RF and Microwave Electromagnetism
RF and Microwave Electromagnetism

Pierre Jarry
Jacques N. Beneat
## Contents

**Preface** ........................................... xi

**INTRODUCTION** ................................. xv

**PART 1. TRANSMISSION LINES** ................. 1

**CHAPTER 1. ELECTROMAGNETIC OF TEM TRANSMISSION LINES** ............................. 3

1.1. General waves .................................. 3
1.2. Transverse electromagnetic (TEM) waves .... 3
1.3. Solutions of the transverse electromagnetic waves ............................................ 7
1.4. Characteristic parameters of the TEM lines. . 8
   1.4.1. Capacitance per unit of length.......... 8
   1.4.2. Characteristic impedance ................. 10
   1.4.3. Conductance per unit of length ......... 11
1.5. The power ....................................... 13
   1.5.1. Density .................................. 13
   1.5.2. Flux .................................... 13
1.6. Problems ....................................... 16
   1.6.1. The band-line ............................ 16
   1.6.2. The coaxial cable ......................... 18
1.7. Bibliography ................................... 21
# CHAPTER 2. LOSSES IN TEM TRANSMISSION LINES

2.1. Introduction ................................................. 23
2.2. Perturbation computing ................................. 23
2.3. Dielectric losses .......................................... 24
    2.3.1. Determination from the dielectric constant ................................. 24
    2.3.2. Determination from the Maxwell–Ampere relation ............................ 25
2.4. Metallic losses ............................................. 27
2.5. General case: dielectric losses and metallic losses ............................. 30
2.6. Problems .................................................. 32
    2.6.1. The transmission line with low losses ..................................... 32
    2.6.2. Coaxial cable with losses ............................................. 39
2.7. Bibliography ................................................. 49

# CHAPTER 3. DETERMINATION OF THE CHARACTERISTICS OF TEM LINES

3.1. Introduction ................................................ 51
3.2. Conform transformations .................................. 51
    3.2.1. Determination of the capacity ............................................. 51
    3.2.2. Transformation in the complex plane ..................................... 52
    3.2.3. Orthogonality ............................................... 54
    3.2.4. Position of $\nabla u$ versus $\nabla v$ ............................................. 55
    3.2.5. Recapitulation ............................................... 56
    3.2.6. Example of computation .......................................... 56
3.3. Finite differences method ................................... 57
    3.3.1. Example of the finite differences method ............................... 59
3.4. Problems .................................................. 61
    3.4.1. Conform transformations ............................................. 61
    3.4.2. Eccentric coaxial line using conform transformations ...................... 65
3.5. Bibliography ................................................. 74
PART 2. GUIDES ............................................. 77

CHAPTER 4. ELECTROMAGNETIC IN LINEAR, HOMOGENEOUS, ISOTROPIC AND LOSSLESS GUIDES . 79

4.1. Introduction ............................................. 79
4.2. Equations for a lossless medium .................... 80
4.3. Limiting conditions ................................ 84
4.4. Progressive and evanescent waves ............... 86
4.5. Propagating waves .................................. 89
4.6. Group speed .......................................... 92
4.7. Average power flux ................................... 93
  4.7.1. Stokes’ theorem .................................. 95
  4.7.2. Ostrogradsky’s theorem ......................... 96
4.8. Power density ........................................... 97
4.9. Energy speed .......................................... 98
4.10. First example of TE waves ......................... 99
4.11. Second example of TM waves .................. 101
4.12. Inverse waves ....................................... 102
4.13. Behavior of the TE and TM waves versus the position of frequency in connection with the cutoff ................................. 103
  4.13.1. Above the cutoff $\omega > \omega_c$ ................ 103
  4.13.2. At the cutoff $\omega = \omega_c$ .................... 104
  4.13.3. Under the cutoff $\omega < \omega_c$ ............... 104
  4.13.4. Summary ........................................ 105
4.14. Bibliography .......................................... 105

CHAPTER 5. LOSSES IN GUIDES ....................... 107

5.1. Introduction ........................................... 107
5.2. TE waves .............................................. 108
5.3. TM waves ............................................... 109
5.4. Attenuation in the cases of TM and TM waves . 110
5.5. Problem .................................................. 111
  5.5.1. Waves between two parallel metallic and lossy planes ................................ 111
5.6. Bibliography ............................................ 121
CHAPTER 6. RECTANGULAR TM AND TE GUIDES . . . 123

