such as to make it fall in between the two extreme cases pointed out[10]. The in-between cases of beams, where the effect of the shear force cannot be neglected in the calculation of the total deflection, are called Timoshenko beams in honour of that outstanding engineer-scientist S.P. Timoshenko, because it was he who showed how the effect of shear forces may be reckoned.

Let us also look at the same equation, and examine what it entails. We have added two components of the deflection to obtain the total deflection. It is equivalent to adding the strain energies U_M and U_S, calculated one at a time, to get the total strain energy. This raises an important question: are we permitted to add the strain energy due to each component of the load, and add them to obtain the total strain energy? In other words, can such strain energies be superposed? We shall examine this question.

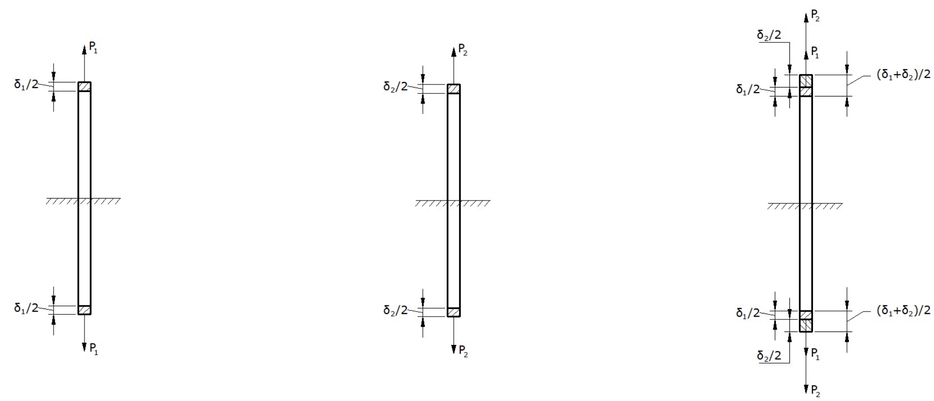
[10] Students are urged to do these calculations. The experience will improve our intuitive understanding of what, and what not, are important in engineering calculations. We learn some topics, and forget the details; still there is some residue left over. It is this residue that adds to our qualitative understanding. This collective understanding gained over the years forms what can vaguely be referred to as engineering judgement. Engineering judgement is no small matter; it is a component of overriding importance in the professional competence of engineers. This is to be consciously cultivated.

The answer (in anticipation of the result of this inquiry) is: superposition of strain energies is not permitted in general, but in certain special situations, it is permissible. What we have seen above in Eq. (10.17) is one of such special situations.

Superposition of Strain Energies

We shall take up this matter in some detail, and come to a proper conclusion. First we shall consider as an example a simple, special case of two axial (tensile) loads applied on a uniform bar, one at a time, and later both together.

Why strain energies cannot be obtained by superposition:



(a) Tensile load $P_1 - P_1$ (b) Tensile load $P_2 - P_2$ (c) Loads $(P_1 + P_2) - (P_1 + P_2)$

Figure 10.4: The strain energies, U_1 due to the load P_1 and U_2 due to the load P_2 cannot be added (superposed) to obtain the total strain energy, U due to $(P_1 + P_2)$: $U \neq U_1 + U_2$!

Let us consider the case of a uniform bar loaded by $P_1 - P_1$ as shown in Fig. 10.4a. The elongation of the bar and the strain energy stored are,

$$\text{stress: } \sigma_1 = \frac{P_1}{A}; \qquad \text{strain: } \epsilon_1 = \frac{P_1}{AE}; \qquad \text{axial elongation: } \delta_1 = \frac{P_1 L}{AE}$$

$$\text{strain energy stored: } U_1 = \frac{P_1^2 L}{2AE}. \tag{10.18}$$

In exactly the same way, if an axial load P_2 alone acts on the bar, the strain energy U_2 due to this load P_2 alone is

$$U_2 = \frac{P_2^2 L}{2AE}. \tag{10.19}$$

Now, if the two loads P_1 and P_2 act together, that is, if the load is $P_1 + P_2$, the strain energy U is similarly

$$U = \frac{(P_1 + P_2)^2}{2AE} \neq \frac{P_1^2 L}{2AE} + \frac{P_2^2 L}{2AE} \tag{10.20}$$

showing that the total strain energy due to two loads is *not equal to* the sum of the strain energies due to each load. Why is this so? Mathematically, this is because $(P_1 + P_2)^2 =$

$P_1^2 + P_2^2 + 2P_1P_2 \neq P_1^2 + P_2^2$, which is almost a silly explanation. The expression for the strain energy is *nonlinear* (the square of the load appearing in the expression); we know that superposition is not valid when there is nonlinearity.

This explanation is fine as far as it goes. But it is insightful when we examine this from a physical point of view. The 'cross-term' $2P_1P_2$ is the villain; it is the presence of this 'cross-term' that violates or spoils the property of superposition. The cross-term corresponds to the work done by the first load during the additional displacement caused when the second load is applied!

The moral of this demonstration is this. The total strain energy corresponding to two (or more) loads *cannot be obtained* by calculating the strain energy corresponding to each load, acting only one at a time, and adding them together.

If this is so, what shall we do to obtain the total strain energy?
Fortunately, the situation is not as hopeless as this demonstration seems to show. When an axial load acts, there is an axial displacement, and there is a resulting strain energy. Now if a bending moment is additionally applied, there is a corresponding 'work absorbing displacement' which, for the case of a bending moment, is the rotation of the cross-section. During this rotation, the axial load *does not* do any work. There is no 'cross effect' now, and it is, therefore, perfectly legitimate to add the two strain energies, calculated one at a time, to obtain the total strain energy! Superposition then does hold if the loads are such! The reason is, to repeat: when the second load acts, and a corresponding 'work absorbing displacement' results, the first load does not do any work!

Thus, in spite of the objection raised, it is indeed permissible to calculate the strain energy because of each of the separate loads, and add them up (superpose them) to obtain the total strain energy *if there are no 'cross-effects'*.

We have answered the question raised above. We shall proceed to the next topic in the next section. This concerns the potential of the external loads.

Potential of the External Loads (External Potential Energy)

We have discussed the topic of strain energy above. Let us note once again that, when the external forces do work on a conservative system — bars in tension, shafts in torsion, beams in bending, etc. are conservative systems; there is no energy dissipation — the entire work done is stored as internal energy inside the body. The strain energy is the internal energy stored in the structural members that we discussed above.

Strain energy is an important topic; some very useful theorems are based on this. There are other theorems that are based on the total potential. The total potential (energy) is the sum of internal potential energy (strain energy), and the potential (energy) of the external loads. Therefore, we shall now see what the potential of the external loads is.

We shall explain the concept of the external potential energy, or the potential of the external forces, with reference to the example of a cantilever with an end load P producing an end deflection δ. There is only one applied external load, P. This load moves through the distance δ (deflection of the free end). The external potential is $-P\delta$. For this cantilever,

the various terms are the following.

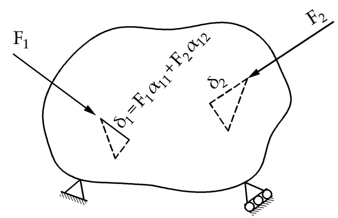
$$\text{strain energy (internal energy):} \quad U = \int_0^L \frac{M^2}{2EI}\,dx$$

$$\text{external potential (energy):} \quad V = -P\delta$$

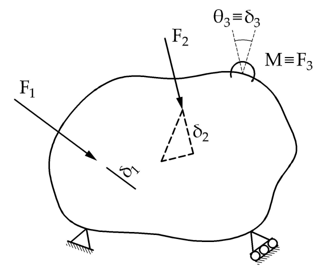
$$\text{total potential (energy):} \quad \pi = V + U = -P\delta + \int_0^L \frac{M^2}{2EI}\,dx$$

Now there are a few questions here[11].

Influence Coefficients



(a) F_1 and F_2 acting at 1 and 2



(b) Several forces and moments

Figure 10.5: Two forces F_1 and F_2 are acting on a body at 1 and 2 producing deflections (total) δ_1 and δ_2 [Fig. 10.5a]. Fig. 10.5b shows several forces and moments on the body.

Consider a linearly elastic body in equilibrium acted upon by a set of forces and moments, say, F_1, $F_2, \cdots$; M_1, $M_2, \cdots$ [Fig. 10.5b]. These produce linear and angular displacements. To express these, it is convenient to use influence coefficients α_{ij}.

The influence coefficient α_{ij} is defined as the deflection (linear or angular, as the case may be) at the point i when a unit load (force or moment, as the case may be) acts at the point j. Thus, if a force F_1 acts at the point 1, and another force F_2 at the point 2, the total deflections at the points 1 and 2 are expressed as $\delta_1 = \alpha_{11} F_1 + \alpha_{12} F_2$ and $\delta_2 = \alpha_{21} F_1 + \alpha_{22} F_2$.

We must realise that the δ_1 is the 'work absorbing component' of the force F_1. That is, δ_1 is the component (of the total displacement at 1) in the direction of F_1. The work done by a force F corresponding to the 'work absorbing component' δ of the displacement is $F\,\delta_1$ if the full force moves through the full displacement δ. If, on the other hand, the force F is gradually applied, then the work done is $(1/2) \times F \times \delta$. Whether or not the factor 1/2 is present in the expression must be understood clearly. These influence coefficients enjoy the important property of symmetry, i.e., $\alpha_{ij} = \alpha_{ji}$, called the reciprocal relations.

[11] Firstly, why the minus sign? After all the force P and its displacement δ are in the same direction. Secondly, it would appear that the work done by the external force is $(1/2)P\delta$. Some explanation seems to be necessary to understand these confusing issues. The trouble arises because we call this the work done. If we use the term potential, we do not have these difficulties. There is a strong case to use the term (gravitational) potential instead of the more popular term potential energy.

Betti-Maxwell Reciprocal Relation

We shall give presently a simple proof of this Betti-Maxwell reciprocal theorem[12].

Proof:

First apply a load F_1 at a point 1 on the body [Fig. 10.5a]. Calculate the work done by this load during the resulting displacement δ_1 which is stored inside the body as elastic strain energy. Next apply a second load F_2 at another point 2 [Fig. 10.5a]. Calculate the work done by the load F_2 and also by F_1, and the total strain energy U_1 stored in the body.

Now start all over again, and apply first a load F_2 at 2, and then the load F_1 at 1. Calculate the total work done and hence the strain energy stored. Now we argue that the total strain energy stored must be the same independent of the order of application of the loads. Equating these two expressions for the total energy in the body, we arrive at the desired result $\alpha_{12} = \alpha_{21}$. This is the procedure. We shall now carry out the calculations.

Step 1

When the first load F_1 alone acts, the displacement is δ_1, and hence the work done (equal to the strain energy stored) is $1/2 \times F_1 \times \delta_1$ because the load F_1 is applied gradually.

$$U_1 = \frac{1}{2}\, F_1\, \delta_1 = \frac{1}{2}\, F_1(\alpha_{11}\, F_1) = \frac{1}{2}\, \alpha_{11}\, F_1^2$$

Now the second load F_2 is applied gradually at the point 2. This produces a displacement of (i) $\alpha_{22}\, F_2$ at the point 2, and (ii) $\alpha_{12}\, F_2$ at the point 1. During these displacements, the work done by F_2 is $(1/2)F_2(\alpha_{22}\, F_2)$, while that done by F_1 (which has its full presence, not gradual) is $F_1(\alpha_{12}\, F_2)$. Thus,

$$U_2 = \frac{1}{2}\, F_2(\alpha_{22}\, F_2) + F_1(\alpha_{12}\, F_2).$$

(Note the presence and absence of the factor 1/2 in the first and second terms, respectively.)

Step 2

Now start all over again. Apply on the virgin unloaded body, first F_2 at the point 2, and then F_1 at the point 1. The work done during the first loading is

$$U_1' = \frac{1}{2}\, F_2\, \delta_2 = \frac{1}{2}\, F_2(\alpha_{22}\, F_2).$$

The work done during the application of F_1 is

$$U_2' = \frac{1}{2}\, F_1(\alpha_{11}\, F_1) + F_2(\alpha_{21}\, F_1).$$

(Note again the presence and absence of the factor 1/2 in the first and second terms, respectively.)

Step 3

[12] Opinion is divided about the priority. A.E.H. Love attributes this to Enrico Betti (1872), but states that this is a special case of a more general theorem due to Lord Rayleigh (1873). Lord Rayleigh himself acknowledges the reciprocal theorem to Betti. Neither Love not Rayleigh mentions Maxwell's (1864) name. The theorem is now generally known as the Betti-Maxwell reciprocal theorem.

Now arguing that the total elastic energy stored in the body cannot depend on the order of application of the two loads F_1 and F_2, we write

$$U = U_1 + U_2 = U_1' + U_2' = U';$$
$$\text{i.e.,}\ \frac{1}{2}\,\alpha_{11}\,F_1^2 + \alpha_{22}\,F_2' + \alpha_{12}\,F_1F_2 = \frac{1}{2}\,\alpha_{22}\,F_2^2 + \alpha_{11}\,F_1' + \alpha_{21}\,F_1F_2,$$

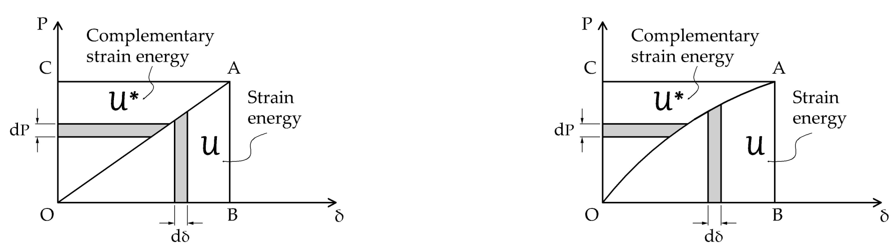
from which we conclude that $\alpha_{12} = \alpha_{21}$.

More generally, when we have several loads some of which may be moments, $\alpha_{ij} = \alpha_{ji}$. We emphasise that the displacements concerned are the work absorbing displacements. [We must be careful not to be trapped into concluding the following. The angular displacement (rotation) θ_1 at the point 1 when a moment M_2 is applied at the point 2 = the deflection at the point 2 when a force F_1 is applied at the point 1. This is completely wrong. Why? The angular displacement θ is not a work absorbing component displacement of the force!]

LOOKING AHEAD

So far we have seen some basic concepts. From here on the subject becomes abstract and difficult. We shall, therefore, just indicate the dual formulation, and then quickly pass on to a simpler treatment.

Dual Formulation



(a) Linear load-deflection diagram (b) Nonlinear load-deflection diagram

Figure 10.6: Strain energy, U and complementary energy, U^*. Load-deflection diagrams: linear [Fig. 10.6a] and nonlinear [Fig. 10.6b]. Note that $P\,d\delta = \delta U$, and that $\delta\,dP = \delta U^*$ for both the linear and nonlinear cases.

We shall discuss this dual formulation briefly. In energy methods this is very important. For example, corresponding to the principle of virtual work, there is its dual, the principle of complementary virtual work. There are more of such pairs. We shall see the simplest of them, namely, strain energy and complementary energy. The complementary quantities are shown with stars / asterisks (*).

A load-extension diagram is shown in Fig. 10.6. In general it is nonlinear even though the material is still elastic. The area under the curve is the strain energy, U. The area above the curve is called the complementary energy, U^*. For a linear material, $U = U^*$, because the two areas are equal. Still it is not quite appropriate to write $U = U^*$, even though the

two areas are numerically equal. The strain energy U is a function of the deflection d, while the complementary energy U^* is a function of the load, P. Let us also note that U^* is not an energy, even though it has the same dimension as an energy term.

Referring to the above load-extension diagram [Fig. 10.6] (which may be linear or nonlinear as we have shown here in the two diagrams), we note that

$$\delta U = P\,\delta d \qquad \longrightarrow \qquad \frac{dU}{dd} = P; \text{ and} \tag{10.21}$$

$$\delta U^* = d\,\delta P \qquad \longrightarrow \qquad \frac{dU^*}{dP} = d. \tag{10.22}$$

Note particularly that, to apply Eq. (10.21), U is to be expressed as a function of the deflection (extension) d, while, to apply Eq. (10.22), U^* is to be expressed as a function of the load P. It is not quite appropriate to write $U = U^*$, even in the linear case (when the load-deflection curve is a straight line passing through the origin), and even if the two areas are equal. This is because of the different functional relationships. These are (special, simple, cases of) Castigliano's first and second theorems.

Let us consider a relationship like $W = P \times d$. We can (i) keep P unchanged and vary d, or (ii) keep d unchanged and vary P, giving us

$$\delta W = P\,\delta d, \qquad \text{or} \qquad \delta W^* = d\,\delta P.$$

We shall call them the (variations of the) virtual work and the complementary work. Similarly, in the expression

$$U = \frac{1}{2}\iiint_{\mathcal{V}} (\sigma_{xx}e_{xx} + \sigma_{yy}e_{yy} + \sigma_{zz}e_{zz} + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})\,dx\,dy\,dz,$$

we can keep the stress components σ_{ij} unchanged, and apply a strain field variation as

$$\delta U = \frac{1}{2}\iiint_{\mathcal{V}} (\sigma_{xx}\,\delta e_{xx} + \sigma_{yy}\,\delta e_{yy} + \sigma_{zz}\,\delta e_{zz} + \tau_{xy}\,\delta\gamma_{xy} + \tau_{yz}\,\delta\gamma_{yz} + \tau_{zx}\,\delta\gamma_{zx})\,dx\,dy\,dz,$$

or, keep the strain components ϵ_{ij} unchanged, and apply a stress field variation as

$$\delta U^* = \frac{1}{2}\iiint_{\mathcal{V}} (\delta\sigma_{xx}\,e_{xx} + \delta\sigma_{yy}\,e_{yy} + \delta\sigma_{zz}\,e_{zz} + \delta\tau_{xy}\,\gamma_{xy} + \delta\tau_{yz}\,\gamma_{yz} + \delta\tau_{zx}\,\gamma_{zx})\,dx\,dy\,dz.$$

We can call them (variations of) the strain energy and the complementary energy[13].

[13] To understand these matters properly we need to learn the Calculus of Variations and Legendre transformations. We can see that the complementary energy U^* is obtained from the strain energy U by a Legendre transformation. Corresponding to the principles of virtual work and total potential energy, we can have the principles of complementary virtual work and total complementary energy. These are very powerful methods. Several approximate methods are based on these.

Ambitious readers are advised to see [9], where several good books are referred to.

Comments

In every problem, two conditions must be uncompromisingly satisfied: (a) the forces / stresses must satisfy the equations of equilibrium, and (b) the displacements / strains must satisfy the compatibility conditions. Correspondingly, we can have two approaches. (a) We can have several force distributions, all of them satisfying the equations of equilibrium, but among them only one can be correct (a consequence of the uniqueness theorem), the correct one corresponding to displacements that are compatible. Alternatively, (b) we can have different displacement distributions, all of them satisfying the compatibility conditions, but only one of them can be correct (again, a consequence of the uniqueness theorem), the correct one corresponding to forces / stresses that satisfy the equations of equilibrium. Energy theorems are of great help to us to choose the correct one directly without having to compute (a) the displacements, or (b) the forces / stresses. If we keep this in mind, we can have a better appreciation of the essential difference between the force methods and the displacement methods. Corresponding to these two approaches, we have the dummy displacement method and the dummy load method, and the two Castigliano's theorems.

SIMPLER ASPECTS OF ENERGY METHODS

We shall now abandon these advanced concepts and theorems and take up the simpler aspects of energy methods. Fortunately, many technically important problems can be solved by using these relatively simpler methods.

We shall see three important results: (i) Castigliano's[14] first theorem; (ii) Castigliano's second theorem; (iii) the theorem of minimum strain energy; and (iv) the theorem of minimum total potential[15].

Castigliano's First Theorem

If, in an elastic body in equilibrium under the influence of several (generalised) forces Q_i, the displacement field is changed (varied), the work done by these generalised forces during these generalised displacements δd_i is equal to the increase in the strain energy dU. The theorem states that

$$dU = \sum_{i=1}^{n} Q_i \, \delta d_i \qquad \longrightarrow \qquad \frac{\partial U(d_i)}{\partial d_i} = Q_i. \tag{10.23}$$

[14] Carlo Alberto Castigliano (Nov. 1847 - Oct. 1884) was an Italian engineer and mathematician.

[15] The terminology used in the literature is not consistent; different authors use different names for these theorems. Theorem of work, theorem of least work, etc. are commonly used, though often inconsistently. Some authors associate the names of Menebrea, Engesser and Crotti-Engesser with these theorems. There were also priority disputes between Castigliano and Menebrea.

To see the logical reasoning, we need to go the starting point, which is the principle of virtual work applied to a deformable body, and proceed step by step. We come to the principle of minimum total potential, and then to Castigliano's first theorem $\partial U(d)/\partial d_J = Q_j$, on the one hand, and on the other, using the dual formulations, start from the principle of complementary virtual work and come to the principle of minimum total complementary energy, and then to Castigliano's second theorem $\partial U^*(Q)/\partial Q_J = d_j$. Here all we can do is to be aware that different authors use different names, but to stick to our own terminology consistently.

The strain energy U is now to be expressed in terms of the displacements. [See Figs 10.6a, 10.6b, Eq. 10.21].

Castigliano's Second Theorem

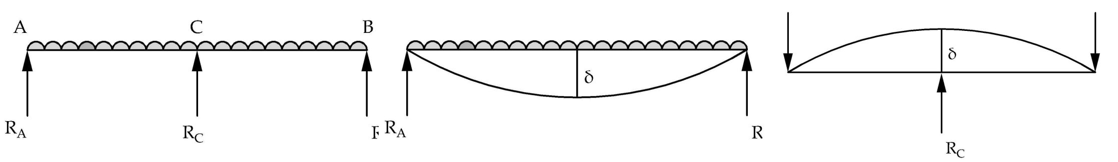
This is confusingly similar to the first theorem. But as we have explained the difference earlier, we shall merely state the theorem without much explanation.

If the strain energy stored U is expressed as a function of the (generalised) forces Q_i, then the work absorbing components of the corresponding displacements d_i are related by

$$d_i = \frac{\partial U(Q_i)}{\partial Q_i}.$$

To apply this theorem, the strain energy is to be expressed in terms of the (generalised) forces Q_i.

Minimum Strain Energy Theorem



(a) A continuous beam (b) The beam without R_C (c) Load R_C on the beam AB

Figure 10.7: A continuous beam with a uniformly distributed load. If we remove the support at C, we will have a simply supported beam as shown in Fig. 10.7b. R_C can be considered to be an applied load of the right magnitude as to make the total deflection at C to be zero.

This powerful theorem may be stated in a simple way as follows. Among all the stress / load distributions that satisfy the equations of equilibrium, but not necessarily the equations of compatibility, the true or compatible stress[16] / load distribution is that which makes the strain energy a minimum, the non-compatible stress / load distributions containing higher strain energy than the true or correct one.

Let us take a concrete example to explain this. Consider a continuous beam with a uniformly distributed load [Fig. 10.7a]. This is a statically indeterminate problem; the equations of equilibrium alone cannot give us the solution[17]. This means that the equations of equilibrium are to be supplemented by an equation of consistent deformations. How shall we obtain this?

Well, one way to obtain this is the following. Imagine that the support R_C is removed. Now the central deflection is $(5\,wl^4)/(384\,EI)$ [Fig. 2.13b]. The support reaction is such as

[16] By the term a compatible stress distribution is meant a stress distribution that leads to a compatible displacement field.

[17] If we draw a free-body diagram, we can see that this is a concurrent, coplanar system of forces. There are three unknowns, R_A, R_B, R_C, but only two separate (linearly independent) equations of equilibrium available. Sum of the forces in the horizontal direction is zero ($\sum F_x = 0$) is a correct, but useless equation!

to make the net deflection at $C = 0$. Thus,

$$\frac{5}{384}\,\frac{wl^4}{EI} = \delta = \frac{R_c l^3}{48\,EI} \qquad \longrightarrow \qquad R_C = 0.625\,wl.$$

Having obtained R_C it is easy to obtain the other reactions. Each of them turns to be $(1 - 0.625)/2 = 0.188\,wl$.

Let us now argue this way. Let us set $R_C = 0$. We can now calculate R_A, R_B and the total strain energy U_1. Now set $R_C = 1$. Calculate again R_A, R_B and the total strain energy U_2. Thus, we set different values for R_C and calculate the strain energy U in each case. If we compare them, all these load distributions satisfy the equations of equilibrium. If we calculate the net deflection at C, we will get the correct answer of $\delta = 0$ only in one case. That case, that case alone, is the correct one. That alone satisfies the condition of compatibility, or the condition of consistent deformation. Here are, thus, various load distributions. All of them satisfy the equations of equilibrium. Among them only one leads to the correct displacement of zero at the point C. According to the theorem, this correct load distribution has the minimum strain energy!

This, then, is a pointer. Let us call the central load as R_C and calculate the reactions and the strain all in terms of the (unknown) reaction R_C. Thus, we obtain an analytical expression for U in terms of R_C: $U = U(R_C)$. Now if we invoke the theorem, R_C is such as to make U a minimum, leading to $dU(R_C)/dR_C = 0$. The energy theorem has an in-built mechanism to make sure that the compatibility condition is satisfied.

[Let us have a slight digression. In the same way, we can conceive of different displacement distributions, all of them satisfying the compatibility condition. Among them, one and only one, is correct. Which one is that? Physically, the one that leads to the equilibrium requirement — the equation(s) of equilibrium must be satisfied. This corresponds to the minimum of the complementary energy! Corresponding to every result, there is a result in the dual representation also! We can see, understand, appreciate, and admire the beauty and the structure of the energy methods. Unfortunately, that calls for a higher level of sophistication. Some other time perhaps!]

Here is another interesting way of looking at the result. We used the third theorem here. We could equally well have looked at the problem, argued that the strain energy $U = U(R_C)$, and applied the Castigliano's second theorem which says that $dU(R_C)/dR_U = \delta_C = 0$ (the deflection at C in the work absorbing vertical direction is zero!) The equation that we use is the same, but our argument is different! In the first case, we minimise the expression for the strain energy, while in this second case, we calculate the expression for the deflection using Castigliano's second theorem and set it equal to zero!

A similar situation arises when we try to determine the reaction of a propped cantilever [Figs 12.6a, 12.6b, 12.6c]. If we call the (unknown) prop reaction as R, and if we desire to determine its value, we can write down the total bending moment, and hence the total strain energy U in terms of this unknown R as $U = U(R)$. We can write $dU(R)/dR = 0$. The equation is the same, but we can argue differently. One way is to understand that $U = U(R)$ reaches its minimum value for the correct value of R, using the minimum strain

energy theorem. The other way is to find the deflection at the end using Castigliano's second theorem. In either case, the equation that we obtain is the same. The case of a yielding support also is not difficult to handle.

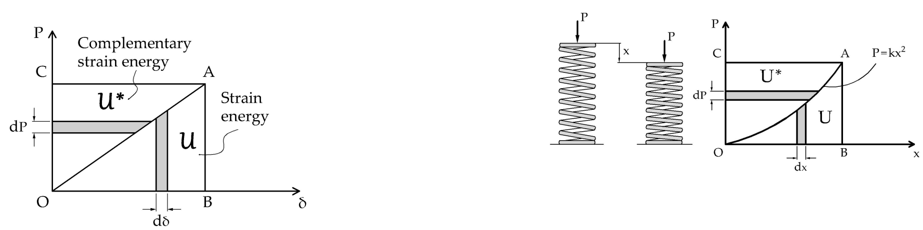
Minimum Total Potential Theorem

The strain energy and the potential of the external loads make up the total potential. The stress components are generated from the strain energy, U (which is the potential of the internal stresses), and the external loads from the potential, V of the external loads. In this way the total potential of the system, $\pi = U + V$ is introduced into the formulation. Then we obtain the principle of total potential ready and convenient for applications[18].

$$\pi = U + V \qquad \delta\pi = \delta U + \delta V = 0.$$

We may emphasise here that (i) the external forces must be statically compatible — this is obvious; otherwise the body will not be in static equilibrium — and (ii) the constitutive equation — elastic, not necessarily linear — is incorporated into this formulation.

A Simple Example of a Simple Linear Spring



(a) Linear load-deflection diagram (b) Nonlinear load-deflection diagram

Figure 10.8: Strain energy, U and complementary energy, U^*. Load-deflection diagrams: linear [Fig. 10.8a] and nonlinear [Fig. 10.8b]. Note that $P\,d\delta = \delta U$, and that $\delta\,dP = \delta U^*$ for both the linear and nonlinear cases. A spring also is shown alongside.

We shall consider a simple linear spring with a spring constant k [Fig. 10.8b — this is a linear spring for this example] so that the load P acting on it, and the resulting compression x are related by the relationship $P = kx$. We shall demonstrate how the theorems are used in this simple case, perhaps the simplest case that can be imagined.

[18] This follows from the principle of virtual work, which is valid for all materials — the constitutive equation never enters into the principle of virtual work — restricted to elastic materials.

We can also argue backwards and show, using the calculus of variations, that the principle of total potential is *both necessary and sufficient* for equilibrium. Actually Castigliano's first theorem follows from the principle of minimum total potential. As the dual formulation, we can define a complementary potential $\pi^* = U^* + V^*$ also, and we obtain the principle of minimum total complementary energy ($\delta\pi^* = 0$). From this follows Castigliano's second theorem.

The strain energy in the spring, $U = \frac{1}{2}\,kx^2$. The load P moves down by the amount of the deflection x. Thus, the potential of the external load, $\pi = -P\,\delta$. (Note carefully that (i) there is no factor 1/2, and that (ii) there is a negative sign.) To this system we shall apply the various energy theorems. We shall obtain the (expression for) the deflection (compression) of the spring, x.

Castigliano's first theorem

To apply this, the strain energy, U is to be expressed in terms of x. Thus, we have

$$U = \frac{1}{2}\,kx^2;$$
$$\frac{dU}{dx} = P; \qquad \longrightarrow \qquad kx = P; \qquad \longrightarrow x = \frac{P}{k}.$$

Castigliano's second theorem

To apply this, the strain energy, U is to be expressed in terms of P. Thus, we have

$$U = \frac{1}{2}\,kx^2 = \frac{1}{2}\,k\left(\frac{P}{k}\right)^2 = \frac{1}{2}\,\frac{P^2}{k};$$
$$\frac{dU}{dP} = x; \qquad \longrightarrow \qquad \frac{P}{k} = x; \qquad \longrightarrow \qquad x = \frac{P}{k}.$$

The expression for the strain energy U expressed as a function of x is $U = U(x) = kx^2$. [When this is expressed as a function of P (so as to be in good shape for differentiation w.r.to P), it is $U = P^2/(2k)$. Thus, $U = U(x) = U[x(P)] = U_1(P)$. We write U_1 instead of U here, because the functional form is different! As $U = U(x) = kx^2/2$, $U = U(P)$ stands for $U = kP^2/2$ which is clearly wrong!][19]

Minimum strain energy theorem

In the context of this problem, this theorem does not give us any fresh insight. Thus, the application of this theorem is not discussed here.

Minimum total potential theorem

The total potential, π of the system is the sum of (i) the internal potential which is the strain energy, U and (ii) potential of the external load, V. They are

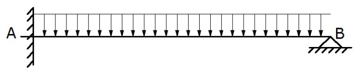
$$U = \frac{1}{2}\,kx^2; \qquad V = -Px; \qquad \pi = U + V = -Px + \frac{1}{2}\,kx^2.$$

$$\frac{d\pi}{dx} = 0 \qquad \longrightarrow \qquad -P + kx = 0; \qquad \longrightarrow \qquad x = \frac{P}{k}.$$

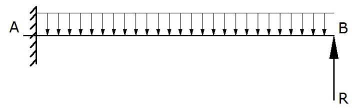
Castigliano's Second Theorem: An Example

The figure [Fig. 10.9a] shows a uniform (EI constant) cantilever of length l with a uniformly distributed load w/unit length. It is supported by a rigid prop. We desire to calculate the reaction R at B. We shall use Castigliano's second theorem to solve this problem.

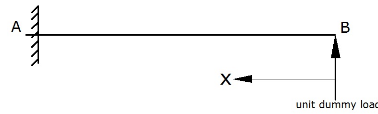
[19] Even some learned authors do not distinguish between the two different functional forms. Granted that they have no confusion. When they use it, we hesitate to call it a mistake. It is only a carelessly chosen notation. But we unsuspecting readers may be misled.



(a) A propped cantilever loaded by a u.d.l.



(b) Support removed and replaced by a reaction R



(c) A concentrated load R at the end

Figure 10.9: A propped cantilever AB with a uniformly distributed load, w per unit length propped by a rigid support. We shall use Castigliano's second theorem to calculate the support reaction, R.

To apply Castigliano's second theorem, we need to calculate the strain energy U expressed as a function of the (unknown) reaction R. The strain energy U in the cantilever is given by

$$\begin{aligned} U &= \int_0^l \frac{M^2\,dx}{2\,EI} = \int_0^l \frac{[Rx - \frac{w\,x^2}{2}]\,dx}{2\,EI}; \\ &= \frac{1}{E}\left[\frac{R^2 l^3}{6} + \frac{w^2 l^5}{40} - \frac{Rwl^4}{8}\right]. \end{aligned}$$

As the deflection at the rigid support $B = 0$, we obtain on applying Castigliano's second theorem,

$$\text{deflection at } B \ = 0 = \frac{dU(R)}{dR} = \frac{Rl^3}{6\,EI} - \frac{wl^4}{8\,EI} \quad \longrightarrow \quad R = \frac{3}{8}wl.$$

This completes the solution.

Among these four theorems stated here, the last one is the most general and powerful one. The other fact is that these four results — or, theorems if we like to call them so — are not independent. To see, understand and appreciate these facts, we have to go to the fundamentals, which is difficult with the limitations that we have. The power of the theorem can be demonstrated only when it is applied to more important and more difficult problems. We regret our inability to pursue these matters further.

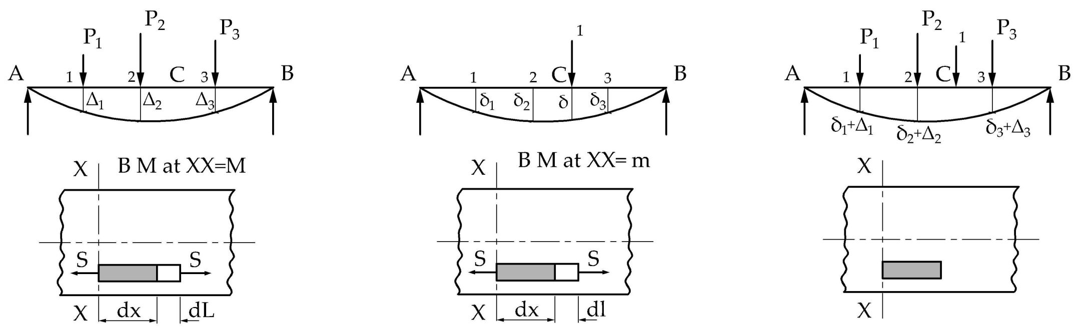
UNIT-LOAD METHOD

The unit-load method is one of the useful energy methods by which the deflection and the slope of a bent beam can be calculated relatively easily. (This method is applicable for other structures also.) We shall first derive the basic working formula, and then illustrate the method by a few illustrative examples in a later chapter.

We shall take up the case of a simply supported beam to explain this method. Shown in Figs 10.10a, 10.10b and 10.10c is a beam simply supported at A and B. We shall consider three sets of loading and calculate the corresponding (i) work done by the external loads, and (ii) the internal energy (strain energy).

Case (a) Loads P_1, P_2, P_3 only

Three vertical loads are gradually applied on this beam at the points 1, 2, 3 respectively.



(a) Three loads (b) A unit load at C (c) Four $(3+1)$ loads

Figure 10.10: A simply supported beam with three sets of loading is shown: (a) three actual loads; (b) a unit load at C; and (c) three actual loads and the unit load.

The corresponding deflections — the work absorbing deflections — are Δ_1, Δ_2 and Δ_3 [Fig. 10.10a]. The work done by these external forces on the beam are

$$\frac{1}{2}\sum_{i=1}^{3} P_i\,\Delta_i \equiv \frac{1}{2}\,P_1\,\Delta_1 + \frac{1}{2}\,P_2\,\Delta_2 + \frac{1}{2}\,P_3\,\Delta_3. \tag{10.24}$$

What happens inside the beam? Well, stresses are developed inside the beam. There are strain energies associated with these stresses. These are of the form $1/2\sum S\,dL$ where S is the axial force and dL is the associated elongation / contraction. As the system is conservative, the work done by the externally applied forces will be, must be, equal to the internal energy (which is the same as the strain energy) stored in the beam. Thus, we have

$$\frac{1}{2}\,P_1\,\Delta_1 + \frac{1}{2}\,P_2\,\Delta_2 + \frac{1}{2}\,P_3\,\Delta_3 = \frac{1}{2}\sum S\,dL. \tag{10.25}$$

Case (b) Unit load only

If, in the beginning, only a unit load 1 is applied gradually and vertically at the point C, what happens now? This unit load produces a vertical deflection at C, and δ_1, δ_2, δ_3 at the points 1, 2, 3, respectively [Fig. 10.10b]. Again, as before, the work done by the externally applied unit load is $1/2 \times (1) \times dl$. The corresponding internal energy (strain energy) is now $1/2\sum s\,dl$. Conservation of energy demands that

$$\frac{1}{2} \times (1) \times \delta = \frac{1}{2}\sum s\,dl. \tag{10.26}$$

Case (c) The unit load first, and then the three loads P_1, P_2, P_3

Let us consider the case (b) (only the unit load applied), and additionally apply the external loads P_1, P_2, P_3 at the points 1, 2, 3, respectively [Fig. 10.10c]. The total deflections now are:

at 1: $(\delta_1 + \Delta_1)$ at 2: $(\delta_2 + \Delta_2)$ at 3: $(\delta_3 + \Delta_3)$ at C: $(\delta + \Delta)$.

The total work done by the external forces including the unit load is

$$\left(\frac{1}{2}\times\delta\right)+\left(\frac{1}{2}\times P_1\times\Delta_1\right)+\left(\frac{1}{2}\times P_2\times\Delta_2\right)+\left(\frac{1}{2}\times P_3\times\Delta_3\right)+(1\times\Delta)\,. \qquad (10.27)$$

Note carefully that there is no factor 1/2 in the last term. This is because the full unit load was acting before P_1, P_2, P_3 were applied. thus, the unit load performs work equal to (i) $(1/2)\times 1\times\delta$, and (ii) $(1\times\Delta)$ when the loads P_1, P_2, P_3 are applied. The total internal energy (strain energy) is

$$\frac{1}{2}\sum s\,dl+\frac{1}{2}\sum S\,dL+\sum s\,dL.$$

Note again that there is no factor 1/2 in the last term for the same reason as explained above. Equating the total external work done to the total internal energy, we obtain

$$\begin{aligned}\left(\frac{1}{2}\times\delta\right)+\left(\frac{1}{2}\times P_1\times\Delta_1\right)+\left(\frac{1}{2}\times P_2\times\Delta_2\right)+\left(\frac{1}{2}\times P_3\times\Delta_3\right)+(1\times\Delta)&\\ =\frac{1}{2}\sum s\,dl+\frac{1}{2}\sum S\,dL+\sum s\,dL.&\end{aligned} \qquad (10.28)$$

[The alert attentive reader would have noted that the small letters such as δ, $\delta_1,\delta_2,\cdots$ and u, dl correspond to the unit load, and that the capital letters such as Δ_1, Δ_2, $\cdots$ and S, dL correspond to the externally applied loads P_1, P_2, P_3.]

From Eqs (10.25), (10.27), and (10.28) we obtain [by subtracting the sum of Eqs (10.25) and (10.27) from Eq. (10.28)],

$$1\times\Delta=\sum s\,dL. \qquad (10.29)$$

S and s are the axial forces on an element; that is, $S=\sigma\,dA$. We shall now apply this equation (10.29) to the problem of determining the deflection of beams.

To Find the Deflection of Beams

Let us apply this unit-load method to find the deflection of beams. To be specific, we consider the problem of finding the vertical deflection at the point C when three given loads P_1, P_2, P_3 act on a simply supported beam [Fig. 10.10a]. Apply a unit load at C in the vertical direction (that is, in the direction of the required deflection) [Fig. 10.10b]. We begin with Eq. (10.29), and process in further to bring it to a ready-to-use form.

Let M and m be the bending moments at a typical cross-section XX for (i) the actual loading [Fig. 10.10a], and when (ii) the unit load alone is applied, respectively. An element MN of original length dx is shown on the cross-section (length exaggerated). The deformed lengths (shown elongated; they are in the tension zone) are $dx+dL$ and $dx+dl$ respectively for the cases (i) [Fig. 10.10a] and (ii) [Fig. 10.10b]. Consider Eq. (10.29) and work out the expressions for s and dL as

$$s=\sigma\,dA=\frac{my}{I}\,dA,\ \text{and}$$

$$\begin{aligned} dL &= \text{strain} \times dx = \frac{\text{stress } \times dx}{E} = \frac{\text{force}}{\text{area}} \times \frac{dx}{E} \\ &= \frac{S}{dA} \times \frac{dx}{E} = \left(\frac{M\,y}{I}\, dA \right) \times \frac{dx}{E\,dA} = \frac{M\,m}{EI}\, dx \end{aligned}$$

Substituting these expressions for s and dL in Eq. (10.29), we obtain

$$\begin{aligned} \Delta &= \sum s\, dL = \sum \left(\frac{m\,y}{I}\, dA \right) \left(\frac{M\,y}{EI}\, dx \right) = \int_0^l \int_A \frac{M m\, y^2}{EI^2}\, dA\, dx \\ &= \int_0^l \frac{Mm}{EI^2}\, dx \int_A y^2\, dA = \int_0^l \frac{Mm}{EI}\, dx. \end{aligned}$$

Thus, we arrive at the working formula for all such deflection calculations as

$$\Delta = \int_0^l \frac{Mm}{EI}\, dx \qquad \text{(unit-load applied).} \tag{10.30}$$

The procedure and the formula remain unchanged when we desire to obtain the slope (rotation) also. Instead of the unit load, we now apply a unit moment at the desired point in the desired direction. (*The work absorbing displacement* is the mantra!)

$$\theta = \int_0^l \frac{Mm}{EI}\, dx \qquad \text{(unit moment applied).} \tag{10.31}$$

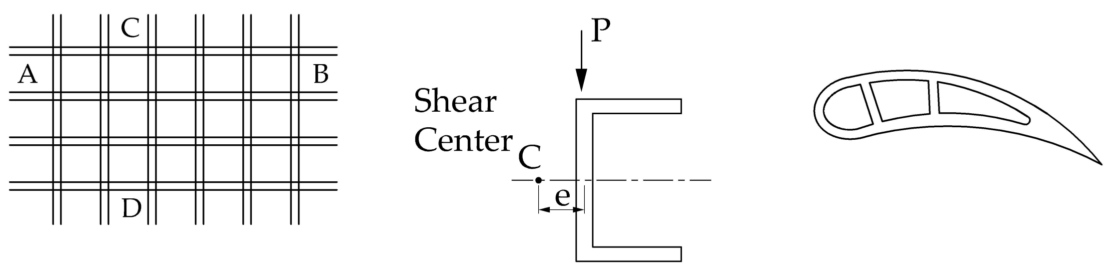
See a later chapter for a few worked out illustrative examples.

In the next chapter, we shall discuss the topic of torsion of non-circular bars.

Chapter 11

TORSION OF NON-CIRCULAR BARS

Torsion of bars is a topic of great relevance to engineers. Torsion of circular prismatic bars (shafts) is the most important topic, but that of non-circular bars also is of much interest to engineers. We shall discuss the simpler aspects of this theory in this chapter.



(a) A grid of horizontal beams (b) A section loaded by P (c) The wing of an aircraft

Figure 11.1: Shown in the figure are three examples of non-circular cross-sections subjected to twisting moments. A slope in the beam AB [Fig. 11.1a] appears as a twist in the beam CD, and vice versa. In Fig. 11.1c, if the load P does not pass through the shear centre C, the channel section will twist. Fig. 11.1c shows the cross-section of an aircraft wing which is multi-cellular subjected to a twisting moment.

Non-circular prismatic bars are not used for transmitting power[1]. But torsion is called into play not only in shafts transmitting power. In a grid of integrally cast (horizontal) beams [Fig. 11.1a], the slope of the beam AB manifests as a twist of a (horizontal) beam like CD. Further, when a verticl load acts on a structural member, if the load does not pass

[1] When a keyway is provided, the cross-section of a circular shaft becomes non-circular. It is of interest to study the effect of a keyway in a circular shaft. The stresses will become very large at the two re-entrant corners, and the torsional rigidity is slightly reduced.

through the shear centre C [Fig. 11.1b], bending of the beam is accompanied by twisting (the twisting moment being equal to $P\,e$ clockwise). Torsion of wings of an aircraft [Fig. 11.1c] as in torsional vibrations is another example when the cross-section is non-circular and multi-cellular. Again, in the case of a balcony beam, the cross- sections are subjected not only to bending moments, but also to twisting moments. The cross-sections in these examples are non-circular. Thus, these examples should suffice to convince ourselves that there is a strong case to learn this important topic of non-circular prismatic bars.

TORSION OF BARS

In general, we have the following cases to consider.

(i) Circular prismatic bars (The cross-section is the same everywhere.)

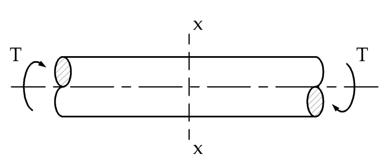
(ii) Non-circular prismatic bars (The cross-section is the same everywhere.)

(iii) Circular non-prismatic bars (For example, conical shafts)

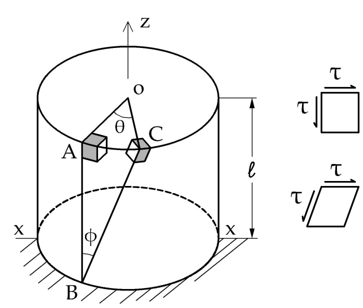
(iv) Non-circular non-prismatic bars

The first is the easiest; Coulomb's theory deals with this case. Analytical solutions are available for the second and the third also. No analytical solution is available for the last case. We shall take up the second case here which the topic of this chapter. However, it is desirable to have a quick review of the Coulomb's theory.

A QUICK REVIEW OF COULOMB'S THEORY



(a) A circular prismatic bar subjected to twisting moments $T - T$



(b) To explain Coulomb's theory

Figure 11.2: A circular prismatic bar in torsion. Fig. 11.2b shows the details of the deformation. A generator like BA deforms to the position BC when the cross-section rotates through the angle θ. This introduces a shear strain ϕ as marked.

Fig. 11.2a shows a circular shaft in torsion.The cross-sections on the left rotate (twist) in one direction, while those on the right rotate (twist) in the other direction. Somewhere in the middle there is a cross-section XX that does not rotate. Let us hold on to this

cross-section and consider it a fixed[2]. A generator BA, on deformation, becomes BC. The straight radial line OA, on deformation, becomes a straight radial line OC. θ is the angle of twist. Small cubes at A and C are shown enlarged.

The Coulomb's theory, we recall, is developed on the basis of two facts, often stated as assumptions[3]: (i) cross-sections do not warp; and (ii) straight radial lines such as OA remain, on deformation, as straight radial lines such as OC; they do not become curved.

From Fig. 11.2b we note that the arc length $AC = R\,\theta = l\,\phi \quad \longrightarrow \quad \phi = (R\,\theta)/l$, and that $DE = r\,\theta = l\,\phi \quad \longrightarrow \quad \phi = (r\,\theta)/l$, where ϕ this time refers to shear strain (corresponding to a general radius r).

This equation $\phi = (r\,\theta)/l$ relates the displacement (angular displacement θ) and the shearing strain ϕ. Using now the constitutive equation, we can obtain the expression for the shear strain τ at any radius on the cross-section — this is really $\tau_{z\theta}$) as

$$\tau = G\,\phi = \frac{G\,r\,\theta}{l} \qquad \longrightarrow \qquad \frac{\tau}{r} = \frac{G\,\theta}{l}. \tag{11.1}$$

Here we have used the constitutive equation. Now we can consider equilibrium. The condition of equilibrium demands that the applied twisting moment T is related to the resisting shear stress τ by the equation

$$(\tau) \times (2\,\pi\,r\,dr) \times (r) = dT \quad \longrightarrow \quad T = \int_A \tau\,r\,dA = \int_A \frac{\tau}{r} r^2\,dA \tag{11.2}$$

$$= \int_A \frac{G\,\theta}{l}\,r^2\,dA = \frac{G\,\theta}{l}\int_A r^2\,dA = \frac{G\,\theta}{l}J. \tag{11.3}$$

Combining Eqs (11.1) and (11.3, we obtain the fundamental equation governing torsion of circular bars (shafts) as[4]

$$\frac{\tau}{r} = \frac{G\,\theta}{l} = \frac{T}{J}, \tag{11.4}$$

where J is the polar second moment of area (the polar area moment of inertia)$=\pi\,d^4/32$. Note in particular that the points farthest from the axis Oz are stressed the most. We emphasise that τ refers to the (torsional) shear stress on the cross-section ($\tau_{z\theta}$).

Argument Based on Circular Symmetry

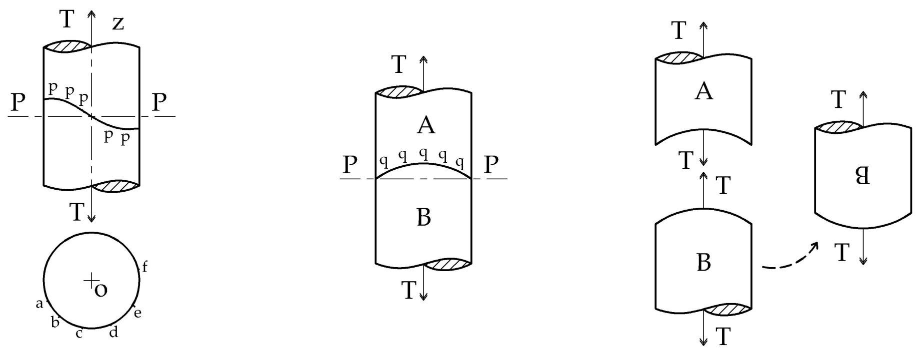
We had stated that, for the case of torsion of circular (prismatic) shafts, (i) cross-sections do not warp, and that (ii) straight radial lines remain straight radial even after deformation.

---

[2] Fixed in this limited sense. This is the cross-section marked XX in Fig. 11.2b. Actually when the shaft rotates — rigid body rotation — *every* cross-section, including this 'fixed' cross-section XX, rotates. Cross-section near XX rotate — we do not mean rigid body rotation — only a little bit; the farther the cross-section is from XX, the more it rotates. We repeat for emphasis: we are not talking about the rigid body rotation of the shaft as a whole.

[3] Actually these are facts and not assumptions. Using an argument based on axisymmetry (rotational symmetry), we can see the truth of these statements. See below [Fig. 13.7] and Den Hartog [3] for details.