6.1. Introduction ....................................... 123
6.2. TM rectangular guide ............................... 124
  6.2.1. The fields .................................... 124
  6.2.2. The dispersive relation ...................... 126
  6.2.3. The power flux ............................... 128
  6.2.4. Attenuation .................................. 129
  6.2.5. Field lines ................................... 131
6.3. TE rectangular guide ............................... 136
  6.3.1. The fields .................................... 136
  6.3.2. The dispersive relation ...................... 139
  6.3.3. The power flux ............................... 140
  6.3.4. Attenuation of the fundamental
  $m = 0$ and $n = 1$ .................................. 141
6.4. Problems ........................................ 142
  6.4.1. The fundamental $\text{TE}_{01}$ mode of the
  rectangular guide .................................. 142
  6.4.2. Rectangular $\text{TE}_{01}$ guide with dielectric . 146
6.5. Bibliography .................................... 150

CHAPTER 7. CIRCULAR TM AND TE GUIDES .......... 151

7.1. Introduction ....................................... 151
7.2. Properties of the TE and TM circular
  waveguide ............................................ 151
7.3. TM circular waveguide ............................ 154
7.4. TE circular waveguide ............................ 156
7.5. Fundamental mode and classification
  of the modes ....................................... 158
7.6. Utilization band of the fundamental
  mode $\text{TE}_{11}$ ..................................... 161
7.7. Field lines of the first modes .................... 162
  7.7.1. The fundamental $\text{TE}_{11}$ .................. 162
  7.7.2. $\text{TM}_{01}$ Symmetry of revolution ........ 162
  7.7.3. $\text{TE}_{21}$ Quadrupolar mode ................ 162
  7.7.4. $\text{TE}_{01}$ Degenerated mode with the $\text{TM}_{11}$ 163
  7.7.5. $\text{TM}_{11}$ Degenerated mode with the $\text{TE}_{01}$ 163
7.8. Power flux and attenuations .................. 163
7.9. Problems .................................. 165
  7.9.1. Semi-circular and quadrantl guide .... 165
  7.9.2. Angle $\alpha$ guide ..................... 168
  7.9.3. Computing the power flux and the
         attenuations for TM and TE fields ...... 170
7.10. Bibliography ................................ 170

**PART 3. CAVITIES** ............................. 173

**CHAPTER 8. RECTANGULAR TE$_{011}$ CAVITY** ........ 175
  8.1. Introduction ............................. 175
  8.2. The fundamental waves ................... 175
  8.3. Construction of the cavity ................ 176
  8.4. The cavity ................................ 178
  8.5. The waves in the cavity .................. 179
  8.6. Electric and magnetic energies in the cavity .. 181
     8.6.1. Electric energy ..................... 181
     8.6.2. Magnetic energy ................... 182
  8.7. Quality factor $Q$ of the cavity .......... 184
  8.8. Bibliography .......................... 189

**CHAPTER 9. CIRCULAR TE$_{mnp}$ AND TM$_{mnp}$
          CAVITIES** ................................ 191
  9.1. Introduction ............................. 191
  9.2. The fundamental propagative
       TM$_{m,n}$ and TE$_{m,n}$ waves ............ 191
  9.3. TE and TM stationary waves ............. 192
  9.4. Realization of a cavity .................. 193
  9.5. The cavity ................................ 194
  9.6. Curve representations ................... 195
  9.7. Frequent and particular examples of modes . 197
  9.8. Examples of the fields of current modes . 198
  9.9. Bibliography .......................... 199

**INDEX** ....................................... 201
Preface

Microwave and Radio Frequency (RF) elements play an important role in communication systems, and due to the proliferation of radar, satellite and mobile wireless systems, there is a need for the study of electromagnetism. This book provides basic knowledge of the microwave and RF range. It has grown from the authors’ own teaching and as such has a unity of methodology and style, essential for a smooth reading.

The book is intended for microwave engineers and advanced graduate students.

Each of the nine chapters provides a complete analysis and modeling of the microwave structure used for emission or reception technology. We hope that this book will provide students with a set of approaches that they could use for current and future RF and microwave circuit designs. We also emphasize the practical nature of the subject by summarizing the analysis steps and giving numerous examples of problems and exercises with solutions so that RF and microwave students can have an appreciation of each aspect. The book is therefore theoretical but also experimental with 17 microwave problems and examples. The exercises occupy about 40% of the book. This approach,
we believe, has produced a coherent, practical and real-life treatment of the subject.