[4] This is analogous to the Euler-Bernoulli equation governing the bending of beams. The consequences are also analogous: the material far from the axis is more effectively utilised. For this reason, hollow shafts are much better. (In some cases, the gain may not be worth the extra effort and cost of having a hollow cross-section.)

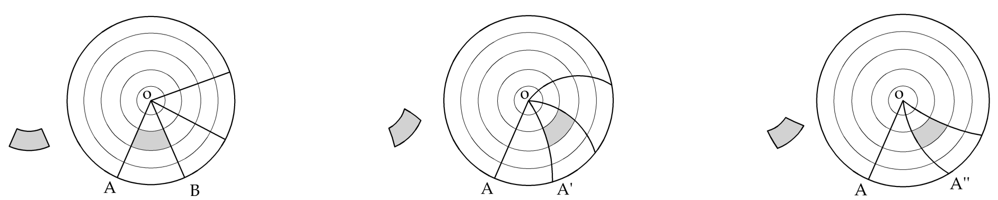


(a) Such warping is impossible. Why?
(b) Such axisymmetric warping also is impossible.
(c) Compare the two identical parts A and B identically loaded.

Figure 11.3: Such warping as in Fig. 11.3a is clearly impossible. Even the axisymmetric warping as shown in Fig. 11.3b is impossible. Why? In Fig. 11.3c are shown two identical parts identically loaded. It follows that the deformations must be identical. This argument shows us that cross-sections cannot warp in such axisymmetric cases.

Let us examine these facts in the light of the axisymmetry (circular symmetry) that this problem enjoys.

Fig. 11.3a shows a circular (prismatic) shaft subjected to $T-T$ represented by double-headed arrows. A typical cross-section like PP cannot possibly warp and become pppppp. Why? Because there is axisymmetry, all the points like a, b, c, $d, \cdots$ can deform only identically. If a moves up, e cannot move down. However, it may appear that the cross-section can warp axisymmetrically and become like $PqqqqqP$ which surface is still axisymmetric [Fig. 11.3b]. If we imagine that the cylindrical shaft is cut into two parts A and B, and take the lower part B, rotate it and keep it by the side of part A [Fig. 11.3c], we find an anomaly. Here are two identical parts A and B, identically loaded. Can they deform differently? This is impossible. Hence we conclude that cross-sections cannot warp!



(a) Straight radial lines like OA remain straight like OB.
(b) Straight radial lines like OA cannot curve in like OA'.
(c) Straight radial lines like OA cannot curve out like OA''.

Figure 11.4: Straight radial lines like OA on the cross-section remain straight even after deformation; they do not curve in like OA' [Fig. 11.4b] or curve out like OA'' [Fig. 11.4c].

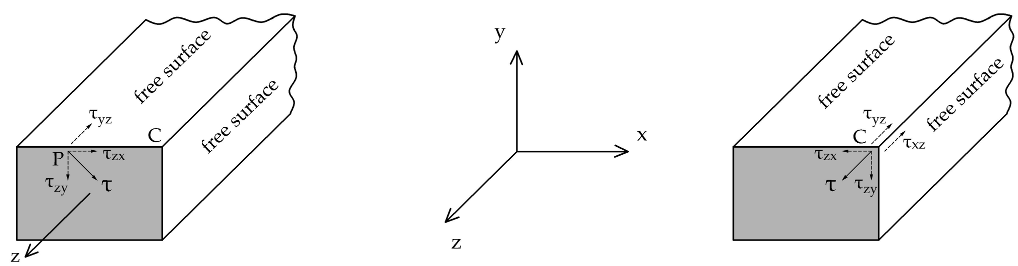
A similar argument leads us to conclude that straight radial lines like OA [Fig. 11.4a] can only rotate to become another straight radial line like OB, but cannot curve in like OA'

[Fig. 11.4b][5], or curve out like OA'' [Fig. 11.4c]. We can, thus, infer that cross-sections do not distort in their own planes. It means that there is no shear stress τ_{xy} producing a shear strain γ_{xy}; that is, the right angle in the shaded small block in Fig. 11.4a will be preserved.

Torsion of Non-circular Prismatic Bars

Some of the early investigators tried to develop this theory based on the assumption that cross-sections do not warp trying to imitate Coulomb's theory, but they ran into difficulties. We now know that this is only natural. For the present case of non-circular prismatic bars, cross-sections are required to warp; they will, they must, warp[6].

Cross-sections Will Have to Warp!



(a) Shear stress components on the cross-section at a boundary point P

(b) The coordinate system xyz used

(c) Shear stress components on the cross-section at a corner point C

Figure 11.5: Our objective is to show that cross-sections will warp. As a first step, we show that the resultant shearing stress on the cross-section at a boundary point like P is tangential to the boundary. We are referring to the shear stress component τ_{zx}. Fig. 11.5c is to show that C is a dead corner with no shear stress like τ_{zx} or τ_{zy}.

We shall show this in three stages taking the example of a rectangular cross-section.

Step (1)

First we note that the resultant shearing stress on the cross-section at a boundary point must be entirely tangential to the boundary. P is a boundary point on the cross-section [Fig. 11.5a]. The cross-section is the $z-$plane [Fig. 11.5b]. If possible, let the shear stress τ on the $z-$plane (cross-section) be inclined. Then it will have two components, the tangential component τ_{zx} and the normal component τ_{zy}. The normal component, if it exists, will be

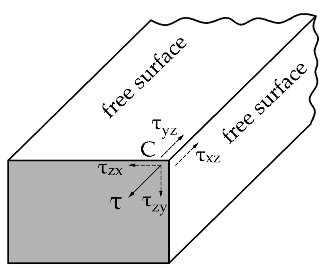
[5] The straight radial line OA cannot curve in like OA' (inswinger), or curve out like $OA'^{\prime}$ (outswinger), to use the popular cricket terminology!

[6] Straight radial lines will still remain straight radial lines. There is no difference here. We say that cross-sections do not distort in their own planes. In other words, there is no shearing strain like $\tau_{r\theta}$ or τ_{xy} where the $z - axis$ is along the axis of the prismatic bar. But we can make this statement only after the problem is solved. (In the case of a circular non-prismatic bar like a conical shaft, cross-sections do not warp — they cannot; remember the argument based on axisymmetry — but straight radial lines will become curved. In other words, cross-sections do distort in their own planes. This problem can be solved analytically, but we do not do this here. Den Hartog [3] gives a delightful treatment of this case including the so-called razor blade analogy.

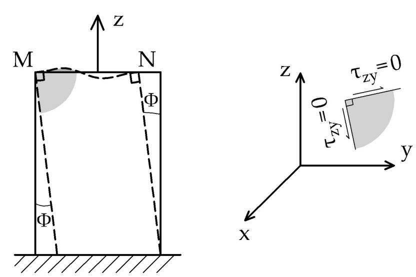
accompanied by its complement τ_{yz} ('shear and complementary shear') acting on the $y-$plane. But the $y-$plane is a free surface. On a free surface, there cannot be any shear stress[7]. Thus, τ_{yz} does not exist and, consequently, τ_{zy} cannot exist either. Thus, we conclude that the resultant shearing stress on the cross-section at a boundary point will be, must be, tangential!

Step (2)
With this understanding, let us consider a corner point, say, C on the cross-section. If there is a shear stress on the cross-section (on the $z-$plane) at C, it will have the components τ_{zx} and / or τ_{zy}. These, if they exist, will be accompanied by their complementary shear shear stresses τ_{xz} and / or τ_{yz} which cannot exist because the $x-$plane and the $y-$plane are free surfaces. Thus, we conclude that there cannot be a shear stress on the cross-section at a corner point!

Let us not fail to notice this important conclusion. Coulomb's theory seems to indicate that the shear stress on the cross-section is a maximum at the farthest point. But no, it is not a maximum; it is, in fact, zero! Thus, we have the eleventh commandment: "Thou shalt not apply Coulomb's theory to non-circular cross-sections!" It is not a minor mistake, nor is it an approximation, but a major seriously wrong mistake to apply Coulomb's theory to non-circular cross-sections!



(a) Shear stress components on the cross-section at a corner point C



(b) The shear strain at M and N is zero. Cross-sections must necessarily warp!

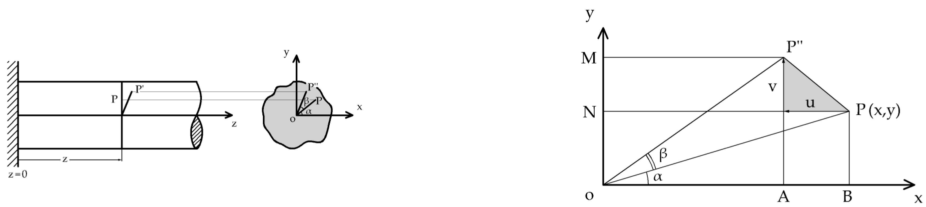
Figure 11.6: To show that cross-sections must necessarily warp: careful examination of the deformation leads us to conclude that the cross-sections have no choice except to warp!

Step (3)
We have seen that the corners are free of shear stresses. That is, $\tau_{zy} = \tau_{yz} = 0$ at the corner M [Fig. 11.6a]. Now if the shear stress on the cross-section at $M = 0$, the corresponding shear strain also is zero. The twisting introduces an angle ϕ as shown in Fig. 11.6b. If the shear strain is zero, the angle of 90° will have to be preserved. Thus, the cross-section will have to tilt up by an angle ϕ. A similar situation arises at the corner N too. The angle at N will have to be preserved as 90°, making it necessary for the surface to tile down by the angle ϕ. Thus the cross-section has no choice except to warp!

[7] unless somebody is rubbing vigorously on the free surface as in massaging!

SAINT-VENANT'S THEORY

It was Barre de Saint-Venant who solved the problem correctly in 1855. He used a semi-inverse method[8]. That is, after examining the deformation, he makes a guess, an intelligent guess, leaving a function as unspecified. We shall see this method below. The prismatic



(a) A cross-section at z from the 'fixed' cross-section (b) The u and v displacements of P

Figure 11.7: To make 'an intelligent guess' about the displacements u, v, w

bar is 'fixed'[9] at the cross-section[10] $z = 0$. A typical point P moves to the point P'. Because the cross-section warps, the point P' moves out of the plane. This is the warping displacement w which is a function of x and y, but not of z: $w = w(x, y)$. The fact that w does not depend on z means that all cross-sections warp identically. The projection of P' on the cross-section is the point P'' [Fig. 11.7b]. The angle through which the cross-section rotates is β which is very small; cross-sections twist only by a small, very small, angle. This is assumed to be proportional to z. That is, the farther the cross-section is from the 'fixed' cross-section, the larger is the angle of twist: $\beta = \theta\, z$, where θ is the angle of twist per unit length. The angle α may, or may not, be small. (It is small for points like P near the x axis, but is not small for points farther from the x axis.) We do such an exercise analysis to make 'an intelligent guess' about the displacements. If the cross-sections rotate anti-clockwise, the u displacement is negative (i.e., to the left), while the v displacement is positive (i.e., upwards).

Let us calculate the expressions for the displacements u and v.

$$\begin{aligned}\text{Magnitude of } u &= OB - OA = OP\, \cos\alpha - OP''\, \cos(\alpha+\beta)\\ &= x - OP(\cos\alpha\, \cos\beta - \sin\alpha\, \sin\beta)\\ &= x - OP\, \cos\alpha + (OP\, \sin\alpha)\beta = \theta\, yz.\end{aligned}$$

$$\text{Magnitude of } v = P''A - PB = OP''\, \sin(\alpha+\beta) - OP\, \sin\alpha$$

[8] The inverse method is to begin with an assumed solution and later to find out what problem it solves. Here Saint-Venant employs this method, but with some freedom in the form of an unspecified function, so that he can choose this function later in order that the equations of equilibrium are satisfied.

[9] Recall what fixed means. See Fig. 11.2a and its explanation to understand what is meant by a fixed cross-section.

[10] Or rather, z is measured from this 'fixed' cross-section.

$$= OP(\sin\alpha\,\cos\beta + \cos\alpha\,\sin\beta) - OP\,\sin\alpha$$
$$= OP\,\sin\alpha + (OP\,\cos\alpha)\beta - OP\,\sin\alpha = \theta\,yz.$$

(Note that OP'' is almost equal to OP, because the warping displacement w is very small. The angle of twist β per unit length is very small; hence $\sin\beta \approx \beta$ and $\cos\beta \approx 1$.)

These expressions for the displacements do not seem to be unreasonable. Following Saint-Venant, let us assume the displacements as:

$$u = -\theta\,yz \tag{11.5a}$$
$$v = \theta\,xz \tag{11.5b}$$
$$w = \theta\,\phi(x,y) \tag{11.5c}$$

Saint-Venant at this stage leaves $\phi = \phi(x,y)$ as unspecified. He still retains the freedom to choose this function ϕ so that the equations of equilibrium are satisfied. Hence this method is known as a semi-inverse method. The assumption behind this — ϕ being independent of z — is that all cross-sections warp identically.

First Formulation

Let us begin by assuming the displacements as

$$u = -\theta\,yz; \qquad v = \theta\,xz; \qquad w = \theta\,\phi(x,y).$$

The strains, obtained from the strain-displacement relations $e_{ij} = (1/2)[u_{i,j} + u_{j,i}]$, are

$$e_{xx} = e_{yy} = e_{zz} = 0;\ \gamma_{xy} = 0;$$
$$\gamma_{zx} = \gamma_{xz} = \theta\left(\frac{\partial\phi}{\partial x} - y\right); \quad \gamma_{zy} = \gamma_{yz} = \theta\left(\frac{\partial\phi}{\partial y} + x\right).$$

The stress components are computed from the constitutive equations as

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0;\ \tau_{xy} = 0;$$
$$\tau_{zx} = \tau_{xz} = G\theta\left(\frac{\partial\phi}{\partial x} - y\right); \quad \tau_{zy} = \tau_{yz} = G\theta\left(\frac{\partial\phi}{\partial y} + x\right).$$

Substituting these expressions in the equations of equilibrium $\sigma_{ij,l} = 0$, we find that

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} = 0 \qquad \text{(satisfied)};$$
$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} = 0 \qquad \text{(satisfied)};$$
$$\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} = 0 \quad \longrightarrow \quad \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \text{ in } \mathcal{R}.$$

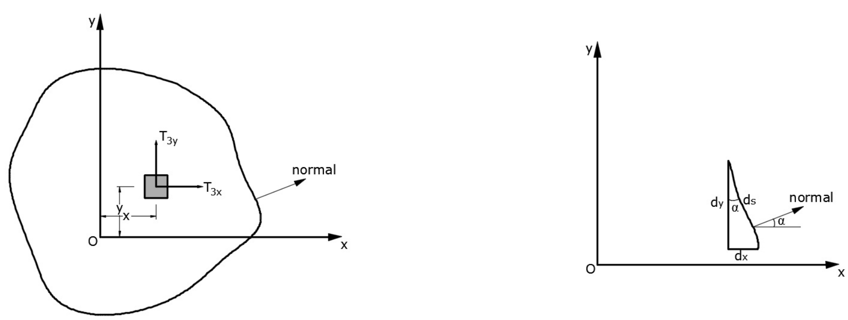
Thus, the governing differential equation and the associated boundary condition are

$$\text{Laplace's equation:} \quad \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \text{ in } \mathcal{R}, \tag{11.6}$$

$$l\left(\frac{\partial\phi}{\partial x} - y\right) + m\left(\frac{\partial\phi}{\partial y} + x\right) = 0 \text{ on the boundary } \mathcal{C}. \tag{11.7}$$

Saint Venant thus succeeded in converting the torsion problem to a boundary value problem (a problem in partial differential equations to solve). Once the function ϕ is obtained, the torsion problem is as good as solved, because the stresses, the strains and the displacements are known. The twisting moment T and, thus, the torsional rigidity GJ, are also easily calculated; they are only one integration away as given below [Fig. 11.8a].

$$\begin{aligned}
\text{torque: } T &= \iint_{\mathcal{R}} (x\,\tau_{zy} - y\,\tau_{zx})\,dx\,dy; \qquad \left(\text{torsional rigidity: } GJ = \frac{T}{\theta}\right). \\
&= G\theta \iint_{\mathcal{R}} \left(x\,\frac{\partial\phi}{\partial y} - y\,\frac{\partial\phi}{\partial x} + x^2 + y^2\right)\,dx\,dy; \\
J &= \iint_{\mathcal{R}} \left(x\,\frac{\partial\phi}{\partial y} - y\,\frac{\partial\phi}{\partial x} + x^2 + y^2\right)\,dx\,dy.
\end{aligned}$$



(a) The region $\mathcal{R}$ (b) ds, normal and l, m

Figure 11.8: The region $\mathcal{R}$; a small part of the arc length (elemental arc length ds, normal and the direction cosines l, m. Let us note that $l = \cos(n,x) = \partial y/\partial s$; $m = \cos(n,y) = -\partial x/\partial s$. The shear stresses τ_{zx} and τ_{zy} are marked on an elemental area $dx\,dy$; we can then calculate the torque and thereby the torsional stiffness GJ.

Second Formulation

In the first formulation, the governing differential equation is the well known Laplace's equation (11.6); that creates no problem. But the associated boundary condition (11.7) is not easy to satisfy. Thus, we can have another, a second, formulation in terms of an auxiliary function $\psi = \psi(x,y)$, so that the boundary condition is easier to satisfy.

Towards this objective, let us change over from the function ϕ to ψ such that $\phi + i\,\psi$ is an analytic function of the complex variable $x + i\,y$. These two functions are related by the Cauchy-Riemann equations

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}; \qquad \text{and} \qquad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}. \tag{11.8}$$

Replacing ϕ by ψ as given above by Eqs (11.8), we obtain the governing differential equation

$$\text{Laplace's equation:} \quad \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \text{ in } \mathcal{R}. \tag{11.9}$$

[We can write down the above equation without any calculation, as we know that the real and imaginary parts of an analytic function separately satisfy the Laplace's equation.]

We shall see in a minute that the boundary condition in terms of ψ will be much simpler and easier to deal with. In Eq. (11.7), let us make the following replacements.

$$l = \cos(n, x) = \frac{\partial y}{\partial s}; \qquad m = \cos(n, y) = -\frac{\partial x}{\partial s};$$
$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}; \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

With these replacements, the boundary condition (11.7) becomes

$$\frac{\partial y}{\partial s}\left(\frac{\partial \psi}{\partial y} - y\right) - \frac{\partial x}{\partial s}\left(-\frac{\partial \psi}{\partial x} + x\right) = 0 \text{ on } \mathcal{C};$$
$$\text{i.e.,} \quad \frac{\partial \psi}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial \psi}{\partial y}\frac{\partial y}{\partial s} - \left(x\,\frac{\partial x}{\partial s} + y\,\frac{\partial y}{\partial s}\right) = 0 \text{ on } \mathcal{C};$$
$$\text{i.e.,} \quad \frac{\partial \psi}{\partial s} = x\,\frac{\partial x}{\partial s} + y\,\frac{\partial y}{\partial s} = \frac{1}{2}\,\frac{\partial}{\partial s}(x^2 + y^2) \text{ on } \mathcal{C};$$
$$\text{i.e.,} \quad \psi = \frac{1}{2}\,(x^2 + y^2) + \mathit{constant} \text{ on } \mathcal{C}.$$

Thus, the governing differential equation in this formulation is

$$\text{Laplace's equation:} \quad \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \text{ in } \mathcal{R}, \tag{11.10}$$

and the associated boundary condition is

$$\psi = \frac{1}{2}\,(x^2 + y^2) + \mathit{constant} \text{ on the boundary } \mathcal{C}. \tag{11.11}$$

If there is only one boundary — that is, if there are no holes, or which comes to the same thing as stating that the region is simply connected — this constant can be chosen conveniently as 0. This is because the slopes of the ψ hill are all that matter, and the slopes are unaffected by adding a constant to ψ. With reference to the physical picture of the ψ hill, we can readily see that the effect of adding a constant is to lift or lower the ψ hill, which process will never change the slopes. However, if there are two or more boundaries — i.e., if the region is multiply connected — the constant on any one can be chosen at will (say, as zero), but the constants (different values on the different boundaries) are to be chosen so that all is well[11]. [This formulation is the basis of an electrical analogy[12].]

[11] We avoid discussing the difficulties by stating that all is well. Additional conditions are necessary to make sure that the displacements are single valued. There are special difficulties associated with multiply connected regions.

[12] We can exploit this analogy and obtain experimentally the torsional shear stresses on the cross-sections

Third Formulation

We have seen [page 11-8] that the first two of the three differential equations of equilibrium are automatically satisfied, and that the third one becomes

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0.$$

In such cases, it is a usual technique to define a function $F = F(x, y)$ so that

$$\tau_{zx} = \tau_{xz} = \frac{\partial F}{\partial y}; \quad \text{and} \quad \tau_{zy} = \tau_{yz} = -\frac{\partial F}{\partial x}.$$

Then we have

$$\frac{\partial F}{\partial x} = -\tau_{zy} = -G\theta\left(\frac{\partial \phi}{\partial y} + x\right); \quad \text{and} \quad \frac{\partial F}{\partial y} = +\tau_{zx} = +G\theta\left(\frac{\partial \phi}{\partial x} - y\right),$$

from which we see that $\bigtriangledown^2 F = -2G\theta$.

Thus, the governing differential equation and the associated boundary condition are

$$\text{Poisson's equation:} \quad \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = -2G\theta \ \text{ in } \mathcal{R}, \tag{11.12}$$

$$F = \textit{constant} \ \text{ on the boundary } \mathcal{C}. \tag{11.13}$$

If there is only one boundary, this C can be chosen as zero. Otherwise, the constant value on any one can be chosen at will (as, say, zero), but the other ones have to be chosen so that the displacements are single valued.

Alternative Poisson's equation:
The problem may also be formulated by the Poisson's equation and the boundary condition

$$\text{Poisson's equation:} \quad \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = -1 \ \text{ in } \mathcal{R}, \tag{11.14}$$

$$\Phi = \textit{constant} \ \text{ on the boundary } \mathcal{C}. \tag{11.15}$$

This equation (11.14) can be brought to the form [Eq. (11.12)] by the change of variables $F = 2G\theta\,\Phi$. See the illustrative examples given in a later chapter 13.

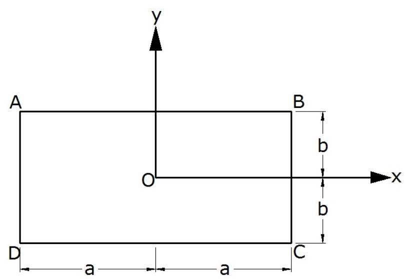
TORSION OF A BAR OF RECTANGULAR CROSS-SECTION

The solution of this problem using a double infinite series is not easy. We shall not discuss this. But we can obtain very satisfactory results by approximate methods. One such method is shown, but not worked out in detail, below. This is not very easy either.

Let us obtain an approximate solution of the torsion problem — torsion of non-circular, prismatic bars — for a rectangular cross-section [Fig. 11.9], governed by the Poisson's equation[13]

and the torsional rigidities. This is done by applying, on an electrically conducting sheet cut in the shape of the cross-section to be investigated, voltages as demanded by the boundary condition (11.11), and measuring the voltage gradients. These are interesting, but we do not propose to discuss here this and other beautiful analogies in torsion.

[13] This is based on the third formulation just given above.



The rectangular region inside $ABCD$ is the domain concerned $\mathcal{D}$; $\mathcal{C}$ is its boundary. The coordinate system is chosen as shown in order to exploit the inherent symmetry in the problem. To solve the torsion problem, it is necessary to solve the Poisson's equation subject to the boundary condition $F(x, y) = 0$ on $\mathcal{C}$.

Figure 11.9: A rectangular cross-section

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial x^2} = -2G\theta \text{ in } \mathcal{D} \qquad (-a < x < a; -b < y < b)$$

subject to the boundary condition

$$F = F(x, y) = 0 \text{ on } \mathcal{C}.$$

The corresponding functional to be minimised is

$$I[F(x, y)] = \iint_{\mathcal{D}} \left\{ \frac{1}{2} \left[\left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2 \right] - 2G\theta\, F \right\} dx\, dy \tag{11.16}$$

We shall assume the solution in the form

$$F = F(x, y) = (x^2 - a^2)(y^2 - b^2) \sum_m \sum_n c_{mn}\, x^m\, y^n, \tag{11.17}$$

where the constants c_{mn} are unknown. As we desire an approximate solution, we retain only a few terms. Let us note that all these coordinate functions (i) satisfy the boundary conditions, and (ii) are linearly independent. Because of the inherent symmetry in the problem, m and n are to be even integers.

The method to determine the unknown coefficients c_{mn} is as follows. Let us substitute Eq. (11.17) in Eq. (11.16) and obtain an approximate functional as

$$I[\tilde{F}(x, y)] = \iint_{\mathcal{D}} \left\{ \frac{1}{2} \left[\left(\frac{\partial \tilde{F}}{\partial x} \right)^2 + \left(\frac{\partial \tilde{F}}{\partial y} \right)^2 \right] - 2G\theta\, \tilde{F} \right\} dx\, dy. \tag{11.18}$$

Now I is a function of the unknown coefficients c_{mn}. To minimise the expression (11.18), we need to differentiate the expression with respect to each of these coefficients c_{mn}, and set each of the derivatives to zero. Although we retain several terms in the assumed form of the solution, in practice, we retain only a few terms, one, two or three. That is usually quite sufficient.

To take a simple case for illustration, let us take a two-term approximation[14] as

$$\tilde{F}(x, y) = (x^2 - a^2)(y^2 - b^2)\left[c_0 + c_1(x^2 + y^2)\right] \tag{11.19}$$

[14] Actually, there are three terms; but it is perhaps less misleading to refer to this as a two-term approximation, because there are only two constants, c_0 and c_1 to be determined.

The two (necessary) conditions to minimise the functional (11.18) are

$$\frac{\partial}{\partial c_0} I[c_0, c_1] = 0; \qquad \frac{\partial}{\partial c_1} I[c_0, c_1] = 0.$$

Carrying out the calculations using MAPLE, we obtain the results easily. For our purposes here, the problem is solved completely[15].

There are some beautiful analogies in torsion. It is necessary for us to be familiar with at least a few of them. We shall discuss them below.

ANALOGIES IN TORSION

There are a few analogies that are relevant here. The basis of the analogies is the mathematical similarity of the governing differential equations and the boundary conditions. These analogies are useful in several ways. It is always academically exciting to see common threads that run through apparently unrelated areas. Apart from this intellectual thrill and pleasure, they are also useful in a practical sense. Sometimes an analogy helps us to visualise the solution of a problem. For example, there is no way we can visualise the variation of the shear stresses or the torsional rigidity of a cross-section directly. $\Phi = \Phi(x, y)$ (third formulation) can, however, be conceived of an a Φ hill by appealing to the membrane analogy. Additionally, theses analogies help us to have new experimental techniques to solve torsion problems. Electrical analogy, fluid flow analogies (there are different analogies), membrane analogy (soap film analogy), sand heap analogy, and razor blade analogy are well developed. Several important results were obtained in the early years using experimental techniques based on these analogies. We shall see some of them briefly.

Electrical Analogy

Let us refer to the second formulation. The governing differential equation and the boundary conditions are

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \text{ in } \mathcal{R},$$

[15] However, if we are interested in the topic of torsion, it would be nice to calculate the maximum shear stress on the cross-section, and the torsional rigidity of the cross-section, and compare them with (i) the exact values and with (ii) one-term and three-term approximations.

We recall from our earlier studies that the torque, T is given by

$$T = 2 \iint_{\mathcal{D}} \tilde{F}(x, y)\, dx\, dy.$$

Although detailed calculations and comparisons cannot be made here, we shall still indicate that the error is only of the order of 1 %, which is eminently satisfactory for such engineering calculations.

Even a one term approximation $\Phi(x, y) = c_1(a^2 - x^2)(b^2 - y^2)$ gives us

$$c_1 = \frac{5}{4(a^2 + b^2)}; \quad \frac{T}{\theta} = \cdots$$

excellent results. This is a measure of how good the approximation is.

$$\psi = \frac{1}{2}(x^2 + y^2) + C, \text{ a constant on the boundary } \mathcal{C}.$$

We shall consider only simply connected domains[16], that is, cross-sections that have only one boundary. Then the only constant can be chosen at will as zero. The effect of this is only to raise or lower the ψ hill. The absolute value of ψ has no importance; only its derivatives (that stand for the slopes) are important, being related to the shear stress components. Once $\psi = \psi(x, y)$ is known, all the relevant information such as the shear stress, the torque, and the torsional rigidity can be computed.

The basis of the analogy is the realisation that that the voltage distribution in an electrically conducting sheet (homogeneous and isotropic) also satisfies the Laplace's equation. Thus, if an electrically conducting sheet is cut in the shape of the cross-section to be investigated, and the voltages applied on the boundary in accordance with the boundary condition $V = (1/2)(x^2 + y^2)$, and if the voltage gradients $\partial V/\partial x)$ and $\partial V/\partial y$ are measured with a two-point probe, we can experimentally solve the torsion problem.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \text{ in } \mathcal{R}, \qquad V = \frac{1}{2}(x^2 + y^2) \text{ on the boundary } \mathcal{C}.$$

Electrolytic tanks and commercially available conducting sheets were used in the past. There are several details that cannot be discussed here.

Fluid Flow Analogy

We shall discuss a fluid flow analogy. The contour lines of the stress function for twist [Fig. 11.10][17] were interpreted to be analogous to the streamlines of an incompressible fluid[18]. From fluid mechanics we know that the velocity components v_x and v_y are related to the stream function $\Psi = \Psi(x, y)$ by the relation

$$v_x = -\frac{\partial \Psi}{\partial y}; \qquad v_y = \frac{\partial \Psi}{\partial x}, \tag{11.20}$$

so that the continuity equation $\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} = 0$ is automatically satisfied. If the expression for the vorticity ω of the fluid

$$\omega = \frac{1}{2}\left[\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x}\right]$$

[16] Simply connected domains have only one boundary. They correspond to cross-sections with no holes. In general, there can be holes and, therefore, several closed boundaries. The constant C will have to be different on each boundary and in such a way that all is well (that is, the displacements are single-valued). Some additional conditions are to be satisfied. These are not always easy. Consideration of multiply connected regions entails complexities that we cannot handle here. The ambitious reader can begin by examining the behaviour of a function like $\log z$, where z is a complex number.

[17] These figures and the following explanatory caption are taken from Den Hartog [3]. "Three cross sections with the contour lines of the stress functions for twist. In the corner points marked A the stress is zero; these corners can be pared away without changing the stiffness of the section; the points marked B are those of maximum stress; the bottom of the keyway has a stress which depends vitally on the fillet radius, and is sure to reach the fatigue limit for even a small alternating torque in the case of a sharp corner."

[18] This analogy was proposed by William Thompson (June 1824 - Dec. 1907), better known as Lord Kelvin. He was a famous British mathematician, mathematical physicist, and engineer of Irish origin.

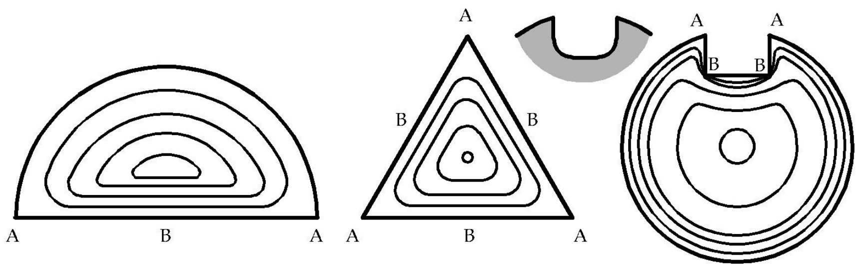


Figure 11.10: Three cross-sections with the contour lines of the stress function for twist. The corners A in all three cases are dead or stress-free. The 'mid-points' B on the boundary 'closest to the centre' have the maximum shear stresses. At the reentrant corners B of the keyway the stresses are theoretically infinite, but actually very large. They are potential sources of failure unless sufficiently generous fillets are provided. It is insightful to relate these facts to the velocities in the fluid flow analogy.

is used along with Eq. (13.43), we obtain

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = -2\,\omega \qquad \text{which is analogous to} \qquad \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = -2\,G\theta$$

governing torsion (third formulation). We also know from fluid mechanics that the lines Ψ = constant are the streamlines. It is thus clear that Ψ = constant along the boundary. Thus, the differential equations and the associated boundary conditions are similar, and the Kelvin fluid flow analogy is established. Its primary use is not as a computational tool or as a basis to perform experiments, but as a means of visualisation. Where the streamlines are crowded together, and where they are well separated, $\cdots$ [What happens? Complete the sentence.]

Prandtl's Membrane Analogy (Soap Film Analogy)

This is by far the most important and useful analogy. This is due to Prandtl[19]. We shall discuss this in some detail. This is based on the third formulation.

The governing equation and the boundary conditions are

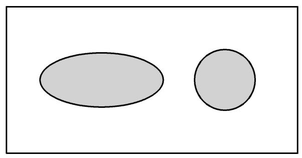
$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = -2\,G\theta \text{ in } \mathcal{R}, \text{ and} \tag{11.21}$$

$$\Phi = \text{constant on } \mathcal{C}; \; = 0 \text{ if there is only one boundary.} \tag{11.22}$$

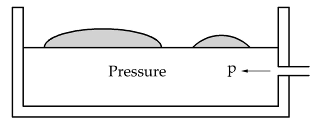
If there is only one boundary (simply connected region, cross-section with no holes), the constant may be chosen at will, say, 0 for convenience. However, if there are several boundaries, the constant C will, in general, be different for different boundaries. These different constant values on the different boundaries (multiply connected regions, cross-sections with

[19] The great Ludwig Prandtl (Feb. 1875 - Aug. 1953) of Göttingen, often considered as the father of aerodynamics, and famous for his work in fluid dynamics (boundary layer theory)!

holes) have to be chosen so that all is well (the displacements are single valued). As these calculations are not always easy, we shall consider only singly connected regions (cross-sections with no holes) for the most part.



(a) Holes cut out on a flat plate



(b) A schematic arrangement for the soap film

Figure 11.11: The cross-section to be analysed (say, an elliptical hole) and a circular hole are cut out from a flat plate [Fig. 11.11a]. A small air pressure is applied below the same soap film stretched across the two holes [Fig. 11.11b]. Optical methods are used to make the necessary measurements (slopes, and volumes under the soap films).

Our first job is to establish the analogy. A hole is cut out on a flat plate in the shape of the section to be investigated, and a membrane — a soap film — is stretched across the hole, and a slight air pressure applied from below [Fig. 11.11b]. The soap film bulbs up. Alongside the (say, elliptical) cross-section, a circular hole also is cut out and the same soap film is stretched across this circular cross-section also.

We shall show that (i) the differential equation governing the shape of the soap film and (ii) the boundary condition are, respectively,

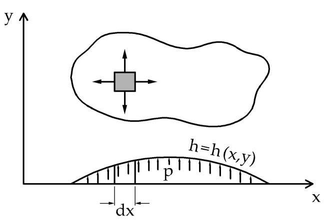
$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = -\frac{p}{S}, \text{ and } h = 0 \text{ on the only boundary.} \tag{11.23}$$

The boundary condition is obvious; the soap film cannot have any height on the boundary. We shall now derive the above equation (Poisson's equation) for the shape of the soap film. Considering an element of the membrane (soap film) [Fig. 11.12], we note that it is acted upon by (i) a surface tension T per unit length on all the edges, and (ii) an air pressure p. If we assume that the surface tension T is relatively large, and that the height of the soap film $h = h(x, y)$ is small, we can see that the surface tension T can be regarded as a constant. However, the slopes are not equal on all the edges and, therefore, the equation of equilibrium in the vertical direction can be written down taking into account the slopes also. The terms in the equation of equilibrium are [Fig. 11.12b]

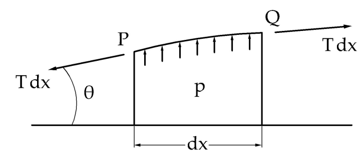
$$-(T\,dy)\frac{\partial h}{\partial x} + (T\,dy)\left(\frac{\partial h}{\partial x} + \frac{\partial}{\partial x}\frac{\partial h}{\partial x}\,dx\right).$$

A similar pair of terms also is present. This is

$$-(T\,dx)\frac{\partial h}{\partial y} + (T\,dx)\left(\frac{\partial h}{\partial y} + \frac{\partial}{\partial y}\frac{\partial h}{\partial y}\,dy\right).$$



(a) Forces acting on an element



(b) Forces on an element of the soap film

Figure 11.12: The forces on an element of the soap film are shown. There is an air pressure $p\,dx\,dy$ acting upwards. The surface tension has vertical components. These are the two forces to be reckoned when we derive the (differential) equation for the shape of the film.

Additionally, there is the upward force acting on the element due to the air pressure $p\,dx\,dy$. When all these terms are reckoned, the equation of equilibrium in the vertical direction is

$$\left[-(T\,dy)\frac{\partial h}{\partial x}+(T\,dy)\left(\frac{\partial h}{\partial x}+\frac{\partial}{\partial x}\frac{\partial h}{\partial x}\,dx\right)\right]+$$
$$\left[-(T\,dx)\frac{\partial h}{\partial y}+(T\,dx)\left(\frac{\partial h}{\partial y}+\frac{\partial}{\partial y}\frac{\partial h}{\partial y}\,dy\right)\right]+p\,dx\,dy=0,$$

which on cleaning up appears as

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=-\frac{p}{S}. \tag{11.24}$$

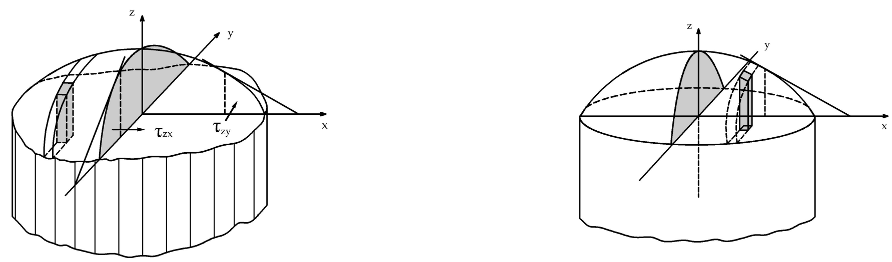
This is the governing equation defining the shape of the blown up or bulbed up soap film. The boundary condition is $h = h(x, y) = 0$ on the boundary. The soap film cannot have any height on the (only) boundary. [When there are several boundaries as in a multiply connected region corresponding to a cross-section with holes, there are changes as explained earlier. We shall for the time being consider only simply connected regions.] Eq. (11.24) may be interpreted as: the sum of the curvatures (approximate expressions, because the height h and the slopes are small) is a constant everywhere on the membrane.

The analogy is established if the differential equations and the boundary conditions in the two cases [the torsion problem, Eq. (11.21), and the soap film, Eq. (11.24)] are similar. The analogous quantities are

(i) $\dfrac{p}{S} \quad\rightarrow\quad 2\,G\theta$ (ii) $h \quad\rightarrow\quad \Phi$ (iii) slope of the soap film $\quad\rightarrow\quad$ shear stress.

The slope of the soap film in any direction is analogous to the shear stress (on the cross-section) in the perpendicular direction. For example, the slope in the x direction $\partial h/\partial x \quad\rightarrow$ τ_{zy} and the slope in the y direction $\partial h/\partial y \quad\rightarrow\quad \tau_{zx}$. In addition, (iv) twice the volume under the soap film $\quad\rightarrow\quad$ torque, T. We know that

$$T=2\iint_{\mathcal{R}}\Phi\,dx\,dy \qquad \text{which is analogous to} \qquad 2\iint_{\mathcal{R}}h\,dx\,dy,$$



(a) Soap film (elliptical section) (b) Soap film (circular section)

Figure 11.13: The soap films corresponding to (a) an elliptical and (b) a circular cross-sections. The slope of the soap film is analogous to the shear stress in the perpendicular direction, and twice the volume to the torque. The same soap film is used for both sections so that the surface tension and the air pressure are the same for both. The circular section serves as a reference for comparison, because its analytical solution is easy and known.

which is twice the volume under the soap film.

This analogy helps us greatly in visualising the slope of the Φ hill, and to have an excellent understanding of the qualitative nature of the shear stresses and the torsional rigidity even without making any calculation. [For example, we know that the resultant shearing stress on the cross-section at a boundary point must be tangential to the boundary. This fact can be readily seen. Along the boundary, the slope $\partial h/\partial s = 0$ which, when translated to the torsion problem gives us the aforementioned result.] Additionally simple calculations on the shape of the soap film give us valuable information on the corresponding torsion problem. In the early days[20], several important cases were solved experimentally.

Sand Heap Analogy

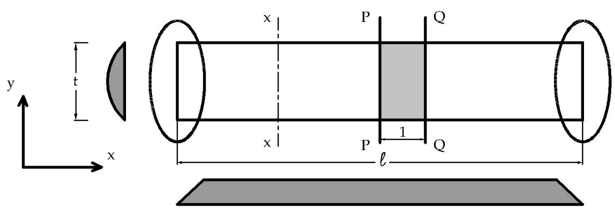
This refers to the case of plasticity when the entire cross-section has gone into yield. This analogy is not difficult to understand. It is an extension of the idea of the membrane analogy. The cross-section to be investigated is cut on a plate — this section is not removed as for the membrane — and sand is heaped on the cross-section until the sand spills over to the ground or to a base plate placed below. The slope of the sand heap will be always the same everywhere; that is a property of all soils including sand. The slope corresponds to the yield stress in shear. If the same experiment is repeated on a circular plate, the ratio of the weights of the sand collected on the two cross-sections gives the ratio of the torques and, thus, of the torsional rigidities of the two cross-sections. As the torsional rigidity of the circular cross-section is known, that of the given cross-section, say, elliptical, can be thus

[20] There are several experimental details which are of decisive importance when actual experiments are conducted. The usual soap that we use when bathing is not suitable; the film is not stable and it soon collapses. Suitable substitutes were developed. The slopes are measured by optical methods. Experiments were developed to cover the more difficult cases of multiply connected regions.

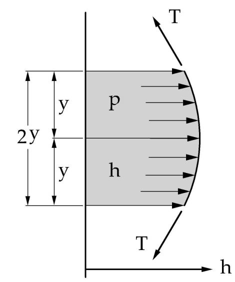
Those were the early days of aeronautics when Prandtl was in Göttingen, the days of furious academic activity, the golden years of Göttingen. G.I. Taylor & Co. in England also contributed substantially to these developments.

experimentally determined. This is known as Nadai's[21] sand heap analogy. It is possible to cover the case of partially plastic bars in torsion also[22].

A Long Narrow Rectangular Cross-section



(a) The narrow long rectangle and the soap film



(b) The air pressure and the surface tension on the soap film

Figure 11.14: A long narrow rectangular cross-section and the shape of the corresponding soap film are shown. All the cross-sections, except at the two ends, are identical. On the right is shown [Fig. 13.8b] a section to enable us to write the equation of equilibrium.

We shall demonstrate how the membrane analogy may be made use of to analyse the torsion problem of a long narrow rectangular cross-section. The solution of this problem is of great practical importance. Most structural steel cross-sections (except those in the form of a box) can be analysed using the results of this problem.

We wish to examine a long narrow rectangle [Fig. 13.8a] (length l and breadth (thickness) t with $l >> t$) using the membrane analogy. The shape of the corresponding soap film is shown shaded. Two views of this are shown, both shaded. We can see that the soap film has the same cross-sectional shape everywhere except at the two ends. This fact makes the problem one-dimensional (if we disregard the two ends), and the governing differential equation becomes simpler: from a partial differential equation [two independent variables x and y; $h = h(x, y)$] to an ordinary differential equation [only one independent variable y; $h = h(y)$]. We shall now derive this (ordinary) differential equation.

Consider a part of the soap film between the sections PP and QQ of width $y + y = 2y$ separated by a unit distance. We note that this element of the soap film of length unity along the $x-$axis (between the sections PP and QQ) and of width $2y$ is acted upon by (i) the air pressure to the right [Fig. 13.8b] (actually upwards), and (ii) the component of the surface tension S per unit length acting on the two edges to the left [Fig. 13.8a] (actually downwards). Writing down the equation of equilibrium, we obtain

$$p \times 2y \times 1 = -2S \times \frac{dh}{dy} \times 1 \qquad \longrightarrow \qquad \frac{dh}{dy} = -\tfrac{p}{S}\, y.$$

[21] Named after Arpad Ludwig Nadai (April 1883 - 1963) — born in Hungary, higher education in Germany, settled down in the US — was a Hungarian American mathematician and engineer.

[22] "It is only necessary to erect a fixed roof of constant slope over the membrane with the outer boundary of the section as its base", writes Rodney Hill, the famous author of the famous book, *The Mathematical Theory of Plasticity*, Oxford University Press, (1950).

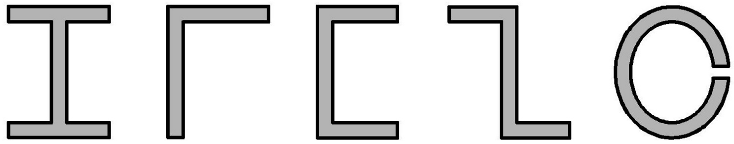


Figure 11.15: The theory developed for a long narrow rectangle is applicable to all open sections (with appropriate minor modifications depending on the context). Five examples are shown: I, angle iron, channel, Z and a thin ring with a small slit.

On integrating this, we have $h = -(p/S)\int y\ dy = -(py^2)/(2S) +$ a constant. This constant is determined using the boundary condition: at $y = \pm(t/2)$, the height $h = 0$. Thus, the constant is evaluated as $p/(2S)\,(t^2)/(4)$. The shape is a parabola, obtained as the solution

$$h = \frac{p}{2S}\left[\frac{t^2}{4} - y^2\right]. \tag{11.25}$$

The maximum slope / the maximum shear stress is clearly at the edges $y = \pm t/2$.

$$\left(\frac{dh}{dy}\right)_{\max} = \frac{p}{S}\,\frac{t}{2} \quad\longrightarrow\quad \tau_{\max} = G\theta\, t, \tag{11.26}$$

invoking the membrane analogy to refer to the torsion problem. The volume under the soap film =(2/3)$\times$ base $\times$ height (parabola) $\times$ length is

$$= \frac{2}{3}t\left(\frac{p}{2S}\,\frac{t^2}{4}\right)l \qquad\longrightarrow\qquad \frac{T}{2} = \frac{2}{3}\,t\left(\frac{2G\theta}{2}\,\frac{t^2}{4}\right)l.$$

The torsional rigidity (or torsional stiffness) C is, accordingly,

$$C = \frac{T}{\theta} = G\,\frac{t^3\,l}{3}. \tag{11.27}$$

Eliminating $G\theta$ using Eqs (11.26) and (11.27), we obtain

$$\tau_{\max} = G\theta\, t = \frac{3\,T}{t^3\,l} \qquad\longrightarrow\qquad \tau_{\max} = \frac{3\,T}{t^3\,l}. \tag{11.28}$$

Application to Structural Members

We had stated that the above problem of torsion of a long narrow rectangle is of much practical importance. It is applicable for all open sections such as I, angle iron and channel sections used widely as structural members [Fig. 13.29], but not for closed nor for box sections. The length l referred to above is now to be interpreted as the total length of all the legs comprising the open section. Visualisation of the corresponding soap film will help us understand all this. There will surely be stress concentration, and a corresponding slight reduction in the height and volume of the membrane (soap film) at the ends, but these

factors do not prohibit using the theory developed to all such open sections. [While on this topic let us remark in passing that we should distinguish between two kinds of corners: (i) reentrant corners where the stress is very high (severe stress concentration), and (ii) dead corners which are free of stresses[23]. We are talking about the shear stresses on the cross-section due to torque. At the 'dead corners' there can be bending stresses (which are normal stresses, tensile or compressive). The fluid flow analogy also helps us understand the difference between reentrant and dead corners. The velocity at a reentrant corner is very high; the streamlines close in there. In contrast to this situation, the velocity at a 'dead corner' is zero. It is also possible, on the basis of some reasonable assumptions, to analyse the state of stress when a fillet is provided at a reentrant corner.]

What happens if the various legs in the sections shown in Fig.13.29? Well, Eq. (11.27) still applies, but the torque (as well as the torsional rigidity) has to be calculated separately for each leg and added up. A little reflection on the shape of the soap film will tell us what to do in all such cases. The concept of shear flow based on the fluid flow analogy also will help us.

The results $C = T/\theta = (G\,l\,t^3)/3$ and $\tau = 3\,T/(l\,t^2)$ are significant, and are pointers to a better qualitative understanding of the problem. It is the points *nearest to the centre* that have the largest shear stress, and *not the farthest points from the centre*! This conclusion is completely at variance with the implications of the Coulomb's theory that it is the farthest points that are stressed the most! Equally important is the fact that the torsional rigidity (stiffness) $C = T/\theta = G\,lt^3/3$ increases only as the first power of l. The polar second moment of area increases as the cube of l, which fact warns us not to apply or extrapolate the conclusions from the Coulomb's theory to long rectangular cross-sections.

HOLLOW THIN-WALLED SECTIONS

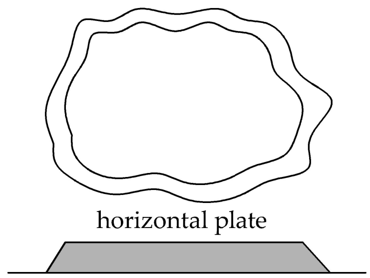
From a practical point of view, hollow thin-walled sections are very important. Steel box girders, thin-walled closed tubes, wings and fuselages of aircrafts are but a few examples. The theory now is relatively simple. Fig. 11.16a shows a thin-walled closed section and the corresponding membrane. The plate is to be horizontal because the height h (analogous to the function Φ in the third formulation) must be a constant on every boundary. As the wall is thin, we may regard the slope of the soap film to be almost constant across the wall thickness. Thus, it follows that $\tau \times t =$ constant. (Why? The slope $\approx h/t \quad \rightarrow \quad \tau$.) This relationship leads us to conclude immediately: that where the thickness is small, the shear stress τ is large, and vice versa. This is analogous (similar) to the case of an incompressible fluid flowing in a channel (of constant depth) when the continuity equation demands that the velocity, $v \times t$, thickness = constant. Thus arises the concept of shear flow.

Considering the (vertical) equilibrium of the horizontal plate [Fig. 11.16a], we have

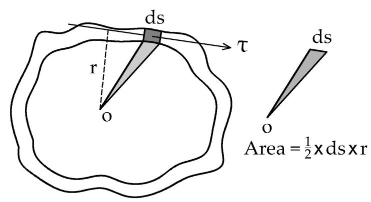
$$pA = \oint_C S\,\frac{h}{t}\,ds \quad \longrightarrow \quad \frac{p}{S} = \oint_C \frac{h}{t}\,ds$$

($S \times ds$ is the surface tension; h/t is the slope.)

[23] This book is not on design; it is on analysis. Even so, let us point out that designers shall be careful about reentrant corners, and that generous fillets at such vulnerable places should be provided.



(a) A thin-walled closed section



(b) To derive the governing equation

Figure 11.16: A thin-walled closed section and the corresponding membrane are shown in Fig. 11.16a. The plate is maintained horizontal. The expression for the elemental shear force is [Fig. 11.16b] $\tau \times ds$, and its moment about O is $\tau \times ds \times n$. Let us also note that the area of the elemental triangle shown shaded is half the base multiplied by the relevant perpendicular distance $= 1/2 \times ds \times n$.

This equation, when translated to the torsion problem, yields

$$2\,G\theta = \frac{1}{A}\oint_{\mathcal{C}} \tau\, ds. \tag{11.29}$$

The torque T is [Fig. 11.16b] the moment of the elemental force $(\tau \times t \times ds)$ about the point O integrated along the cell. Accordingly,

$$T = \oint_{\mathcal{C}} (\tau \times t) \times n\, ds = (\tau \times t) \oint_{\mathcal{C}} n\, ds = (\tau \times t) \times (2A).$$

Although τ and t may vary from point to point, the product $(\tau \times t)$ is a constant (equal to the height of the horizontal plate). Further $(n\, ds)$ is twice the area of the shaded triangle [Fig. 11.16b]. Thus,

$$T = (\tau \times t)\, 2A \qquad \longrightarrow \qquad \tau = \frac{T}{2\,At}, \tag{11.30}$$

where A is the area of the plate. Infinitesimal areas like $(n \times ds)$ represented by the shaded triangle, when integrated (added up), give us the area of the plate.

Using Eqs (11.29) and (11.30), we find

$$\theta = \frac{T}{4\,G\,A^2}\oint_{\mathcal{C}} \frac{ds}{t} \qquad \longrightarrow \qquad C = \frac{T}{\theta} = \frac{4\,G\,A^2}{\oint \frac{ds}{t}}. \tag{11.31}$$

If the wall thickness t is a constant which is often the case, this equation is simplified as

$$\theta = \frac{T\,L}{4\,G\,A^2 t} \qquad \longrightarrow \qquad C = \frac{T}{\theta} = \frac{4\,G\,A^2 t}{L}, \tag{11.32}$$

where L is the total length — perimeter – of the cell. These equations suffice to solve torsion problems of hollow thin-walled sections[24].

[24] The result $\tau = T/(2\,At)$ is known as Bredt's formula, named after the German engineer Rudolph Bredt (April 1842 - May 1900).

Sections with Two or More Cells (Multicellular Sections)

(a) A two-cell closed section

(b) A symmetrical two-cell section

Figure 11.17: A multicellular (two cells) cross-section. The shape of the corresponding membrane and the shear flows are shown. Fig. 11.17b shows a symmetrical cross-section. Now there is no shear flow, and no shear stress in the middle leg (wall).

Let us now investigate the case of multicellular cross-sections. We shall consider a two-cell section (triply connected region; there are three boundaries) [Fig. 11.17]. The (two) horizontal weightless plates are now at different heights[25]. This is because there are three boundaries for the section, as the region is multiply (triply) connected; the constant values on the boundary, we recall from the boundary condition in the third formulation, will be different. The shear stress in the three walls are

$$\tau_1 = \frac{h_1}{t_1}, \qquad \tau_1 = \frac{h_2}{t_2}, \qquad \text{and} \qquad \tau_3 = \frac{h_2 - h_1}{t_1}, \tag{11.33}$$

where h_1 and h_2 are the unknown heights of the two plates. (The constant value on the outermost boundaries is, as always, conveniently taken as zero.)

The torque T given by twice the volume under the membrane including the horizontal plates is

$$\begin{aligned} T &= 2 \times \text{volume} = 2(A_1 h_1 + A_2 h_h 2) \\ &= 2(A_1\,\tau_1 + A_2\,\tau_2), \text{ using Eq. (11.33)}. \end{aligned} \tag{11.34}$$

The equation

$$\oint \tau\, ds = 2\, G\theta \tag{11.35}$$

may be applied around (i) the first cell A_1, (ii) the second cell A_2, and (iii) the full two-cell section $(A_1 + A_2)$. When applied to the paths enclosing $(A_1 + A_2)$ and A_1, we obtain the following equations.

$$\tau_1\, s_1 + \tau_2\, s_2 = 2\, G(A_1 + A_2)\,\theta \qquad \text{around } A_! + A_2 \tag{11.36}$$

$$\tau_2\, s_2 + \tau_3\, s_3 = 2\, G\, A_2\,\theta \qquad \text{around } A_2 \tag{11.37}$$

[25] If the section is symmetrical as in Fig. 11.17b, both the plates will be at the same level. We can then conclude that the middle leg (the middle wall, the wall that separates the two cells) carries no shear stress.

Subtracting the latter from the former, we obtain

$$\tau_2\, s_2 + \tau_3\, s_3 \;=\; 2\, G\, A_2\, \theta \tag{11.38}$$

which is what we obtain when Eq. (11.35) is applied around the first cell A_1. Notice the negative sign and the shear flow in the wall between the two cells A_1 and A_2. Appealing to the fluid flow analogy, we can see that the directions of the velocity in this wall (middle leg) are opposite to each other. In other words,

$$\tau_3 = \tau_2 - \tau_1. \tag{11.39}$$

We may formulate this problem alternatively in terms of the two unknown heights h_1 and h_2. The net shear flow $\tau_3\, t_3$ is the difference between the shear flows $\tau_3\, t_3 = \tau_2\, t_2$ (downwards) and $\tau_3\, t_3 = \tau_1\, t_1$ (upwards).

The above equations are sufficient to solve the problem. See the worked out examples on thin-walled closed sections.

It is interesting to see what happens when the cross-section is symmetrical as shown in Fig. (11.17b). Now because of symmetry, $h_1 = h_2$ $(= h)$, and $\tau_1 = \tau_2$ $(= \tau)$. The consequence is that the middle leg (the wall between the two cells) is stress-free! This leg is entirely useless to resist torsional moments. Still such a leg (wall) is provided, because it serves other functions. The cross-section in general is subjected to loads other than a torsional moment.

Closing Remarks

We have discussed some aspects of the theory of torsion of non-circular prismatic bars. Actually it is better to discuss the general bending problem, and to treat torsion as a sub-problem in the theory of bending. We know that when the load does not pass through the shear centre of the cross-section, bending is accompanied by torsion. However, such an approach is more difficult mathematically. There are many more aspects that we could not discuss here; this is but natural. With these few remarks we come to the end of this chapter.

In the next chapter, we shall see the field equations of the theory of elasticity.

Chapter 12

FIELD EQUATIONS OF THE THEORY OF ELASTICITY

The field equations of elasticity are the basis of solution of problems, not only of mathematical / analytical solution, but also of approximate and numerical methods like the finite element method FEM. These play a crucial role in the formulation (and later solution also) of all stress analysis problems. Thus, it is necessary to discuss these equations in fair detail.

We have now seen (i) the nature of the state of stress at a point, (ii) some details of the state of strain and deformation at a point, and (iii) the constitutive equations (material laws) for an elastic material. These are the fundamental prerequisites before the topic of this chapter can be discussed.

FUNDAMENTAL EQUATIONS

We shall present in this chapter all the fundamental equations necessary for the formulation of a general theory. Stress analysis problems are formulated and later solved, exactly or approximately, using these equations. With these few words we shall begin the discussions on the three pillars of the theory of elasticity.

The Three Pillars of the Theory of Elasticity

The fundamental equations are

(i) the equations of equilibrium;

(ii) the strain-displacement relations (also known as the kinematic relations); and

(iii) the constitutive equations (material laws).

These are the three pillars on which the edifice of the theory of elasticity and the science of stress analysis is built. Thus, these three sets of equations are of great importance.

The first set depends only on the equilibrium of forces (associated with the stress components and the applied external forces); it has nothing to do with the material properties.

The second set depends only on the geometry of deformation; it has nothing to do with the material properties either. Thus, these two sets are equally applicable for other materials also. Inasmuch as these two sets are independent of the material properties, these are applicable in the theories of plasticity[1], viscoelasticity, thermoelasticity, etc. Only the third set involves the material properties (constitutive equations). It is only the third set of equations that restricts the analysis to the particular material represented by the material law (constitutive equation)[2].

Other Equations

In addition, there are other important equations: the compatibility equations; Navier's equations; Beltrami-Michell equations, etc. Some of the best brains have laboured and done great work in these areas during the last three hundred years or more[3]. There are many, very many, equations that are derived and used. Only the most important ones can be considered here.

The principle of virtual work applied to the case of a deformable body is also of great importance. The variational methods of solid mechanics are based on this principle. Thus, we shall discuss this also, though only briefly.

Some of these equations referred to above are differential equations. Thus, they must be accompanied by the appropriate boundary conditions. Even though the boundary conditions are not among the field equations, we shall still discuss them in this chapter.

Number of Unknowns and Equations Available

In a typical stress analysis problems, it is desired to determine (i) the stress components, (ii) the strain components, and (iii) the displacement components at any point inside the body. These are the unknowns: the six (6) stress components $\sigma_{ij}\ \ (i,\, j = 1,\, 2,\, 3)$; the six (6) strain components $e_{ij}\ \ (i,\, j = i,\, 2,\, 3)$; and the three (3) displacement components $u_i\ \ (i = 1,\, 2, 3)$ making up a total of $6+6+3 = 15$ unknowns. Here we have used the symmetry of both the stress and the strain matrices. The available equations to determine them are: the three

[1] Even though these remarks are valid, some changes may have to be made depending on the context. For example, in the theory of elasticity, we can almost always assume that the strains are small. In the theory of plasticity, however, the strains may not be small; we may have to account for large strains. Accordingly, the actual strain-displacement equations used may have to be modified accordingly.

[2] Let us emphasise that by a material we mean not quite the *material*, but the *model* we choose in a given context. For example, concrete may be considered as an elastic material for many applications. However, it can be considered as perfectly rigid in some situations where the deformations of the other components are so large in comparison that the concrete foundation on which these are erected may be regarded as rigid. On the other hand, when creep is to be calculated — long term creep effects are quite significant for concrete — the concrete, the very same concrete, is to be regarded as viscoelastic. Similarly, the (clayey) soil on which buildings are erected may be treated as elastic in some cases. But, if the tall buildings settle down sinking more and more into the ground — the soil in Mexico city is said to be so notoriously yielding that one or two storeys of buildings are said to sink down into the clayey soil — it may be necessary to treat the soil as viscoelastic. Only viscoelasticity, and not elasticity, can account for such time dependent displacements as in this example and creep.

[3] Readers are advised to be familiar with the historical development of our subject. Timoshenko's *History of the Strength of Materials* is recommended.

(3) differential equations of equilibrium $\sigma_{ji,j} + F_i = 0 \ \ (i,\, j = 1,\, 2,\, 3)$; the six (6) strain-displacement relations $e_{ij} = (1/2)(u_{i,j} + u_{j,i}) \ \ (i,\, j = 1,\, 2,\, 3)$; and the six (6) stress-strain relations (generalised Hooke's law) $\sigma_{ij} = \lambda\, \delta_{ij}\, e_{kk} + 2G\, e_{ij} \ \ (i,\, j,\, k = 1,\, 2,\, 3)$; making up a total of $3 + 6 + 6 = 15$ equations.

Looked at differently, we can regard that there are nine (9) stress components, in which case there are six (6) equations of equilibrium (the three moment equations of equilibrium also are available). Now there are $9 + 6 + 3 = 18$ unknowns and $6 + 6 + 6 = 18$ equations available.

These are far too many equations to be solved. Thus, the general problem in the theory of elasticity is a hopelessly difficult problem. Let us consider first set of equations, viz., the (differential) equations of equilibrium.

DIFFERENTIAL EQUATIONS OF EQUILIBRIUM

This topic was taken up earlier and discussed in some detail in the earlier chapters. These will not be repeated here. Generally these equations are linear, but it is possible to have nonlinearities and other complications when complicated problems or more general formulations are considered.

As we have remarked elsewhere, these equations alone are not sufficient to solve any problem in stress analysis because there are more unknowns than there are equations. In other words, *all* problems in the mechanics of solids and / or the theory of elasticity are statically indeterminate internally. Hence we must have other equations to supplement.

These are the constitutive equations and the strain-displacement relations. We shall take the latter for discussion below.

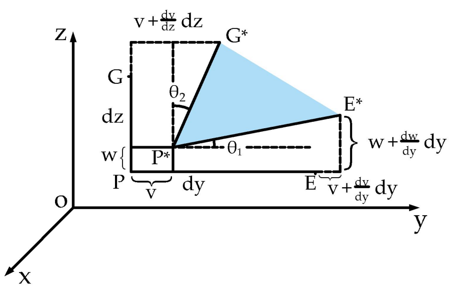
STRAIN-DISPLACEMENT RELATIONS

We shall now take up the next important set of equations, viz., the strain-displacement relations. These are also called the kinematic relationships. Although we had seen these earlier, let us derive these equations in a slightly different way.

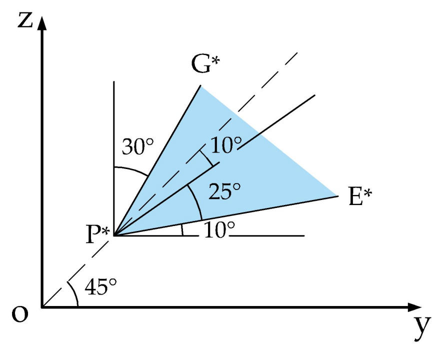
Shear Strains and Rotations

Fig. 12.1 shows the details of the deformation in a neighbourhood of a point. The point P moves to P^*. The displacement components are u, v, w in the x, y, z directions, respectively. A neighbouring point E moves to E^*. Similarly, the point G moves to G^*. The displacements concerned are:

$$\begin{aligned}
&\text{of } P \text{ along the } y \text{ axis:} && = v; \\
&\text{of } E \text{ along the } y \text{ axis:} && = v + \frac{\partial v}{\partial y}\, dy; \\
&\text{of } G \text{ along the } y \text{ axis:} && = v + \frac{\partial v}{\partial z}\, dz; \\
&\text{of } P \text{ along the } z \text{ axis:} && = w;
\end{aligned}$$



(a) Deformation in the neighbourhood of a point P



(b) The shear strain and the rotation

Figure 12.1: The details of deformation in the neighbourhood of a typical point P are shown in Fig. 12.1a. Fig. 12.1b helps us understand how the shear strain and the rotation are separated. Some numerical values are taken to understand this. The values of 30 °, etc. are unrealistically high. The changes in the angles are very small indeed.

$$\text{of } E \text{ along the } z \text{ axis:} \quad = w + \frac{\partial w}{\partial y}\, dy;$$
$$\text{of } G \text{ along the } z \text{ axis:} \quad = w + \frac{\partial w}{\partial z}\, dz.$$

Fig. 12.1b shows the shaded wedge shaped element after deformation[4]. We can notice that the change in the angle is $30 + 10 = 40\,°$, while the rotation of the element is $(30 - 10)/2 = 10\,°$. This example will probably help us understand that

$$e_{xy} = \frac{1}{2}\gamma_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right); \qquad \omega_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right).$$

From Fig. 12.1a, it is clear that the angles θ_1, θ_2 and the shear strain, γ_{yz} and the rotation ω_{yz} are given by

$$\theta_1 = \frac{\partial w}{\partial y}\, dy; \qquad \theta_2 = \frac{\partial v}{\partial z}\, dz;$$
$$e_{yz} \equiv \frac{1}{2}\gamma_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right); \qquad \omega_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right).$$

Let us also note that the linear strain along the y direction Fig. 12.1a is

$$e_{yy} = \frac{\left(v + \frac{\partial v}{\partial y}\, dy\right) - v}{dy} = \frac{\partial v}{\partial y}.$$

Writing the companion equations (from cyclic permutation and by noting the pattern), we obtain all the strain displacement relations.

$$e_{xx} = \frac{\partial u}{\partial x}; \qquad e_{yy} = \frac{\partial v}{\partial y}; \qquad e_{zz} = \frac{\partial w}{\partial z};$$

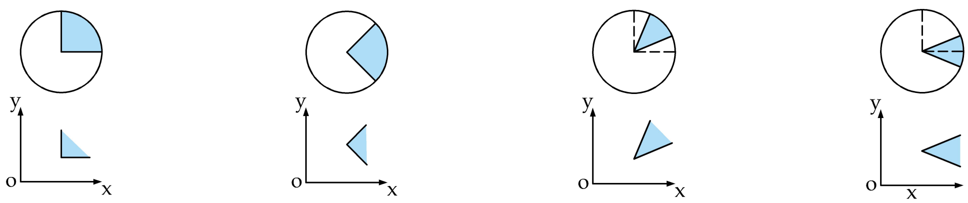
[4] The angles are unrealistically high; in a real engineering structure, these angles can never be as high as 30 °; the strains, we know, are only of the order of 1 part in 1000.

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right); \quad e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right); \quad e_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial x}{\partial z}\right);$$

$$\omega_{xx} = 0; \qquad \omega_{yy} = 0; \qquad \omega_{zz} = 0;$$

$$\omega_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right); \quad \omega_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right); \quad \omega_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial x}{\partial z}\right).$$

Using the index notation we can write as $e_{ij} = \dfrac{1}{2}(u_{i,j} + u_{j,i})$; $\omega_{ij} = \dfrac{1}{2}(u_{i,j} - u_{j,i})$.



(a) An element before deformation (b) No shear strain, only rotation (c) No rotation, only shearing strain (d) Shear strain and rotation

Figure 12.2: An undeformed element [Fig. 12.2a may undergo either a shearing alone, or a rotation alone, or both a shearing strain and a rotation.

We can write $u_{i,j} = \left[\dfrac{1}{2}(u_{i,j} + u_{j,i})\right] + \left[\dfrac{1}{2}(u_{i,j} + u_{j,i})\right] = e_{ij} + \omega_{ij}$. Let us emphasise that the strain matrix e_{ij} is symmetric ($e_{ij} = e_{ji}$), while the rotation matrix ω_{ij} is skew-symmetric ($\omega_{ij} = -\omega_{ji}$). Such a decomposition into a strain tensor and a deformation tensor can be understood physically with the help of Fig. 12.2. A solid element shown in Fig. 12.2a may undergo a change in shape and / or a rotation. Fig. 12.2b shows the same element after it rotates, but without any shear strain, while Fig. 12.2c shows the element with only a shear strain, but no rotation. The general case is represented by Fig. 12.2d involving both a shearing strain and a rotation.

These rotations are not related to the stress components in Solid Mechanics and, therefore, they have only a secondary role here. In Fluid Mechanics, on the other hand, they play a crucial role.

The reason why we define e_{xy} as half the corresponding shear strain is this. Let us compare the following two matrices.

$$\begin{bmatrix} e_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & e_{yy} & \gamma_{yz} \\ \gamma zx & \gamma_{zy} & e_{zz} \end{bmatrix} \quad \text{(\textbf{does not have} the required transformation property.)}$$

$$\begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{bmatrix} \quad \text{(\textbf{does have} the required transformation property.)}$$

The first one does not transform as a (second order) tensor; hence it is not acceptable, while the second one does and is, therefore, acceptable. On comparison with the stress matrix,

we notice that the strain matrix is the matrix representation of the invariant, symmetric strain tensor, and that the analogous quantities are

$$\begin{array}{llllllll} e_{xx} & \longrightarrow & \sigma_{xx}; & e_{yy} & \longrightarrow & \sigma_{yy}; & e_{zz} & \longrightarrow & \sigma_{zz}; \\ \gamma_{xy} & \longrightarrow & 2\times\tau_{xy}; & \gamma_{yz} & \longrightarrow & 2\times\tau_{yz}; & \gamma_{zx} & \longrightarrow & 2\times\tau_{zx}. \end{array}$$

As a further vindication of this position, we recall the formulae for the principal stresses and principal strains (for the simplified two-dimensional case).

$$\sigma_{11},\ \sigma_{22} = \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + (\tau_{xy})^2};$$

$$e_{11},\ e_{22} = \frac{e_{xx}+e_{yy}}{2} \pm \sqrt{\left(\frac{e_{xx}-e_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

We can see that the formulae are entirely similar or analogous, if τ_{xy} is replaced by $\gamma_{xy}/2$!

Engineers' Strain and True (or Natural) Strain

The simplest strain-displacement relation is the engineers' definition of strain,

$$e = \frac{\text{change in length}}{\text{original length}} = \frac{l_f - l_i}{l_i} = \frac{\Delta l}{l_i}.$$

This is fine as long as the strains are small. This is usually the case. The strain of, say, steel at the yield point is of the order of 10^{-3}. This is quite small indeed (compared to 1). However, when large strains are encountered as, say, in plastic flow there are difficulties. This definition is not adequate. Let us take a simple, hypothetical example of a (uniform) bar of initial length 100 mm when it is stretched to the final length of $l_f = 200$ mm. By the engineers' definition given above, the strain is $(200-100)/100 = 1.0 = 100\,\%$. Let us imagine that the elongation has taken place in two stages, say, first (i) from 100 mm to 150 mm, and next (ii) from 150 mm to 200 mm. What are the strains, in the first stage, second stage, and in the final state? Well, in the first stage, $e_{(1)} = (150-100)/100 = 0.5 = 50\,\%$, while in the second stage, $e_{(2)} = (200-150)/150 = 0.333 = 33.3\,\%$. When we add up, $e_{(1)} + e_{(2)} = 50\,\% + 33.3\,\% \neq 100\,\%$.

What we have done is only imagination. We can also imagine, just as well and justifiably, that the two stages are (i) from 100 mm to 120 mm, and in the next stage, $e_{(2)} = (200 - 120)/120 = 0.666 = 66.6\,\%$. We again see not only that (a) they do not add up to $100\,\%$, but also that (ii) the sum is now different: $20\ \% + 66.6\,\% = 86.6\,\%$. We can equally well imagine that the total elongation has taken place in three stages. The answer will then be again different.

There is no consistency. This lack of consistency is not only inconvenient, but is more serious. What shall we do to clean up the resulting confusion?

We can imagine that the elongation has taken place not in one, two or three stages, but in infinite stages. Then the total strain will be

$$e = \int_{l_i}^{l_f} \frac{dl}{l} = \log\frac{l_f}{l_i} = \log\left(1 + \frac{l_f - l_i}{l_i}\right) = \log\left(1 + \frac{\Delta l}{l_i}\right). \tag{12.1}$$

This is called the true strain or natural strain. We can see that at least the inconsistency is now removed; everybody will arrive at the same answer! We can also see that this definition (or measure or index) of strain is consistent with the engineers' definition when the change in length Δl is very small[5] compared to the initial length l_i.

This definition was popular at one time. Really, this is only one of several possibilities. There is nothing particularly true about this definition, nor anything particularly false about other measures of strain. For example, we can surely define engineers' strain as (change in length)/(final length), which agrees with the traditional definition whenever the change in length is very small. [Recall the definitions of the Lagrangian and Eulerian measures of strain.]

Compatibility Equations: Integrability

The strain-displacement relations, we have seen, are the following.

$$e_{xx} = \frac{\partial u}{\partial x}; \qquad e_{yy} = \frac{\partial v}{\partial y}: \qquad e_{zz} = \frac{\partial w}{\partial z}; \tag{12.2a}$$

$$e_{xy} = \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]; \quad e_{yz} = \frac{1}{2}\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right]; \quad e_{zx} = \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right]. \tag{12.2b}$$

The equations (12.2a) relate the normal strains e_{xx}, e_{yy} and e_{zz} to the (partial derivatives of) the displacement components u, v and w, while the ones (12.2b) relate half[6] the shearing strains to the (partial derivatives[7] of) the displacement components.

The six strain components have common parentage; they have all come from the three displacement components (u, v, w). Thus, the six e_{ij}'s are not independent; they are cousins or half brothers. The e_{ij}'s are, therefore, related to one another. These relationships among the six strain components are the compatibility equations. Mathematically speaking, the above six strain-displacement relations may be regarded as a system of six partial differential equations for the determination of only three unknown functions (displacement components) u, v and w. Thus, this is an overdetermined system, and consequently this system cannot have any solution in general; certain conditions must be satisfied so that there can be admissible (single-valued, continuous) functions $u.v, w$. These integrability condition are the compatibility equations (or conditions).

It is clear from the six (6) strain-displacement relations [Eqs (12.2a), (12.2b)] that the six strain components *cannot* be independent, because they have come from only three three displacement functions u, v, w. This fact shows that the six strain components are

[5] We can see this by expanding $\log(1+x) \equiv \log\left(1 + \dfrac{\Delta l}{l_i}\right)$ in an infinite series.

[6] The learned professor, Dr Bhoj Raj Seth (B.R. Seth), for long at IIT, Kharagpur, used to emphasise the absolute necessity of introducing the factor of half in these equations. We should realise that without this factor of half, the strain matrix will not have the transformation properties enjoyed (and required) by the strain tensor. A comparison of the transformation equations of stress components and strain components would give us this insight. Let me pause here to pay homage to this great teacher of ours with much pleasure and gratitude. What a great inspiration even his mere presence was!

[7] 'cross' derivatives: u with y, v with x, and so on

related to one another. We shall now obtain these relations. These equations are called the compatibility equations. There are six of them which are independent. They are of two kinds, each of three equations.

Three equations of one (the first) kind:

These can be obtained as follows. Starting from the strain displacement relations

$$e_{xx} = \frac{\partial u}{\partial x}; \qquad e_{yy} = \frac{\partial v}{\partial y}; \qquad \gamma_{xy} \equiv 2\, e_{xx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},$$

we obtain on performing the indicated differentiations,

$$\frac{\partial^2 e_{xx}}{\partial y^2} = \frac{\partial^2 u}{\partial y^2\, \partial x}; \qquad \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 v}{\partial x^2\, \partial y}; \qquad \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial x\, \partial y^2} + \frac{\partial^2 v}{\partial x^2\, \partial y},$$

which leads to the first of the three compatibility equations.

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y}.$$

The other two companion equations of this kind are obtained by cyclic change:
$x \;\to\; y; \quad y \;\to\; z; \quad z \;\to\; x.$

Three equations of the other (second) kind:

The other three equations of the other (second) kind can be obtained as shown below. Starting from the indicated derivatives

$$\frac{\partial^2 e_{xx}}{\partial y\, \partial z} = \frac{\partial^3 u}{\partial x\, \partial y\, \partial z}; \;\; \frac{\partial \gamma_{yz}}{\partial x} = \frac{\partial^2 v}{\partial x\, \partial z} + \frac{\partial^2 w}{\partial x\, \partial y}; \;\; \frac{\partial \gamma_{xz}}{\partial y} = \frac{\partial^2 u}{\partial y\, \partial z} + \frac{\partial^2 w}{\partial x\, \partial y}; \;\; \frac{\partial \gamma_{xy}}{\partial z} = \frac{\partial^2 u}{\partial y\, \partial z} + \frac{\partial^2 v}{\partial x\, \partial z},$$

we note that

$$2\, \frac{\partial^2 e_{xx}}{\partial y\, \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}.\right).$$

As before, the two companion equations are obtained by cyclic change:
$x \;\to\; y; \quad y \;\to\; z; \quad z \;\to\; x.$

All these six (6) compatibility equations, which are all independent, are displayed below.
Equations of compatibility:

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y}; \tag{12.3a}$$

$$\frac{\partial^2 e_{yy}}{\partial z^2} + \frac{\partial^2 e_{zz}}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y\, \partial z}; \tag{12.3b}$$

$$\frac{\partial^2 e_{zz}}{\partial x^2} + \frac{\partial^2 e_{xx}}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z\, \partial x}; \tag{12.3c}$$

$$2\, \frac{\partial^2 e_{xx}}{\partial y\, \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right); \tag{12.3d}$$

$$2\, \frac{\partial^2 e_{yy}}{\partial z\, \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{yx}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x}\right); \tag{12.3e}$$

$$2\, \frac{\partial^2 e_{zz}}{\partial x\, \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{zy}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y}\right). \tag{12.3f}$$

There is a physically more meaningful way of looking at the compatibility equations. Let us imagine that the undeformed body is divided into a grid-work of rectangular blocks (or brick elements). If arbitrary strains are imposed on these elements, they would go out of shape as demanded by the strain field imposed. If the strain field is recklessly specified, the deformed shapes will not close to form the deformed shapes with no tears or cracks. The jig-saw puzzle cannot be solved. The deformed shapes cannot combine to be a continuous (deformed) body any more. This makes it abundantly clear that the strains imposed cannot be arbitrary; some conditions will have to be satisfied. These conditions are the compatibility conditions. These, as stated earlier, may also be regarded as the integrability conditions for the overdetermined system of six (6) partial differential equations for the determination of the three (3) displacements.

In index notation these compatibility conditions are written as

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0, \qquad (i,\, j,\, k,\, l = 1, 2, 3). \tag{12.4}$$

Although there are $3 \times 3 \times 3 \times 3 = 81$ equations here, only six (6) of them are independent. [The explicit forms of the compatibility equations in a general curvilinear system are long and complicated. They can be worked out explicitly by interpreting the comma (,) as covariant differentiation using the methods of the general tensor analysis.]

We can verify the correctness of Eq. (12.4) by calculating the indicated derivatives. Let us calculate the various derivatives which appear in Eq. (12.4). From $e_{ij} = (1/2)[u_{i,j} + u_{j,i}]$, we obtain

$$\begin{aligned}
e_{ij,kl} &= \frac{1}{2}\left[u_{i,jkl} + u_{j,ikl}\right]; \\
e_{kl,ij} &= \frac{1}{2}\left[u_{k,lij} + u_{l,kij}\right]; \\
e_{ik,jl} &= \frac{1}{2}\left[u_{i,kjl} + u_{k,ijl}\right]; \\
e_{jl,ik} &= \frac{1}{2}\left[u_{j,lik} + u_{l,jik}\right].
\end{aligned}$$

The order of differentiation is immaterial; i.e., $u_{k,lij} = u_{kijl}$, etc. Then we can see that

$$\begin{aligned}
e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} =& \frac{1}{2}\left[u_{i,jkl} + u_{j,ikl} + u_{k,lij} + u_{l,kij}\right] \\
&- \frac{1}{2}\left[u_{i,kjl} + u_{k,ijl} + u_{j,lik} + u_{l,jik}\right] = 0.
\end{aligned}$$

The Role of Compatibility Equations

The role of the compatibility equations is merely to guarantee the existence of admissible displacements. By admissible we mean continuous displacements that are compatible. If the displacements are specified (that do not violate the obvious conditions of continuity, etc.), the compatibility conditions have no further role to play. In other words, the compatibility conditions will now be surely satisfied.

On the other hand, if we work with stresses, we have no direct information on whether the associated displacements (corresponding to the strain components that correspond to these stress components) will turn out to be compatible. Now the compatibility conditions do step in to stop or prevent inadmissible stress distributions (that correspond to strain distributions that correspond to impossible displacement fields).

Even at the risk of repetition, let us emphasise the following very important fact. Both the equations of equilibrium and the compatibility of displacements must be simultaneously satisfied. The equations of equilibrium are usually specified in terms of stress components, while the compatibility refers to the displacements[8]. For example, in solving the torsion problem (of non-circular, prismatic bars), the displacements are assumed in the Saint-Venant's theory. Then there is no further role for the compatibility equations here. On the other hand, we may decide to work in terms of Airy's stress function. When we choose an Airy's stress function, effectively, we choose the stress field. The equations of equilibrium are automatically satisfied, because that is how the Airy's stress function is cleverly defined. Now the compatibility condition does step in. This condition leads to the governing equation which is the biharmonic equation.

In conclusion, the compatibility conditions have only a passive role — their presence is not even felt or recognised at the operational level — when the displacement field is assumed. On the other hand, they have an active role when a stress field is assumed[9].

STRESS-STRAIN RELATIONS

We shall take up this topic again to give more details using the index notation. The nine stress components are related to the nine strain components by the equations

$$\sigma_{ij} = D_{ijkl}\, e_{kl} \quad \text{(D: stiffness tensor)}, \qquad e_{ij} = C_{ijkl}\, \sigma_{kl} \quad \text{(C: compliance tensor)},$$

where D and C are tensors of order four (4), called the stiffness tensor and the compliance tensor, respectively.. There are, thus, $3 \times 3 \times 3 \times 3 = 81$ elastic constants[10]. We may mention, in passing, that being tensors, both the stiffness tensor and the compliance tensor — both these qualify to be called constitutive tensors — obey the transformation laws

$$D'_{ijkl} = a_{ip}\, a_{jq}\, a_{kr}\, a_{ls}\, D_{pqrs}; \;\text{ and}$$

[8] There is, thus, a case for formulating the equations of equilibrium also in terms of displacements so that we can work entirely with displacement fields. This is the motivation for considering the Navier's (or Lamé-Navier) equations of equilibrium in terms of the displacements. On the other hand, we can decide to work entirely with the stress components, in which case the compatibility equations play an active role. This is the motivation for developing the Beltrami-Michell equations of compatibility in terms of the stress components.

[9] Readers who have difficulty to follow these statements are advised to read a good book on the theory of elasticity.

[10] If we assume that the body is *homogeneous*, the elastic properties are the same; they do not change from point to point. They are, thus, constants everywhere. More generally, they may be functions of position. Assumptions of homogeneity and isotropy are often made together, sometimes tacitly, in some books written in the strength of materials style. But we should not fail to see that these are quite different, and that the consequences are also, naturally enough, different.

$$C'_{ijkl} = a_{ip}\, a_{jq}\, a_{kr}\, a_{ls}\, C_{pqrs}.$$

Reduction from 81 to 36 Elastic Constants: Symmetry

The stress tensor, we know, is symmetric. Hence D and C are symmetric w.r.to the first two indices of D, and w.r.to the last two indices of C. The strain tensor is symmetric too. Hence D and C are symmetric w.r.to the last two indices of D, and w.r.to the first two indices of C. The number of elastic constants thus becomes 36.

$$\sigma_{ij} = \sigma_{ji} \quad \longrightarrow \quad D_{ijkl} = D_{jikl} \qquad \text{and} \qquad e_{ij} = e_{ji} \quad \longrightarrow \quad D_{ijkl} = D_{ijlk}$$

Strain Energy Density Function: 36 to 21 Elastic Constants

A strain energy density function $\mathcal{U} = \mathcal{U}(e_{ij})$ can be defined for a given state of stress and strain such that the stress and strain components are related by the equation

$$\sigma_{ij} = \frac{\partial \mathcal{U}}{\partial e_{ij}}, \quad (i,\, j = 1,\, 2,\, 3). \tag{12.5}$$

From $\mathcal{U} = \mathcal{U}(e_{ij})$, we may also write

$$\delta\mathcal{U} = \frac{\partial \mathcal{U}}{\partial\, \delta e_{ij}}\, \delta e_{ij} = \sigma_{ij}\, \delta e_{ij}. \tag{12.6}$$

On comparing Eqs (12.5) and (12.6),

$$\frac{\partial^2 \mathcal{U}}{\partial e_{ij}\, \partial e_{kl}} = \frac{\partial}{\partial e_{ij}} \left(\frac{\partial \mathcal{U}}{\partial\, \delta e_{kl}} \right) = \frac{\partial \sigma_{kl}}{\partial e_{ij}} = \frac{\partial^2 \mathcal{U}}{\partial e_{kl}\, \partial e_{ij}} = \frac{\partial}{\partial e_{kl}} \left(\frac{\partial \mathcal{U}}{\partial\, \delta e_{ij}} \right) = \frac{\partial \sigma_{ij}}{\partial e_{kl}}.$$

The equation

$$\frac{\partial \sigma_{kl}}{\partial e_{ij}} = \frac{\partial \sigma_{ij}}{\partial e_{kl}}$$

implies that the 6×6 constitutive matrix is symmetric. Putting, for example, $i = 1$, $j = 2$, $k = 3$, $l = 1$, we obtain

$$\frac{\partial \sigma_{31}}{\partial e_{12}} = \frac{\partial \sigma_{12}}{\partial e_{31}}.$$

Referring to Eq. (7.6) [p. 7-4], we can recognise the derivative on the left hand side to be the elastic constant relating $\sigma_{31} \equiv \tau_{zx}$ and $e_{12} \equiv e_{xy}$ which is D_{64}. The derivative on the right hand side can similarly be seen to be the elastic constant relating $\sigma_{12} \equiv \tau_{xy}$ and $e_{31} \equiv e_{zx}$ which is D_{46}. Thus, $D_{64} = D_{46}$. In this way, the number of elastic constants is further reduced to 21. The general anisotropic elastic material has 21 elastic constants.

Isotropy

Further reduction for the most important case of isotropic elasticity can be achieved in the following way. Firstly, let us note that the constitutive tensor, being of order four (4), will transform as

$$D'_{ijkl} = a_{ip}\, a_{jq}\, a_{kr}\, a_{ls}\, D_{pqrs}. \tag{12.7}$$

Secondly, let us realise that isotropic elasticity demands that the constitutive tensor is not affected at all by any rotation of the reference coordinate axes. That is to say,

$$D'_{ijkl} = D_{ijkl}. \tag{12.8}$$

Eqs (12.7) and (12.8) lead to the requirement

$$D_{pqrs} = a_{ip}\, a_{jq}\, a_{kr}\, a_{ls}\, D_{pqrs}. \tag{12.9}$$

Eq. (12.9) can be satisfied only if

$$D_{pqrs} = \lambda\, \delta_{pq}\, \delta_{rs} + \mu\, \delta_{pr}\, \delta_{qs} + \gamma\, \delta_{ps}\, \delta_{qs}, \tag{12.10}$$

where λ, μ and γ are elastic constants. We can verify the correctness of this statement by substituting Eq. (12.10) in Eq. (12.9).

[Here is another instance where the use of the index notation is not only convenient, but also almost indispensable in discussions of the kind shown above. Thus, there is a strong case for the young readers of this book to be comfortable with the index notation. The effort taken will not be wasted.]

SOME COMMENTS

We note that *every* problem in the mechanics of solids (and in the theory of elasticity) is statically indeterminate internally. That is to say, the equations of equilibrium alone will not be able to give us the solution[11]. Two requirements must always be uncompromisingly met: (i) each and every part of the body must be in equilibrium[12]; (ii) furthermore, the displacements (as a result of the deformations[13]) must be compatible, so that the deformed body stays as a whole without tears or cracks, or without one part of the body intruding into another.

The first condition (of equilibrium) is expressed naturally and conveniently in terms of stress components. The second condition (of compatibility of displacements) is on the displacements that correspond to the strain components (related by the strain-displacement relations) that correspond to the stress components (related by the stress-strain relations). One wonders if the solution of problems in the mechanics of solids would be less difficult if both conditions are written either (a) all in terms of the stress components, or (b) all in terms of the displacement components. Would this approach help?

[11] We may be tempted to state that only the equation of equilibrium is used to obtain the (uniformly distributed) stress $\sigma = P/A$ in the case of a one-dimensional uniform bar subjected to end loads $P - P$. Actually, among the various possibilities of stress distribution across the cross-section, all of which satisfying the equations of equilibrium (such as, say, (i) triangular, (ii) parabolic, (iii) sinusoidal, (iv) uniform stress on only *part* of the cross-section), only the uniform distribution is correct, because that is the *only* case for which the resulting displacements are compatible. It is edifying to relate the two prime, essential requirements, and see how these are met directly or indirectly by the various methods used. The educated reader would probably like to link the force and displacement methods of structural analysis, the strain energy and complementary energy methods, and the load bounding (upper and lower bound theorems in the theory of plasticity) techniques to what is stated above.

[12] It is not sufficient if the equilibrium condition is globally satisfied; that is, if the body as a whole is in equilibrium. Every tiny bit of the body must be in equilibrium.

[13] or when a displacement field is prescribed in advance

DISPLACEMENT FORMULATION

In the displacement formulation, the unknown field variables are the displacements. Thus, in this approach, we have to recast the differential equations of equilibrium in terms of displacements. The broad general procedure is as follows.

(i) As the first step, let us replace the stress components in the equations of equilibrium by the strain components (using the constitutive equations).

(ii) In the second step, we shall replace the strain components by the displacement components (using the strain-displacement relations).

Thus we shall obtain the differential equations of equilibrium in terms of the displacements, known as Navier's (or Lamé-Navier) equations. We shall now carry out this procedure.

Navier's (Lamé-Navier) Equations

As outlined above, let us first substitute the stress-strain relations (constitutive equations, generalised Hooke's law) $\sigma_{ij} = \lambda\,\delta_{ij}\,e_{kk} + 2\mu\,e_{ij}$ into the differential equations of equilibrium $\sigma_{ji,j} + X_i = 0$. This gives us

$$(\lambda\,\delta_{ij}\,e_{kk} + 2\mu\,e_{ij})_{,j} + X_i = 0,$$

which is rewritten as

$$\lambda\,\delta_{ij}\,e_{kk,j} + 2\mu\,e_{ij,j} + X_i = 0.$$

That is, using the substitution property $\delta_{ij}\,e_{kk,j} = e_{kk,i}$,

$$\lambda\,e_{kk,i} + 2\mu\,e_{ij,j} + X_i = 0, \tag{12.11}$$

which are the differential equations of equilibrium *in terms of the strain components.*

Now the strain components are all replaced by the displacement components using the strain-displacement relations $e_{ij} = \dfrac{1}{2}(u_{i,j} + u_{j,i})$ to give us the desired equation in the desired form after a few simplifications shown below.

$$\lambda\frac{1}{2}\left(u_{k,k} + u_{k,k}\right)_{,i} + 2\mu\frac{1}{2}\left(u_{i,j} + u_{j,i}\right)_{,j} + X_i = 0; \text{ that is,}$$

$$\lambda\,u_{k,ki} + \mu\,(u_{i,jj} + u_{j,ij}) + X_i = 0.$$

Changing the dummy index k in the first term to j, this equation appears as

$$\lambda\,u_{j,ji} + \mu\,u_{i,jj} + \mu\,u_{j,ij} + X_i = 0,$$

and as $u_{j,ji} = u_{j,ij}$, the order of differentiation being immaterial, the well known Navier's equation is obtained in the form

$$(\lambda + \mu)\;u_{j,ij} + \mu\,u_{i,jj} + X_i = 0. \tag{12.12}$$

This, in content, is the set of 3 (the free index $i = 1, 2, 3$) equations of equilibrium, but in terms of the displacements u_i which are functions of position.

Equations of Equilibrium in Longhand Notation

We shall obtain the equations of equilibrium in terms of the displacement components u, v, w. We shall now carry out this exercise below.

Consider the differential equations of equilibrium and the associated boundary conditions in terms of stresses (Cauchy's result).

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = 0; \qquad \overset{(\nu)}{T_x} = l\sigma_{xx} + m\tau_{yx} + n\tau_{zx};$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + F_y = 0; \qquad \overset{(\nu)}{T_y} = l\tau_{xy} + m\sigma_{yy} + n\tau_{zy};$$
$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z = 0; \qquad \overset{(\nu)}{T_y} = l\tau_{xz} + m\tau_{yz} + n\sigma_{zz}.$$

Let us substitute the expressions (from the constitutive relations) for the stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$

$$\sigma_{xx} = \lambda e + 2Ge_{xx}; \qquad \sigma_{yy} = \lambda e + 2Ge_{yy}; \qquad \sigma_{zz} = \lambda e + 2Ge_{zz} \qquad (e = e_{xx} + e_{yy} + e_{zz})$$

$$\tau_{xy} = \tau_{yx} = G\gamma_{xy} = G\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right); \qquad \tau_{xz} = \tau_{zx} = G\gamma_{xz} = G\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

Substituting these expressions for the stresses, the first equation of equilibrium becomes

$$(\lambda + G)\frac{\partial e}{\partial x} + G\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + F_x = 0.$$

Navier's (Lamé-Navier) equations:

The two companion equations also may be arrived at similarly. Thus, the three Navier's (or Lamé-Navier) equations — the equations of equilibrium in terms of the displacement components — are

$$(\lambda + G)\frac{\partial e}{\partial x} + G\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + F_x = 0; \tag{12.14a}$$
$$(\lambda + G)\frac{\partial e}{\partial y} + G\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + F_y = 0; \tag{12.14b}$$
$$(\lambda + G)\frac{\partial e}{\partial z} + G\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + F_z = 0. \tag{12.14c}$$

Cauchy's result:

Cauchy's result usually written in terms of the stress components now appears in terms of the displacement components as

$$\overset{(\nu)}{T_x} = \lambda e\, l + G\left(l\,\frac{\partial u}{\partial x} + m\,\frac{\partial u}{\partial y} + n\,\frac{\partial u}{\partial z}\right) + G\left(l\,\frac{\partial u}{\partial x} + m\,\frac{\partial v}{\partial x} + n\,\frac{\partial w}{\partial x}\right); \tag{12.15a}$$

$$\overset{(\nu)}{T}_y = \lambda e\, m + G\left(l\,\frac{\partial v}{\partial x} + m\,\frac{\partial v}{\partial y} + n\,\frac{\partial w}{\partial z}\right) + G\left(l\,\frac{\partial u}{\partial y} + m\,\frac{\partial v}{\partial y} + n\,\frac{\partial w}{\partial y}\right); \tag{12.15b}$$

$$\overset{(\nu)}{T}_z = \lambda e\, n + G\left(l\,\frac{\partial w}{\partial x} + m\,\frac{\partial w}{\partial y} + n\,\frac{\partial w}{\partial z}\right) + G\left(l\,\frac{\partial u}{\partial z} + m\,\frac{\partial v}{\partial z} + n\,\frac{\partial w}{\partial z}\right). \tag{12.15c}$$

Special simplified case, no body force:
The Laplacian in the above equations may be written as $\bigtriangledown^2$. If the body forces are absent, these equations are simplified as

$$(\lambda + G)\frac{\partial e}{\partial x} + G \bigtriangledown^2 u = 0; \tag{12.16a}$$

$$(\lambda + G)\frac{\partial e}{\partial y} + G \bigtriangledown^2 v = 0; \tag{12.16b}$$

$$(\lambda + G)\frac{\partial e}{\partial z} + G \bigtriangledown^2 w = 0. \tag{12.16c}$$

An important relationship:
We may also arrive at an important relationship. The volume dilatation (volume expansion) e satisfies the (3-dimensional) Laplace's equation

$$(\lambda + 2G) \bigtriangledown e = 0 \quad \longrightarrow \quad \bigtriangledown e \equiv \frac{\partial^2 e}{\partial x^2} + \frac{\partial^2 e}{\partial y^2} + \frac{\partial^2 e}{\partial z^2} = 0. \tag{12.17}$$

We can see this by differentiating Eq. (12.16a) w.r.to x, Eq. (12.16b) w.r.to y, Eq. (12.16c) w.r.to z, and adding them together. The result is valid not only if the body forces are absent, but also when they are constant throughout the volume of the body.

BELTRAMI-MICHELL EQUATIONS

We shall now see the Beltrami-Michell equations which are the compatibility equations expressed in terms of the stress components, convenient in an all-stress formulation.

Introduction

We had stated that, among others, there are two possibilities: (i) recast all the governing equations so that all the unknowns are in terms of (i) the displacements u, v, w, and (ii) the stress components. We have seen just now the first possibility (i). We shall presently examine the second one (ii).

The equations of equilibrium are written in terms of the stress components. There is nothing more to be done about them; they are already in good shape. However, the compatibility conditions, which are in terms of the strain components, are to be processed and to be recast in terms of the stress components. The Beltrami-Michell equations, the topic here, are the compatibility equations in terms of the stress components.

Compatibility Equations in Terms of the Stress Components

Our objective is to obtain the compatibility equations in terms of the stress components. We can accomplish this task in the following way.

The compatibility equations are usually written in terms of the strain components. However, it is possible to recast them in terms of stresses. We shall take up one compatibility equation, viz.,

$$\frac{\partial^2 e_{yy}}{\partial z^2} + \frac{\partial^2 e_{zz}}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y\, \partial z}, \tag{12.18}$$

and show how it is processed. In this equation, we replace the strain components by the stress ones (stress components) using the constitutive relations

$$\begin{aligned} e_{yy} &= \frac{1}{E}\left[(1+\nu)\sigma_{yy} - \nu\Theta\right]; \qquad \Theta = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}; \\ e_{zz} &= \frac{1}{E}\left[(1+\nu)\sigma_{zz} - \nu\Theta\right]; \\ \gamma_{yz} &= \frac{2(1+\nu)\tau_{yz}}{E}, \end{aligned}$$

where $\Theta \equiv I_1$, is the first stress invariant. On substitution, the above compatibility equation appears as

$$(1+\nu)\left(\frac{\partial^2 \sigma_{yy}}{\partial z^2} + \frac{\partial^2 \sigma_{zz}}{\partial y^2}\right) - \nu\left(\frac{\partial^2 \Theta}{\partial z^2} + \frac{\partial^2 \Theta}{\partial y^2}\right) = 2(1+\nu)\frac{\partial^2 \tau_{yz}}{\partial y\, \partial z}. \tag{12.19}$$

We have achieved what we set out to do. However, we proceed further and present this in the more familiar convenient form. Towards this end, let us process the right hand side of this equation using the equations of equilibrium.

$$\begin{aligned} \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{xz}}{\partial x} + F_z = 0 \quad &\longrightarrow \quad \frac{\partial \tau_{yz}}{\partial y} = -\frac{\partial \sigma_{zz}}{\partial z} - \frac{\partial \tau_{xz}}{\partial x} - F_z; \\ \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z} + F_y = 0 \quad &\longrightarrow \quad \frac{\partial \tau_{yz}}{\partial z} = -\frac{\partial \sigma_{yy}}{\partial y} - \frac{\partial \tau_{xy}}{\partial x} - F_y. \end{aligned}$$

Differentiating the first w.r.to z, and the second w.r.to y, and adding them, we obtain

$$\begin{aligned} 2\frac{\partial^2 \tau_{yz}}{\partial y\, \partial z} &= -\frac{\partial^2 \sigma_{zz}}{\partial z^2} - \frac{\partial^2 \sigma_{yy}}{\partial y^2} - \frac{\partial}{\partial x}\left(\frac{\partial \tau_{xz}}{\partial z} + \frac{\partial \tau_{xy}}{\partial y}\right) - \frac{\partial F_z}{\partial z} - \frac{\partial F_y}{\partial y} \\ &= \frac{\partial^2 \sigma_{xx}}{\partial x^2} - \frac{\partial^2 \sigma_{yy}}{\partial y^2} - \frac{\partial^2 \sigma_{zz}}{\partial z^2} + \frac{\partial F_x}{\partial x} - \frac{\partial F_y}{\partial y} - \frac{\partial F_z}{\partial z}. \end{aligned}$$

In the last step we have used the first equation of equilibrium (in the x-direction). Now on substituting this expression in the right hand side of Eq. (12.19), we obtain

$$(1+\nu)\left(\nabla^2\Theta - \nabla^2\sigma_{xx} - \frac{\partial^2 \Theta}{\partial x^2}\right) - \nu\left(\nabla^2\Theta - \frac{\partial^2 \Theta}{\partial x^2}\right) = (1+\nu)\left(\frac{\partial F_x}{\partial x} - \frac{\partial F_y}{\partial y} - \frac{\partial F_z}{\partial z}\right) \tag{12.20}$$

To make the equation short, we have used the notation

$$\nabla^2 \equiv \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}.$$

Two more equations of the same kind, the companion equations, can be obtained on similar lines. If we add up these three equations of the kind of Eq. (12.20), we obtain

$$(1-\nu)\,\bigtriangledown^2\,\Theta = -(1+\nu)\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right). \tag{12.21}$$

These correspond to the three compatibility conditions of the first kind. This last equation (12.21) gives the expression for $\bigtriangledown^2\Theta$. If we substitute this expression for $\bigtriangledown^2\Theta$ in Eq. (12.20), we obtain

$$\bigtriangledown^2\sigma_{xx}+\frac{1}{(1+\nu)}\frac{\partial^2\Theta}{\partial x^2} = -\frac{\nu}{(1-\nu)}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right)-2\frac{\partial F_x}{\partial x} \tag{12.22}$$

There are, of course, the two companion equations. These three correspond to the three compatibility conditions of the first kind. The other three compatibility conditions (of the second kind) can also be processed similarly, and recast in terms of the stress components. We obtain

$$\bigtriangledown^2\tau_{yz}+\frac{1}{(1+\nu)}\frac{\partial^2\Theta}{\partial y\,\partial z} = -\left(\frac{\partial F_z}{\partial y}+\frac{\partial F_y}{\partial z}\right), \tag{12.23}$$

and its two companion equations. If the body forces are absent, or if they are constant throughout the volume of the body, there is some simplification. The six equations of compatibility, now expressed in terms of the stress components appear as follows.

$$(1+\nu)\,\bigtriangledown^2\,\sigma_{xx}+\frac{\partial^2\Theta}{\partial x^2} = 0; \tag{12.24a}$$

$$(1+\nu)\,\bigtriangledown^2\,\sigma_{yy}+\frac{\partial^2\Theta}{\partial y^2} = 0; \tag{12.24b}$$

$$(1+\nu)\,\bigtriangledown^2\,\sigma_{zz}+\frac{\partial^2\Theta}{\partial z^2} = 0; \tag{12.24c}$$

$$(1+\nu)\,\bigtriangledown^2\,\tau_{yz}+\frac{\partial^2\Theta}{\partial y\,\partial z} = 0; \tag{12.24d}$$

$$(1+\nu)\,\bigtriangledown^2\,\tau_{xz}+\frac{\partial^2\Theta}{\partial x\,\partial z} = 0; \tag{12.24e}$$

$$(1+\nu)\,\bigtriangledown^2\,\tau_{xy}+\frac{\partial^2\Theta}{\partial x\,\partial y} = 0. \tag{12.24f}$$

Comments

In this all-stress formulation, the governing equations are (i) the equations of equilibrium and (ii) these six (6) compatibility equations in terms of stresses. The (traction) boundary conditions (Cauchy's results) must be satisfied. With these the stress components in a linearly elastic, isotropic body can be determined. These equations are *generally, but not always*[14], sufficient.