We have decided to successively study the functions that allow the reception and the emission of a signal in the cases of Earth stations, of satellites and of RF (mobile phones):

– the transmission lines;
– the guides;
– the cavities.

For all these three functions, we give their principal properties in several chapters mixed with exercises and problems.

Figure 1. Organization of the book

The book is divided into three parts and nine chapters:

i) Lines + Problems
ii) Guides + Problems
iii) Cavities + Problems
Part 1 is entirely devoted to the introduction of transmission lines. The goal of Chapter 1 is to give the characteristic parameters of the TEM lines (capacitance, conductance per unit of length and the characteristic impedance). First, we discuss power. Then we give examples and problems (with solutions) of the band-line and of the coaxial cable.

Chapter 2 gives the losses in the TEM transmission line. The metallic and the dielectric losses are computed from a method of perturbation. These results are applied to the electromagnetic of a classical transmission line and to a coaxial cable.

In Chapter 3, we describe different methods of determining the characteristics of TEM lines as conformal transformation and the finite differences method. In the problems with solutions, we study classical conformal transformations and the case of the eccentric coaxial.

In Part 2, we consider what the properties of the guides are.

The first chapter of this second part (Chapter 4) is devoted to the determination of the waves starting from Maxwell equations. We compute the energy speed from Ostrogradsky's and Stokes's theorems. We consider the cases of TE or TM waves and of waves above the cutoff, at the cutoff and under the cutoff (evanescent). A summary table is given at the end of this chapter.

Chapter 5 is devoted to the determination of losses in the cases of TE or TM waves. We give, as a problem with a solution, the losses for a guide composed of parallel metallic and lossy planes.

Now we consider, in particular (Chapter 6), rectangular TE and TM guides. We give magnetic field lines and electric
field lines for a $TM$ rectangular guide and in the plane $x = Cte$. In the case of $TE$ propagation, we consider the fundamental $TE_{01}$ as a problem. In the two cases, we also give the dispersive relation, the power flux and the attenuation. We do the same for circular $TE$ and $TM$ guides (Chapter 7) and we show that the Bessel functions apply. As problems, we consider the semi-circular and quadrantial guide and also the $\alpha$ angle guide.

In the last part, Part 3, we show how to realize microwave and RF cavities.

First, in the case of the rectangular cavity (Chapter 8), we construct the fundamental which is the $TE_{011}$. We give the waves in the cavity, the electric energy and the magnetic energy, and we define the quality factor of the cavity $Q$. This chapter is easy to understand and can be considered as an entire problem. Chapter 9 is more complicated because we consider the general cases $TE_{mnp}$ and $TM_{mnp}$. For the different values of $(m,n,p)$, we give the waves in the constructed cavity.

The aspects and the corresponding problems are given during the fourth year of university and at specialist engineering schools.

Professor Pierre JARRY
France
Professor Jacques N. BENEAT
USA
April 2014
I.1. Introduction

The first microwaves were reserved for radar and telecommunications. Microwave development is now increasing by about 15% per year and we find microwaves in various applications:

− satellite equipment;
− Hertzian equipment;
− mobile phones;
− medical applications;
− astronomic radio;
− numerical transmission systems;
− heating, etc.

In the 1950s, we began by using tube generators. In the 1970s, with the emergence of microwave transistors, circuits became increasingly compact (e.g. the bipolar and the field-effect transistors (FET)). Then it was possible to integrate the active components using strip, and then microstrip, lines. From 1990 to now, the complete integration of active components has been made possible using
microwave amplifiers, couplers, filters, diodes, attenuators, commutators, phasors, etc.

## I.2. The electromagnetic spectrum

Electromagnetic waves are characterized by electric and magnetic fields, and it is said that we are in a microwave domain if we work between two frequencies from 300 MHz (\(M = 10^6\)) to 300 GHz (\(G = 10^9\)).