[14] These exceptional cases where they are not sufficient cannot be discussed here; the discussion will be at a much higher level, too high for this book.

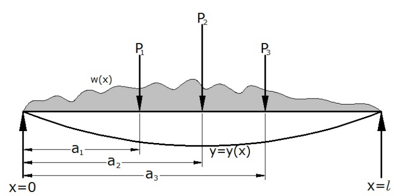
Let us note further these compatibility conditions contain no higher derivatives of the stress components than the second. What this means is that the stress field (satisfying the equations of equilibrium and the boundary conditions) gives the correct solution if the stress components are either constants or linear functions of the coordinates.

BOUNDARY CONDITIONS

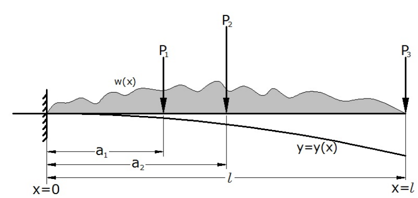
So far we have seen the field equations. These are the basis of the formulation of a stress analysis problem. However they are not all; they are only one part of the story. These equations are to be supplemented by the relevant boundary conditions which are almost as important. We shall discuss the boundary conditions here.

A comprehensive coverage of this topic may not be possible. We will be able to discuss only simple cases at an elementary level. First we shall see one-dimensional problems like a beam.

One-dimensional Problems: Beams

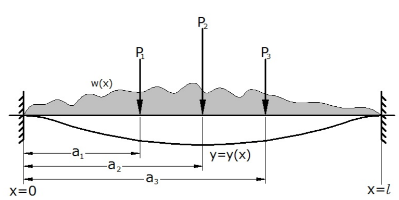


(a) A simply supported beam

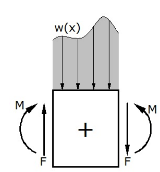


(b) A cantilever

Figure 12.3: The beam in Fig. 12.3a is simply supported at both ends, while that Fig. 12.3b, a cantilever, is fixed at the left end and free at the other. The deflection at a (i) supported end and (ii) a fixed end shall be zero. The slope at a fixed end is zero.



(a) A beam fixed at both ends



(b) An element of a beam

Figure 12.4: A fixed-fixed (fixed at both ends) beam is shown in Fig. 12.4a. The deflection and slope at each fixed end is zero. Fig. 12.4b shows a beam element with the load, the shear forces, and the bending moments marked on it in their respective positive directions.

Let us consider a beam loaded by a transverse load. Three cases are shown in Figs 12.3a (a simply supported beam), 12.3b (a cantilever), and 12.4a (a fixed-fixed beam). They are

all governed by the same linear differential equation[15].

$$EI\,\frac{d^2y}{dx^2} = -M, \quad (0 < x < l). \tag{12.25}$$

The two boundary conditions of this second order differential equation are:

$$\begin{aligned} \text{simply supported: [Fig. 12.3a]} &\quad y(0) = 0; \quad y(l) = 0; && (12.26a)\\ \text{cantilever: [Fig. 12.3b]} &\quad y(0) = 0;\; y'(0) = 0; && (12.26b)\\ \text{fixed-fixed: [Fig. 12.4a]} &\quad y(a) = 0; \quad y(b) = 0; \;\; y'(0) = 0; \;\; y'(l) = 0.. && (12.26c) \end{aligned}$$

The physical meaning of these boundary conditions are:

at a support (which is assumed to be rigid, non-yielding) the deflection $y = 0$; and
at a fixed end, the deflection and the slope are both zero: $y(0) = 0;\; y'(0) = 0$.

We know from our earlier study of the mechanics of solids that

$$\frac{dF}{dx} = -w; \qquad \frac{dM}{dx} = F; \qquad EI\,\frac{d^2y}{dx^2} = -M, \tag{12.27}$$

and that the bending of the beam is governed by the equation

$$\frac{d^2y}{dx^2}\left[EI\,\frac{d^2y}{dx^2}\right] = w, \tag{12.28}$$

where w, F, M are, respectively, the rate of loading, the shear force, and the bending moment. The differential equation (still linear, but could be a variable coefficient one) is now of order four (4). It can now take four (4) boundary conditions. What are these four boundary conditions? We shall discuss these with reference to a beam on an elastic foundation [Fig. 12.5].

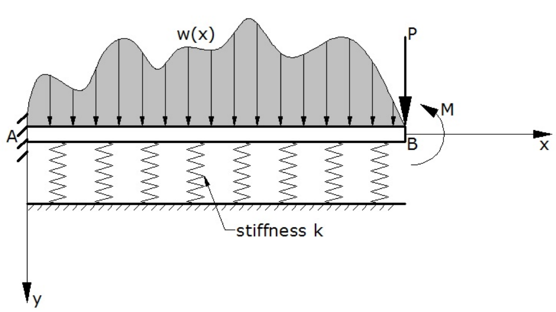


Figure 12.5: A beam on an elastic foundation

Fig. 12.5 shows a beam on an elastic foundation. The left end A $(x = 0)$ is fixed. Thus, the deflection and the slope there are both zero. Thus, these are two of the (four) boundary conditions, viz.,

$$y(0) = 0; \qquad y'(0) = 0.$$

The other two conditions refer to the bending moment (M) and the shearing force (F) at the free end B $(x = l)$.

[15] The right hand side may be written as M or as $-M$, depending on the sign convention used. Here we choose $-M$. This is consistent with the marking in Fig. 12.4b. The deflection is positive downwards.

The differential equation governing bending is

$$\frac{d^2y}{dx^2}\left[EI\,\frac{d^2y}{dx^2}\right] + ky = w(x) \quad (0 < x < l), \tag{12.29}$$

where k is the 'spring constant' ('Winkler constant' of the 'Winkler foundation').

Now we shall consider the four boundary conditions. These boundary conditions at the end B $(x = l)$ are:

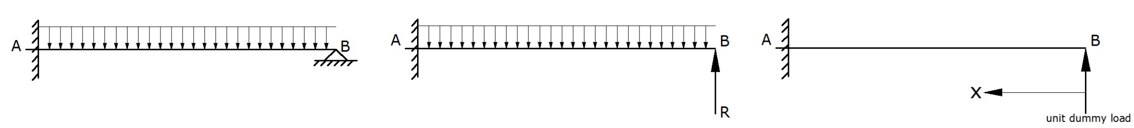
$$\left(EI\,\frac{d^2y}{dx^2}\right)\bigg|_{x=l} = -M \qquad \text{applied B.M. at } B \tag{12.30}$$

$$\frac{d}{dx}\left(EI\,\frac{d^2y}{dx^2}\right)\bigg|_{x=l} = -P \qquad \text{applied concentrated force at } B \tag{12.31}$$

These four (4) boundary conditions can, thus, be written down easily using the physical facts of the problem. We used only elementary facts to obtain these conditions.

There is a more insightful way of obtaining these boundary conditions and of looking at them. We can see that they are of two different kinds: (i) on the displacement (deflection) y and its derivative (slope) y', and (ii) on the bending moment (M) and its derivative (shear force) (F). These two kinds may be regarded as the primary and the secondary variables. To have this more enlightened view, we need to have a fair understanding of the calculus of variations where the two kinds of boundary conditions are referred to as (i) the prescribed and (ii) the natural boundary conditions. [There is nothing natural about these, nor anything artificial about those of the other kind; they are both technical terms. See [9] and [10], or better still, the series of books by the learned Professor J.N. Reddy.]

A Propped Cantilever: Rigid and Yielding Props



(a) A propped cantilever loaded by a u.d.l.

(b) Support removed and replaced by a reaction R

(c) A concentrated load R at the end

Figure 12.6: A propped cantilever AB with a uniformly distributed load, w per unit length propped by a rigid or yielding support. We use the principle of consistent deformations (compatibility of deformations) to calculate the support reaction, R.

It is not possible, or permissible, to specify both the applied force and the corresponding (work absorbing) displacement at the same point. In the language of the calculus of variations, both the prescribed and the corresponding natural boundary conditions cannot be specified at the same point. On the physical side this may be understood in the following way.

We have before us, let us say, a horizontal beam loaded by its self-weight. At a certain point, say at its midpoint, we can apply an upward load of, say, 1 kN from below to prop

it up. This may result in a reduced deflection. Instead of applying the specified 1 kN load, we may lift the beam at the same point by, say, 2 mm. This may require a force of, say, 0.5 kN. We cannot, can we, specify *both the force and the displacement* at the same point. If a load of 1 kN is applied, the stiffness of the structure decides how much the resulting displacement is. We cannot specify or realise both the applied load and the resultant displacement separately and independently. This is very clear from the physics of the problem; every engineer can understand it easily. But the interesting part of this is that these facts will be obtained automatically (by applied mathematical techniques) without appealing to the physical understanding of the problem. Another example of the beauty of applied mathematics in action! [An engineer can write down the boundary conditions correctly from his physical understanding of a problem. An applied mathematician also will be able to arrive at these conditions correctly. We who are bilingual — we speak both languages, the languages of both engineers and applied mathematicians — can not only understand both, but can really enjoy and appreciate how applied mathematics and engineering science support each other.]

The case of a propped cantilever is taken up in this sub-section. Fig. 12.6a shows a cantilever AB of length l, fixed at the end A and loaded by a uniformly distributed load w per unit length. It is propped at the end B. There are two cases to consider: (a) a rigid (unyielding) support, and (b) a yielding support which is the same as a spring of stiffness (spring constant) k. We need to discuss the boundary conditions at the ends.

Actually, this is not a question of writing the boundary conditions. Both cases are statically indeterminate. We need to make use of the principle of consistent deformations (compatibility of displacements) to solve the problem. Let us remove the support and introduce a concentrated force, which is the support reaction R.

Case (a) Rigid support:

Remove the force and calculate the end deflection due to the uniformly distributed load w per unit length. This is $w\,l^4/(8\,EI)$ downwards. Now the force R is such as to produce an end deflection $w\,l^4/(8\,EI)$ upwards, so that the total deflection is zero, as demanded by the condition of an unyielding support. Recalling that the end deflection of a cantilever due to an end load R is $R\,l^3/(3\,EI)$, we obtain

$$\frac{w\,l^4}{8\,EI} = \frac{R\,l^3}{3\,EI} \qquad \longrightarrow \qquad R = \frac{3}{8}\,wl.$$

This is the principle of consistent deformation. The problem is, thus, solved.

Case (b) Yielding support (spring support):

Let the end deflection be δ. This is now unknown. It is equal to R/k, where k is the spring constant. This deflection δ is the difference between the downward deflection due to the u.d.l., and the upward deflection because of the unknown support force R. Thus, the principle of consistent deformation is

$$\frac{w\,l^4}{8\,EI} - \frac{R\,l^3}{3\,EI} = \delta = \frac{R}{k},$$

from which the support force R can be readily calculated. This completes the solution of the problem stated.

Our last example in this set refers to the axial deformation of a one-dimensional bar.

Axial Deformation of a One-dimensional Bar

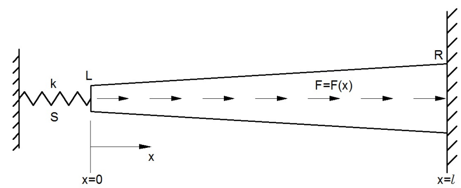


Figure 12.7: A one-dimensional bar with a spring S (stiffness k) at the left end $L\,(x=0)$, and fixed at the right end $R\,(x=l)$. The area of cross-section A (or more generally, EA) is variable. An external axial load $F=F(x)$ acts on the bar. An example of such an axial load might be the self-weight if the bar is vertical with its support at $R\,(x=l)$.

We had discussed this problem earlier [p. 5-10]. We desire to formulate this problem in terms of the axial displacement $u(x)$, and discuss the boundary conditions of the problem. The differential equation of equilibrium (5.15), we recall, was obtained as

$$\frac{d}{dx}\,(\sigma\,A) + F = 0, \qquad (0 < x < l). \tag{12.32}$$

We desire to recast this differential equation of equilibrium[16] in terms of the displacement u. We know that

$$\text{stress, } \sigma = E \times \text{ strain} = E\,\frac{du}{dx}.$$

If the stress σ in the above equation (12.32) is replaced by this expression in terms of the strain, we obtain the governing differential equation in the desired form as

$$\frac{d}{dx}\left(AE\,\frac{du}{dx}\right) + F = 0, \qquad (0 < x < l). \tag{12.33}$$

$$\text{Natural boundary condition:} \qquad AE\left(\frac{du}{dx}\right) - ku(0) = 0 \;\text{ at }\; x = l \tag{12.34}$$

$$\text{Prescribed boundary condition:} \qquad u(l) = 0. \tag{12.35}$$

The prescribed boundary condition (12.35) states that the right end of the bar is fixed and, consequently, there can be no (axial) displacement there; that is, $u(l) = 0$. The natural boundary condition (12.34) states that, at the left end of the bar, the force on the bar is the spring force.

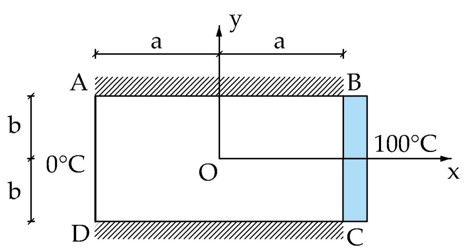
$$\text{spring force:} \quad = ku(0); \quad \text{axial force:} \quad = AE \times \text{strain} = AE\frac{du}{dx}; \quad AE\frac{du}{dx} = ku(0).$$

[16] Recall that we had called attention to such cases where the differential equation(s) is / are recast in terms of the displacement(s). This is one such case.

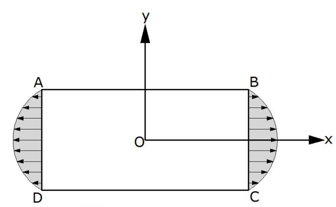
As indicated elsewhere these problems can be formulated using variational methods. The prescribed boundary conditions are easy to write, but the natural boundary conditions, which can be hard to write sometimes, emerge beautifully and effortlessly from a variational formulation.

Next we shall see the boundary conditions in two-dimensional problems.

Two-dimensional Problems



(a) Steady-state heat transfer in a plate



(b) A plate $ABCD$ subjected to stresses

Figure 12.8: A rectangular plate $ABCD$ is shown. In the first case [Fig. 12.8a] a steady-state heat transfer is considered, while in the second [Fig. 12.8b] the rectangular plate is subjected to tensile stresses as shown. The biharmonic equation is to be solved in the region shown.

Two-dimensional boundary values problems are associated with two-dimensional regions. There are several cases to be considered here, but we do not propose to be comprehensive. We shall consider only two examples of how the boundary conditions are written for such problems.

Biharmonic Equation: Two-dimensional Stress Analysis

We know from our earlier study of two-dimensional elasticity problems that the Airy's stress function is sometimes useful. The problem that we take up is to determine the stresses in a rectangular plate $ABCD$ subjected to a given stress distribution on the edges. [Fig. 12.8b] gives the details as a stress analysis problem, while [Fig. 12.8a] is to explain the corresponding boundary value problem in terms of the biharmonic function $\phi = \phi(x, y)$ in $\mathcal{R}$ when the boundary conditions are specified on $\mathcal{C}$.

We also know that the Airy's stress function $\phi = \phi(x, y)$ satisfies the biharmonic equation, and that the stress components are related to ϕ by

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}; \qquad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}; \qquad \tau_{xy} = \tau_{yx} = \frac{\partial^2 \phi}{\partial x\, \partial y} = \frac{\partial^2 \phi}{\partial y\, \partial x};$$

$$\bigtriangledown^2 \left(\bigtriangledown^2 \phi\right) = \bigtriangledown^4 \phi \equiv \frac{\partial^4 \phi}{\partial x^4} + 2\, \frac{\partial^4 \phi}{\partial x^2\, \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0.$$

The stress analysis problem is thus converted into, or reformulated as, a boundary value problem for the determination of $\phi = \phi(x, y)$ in the region $\mathcal{R}$.

$$\text{differential equation:} \quad \frac{\partial^4 \phi}{\partial x^4} + 2\,\frac{\partial^4 \phi}{\partial x^2\,\partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \ \text{ in } \ \mathcal{R}:$$

$$\begin{aligned}
\text{boundary conditions:} \quad & \text{edge AB} \quad \phi_{xx} = 0 \quad (\sigma_{yy} = 0);\\
& \text{edge AB} \quad \phi_{xy} = 0 \quad (\tau_{yx} = 0);\\
& \text{edge BC} \quad \phi_{xy} = 0 \quad (\tau_{xy} = 0);\\
& \text{edge BC} \quad \phi_{yy} = S\left(1 - \frac{y^2}{b^2}\right) \quad (\sigma_{xx} = S\left(1 - \frac{y^2}{b^2}\right));\\
& \text{edge CD} \quad \phi_{xy} = 0 \quad (\tau_{xy} = 0);\\
& \text{edge CD} \quad \phi_{xx} = 0 \quad (\sigma_{yy} = 0);\\
& \text{edge DA} \quad \phi_{xy} = 0 \quad (\tau_{xy} = 0);\\
& \text{edge DA} \quad \phi_{yy} = S\left(1 - \frac{y^2}{b^2}\right) \quad (\sigma_{xx} = S\left(1 - \frac{y^2}{b^2}\right));
\end{aligned}$$

We have written all the boundary conditions. This exercise is over at this stage[17].

Traction and Displacement Boundary Conditions

Stress analysis problems are generally formulated as boundary value problems. Thus, the boundary conditions expressed in terms of the unknown dependent variable are to be specified. In several, perhaps most, cases either the traction or the displacement is specified on the boundary. Sometimes, traction is specified on part of the boundary, and displacement(s) on the remaining part. It is not possible, or permissible, to specify both the traction and the (corresponding work absorbing) displacement at the same point.

If, for example, part of the boundary, say $\mathcal{C}_1$ is fixed, then the displacement boundary condition there is $u = v = w = 0$ on $\mathcal{C}_1 = 0$. If the traction is specified on another part, say $\mathcal{C}_2$, the traction boundary condition there is written using Cauchy's result as

$$\overset{(\nu)}{T_x} \text{ specified} = l\sigma_{xx} + m\tau_{yx} + n\tau_{zx};$$

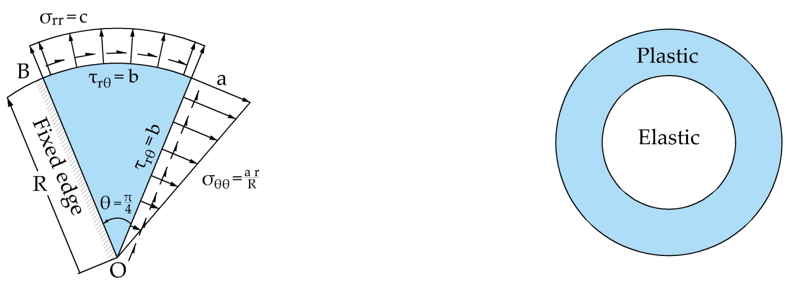
$$\overset{(\nu)}{T_y} \text{ specified} = l\sigma_{xy} + m\sigma_{yy} + n\tau_{zx};$$

$$\overset{(\nu)}{T_z} \text{ specified} = l\tau_{xz} + m\tau yz + n\sigma_{zz}.$$

A Sector of a Circle: Polar Coordinates

Next we shall consider a sector of a circle [Fig. 12.9a]. We shall use convenient polar coordinates (r, θ). The edge OB is fixed. The loadings on the straight edge OA and the curved edge AB are shown. The boundary conditions are written as follows.

[17] The boundary conditions are not all homogeneous; this is a little inconvenient in some cases as, for example, in seeking approximate solutions. We can make all the boundary conditions homogeneous by defining some quantities differently. We do not discuss them here as they are unnecessarily distracting.



(a) A sector OAB of a circle (b) An elastic-plastic interface

Figure 12.9: A circular sector OAB is shown [Fig. 12.9a]. The edge OB is fixed. The loadings on the other edges are shown. In the second case [Fig. 12.9b] the circular cylinder is partly, but not fully, in the plastic regime. There is an inner elastic core surrounded by a plastic region with an elastic-plastic interface.

i) **On the edge OB:** The edge OB is fixed. Hence the displacements u and v are both zero at all points on the edge.

$$u(r,\theta)\Big|_{\theta=\frac{\pi}{4}} \equiv u\left(r,\frac{\pi}{4}\right) = 0 \text{ for all } r \ \ (0 \le r \le R);$$
$$v(r,\theta)\Big|_{\theta=\frac{\pi}{4}} \equiv v\left(r,\frac{\pi}{4}\right) = 0 \text{ for all } r \ \ (0 \le r \le R).$$

ii) **On the edge OA:** The normal stress $\sigma_{\theta\theta}$ is linearly distributed as shown. The shear stress $\tau_{r\theta} = b$, a constant.

$$\sigma_{\theta\theta}(r,\theta)\Big|_{\theta=0} \equiv \sigma_{\theta\theta}(r,0) = a\,\frac{r}{R} \ \text{ for all } r \ \ (0 \le r \le R);$$
$$\tau_{r\theta}(r,\theta)\Big|_{\theta=0} \equiv \tau_{r\theta}(r,0) = b \ \text{ for all } r \ \ (0 \le r \le R).$$

iii) **On the edge AB:** The normal stress $\sigma_{rr} = c$, a constant. The shear stress $\tau_{r\theta} = b$, a constant.

$$\sigma_{rr}(r,\theta)\Big|_{r=R} \equiv \sigma_{rr}(R,\theta) = c \ \text{ for all } \theta \ \ (0 \le \theta \le \frac{\pi}{4});$$
$$\tau_{r\theta}(r,\theta)\Big|_{r=R} \equiv \tau_{r\theta}(R,\theta) = b \ \text{ for all } \theta \ \ (0 \le \theta \le \frac{\pi}{4}).$$

All the boundary conditions have now been written; this exercise is over here.

Special Cases and Concluding Remarks

There can be special cases not covered so far. One is when a concentrated load, say, P is applied at a point $(x = a)$. For this case, the rate of loading w can be written as

$$w = P\,\delta(a), \qquad \delta = 0 \ \text{ for } \ x \ne a, \qquad \int_{-\infty}^{+\infty} \delta(x)\,dx = 1,$$

where $\delta(a)$ is the Dirac delta 'function' (also called an impulse function). [This is not a function at all in the traditional sense. Mathematicians continued to object, while physicists continued to use it. How can a mathematicians accept the validity of this integral?

$$\int_{-\infty}^{+\infty} \delta(x)\, dx = 1, \qquad \delta = 0 \;\text{ for }\; x \neq a.$$

Now the use of the Dirac delta function is mathematically legitimatised by the relatively recent theory of distributions.]

Here is a second case [Fig. 12.9b] of an elastic-plastic interface. A circular, prismatic shaft is subjected to a torque T. When T is increased, the maximum shear stress on the cross-section at the outermost fibre reaches its elastic limit. If T is increased, plasticity will creep in; on further increase plasticity creeps further inside (with an outer plastic region surrounding an inner elastic core). For such a boundary value problem with an (unknown) interface, we have two sub-regions: (i) an outer ring where the material has already yielded and, consequently, plasticity conditions prevail, and (ii) an inner elastic core. Now what are the boundary conditions at the interface?

There can be other complications when we have an inclusion surrounded by an outer matrix with different properties. Yet another case is the problem of contact stresses (referred to as Hertzian stresses). We cannot obviously discuss all these cases here. At this stage we close this chapter.

In the next chapter we shall discuss some illustrative examples.

Chapter 13

A FEW ILLUSTRATIVE EXAMPLES

We shall work out several illustrative examples[1]. In some cases, only the conceptual part is discussed, and long laborious calculations are partially, or even entirely, left out. Use of MATLAB, MAPLE and / or MATHEMATICA is encouraged.

A FEW WORKED OUT EXAMPLES

A few examples — not all numerical ones — are worked out below for illustration. These are somewhat jumbled, not always in a logical order. It may help if the full procedure is explained and the steps shown before the detailed calculations are taken up.

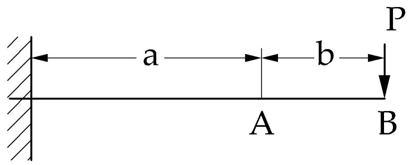
1. Example: Deflection of a Cantilever

Calculate the deflection at A when a concentrated force P acts at B on a cantilever.

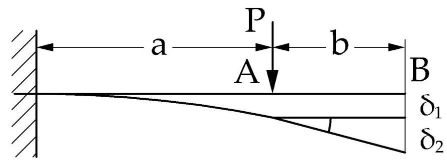
We know from the Betti-Maxwell reciprocal theorem[2] that this is equal to the deflection at B when the force P acts at A [Fig. 13.1b]. This deflection has two components δ_1 and

[1] Students, when writing the all important(!) 'university examinations' seem to be obsessed with the idea that numerical correctness of the final answer is all that matters. They have a good reason to think so. Over the years examiners often appear to attach too much importance for the correctness of the final answers, and too little for the correctness of the intermediate steps, and nothing at all for the conceptual understanding of the fundamentals. Furthermore, these questions seem to be a speed test. Actually, speed within reasonable limits should not be a factor at all. All this has created a warped thinking on the part of most students. Students are advised to pause, think, and write down the various steps before they begin the detailed calculations. It is natural for everybody, particularly the young students, to be nervous in the examination hall. They often spend too much time struggling with the detailed calculation of a relatively unimportant first step, and end up losing time. Students are advised to write down first the overall procedure before undertaking the detailed calculations. In this way they can guard against the pressure created by lack of time. Some of the problems take too long to work out fully. We hope that the examiners will be kind and considerate, and remember the days when they were students.

[2] The deflection at A due to a force acting at B is equal to the deflection at B due to the same force acting at A. The word force is used in a general sense; it can include moments also just as in the statement *Man is mortal*, the word man includes woman also! If the force is a moment, the corresponding work absorbing displacement is a rotation.



(a) A cantilever loaded at B



(b) The same cantilever loaded at A

Figure 13.1: The deflection at A of a cantilever loaded at B [Fig. 13.1] is obtained, using the Betti-Maxwell reciprocal theorem, by calculating the deflection at B when the load acts at A [Fig. 13.1b].

δ_2. The second component arises because of the slope at A. The deflected cantilever has no curvature in the region AB — it is a straight line set at an angle equal to the slope at A — because the bending moment in the region between A and B is zero[3]. Thus, the required deflection is $\delta = \delta_1 + \delta_2 = \dfrac{P\,a^3}{3\,EI} + \dfrac{P\,a^2}{2\,EI}\,b$. With a little experience these steps are gone through in the mind almost effortlessly, and the final expression can be written in just about the same time as it takes to write your name!

2. Example: Hydrostatic and Pure Shear States

Let us split the following stress matrix into two matrices representing (a) a hydrostatic state of stress, and (b) a state of pure shear. All the values are in MPa (N/mm^2).

$$\begin{bmatrix} 10 & 12 & 8 \\ 12 & 15 & -5 \\ 8 & -5 & -7 \end{bmatrix}$$

Let us decompose this as

$$\begin{bmatrix} 10 & 12 & 8 \\ 12 & 15 & -5 \\ 8 & -5 & -7 \end{bmatrix} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} + \begin{bmatrix} 10-p & 12 & 8 \\ 12 & +15-p & -5 \\ 8 & -5 & -7-p \end{bmatrix}.$$

The first matrix represents a state of hydrostatic stress for all values of p, but the second one represents a state of pure shear only if its first invariant is zero. Thus, we require that $[10-p]+[15-p]+[-7-p]=0$, giving us $p=6$. The required decomposition is

$$\begin{bmatrix} 10 & 12 & 8 \\ 12 & 15 & -5 \\ 8 & -5 & -7 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 12 & 8 \\ 12 & 9 & -5 \\ 8 & -5 & -13 \end{bmatrix}.$$

This completes the solution. But there are some important comments to make. Let us note that the first matrix represents a state of hydrostatic stress[4], and verify that the second one a state of pure shear: its first invariant is $[4+9-13=0]$. Let us note further

[3] Recall the formula $\dfrac{M}{I} = \dfrac{E}{R} \quad \longrightarrow \quad \dfrac{1}{R} = \dfrac{M}{EI}$. Where M is zero, the curvature $1/R$ is zero.

[4] Here it is tensile; we still use the word hydrostatic stress, because it is used in a figurative sense.

that normal stress elements are present in the second matrix even though it represents a state of pure shear. We can see — we can prove this if we desire, although we will not do it here — that there exists a (right handed cartesian) coordinate system (u, v, w) in which the second matrix appears with no normal stress as

$$\begin{bmatrix} 0 & \tau_{uv} & \tau_{uw} \\ \tau_{vu} & 0 & \tau_{vw} \\ \tau_{wu} & \tau_{wv} & 0 \end{bmatrix} \qquad \text{with } \tau_{uv} = \tau_{vu}, \, \tau_{vw} = \tau_{wv}, \tau_{wu} = \tau_{uw}.$$

3. Example: Expressions for σ and τ on Octahedral Planes
Obtain the expressions for the normal and shear stresses on the octahedral planes and, thereby, derive the expressions [Eqs (4.49a - 4.49d), (4.50a - 4.50c), p. 4-35]. We leave this for the readers to work out.

4. Example: Tresca's and von Mises' Criteria
Let us calculate the maximum torque that may be applied on a steel shaft of uniform diameter $d = 20$ mm based on the two criteria, Tresca's and von Mises'. The stress at which yielding — the onset of yielding is reckoned to be the threshold of failure — in a simple tension test is 150 MN/m^2. [We shall demonstrate through this example that the Tresca's criterion is about 15% more conservative than the von Mises' one.]
Tresca's criterion:
We know that the yield stress in shear, τ_y is one half the yield stress σ_y in tension. Equating the maximum shear stress in torsion to the yield strength in shear, we have

$$\tau_{max} = \frac{16\,T}{\pi\, d^3} = \frac{1}{2}\,\sigma_y.$$

Solving for the desired maximum torque, T, we obtain

$$T = \frac{\pi\, d^3}{32}\,\sigma_y = \frac{\pi(0.020)^3}{32} \times 150 \times 10^6 = 117.81 \text{ Nm} \quad \text{(Tresca).}$$

Von Mises' criterion:
The state of stress in torsion is

$$\begin{bmatrix} 0 & (\tau_{xy} = \tau_y) & 0 \\ (\tau_{yx} = \tau_y) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \longrightarrow \qquad \begin{bmatrix} \tau_y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tau_y \end{bmatrix}$$

The three principal stresses are $\sigma_{11} = \tau_y$; $\sigma_{22} = 0$; $\sigma_{33} = -\tau_y$. With these values of the principal stresses, Eq. 4.59 gives us

$$3\tau_y^2 = \sigma_y^2 \qquad \tau_y = 0.577\sigma_y. \tag{13.1}$$

Let us note that the curve bbbb in Fig. 4.20b cuts the vertical axis (y-axis, τ axis) at this value of 0.577. This is a little more that 15% higher than the value of 0.500, showing that the Tresca's criterion is about 15% more conservative than the von Mises' one.

The maximum torque T is, accordingly,

$$T = \frac{(0.577)\,\pi}{16}\, d^3\, \sigma_y = \frac{0.577\,\pi(0.020)^3}{16} \times 150 \times 10^6 = 117.81 \text{ Nm} \quad \text{(von Mises)}.$$

This vindicates the statement made earlier that the Tresca's criterion is about 15% more conservative than the von Mises' one.

5. Example: Stress Vector on an Inclined Plane

The state of stress at a point P in a stressed body is given by the stress matrix

$$\begin{bmatrix} 40 & 20 & 30 \\ 20 & 10 & -30 \\ 30 & -30 & -10 \end{bmatrix},$$

all in MPa. Calculate the stress vector (magnitude and direction) on a plane passing through P parallel to the plane $x_1 + 2x_2 - x_3 = 2$.

From the equation $x_1 + 2x_2 - x_3 = 2$, we obtain the direction ratios as $(1,\, 2,\, -1)$, and hence the direction cosines as

$$n_1 = \frac{1}{\sqrt{1^2 + 2^2 + (-1)^2}} = 0.408; \quad n_2 = \frac{2}{\sqrt{6}} = 0.816; \quad n_3 = \frac{-1}{\sqrt{6}} = -0.408.$$

[We do not really need to check if these are orthonormal ($n_1^2 + n_2^2 + n_3^2 = 0$), but it is always a good habit to check to be sure that there are no mistakes in the calculations.]

The stress resultants $\overset{(\nu)}{T_1}$, $\overset{(\nu)}{T_1}$, $\overset{(\nu)}{T_1}$ are, using Eq. (4.3) (Cauchy's result),

$$\begin{Bmatrix} \overset{(\nu)}{T_1} \\ \overset{(\nu)}{T_2} \\ \overset{(\nu)}{T_3} \end{Bmatrix} = \begin{bmatrix} 40 & 20 & 30 \\ 20 & 10 & -30 \\ 30 & -30 & -10 \end{bmatrix} \begin{Bmatrix} 0.408 \\ 0.816 \\ -0.408 \end{Bmatrix} = \begin{Bmatrix} 20.412 \\ 12.247 \\ -8.165 \end{Bmatrix} \quad \text{all in MPa.}$$

Thus, the stress vector on the plane is $\overset{(\nu)}{\boldsymbol{T}} = 20.412\,\boldsymbol{e}_1 + 12.247\,\boldsymbol{e}_2 - 8.165\,\boldsymbol{e}_3$ MPa.

This defines the resultant stress vector completely. Even so, let us calculate its magnitude and the angle that this makes with the normal to the plane. The magnitude is

$$\left|\overset{(\nu)}{\boldsymbol{T}}\right| = \sqrt{20.412^2 + 12.247^2 + (-8.165)^2} = 25.166 \text{ MPa}.$$

The angle between this resultant stress vector and the normal to the plane is given by

$$\begin{aligned} \cos^{-1} \frac{\overset{(\nu)}{\boldsymbol{T}} \,.\, \boldsymbol{\nu}}{\left|\overset{(\nu)}{\boldsymbol{T}}\right| |\boldsymbol{\nu}|} &= \cos^{-1} \frac{(20.412\,\boldsymbol{e}_1 + 12.247\,\boldsymbol{e}_2 - 8.165\,\boldsymbol{e}_3) \,.\, (0.408\,\boldsymbol{e}_i + 0.816\,\boldsymbol{e}_2 - 0.408\,\boldsymbol{e}_3)}{25.166} \\ &= 0.861 \qquad \longrightarrow \qquad \text{the angle } = 30.57^\circ. \end{aligned}$$

The resultant stress is not along the normal. This fact does not surprise us at all.

6. Example: Normal and Shearing Stresses on an Inclined Plane

The state of stress at a point P in a stressed body w.r.to the xyz coordinate system is given by the stress matrix

$$\begin{bmatrix} 30 & 15 & -10 \\ 15 & 20 & 5 \\ -10 & 5 & 10 \end{bmatrix},$$

all in MPa. Determine the normal stress, the shear stress, and the magnitude of the resultant stress vector at this point on the plane $3x - y + 2z = 5$.

Procedure:

(a) First calculate the direction ratios and then the direction cosines $l.m, n$ of the plane.
(b) Calculate $\overset{(\nu)}{T}_x$, $\overset{(\nu)}{T}_x$, $\overset{(\nu)}{T}_x$ using Cauchy's result.
(c) Calculate the magnitude of the resultant stress vector.
(d) Calculate the normal stress on the ν plane.
(e) Calculate shearing stress on the ν plane.

We shall now carry out the calculations.

(a) If the equation to the plane is $x - 2y + z = 5$, we know that the direction ratios are $(3, -1, 2)$. The direction cosines l, m, n of the normal ν to the plane are

$$l = \frac{3}{\sqrt{3^2 + (-1)^2 + 2^2}} = 0.802; \qquad m = \frac{-2}{\sqrt{14}} = -0.535; \qquad n = \frac{2}{\sqrt{14}} = 0.267.$$

(b) Using Cauchy's result

$$\overset{(\nu)}{T}_x = l\sigma_{xx} + m\tau_{yx} + n\tau_{zx} = [0.802 \times 30] + [(-0.267) \times 15] + [0.535 \times (-10)] = 16.699 \text{ MPa};$$
$$\overset{(\nu)}{T}_y = l\tau_{xy} + m\sigma_{yy} + n\tau_{zy} = [0.802 \times 15] + [(-0.267) \times 20] + [0.535 \times 5] \qquad = 9.354 \text{ MPa};$$
$$\overset{(\nu)}{T}_z = l\tau_{xz} + m\tau_{yz} + n\sigma_{zz} = [0.802 \times (-10)] + [(-0.267) \times 5] + [0.535 \times 10] \quad = -4.009 \text{ MPa}.$$

(c) Magnitude of the resultant stress vector

$$= \sqrt{\left[\overset{(\nu)}{T}_x\right]^2 + \left[\overset{(\nu)}{T}_y\right]^2 + \left[\overset{(\nu)}{T}_z\right]^2} = \sqrt{(16.699)^2 + (9.354)^2 + (-4.009)^2} = 17.878 \text{ MPa}.$$

(d) Magnitude of the normal stress $\sigma_{\nu\nu} = l^2\,\sigma_{xx} + m^2\,\sigma_{yy} + n^2\,\sigma_{zz} + 2lm\,\tau_{xy} + 2mn\,\tau_{yz} + 2nl\,\tau_{zx} = [0.802^2 \times 30] + [(-0.267)^2 \times 20] + [0.535^2 \times 10] = 7.150$ MPa.

(e) Magnitude of the shear stress $\tau_{(\nu)} = \sqrt{17.878^2 - 7.150^2} = 16.386$ MPa.

7. Example: Principal Stresses

The state of stress at a point P in a stressed material is given by the stress matrix

$$\begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 45 & 15 & 15 \\ 15 & 0 & 30 \\ 15 & 30 & 0 \end{bmatrix} \text{ MPa.}$$

Determine the principal stresses at the point.

Procedure:
(a) Evaluate the three stress invariants I_1, I_2, I_3.
(b) Write down the characteristic equation and find its roots.
(c) Arrange them in the algebraically decreasing order. These are the principal stresses.
(d) Calculate the sum of the principal stresses and verify that it is an invariant.
(e) Solve using MAPLE and verify that the answers obtained are correct.

We shall now carry out the calculations.

(a) The three stress invariants are

$$\begin{aligned}
I_1 &= \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 45 + 0 + 0 = 45 \text{ MPa};\\
I_2 &= \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2\\
&= (45 \times 0) + (0 \times 0) + (0 \times 45) - 15^2 - 30^2 - 15^2 = -1350 \text{ (MPa)}^2;\\
I_3 &= \sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx}\\
&= (45 \times 0 \times 0) - (45 \times 30 \times 30) - (0 \times 15 \times 15) - (0 \times 15 \times 15) - (2 \times 15 \times 30 \times 15)\\
&= 27000 \text{ (MPa)}^3.
\end{aligned}$$

The characteristic equation of the stress matrix is

$$\begin{aligned}
&\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0;\\
\text{i.e.,}\quad &\sigma^3 - 45\sigma^2 - 1350\sigma + 27000 = 0 \qquad \longrightarrow \qquad (\sigma - 15)(\sigma^2 - 30\sigma - 1800) = 0.
\end{aligned}$$

The roots of this characteristic equation (which are the eigenvalues of the stress matrix) are 15, 60 and −30, all in MPa.

The principal stresses are the eigenvalues of the stress matrix. These are

$$\sigma_1 = 60 \text{ MPa}; \quad \sigma_2 = 15 \text{ MPa}; \quad \sigma_3 = -30 \text{ MPa}; \quad \text{(algebraically descending order).}$$

(Generally, the *algebraically* largest — here it also happens to be the numerically largest — principal stress is designated as σ_1, and the *algebraically* lowest designated as σ_3.)

8. Example: Principal Planes

Find the principal planes of the principal stress of the problem above.

The problem of locating the principal planes is the same as finding the eigenvectors of the given stress matrix. This is the same as finding the non-trivial solutions for l, m, n from the linear, simultaneous, algebraic, homogeneous equations

$$\begin{bmatrix} (\sigma_{xx} - \sigma) & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & (\sigma_{yy} - \sigma) & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & (\sigma_{zz} - \sigma) \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{13.2}$$

Equivalently,

$$(\sigma_{xx} - \sigma)\, l + \tau_{yx}\, m + \tau_{zx}\, n = 0;$$

$$\tau_{xy}\, l + (\sigma_{yy} - \sigma)\, m + \tau_{zy}\, n = 0;$$
$$\tau_{xz}\, l + \tau_{yz} + (\sigma_{zz} - \sigma)\, n = 0.$$

For our problem, the eigenvalues have already been calculated as

$$\sigma_1 = 60 \text{ MPa}; \quad \sigma_2 = 15 \text{ MPa}; \quad \sigma_3 = -30 \text{ MPa}; \quad \text{(algebraically descending order)}.$$

Eigenvector corresponding to $\sigma_1 = 60$ MPa.
The relevant linear, simultaneous, algebraic, homogeneous equations are

$$(45 - 60)\, l + 15\, m + 15\, n = 0;$$
$$15\, l + (0 - 60)\, m + 30\, n = 0;$$
$$15\, l + 30\, m + (0 - 60)\, n = 0.$$

First note that we can obtain only the ratios. Why? Because now the solution is not unique. We can know this from linear algebra in a proper way. A less satisfactory, though effective, method is to note that, if a, b, c is a solution, $2a, 2b, 2c$ also is a solution. And so is any multiple of a, b, c. If we can obtain only the ratios, we can assume $l = 1$, and find the corresponding values of m and n.

If we substitute $l = 1$ in these equations, we obtain

$$-15 + 15\, m + 15\, n = 0;$$
$$15 - 60\, m + 30\, n = 0;$$
$$15 + 30\, m - 60\, n = 0.$$

Here are three equations in only two unknowns m and n. Now there arises the question of consistency. Are these equations consistent[5]? From the first two equations we find

$$-15 + 15\, m + 15\, n = 0;$$
$$15 - 60\, m + 30\, n = 0.$$

These equations, when solved (which is very easy), give us $m = n = 1/2$. Thus, the direction ratios are (1, 0.5, 0.5), which when normalised give us the first eigenvector as $[0.816 \quad 0.408 \quad 0.408]^T$.

Eigenvector corresponding to $\sigma_2 = 15$ MPa.
The relevant linear, simultaneous, algebraic, homogeneous equations now are

$$(45 - 15)\, l + 15\, m + 15\, n = 0;$$
$$15\, l + (0 - 15)\, m + 30\, n = 0;$$
$$15\, l + 30\, m + (0 - 15)\, n = 0.$$

[5] Sure, they are? How can we be sure? Consider the condition for consistency. The rank of the coefficient matrix must be equal to the rank of the augmented matrix. Readers are advised to revise linear algebra. Such situations occur again and again. It, therefore, pays to learn linear algebra in the right royal way.

Again, as before, let us put $l = 1$ in these equations. Taking, say, the first two equations we find

$$30 + 15\,m + 15\,n = 0;$$
$$15 - 15\,m + 30\,n = 0.$$

These equations, when solved (which is very easy), give us $m = n = -1$. Thus, the direction ratios are (1, −1, −1), which when normalised give us the second eigenvector as $[-0.577 \quad 0.577 \quad 0.577]^T$.

Eigenvector corresponding to $\sigma_3 = -30$ MPa.
The relevant linear, simultaneous, algebraic, homogeneous equations now are

$$\begin{aligned}(45 + 30)\,l + 15\,m + 15\,n &= 0;\\ 15\,l + (0 + 30)\,m + 30\,n &= 0;\\ 15\,l + 30\,m + 30\,n &= 0.\end{aligned}$$

This time, let us put $n = 1$[6]. Taking, say, the first two equations we find

$$75\,l + 15\,m = -15;$$
$$15\,l + 30\,m = -30.$$

These equations, when solved (which is very easy), give us $l = 0, m = -1$. Thus, the direction ratios are (0, −1, 1), which when normalised give us the third eigenvector as $[0 \quad -0.707 \quad 0.707]^T$.

Now that the eigenvalues (principal stresses) are distinct ($\sigma_1 \neq \sigma_2 \neq \sigma_3$), the corresponding eigenvectors are mutually orthogonal. Physically, this means that now the principal planes are mutually perpendicular[7].

It is desirable — soon it would become essential — for students to learn how to use Computer Algebraic Systems like MATHEMATICA and MAPLE.
As a motivation to become familiar with Computer Algebraic Systems, the solution by MAPLE is given below.

Solution (of the Last Two Examples) by MAPLE

If we use MAPLE, the solution of the last two examples — finding the principal stresses and the principal planes, which is the same as finding the eigenvalues and eigenvectors of the given stress matrix — is only one click away. The answers, given below, are displayed in a screen shot [Fig. 9.20]. The numbers are to be interpreted properly. What is obtained are the direction ratios; the direction cosines can be easily calculated from them.

$$\sigma_1 = 60 : \{2 \quad 1 \quad 1\}^T; \qquad \sigma_2 = 15 : \{-1 \quad 1 \quad 1\}^T; \qquad \sigma_3 = -30 : \{0 \quad -1 \quad 1\}^T.$$

[6] We can give any value to any one of l, m, n. What would happen if we put $l = 1$? Examine and find out.

[7] If the eigenvalues are not distinct, the corresponding eigenvectors may not be orthogonal. However, it is always possible (by the Gram-Schmidt procedure) to find a set of three mutually orthogonal principal directions. These are important; students are advised to look up all this in two or three *good* books. Geometrical visualisation (say, using Lamé's ellipsoid) will also be insightful. See [10] or, better still, [5].

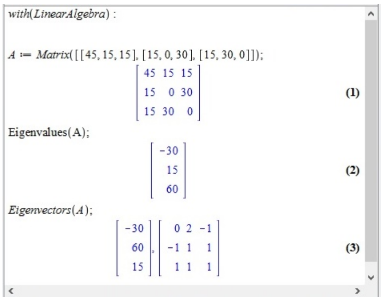


Figure 13.2: MAPLE solution: screen display. The eigenvalues are: $\sigma_1 = 60$; $\sigma_2 = 15$, $\sigma_3 = -20$, all in MPa.

Fig. 13.2 shows a the screen display of the MAPLE solution. The eigenvectors are the direction ratios; they are to be normalised to obtain the direction cosines. Further, as the eigenvalues are distinct (are all different) ($\sigma_1 \neq \sigma_2 \neq \sigma_3$, arranged in the algebraically decreasing order), the corresponding eigenvectors (representing the principal planes) are mutually orthogonal. That is, the principal planes are mutually orthogonal! Let us also note that $(0 \quad -1 \quad 1)^T$ is the same eigenvector as $(0 \quad 1 \quad -1)^T$.

The direction cosines are the normalised direction ratios. Thus, we have

$$\frac{2}{\sqrt{2^2+1^2+1^2}} = 0.816, \text{etc.} \qquad \sigma_1 = 60 : \{0.816 \quad 0.408 \quad 0.408\}^T;$$
$$\sigma_2 = 15 : \{-0.577 \quad 0.577 \quad 0.577\}^T; \qquad \sigma_3 = -30 : \{0 \quad -0.707 \quad 0.707\}^T.$$

As the principal stresses are distinct, the corresponding direction ratios will be mutually orthogonal. Thus, we verify that

$$\begin{array}{llll} \text{(a):} & \mathbf{1.2}: \longrightarrow & [(0.816)\times(-0.577)] + [0.408\times 0.577] + [0.408\times 0.577] & = 0; \\ \text{(b):} & \mathbf{2.3}: \longrightarrow & [(-0.577)\times 0] + [0.577\times(-0.707)] + [0.577\times 0.707] & = 0; \\ \text{(c):} & \mathbf{3.1}: \longrightarrow & [0\times 0.816] + [(-0.707\times 0.408] + [0.707\times 0.408] & = 0. \end{array}$$

9. Example: Strain Transformation Equations

If the coordinate axes are changed from (x, y) to (x', y') — θ is the angle between the x and the x' axes — show by coordinate transformation that

$$e_{x'x'} = e_{xx}\cos^2\theta + e_{yy}\sin^2\theta + e_{xy}\sin 2\theta.$$

The old and the new axes related by[8]

$$\begin{array}{ll} x = x'\cos\theta - y'\sin\theta; & \qquad u' = u\cos\theta + v\sin\theta; \\ y = x'\sin\theta + y'\cos\theta; & \qquad v' = -u\sin\theta + v\cos\theta. \end{array}$$

The strain-displacement relations are $e_{xx} = \partial u/\partial x$, $e'_{x'x'} = \partial u'/\partial x'$, etc. Thus, by the chain rule of partial differentiation, we have

$$e_{x'x'} = \frac{\partial u'}{\partial x'} = \frac{\partial u'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial x'};$$

[8] Draw two sets of axes $(x, y$ and (x', y') and obtain the given equations. (u, v) and (u', v') are the displacement components in the old and new coordinate systems. The results follow from the geometry.

$$e_{y'y'} = \frac{\partial v'}{\partial y'} = \frac{\partial v'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial y'};$$
$$e_{x'y'} = \frac{1}{2}\left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}\right) = \frac{1}{2}\left(\frac{\partial v'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial x'}\right) + \frac{1}{2}\left(\frac{\partial u'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial y'}\right).$$

Thus, we obtain

$$\begin{aligned} e_{x'x'} &= \left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial v}{\partial x}\sin\theta\right)\cos\theta + \left(\frac{\partial u}{\partial y}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right)\sin\theta \\ &= \frac{\partial u}{\partial x}\cos^2\theta + \frac{\partial v}{\partial y}\sin^2\theta + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\sin\theta\cos\theta \\ &= e_{xx}\cos^2\theta + e_{yy}\sin^2\theta + e_{xy}\sin 2\theta. \end{aligned}$$

10. Example: Strains from Stresses

Calculate the strain matrix corresponding to the given stress matrix (all in MPa), if $E = 200$ GPa and $\nu = 0.3$.

$$\begin{bmatrix} 150 & 50 & -60 \\ 50 & 100 & -70 \\ -60 & -70 & -80 \end{bmatrix} \longrightarrow \begin{bmatrix} 0.720 & 0.325 & -0.390 \\ 0.325 & 0.395 & -0.455 \\ -0.390 & -0.455 & -0.775 \end{bmatrix} \times 10^{-3}.$$

The strain components are calculated from the generalised Hooke's law. $G = E/[2(1+\nu)] = 200/2.6 = 76.923$ GPa.

$$\begin{aligned} e_{xx} &= \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] = \frac{1}{200\times 10^3}[150 - 0.3(100 - 80)] = 0.72\times 10^{-3}; \\ e_{yy} &= \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})] = \frac{1}{200\times 10^3}[100 - 0.3(150 - 80)] = 0.395\times 10^{-3}; \\ e_{xx} &= \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{zz} + \sigma_{xx})] = \frac{1}{200\times 10^3}[-80 - 0.3(150 + 100)] = -0.775\times 10^{-3}; \\ e_{xy} &= \frac{1}{2}\gamma_{xy} = \frac{\tau_{xy}}{2G} = \frac{50}{2\times 76.923} = 0.325\times 10^{-3}; \\ e_{yz} &= \frac{1}{2}\gamma_{yz} = \frac{\tau_{yz}}{2G} = \frac{-70}{2\times 76.923} = -0.455\times 10^{-3}; \\ e_{zx} &= \frac{1}{2}\gamma_{zx} = \frac{\tau_{zx}}{2G} = \frac{-60}{2\times 76.923} = -0.390\times 10^{-3}. \end{aligned}$$

The strain matrix, obtained from the generalised Hooke's law, is displayed alongside the given stress matrix.

11. Example: Stresses from Strains

Given the strain matrix shown in the example above, calculate the stress matrix. If our calculations are correct, we must obtain the stress matrix given in the previous example.

Now we know the strain components. We again use the generalised Hooke's law, but in the form given below, and calculate the stress components. Let us first calculate λ, e, and G.

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} = 115.385\,\text{MPa};$$

$$e = e_{xx} + e_{yy} + e_{zz} = 0.720 = 0.395 - 0.775 = 0.340 \times 10^{-3};$$
$$G = \frac{E}{2(1+\nu)} = 76.923\,\text{GPa}.$$

With these values, the stress components are worked out as below.

$$\begin{aligned}
\sigma_{xx} &= \lambda e + 2Ge_{xx} = 150\,\text{MPa};\\
\sigma_{yy} &= \lambda e + 2Ge_{yy} = 100\,\text{MPa};\\
\sigma_{zz} &= \lambda e + 2Ge_{zz} = -80\,\text{MPa};\\
\tau_{xy} &= 2Ge_{xy} = 50\,\text{MPa};\\
\tau_{yz} &= 2Ge_{yz} = -70\,\text{MPa};\\
\tau_{zx} &= 2Ge_{zx} = -60\,\text{MPa}.
\end{aligned}$$

We obtain the original stress matrix that we started out with in the example above. We note that e is the first strain invariant, and that λ and G are the Lamé's constants.

12. Example: Transformation of the Strain Matrix

The strain matrix referred to the (x, y, z) coordinates is given in the previous example. Find the strain matrix in the 'new' coordinate system specified by the angles in the table below which shows how the new axes are disposed relative to the old ones.

New coordinates	Old coordinates x_1	x_2	x_3
x'_1	30° $\left(\boldsymbol{\frac{\sqrt{3}}{2}}\right)$	60° $\left(\boldsymbol{\frac{1}{2}}\right)$	90° **(0)**
x'_2	120° $\left(\boldsymbol{-\frac{1}{2}}\right)$	30° $\left(\boldsymbol{\frac{\sqrt{3}}{2}}\right)$	90° **(0)**
x'_3	90° **(0)**	90° **(0)**	0° **(1)**

In three dimensions it is more convenient to work with the direction cosines and not with the angles. The direction cosines in bold within brackets are shown in the table.

13. Example: Compatibility

Is the given two-dimensional strain field possible?

$$e_{xx} = (x^2 + 2y^2 + 3xy + 2) \times 10^{-3}; \qquad e_{yy} = (2x^2 + y^3 + x^2y^2 + 3) \times 10^{-3};$$
$$\gamma_{xy} = [2xy(x^2y + xy^2) + 1] \times 10^{-3}$$

To answer this question, we need to check if the following compatibility equation is satisfied.

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

Let us calculate the various partial derivatives in the compatibility equation.

$$\frac{\partial^2 e_{xx}}{\partial y^2} = 4 \times 10^{-3}; \quad \frac{\partial^2 e_{yy}}{\partial x^2} = (4 + 2y^2) \times 10^{-3}; \quad \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = (12x^2 y + 12xy^2) \times 10^{-3}.$$

As the compatibility equation is not satisfied, the given strain field is not possible.

14. Example: Compatibility

Is the strain field given below possible?

$$e_{xx} = 4x^2 y; \quad e_{yy} = 2y^2 x; \quad e_{xy} = 3xy + 2x^3.$$

To answer this, we check if the compatibility equation is satisfied. Let us calculate the terms in the compatibility equation

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}.$$

$$\frac{\partial^2 e_{xx}}{\partial y^2} = 0; \quad \frac{\partial^2 e_{yy}}{\partial x^2} = 0; \quad \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 6.$$

The compatibility equation is not satisfied; it is impossible to have the given strain field.

15. Example: Rigid Body Displacements

The displacement field is given by the following equations. What do the last terms within the flower brackets $\{\cdots\}$ represent? The C's and u_0, v_0, w_0 are constants. Refer to Fig. 11.7b, p. 11-7.

$$u = \frac{P}{A}\left[C_{13}\, x + 2\, C_{34}\, y + 2\, C_{36}\, z\right] + \{(+\omega_2\, z - \omega_3\, y) + u_0\}\,; \tag{13.3a}$$

$$v = \frac{P}{A}\left[C_{23}\, y + 2\, C_{35}\, z\right] + \{(+\omega_3\, x - \omega_1\, z) + v_0\}\,; \tag{13.3b}$$

$$w = \frac{P}{A}\left[C_{33}\right] + \{(+\omega_1\, y - \omega_2\, x) + w_0\}\,. \tag{13.3c}$$

The quantities u_0, v_0, w_0 in the above equations are constants. They represent translations of the body in the x, y, z directions, respectively. Consider the term $-\omega_3\, y$ in Eq. (13.3a). This represents a rotation about the z axis. Refer to Fig. 11.7a, p. 11-7. If we draw line, say, OP and rotate it about the z axis, we notice that the rotation introduces u and v displacements. A little reflection would convince us that this ω_3 represents a rotation about the z axis. In the same manner we can see that ω_1, ω_2 represent rotations about the x and y axes, respectively. It is useful to be aware of these facts.

16. Example: Rotation and Strains

When a (deformable) body rotates about an axis (say, the z axis) through an angle θ — this is a rigid body rotation — there are displacements u and v in the x and y directions, respectively. Calculate the strains and rotation ω_z if the displacements are $u = u(x, y) = (\cos\theta - 1)\, x - (\sin\theta)\, y; \quad v = v(x, y) = (\sin\theta)\, x + (\cos\theta - 1)\, y.$

The strains and rotations are

$$\begin{aligned}
e_{xx} &= \frac{\partial u}{\partial x} = (\cos\theta - 1);\\
e_{yy} &= \frac{\partial v}{\partial y} = (\cos\theta - 1);\\
e_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\sin\theta + \sin\theta = 0;\\
\omega_z &= \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = -\sin\theta - \sin\theta = -2\,\sin\theta.
\end{aligned}$$

The shear strain is zero, but the rotation is not.

17. Example: Strains and Rotations

Verify that the displacement field given corresponds to no strain, but only to rotations.

$$u = 3y + 2z: \quad v = -3x + z; \quad w = -2z - y.$$

Let us calculate the strain components given by:

$$e_{xx} = \frac{\partial u}{\partial x} = 0; \quad e_{yy} = \frac{\partial v}{\partial y} = 0; \quad e_{zz} = \frac{\partial w}{\partial z} = 0; \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 3 - 3 = 0;$$
$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial x} = -2 + 2 = 0; \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial y} = 1 - 1 = 0.$$

The strains are all zero. We shall presently see that the rotations are not zero.

$$\omega_{xy} = \frac{1}{2}\left[\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right] = 3 \neq 0; \quad \omega_{yz} = \frac{1}{2}\left[\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right] = 1 \neq 0;$$
$$\omega_{zx} = \frac{1}{2}\left[\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right] = -2 \neq 0.$$

18. Example: Some Special Cases of Displacement Fields

Four different displacement fields are given below. Calculate the corresponding strains / strain fields. What special cases do they represent? (a and λ are constants.)

$$\begin{aligned}
&\text{(a):} \quad u = \lambda x; \;\; v = w = 0;\\
&\text{(b):} \quad u = 2ay; \;\; v = w = 0;\\
&\text{(c):} \quad u = \lambda x; \;\; v = \lambda y; \;\; w = \lambda z;\\
&\text{(d):} \quad u = u(x, y); \;\; v = v(x, y); \;\; w = 0.
\end{aligned}$$

To answer this question, let us first calculate the various strain components. Then we can see what special case each of them represents.

Case (a):

$$e_{xx} = \frac{\partial u}{\partial x} = \lambda; \;\; e_{yy} = \frac{\partial v}{\partial y} = 0; \;\; e_{zz} = \frac{\partial w}{\partial z} = 0;$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 : \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 : \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0.$$

This, then, represents a simple extension in the x direction.
Case (b):

$$e_{xx} = \frac{\partial u}{\partial x} = 0; \quad e_{yy} = \frac{\partial v}{\partial y} = 0; \quad e_{zz} = \frac{\partial w}{\partial z} = 0;$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2a : \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 : \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0.$$

This, therefore, represents a simple case of shearing.
Case (c):

$$e_{xx} = \frac{\partial u}{\partial x} = \lambda; \quad e_{yy} = \frac{\partial v}{\partial y} = \lambda; \quad e_{zz} = \frac{\partial w}{\partial z} = \lambda;$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0; \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 : \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0.$$

What case does this case represent? Well, a uniform dilatation $e_{xx} + e_{yy} + e_{zz} = 3\lambda$. Each (normal) strain components is equal to $\lambda/3$.
Case (d):

$$e_{xx} = \frac{\partial u}{\partial x}; \quad e_{yy} = \frac{\partial v}{\partial y}; \quad e_{zz} = \frac{\partial w}{\partial z} = 0;$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}; \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 : \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0.$$

Now e_{xx}, e_{yy} and γ_{xy} are all present in general. The components e_{zz}, γ_{xz}, γ_{yz} are all zero. This is the general case of a state stress known as plane strain.

19. Example: Solution of an Elasticity Problem?
Examine if the following stress field represents the solution of a problem in elasticity.

$$\sigma_{xx} = a\left[y^2 + \nu(x^2 - y^2)\right]; \quad \sigma_{yy} = a\left[x^2 + \nu(y^2 - x^2)\right]; \quad \sigma_{zz} = a\nu(x^2 + y^2);$$
$$\tau_{xy} = -2a\nu\, xy; \quad \tau_{yz} = \tau_{zx} = 0.$$

Let us first check if these stress components satisfy the equations of equilibrium (with no body forces).

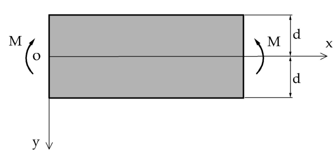
$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0 \quad \longrightarrow \quad 2a\nu\, x - 2a\nu\, x + 0 = 0 \quad \text{(satisfied)};$$
$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \quad \longrightarrow \quad 2a\nu\, y - 2a\nu\, y + 0 = 0 \quad \text{(satisfied)};$$
$$\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \quad \longrightarrow \quad 0 + 0 + 0 = 0 \quad \text{(satisfied)}.$$

Yes, the equations of equilibrium are satisfied. However, it is premature to make any conclusion yet. This is a stress formulation. The compatibility conditions play a vital role; their role is not passive now. The compatibility conditions are written in terms of the strain components. To check if the compatibility equations are satisfied, we therefore need to work out the expressions for the strain components. We shall obtain the expressions for the strain components using the constitutive equations (generalised Hooke's laws).

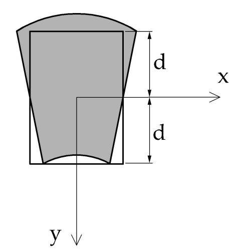
$$
\begin{aligned}
e_{xx} &= \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz}] = \cdots ; \\
e_{yy} &= \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx}] = \cdots ; \\
e_{zz} &= \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy}] = \cdots ; \\
\cdots &= \cdots
\end{aligned}
$$

Now we can check if the compatibility equations are satisfied. We can see that *all* the compatibility equations are not satisfied. Hence we conclude that the given stress field cannot represent the solution of any elasticity problem. Readers are advised to complete the solution and to convince themselves that the statement made above is correct.

20. Example: Pure Bending of a Uniform Beam of Rectangular Cross-section



(a) A uniform beam in pure bending



(b) The cross-section

Figure 13.3: A uniform beam of rectangular cross-section in pure bending. Fig. 13.3b shows the cross-section before and after bending.

Solve the problem of pure bending of a uniform beam of rectangular cross-section [Fig. 13.3]. The usual notations are followed.

From our earlier knowledge of the theory of bending, the stress components are

$$\text{bending stress } \sigma_{xx} = \frac{My}{I}; \quad \sigma_{yy} = \sigma_{zz} = 0; \quad \tau_{xy} = \tau_{yz} = \tau_{zx} = 0.$$

From the generalised Hooke's law, we find that

$$e_{xx} = \frac{My}{I}; \qquad e_{yy} = e_{zz} = -\nu\frac{My}{EI}; \qquad e_{xy} = e_{yz} = e_{zx} = 0.$$

Let us take this as a guess of the displacement field. We can then use the strain-displacement equations and calculate the displacements.

$$e_{xx} = \frac{\partial u}{\partial x} = \frac{My}{EI}; \qquad e_{yy} = \frac{\partial v}{\partial y} = -\nu\frac{My}{EI}; \qquad e_{zz} = \frac{\partial w}{\partial z} = -\nu\frac{My}{EI}; \tag{13.4}$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0; \quad e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0; \quad e_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = 0. \tag{13.5}$$

These equations are now integrated to yield

$$u = \frac{M}{EI}xy \qquad\qquad +\{(\omega_2\, z - \omega_3\, y) + u_0\}; \tag{13.6a}$$

$$v = -\frac{M}{2\,EI}[x^2 + \nu(y^2 - z^2)] + \{(\omega_3\, x - \omega_1\, z) + v_0\}; \tag{13.6b}$$

$$w = -\frac{\nu M}{EI}yz \qquad\qquad +\{(\omega_1\, y - \omega_2\, x) + w_0\}. \tag{13.6c}$$

We note that the displacements can be determined only to within rigid body displacements that do not contribute to strains and, therefore, to stresses. The terms $\{\cdots\}$ are these rigid body displacements, u_0, v_0, w_0 representing translations, and $\omega_1, \omega_2, \omega_3$ rotations, along / about the x, y, z axes, respectively. This is the general form representing rigid body displacements. We have seen this before (p. 13-12).

The rigid body translations can be arrested by forcing the point O to be fixed; that is, by demanding that $u = v = w = 0$ there at O ($x = y = z = 0$). Furthermore, we can assume that (i) an element of the $x-$axis at the fixed end is zero, and that (ii) an element of the xy plane is also fixed. These conditions at $x = y = z = 0$,

$$u = v = w = 0; \qquad \frac{\partial v}{\partial x} = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0$$

give the displacement field — the rigid body displacements are now arrested — as

$$u = \frac{M}{EI}xy;$$
$$v = -\frac{M}{2\,EI}[x^2 + \nu(y^2 - z^2)];$$
$$w = -\frac{\nu M}{EI}yz.$$

We have thus arrived at a displacement field that appears to be reasonable. We can be sure that this gives us a correct solution if all the governing equations are satisfied[9]. By the uniqueness theorem we are also sure that this is the *only* correct solution.

[9] (i) Calculate the strain field using the strain-displacement relations.
(ii) Calculate the stress field using the constitutive equations (generalised Hooke's law).
(iii) Check to make sure that all the equations of equilibrium are satisfied.
The boundary conditions must be satisfied too. The stress distributions at the ends must lead to a moment M. All these can be seen to be satisfied.

21. Strain: Change in Length

A rectangular plate $ABCD$ of length 400 mm (along AB) and width 300 mm (along AD is subjected to a uniform two-dimensional strain field: $e_{xx} = 0.002$; $e_{yy} = 0.003$; $\gamma_{xy} = 0.001$. The x and y coordinate axes are along the edges AB and AD of the plate with its origin at the lower left corner A of the plate. Calculate the length of the line element AC (diagonal of the block) before and after deformation.

Length of the diagonal AC (before deformation) is obviously $\sqrt{4^2 + 3^2} = 500$ mm. It is evident that the direction cosines (l, m, n) of the line element AC are $l = \cos 30\,^\circ = 0.866$; $m = \cos 60\,^\circ = 0.500$; $n = \cos 90\,^\circ$. We use the strain transformation equation to calculate the strain along AC. This gives us

$$\begin{aligned} e_{AC} &= l^2\, e_{xx} + m^2\, e_{yy} + n^2\, e_{zz} + 2lm\, e_{xy} + 2mn\, e_{yz} + 2nl\, e_{zx} \\ &= 0.866^2 \times 0.002 + 0.500^2 \times 0.003 + 2 \times 0.866 \times 0.500 \times \left(\frac{0.001}{2}\right) \\ &= 0.00268. \\ \text{length } A'C' &= 500 \times 1.00268 = 501.341 \text{ mm.} \end{aligned}$$

The length of the line element $A'C' = 501.341$ mm. This is the length after deformation. (The line AC before deformation becomes $A'C'$ after deformation.)

22. Example: Taylor & Quinney Experiments

We have already seen the results of the famous Taylor & Quinney experiments [Fig. 4.20b]. The experimental results lie between the predictions of the Tresca's (curve aaaa) and the von Mises' (curve bbbb) criteria. Both these curves aaaa and bbbb are ellipses. We shall now obtain the equations to these ellipses.

For the test specimen in Taylor & Quinney experiments on thin-walled tubes, if we take the x- and y-axes along the axis of the tube and perpendicular to it [Fig. 4.20a] respectively, the stress components are

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}.$$

The three principal stresses and the maximum shear stress are

$$\begin{aligned} \sigma_{11} &= \frac{\sigma_{xx}}{2} + \sqrt{\left(\frac{\sigma_{xx}}{2}\right)^2 + \tau_{xy}^2}; \\ \sigma_{22} &= 0; \\ \sigma_{33} &= \frac{\sigma_{xx}}{2} - \sqrt{\left(\frac{\sigma_{xx}}{2}\right)^2 + \tau_{xy}^2}; \\ \tau_{max} &= \frac{\sigma_{11} - \sigma_{33}}{2} = \sqrt{\left(\frac{\sigma_{xx}}{2}\right)^2 + \tau_{xy}^2}. \end{aligned}$$

(a) Tresca's criterion:
Thus, the equation that governs the Tresca's criterion is

$$\sqrt{\left(\frac{\sigma_{xx}}{2}\right)^2 + \tau_{xy}^2} = \frac{\sigma_y}{2},$$

which gives the equation to the ellipse as

$$\left(\frac{\sigma_{xx}}{\sigma_y}\right)^2 + 4\left(\frac{\tau_{xy}}{\sigma_y}\right)^2 = 1 \qquad \text{(curve aaaa in Fig. 4.20b).} \tag{13.7}$$

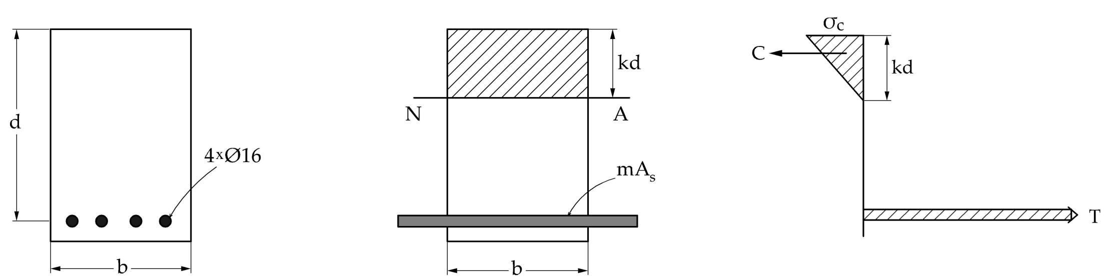
(b) Von Mises' criterion:
The equation that represents the von Mises' criterion is likewise

$$\left[\frac{\sigma_{xx}}{2} + \sqrt{\left(\frac{\sigma_{xx}}{2}\right)^2 + \tau_{xy}^2}\right]\left[\frac{\sigma_{xx}}{2} - \sqrt{\left(\frac{\sigma_{xx}}{2}\right)^2 + \tau_{xy}^2}\right] + \left[2\sqrt{\left(\frac{\sigma_{xx}}{2}\right)^2 + \tau_{xy}^2}\right]^2 = 2\sigma_y^2,$$

which, on simplification, is the equation to the ellipse as

$$\left(\frac{\sigma_{xx}}{\sigma_y}\right)^2 + 3\left(\frac{\tau_{xy}}{\sigma_y}\right)^2 = 1 \qquad \text{(curve bbbb in Fig. 4.20b).} \tag{13.8}$$

23. Example: A Beam of Two Materials



(a) Cross-section (b) Tension and compression areas (c) Force resultants T and C

Figure 13.4: The cross-section of an RCC beam (dimensions in mm). Four (4) reinforcing steel rods, the neutral axis, the 'transformed area' $m\,A_s$, and the force resultants (T, C) are shown. The last figure on the right shows that the tension in the concrete is neglected. The strain distribution across the cross-section is still linear [Fig. 2.5b, p. 2-7].

Examine the behaviour of a reinforced cement concrete (RCC) beam in bending[10]. The dimensions of the cross-section are $d = 500$ mm (effective depth) and $b = 230$ mm (width). The concrete is M20; there are four (4) steel bars, each of 16 mm diameter. The bending moment applied is $M = 50$ kNm.

[10] I am grateful to Professors K.V. Leela of Vidya and Devdas Menon of IIT Madras, both Professors of Structural Engineering for their help in this example.

The beam (cross-section) is clearly of two materials, viz., concrete and steel. The cross-sectional area A_s of the steel bars is multiplied by a factor $m = E_s/E_c$ to obtain the 'transformed area' (or the 'equivalent area') reckoned from the point of view of that there is only one material[11]. Generally the presence of tensile stress in the concrete (in the tension zone) is disregarded. The neutral axis will not pass through the mid-section; it will be at a depth of kd as shown in the figure.

As always the equilibrium requirements are:

i) $\int_A \sigma \, dA = 0 \quad \longrightarrow \quad$ (compressive force=tensile force; net axial force = 0)

ii) $\int_A \sigma \, y \, dA = M \quad \longrightarrow \quad$ (moment = applied bending moment)

Equating the moments about the neutral axis, we obtain

$$b \times k\,d \times \frac{k\,d}{2} = m\,A_s(d - k\,d), \tag{13.9}$$

where $m\,A_s$ is the 'transformed area' (or the 'equivalent area'), the value of m as per codal provisions being

$$m = \frac{280}{3\,(\sigma_c)_{bc}},$$

where $(\sigma_c)_{bc}$ is the allowable stress in concrete in bending compression. For $M20$ concrete this is (from $IS\,456$) $(\sigma_c)_{bc} = 7\,\mathrm{N/mm^2}$.

$$m = \frac{280}{3 \times 7} = 13.33 \text{ (which may be taken as 13 as per the relevant code.)}$$
$$k = 0.3485 \text{ which defines where the neutral axis is.}$$

The equilibrium condition, total compressive force = total tensile force, gives us

$$\frac{\sigma_c}{2} \times k\,d \times b = \sigma_s \times A_s \tag{13.10}$$

$$\frac{\sigma_c}{(\sigma_s/m)} = \frac{k\,d}{d - k\,d}. \tag{13.11}$$

The equation, moment of resistance $= M$, gives us

$$\sigma_s \times A_s\left(d - \frac{kd}{3}\right) = M = 50\,\mathrm{kNm}. \tag{13.12}$$

From these equations (13.9) or [(13.10) and (13.11)], the factor k can be found out. Its value is obtained as $k = 0.3485$. Using Eq. (13.12) we obtain the stresses as

$$\sigma_c = 5.78\ \mathrm{N/mm^2}, \quad \sigma_s = 140.68\ \mathrm{N/mm^2}.$$

[11] This ratio is nominally $m = E_s/E_c$. In reinforced cement concrete calculations, some modification is made as specified by the code: $m = \dfrac{280}{(\sigma_c)_{bc}}$. See the comment at the end of this example.

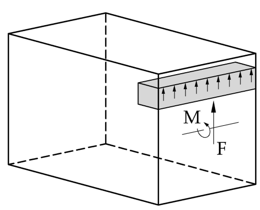
Comments: Reinforced concrete calculations are not quite the same as those in the mechanics of solids. There are several details that cannot be discussed here. There are codal provisions. These codes are arrived at, recommended, and incorporated as mandatory provisions by official bodies with the support of subject experts based on (i) deep analytical studies, (ii) experimental results, (iii) professional experience, and (iv) sound engineering judgement.

If we use the concept of a transformed area, its width will exceed the width of the concrete beam. The empirical formula for m is to accommodate for creep in concrete which effectively enhances the modular ratio with time. In structural design practice, this working stress design approach is outdated. A limit state design approach at collapse involving nonlinear stress-strain behaviour of concrete and steel is used.

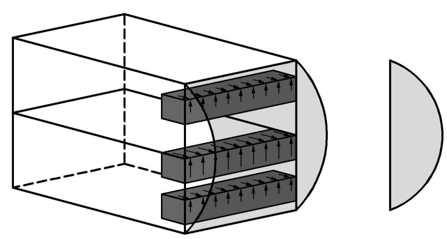
For example, we know that the stress-strain relationship of concrete is nonlinear. That is to say, the slope of the stress-strain curve representing the modulus of elasticity is different at different points (at different stress levels). The factor m, which is theoretically the ratio $m = E_s/E_c$ is, thus, dependent on the stress level. These aspects are all acknowledged and looked at by the experts. Although there are several such modifications specified by the codes, the science of the mechanics of solids is still the guiding light and the fundamental basis of these calculations.

24. Example: Cross-shear in a Rectangular Cross-section

Find the shear stress distribution on the rectangular cross-section of a beam in bending.



(a) A beam element with a $b \times d$ section



(b) Parabolic shear stress distribution

Figure 13.5: The figures show how the shear stress in bending (sometimes called cross-shear) occurs. Fig. 13.5a shows a rectangular ($b \times d$) cross-section. Fig. 13.5b shows the parabolic shear stress distribution across the rectangular cross-section.

Fig. 13.5a shows the rectangular cross-section of the beam. An element $abcd$ is shown. From Eq. (2.22, p. 2-16) we borrow the result

$$\begin{aligned}
\tau = \frac{F\,Q}{I\,b} &= \frac{F}{I\,b}\int_{abcd} y\,dA = \frac{F}{I\,b}\int_{y_1}^{d/2} y\,(b\,dy) \\
&= \frac{F}{I}\frac{y^2}{2}\Big|_{y_1}^{d/2} = \frac{F}{2\,I}\left[\left(\frac{d}{2}\right)^2 - y_1^2\right]
\end{aligned}$$

This is the distribution of the shear stress (parabolic). The maximum is at the neutral axis. To find this value, we note that $I = bd^3/12$ and $y_1 = 0$. The maximum is thus

$$\tau_{max} = \frac{F}{2 \times bd^3/12}\left[\frac{d^2}{4}\right] = \frac{3\,F}{2\,bd} = \frac{3\,F}{2\,A}. \tag{13.13}$$

F/A is the 'average' shear stress on the cross-section. The maximum is 50% higher than this average value[12]. This result is important.

25. Same Example: Alternative Method

We shall solve this problem again by an alternative method. We begin with the (differential) equation of equilibrium with no body force.

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0 \qquad \longrightarrow \qquad \frac{\partial \tau_{yx}}{\partial y} = -\frac{\partial \sigma_{xx}}{\partial x}. \tag{13.14}$$

But we already know how σ_{xx}, the bending stress, is distributed. It is given by (the Euler-Bernoulli equation)

$$\sigma_{xx} = \frac{M\,y}{I} \qquad \longrightarrow \qquad \frac{\partial \sigma_{xx}}{\partial x} = \frac{\partial M}{\partial x}\,\frac{y}{I} = \frac{F\,y}{I} \qquad \left(\frac{\partial M}{\partial x} = F, \text{ the shear force}\right).$$

Substituting this in Eq. (13.14), we obtain

$$\frac{\partial \tau_{yx}}{\partial y} = -\frac{\partial \sigma_{xx}}{\partial x} = \frac{y}{I} = -\frac{F\,y}{I}. \tag{13.15}$$

On integration, this yields

$$\tau_{yx} = -\frac{F\,y^2}{2\,I} + C. \tag{13.16}$$

This constant of integration C is evaluated using the boundary condition

$$\tau_{yx}\Big|_{(y=\frac{d}{2})} = \tau_{yx}\Big|_{(y=-\frac{d}{2})} = 0, \tag{13.17}$$

because at a boundary point — (at the end $y = d/2$ and also at $y = -d/2$) — the shear stress τ_{yx} complementary to τ_{xy} cannot exist on a free surface. Applying the boundary condition in Eq. (13.16), we obtain

$$\tau_{yx}\Big|_{(y=\pm\frac{d}{2})} = 0 \qquad \longrightarrow \qquad 0 = -\frac{F(d/2)^2}{2\,I} + C \qquad \longrightarrow \qquad C = \frac{F\,d^2}{8\,I}.$$

[12] Popov [11] remarks: "Eq. (13.13) is very useful. It is widely used in the design of wooden beams since the shear strength of wood on planes parallel to the grain is small. Thus, although equal shear stresses exist on mutually perpendicular planes, wooden beams have a tendency to split longitudinally along the neutral axis. Note that the maximum shear stress is 1.5 times as great as the *average shear stress* F/A. Nevertheless, in the analysis of bolts and rivets, it is customary to determine their shear strengths by dividing the shear force F by the cross-sectional area A. $\cdots$".

The final answer is that the shear stress on the cross-section $\tau_{xy} = \tau_{yx}$ is distributed as

$$\tau_{xy} = \tau_{yx} = \frac{F}{2\,I}\left[\left(\frac{d}{2}\right)^2 - y^2\right] \qquad \text{(parabola)}$$

with its maximum at the neutral axis as

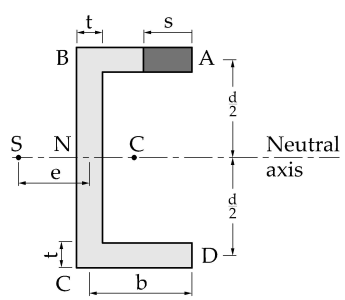
$$\tau_{max} = \frac{3}{2}\frac{F}{A} \quad \text{as before} \quad (I = bd^3/12;\ A = b\,d).$$

Comment: We know that *all* problems in the theory of elasticity are statically indeterminate internally. But here we obtained the solution using only the equation of equilibrium. How was this possible? Where is the mistake? What is the explanation?

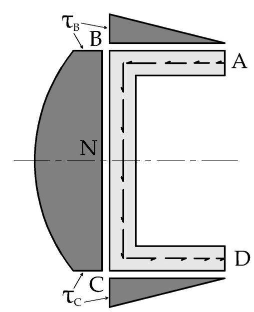
Here we are not solving the problem in the right royal way of the mathematical theory of elasticity. Instead we are following an engineers' approach. We recall that we made a crucial assumption that plane cross-sections remain plane even after bending. This is the Love-Kirchhoff assumption which is an assumption on the kinematics of deformation. Such assumptions simplify the solution of problems and are, therefore, often employed in developing an engineers' theory. Thus, the apparent anomaly is amicably settled.

26. Example: Shear Centre

Locate the shear centre of a channel section shown in Fig. 13.6a.



(a) A channel section



(b) The shear stress distribution

Figure 13.6: Fig. 13.6a shows a channel section with a uniform thickness t. C is the centroid, and S is the shear centre to be located. Fig. 13.6b shows the distribution of the shear stress in the two horizontal flanges and in the vertical web.

The figure shows a channel section with a uniform thickness t. C is the centroid, and S the shear centre to be located. If V is the vertical shear force on a typical cross-section, there will be shear stresses on the cross-section in the two horizontal flanges and in the vertical web. [If there is no shear force (that is, if this is a case of pure bending in the absence of any shear force), the shear centre has no significance.] The shear stress in the flanges and in the web are distributed [Fig. 13.6b], growing from 0 at A and reaching a maximum at D on the neutral axis and then gradually[13] reducing to 0 at D.

[13] If the thickness changes abruptly, the shear stress also changes abruptly. Really, however, the stress cannot

The method to locate the shear centre is (i) first, to recognise the distribution of the shear stress in the flanges and in the web, and (ii) then to find the magnitude of the moment due to this shear stress distribution. If the external vertical load passes through the shear centre S at a distance of e from the web as marked in Fig. 13.6a, the net moment shall be zero if the beam should only bend without twisting. This condition enables us calculate the moment arm of the external load and, thus, to locate the shear centre S. We shall now undertake the detailed calculations.

We recall how the shear stress (cross-shear) distribution is calculated. Consider the shaded area in the top flange of length s in Fig. 13.6a. The moment of this area about the neutral axis is area $\times$ moment arm $= (s \times t)d/2$. The shear flow q is given by

$$q = \frac{V\,Q}{I} = \frac{V(s \times t)\frac{d}{2}}{I}.$$

Here V is the vertical shear force on the cross-section, and I the second moment of the area about the neutral axis.

With this information we are ready to calculate the total horizontal shear force V_1 (to the left on the top flange).

$$V_1 = \int_0^b \frac{V(s \times t \times d)}{2\,I}\,ds = \frac{V\,t\,d}{2\,I}\int_0^b s\,ds = \frac{V\,t\,d\,b^2}{4\,I}.$$

An equal force V_2 acts to the right on the bottom flange. Further, the vertical shear force V_2 on the web (acting downwards) is the same as V; there is no need for any calculation. Now this force system of V_1, V_2 and V_3 leads to (i) no horizontal resultant, (ii) the vertical force $V = V_3$ (downwards), and (iii) an anti-clockwise moment $V_1 \times d$.

If the beam should only bend without any twisting, there should be no net twisting moment. Thus, this anti-clockwise moment $V_1 \times d$ should balance the clockwise moment of the vertical shear force $V = V_3$. Thus, we have

$$V \times e = V_1 \times d \quad \longrightarrow \quad e = \frac{V_1\,d}{V} = \frac{V\,t\,d\,b^2}{4\,I}\,\frac{d}{V} = \frac{t\,d^2\,b^2}{4\,I}. \tag{13.18}$$

The second moment of the area of this channel section is

$$\begin{aligned} I = I_{web} + 2\,I_{flange} &= \frac{t\,d^3}{12} + 2\left[\frac{b\,t^3}{12} + b\,t\left(\frac{d}{2}\right)^2\right] \\ &\approx \frac{t\,d^3}{12} + \frac{1}{2}t\,b\,d^2 = \frac{1}{12}\,t\,d^2(6\,b + d). \end{aligned} \tag{13.19}$$

The first term within the brackets $[\cdots]$ is negligibly small in comparison to the other term (the cube of a small quantity t being very small). Substituting this expression for for I in Eq. (13.18), we obtain the distance e as

$$e = \frac{3\,b^2}{6\,b + d} = \frac{b}{2 + \frac{d}{3\,b}}. \tag{13.20}$$

change discontinuously; there will be local redistribution of stresses. In a simplified theory as we consider now, these finer details cannot be taken into account.

This locates the shear centre. If the numerical values of the dimensions b and d are given, we can find the location of the shear centre. For example, if $d = 160$, $b = 90$, $t = 3$ (all in mm), the distance e works out to

$$e = \frac{b}{2 + \frac{d}{3b}} = \frac{90}{2 + 160/270} = 34.71\,\text{mm}.$$

We may note that the uniform thickness does not figure in the final result.

27. Example: Unsymmetrical Bending

Fig. 13.7 shows a cantilever with an end load $P = 3,000$ N as shown. The dimensions

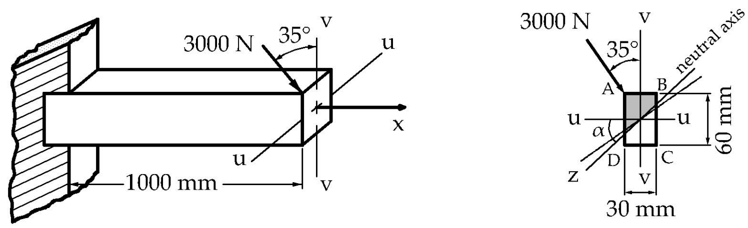


Figure 13.7: A beam in unsymmetrical bending

are given. Locate the neutral axis and find the maximum tensile stress.

Procedure:

We shall first see the steps in fair detail before we undertake the detailed calculations.

(i) Understand the problem. The line of the load is not along, nor parallel to, any principal axis. Hence this is a problem in unsymmetrical bending.

(ii) Convert this one problem in unsymmetrical problem into two problems, each of which is a case of symmetrical bending.

(iii) To do this, the load P is resolved along uu and vv which are the principal axes of inertia. (Why? Because axes of symmetry are automatically principal axes of inertia.)

(iv) Note that the maximum stress occurs on the cross-section that has the largest bending moment. This is at the fixed end.

(v) The bending moments at the fixed end are

$$M_u = P\,\cos\theta \times l = 3000\,\cos 35^\circ \times 1 = 2475.5 \text{ Nm}; \qquad M_v = P\,\sin\theta \times l = 3000\,\sin 35^\circ \times 1 = 1720.7 \text{ Nm}.$$

(vi) The principal second moments of area (area moments of inertia) are

$$I_{uu} = \frac{1}{12}(30)(60^3) = 0.54 \times 10^{-6}\ \text{m}^4; \qquad I_{vv} = \frac{1}{12}(60)(30^3) = 0.135 \times 10^{-6}\ \text{m}^4.$$

(vii) Use the Euler-Bernoulli equation to calculate the stresses.

(viii) Recognise that A is the point where the tensile stress is the largest. Why? A is in the tension zone for both M_u and M_v, and farthest from the neutral axis.

(ix) Let us note that the neutral axis is characterised by the fact that the bending stress at all points on it is zero.

$$\sigma = \frac{M_v\, u}{I_v} + \frac{M_u\, v}{I_u} \qquad\longrightarrow\qquad \frac{u}{v} = -\frac{M_u}{M_v}\frac{I_{vv}}{I_{uu}}.$$

The angle α that defines the neutral axis is given by

$$\tan\alpha = \frac{u}{v} = -\frac{M_u}{M_v}\frac{I_{vv}}{I_{uu}}.$$

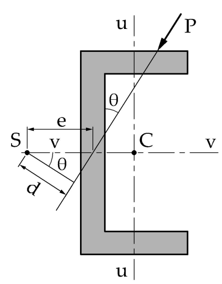
We can now complete the solution by working out these quantities numerically. The largest (maximum) tensile stress is at A; its numerical value is

$$\sigma_{max} = \frac{M_u}{I_u}\times\frac{60}{2} + \frac{M_v}{I_v}\times\frac{30}{2} = \frac{2457.5\times 0.03}{0.54\times 10^{-6}} + \frac{1720.7\times 0.015}{0.135\times 10^{-6}} = 327.6 \text{ MPa}.$$

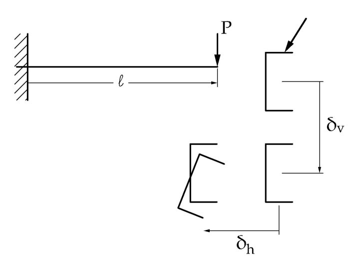
$\tan\alpha = 0.357;\ \alpha = 19.6\,^\circ$.

Note particularly that $\alpha \neq 35\,^\circ$; that is, the neutral axis is not perpendicular to the line of the applied load! Note further that, if $I_{uu} = I_{vv}$, the angle $\alpha = \theta = 35\,^\circ$. Why is this so? If $I_{uu} = I_{vv}$ (as, for example, for a square cross-section, then every direction is a principal direction! The inertia tensor then becomes an isotropic tensor! Note further that, if $I_{uu} = I_{vv}$ (as for example, for a square cross-section), then every direction is a principal direction! The inertial tensor then becomes an isotropic tensor!

28. Example: Bending and Twisting of a Channel Section



(a) A channel section: inclined load



(b) Deflections at the end of a cantilever

Figure 13.8: A load P acts at the end of a cantilever as shown [Figs 13.8a, 13.8b]; it does not pass through the shear centre.

A load P acts at the end of a cantilever with a uniform channel section as shown [Figs 13.8]. We desire to calculate (i) the stresses, and (ii) the deflection at the end.

Let us discuss the procedure before we do the calculations. First we must understand the problem. A few points are raised below.

(i) uu and vv are the principal axes of inertia passing through the centroid C.

(ii) As the line of the load is not parallel to uu nor to vv, this is a case of unsymmetrical bending.

(iii) We can replace this one problem of unsymmetrical bending by two sub-problems, each of symmetrical bending. This is done by resolving the force P along uu and vv ($P\,\cos\theta$ and $P\,\sin\theta$, respectively). The bending moments at the fixed end (which is the vulnerable cross-section because it carries the largest bending moments) are $M_{vv} = (P\,\cos\theta)\,l$, and $M_{uu} = (P\,\sin\theta)\,l$. As these sub-problems are of symmetrical bending, the bending stresses can be calculated using the Euler-Bernoulli equation $\sigma/y = M/I = E/R$ interpreting M and I correctly. Thus,

$$\frac{M_{vv}}{I_{vv}} = \frac{(P\,\cos\theta)\,l}{I_{vv}} \qquad \text{and} \qquad \frac{M_{uu}}{I_{uu}} = \frac{(P\,\sin\theta)\,l}{I_{uu}}.$$

(iv) The vertical and horizontal deflections (δ_v and δ_h, respectively) at the end of the cantilever can be calculated as

$$\delta_v = \frac{P\,\cos\theta\,l^3}{3\,EI_{vv}} \text{ downwards, and} \qquad \delta_h = \frac{P\,\sin\theta\,l^3}{3\,EI_{uu}} \text{ to the left.}$$

(v) Notice that the load does not pass through the shear centre. We need to locate the shear centre S by a separate calculation.

(vi) As the load does not pass through the shear centre S, the cross-section is subjected to a clockwise torque of magnitude $P \times e$, where e is the eccentricity to be calculated as in item (v) above.

(vii) Recall the topic of torsion of open cross-sections based on the solution for long narrow rectangular sections.

(viii) Calculate the torsional shear stresses on the cross-section, and the torsional rigidity of the channel section. As the thicknesses of the legs are different in general (the two flanges are identical), the torsional rigidity of each leg is to be calculated, and added up.

(ix) The total stresses can be found by superposition. The nature of the stress components must be understood clearly before they can be 'added up' (superposed).

(x) The critical section (the most vulnerable) is at the support. There are uncertainties at the fixed support. Besides and more importantly, the warping of the cross-section is prevented at the fixed end. Here the theory of torsion is to be modified[14] and applied. For the limited purpose of working out this illustrative example at this level, we need not be concerned above these refinements. These must be regarded as limitations of the method used.

[14] In the early days of Timoshenko's teaching career, he used to spend his vacation in Germany. On one such occasion when he was in Göttingen, he requested Prandtl (who had by then left this area and turned to fluid mechanics and his famous boundary layer theory) to suggest a research topic. Prandtl suggested the topic of torsion when warping is thus restrained. A little later when someone asked Prandtl what he thought of young Timoshenko, Prandtl is said to have remarked that "he must be good, because after getting the problem, he never came to me."

To carry out the subsequent calculations, let us take the following numerical values: width, $b = 90$; depth, $d = 160$; thickness, $t = 3$ (uniform throughout) (all in mm); end load, $P = 1.0$ kN; span, $l = 1000$ mm.

The calculations are simple and straightforward. Nevertheless, we shall identify the various steps.

Steps

1. Find locate the centroid of the cross-section.
2. Next calculate the second moments of area I_{xx} and I_{yy} about the centroidal axes (horizontal x, and vertical y).
3. Resolve the load P along the horizontal and the vertical directions, P_x and P_y, respectively.
4. Calculate the end deflections $\delta_{vert} = \dfrac{P_y l^3}{3EI_{xx}}$ and $\delta_{horiz} = \dfrac{P_x l^3}{3EI_{yy}}$.
5. Calculate the twist at the end. After calculating the twist θ per unit length, this is to be multiplied by the length, viz., l. The theory of a long narrow rectangle is to be used here as this (channel) is an open section. The length is the sum of the three legs; here it is $2b + d$. The twisting moment is equal to the load P multiplied by the distance between the shear centre and the line of application of the load P. (When the load does not pass through the shear centre, a twisting moment is called into play. This is why the cross-section twists.)
6. Finally we should recognise that there will be some error, but that this will be small. The error arises because we did not use Timoshenko's method when the end is fixed. The warping is prevented at the fixed end, but we disregarded this in our solution. In any case, there is some approximation in using the theory of a long narrow section.

The distance e, given by $e = 3b^2/(6b + d) = 34.71$ mm [Eq. (13.20) as calculated in an earlier example. The remaining part is left as an exercise for the young readers.

29. Example: A Given Channel Section

Work out the above problem for a beam with the channel section shown. The load is $P = 1$ kN, and the span $l = 1.5$ m. $E = 200$ GPa; $\theta = 30°$.

[The one difference is that we can, if we desire, include the self-weight also. This is a uniformly distributed load. Thus, the vertical deflection will be increased by the amount $wl^4/(8EI_{xx})$. The calculation of the torque is a little tricky; it changes from section to section along the length of the cantilever. However, if we take a specific case like, say, the channel section ISJC100, we can see that the self weight is only 56.90 N per metre. This is negligibly small. Hence the self-weight is neglected.]

Preliminaries

Location of the centroid: We can break the cross-section into three rectangles A_1, A_2, A_3 and locate the centroid as shown below.

$$A_1 = 90 \times 3 = 270 \text{ mm}^4; \qquad A_2 = 154 \times 3 = 462 \text{ mm}^4; \qquad A_3 = 90 \times 3 = 270 \text{ mm}^4;$$

$$\bar{x} = \frac{A_1\,x_1 + A_2\,x_2 + A_3\,x_3}{A_1 + A_2 + A_3} = \frac{270 \times 45 + 462 \times 1.5 + 270 \times 45}{270 + 462 + 270} = 24.94 \text{ mm};$$

$$\bar{y} = 80 \text{ mm}; \qquad I_{xx} = \left(\frac{90 \times 3^3}{12} + 270 \times (80 - 1.5^2)\right) + \cdots = 4.241 \times 10^6 \text{ mm}^4;$$

$$I_{yy} = \cdots = 0.8357 \text{ mm}^4.$$

Calculations
Vertical deflection, δ_v

$$\delta_v = \frac{P_y\,l^3}{3\,EI_{xx}} = \frac{1000 \times \cos 30^\circ \times 1500^3}{3 \times 200 \times 10^9 \times 4.241 \times 10^6}$$
$$= 1.147\,\text{mm}.$$

Horizontal deflection, δ_h

$$\delta_h = \frac{P_x\,l^3}{3\,EI_{yy}} = \frac{1000 \times \sin 30^\circ \times 1500^3}{3 \times 200 \times 10^9 \times 0.8357 \times 10^6}$$
$$= 3.365\,\text{mm}.$$

Rotation, θ
This is the twist because of the twisting moment. We note that the load P does not pass through the shear centre. The twisting moment, $T = P \times 34.71 \times \cos 30^\circ = 30059.7$ N mm. As we had explained earlier, this is an open section and, therefore, is a case of a long narrow rectangle in torsion. The torsional rigidity, we recall, is given by

$$\frac{T}{\theta} = \frac{Gt^3 l}{3},$$

where this l is the length of the open section $= (90 - 1.5) + (160 - 3) + (90 - 1.5) = 334$ mm, and $t = 3$ mm (thickness of the 'long, narrow rectangle' in torsion).

$$\theta = \frac{3T}{Gt^3 l} = [2.864 \times 10^{-3}]^\circ/\text{mm}.$$

This is the twist per unit length. Hence the total twist of the end section $= 2.864 \times 1.5 = 4.296^\circ$. Fig. 13.8b shows what these deflections and twist.

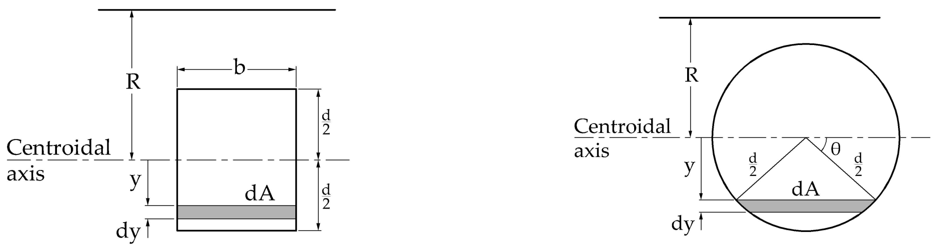
30. Example: Z for a Rectangular Cross-section
Calculate Z for a rectangular cross-section.

Fig. 13.9a shows a rectangular cross-section. The relevant dimensions are all marked in the figure. The property Z is defined as

$$Z = -\frac{1}{A}\int_A \frac{y}{R+y}\,dA = -\frac{1}{bd}\int_{-d/2}^{d/2} \frac{y}{R+y}\,b\,dy = -\frac{1}{d}\int_{-d/2}^{d/2} \frac{y}{R+y}\,dy.$$

The fraction $1/(R+y)$ may be written and expanded in an infinite series using the binomial theorem, and the expression for Z may be obtained by direct integration.

$$\frac{1}{R+y} = (R+y)^{-1} = \frac{1}{R}\left(1 + \frac{y}{R}\right)^{-1}$$



(a) A rectangular cross-section

(b) A circular cross-section

Figure 13.9: Two cross-sections, (rectangular and circular) with the relevant dimensions are shown. We wish to calculate Z for these two cross-sections.

As $|y/R| < 1$, we may use the binomial expansion. Then by direction integration and substituting the limits, we can work out the expression for Z as shown below.

$$\begin{aligned} Z &= -\frac{1}{d}\int_{-d/2}^{d/2}\left[\left(\frac{y}{R}\right) - \left(\frac{y}{R}\right)^2 + \left(\frac{y}{R}\right)^3 - \left(\frac{y}{R}\right)^4 + \cdots\right] dy \\ &= \frac{1}{3}\left(\frac{d}{2\,R}\right)^2 + \frac{1}{5}\left(\frac{d}{2\,R}\right)^4 + \frac{1}{7}\left(\frac{d}{2\,R}\right)^6 + \frac{1}{9}\left(\frac{d}{2\,R}\right)^8 + \cdots \end{aligned}$$

This infinite series converges fast; there is no difficulty in finding the answer with sufficient accuracy.

To get a feel for numbers, it is desirable to work out the numerical value of Z if, for example, $b = 10$, $d = 14$, and $R = 17$, all in mm. The value of Z is

$$Z = \frac{1}{3}\left(\frac{14}{34}\right)^2 + \frac{1}{5}\left(\frac{14}{34}\right)^4 + \frac{1}{7}\left(\frac{14}{34}\right)^6 + \frac{1}{9}\left(\frac{14}{34}\right)^8 + \cdots = 0.0630.$$

Note that this series converges fast. More importantly, note that Z is dimensionless; its numerical value is the same in all units.

An alternative method:
An alternative method is to integrate directly without the infinite series expansion. This gives us the value of Z as

$$\begin{aligned} Z &= -\frac{1}{bd}\int_A \frac{y}{R+y}\,dA = -\frac{1}{bd}\int_A \frac{R+y-R}{R+y}\,dA \\ &= -\frac{1}{d}\int_{-d/2}^{d/2}\left(1 - \frac{R}{R+y}\right) dy = -\frac{1}{d}\left[y - R\,\ln(R+y)\right]_{-d/2}^{d/2} \\ &= -1 + \frac{R}{d}\,\log\left(\frac{R+\frac{d}{2}}{R-\frac{d}{2}}\right) = -1 + \frac{R}{d}\,\ln\left(\frac{2\,R+d}{2\,R-d}\right). \end{aligned}$$

For the numerical values of $b = 10$, $d = 14$, and $R = 17$, all in mm,

$$Z = -1 + \frac{17}{14}\,\ln\left(\frac{17+7}{17-7}\right) = 0.0630.$$

31. Example: Z for a Circular Cross-section

The circular cross-section and the relevant dimensions and markings are shown in Fig. 13.9b. The diameter of the circular cross-section is d, and the radius of curvature is R. Then,

$$\begin{aligned} Z &= -\frac{1}{A}\int_A \frac{y}{R+y}\,dA \\ &= -\frac{1}{\pi\frac{d^2}{4}}\int_{-\pi/2}^{\pi/2} \frac{\frac{d}{2}\sin\theta}{R+\frac{d}{2}\sin\theta}\,2\left(\frac{d}{2}\right)^2\cos\theta\,d\theta \\ &= -\frac{d}{\pi}\int_{-\pi/2}^{\pi/2}\frac{\cos^2\theta\,\sin\theta}{R+\frac{d}{2}\sin\theta}\,d\theta. \end{aligned}$$

As before, let us expand

$$\frac{1}{R+\frac{d}{2}\sin\theta} = \left(R+\frac{d}{2}\sin\theta\right)^{-1} = R\left(1+\frac{d}{2R}\sin\theta\right)^{-1}$$

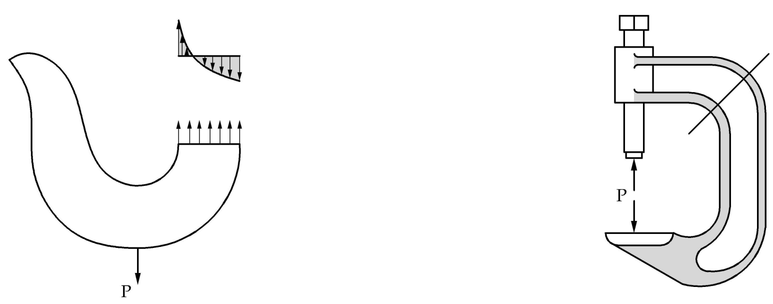
in an infinite series. The result is that the expression for Z is obtained as

$$Z = \frac{1}{4}\left(\frac{d}{2\,R}\right)^2 + \frac{1}{8}\left(\frac{d}{2\,R}\right)^4 + \frac{5}{64}\left(\frac{d}{2\,R}\right)^6 + \frac{7}{128}\left(\frac{d}{2\,R}\right)^8 + \cdots$$

For the numerical values of $d = 14$, and $R = 17$, all in mm,

$$Z = \frac{1}{4}\left(\frac{14}{34}\right)^2 + \frac{1}{8}\left(\frac{14}{34}\right)^4 + \frac{5}{64}\left(\frac{14}{34}\right)^6 + \frac{7}{128}\left(\frac{14}{34}\right)^8 + \cdots = 0.04640.$$

32. Example: Stresses in a Curved Beam



(a) A curved beam with a load W (b) Cross-section and stresses

Figure 13.10: A vertical load W acts as shown. The dimensions of the cross-section and the stress distributions are shown alongside.

A curved beam in the shape of a crane hook with the relevant dimensions is shown in Figs 13.10. Calculate the maximum load so that the stress does not exceed MPa.

The critical cross-section is $X-X$. There are (i) bending stresses and (ii) direct stresses. The bending stresses are given by the Winkler-Bach formula. These (circumferential) stresses are distributed hyperbolically: largest at the inner point A (tensile). At B the compressive bending stress is a maximum. The direct stresses on the cross-section is tensile, and is (assumed to be) uniform, $[W/(\text{area of c.s.})]$. The total stresses are obtained by adding up the stresses due to (i) bending, and (ii) direct tension. Thus, on the most vulnerable cross-section $X-X$, the maximum stress (tensile) is at A.

Now we can make the detailed calculations. The value of Z for this case was already worked out earlier; it is $Z = 0.0630$. The area of the cross-section $A = b \times d = 140$ mm^2. $W = 10$ kN. The bending moment $= -WR = -170$ Nm. The distance of the point A from the *centroidal* axis is $y_A = 14/2 = 7$ mm $= 7 \times 10^{-3}$ m.

Total stresses at A and B on the cross-section are

$$\begin{aligned}
\sigma_A &= \frac{W}{A} + \frac{M}{AR}\left[1 + \frac{1}{Z}\,\frac{-y_A}{R+(-y_A)}\right];\\
&= \frac{10 \times 1000}{140 \times 10^{-6}} - \frac{170}{140 \times 10^{-6} \times 17 \times 10^{-3}}\left[1 + \frac{1}{0.0630}\left(\frac{-7}{17-7}\right)\right]\\
&= 71.428 + 722.22 = 793.65 \text{ MPa.}
\end{aligned}$$

$$\begin{aligned}
\sigma_A &= \frac{W}{A} + \frac{M}{AR}\left[1 + \frac{1}{Z}\,\frac{y_B}{R+y_B}\right];\\
&= \frac{10 \times 1000}{140 \times 10^{-6}} - \frac{170}{140 \times 10^{-6} \times 17 \times 10^{-3}}\left[1 + \frac{1}{0.0630}\left(\frac{7}{17+7}\right)\right]\\
&= 71.428 - 402.116 = -330.69 \text{ MPa.}
\end{aligned}$$

Actually it is not necessary to calculate the stress at B, σ_B. This is done merely to clarify matters.

33. Example: Equations of Equilibrium by Coordinate Transformation

Starting from the differential equations of equilibrium in rectangular coordinates, obtain the corresponding equations in polar coordinates.

The two systems of coordinates (x, y) and (r, θ) are related by

$$\begin{aligned}
x &= r\,\cos\theta; \qquad & r &= \sqrt{x^2+y^2};\\
y &= r\,\sin\theta; \qquad & \theta &= \tan^{-1}\left(\frac{y}{x}\right).
\end{aligned}$$

The derivatives are calculated as

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{x}{r} = \cos\theta; \qquad & \frac{\partial \theta}{\partial x} &= -\frac{y}{r^2} = -\frac{\sin\theta}{r};\\
\frac{\partial r}{\partial y} &= \frac{x}{r} = \sin\theta; \qquad & \frac{\partial \theta}{\partial y} &= \frac{x}{r^2} = \frac{\cos\theta}{r}.
\end{aligned}$$

From the stress transformation laws we obtain the two following sets of results.

(a)
$$\begin{aligned}\sigma_{rr} &= \sigma_{xx}\,\cos^2\theta + \sigma_{yy}\,\sin^2\theta + \tau_{xy}\,\sin 2\theta;\\ \sigma_{\theta\theta} &= \sigma_{xx}\,\sin^2\theta + \sigma_{yy}\,\cos^2\theta - \tau_{xy}\,\sin 2\theta;\\ \tau_{r\theta} &= (\sigma_{yy} - \sigma_{xx})\,\sin\theta\,\cos\theta + \tau_{xy}\,\cos 2\theta.\end{aligned}$$

(b)
$$\begin{aligned}\sigma_{xx} &= \sigma_{rr}\,\cos^2\theta + \sigma_{\theta\theta}\,\sin^2\theta - \tau_{r\theta}\,\sin 2\theta;\\ \sigma_{yy} &= \sigma_{rr}\,\sin^2\theta + \sigma_{\theta\theta}\,\cos^2\theta + \tau_{r\theta}\,\sin 2\theta;\\ \tau_{xy} &= (\sigma_{rr} - \sigma_{\theta\theta})\,\sin\theta\,\cos\theta + \tau_{r\theta}\,\cos 2\theta.\end{aligned}$$

Let us now compute the three terms in the differential equations of equilibrium [Eqs 5.6a, 5.6b; p. 5-7].

$$\begin{aligned}\frac{\partial\sigma_{xx}}{\partial x} &= \cos\theta\,\frac{\partial}{\partial r}\left[\sigma_{rr}\,\cos^2\theta + \sigma_{\theta\theta}\,\sin^2\theta - \tau_{r\theta}\,\sin 2\theta\right] -\\ &\quad \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}\left[\sigma_{rr}\,\cos^2\theta + \sigma_{\theta\theta}\,\sin^2\theta - \tau_{r\theta}\,\sin 2\theta\right];\\ \frac{\partial\sigma_{yy}}{\partial y} &= \sin\theta\,\frac{\partial}{\partial r}\left[\sigma_{rr}\,\sin^2\theta + \sigma_{\theta\theta}\,\cos^2\theta + \tau_{r\theta}\,\sin 2\theta\right] +\\ &\quad \frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta}\left[\sigma_{rr}\,\sin^2\theta + \sigma_{\theta\theta}\,\cos^2\theta + \tau_{r\theta}\,\sin 2\theta\right];\\ \frac{\partial\sigma_{xy}}{\partial x} &= \cdots;\\ \frac{\partial\sigma_{xy}}{\partial y} &= \cdots.\end{aligned}$$

When these expressions are substituted in the two equations of equilibrium [Eqs 5.6a, 5.6b; p. 5-7], we obtain

$$\begin{aligned}\frac{\cos\theta}{r}\left(\sigma_{rr} - \sigma_{\theta\theta} + \frac{\partial\tau_{r\theta}}{\partial\theta}\right) + \cos\theta\,\frac{\partial\sigma_{rr}}{\partial r} - \sin\theta\,\frac{\partial\tau_{r\theta}}{\partial r} - \frac{\sin\theta}{r}\left(\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + 2\tau_{r\theta}\right) &= 0;\\ \frac{\sin\theta}{r}\left(\sigma_{rr} - \sigma_{\theta\theta} + \frac{\partial\tau_{r\theta}}{\partial\theta}\right) + \sin\theta\,\frac{\partial\sigma_{rr}}{\partial r} + \cos\theta\,\frac{\partial\tau_{r\theta}}{\partial r} + \frac{\cos\theta}{r}\left(\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + 2\tau_{r\theta}\right) &= 0.\end{aligned}$$

These two equations are simplified, and we arrive at the desired form as

$$\begin{aligned}\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\,\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} &= 0;\\ \frac{1}{r}\,\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\tau_{r\theta}}{\partial r} + \frac{2\,\tau_{r\theta}}{r} &= 0.\end{aligned}$$

If the body forces are also to be included, the equations are

$$\begin{aligned}\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\,\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + F_r &= 0;\\ \frac{1}{r}\,\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\tau_{r\theta}}{\partial r} + \frac{2\,\tau_{r\theta}}{r} + F_\theta &= 0.\end{aligned}$$

34. Example: Airy's Stress Function

Examine what problems are solved by the following Airy's stress functions.

$$\begin{array}{ll} \text{(i)} & \phi = C_1; \\ \text{(ii)} & \phi = C_1 x^2; \\ \text{(iii)} & \phi = C_2 x^2 + C_3 xy + C_1 y^2 \\ \text{(iv)} & \phi = C_2 x^2 + C_3 x^2 y + C_1 xy^2 + C_4 y^3 \end{array}$$

C_1, C_2, C_3 C_4 are all constants.

First of all, let us check and make sure that the governing equation

$$\nabla^4 \phi \equiv \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

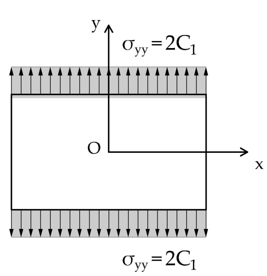
(which is the compatibility equation) is satisfied by all these four functions $\phi = \phi(x, y)$. We shall now discuss these four cases one by one.

(i) $\phi = C_1$. This case is trivial; all the stress components are zero everywhere.
(ii)$\phi = C_1 x^2$.

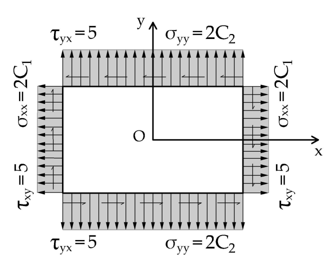
$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = 0; \qquad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 2C_1; \qquad \tau_{xx} = -\frac{\partial^2 \phi}{\partial x \partial y} = 0.$$

This can represent a homogeneous (uniform) state of stress (tensile if $C_1 > 0$, compressive if $C_1 < 0$) in a bar or a plate [Fig. 13.11a].
(iii) $\phi = C_2 x^2 + C_3 xy + C_1 y^2$.

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = 2C_1; \qquad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 2C_2; \qquad \tau_{xx} = -\frac{\partial^2 \phi}{\partial x \partial y} = -C_3.$$



(a) A plate loaded by $\sigma_{yy} = 2C_1$



(b) A plate loaded as shown

Figure 13.11: The figures are drawn for the numerical values of $C_1 = 10$, $C_2 = 15$, $C_3 = 5$, all in MPa. Note particularly that the shear stress $\tau_{xy} = -5$ is shown as marked. The negative sign is already reckoned by reversing the direction; the magnitude, therefore, is to be marked as 5 (and not as -5).

This can represent a loading as shown in Fig. 13.11b.
(iv) $\phi = C_2x^3 + C_3x^2y + C_1xy^2 + C_4y^3$.

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = 2C_1x + 6C_4y; \qquad \sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = 6C_2x + 2C_3y;$$
$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -2C_3x - 2C_1y.$$

All the stress components have linear variations w.r.to x and y. If $C_1 = C_2 = C_3 = 0$ and $C_4 \neq 0$, that is, if $\phi = C_4y^3$, this stress function solves the problem of bending of a beam. We have seen this earlier.

35. Example: Mohr's Circle

Given that $\sigma_{11} = 50$, $\sigma_{22} = 30$, $\sigma_{33} = -10$ (all in MPa), let us use the Mohr's circle construction to find out the normal stress $\sigma_{(\nu)}$ and the shear stress $\tau_{(\nu)}$ on an octahedral plane — equally inclined to the three principal planes — with the direction cosines $l = m = n = 1/\sqrt{3}$. We shall also calculate these analytically for comparison.

Graphical construction:

Following the steps given on p. 5-40, let us proceed as follows [Fig. 13.12]. All numerical values of stresses are in MPa.

(i) Draw to a convenient scale the $\sigma_{(\nu)}\,\tau_{(\nu)}$ axes, and mark off the principal stresses ($\sigma_{11} = 50$, point A), ($\sigma_{22} = 30$, point B), ($\sigma_{33} = -10$, point C, 10 units to the left of the origin O).

(ii) Draw three semi-circles aaaa, bbbb, cccc with the centres and radii as follows.

circle aaaa:	centre D: $\dfrac{\sigma_{11}+\sigma_{33}}{2} = 20$	radius $\left\lvert\dfrac{\sigma_{11}-\sigma_{33}}{2}\right\rvert = 30;$
circle bbbb:	centre E: $\dfrac{\sigma_{22}+\sigma_{11}}{2} = 40$	radius $\left\lvert\dfrac{\sigma_{22}-\sigma_{11}}{2}\right\rvert = 10;$
circle cccc:	centre F: $\dfrac{\sigma_{33}+\sigma_{22}}{2} = 10$	radius $\left\lvert\dfrac{\sigma_{33}-\sigma_{22}}{2}\right\rvert = 20.$

(iii) The angles $(\nu,\, x)$, $(\nu,\, z)$ corresponding to the direction cosines $1/\sqrt{3}$, $1/\sqrt{3}$, $1/\sqrt{3}$ are calculated as $\cos^{-1}(1/\sqrt{3}) \equiv (\nu,\, x) = 54.74\,°$.

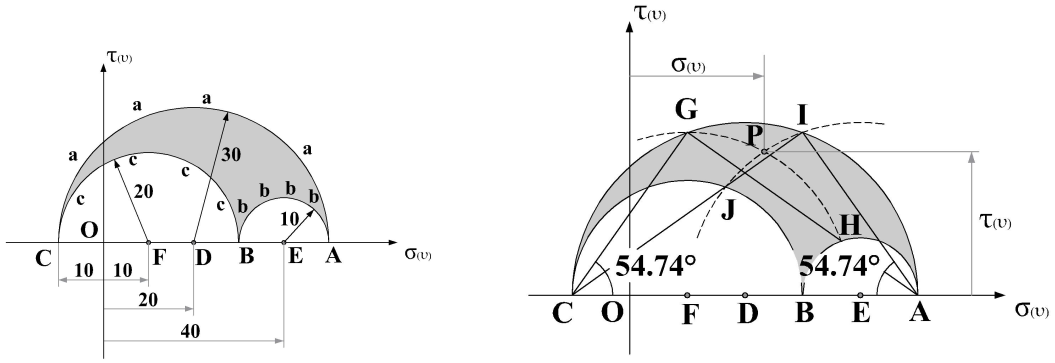
(iv) Draw a straight line CG at this calculated angle of $(\nu,\, x) = 54.74\,°$, and the straight line AG to meet the circle aaaa at G. This line cuts the circle bbbb at the point H [Fig. 5.30b].

(v) With F as the centre, draw a circular arc through G. This will necessarily pass through E.

(vi) In the same way draw a straight line AI at the calculated angle of $(\nu,\, z) = 54.74\,°$, and the straight line CI to meet the circle aaaa at J. This line cuts the circle cccc at the point J.

(vii) With E as the centre, draw a circular arc through I. This will necessarily pass through J.

(viii) These two circular arcs intersect at the point P. The coordinates of P give us the sought after values of the normal $\sigma_{(\nu)}$ and shearing $\tau_{(\nu)}$ stresses. These values are obtained by measuring the coordinates. The values thus obtained [$\sigma_{(\nu)}$ = 23.33 MPa, $\tau_{(\nu)}$ = 24.94 MPa] can be compared with the values obtained by calculations shown below. It will be seen that the values match if due allowance is given for unavoidable errors in any graphical construction[15].



(a) Mohr's circles: step 1

(b) Mohr's circles: step 2

Figure 13.12: The various steps to solve this numerical problem by drawing the Mohr's circles are explained in the text with reference to this figure.

Answers using analytical / mathematical equations:

The resultant stress $\left|\overset{(\nu)}{\boldsymbol{T}}\right|$, the normal stress σ_{oct}, and the shear stress τ_{oct}, are given by the following expressions. We obtain the answers by substituting the numerical values.

$$
\begin{aligned}
|\overset{(\nu)}{\boldsymbol{T}}| &= \sqrt{\frac{\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{11}^2}{3}} = \sqrt{\frac{50^2 + 30^2 + 10^2}{3}} = 34.16 \text{ MPa};\\
\sigma_{oct} &= \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}\\
&= \frac{50 + 30 - 10}{3} = 23.33 \text{ MPa};\\
\tau_{oct} &= \frac{1}{3}\sqrt{(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2}\\
&= \frac{1}{3}(20^2 + 40^2 + 60^2) = 24.94 \text{ MPa}.
\end{aligned}
$$

$$
\text{verify:} \quad \sigma_{oct}^2 + \tau_{oct}^2 = |\overset{(\nu)}{\boldsymbol{T}}|^2 \quad \longrightarrow \quad 544.44 + 622.21 \approx 1166.67.
$$

[15] This is usually the case, but here the values obtained are remarkably and unbelievably the same as the ones obtained by analytical methods!

36. Example: Strains from Displacements

The displacement field in a body is given as $\boldsymbol{u} = u\,\boldsymbol{i} + v\,\boldsymbol{j} + w\,\boldsymbol{k} = (2x + y^2)\,\boldsymbol{i} + (x^3 + 2z^2)\,\boldsymbol{j} + (xy^2 - xz)\,\boldsymbol{k}$, find (i) the initial and final position vectors and (ii) the displacement vector $\boldsymbol{u}$ and the strain components at the point $P(1,2,3)$. All are in 10^{-3} mm.

$$\begin{aligned}
\text{Initial position vector } \boldsymbol{r} &= 1\,\boldsymbol{i} + 2\,\boldsymbol{j} + 3\,\boldsymbol{k} \\
\text{Displacement vector } \boldsymbol{u} \text{ at } P(1,2,3) &= 1\,\boldsymbol{i} + 2\,\boldsymbol{j} + 3\,\boldsymbol{k} \\
&= (2\times 1 + 2^2)\,\boldsymbol{i} + (1^3 + 2\times 3^2)\,\boldsymbol{j} + (1\times 2^2 - 1\times 3)\,\boldsymbol{k} \\
&= 6\,\boldsymbol{i} + 19\,\boldsymbol{j} - \boldsymbol{k} \\
\text{Final position vector } \boldsymbol{r}^* &= \boldsymbol{r} + \boldsymbol{u} = (1\,\boldsymbol{i} + 2\,\boldsymbol{j} + 3\,\boldsymbol{k}) + (6\,\boldsymbol{i} + 19\,\boldsymbol{j} - \boldsymbol{k}) \\
&= 7\,\boldsymbol{i} + 21\,\boldsymbol{j} + 2\,\boldsymbol{k}
\end{aligned}$$

$$\begin{aligned}
e_{xx} &= \frac{\partial u}{\partial x} = 2; \qquad e_{yy} = \frac{\partial v}{\partial y} = 0; \qquad e_{zz} = \frac{\partial w}{\partial z} = -x = -1; \\
e_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}(2y + 3x^2) = \frac{1}{2}(2\times 2 + 3\times 1^2) = 3.5; \\
e_{yz} &= \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}(4z + 2xy) = \frac{1}{2}(4\times 3 + 2\times 1\times 2) = 8; \\
e_{zx} &= \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \frac{1}{2}(2y - z + 0) = 0.5, \text{ all in } 10^{-3}.
\end{aligned}$$

37. Example: Theories of Failure

Determine the diameter of a steel shaft on the basis of (i) the maximum shear stress theory, and (ii) the distortion energy theory if the shaft is subjected to a combined load of a torque $T = 20,000$ Nm and a bending moment $M = 10,000$ Nm. Assume that the yield stress, $\sigma_{y.p.} = 400$ MPa (obtained from a simple tension test), and a factor of safety of 1.8.

(a) Preliminaries

$$\begin{aligned}
\text{Second moment of area (for bending), } I &= \frac{\pi\, d^4}{64}; \\
\text{Polar second moment of area (for bending), } J &= \frac{\pi\, d^4}{32}.
\end{aligned}$$

As there are only the normal stress σ due to bending, and the shear stress τ due to twisting, the principal stresses — this is a two-dimensional case — and the maximum shear stress are:

$$\begin{aligned}
\sigma_1,\, \sigma_2 &= \frac{\sigma}{2} \pm \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}; \\
\tau_{max} &= \frac{1}{2}(\sigma_1 - \sigma_2) = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2}.
\end{aligned}$$

$$\text{normal stress in bending, } \sigma = \frac{M(\frac{d}{2})}{I} = \frac{M}{\frac{\pi d^4}{64}}\,\frac{d}{2} = \frac{32\,M}{\pi d^3}$$

$$\text{shear stress in twisting, } \tau = \frac{T(\frac{d}{2})}{J} = \frac{T}{\frac{\pi d^4}{32}}\,\frac{d}{2} = \frac{16\,T}{\pi d^3}.$$

(b) Maximum shear stress theory
Maximum shear stress, $\tau_{max} = (1/2)\sqrt{\sigma^2 + 4\tau^2}$. The maximum shear at the point of yielding in a simple tension test $= (1/2)\sigma_{y.p.}$. Thus, the design equation is

$$\frac{1}{2}\sqrt{\left(\frac{32\,M}{\pi d^3}\right)^2 + 4\left(\frac{16\,T}{\pi d^3}\right)^2} = \frac{1}{2}\,\frac{\sigma_{y.p.}}{N},$$

where N is the 'factor of safety'[16]. Thus,

$$\frac{1}{2}\sqrt{\left(\frac{32\,M}{\pi d^3}\right)^2 + 4\left(\frac{16\,T}{\pi d^3}\right)^2} = \frac{1}{2}\,\frac{\sigma_{y.p.}}{1.8}; \quad \text{i.e.,} \quad \frac{16}{\pi d^3}\sqrt{M^2 + T^2} = \frac{1}{2}\,\frac{\sigma_{y.p.}}{1.8};$$

$$\frac{160000 \times 1000}{\pi d^3}\sqrt{1^2 + 2^2} = \frac{1}{2}\,\frac{(400)}{1.8}; \qquad d = 100.8\,\text{mm}.$$

We shall see that this theory is a little more conservative than the distortion energy theory which we shall see below.

(c) Distortion energy theory
The design equation is

$$\sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]} = \frac{\sigma_{y.p.}}{N};$$

$$\text{i.e.,} \quad \sqrt{\sigma^2 + 3\tau^2} = \frac{\sigma_{y.p.}}{N}; \qquad \text{i.e.,} \qquad \frac{16}{\pi d^3}\sqrt{4M^2 + 3T^2} = \frac{\sigma_{y.p.}}{N};$$

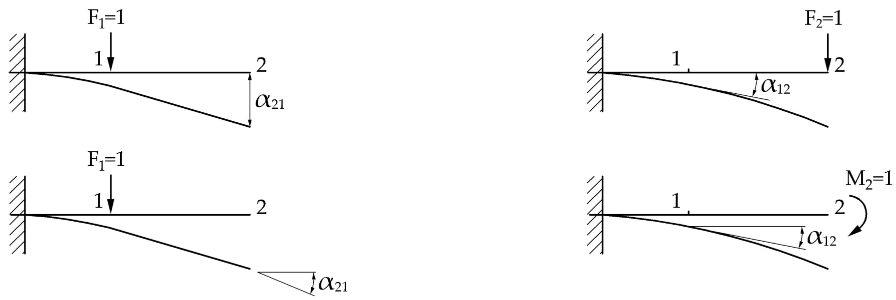
$$\frac{160000 \times 1000}{\pi d^3}\sqrt{4(1)^2 + 3(2)^2} = \frac{400}{1.8}; \qquad d = 97.1\,\text{mm}.$$

The remark that we made above (that the maximum shear stress theory is a little more conservative than the distortion energy theory) is vindicated.

38. Example: Maxwell Reciprocal Theorem

The displacement at a point i because of a unit force applied at a point j (another point or even the same point) is equal to the displacement at the point j because of a unit force applied at the point i. It is emphasised that (i) the displacement may be linear or angular (rotational), and that the force may be a force or a moment; and that (ii) the displacement and the force correspond to each other reckoned at each point in the same direction — work absorbing component!

[16] This is called a factory of safety because, as Den Hartog [3] remarks, 'factor of ignorance' sounds too cynical.



(a) A load applied at 1 (b) A force and a moment applied at 2

Figure 13.13: Fig. 13.13 shows a cantilever. The loadings (force / moment) and the displacements (deflection / slope) are shown. We can see verify the Maxwell reciprocal theorem.

(i) When the forces are applied in the order $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ — first $\boldsymbol{F}_1$ and then $\boldsymbol{F}_2$ — the work done W (and, therefore, the strain energy stored U) is

$$W_1 = U_1 = \frac{1}{2}(\alpha_{11}\,\boldsymbol{F}_1)\boldsymbol{F}_1 + \frac{1}{2}(\alpha_{22}\,\boldsymbol{F}_2)\boldsymbol{F}_2 + \alpha_{12}\,\boldsymbol{F}_1\,\boldsymbol{F}_2.$$

(ii) When the order of application of the forces is reversed — first $\boldsymbol{F}_2$ and then $\boldsymbol{F}_1$ — the work done W (and, therefore, the strain energy stored U) is

$$W_2 = U_2 = \frac{1}{2}(\alpha_{22}\,\boldsymbol{F}_2)\boldsymbol{F}_2 + \frac{1}{2}(\alpha_{11}\,\boldsymbol{F}_1)\boldsymbol{F}_1 + \alpha_{21}\,\boldsymbol{F}_1\,\boldsymbol{F}_2.$$

As $W_1(= U_1)$ and $W_2(= U_2)$ are equal — the total work done is independent of the order of application of the forces — we conclude that

$$W_1 = W_2 \qquad \alpha_{12} = \alpha_{21}.$$

This symmetrical relationship can also be stated differently[17] which is sometimes more convenient to apply.

39. Example: An Application of Betti-Maxwell Reciprocal Theorem

An elastic body (E, ν) is acted upon by a pair of forces $F - F$ acting at the points A and B. If d is the distance between A and B, calculate the change in volume of the body.

At first sight this would appear to be a very difficult problem. However, there is a trick. We can apply the Betti-Maxwell theorem and obtain the solution very easily.

How do we do that? Well, let us subject this body to a hydrostatic state of stress as shown in Fig. 13.14b. Now we have two systems: (a) the body with the two forces

[17] Fung [6] refers to the symmetrical relationship $\alpha_{ij} = \alpha_{ji}$ as Maxwell's theorem, and the different form given below as the Betti-Maxwell reciprocal theorem.

(a) A body with two forces $F - F$ (b) A block in a hydrostatic state of stress

Figure 13.14: Fig. 13.14a shows an elastic body acted upon by a pair of forces $F - F$ acting at the points A and B separated by a distance d. Fig. 13.14b shows a block acted upon by a hydrostatic state of stress represented by $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$.

$F - F$, and the desired change in volume ΔV; and (b) the body in a state of hydrostatic stress, and the change in the distance Δd between A and B.

The strain in the horizontal direction in system, $e_{xx} = \dfrac{p}{E}(1 - 2\nu)$. The points A and B move towards each other by $\delta d = d\dfrac{p}{E}(1 - 2\nu)$.

Now we apply the Betti-Maxwell reciprocal theorem: the work done by the force system in (a) on the displacements in (b)= the work done by the force system in (b) on the displacements in (a). Thus, we have

$$F \times \Delta d = \sigma \times \Delta V; \quad \longrightarrow \quad \Delta V = \frac{F}{\sigma} \times \Delta d = \frac{F\,d}{E}(1 - 2\nu).$$

If $\nu = 0.5$, the change in volume is zero. This is understandable; because if the body is incompressible — $\nu = 0.5$ corresponds to incompressibility — there cannot be any change in the volume.

We can appreciate the power of this theorem. If we are a little innovative, this theorem helps us to solve problems that may appear to be difficult.

40. Example: Castigliano's First and Second Theorems

Find the end deflection of a uniform cantilever subjected to and end load P using Castigliano's first and second theorems[18].

The strain energy U is given by

$$U = \int_0^l \frac{M^2}{2\,EI}\,dx = \int_0^l \frac{(Px)^2}{2\,EI}\,dx = \frac{P^2 l^3}{6\,EI}.$$

U is now expressed in terms of the load P. Thus, Castigliano's second theorem can be applied readily as

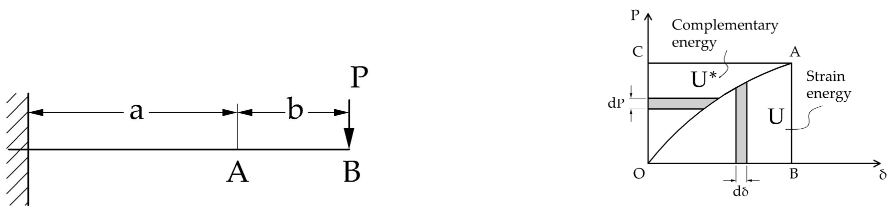
$$\text{end deflection, } \delta = \frac{dU}{dP} = \frac{Pl^3}{3\,EI}.$$

[18]Den Hartog [4] calls them the theorem of work or virtual-work theorem and Castigliano's theorem. He mentions only one Castigliano's theorem, which is what we call Castigliano's second theorem.

To apply Castigliano's first theorem, we must express the strain energy U in terms of δ. Thus, writing $P = k\delta$,

$$U = \frac{(k\delta)^2 l^3}{6\,EI}; \quad \longrightarrow \quad \frac{dU}{d\delta} = \frac{k^2 \delta l^3}{3\,EI} = P \quad \longrightarrow \quad \delta = \frac{Pl^3}{3\,EI}.$$

41. Example: Castigliano's First Theorem



(a) To find the deflection at C (b) Nonlinear case: to find the deflection

Figure 13.15: We desire to calculate the end deflection of a stepped cantilever [Fig. 13.15a]. Fig. 13.15b shows U and U^* for a nonlinear material. We shall see that Castigliano's first theorem will give a wrong result, but if it is applied to U^* we would get the correct result.

We are required to find the deflection at the end C of the stepped cantilever shown.

The strain energy U (neglecting the shear effects which is justified because this is a slender beam) is given by

$$U = \frac{1}{2}\int_0^l \frac{M^2}{EI}\,dx = \int_0^b \frac{(Wx)^2}{2\,EI_2}\,dx + \int_b^l \frac{(Wx)^2}{2\,EI_1}\,dx = \frac{W^2 b^3}{6\,EI_2} + \frac{W^2}{6\,EI_1}(l^3 - b^3).$$

Using Castigliano's (first) theorem, the deflection at C is calculated as

$$\delta_C = \frac{dU}{dW} = \frac{1}{2}\int_0^l \frac{M^2}{EI}\,dx = \int_0^b \frac{(Wx)^2}{2\,EI_2}\,dx + \int_b^l \frac{(Wx)^2}{2\,EI_1}\,dx = \frac{Wb^3}{3\,EI_2} + \frac{W}{3\,EI_1}(l^3 - b^3).$$

42. Example: Castigliano's Theorems

Fig. 13.15b shows the load-extension diagram of a nonlinear spring given by $P = \alpha x^2$. The areas below and above the curve are, as we already know, U (strain energy) and U^* (complementary energy). Let us note clearly that $U = U(x)$, while $U^* = U^*(P)$, and that now U and U^* are not even numerically equal[19].

The strain energy, U and the complementary energy, U^* are given by

$$U = U(x) = \int_0^x P\,dx = \int_0^x k\xi^2\,d\xi = \frac{kx^3}{3};$$

[19] We had emphasised that even in the linear case, when U and U^* are numerically equal, it is not proper to write $U = U^*$, because $U = U(x)$, while $U^* = U^*(P)$. Some authors write $U(x)$ and $U(P)$, instead of $U(x) = U[x(P)] = U_1(P)$. Note carefully that the functional forms U and U_1 are different!

$$U^* = U^*(P) = \int_0^P x\,dP = \int_0^P \left(\frac{P}{k}\right)^{\frac{1}{2}} dP = \frac{1}{k^{\frac{1}{2}}}\left[\frac{2}{3}P^{\frac{3}{2}} - 1\right];$$
$$\frac{dU^*}{dP} = \frac{1}{k^{\frac{1}{2}}}P^{\frac{1}{2}} = x.$$

Expressing U as a function of P, we obtain

$$U = U(x) = U[x(P)] = U_1(P) = \frac{kx^3}{3} = \frac{k}{3}\left(\frac{P}{k}\right)^{\frac{3}{2}},$$

from which we can compute dU_1/dP, but it does not give the correct answer.

Note the difference between the linear and the nonlinear cases.

Linear case:
Let us consider a linear spring specified by force-displacement relationship $P = kx$. Now the strain energy, U and the complimentary energy, U^* are

$$P = kx; \qquad U = U(x) = \frac{1}{2}kx^2; \qquad U^* = U^*(P) = \frac{1}{2}k\frac{P^2}{k^2};$$
$$\frac{dU}{dx} = kx = P; \qquad \frac{dU^*}{dP} = \frac{P}{k} = x;$$
$$dU = P\,dx \quad \longrightarrow \quad \frac{dU}{dx} = P; \qquad dU^* = x\,dP \quad \longrightarrow \quad \frac{dU^*}{dP} = x.$$

43. Example: Timoshenko's Method of Trigonometric Series
We shall now illustrate Timoshenko's method of trigonometric series of obtaining the deflection of beams. Let us, for simplicity, consider a uniform cantilever of length l loaded by a concentrated end load P. We desire to obtain the deflection curve $y = y(x)$.

Assumed deflection curve:
Let us assume the deflection curve in the form of a trigonometric series

$$y = y(x) = c_1\,\phi_1(x) + c_2\,\phi_2(x) + c_3\,\phi_3(x) + \cdots + c_n\,\phi_n(x) + \cdots,$$

where each of the coordinate functions $\phi_i(x)$ satisfies the (prescribed) boundary conditions

deflection at the fixed end:	$y(0) = 0;$
slope at the fixed end:	$y'(0) = 0.$

Let us choose the coordinate functions as

$$\phi_1 = 1 - \cos\frac{\pi x}{2l}; \quad \phi_2 = 1 - \cos\frac{3\pi x}{2l}; \cdots; \quad \phi_n = 1 - \cos\frac{(2n-1)\pi x}{2l}; \cdots.$$

Boundary conditions:
We can see that the prescribed boundary conditions are satisfied for each of these functions. Furthermore, each one of them satisfies also the condition of zero bending moment $M = EI\,y''$ at the free end.

$$\phi_n(0) = 0; \qquad \phi_n'(x) = 0; \qquad \phi_n''(l) = 0.$$

Thus, we can see that these are excellent choices, and we may anticipate that the result also will be excellent. Accordingly, the deflection curve is assumed in the form

$$y = y(x) = c_1\left[1-\cos\frac{\pi x}{2l}\right] + c_2\left[1-\cos\frac{3\pi x}{2l}\right] + \cdots + c_n\left[1-\cos\frac{(2n-1)\pi x}{2l}\right] + \cdots, \tag{13.21}$$

where the c_i's are the unknown coefficients to be determined.

Procedure:
Timoshenko's method of determining these c_i's is the following. When any one coefficient, say, c_n is varied — all the other coefficients are kept unchanged — there will be a corresponding change in the end deflection. Thus, there is some external work done by the end force P. Correspondingly, there will be some change in the strain energy too. Equating these two, one obtains the value of this coefficient c_n. We shall now carry out this procedure.

Calculations:
The approximate expression y'' for the curvature is obtained from Eq. (13.21) as

$$y'' = \sum_{1,2,3,\cdots} c_n\left(\frac{n\pi}{2l}\right)^2 \cos\frac{n\pi x}{2l}.$$

The strain energy U is, therefore, given by

$$\begin{aligned} U &= \int_0^l \frac{EI}{2}(y'')^2\,dx \\ &= \frac{EI}{2}\int_0^l \left[\sum_{1,2,3,\cdots} c_n\left(\frac{n\pi}{2l}\right)^2\cos\frac{n\pi x}{2l}\right]^2 dx \\ &= \frac{EI}{2}\sum_{1,2,3,\cdots}\int_0^l c_n^2\left(\frac{n\pi}{2l}\right)^4\cos^2\frac{n\pi x}{2l}\,dx \\ &= \sum_{1,2,3,\cdots} EI\,c_n^2\,\frac{n^4\pi^4}{64l^3}, \end{aligned} \tag{13.22}$$

where the orthogonality of the cosine functions

$$\int_0^l \cos\frac{m\pi x}{2l}\cos\frac{n\pi x}{2l} = \begin{cases} 0 & \text{if } m\neq n \\ \frac{l}{2} & \text{if } m = n, \end{cases}$$

is exploited and a few steps of simplification are omitted.

We shall now take Eq. (13.22), and change only one coefficient c_n to $c_n + dc_n$, while all the other coefficients are left unchanged. Then the strain energy U will change to $U + dU$, and the deflection y to $y + dy$. The end deflection will change slightly, and the external force P will do some work. This work must be equal to the change dU in U.

$$c_n \to c_n + dc_n \qquad \longrightarrow \qquad U \to U + dU \qquad \longrightarrow \qquad y \to y + dy.$$

The change dy in the deflection y at the end of the cantilever $x = l$ is

$$dy = dc_n \left(1 - \cos\frac{n\pi x}{2l}\right)\Big|_{x=l} = dc_n$$

Equating dU and the external work $P\,dc_n$, we have

$$P\,dc_n = dU \qquad \longrightarrow \qquad P = \frac{\partial U}{\partial c_n} = 2c_n\,\frac{n^4\pi^4}{64l^3}\,EI.$$

Thus, we obtain

$$c_n = \frac{Pl^3}{EI}\,\frac{32}{\pi^4 n^4}.$$

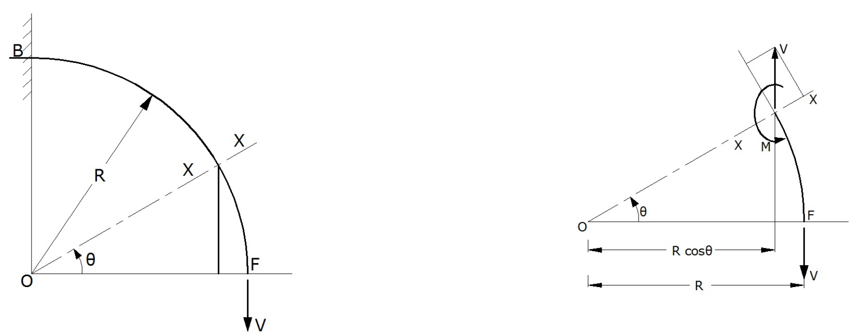
Such a result holds obviously for every coefficient. We, therefore, obtain the deflection of the cantilever as

$$y = \frac{32\,Pl^3}{\pi^4\,EI}\sum_{1,3,5,\cdots}\frac{1}{n^4}\left(1 - \cos\frac{n\pi x}{2l}\right).$$

This rapidly converging infinite series gives the exact solution. The end deflection is of special interest. It can be calculated (if three terms are taken for the calculation) as

$$\delta = y\Big|_{x=l} = \frac{32\,Pl^3}{\pi^4\,EI}\left[1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right] = \frac{Pl^3}{3.001\,EI}.$$

44. Example: Curved Beam - Castigliano's Theorem



(a) A curved beam: a vertical load at the end (b) Free-body diagram: part of the beam

Figure 13.16: A curved beam BF [Fig. 13.16a] — a quarter of a circle — is loaded by a vertical end load V. It is desired to find the vertical deflection δ_v at the free end F. The free-body diagram of a part of the beam is shown [Fig. 13.16b].

We shall see the use of Castigliano's theorem to calculate the deflection of beams. The curved beam of uniform cross-section shown [Fig. 13.16a] is a quarter of a circle of radius R built in at the end B, and free at the end F. It is loaded by a vertical load V at the end F. We desire to find the vertical deflection δ_v at the end of the curved cantilever.

Vertical Deflection Due to a Vertical Load

Castigliano's theorem gives the deflection by the formula $\delta_v = \dfrac{dU}{dV}$, where U is the strain energy of the beam expressed in terms of the load V in good shape to carry out the indicated differentiation. Thus, we need to express U as a function of V in the form $U = U(V)$. We know from our earlier study of the Mechanics of Solids that the strain energy stored in a bent beam is given by

$$U = \int_0^L \frac{M^2}{2EI}\, ds,$$

with the usual notation. Once the bending moment is calculated in terms of the applied load V, the exercise reduces to some elementary mathematical manipulations. We shall now carry out the calculation of the bending moment as a function of the angle θ. Referring to the free-body diagram [Fig. 13.16b], we find that the bending moment M at a typical section $X - X$ at an angular distance θ from the horizontal line is[20]

$$M = V(R - R\cos\theta) = VR(1 - \cos\theta).$$

For this problem, we proceed as follows.

$$U = \int_0^L \frac{M^2}{2EI}\, ds = \int_0^{\frac{\pi}{2}} \frac{M^2}{2EI}\, R\, d\theta \tag{13.23}$$

$$\delta_v = \frac{dU}{dV} = \frac{d}{dV}\int_0^{\frac{\pi}{2}} fracM^2 2EI\, R\, d\theta = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial V}\left(\frac{M^2}{2EI}\right) R\, d\theta \tag{13.24}$$

$$= \frac{1}{EI}\int_0^{\frac{\pi}{2}} M\frac{\partial M}{\partial V}\, R\, d\theta = \frac{1}{EI}\int_0^{\frac{\pi}{2}} [VR(1-\cos\theta)][R(1-\cos\theta)]\, R\, d\theta \tag{13.25}$$

$$= \frac{\pi}{2}\,\frac{VR^3}{EI}.$$

We have obtained the required answer. However, a few remarks seem to be in order. In the first place, we need to form a habit of checking the dimensions. Let us make sure that the final answer obtained does, indeed, have the proper dimensions (m, because this is the expression for the deflection). In addition, let us note that in Eq. (13.24), two operations are involved, integration (with respect to θ) first, and then differentiation (with respect to V). The order of these operations can be interchanged as first differentiation, and then integration. This is mathematically justified. Such an interchange is convenient, because we will have simpler expressions for integration. The integrand in Eq. (13.24) is a function not merely of V, but also of other variables. Thus, it is necessary to write as a partial derivative as indicated in Eq. (13.25). We may also clarify that the integration in Eq. (13.23) is along the full length $0 - L$ of the beam.

[20] The sign of M, positive or negative, depends on the sign convention employed. Here this is of no consequence, because we need the expression of only its square (M^2) in the above integrand. Even so, it is better to stick to one's own convention as a matter of good habit. Yet, it is also desirable to spurn this once in a way, to emphasise or remind ourselves the fact that it is, after all, only a convention and not a binding command.

This exercise is over at this stage. But we can discuss some more aspects regarding such problems. The following points are worth noting.

(i) Although this is a 'curved' beam, we have not used the curved beam theory (the so-called Winkler-Bach equation) here. The reason is that the radius of curvature is very large compared to the depth of the beam. Thus, even though this is a curved beam for external appearances, this is essentially a straight beam; curvature effects need not be taken into account.

(ii) We can notice that on a typical cross-section $X-X$, there is a normal stress N, and a shear force, S, in addition to the bending moment. Thus, there are normal (taken to be uniformly distributed) stresses due to this N, and shear stresses (cross-shear) due to the shear force F. Associated with these stresses, there are contributions to the strain energy, U. We did not take them into account.

This may appear to be a lapse on our part. But, no. For such a this beam as we have here, the contributions of these two components are very small. We may, thus, disregard them in this example. However, when thick beams are considered, we must include the contributions from the direct normal stresses and the cross-shear stresses. We shall consider this more complicated case in a later example.

(iii) If we desire to calculate the horizontal deflection at the free end[21], we need to pretend that there is an external load, say, H at the free end, calculate the strain energy in terms of both V and H, and use the Castigliano's theorem in the form

$$\delta_h = \frac{\partial U}{\partial H} \text{ where } U = U(V, H)$$

and, then after the differentiation is carried out, wake up and realise that no such horizontal force is acting. Thus, substitute $H = 0$ in the final formula to obtain the horizontal deflection δ_h.

We shall illustrate this case below.

Horizontal Deflection Due to a Vertical Load

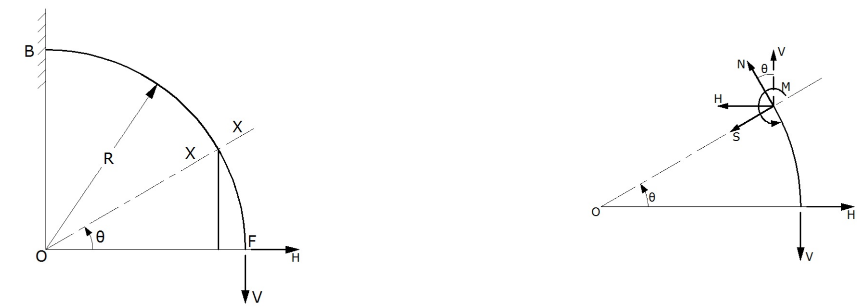
Shown in Fig. 13.17a is the same curved beam, but this time we desire to calculate the horizontal deflection, δ_h, at the free end F. We pretend that a horizontal force H also is acting, and follow the same procedure as in the previous example. We need to do this, because Castigliano's theorem is concerned with the 'work absorbing' components of forces and displacements. The desired deflection is the horizontal deflection, and the associated 'work absorbing' force is a horizontal force H at the free end F.

The horizontal deflection, δ_h is given by

$$\delta_h = \left(\frac{\partial U}{\partial H}\right)_{H=0} = \frac{\partial}{\partial H}\int_0^{\frac{\pi}{2}} \frac{M^2 R}{2EI}\, d\theta = \frac{1}{EI}\int_0^{\frac{\pi}{2}} M \frac{\partial M}{\partial H}\, R\, d\theta\Big|_{H=0}. \tag{13.26}$$

The bending moment on a typical cross-section $X-X$ is $M = VR(1-\cos\theta) - HR\sin\theta$. The derivative of M w.r.to the horizontal load H is $\partial M/\partial H = -R\sin\theta$. Substituting

[21] Some students seem to have difficulty in realising that there will be a horizontal deflection δ_h at the free end, even though the applied load is entirely vertical.



(a) A curved beam: vertical load at the end (b) Free-body diagram: part of the curved beam

Figure 13.17: A curved beam BF [Fig. 13.17a] — a quarter of a circle — is loaded by a vertical end load V. It is desired to find the horizontal deflection δ_h at the free end F. We pretend that a horizontal load H is also acting at the free end. The free-body diagram of a part of the beam is shown [Fig. 13.17b].

these expressions for the bending moment, M, and the derivative $\partial M/\partial H$ in Eq. (13.26), we obtain the required horizontal deflection as

$$\delta_h = \frac{1}{EI}\int_0^{\frac{\pi}{2}} [VR(1-\cos\theta) - HR\,\sin\theta][-R\,\sin\theta]\,R\,d\theta = \frac{VR^3}{4EI}.$$

45. Example: A Truss Problem

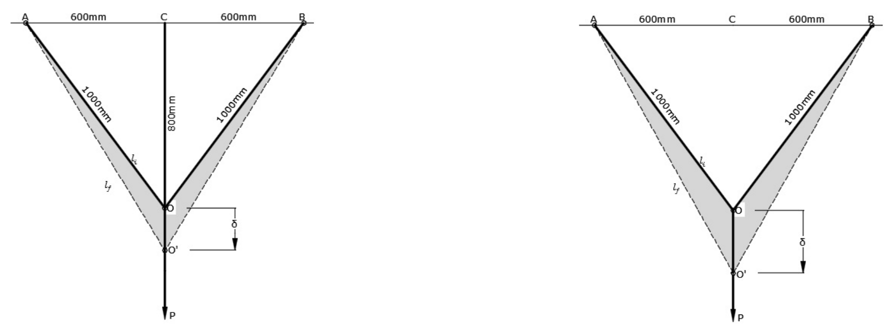
We shall solve a truss problem using Castigliano's theorem[22] in its simplest form.

Shown in Fig. 13.18a is a simple, planar, three-member truss $OACB$. Shown alongside is a similar truss, but without the member OC. Let us calculate the force P to be applied to obtain a specified vertical deflection δ of the joint O for both these cases. The dimensions are $AC = CB = 600$, $CO = 800$, $OA = OB = 1000$, all in mm. The joint O is brought down to the position O'. If the vertical deflection OO' is specified as δ, we are required to calculate the vertical load P to be applied at O. All the bars have the same diameter of 1.5 mm; Young's modulus of elasticity (steel), $E = 200$ GPa. There is no member CO for the second truss.

For convenience, let us designate the length $OA\,(= OB)$ as l, the area of cross-section as A (the same for all the members), the semi-vertical angle COA as α, and the vertical length OC as h. As long as δ is small, which is almost always the case, there is no significant difference between (i) the angles COA and $CO'A$, and (ii) the vertical lengths OC and $O'C$. Both the problems (a) and (b) are exactly similar as far as the geometry of deformation is concerned.

[22] Castigliano's theorem: $\dfrac{\partial U(\delta)}{\partial \delta} = P$

Some authors, for example, J.P. Den Hartog [4], refer to this as the theorem of work. Some others, for example, I.H. Shames & C.L. Dym, *Energy and Finite Elements in Structural Mechanics*, Wiley Eastern Limited,(1950) call this Castigliano's first theorem.



(a) A three-member planar truss (b) A similar two-member truss

Figure 13.18: Two similar planar trusses. The only difference is that in the truss on the right hand side, there is no member OC. The members meet at the point O where a displacement δ is imposed. It is desired to calculate the load P needed for this.

Solution from first principles:
We can surely solve these simple problems from first principles. If the vertical deflection $OO' = \delta$, then the new length AO' can be calculated in terms of δ, h and $\cos\alpha$ by solving the shaded triangle AOO'. Then we can calculate the forces in the members — three, viz., OA, OB and OC in (a); and two, viz., OA and OB in (b) — and obtain the vertical load to be applied in each of the two cases by considering the equilibrium of the joint.

We, however, desire to work out this problem to demonstrate how Castigliano's (first) theorem is used in such cases.

Solution using Castigliano's first theorem:
To be able to use the Castigliano's first theorem, we need to express the energy U as a function of δ. Let us, therefore, calculate the new length $O'A$. This ($O'A$) can be seen to be (by dropping a perpendicular from O to the side $O'A$ of one of the shaded triangles) stretched by the amount $\delta\cos\alpha$. Let us observe a couple of relationships from the geometry.

$$\cos\alpha = \frac{h}{l}; \quad \text{change in length, } O'A - OA = \delta\,\cos\alpha$$

The strain in the bar $O'A$ — the same everywhere in the bar — is

$$\epsilon = \frac{O'A - OA}{O'A} = \frac{\delta\,\cos\alpha}{l} = \frac{\delta\,\cos\alpha}{(h/\cos\alpha)} = \frac{\delta\left(\frac{h}{l}\right)}{h/\left(\frac{h}{l}\right)} = \frac{\delta h}{l^2}.$$

The strain energy in the bar $O'A$ is calculated as shown as

$$\iiint_{\mathcal{V}} \left[\int_{\epsilon} \sigma\, d\epsilon\right] dv = \left[\int_{\epsilon} \sigma\, d\epsilon\right] Al = Al \int_{\epsilon} E\epsilon\, d\epsilon = E\epsilon^2 Al.$$

The triple integral need not scare us; the integrand is a constant at all points in $\mathcal{V}$. All we need to do is to calculate the volume as Al and multiply with the constant integrand.

Case (b): only two members OA and OB:
Let us now consider case (b) first where there are only the two members OA and OB. Now the total strain energy, U is

$$U = U(\delta) = 2 \times E\epsilon^2 Al = 2 \times E\left(\frac{\delta h}{l^2}\right)^2 A\left(\frac{h}{\cos\alpha}\right)$$
$$= \frac{2EA\,\delta^2\,\cos^3\alpha}{h} \tag{13.27}$$

To obtain the desired load, we only need to differentiate this expression (13.27) for $U = U(\delta)$ w.r.to δ.

$$P = \frac{\partial U(\delta)}{\partial\delta} = \frac{2EA\,\cos^3\alpha}{h}\,\delta.$$

When the given numerical values are substituted, we obtain the desired result as

$$\begin{aligned} P &= \frac{2EA\,\cos^3\alpha}{h}\,\delta \\ &= \frac{2(200\times 10^9)(\frac{\pi\,1.5^2}{4}\times 10^{-6})\times(0.8^3)}{0.8}\,\delta \\ &= 452.4\,\delta \text{ kN}; \quad \text{i.e., } 0.4524 \text{ kN for every mm of vertical deflection.} \end{aligned}$$

Being a linear structure, the load P will be proportional to the deflection δ. Now, to complete the problem, we shall take up the first case (a).

Case (a): three members OA,OB and OC:
In this case, the vertical load should, additionally, be able to stretch the vertical member OC also by the same amount, δ. Thus, we may compute the axial tensile force necessary to stretch the member OC — a uniform bar of length 800 mm — by δ, and add this amount to the answer obtained for the case (b) above.

$$\begin{aligned} \text{additional load} \quad &= \frac{AE}{l}\,\delta \\ &= \frac{\pi\,1.5^2(200\times 10^9)}{4\times 8}\,\delta \\ &= 0.4418 \text{ kN per mm of vertical deflection.} \end{aligned}$$

The answer now is $P = 0.4524 + 0.4418 = 0.8942$ kN per mm of vertical deflection.

We can include the additional strain energy in the member OC in Eq. (13.27) and obtain the same answer using Castigliano's first theorem.

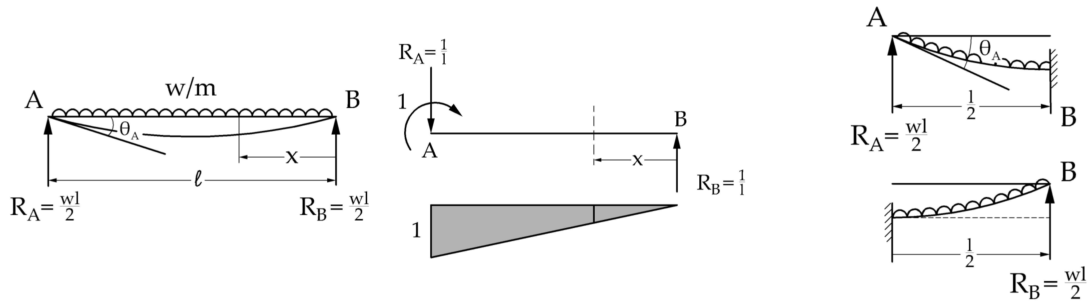
$$\text{total strain energy now} \quad U = U(\delta) = \left[\frac{2EA\,\delta^2\,\cos^3\alpha}{h}\right] + \left[\frac{AE}{2l}\,\delta^2\right]$$

Applying Castigliano's first theorem,

$$\frac{\partial U(\delta)}{\partial\delta} = P.$$

When we substitute numerical values, we again obtain the same answer: $P = 0.4524 + 0.4418 = 0.8942$ kN per mm of vertical deflection.

46. Example: Unit-load Method



(a) To find the slope at A (b) A unit moment applied at A (c) Half the beam: a cantilever

Figure 13.19: Fig. 13.19 shows a simply supported beam with a uniformly distributed load w /unit length. Fig. 13.19b shows a unit moment applied at the end A where we desire to find out the slope (rotation). The bending moment diagram for this loading of a unit-moment is also shown there. Fig. 13.19c shows how this problem may be solved easily by an alternative method which Den Hartog calls the Myosotis method.

Using the unit-load method, calculate the slope θ_A of a simply supported beam AB of length l with a uniformly distributed load w / unit length [Fig. 13.19a].

The bending moment M for this uniformly distributed load is

$$M = M(x) = \frac{1}{2}wlx - \frac{1}{2}wx^2 \qquad (0 \le x \le l).$$

As we desire to determine the slope (rotation) θ_A at the end A, we apply a unit moment at the point A as shown in Fig. 13.19b. The bending moment diagram due to this applied unit moment is shown in the figure. Its expression is given by

$$m = m(x) = \frac{x}{l} \qquad (o < x \le l).$$

Applying the equation

$$\Delta = \int_0^l \frac{m}{EI}\,dx,$$

we obtain

$$\theta_A = \int_0^l \frac{(\frac{wlx}{2} - \frac{wx^2}{2})(\frac{x}{l})}{EI}\,dx = \frac{1}{E}\int_0^l \left(\frac{wx^2}{2} - \frac{wx^3}{2l}\right)dx = \frac{w\,l^3}{24\,EI}.$$

A positive value shows that the slope (rotation) is in the direction of the applied moment (that is, clockwise). Thus, this rotation is clockwise.

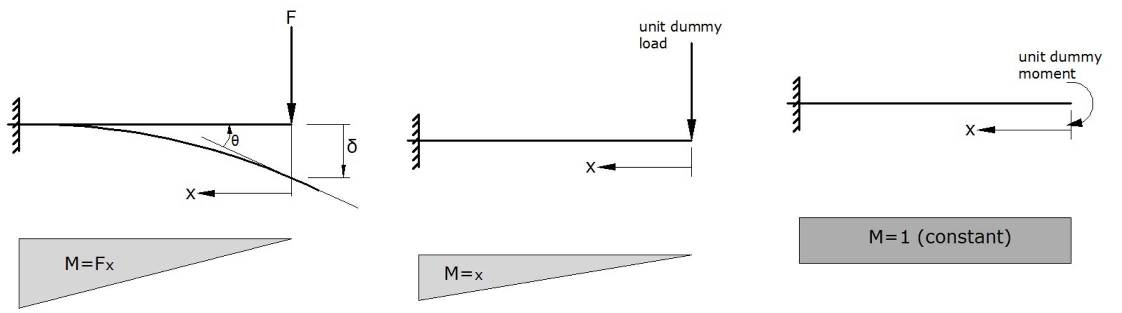
(It is interesting to note that this problem can be solved just by inspection using the 1 2 2 3 6 8 method (Myosotis method). Fig. 13.19c shows a cantilever which is really only half the beam [Fig. 13.19a]. The slope θ_A (which happens to be equal to θ_B because

of the symmetry) can be seen to be the superposition (taking into account the proper signs) of (i) a uniformly distributed load w / unit length, and (ii) a concentrated load of $wl/2$. Accordingly,

$$\theta_A = \theta_B = \frac{w(\frac{l}{2}^3)}{6\,EI} - \frac{(\frac{wl}{2})(\frac{l}{2}^2)}{2\,EI} = \frac{wl^3}{48\,EI} - \frac{wl^3}{16\,EI} = \frac{wl^3}{32\,EI}.$$

47. Example: Cantilever Loaded by an End Load F

Let us illustrate the dummy load method to calculate the end deflection and the end



(a) A cantilever loaded by an end load, F (b) A unit dummy load applied at the end (c) A unit dummy moment applied at the end

Figure 13.20: A cantilever with an end load, F. To compute (i) the vertical deflection and (ii) the rotation (slope) at the end (i) a unit dummy load and (ii) a unit dummy moment are applied, respectively. The corresponding bending moment diagrams are shown.

slope of a uniform cantilever loaded by an end load F. Referring to Fig. 13.20a, the bending moment M due to the load F is $M = F\,x$.

Now let us apply a unit load at the end of the cantilever where the deflection is desired. The bending moment due to this unit dummy load 1 is $m = x$.

$$\delta = \int \frac{Mm}{EI}\,ds = \int_0^l \frac{(Fx)(1\times x)}{EI}\,dx = \left.\frac{Fx^3}{3EI}\right|_0^l = \frac{Fl^3}{3EI} \text{ downwards.}$$

Now to calculate the end slope θ, we apply a unit dummy moment 1. The bending moment m due to this unit moment 1 is $m = 1$ constant throughout the length.

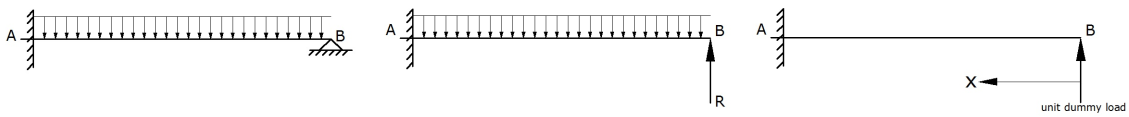
The required slope θ at the end can be calculated as

$$\theta = \int \frac{Mm}{EI}\,ds = \int_0^l \frac{(Fx)(1)}{EI}\,dx = \left.\frac{Fx^2}{2EI}\right|_0^l = \frac{Fl^2}{2EI} \text{ clockwise.}$$

A positive value for the answers (deflection and rotation) signifies that it is in the direction of the applied unit load / applied unit moment.

48. Example: Propped Cantilever

Fig. 13.21 shows a uniform cantilever (constant EI) of length l loaded by a uniformly



(a) A propped cantilever loaded by a u.d.l. (b) Support removed and replaced by a reaction R (c) A unit dummy load applied at the end

Figure 13.21: A propped cantilever with a uniformly distributed load, w per unit length. We employ a unit dummy load method to calculate the support reaction, R.

distributed load w per unit length and supported at the end as shown. We desire to obtain the reaction R at the support using the dummy load method.

Referring to Fig. 13.21b, the bending moment is given by

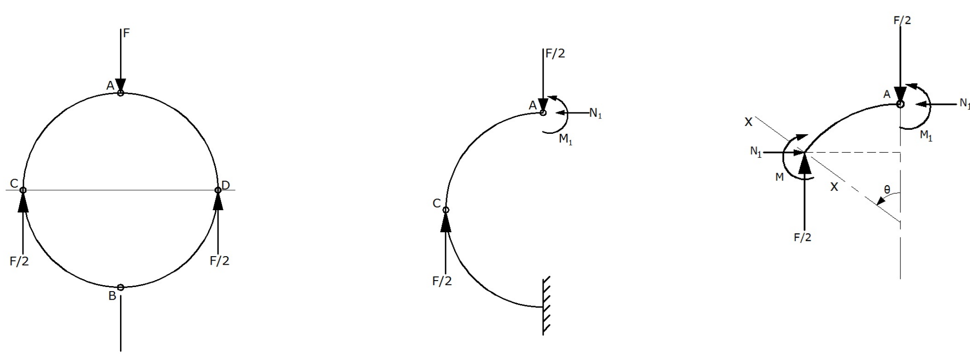
$$M = -\frac{1}{2}wx^2 - Rx.$$

The bending moment due to the unit dummy load applied at the end is $m = x$. Thus, the vertical deflection δ at the end is

$$\text{deflection:} \quad \delta = \int_0^l \frac{Mm}{EI}\,dx = \frac{1}{EI}\int_0^l (-\frac{1}{2}wx^3 + Rx^2)\,dx = \frac{Rl^3}{3EI} - \frac{wl^4}{8EI}. \tag{13.28}$$

But the vertical deflection at the end is zero. Hence setting this expression (13.28) for the vertical deflection $d = 0$, we obtain the desired answer as $R = (3/8)\,wl$.

49. Example: A Large Thin Ring



(a) The full large, thin ring (b) Half the ring (c) Free-body diagram

Figure 13.22: A large, thin ring $ACBD$ is loaded by a vertical downward force F [Fig. 13.22a]. Because of the symmetry about the vertical axis AB, only the left half is considered. It is desired to find the locked-in moment (M_1) and the axial thrust (N_1), both at the point A. The free-body diagram of a part of the ring (beam) is shown [Fig. 13.22c].

(a) A unit dummy load applied at A (b) A unit dummy moment applied at SA

Figure 13.23: To compute the expressions for (i) the vertical deflection and (ii) the rotation of the cross-section at A, (i) a unit dummy load [Fig. 13.23a] and (ii) a unit dummy moment [Fig. 13.23b] are applied respectively. By setting these expressions to zero, we obtain the two equations to obtain the desired locked-in moment M_1 and the axial force N_1.

Fig. 13.22a shows a large, thin ring with a vertical downward force F. We are required to find the locked-in moment (M_1) and the axial thrust (N_1) at the topmost point of the ring where the vertical load is applied.

Let us note that the structure and the loading are symmetrical about the vertical line AB. Consequently, there will be neither a horizontal deflection, nor a rotation, at A. Furthermore, the lowest point B may be regarded as fixed (clamped)[23]. We, therefore, need to consider only one half of the ring as shown in Fig. 13.22b. There will be a locked-in moment (M_1) and an axial thrust (N_1) at A[24].

From Fig. 13.22c we note that the bending moment is given by

$$M = \begin{cases} M_1 + N_1 R(1-\cos\theta) - \frac{F}{2} R \sin\theta & \text{for } 0 < \theta < \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} < \theta < \pi. \end{cases} \tag{13.29}$$

We know from our previous experience that (i) it is sufficient to use the straight beam, Euler-Bernoulli theory, and not the Winkler-Bach theory for curved beams, because the radius of curvature is large compared to the depth of the beam[25], and that (ii) the bending moment is all that contributes to the total strain energy. [In principle, the shear force and the axial thrust do contribute to the strain energy, but these contributions are negligibly small in such cases of large, thin rings.]

Let us note that the two additional equations[26] to determine the unknowns M_1 and N_1 are obtained by calculating the expressions for (i) the horizontal deflection, and (ii) the

[23] Actually the point B may have a vertical deflection, but we also know that a rigid-body displacement does not affect the internal forces or the stresses.

[24] Students often fail to recognise the possible existence of these.

[25] The beam looks curved and, indeed, it is curved. Yet it behaves like a straight beam! The curvature effects are negligible.

[26] in addition to the equations of equilibrium

rotation of the cross-section, both at the point A, and set them equal to zero, because we know from symmetry considerations that they must both be zero.

50. Example: Dummy Load at *A*

Let us apply a unit dummy load at A in the horizontal direction [Fig. 13.23a]. The bending moment m because of the unit dummy load is

$$m = 1 \times R(1 - \cos\theta).$$

Hence, the horizontal deflection at $A \;= 0 = \int \frac{Mm}{EI}\,ds = \frac{R^2}{EI}\int_0^\pi M(1-\cos\theta)\,d\theta$, giving us one equation as

$$\int_0^\pi M(1-\cos\theta)\,d\theta = 0. \tag{13.30}$$

51. Example: Dummy Moment at *A*

Let us now apply a unit dummy moment at A [Fig. 13.23b]. Hence,

$$\text{rotation at } A \;= 0 = \int \frac{Mm}{EI}\,ds = \frac{R}{EI}\int_0^\pi M\,d\theta,$$

giving us the equation as

$$\int_0^\pi M\,d\theta = 0.. \tag{13.31}$$

From Eqs (13.30), (13.31), we obtain

$$\int_0^\pi M\,\cos\theta\,d\theta = 0. \tag{13.32}$$

These two equations (13.31), (13.32) give, when the expression for M is substituted from Eq. (13.29),

$$M_1\int_0^\pi d\theta + N_1 R\int_0^\pi (1-\cos\theta)\,d\theta - \frac{F}{2}R\int_0^{\frac{\pi}{2}} \sin\theta\,d\theta - \frac{F}{2}R\int_{\frac{\pi}{2}}^\pi d\theta = 0, \text{ and} \tag{13.33}$$

$$\begin{aligned} M_1\int_0^\pi \cos\theta\,d\theta + N_1 R\int_0^\pi (1-\cos\theta)\,\cos\theta\,d\theta - \frac{F}{2}R\int_0^{\frac{\pi}{2}} \sin\theta\,\cos\theta\,d\theta \\ - \frac{F}{2}R\int_{\frac{\pi}{2}}^\pi \cos\theta\,d\theta = 0. \end{aligned} \tag{13.34}$$

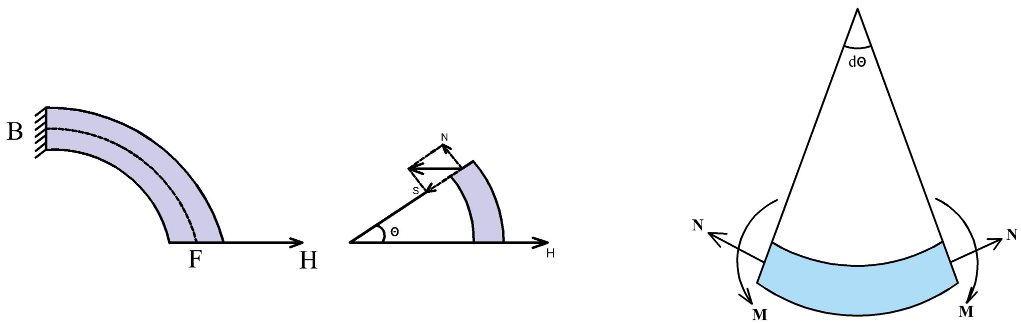
Solving these two equations (13.33, 13.34), we obtain the required result as

$$M_1 = \frac{FR}{4}, \text{ and } N_1 = \frac{F}{2\pi}.$$

This completes the illustrative example. Next we shall consider a thick curved beam.

52. Example: A Thick Curved Beam

We had solved one problem earlier treating the beam as thin [Sec. 44, page 13-43]. We had mentioned that, when the radius of curvature is large compared to the depth of the beam, the curved beam theory need not be used, and that the strain energies due to (i) the direct thrust and (ii) the shear force may be neglected. Let us demonstrate this. Let us solve this problem here in this section, and show that the contributions of (i) and (ii) are indeed negligibly small when the radius of curvature is large, thereby vindicating the position that we took in Sec. 44. Here we shall solve the problem treating the beam as thick. Referring to the free-body diagram, and using the sign conventions as per the



(a) A thick curved beam loaded by a horizontal load at the end

(b) Shear force, thrust and bending moment on a beam element

Figure 13.24: A curved beam BF [Fig. 13.24a] — a quarter of a circle — is loaded by a horizontal end load H. It is desired to find the horizontal deflection δ_h at the free end F. The free-body diagram of a part of the beam is shown alongside. At the right is shown a beam element along with the shear force S, direct thrust N and bending moment M [13.24b].

positive directions marked [Fig. 13.24b, page 13-54], we arrive at the expressions for the bending moment, M; the direct thrust, N; and the shear force, S. These are

$$\text{bending moment:} \quad M = HR\sin\theta; \tag{13.35}$$

$$\text{direct thrust:} \quad N = H\sin\theta; \ \text{ and} \tag{13.36}$$

$$\text{shear force:} \quad S = -H\cos\theta. \tag{13.37}$$

The total strain energy, U is given by

$$\begin{aligned} U &= U_M + U_N + U_S \\ &= \int_0^s \left[\left(\frac{M^2}{2AEeR} \right) + \left(\frac{N^2}{2AE} - \frac{MN}{AER} \right) + \left(\frac{\alpha S^2}{2AG} \right) \right] ds \\ &= \int_0^{\frac{\pi}{2}} \left[\left(\frac{M^2}{2AEeR} \right) + \left(\frac{N^2}{2AE} - \frac{MN}{AER} \right) + \left(\frac{\alpha S^2}{2AG} \right) \right] R\, d\theta. \end{aligned}$$

Substituting the expressions for M, N and S [Eqs (13.35, 13.36 and 13.37)], we obtain the expression for the total strain energy, U as

$$U = \int_0^{\frac{\pi}{2}} \left[\left(\frac{H^2 R^2 \sin^2\theta}{2AEeR} \right) + \left(\frac{H^2 \sin^2\theta}{2AE} - \frac{H^2 R \sin^2\theta}{AER} \right) + \left(\frac{\alpha H^2 \cos^2\theta}{2AG} \right) \right] d\theta.$$

Carrying out the differentiation of U w.r.to the horizontal load, H to obtain the horizontal deflection of the free end, we have

$$\begin{aligned}\delta_h = \frac{dU}{dH} &= \frac{HR}{AE} \int_0^{\frac{\pi}{2}} \left[\left(\frac{R \sin^2\theta}{e} \right) - \left(\sin^2\theta\right) + \left(\frac{\alpha E}{G} \cos^2\theta \right) \right] d\theta \\ &= \frac{\pi HR}{4AE} \left[\frac{R}{e} + \frac{\alpha E}{G} - 1 \right] \\ &= \frac{\pi HR}{4AE} \left[\frac{12R^2}{h^2} + 2.12 \right] \qquad (13.38)\end{aligned}$$

We can see that in Eq. (13.38) the terms 2.12 is really negligible in comparison to the first term within the pair of square brackets, viz., $12R^2/h^2$, whenever h is small compared to R. This vindicates the position that we had taken: that the effects of the direct thrust and the shear forces are negligible for a thin beam.

53. Example: Thick Cylinders - Lamé's Problem

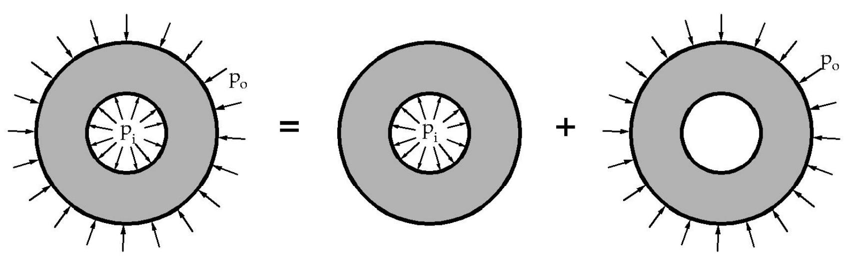


Figure 13.25: The figures show a thick cylinder subjected to pressures. The solution for the case (i) of both an internal pressure p_i and an external pressure p_o can be obtained by superposition of the solutions for the cases of (ii) an internal pressure p_i only, and of (iii) an external pressure p_o only.

Find the stresses in a thick cylinder subjected to an internal pressure p_i and an external pressure p_o. The inner and outer radii are r_i and r_0 respectively.

We know that the radial and tangential stresses are given by

$$\sigma_{rr} = A + \frac{B}{r^2} \qquad \text{and} \qquad \sigma_{\theta\theta} = A - \frac{B}{r^2}.$$

The constants A and B are determined using the known boundary conditions:

$$(a) \;\; \text{at } r = r_i, \quad \sigma_{rr} = -p_i \qquad \longrightarrow \qquad A + \frac{B}{r_i^2} = -p_i;$$

$$(b) \text{ at } r = r_o, \quad \sigma_{rr} = -p_o \qquad \longrightarrow \qquad A + \frac{B}{r_o^2} = -p_o.$$

Solving for A and B — the slightly 'dirty' algebra is left out —

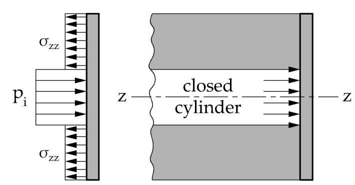
$$A = \frac{p_i\, r_i^2 - p_o\, r_o^2}{r_o^2 - r_i^2}; \qquad B = \frac{(r_i\, r_o)^2 (p_o - p_i)}{r_o^2 - r_i^2}.$$

The expressions for the stress components are

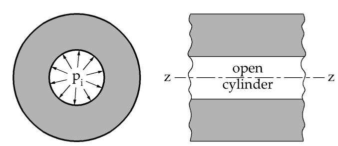
$$\sigma_{rr} = \frac{p_i\, r_i^2 - p_o\, r_o^2 + \left(\frac{r_i\, r_o}{r}\right)^2 (p_o - p_i)}{r_o^2 - r_i^2};$$

$$\sigma_{\theta\theta} = \frac{p_i\, r_i^2 - p_o\, r_o^2 - \left(\frac{r_i\, r_o}{r}\right)^2 (p_o - p_i)}{r_o^2 - r_i^2}.$$

[We may also write Lamé's equations differently, but equivalently, as $\sigma_{rr} = C - D/r^2$ and $\sigma_{\theta\theta} = C + D/r^2$. The constants are now different. It is desirable for us to stick to one set of formulae, but once in a while this habit can be broken to remind ourselves that both sets of formulae are equally correct.]



(a) A closed cylinder



(b) An open cylinder

Figure 13.26: Closed and open thick cylinders subjected to internal pressure. There will be an axial stress σ_{zz} for a closed cylinder, but not for an open one.

Closed cylinder:
In addition, there will be an axial stress σ_{zz} which is uniform on the cross-section if the cylinder is closed [Fig. 13.26a]. The cross-sectional area is $\pi(r_o^2 - r_i^2)$. The fluid pressure acting on the end plate is $p_i\, \pi\, r_i^2$. Hence the axial stress σ_{zz} is

$$\sigma_{zz} = \frac{p_i\, \pi\, r_i^2}{\pi(r_o^2 - r_i^2)} = \frac{p_i\, r_i^2}{r_o^2 - r_i^2}.$$

Open cylinder:
If the cylinder is open [Fig. 13.26b], there is no axial stress: $\sigma_{zz} = 0$.

We may also calculate the stresses for each of the two loadings, viz., (i) p_i only, and (ii) p_o only, and add up the stresses to obtain the stresses for the combined loading of p_i and p_o. As explained earlier, this is the idea of superposition.

Numerical values:

We can work out the numerical values and the distribution of stresses by giving some values, say, $r_i = 100$ mm, $r_o = 140$ mm, $p_i = 2$ MPa, $p_o = 1$ MPa.

54. Example: A Thick Cylinder Subjected to an Internal Pressure

A thick cylinder of steel is subjected to an internal pressure of 2 MPa. Calculate the stresses inside the cylinder and the radial expansion of the inner radius. (The inner and outer radii are $r_i = 100$ mm, $r_o = 140 =$ mm; $E = 200$ GPa; $\nu = 0.3$.)

We can use the formula to obtain the hoop (circumferential) stress $\sigma_{\theta\theta}$:

$$\sigma_{\theta\theta} = p\,\frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} = 2\,\frac{140^2 + 100^2}{140^2 - 100^2} = 6.17\,\text{MPa}.$$

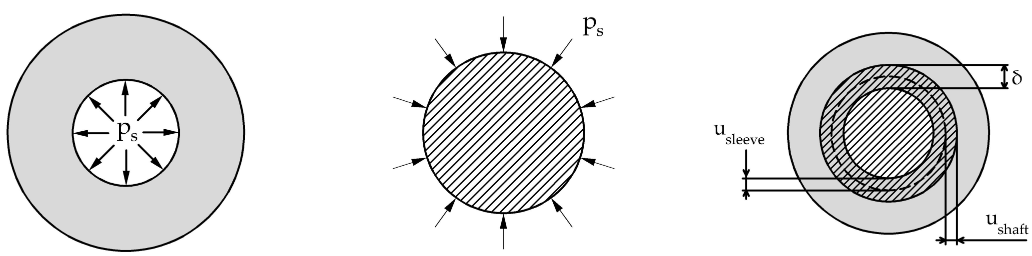
σ_{rr} at the inner radius = 2 MPa.

Closed cylinder

If the cylinder is closed, there will be tensile axial stress. Its magnitude is

$$\sigma_{axial} \equiv \sigma_{zz} = p\,\frac{r_i^2}{r_o^2 - r_i^2} = 20 \times \frac{100^2}{140^2 - 100^2} = 2.08\,\text{MPa}.$$

55. Example: Interference Fit Calculations



(a) An outer brass sleeve (b) The inner solid steel shaft (c) The assembly

Figure 13.27: Figs 13.27 show an interference fit (shrink fit) of a brass sleeve on a steel shaft. Thus, the outer diameter of the solid steel shaft [Fig. 13.27b] is actually a little larger, though nominally the same. When the sleeve is fitted on the shaft, the inner boundary of the sleeve moves up slightly, and the diameter of the shaft correspondingly reduced. Fig. 13.27c shows the various diameters, before and after the assembly.

A brass sleeve is shrunk fitted on a steel shaft. The outer and inner diameters of the sleeve are 280 mm and 200 mm. The shaft has a nominal diameter of 200 mm and an actual diameter of 200.40 mm. (The interference on the radius is $0.40/2 = 0.20$ mm.) Calculate the interference pressure and the stresses in the shaft and in the sleeve. The Young's moduli of elasticity E and the Poisson's ratios ν are $E_{brass} = 100$ GPa; $E_{steel} = 200$ GPa; $\nu_{brass} = 0.33$; and $\nu_{steel} = 0.30$.

Procedure:

(i) Understand the problem: The diameter of the inner steel shaft (nominal diameter 200 mm) is a little more (0.40 mm more) than the inner diameter of the outer brass sleeve. The two are force fitted. There is an interference pressure p_s which acts radially inwards on the inner steel shaft [Fig. 13.27b], and radially outwards on the inner boundary of the outer brass sleeve [Fig. 13.27a]. The interference pressure p_s is unknown.

(ii) This statically indeterminate interference pressure p_s (also called shrinkage pressure) is unknown. We need to use the principle of consistent deformation — the compatibility of displacements — to determine this p_s.

(iii) This is the same problem that was worked out earlier, except that the numerical value of the pressure p_s is now unknown. Thus, we can obtain the (expressions for the) stress components σ_{rr} and $\sigma_{\theta\theta}$ at the interface in both the brass sleeve and the steel shaft.

(iv) The (expression for the) tangential strain

$$e_{\theta\theta} = \frac{u}{r} = \frac{1}{r}[\sigma_{\theta\theta} - \nu\,\sigma_{rr}] \qquad \text{(the axial stress } \sigma_{zz} = 0)$$

can be worked out for both the steel shaft and the brass sleeve in terms of the unknown interference pressure p_s. The appropriate E, E_s (steel) or E_b (brass), is to be used for these calculations.

(v) From the above, u_{shaft} and u_{sleeve} can be calculated at the common boundary (interface) in terms of p_s. The former u_{shaft} is radially inwards, while the latter u_{sleeve} is radially outwards.

(vi) Now the principle of consistent deformation is used. It says [Fig. 13.27c]

$$|u_{shaft}| + |u_{sleeve}| = \delta \qquad \text{(interference on the radius} = 0.40/2 = 0.20 \text{ mm)}$$

As u_{shaft} and u_{sleeve} are of different signs, it is less confusing to consider their numerical values only.

(vii) This equation enables us to determine the interference pressure p_s.

(viii) Having obtained the numerical value of p_s, the (numerical values of the) stresses can be calculated at the interface in both the steel shaft and in the brass sleeve.

(ix) It will add to our understanding to see the nature and distribution of both σ_{rr} and $\sigma_{\theta\theta}$ in (i) the steel shaft, and in the brass sleeve.

Now we can undertake the detailed calculations.

Interference pressure

First we shall calculate the interference (shrinkage) pressure p_s. We have already worked out the expression for the interference pressure p_s earlier in the section on Thick Cylinders. Let us borrow the result:

$$p_s = \frac{\delta}{r_i\left[\frac{1}{E_{brass}}\left(\frac{r_o^2+r_i^2}{r_o^2-r_i^2} + \nu_{brass}\right) + \frac{1}{E_{steel}}\left(1 - \nu_{steel}\right)\right]}.$$

For the numerical values given, viz., $r_o = 140$ mm, $r_i = 100$ mm, $\delta = 0.20$ mm, $E_{steel} =$ 200 GPa and $E_{brass} = 100$ GPa, the above expression gives us $p_s = 26.57$ MPa.

Next, we shall work out the resulting (maximum) stresses in the sleeve and the shaft.
Stresses in the sleeve and the shaft
The maximum stress in the sleeve is the circumferential stress $\sigma_{\theta\theta}$ at the interface radius. This is

$$\sigma_{\theta\theta} = p_s \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} = \times \frac{140^2 + 100^2}{140^2 - 100^2} = 81.92\,\text{MPa}.$$

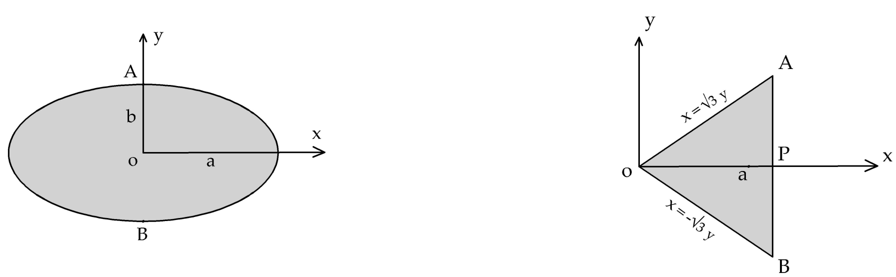
$\sigma_{\theta\theta} = 81.92$ MPa, tensile.

The stress in the shaft is easy to find; in fact, no calculation is needed. Both the circumferential and the radial stresses are equal and, what is more, the same everywhere, and equal to the interference pressure. Thus,

$$\sigma_{\theta\theta} = \sigma_{rr} = -p_s = -26.57\,\text{MPa, compressive.}$$

56. Example: Torsion of an Elliptical Cross-section

Let us solve the torsion problem for a prismatic bar of elliptical cross-section [Fig.



(a) An elliptical cross-section (b) A triangular cross-section

Figure 13.28: An elliptical and a triangular cross-sections. The maximum shear stress on the cross-section is at the boundary points nearest from the centroid.

13.28b]. By solving the torsion problem, we mean to obtain the shear stresses on the cross-section, and the twist θ per unit length, when a known torque T acts on the bar. We use the third formulation.

The governing equation and the boundary conditions are

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = -2\,G\theta \text{ in the region } \mathcal{R},$$

subject to the boundary condition

$$F = 0 \text{ on the only boundary given by the equation } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 0.$$

One convenient and clever technique of satisfying this boundary condition is to assume the desired function $F = F(x, y)$ in the form

$$F = A \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right),$$

where A is an unknown constant. We can obtain the value of this constant by substituting this assumed form of the solution Φ in the governing differential equation $\bigtriangledown^2 F = -2\,G\theta$,

$$\frac{\partial F}{\partial x} = \frac{2\,Ax}{a^2}; \qquad \frac{\partial^2 F}{\partial x^2} = \frac{2\,A}{a^2}; \qquad \frac{\partial F}{\partial y} = \frac{2\,Ay}{b^2}; \qquad \frac{\partial^2 F}{\partial y^2} = \frac{2\,A}{b^2}.$$

$$\bigtriangledown^2 F = -2\,G\theta \qquad \longrightarrow \qquad \frac{2A}{a^2} + \frac{2A}{b^2} = -2\,G\theta \qquad \longrightarrow \qquad A = -G\theta \left(\frac{a^2\,b^2}{a^2 + b^2} \right).$$

Thus,

$$F = F(x, y) = -\frac{a^2\,b^2}{a^2 + b^2}\,G\theta \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \tag{13.39}$$

is the solution of the differential equation. The problem is solved in principle. All that remains to be done is to calculate (i) the shear stresses τ_{zx} and τ_{zy} on the cross-section and, in particular, the maximum values and their locations, and (ii) then torsional rigidity. We shall calculate them below.

The shear stresses are merely the partial derivatives of F. Thus,

$$\tau_{zx} = \frac{\partial F}{\partial y}; \qquad \text{and} \qquad \tau_{zy} = -\frac{\partial F}{\partial x}.$$

These give us the shear stress components in terms of the twist θ per unit length. However, it is more convenient to obtain the results in terms of the applied torque T. To calculate the torsional rigidity $C = T/\theta$ also, we need to work out the expression for T in terms of Φ. The torque T is given by

$$\begin{aligned} T &= 2 \iint_{\mathcal{R}} F\,dx\,dy \\ &= -\frac{2\,a^2\,b^2}{a^2 + b^2}\,G\theta \left[\frac{1}{a^2} \iint_{\mathcal{R}} x^2\,dx\,dy + \frac{1}{b^2} \iint_{\mathcal{R}} y^2\,dx\,dy - \iint_{\mathcal{R}} dx\,dy \right] \end{aligned} \tag{13.40}$$

The three integrals within the square brackets $[\cdots]$ represent, respectively, the second moments of area (of the elliptical cross-section) about the $y-$ and $x-$axes, and the area of the ellipse.

$$I_{yy} = \iint_{\mathcal{R}} x^2\,dx\,dy = \frac{1}{4}\pi a^3 b; \qquad I_{xx} = \iint_{\mathcal{R}} y^2\,dx\,dy = \frac{1}{4}\pi a b^3; \qquad \text{area} = \pi ab.$$

If we make these substitutions in Eq. (13.40), we obtain

$$-2\,G\theta = -\frac{2(a^2 + b^2))\,T}{\pi a^3\,b^3}.$$

Thus, we have

$$F = -\frac{T}{\pi ab}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right). \tag{13.41}$$

The stresses are

$$\tau_{zx} = \frac{\partial F}{\partial y} = -\frac{2\,Ty}{\pi ab^3}; \qquad \tau_{zy} = -\frac{\partial F}{\partial x} = -\frac{2\,Tx}{\pi a^3 b}. \tag{13.42}$$

The resultant shear stress τ is, thus,

$$\tau = \sqrt{\tau_{zx}^2 + \tau_{zy}^2} = \frac{2\,T}{\pi ab}\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4}}. \tag{13.43}$$

The maximum shear stress occurs at the boundary points A and B which as closest to the origin (centre of the section), and not at the farthest points! The maximum shear stress τ_{max} and the torsional rigidity C are obtained as

$$\tau_{max} = \frac{2\,T}{\pi ab^2} \qquad \text{and} \qquad C = \frac{T}{\theta} = \pi a^3 b^3. \tag{13.44}$$

Warning! This technique will not work for all shapes. If, on substitution in the governing equation $\bigtriangledown^2 F = -2G\theta$ (Poisson's equation), A turns out to be not a constant, this method will fail. The idea of using the equation to the boundary to satisfy the boundary condition $F = 0$ on the boundary is applicable in all cases. This idea is exploited in several places, including the methods to obtain approximate solutions.

57. Example: Torsion of a Triangular Cross-section

As another example, let us solve the torsion problem for a triangular cross-section shown in Fig. 13.28b. As before we use the third formulation to solve the problem.

To satisfy the boundary condition, let us write the equations to the three sides OA, OB and AC of the triangle. These are, respectively,

$$x + \sqrt{3}y = 0; \qquad x - \sqrt{3}y = 0; \qquad \text{and} \qquad x - a = 0.$$

Let us, therefore, choose F in the form $F = A\left[(x+\sqrt{3}y)(x-\sqrt{3}y)(x-a)\right]$, so that $F = 0$ at all points on the boundary. Substituting, as before, in the governing differential equation $\bigtriangledown^2 F = -2G\theta$ (Poisson's equation), we find that

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 4\,Aa = -2G\theta \quad \rightarrow \quad A = \frac{-G\theta}{2a} \quad \rightarrow \quad F = \frac{-G\theta}{2a}\left[(x-a)(x^2-3y^2)\right].$$

The torque is calculated as

$$\begin{aligned} T &= 2\iint_{\mathcal{R}} \Phi\,dx\,dy = \frac{-G\theta}{a}\int_{x=0}^{x=a}\int_{y=-x/\sqrt{3}}^{y=x/\sqrt{3}} (x-a)(x^2-3y^2)\,dx\,dy \\ &= -\frac{4\,G\theta}{3\sqrt{3}\,a}\int_{x=0}^{x=a} x^3(x-a)\,dx = \frac{G\theta\,a^4}{15\sqrt{3}} \end{aligned}$$

The angle of twist θ per unit length and the torsional rigidity C are

$$\theta = \frac{15\sqrt{3}}{G\,a^4}\,T; \qquad \text{and} \qquad C = \frac{T}{\theta} = \frac{G\,a^4}{15\sqrt{3}}.$$

The shear stresses are

$$\tau_{zx} = \frac{\partial F}{\partial y} = \frac{3\,G\theta}{a}\,(x-a)y; \qquad \text{and} \qquad \tau_{zy} = -\frac{\partial F}{\partial x} = \frac{G\theta}{2a}\,(3x^2 - y^2 - 2ax).$$

The maximum shear stress on the cross-section is at P which is the boundary point closest to the centre. Its value is $\tau_{max} = (15\sqrt{3}\,T)/(2a^3)$. [As remarked earlier, this technique will not always work.]

58. Example: Torsion of a Circular Prismatic Bar

Obtain the Saint-Venant's solution for the torsion of a circular prismatic bar.

We have already obtained the solution for an elliptical prismatic bar. A circle is a special, simplified, case of an ellipse. Thus, we can surely obtain the required solution by setting $a = b$ in Eq. (13.39), p. 13-60. However, we shall begin afresh. We follow the third formulation.

Let us assume F in the form $F = C(x^2 + y^2 - r^2)$. The boundary condition $F = 0$ is obviously satisfied by this function. We can try[27] to obtain C by substituting this assumed solution in the governing equation $\bigtriangledown^2 = -2G\theta$. This gives us $2C + 2C = -2G\theta$, i.e., $C = -G\theta/2$. Thus, the solution is $F = -(1/2)G\theta(x^2 + y^2 - r^2)$. The torque T is given by

$$\begin{aligned} T &= 2\iint F\,dx\,dy = 2\iint -\frac{1}{2}G\theta(x^2 + y^2 - r^2)\,dx\,dy \\ &= -G\theta\left[\iint x^2\,dx\,dy + \iint y^2\,dx\,dy - r^2\iint dx\,dy\right] \\ &= -G\theta(I_{yy} + I_{xx} - \pi r^4). \end{aligned}$$

For our cross-section, $I_{xx} = I_{yy}$ and, therefore, the polar second moment of area $J = I_{xx} + I_{yy} = \pi r^4/2$. Thus,

$$T = -G\theta(J - 2J) = G\theta J \qquad \longrightarrow \qquad \frac{T}{J} = G\theta.$$

We arrive at the solution as

$$F = \frac{1}{2}\,\frac{T}{J}(x^2 + y^2 - r^2).$$

The shear stress components on the cross-section are obtained by differentiating F w.r.to x and y. Hence we have

$$\tau_{zx} = \frac{\partial F}{\partial y} = -\frac{Ty}{J} \qquad \longrightarrow \qquad \text{when } y = r,\;\; \tau_{zx} = -\frac{Tr}{J};$$

[27] If, on substitution, C turns out to be a constant, then all is well; this method works. On the other hand, if we do not obtain C as a constant, this method fails.

$$\tau_{zy} = -\frac{\partial F}{\partial x} = -\frac{Tx}{J} \qquad \longrightarrow \qquad \text{when } x = r,\ \tau_{zy} = \frac{Tr}{J};$$

These answers agree with the results of Coulomb's theory.

59. Example: Membrane Analogy

We know that a thin-walled tube would lose nearly all of its torsional rigidity if we

(a) A thin-walled closed section (b) The same cross-section but with a thin slit

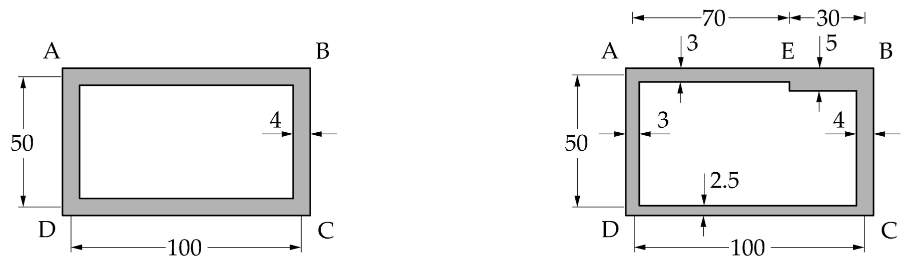
Figure 13.29: A thin-walled closed tube and the shape of the corresponding membrane are shown [Fig. 13.29a]. Fig. 13.29b shows the same section, but with a marked difference: a longitudinal slit is made. Now there is only one boundary; the section has changed from a doubly connected region to a simply connected one. The soap film has all but collapsed. If we compare the volumes in the two figures we can see that there is a drastic reduction in the volume under the membrane. This tells that almost all the torsional rigidity is lost when we introduce a longitudinal slit.

introduce a longitudinal slit [Fig. 13.29]. Let us try to explain and understand this fact in the light of the membrane analogy.

Fig. 13.29a shows a thin-walled cross-section and the corresponding soap film. As the plate is at a certain distance, the soap film has quite some volume under it. When a longitudinal slit is made, the cross-section now becomes singly connected! There is only one boundary. The volume under the soap film is now only a tiny little bit (shown shaded) in Fig. 13.29b telling us unmistakably that the section has lost nearly all of its torsional rigidity. This is our experience, but it is always nice to have a sound theoretical basis and a convincing technical argument supporting our common-sense understanding!

60. Example: Torsion of a Thin-walled Section

Shown in Fig. 13.30 are two hollow thin-walled rectangular box sections, Fig. 13.30a with the same thickness throughout, and Fig. 13.30b with different thicknesses. If a torque of 1.5 kN m is applied, find the shear stress τ and the twist θ per unit length. The dimensions are given in the figures, all in mm. Assume the material to be an aluminium alloy for which the modulus of rigidity G is 26 GPa.



(a) A cellular section (same thickness)   (b) A cellular section (different thicknesses)

Figure 13.30: Two cellular cross-sections

Case (a) Uniform thickness t [Fig. 13.30a]:

The shear stress τ, we know, is given by Eq. (11.30), p. 11-22.

$$\tau = \frac{T}{2\,At} = \frac{1.5 \times 1000 \times 1000}{2(100 \times 50) \times 4} = 37.5\,\mathrm{N/mm}^2.$$

The angle of twist is given by Eq. (11.32) as

$$\theta = \frac{TL}{4\,GA^2t} = \frac{1.5 \times 1000 \times 1000(100 + 50 + 100 + 50)}{4 \times 26 \times 10^3 \times (100 \times 50)^2 \times 4} = 43.27{\times}10^{-6}\,\mathrm{rad} = 2.48{\times}10^{-3}\,{}^\circ.$$

Case (b) Different thicknesses t [Fig. 13.30b]:
As the height of the plate is a constant, the shear stress τ in the various legs will be different. Another way of looking at the situation is to invoke the fluid flow analogy: the velocity in a narrow channel will be higher than that in a wider one. Accordingly, the shear stress τ in the portion AE is given by Eq. (11.30) as

$$\begin{aligned}
\tau \text{ in the leg } AE &= \frac{T}{2\,At_{AE}} = \frac{1.5 \times 1000 \times 1000}{2(100 \times 50) \times 3} &&= 50.0\,\mathrm{N/mm}^2;\\
\tau \text{ in the leg } EB &= \frac{3}{5} \times \tau \text{ in the leg } AE = \frac{3}{5} \times 50.0 &&= 30.0\,\mathrm{N/mm}^2;\\
\tau \text{ in the leg } BC &= \frac{3}{4} \times \tau \text{ in the leg } AE = \frac{3}{4} \times 50.0 &&= 37.5\,\mathrm{N/mm}^2;\\
\tau \text{ in the leg } CD &= \frac{3}{2.5} \times \tau \text{ in the leg } AE = \frac{3}{2.5} \times 50.0 &&= 60.0\,\mathrm{N/mm}^2;\\
\tau \text{ in the leg } DA &= \tau \text{ in the leg } AE &&= 50.0\,\mathrm{N/mm}^2.
\end{aligned}$$

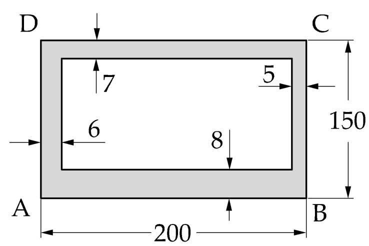
The angle of twist per unit length θ is given by [Eq. (11.31)

$$\begin{aligned}
\theta &= \frac{T}{4\,GA^2} \oint \frac{ds}{t} = \frac{T}{4\,GA^2}\left[\frac{ds}{t_1} + \frac{ds}{t_2} + \frac{ds}{t_3} + \cdots\right]\\
&= \frac{T}{4\,GA^2}\left[\frac{AD}{3} + \frac{DC}{2.5} + \frac{CB}{4} + \frac{BE}{5} + \frac{EA}{3}\right]
\end{aligned}$$

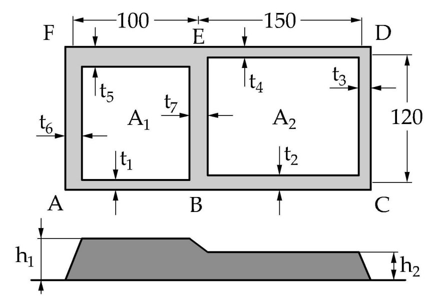
$$= \frac{1.5 \times 1000 \times 1000}{4 \times 26 \times 10^3} \left[\frac{50}{3} + \frac{100}{2.5} + \frac{50}{4} + \frac{30}{5} + \frac{70}{3} \right]$$
$$= 56.827 \times 10^{-6}\,\text{rad/mm} = 56.827 \times 10^{-3}\,\text{rad/m} = 3.26\,^\circ/\text{m}.$$

(Such a section with the thickness varying like this is unlikely to be used in practical applications. This case is taken up merely to illustrate the procedure.)

61. Example: Torsion of Cellular Cross-sections



(a) A cellular cross-section



(b) A two-cell cross-section

Figure 13.31: Two cellular cross-sections. The thicknesses of the various walls are different. The membrane corresponding to the two-cell cross-section is shown in Fig. 13.31b. The heights of the plates are h_1 and h_2.

The cellular cross-section shown [Fig. 13.31a] is subjected to a twisting moment of 5,000 Nm. Find the shear stress in each wall. Evaluate the line integral $\int ds/t$, and obtain the torsional rigidity of the cross-section. Assume that the modulus of rigidity of the material $G = 27$ GPa.

$$\text{Area } A \text{ enclosed by the middle line} = \left(200 - \frac{6}{2} - \frac{5}{2}\right)\left(150 - \frac{7}{2} - \frac{8}{2}\right)$$
$$= 27,716.25\,\text{mm}^2.$$
$$\text{shear stress in the wall } AB\ \tau_{AB} = \frac{T}{2\,A\,t_{AB}} = \frac{5000 \times 1000}{2\,A \times 8} = 11.27\,\text{MPa}.$$
$$\text{shear stress in the wall } BC\ \tau_{BC} = \tau_{AB} \times \frac{8}{5} = 18.03\,\text{MPa}.$$
$$\text{shear stress in the wall } CD\ \tau_{CD} = \tau_{AB} \times \frac{8}{7} = 12.88\,\text{MPa}.$$
$$\text{shear stress in the wall } DA\ \tau_{DA} = \tau_{AB} \times \frac{8}{6} = 15.03\,\text{MPa}.$$

The line integral $\int \frac{ds}{t}$ is calculated as:

$$\int \frac{ds}{t} = \frac{(200 - \frac{6}{2} - \frac{5}{2})}{8} + \frac{(150 - \frac{8}{2} - \frac{7}{2})}{5} + \frac{(200 - \frac{6}{2} - \frac{5}{2})}{7} + \frac{(150 - \frac{8}{2} - \frac{7}{2})}{6} = 104.34.$$

The torsional rigidity C can now be calculated as:

$$C = \frac{T}{\theta} = \frac{4\,G \times A^2}{\int \frac{ds}{t}} = \frac{4 \times (26 \times 10^9) \times (27.72 \times 10^{-3})^2}{104.34} = 0.765 \times 10^6\,\mathrm{Nm/(rad/m)}.$$

62. Example: Torsion of Multi-cellular Cross-sections

Discuss the problem of torsion of multi-cellular sections.

The general approach indicated in the text [p. 11-23] is explained here in a slightly different way for better clarity of understanding. If there are n cells, the heights of the weightless plates held horizontal by pure couples only are, in general, different: $h_i\,(i = 1, 2, 3, \cdots, n)$, i.e., $(h_i,\, h_2,\, \cdots,\, n)$. The slopes of the membranes on the outside walls are h/t. For the walls inside, the slopes are $\Delta h/t$. The thickness t is the appropriate wall thickness which need not be a constant; it can be different at different points. Δh is the difference in the heights of the central horizontal plates concerned. The equation of equilibrium of the i^{th} plate is

$$\text{(equation of equilibrium in the vertical direction:)} \qquad pA_i = S \oint \frac{\Delta h}{t}\,ds. \qquad (13.45)$$

The integral is around the path surrounding the i^{th} plate. As explained earlier, Δh is the difference in the heights of the two adjacent horizontal plates concerned.

There are n such equations $(i = 1, 2, \cdots, n)$, all linear, algebraic, non-homogeneous equations for the n unknown heights $h_1, h_2, \cdots, h_n$. They can be solved. After these heights are determined, we can change over from the membrane to the torsion problem by invoking the membrane analogy:

$$\text{slope} = \frac{\Delta h}{t} \;\rightarrow\; \tau\ \text{(shear stress)}; \qquad \frac{p}{S} \;\rightarrow\; 2\,G\theta; \qquad 2 \times \text{volume} \;\rightarrow\; T.$$

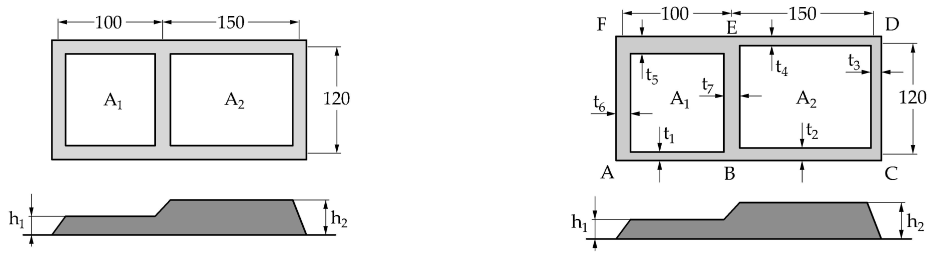
Now the problem is solved in principle. However, the procedure is rather difficult when there are three or more cells. Several special methods and shortcuts are available in the published literature, but we are not concerned with them here. Let us note that we are not required to perform any experiment. The membrane analogy is used for visualisation and a qualitative understanding to work out the solution.

63. Example: Two-cell Cross-sections

What are the shear stresses in the various portions AB, BC, $\cdots$, etc. in each of the two cross-sections when a twisting moment of 30 kNm is applied? What is the twist θ per unit length in each of two cases? Some of the dimensions in mm are given in the figures. For the section on the left, the thickness $t = 3$ mm is the same for all the legs. For the other section (on the right), $t_1 = 3$, $t_2 = 4$, $t_3 = 3.5$, $t_4 = 2.5$, $t_5 = 5$, $t_6 = 4$, $t_7 = 4.5$, all in mm. Take the modulus of rigidity $G = 27$ GPa.

Case (a) Uniform thickness [Fig. 13.32a]

Consider the equilibrium of the two plates held horizontal. For the first plate, there is the air pressure $p \times A_1$ acting upwards, and the vertical component of the surface tension



(a) A two-cell section (same thickness) (b) A two-cell section (different thicknesses)

Figure 13.32: Two two-cell cross-sections. The thickness of the various walls is the same in Fig. 13.32a, while the thicknesses are different in Fig. 13.32b. The membranes corresponding to the two-cell cross-section are also shown in the two figures. The heights of the plates are h_1 and h_2.

all around the plate. Similar is the case for the second plate also. Thus, we obtain for the two plates

$$pA_1 = \frac{S}{t}\left[100 \times h_1 + 120 \times (h_1 - h_2) + 100 \times h_1 + 120 \times h_2\right];$$
$$pA_2 = \frac{S}{t}\left[150 \times h_2 + 120 \times h_2 + 150 \times h_2 + 120 \times (h_2 - h_1)\right].$$

In each of the cases we go around the plate from the left lower point in the anti-clockwise direction. The thickness t, being the same for all the terms, is taken out as a common factor. Note that the slope is $(h_1 - h_2)/t$ in one case, and $(h_2 - h_1)/t$, in the other.

[It is also possible for us to consider the equilibrium of both plates together by going around both A_1 and A_2. But this will not give us a third separate (linearly independent) equation; this would merely be the sum of the two equations already obtained. The contribution from the leg BE common to both cells will be zero.]

Here are two equations — linear, simultaneous, algebraic — in the two unknowns h_1 and h_2. If we solve for them, we obtain

$$h_1 = 116.13\,\frac{p}{S}; \qquad h_2 = 125.81\,\frac{p}{S}.$$

The shear stress is the same in the legs AB, EF, and FA. The shear stress has the same value in the legs BC, CD, and DE (but different from those in the other legs). Which is larger depends on which, between h_1 and h_2, is larger. Here we see that $h_1 < h_2$. The volume V of the membrane is

$$V = A_1 h_1 + A_2 h_2 = [116.13 \times A_1 + 125.81 \times A_2]\frac{p}{S}.$$

Now that we have obtained all the relevant information of the membrane, we can use the membrane analogy and pass on to the torsion problem.

$$\text{slope} = \frac{\Delta h}{t} \;\rightarrow\; \tau \text{ (shear stress)}; \qquad \frac{p}{S} \;\rightarrow\; 2\,G\theta; \qquad 2 \times \text{volume} \;\rightarrow\; T.$$

$$
\begin{aligned}
V =& A_1 h_1 + A_2 h_2 = (100 \times 120 \times 110.11 + 150 \times 120 \times 120.02)\,\frac{p}{S} \\
&= 3.66 \times 10^6\,\frac{p}{S} \qquad\qquad \left(\frac{P}{S} \quad \longrightarrow \quad 2\,G\theta\right) \\
T &= 2V = 2 \times 3.66 \times 10^6\,\frac{p}{S} = 7.32 \times 10^6 \times 2\,G\theta \\
2\,G\theta &= \frac{T}{7.32 \times 10^6} = \frac{30 \times 10^3 \times 10^3}{7.32 \times 10^6} = 4.10 \\
\theta &= \frac{4.10}{2 \times 27 \times 10^3} = 75.9 \times 10^{-6}\ \mathrm{rad/mm} = 75.9 \times 10^{-3}\ \mathrm{rad/m} = 4.35\,^{\circ}/\mathrm{m}.
\end{aligned}
$$

The shear stress is the same in the legs AB, EF and FA.

$$
\begin{aligned}
\tau_{AB} &= \frac{h_1}{t} = \frac{116.13}{3}\,\frac{p}{S} = \frac{116.13}{3} \times 2G\theta = \frac{116.13}{3} \times 4.10 = 158.71\,\mathrm{MPa}; \\
\tau_{BE} &= \frac{h_2 - h_1}{t} = \frac{125.81 - 116.13}{3}\,\frac{p}{S} = \frac{9.68}{3} \times 2G\theta = \frac{9.68}{3} \times 4.10 = 13.22\,\mathrm{MPa}; \\
\tau_{BC} &= \frac{h_2}{t} = \frac{125.81}{3}\,\frac{p}{S} = \frac{125.81}{3} \times 2G\theta = \frac{125.81}{3} \times 4.10 = 171.94\,\mathrm{MPa}.
\end{aligned}
$$

The shear stress is the same in the legs BC, CD and DE.

Case (b) Different thicknesses [Fig. 13.32b]
This case is also similar. Again, as before we write down the equation of equilibrium for each of the two plates. For the first plate, we go around the first plate taking the path AB - BE - EF - FA. This gives us the first equation. For the second plate, the path around it is BC - CD - DE - EB. This gives us the second equation. These two equations (of equilibrium) are the following. (Now that the thicknesses of the various legs are different, the thickness cannot be taken outside as a common factor.)

$$
\begin{aligned}
pA_1 &= S\left[\frac{100}{3} \times h_1 + \frac{120}{4.5} \times (h_1 - h_2) + \frac{100}{5} \times h_1 + \frac{120}{4} \times h_1\right]; \\
pA_2 &= S\left[\frac{150}{4} \times h_2 + \frac{120}{3.5} \times h_2 + \frac{150}{2.5} \times h_2 + \frac{120}{4.5} \times (h_2 - h_1)\right].
\end{aligned}
$$

Here are two equations — linear, simultaneous, algebraic — in the two unknowns h_1 and h_2. If we solve for them, we obtain

$$
h_1 = 142.44\,\frac{p}{S}; \qquad h_2 = 137.57\,\frac{p}{S}.
$$

Here we find that $h_1 > h_2$. Now the heights of the plates are as shown in Fig. 13.31b.

$$
\begin{aligned}
V &= A_1 h_1 + A_2 h_2 = (100 \times 120 \times 142.44 + 150 \times 120 \times 137.57)\,\frac{p}{S} \\
&= 8.37 \times 10^6\,\frac{p}{S} \qquad\qquad \left(\frac{P}{S} \quad \longrightarrow \quad 2\,G\theta\right)
\end{aligned}
$$

$$
\begin{aligned}
T = 2V = 2 \times 4.185 \times 10^6 \, \frac{p}{S} &= 8.37 \times 10^6 \times 2\,G\theta \\
2\,G\theta = \frac{30 \times 10^3 \times 10^3}{8.37 \times 10^6} &= 3.58 \\
\theta = \frac{3.58}{2 \times 27 \times 10^3} &= 66.29 \times 10^{-6}\ \text{rad/mm} = 66.29 \times 10^{-3}\ \text{rad/m} = 3.79\,^\circ/\text{m}.
\end{aligned}
$$

The shear stresses in the various legs can be calculated as explained earlier and as shown below. Usually we are interested in the maximum shear stresses. As h_1 and h_2 are different, we cannot state just by inspection where the maximum shear stress occurs.

$$
\begin{aligned}
\tau_{AB} &= \frac{h_1}{t_1} = \frac{142.44}{3}\,\frac{p}{S} = \frac{142.44}{3} \times 2G\theta = \frac{142.44}{3} \times 3.58 = 169.98\,\text{MPa}; \\
\tau_{EF} &= \frac{h_1}{t_5} = \frac{142.44}{5}\,\frac{p}{S} = \frac{142.44}{5} \times 3.58 = 101.98\,\text{MPa}; \\
\tau_{FA} &= \frac{h_1}{t_6} = \frac{142.44}{4} \times 3.58 = 127.48\,\text{MPa}; \\
\tau_{BE} &= \frac{h_1 - h_2}{t_7} = \frac{142.44 - 137.57}{4.5} \times 3.58 = 3.87\,\text{MPa}; \\
\tau_{BC} &= \frac{h_2}{t_2} = \frac{137.57}{4} \times 3.58 = 123.13\,\text{MPa}; \\
\tau_{CD} &= \frac{h_2}{t_3} = \frac{137.57}{3.5} \times 3.58 = 140.71\,\text{MPa}; \\
\tau_{DE} &= \frac{h_2}{t_4} = \frac{137.57}{2.5} \times 3.58 = 197.00\,\text{MPa}.
\end{aligned}
$$

The maximum shear stress of 197.00 MPa occurs in the leg DE. The stress in the leg BE separating the two cells is, expectedly, very low. This is because the shear flow there is the difference between the two shear flows surrounding the two cells.

What are the comments to be added? Unless there are numerical mistakes (which is indeed a clear possibility!), the shear stresses and the angle of twist seem to be too large for the material suggested by the given value of G. What is a probable material? (Usually we know the material beforehand. Here we are asking this question just to provoke the curiosity of the readers. Solution of numerical problems should not seen as merely plugging in numbers in a set of given formulae. There are deeper issues.) What then? This section seen as a possible design is unsatisfactory; it is to be revised so that the stresses become less and the torsional rigidity is increased.

A Few Suggestions

When we work out problems we may make mistakes. Examiners will, let us hope, understandingly give marks for the various steps and the procedure. They are advised to give about three times the time it takes for them to work out a problem. Yet these suggestions are not always followed. It may be a good habit to check how long it takes to work out a problem.

When we get unreasonable numerical values, we can protect ourselves by writing a comment. For example, we may get a value of, say, 0.6 for the Poisson's ratio. This can be because the values (data) given in the problem are wrong or unrealistic, or more probably, because of the mistakes that we make. Such a value for the Poisson's ratio is clearly impossible. In such cases, it is absolutely necessary to add a comment: that the value of the Poisson's ratio cannot exceed 0.5, and that even this value corresponds to incompressible materials or conditions (as, for example, in plasticity, we assume incompressibility).

It is also a good habit to have an idea of the order of magnitude for each physical quantity. For example, the deflection of an ordinary beam cannot be as large as, say, 2 m. Again, if we obtain an expression for the deflection of a beam as, say,

$$\text{deflection } \delta = \frac{P_1 l^3}{48\,EI} + \frac{P_2 l^2}{3\,EI} + \frac{wl^3}{8\,EI},$$

one close look at the formula tells us that it is wrong. Why? The dimensions do not match. (Technically we say that the equation must be dimensionally homogeneous.) Let us be sensitive to these matters. These are good habits to cultivate.

Finally we need a little bit of luck. Dear young students, best of luck!

In the next and last chapter we shall see a little bit of the early history of our subject.

Chapter 14

EARLY HISTORY

We had made a few indications earlier about the history of our subject. It is desirable, very desirable, to have a historical perspective[1] of the development. This is necessary to have a healthy interest in a subject that is not dead, but vibrantly alive and kicking. However, the scope is too vast and, therefore, it is decided to limit this to the early history.

ANCIENT TIMES

(a) Great Wall of China

(b) The Pont du Gard

(c) A Step-well of Ancient India

Figure 14.1: The Great Wall of China, more than 2300 years old; the Pont du Gard, an ancient ($40-60$ AD) aqueduct in Southern France, 48.8 m tall; one of the famous step-wells (200 AD?) of ancient India

It is known that even long, long ago people did constructions of various kinds: roads, buildings, bridges, canals, towers, etc. Three examples are shown in Fig. 14.1 above. These are clearly impossible without some knowledge of strength of materials. How can people have arrived at the dimensions of the various members in these constructions? It was perhaps only empirical rules gained with experience. Egyptians certainly must have had some knowledge; otherwise it would be impossible to construct not only the great pyramids, but even the less impressive monuments and other structures. The Greeks too possessed considerable knowledge in the art of building. The fundamental science upon which this

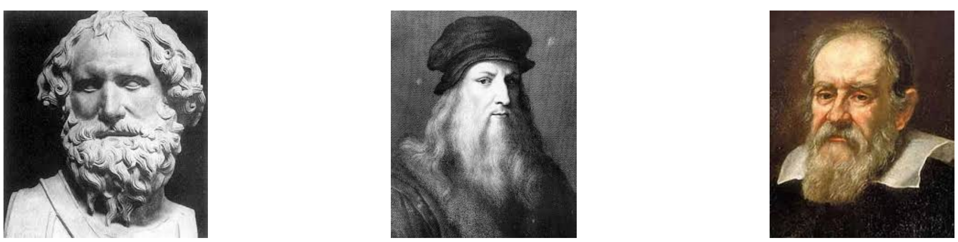
[1] Part of the material for this section is taken from Timoshenko's well known book [14]. Readers are advised to read this book.

knowledge was developed is statics. Archimedes[2] knew a great deal of science. He knew the conditions of equilibrium — he had given rigorous proofs — of levers. He also knew, among many other things, how to locate the centres of gravity of bodies. He had used all this knowledge in constructing different types of hoisting devices. Considering that all this was accomplished in the dim distant past, he must surely rank as one of the greatest scientists.

The Romans also were not far behind in construction activities. Many of their bridges, temples and monuments are a testimony to their knowledge. They may not have developed the science of stress analysis. Nevertheless, they did use arches even though the spans were not too large and the shapes mainly semi-circular. It seems reasonable to assume that great civilisations such as we had in India, China and Mesopotamia could not have evolved and matured without considerable construction activities of various kinds: bridges, canals, columns, dams, domes, forts, fortifications, monoliths, obelisks, roads, roofs, tunnels, $\cdots$[3]. This, in turn, lends credence to supposing that they too knew the essence of what is needed to undertake such building work[4]. There are several cases where we seem to have underestimated the knowledge and skills that our forefathers possessed!

It appears that all these advances were lost. European engineers in the sixteenth century had great difficulty to accomplish engineering tasks that the ancients like the Egyptians must have done rather routinely: to carry heavy stones to the Nile, carrying obelisks to other sites, etc. It was only after the Renaissance that the broken thread was picked up again and progress made.

SIXTEENTH / SEVENTEETH CENTURY: GALILEO



(a) Archimedes of Sicily (b) Leonardo da Vinci (c) Galilei Galileo

Figure 14.2: Three great early scientists

[2] Archimedes of Syracuse (287 BC - 212 BC) born in Sicily, Italy was an ancient Greek scientist, perhaps the greatest among the ancient scientists. He was a mathematician, physicist, astronomer, engineer and inventor.

[3] This list in alphabetical order is borrowed from the internet. This help is gratefully acknowledged.

[4] There probably is no clearly documented evidence. Perhaps whatever was available is lost for ever. Ravages of time and carelessness in preservation are possible reasons. Western scholars are either wholly ignorant of such non-European contributions, or are reluctant to admit that intelligence and creativity are not their monopoly. It is also possible that the colonial powers suppressed and even destroyed such evidence so that they can dominate over the subject race in every sphere.

Finally let us note that absence of proof is not proof of absence!

Leonardo da Vinci[5] was an exceptional genius. His fame is perhaps more as the leading artist of his times, but he was also an outstanding engineer-scientist as revealed in his notebooks. He knew a great deal of mechanics. He was able to obtain the correct solutions of several problems in statics using the method of moments. He had carried out experiments to obtain the strength of iron wires. He had also examined the strength of beams and had enunciated certain general principles. Columns also came within the purview of his investigations. All put together, one wonders whether he was primarily an engineer-scientist who knew a thing or two about art. His statement "Mechanics is the paradise of mathematical science because here we come to the fruits of mathematics" is very significant and relevant even to this day. A truly remarkable person indeed!

The next great name that we find is that of Galileo[6]. In 1586 Galileo measured the density of several substances using a hydrostatic balance that he made. His work on the centre of gravity attracted attention, and he became a professor of mathematics at Pisa when he was not even 26 years old!

It was during the years 1589 - 1592 that he performed the famous experiments on freely falling bodies using the leaning tower of Pisa! His findings were is sharp contrast to the prevailing ideas based on Aristotelian mechanics, and he fell out of favour with the powers that be. He, therefore, had to leave Pisa, but he could move over to the University of Padua. The position was kept vacant until a suitable person like Galileo could be found.

Galileo was very active during the first few years at Padua[7]. He did a tremendous amount of great work there. He became famous; his lectures attracted students in large numbers from several European countries. A very large room had to be used to accommodate the very large number of students.

He wrote the famous treatise *Della Scienza Meccanica* in 1594. The principle of virtual work was used in this to treat several problems in statics. This book gained considerable circulation; copies of the manuscript were widely circulated. At that time, there was shipbuilding activity, and it became necessary to examine problems in strength of materials in this perspective. Galileo became interested in such studies. It was not long before he got

[5] Leonardo di ser Piere da Vinci, better known as Leonardo da Vinci (April 1452 - May 1519)

[6] Galileo Galilei (Feb. 1564 - Jan. 1642) was an Italian astronomer, physicist, engineer, philosopher and mathematician. He played a key role in the scientific revolution of the seventeenth century.

He was born in Pisa. His native place was Florence. He studied Latin, Greek and logic early in his career, and started studying medicine in Pisa University. But he became interested in mathematics, and studied the work of Euclid and Archimedes. He was also attracted by Leonardo da Vinci's work in mechanics. But for want of money he was forced to leave Pisa in 1585 without a degree, but he gave private tuition in mathematics and mechanics. He also continued his own scientific studies. When he became well known for his scientific work, he became a professor, first at Pisa and later at Padua. His scientific output during the early years at Padua was amazing. However, as he supported the Copernicus' theory based on his investigations in astronomy, the Church was not amused; he was summoned to Rome by the Inquisition. He was under house arrest for the last eight years of his life. It may be of interest to the readers to know that Pope Paul II did express, though much belatedly in 1992, regret for the wrong position taken by the Church. Vatican also released two stamps in his honour as an acknowledgement and admission of guilt.

[7] The University of Padua was founded in 1222. This was one of the oldest universities in the world and the second oldest in Italy.

involved with astronomy also. Galileo states in a letter to Kepler that “many years ago I became a convert to the opinions of Copernicus $\cdots$”. Galileo took serious notice of a rumour of the invention of a telescope. This was in 1609. He proceeded to make his own telescope. His telescope had a magnification of 32. Using his telescope, he made important discoveries in astronomy. He could establish important results which had a great impact on further developments in astronomy. These also provided powerful support to Copernicus' theory. It became possible to have direct visual confirmation of several facts. These few years of his work at Padua made him famous. The grand duke of Tuscany nominated him as the ‘philosopher and mathematician extraordinary’.

He now left Padua to return to his native place Florence. His new appointment gave Galileo plenty of time to pursue his scientific work, because it entailed no other official duties. For him this was the time of intense activity in astronomy and astronomical observations. Among his achievements are the discovery of the peculiar shape of Saturn, observation of the phases of Venus, and descriptions of the spots on the sun. The Copernicus' theory now received more and more support from Galileo's findings and vigorous writings. Galileo was indeed a towering figure.

Galileo's Work in Strength of Materials

Galileo's book *Two New Sciences* is very famous and well known. The first two ‘dialogues’ of this book contains his work in the mechanics of materials. He makes several observations about geometrically similar structures. He considers the strength of a bar in direct tension, and concludes that the strength (which he calls the ‘absolute resistance to fracture’) is proportional to the area of the cross-section, and independent of the length. He gives the numerical value of the ultimate strength of copper. He also considers a cantilever loaded at the end, and states where it will fracture, should the beam break. He examines cases of bending and makes important conclusions.

SEVENTEENTH CENTURY

Our subject can properly be considered to have been born in the seventeenth century. The early investigators were primarily French mathematicians. In those days the Church controlled most of the universities, and often stepped well outside their legitimate domain of matters spiritual. Accordingly, the sublime duty of the universities of being centres of knowledge, promoting independent research in a free and fearless atmosphere, could not be fulfilled. Instead scientific societies and national academies of science sprang up in several places in Europe. Italy took the lead in this matter. The *Accademia Secretorum Naturae* was set up in Naples as early as 1560. Rome had its famous *Accademia dei Lincei* in 1603 with Galileo as one of its members. After Galileo died, the *Accdemia del Cimento* came up in Florence. The publications of this academy had discussions on scientifc matters pertaining to thermometers, barometers, and pendula, and various experiments.

In England at about 1645, a set of interested persons used to meet in London every week and discuss the ‘New Philosophy’ or ‘Experimental Philosophy’. These *philosophical inquiries* were related to Physics, Anatomy, Geometry, Astronomy, ‘Natural Experiments’, etc. There were similar activities in France, Germany, and other countries. The famous

Royal Society came into being on July 15, 1662. The prestigious French Academy of Science also had its beginnings at about the same time. or shortly afterwards. The Russian Academy of Science (1725) and the Berlin Academy of Science (1770) were opened later. All these academies and their published transactions played a most significant role in the growth of science in the 18^{th} and the 19^{th} centuries.

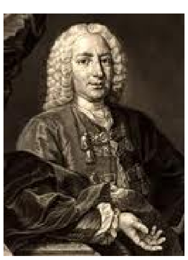
Robert Hooke (July 1635 - Mar. 1703) was a British experimentalist and skilled mechanic. He studied in Oxford and became the curator of the experiments of the Royal Society. There his inventive skill was of much help to the Royal Society. He used springs, and using his innovative ability, carried out the famous experiments that led to the Hooke's law. In 1664 he became Professor of Geometry in Greesham College, but he continued to present to the Royal Society his inventions, descriptions and experiments. His paper *De Potentiâ Restitutiva* (or "Of Spring") published in 1678 was the first published paper that discussed the elastic properties of materials. The linear relation between the force and the resulting elongation is the famous Hooke's law which is one the three pillars of the mechanics of deformable elastic bodies. Hooke also has a clear idea of universal gravitation, and came close to the laws of attraction and motion.

Edme Mariotte (c. 1620 - May 1684) was one of the earliest (1966) members of the famous French Academy of Science. Experimental methods became part of French science mainly because of Mariotte. It was he who studied impact. He used balls suspended by threads, and demonstrated that the momentum is conserved in impacts. The ballistic pendulum was also invented by him.

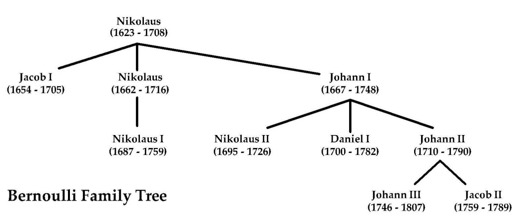
As he had to design pipe lines for water supply, he studied the bending strength of beams. He developed his own theory of bending taking into account the elastic properties of materials. Starting with simple tension tests, he performed several experiments.

The next important contributions are of the Bernoullis. Euler was closely associated with the Bernoullis; they all belonged to Basel, which is almost where France, Germany and Switzerland meet. But Euler lived in the eighteenth century.

Jakob and Johann Bernoullis



(a) Jakob Bernoulli



(b) Bernoulli family tree

(c) Johann Bernoulli

Figure 14.3: Jakob [Fig. 14.3a] and Johann [Fig. 14.3c] Bernoullis, the first two mathematician / scientists of the famous Bernoulli family [Fig. 14.3b]

Jakob (Jacob) (Dec. 1654 - August 1705) was the first famous person in the illustrious Bernoulli family of mathematicians and scientists. Johann (John, also called Jean) (July 1667 - Jan. 1748) was his brother and pupil. They were both elected in 1699 by the French Academy of Sciences as the foreign members. The Bernoulli family was always represented in the Academy until 1790. The calculus[8] progressed rapidly in Europe roughly in the period 1670 - 1720 largely by these two Bernoulli brothers. Johann, the younger Bernoulli was regarded as the greatest mathematician of his time. His lectures led to the publication in 1696 of the first book on the calculus by L'Hôpital[9]. The calculus of variations was born when these two Bernoulli brothers quarrelled about a curve called the brachistochrone[10] which has become historically famous for this reason. Jakob investigated the shape of the deflected curve of a beam. Although he made some erroneous assumption — which is understandable — he did state correctly that the curvature of the deflected curve is proportional to the bending moment. This fact was made use of by Euler and other mathematicians. To Johann Bernoulli must go the credit of formulating the principle of virtual work.

Euler and Daniel Bernoulli

Johann's pupil Euler and son Daniel made far more significant contributions to our subject. Although Daniel Bernoulli's fame rests largely on his book *Hydrodynamica*, he did contribute to the problem of determining the elastic curve. In a letter to Euler, Daniel suggested that Euler should apply the calculus of variations to obtain the deflection curve of loaded beams.

It was Daniel who obtained the differential equation for the lateral vibration of prismatic bars. He studied some particular modes of vibration. Euler integrated this differential equation. Daniel was also good at doing experiments. He performed several experiments for verification, and he was happy that "⋯ I have performed a great many experiments, ⋯". Some of Daniel's experiments were the feeding ground for some of Euler's mathematical problems.

Euler

Euler's contributions are large. He was an exceptional mathematician. He was very versatile, and his output prodigious. It is said that Euler calculated "without any apparent mental effort, just as birds fly and men breathe". He entered the University of Basel, a most important research centre of mathematics, in 1720 at the young age of 13, obtained his master's degree at 16, and published his first scientific research paper and participated in an international competition for a prize offered by the French Academy of Sciences before he was 20. Euler benefitted enormously not only by Johann Bernoulli's brilliant lectures but also from his private tuition classes once a week. When the Russian Academy of Sciences

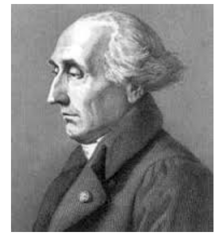
[8] There have always been controversies about the priority of Leibniz — Gottfried Wilhelm Leibniz (July 1646 - Nov. 1716), German mathematician; he is often regarded as the last of the universalists (who 'knew everything') — in continental Europe and Newton in England. These are not settled yet perhaps. Nevertheless it is generally accepted that the calculus was invented by these two scientists independently.

[9] Marquis de l'Hôpital (often spelt as L'Hospital) (Month? 1661 - Feb. 1704).

[10] This curve turns out to be a cycloid.

(a) Daniel Bernoulli

(b) Leonard Euler



(c) Joseph-Louis Lagrange

Figure 14.4: D. Bernoulli, L. Euler and J.L. Lagrange

was opened in 1725 at St. Petersburg, and the Bernoullis, Nicholas and Daniel, both sons of Johann, invited to be members of the new institute, they found a position for Euler there as an associate. Euler went to St. Petersburg and was furiously at work. While working there, he wrote his celebrated book on mechanics *Mechanica sive motus scientia exposita*, 2 vols, St. Petersburg (1736), the first book in which the calculus was extensively used. The traditional geometrical methods of Newton were abandoned; instead differential equations were derived and solved to discuss problems in dynamics. Later developments in mechanics were driven by the tremendous influence that this book had on the growth of mechanics.

Euler turned his attention to the elastic curve, the deflected shape of a loaded beam. Based on the suggestions of Daniel Bernoulli through correspondence, he investigated the problems of bending and of lateral vibration of beams and the associated differential equations. Euler is even more famous for his work on the buckling of columns.

When Frederick II, known as Frederick the Great, became the king of Prussia in 1740, he wanted to promote scientific studies and persuaded Euler to move to Berlin. Euler came to Berlin in 1741 and became associated with the Prussian Academy and the Berlin Academy. For the next 25 years he stayed in Berlin. It was during this period that he wrote first his famous book *Methodus inveniendi lineas curves* $\cdots$ (1944), which was the first ever book written on the calculus of variations. This was followed by *Introduction to Calculus* (1748), *Differential calculus*, 2 vols (1755) and *Integral calculus*, 3 vols (1768 - 1770). All these books were the leading lights for the mathematicians from the last part of the eighteenth and the early part of the nineteenth centuries. In this sense, all the mathematicians may be said to have been Euler's students!

Catherine II became the empress of Russia in 1762, and persuaded Euler to return to St. Petersburg in 1766 by giving him a much better offer. During the last phase of his career, he became completely blind. He had lost one eye as early as in 1735. But it is surprising that his mathematical prowess only increased. During this last phase of his career (1766 - 1783), he published more than 400 papers!

Euler was prolific. He wrote full research papers between the first ("dinner is ready") and the second ("dinner is getting cold") calls from his wife! It is not possible to discuss in detail his many contributions. Above all, he was great as a human being: modest, self-effacing, very appreciative of others and always letting others take the credit of his work! Let us pause here, remember Euler with pleasure, gratitude and admiration.

Lagrange

It is almost a crime not to bring up Lagrange's name when Euler's life and work are mentioned. Lagrange was born in Turin (now in Italy). He became a professor of mathematics when he was only 19! Euler was highly impressed with Lagrange. He helped Euler in building up the calculus of variations. Based on Euler's recommendation, Lagrange was nominated as a foreign member of the Berlin Academy. Lagrange is best known for his famous book *Mécanique analytique* in the preface of which he declares that there are no figures in this book. This book was published in Paris only in 1788. This book exerted a great influence on subsequent developments in mechanics. Here is an approach quite different from the Newtonian mechanics. Lagrangian mechanics, based on generalised coordinates, generalised velocities and forces, and the recently developed variational methods, gave mechanics an entirely different face.

He, in his capacity as the judge of the entries of a prize problem, was the one to present the governing equation (the biharmonic equation) for the bending of plates. Sophie Germain competed for this prize, but there were mistakes, and it was Lagrange who corrected them. In spite of the mistakes Sophie Germain was finally given the prize on her fourth attempt!

Lagrange was in Berlin until 1787 when, consequent on the death of Frederick the Great, the conditions in Berlin became not as congenial, he moved to Paris where he was given a warm welcome. His "most important contribution to the theory of elastic curve", states Timoshenko [14], "is his memoir *Sur la figure des columnes*". Lagrange considers an axially loaded column, and obtains the critical loads. Further, he examines the deflection when the load exceeds the critical load. He considers columns of variable cross-section also.

For a short while because of overwork, he temporarily lost interest in mathematics. It is said that his printed book *Mécanique analytique* lay unopened for two years. He was attracted by other sciences. He also served on a commission that was discussing the possibility of introducing the metric system in France. France then was going through tumultuous times. Following the French revolution, the new government tended to look upon scientists with suspicion. It removed some members of the aforesaid commission. Lavoisier, a chemist and Bailly, an astronomer were executed. Lagrange got disgusted with all this, and was about to leave France. But he was asked to lecture on the calculus at the newly established École Polytechnique. Towards the end of his career, Lagrange tried to revise his book, but he died in 1813 before the revision could be completed.

Hamilton[11] calls Lagrange as the Shakespeare of mathematics!

EIGHTEENTH CENTURY

We are already familiar with the work of Galileo, Hooke and Mariotte in the 17^{th} century. The scientific work of the previous century found applications. Scientific methods were gradually introduced into the various fields of engineering. New advances in military and structural engineering demanded not only practical knowledge, but also the scientific ability

[11] William Rowan Hamilton (Aug. 1805 - Sept. 1865), Irish mathematician who became a professor of mathematics when he was still an undergraduate student!

to analyse new problems. Thus, in this background several engineering schools were opened, and the first books on structural engineering published to serve as the textbooks in these schools. The mechanics of solids and structural engineering got a big boost and progressed considerably. France was well ahead of all other countries.

In 1720 several military schools were opened in France to train qualified engineers for artillery and fortifications. Belidor[12] published a book on mathematics for use in these schools. This book contained not only mathematics, but also its applications to artillery, geodesy and mechanics. Belidor's book contains only elementary mathematics, but he recommends the book *Analyse des infiniment petits* by L'Hôpital. Mathematics was liberally used in more and more engineering problems, and this tendency grew rapidly. This was the general scene in France.

Parent[13] brought forth two memoirs in 1713, both relating to the bending of beams. It took more than fifty years before further progress was made. The reason was perhaps that Parent's writings were not easy to read, and moreover that he was not too popular as he was harshly critical of the work of others.

A great deal of experimental work of direct practical use was carried out in the 18^{th} century. This information is crucial to assessing the strength of structural materials. Among these scientists mention must be made about Réaumur[14] and Mussechenbroek[15]. There was much progress in the theory of retaining walls and arches in the 18^{th} century.

Coulomb

Coulomb[16] had his early education in Paris. Thereafter he joined the military corp of engineers and was sent to an island where he was given the responsibility of various construction activities. This job led him to study the mechanical properties of materials. Various structural engineering problems also engaged his attention. He worked in this island for nine years. During this time he presented a famous paper to the French Academy of Sciences. This was in 1773. After he returned to France, he worked as an engineer. He first won a prize in 1779, jointly with another person (Van Swindes) from the Academy, for the construction of a compass. Again in 1781 he won the Academy prize for his memoir *Theoérie des machines simples*. After this Coulomb was permanently at the Academy in Paris when he was elected as a member of the Academy. The facilities for scientific work there were excellent.

He now turned his attention to electricity and magnetism. He invented a very sensitive torsion balance to measure small electric and magnetic forces. It was this work that led to his investigations on torsion (of circular prismatic bars, Coulomb's theory). He devised an experimental setup to study torsional vibrations.

[12] Bernard Forest de Bélidor (1697 or 1698 - Sept. 1761), French engineer

[13] Antoine Parent (Sept. 1666 - Sept. 1716), French engineer

[14] René Antoine Ferchault de Réaumur (Feb. 1683 - Oct. 1757), Dutch engineer

[15] Petrus (Pieter) van Mussechenbrock (Mar. 1692 - Sept. 1761), French scientist

[16] Charles-Augustine de Coulomb (June 1736 - Aug. 1806), French scientist and engineer

When the French revolution[17] erupted in 1789, Coulomb retired to his place. The Academy was closed in 1793, but was reopened two years later, but with a different name, *L'Institut National des Sciences et des Arts*. Coulomb was elected as a member. There he continued his scientific work. His last papers dealt with the viscosity of fluids and with magnetism. These were published in the memoirs of the new institute in 1801 and 1806. In 1802 he was appointed as one of the inspectors of study which entailed a lot of travel. He was now concerned with the improvement of education.

The greatest contribution in the eighteenth century to the mechanics of solids was by Coulomb. It is remarkable that his theories of friction, torsion and strength of structural materials still stand, and are still used.

École Polytechnique

Figure 14.5: École Polytechnique: three pictures

The École Polytechnique[18] has played a crucial role in the development of our subject. The new institute had a vision quite different from that of the existing technical institutions. Admission was made open to students from all classes of society. To get the best students an entrance examination was introduced. The most accomplished scientists were appointed as the faculty members. Almost all of the famous French scientists were either students, or teachers, or both of this institute which immediately became famous. There were marked revolutionary changes from the existing practices. Great emphasis was given to basic, fundamental sciences. Students were given a very strong dose of mathematics, physics, chemistry and mechanics, and not much of anything else[19]. All were mathematical / analytical subjects taught by the most outstanding scientists. Lectures were delivered to large classes, but they were supplemented by small groups of twenty students supervised by younger teachers. This is the same as the small tutorial classes where the students do work supervised by teachers. The idea is to develop the ability to learn. Learning how to learn

[17] 1789 - 1799; the revolution ended when Napolean Boneparte took power in 1799. King Louis XVI (1754 - 1793) on January 21, 1793 and later the queen, his wife, Marie Antoinette on October 16, 1793 were executed. According to the available records, over 17,000 people were officially tried and executed, 16,594 with the guillotine, and an unknown number of others, perhaps more than 40,000, died in prison or without trial.

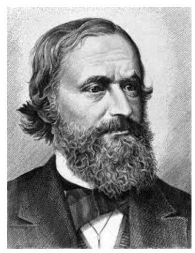
[18] Founded on March 11, 1794 by Gaspard Monge. It was right in Paris. In 1976 it was moved to a Parisian suburb, Palaiseau, about 30 km from Paris. The Laboratorie de Méchanique des Solides (LMS) is perhaps the part of the École Polytechnique that has the greatest relevance to us in this context.

[19] It must be conceded that in those days there was no electrical engineering nor computer science. There were no 'muck' subjects.

(whatever else will be needed in later life) was the *mantra*. École Polytechnique had this unique stamp. History records that the products of this institute became highly successful engineers. The institute was a great training ground not only for engineers, but also for teachers and professors. This general style of focussing on developing the ability to learn has continued to this day. Timoshenko [14] remarks: "It seems that this was the first time that laboratory work was introduced as part of a teaching program." This institute played a pivotal role in the growth of engineering science.

Needless to state, several universities like the Polytechnical Institute in Vienna and the Polytechnical Institute in Zurich followed suit. Russia copied this French system. Many German universities also took the best out of this and adapted this to suit their conditions of giving far greater participation to private industries.

(a) Claude-Louis Navier



(b) Gustav Robert Kirchhoff

(c) Gabriel Lamé

Figure 14.6: Three great scientists who contributed a great deal to our subject

SUBSEQUENT DEVELOPMENTS

Subsequent developments were very rapid, varied and almost explosive[20]. We shall, therefore, refrain from carrying on with the story. We had stated right in the beginning of these historical notes that we propose to have a broad look only of the early progress. Our subject is almost an inseparable mix of strength of materials, structural engineering and the theory of elasticity. We can see that France had played the leading role in the early days; the French scientists were clearly far ahead of other countries. By about the nineteenth century, the scene shifted gradually to Germany. This does not mean that there was nothing of note in France, England and other countries.

Again in recent times, the face of our subject has changed and is still changing. Continuum mechanics, rational mechanics, nonlinear theory of elasticity, micropolar elasticity, thermoelasticity, etc. along one direction; theory of plasticity, viscoelasticity, hyperelasticity, etc. in another; powerful numerical methods like Finite Difference Methods, Finite Element Methods of various colours and hues in yet another; composites, fracture mechanics, etc. in a new direction; and so on. All these place heavy demands on the mathematics needed which becomes more and more abstract. And then there are methods of experimental stress analysis and sophisticated experimental techniques. The subject has opened up

[20] Timoshenko [16] divides into separate chapters the history of strength of materials between 1800 - 1833 and 1833 - 1867, and considers in a separate chapter the history of the theory of elasticity.

immense possibilities. It is vigorously growing like all branches of science. Some familiarity with these various disciplines will be beneficial, although it is clearly impossible to be abreast in all these.

In a sense it is unfair to gloss over the great developments that happened in the later part of the 18^{th} century and later. But one has to make harsh decisions when leaving out important matters and topics. With these words of apology, let us see the names of a few outstanding scientists below and just a few lines about them. Navier[21], Lamé[22] and Kirchhoff[23] were these three outstanding scientists who contributed much to our subject.

Navier

Although Coulomb had given in his famous memoir (1773) the correct solutions of several problems of technical importance, engineers had to wait for a long period of forty years to understand them and use them for practical purposes. Further progress was made by Navier. When he was 14, he lost his father, and then he lived with his uncle Gauthey who was a famous French engineer. The uncle gave his nephew much practical knowledge about bridge and channel construction. Thus, when he graduated first from École Polytechnique in 1804, and then from École des Ponts in 1808, he was in an ideal position to study theoretical matters related to practical engineering problems. When his uncle died in 1807, Navier completed his uncle's incomplete book in three volumes in 1809, 1813 and 1816. He had added several important notes to update his uncle's book. His own (Navier's) famous book on strength of materials appeared in print in 1826. The book expectedly had a profound influence on the course of development of our subject. In this book he had discussed, among many other topics, retaining walls, arches, plates and trusses also. In 1820 he presented a memoir on the bending of plates. In 1821 appeared his famous paper in which all the fundamental equations of the theory of elasticity were given. In 1830 he became a professor of the calculus and mechanics at the École Polytechnique.

Kirchhoff

Gustav Robert Kirchhoff was born in Königsberg. He entered the Königsberg University, where he became a student of Franz Neumann, a great name in those days. He was in Heidelberg along with the chemist Bunsen and the great Helmholtz, and instrumental in establishing Heidelberg as the centre of great scientific activity. In 1845 he announced his famous Kirchhoff's laws, known sometimes as KVL and KCL, which are of fundamental importance in electrical engineering.

His contributions to our subject are many: Love-Kirchhoff assumption, proof of the uniqueness theorem, obtaining the correct boundary conditions for a plate using variational methods ('Kirchhoff shear'). This is only one part of his scientific achievements.

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[21] Claude-Louis Navier (Feb. 1785 - Aug. 1836), French engineer and physicist who specialised in mechanics. His name, sure enough, is included in the 72 names inscribed on the Eiffel Tower.

[22] Gabriel Leon Jean Bapiste Lamé (July 1795 - May 1870) "was the consummate engineer-mathematician." "French mathematicians considered him to be too practical, and French scientists too theoretical."

[23] Gustav Robert Kirchhoff (Mar. 1824 - Oct. 1887), German physicist of Königsberg, Kingdom of Prussia (now in Russia)

He moved to Berlin University in 1857. In 1859 he published some of his well known papers. His famous book on mechanics, which is the first volume of his lectures on theoretical physics *Forlesungen über mathematische Physik, Mechanik* appeared in 1876. He was an excellent teacher, very strong in both theoretical physics and experimental methods. His health was seriously affected when he sustained a leg injury in an accident and had to use a wheel-chair. His collected papers were published in 1882. His contributions are too many to mention here. A great scientist indeed!

Lamé

Gabriel Lamé, like several distinguished French scientists and engineers, studied first at École Polytechnique (1813 - 1817). He wrote research papers even when he was an undergraduate student there. Later he studied at École des Mines graduating in 1820. At this time, he published his new method of calculating the angles between the faces of crystals.

On a request from the Emperor of Russia, Alexander I, France sent Navier and Émile Clapeyron to work in St. Petersburg. The events in France had convinced the Emperor of the importance of scientific research. He went to Russia along with his colleague in 1820. Lamé was appointed as professor and engineer. During his 12 years there, he published several papers in French and Russian, some jointly with Clapeyron. He returned to Paris in 1832, he and Clapeyron joined an engineering firm, but he was also Professor of Physics at École Polytechnique. He was appointed as Chief Engineer in 1836. He was involved in building railway lines in France. In 1836 he moved to the prestigious Sorbonne as Professor of Mathematical Physics and Probability. He was elected to *Academie Sciences* in 1842.

Lamé had made several outstanding contributions including the extensive use of curvilinear coordinates. He was considered as the leading French mathematician by no less a person than the great Gauss and others. He was a great engineer. Perhaps he was even greater as a scientist in engineering mathematics and elasticity.


(a) C.A. Coulomb


(b) Thomas Young


(c) B.P.E. Clapeyron

Figure 14.7: Three great engineer-scientists who contributed a great deal to our subject.

Theodore von Kármán (May 1881 - May 1963), Geoffrey Ingram Taylor (March 1886 - June 1975) and Stephen Prokofyevich Timoshenko (Dec. 22, 1878 - May 29, 1972) are three engineer-mathematician-scientists of relatively recent times.

At this stage we close our excursion to the past. The purpose of these few pages on the historical development is not to produce a complete and comprehensive document giving all

(a) A.L. Cauchy (b) J.V. Poncelet (c) G. Monge (d) B. Saint-Venant

Figure 14.8: Four great engineer-mathematicians

(a) Th. von Kármán (b) G.I. Taylor (c) S.P. Timoshenko

Figure 14.9: Three giants of relatively recent times

the details (who did what, and when). It is merely to take the readers to a bygone era, and to be in the company of the great scientists. Some of the young readers may be inspired to attain higher academic levels. One hopes, therefore, that no apology is needed for not covering important topics, eras, persons and achievements. It is not easy to achieve the right balance in such a discussion.

We have come to the end of the book. At this stage, this author seeks the permission of his readers to sign off.

Best wishes and happy learning!

"I keep six honest serving men
They taught me all I know;
Their names are
What and Why and When
And Where and How and Who."

(Rudyard Kipling)

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