### Length wave classification

<table>
<thead>
<tr>
<th>Frequency Range</th>
<th>Wave Length</th>
<th>Classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>300 MHz – 3 GHz</td>
<td>1 m – 10 cm</td>
<td>Ultra High Frequencies (U.H.F.) Decimetric Waves</td>
</tr>
<tr>
<td>3 GHz – 30 GHz</td>
<td>10 cm – 1 cm</td>
<td>Super High Frequencies (S.H.F.) Centimetric Waves</td>
</tr>
<tr>
<td>30 GHz – 300 GHz</td>
<td>1 cm – 1 mm</td>
<td>Extremely High Frequencies (E.H.F.) Millimetric Waves</td>
</tr>
</tbody>
</table>

### Band classifications

Some bands are allocated to the corresponding waveguides.

- **L**: 1.14 to 1.73 GHz
- **D**: 1.72 to 2.61 GHz
- **S**: 2.6 to 3.95 GHz
- **G**: 3.95 to 5.85 GHz
- **C**: 5.20 to 5.90 GHz
- **J**: 5.30 to 8.20 GHz
- **H**: 7.05 to 10 GHz
- **X**: 8.20 to 12.4 GHz
- **Ku**: 11.9 to 18 GHz
- **Kx**: 17.6 to 26.7 GHz
- **Q (Ka)**: 26.5 to 40 GHz

### A rapid historical glance

1920 was the date of the first generator, the Magnetron (by Hull).
1935, a more sophisticated generator, the Klystron (developed by Russel and Variant).

From 1940 to now, the development of radar (in military and civil applications such as guiding, telecommunications, space, etc.).

1950, ferrite components.

1962, GUNN diode.

1970, microwave transistors such as the FET.

1990 to now, microwave integrated circuits, mobile phones.

I.3. International frequencies

Frequencies (or the band of frequencies) are attributed for specific applications. We present frequencies of some of these applications.

<table>
<thead>
<tr>
<th>Frequency Range</th>
<th>Application</th>
</tr>
</thead>
<tbody>
<tr>
<td>470 to 890 MHz</td>
<td>Television</td>
</tr>
<tr>
<td>890 to 940 MHz</td>
<td>Civil telecommunications</td>
</tr>
<tr>
<td>940 to 1350 MHz</td>
<td>Radiophone, Radar, Mobile, Hertzien, etc.</td>
</tr>
<tr>
<td>1350 to 2700 MHz</td>
<td>Data, meteo, radioprobs, Mobile, etc.</td>
</tr>
<tr>
<td>2.7 to 5 GHz</td>
<td>Satellite, Radioastronomy, Radar, Mobile, Radionavigator</td>
</tr>
<tr>
<td>5 to 20 GHz</td>
<td>Satellite, Mobile, Television, etc.</td>
</tr>
<tr>
<td>from 20 GHz</td>
<td>Mobile, Numerical on optic fibers, etc.</td>
</tr>
</tbody>
</table>

With microwaves we are able to construct systems with broadband and then transport maximum information.
I.4. Bibliography


PART 1
Transmission Lines
Electromagnetic of TEM Transmission Lines

1.1. General waves

Generally, electromagnetic waves are given by the solution of well-known Maxwell–Faraday formulas. These formulas give electric and magnetic fields (\( \bar{E}, \bar{H} \)). In the case without charges we have the following expressions:

\[
\begin{align*}
\text{rot } \bar{E} & = -\mu \frac{\partial \bar{H}}{\partial t} \\
\text{rot } \bar{H} & = \varepsilon \frac{\partial \bar{E}}{\partial t} \\
\text{div } \bar{E} & = 0 \\
\text{div } \bar{H} & = 0
\end{align*}
\]

1.2. Transverse electromagnetic (TEM) waves

The waves are propagating in a medium which is

– linear;

– homogeneous;
– isotropic;
– without loss.

The medium being linear means that the permittivity constant $\varepsilon$ and the permeability constant $\mu$ are independent of the frequency.

In the case of a harmonious state, the electric and magnetic fields are of the form:

$$
\begin{align*}
\mathcal{E}(M,t) &= \mathcal{E}(x,y) e^{j\omega t + \gamma z} \\
\mathcal{H}(M,t) &= \mathcal{H}(x,y) e^{j\omega t + \gamma z}
\end{align*}
$$

where the propagating constant is:

$$
\gamma = \alpha + j \beta
$$

$\alpha$ is the attenuation and $\beta$ is the propagation.

When we consider only positive propagating waves, we have a solution in $-\gamma$, and the resulting electric and the magnetic waves are:

$$
\begin{align*}
\mathcal{E}(M,t) &= \mathcal{E}(x,y) e^{-\alpha z} e^{j(\omega t - \beta z)} \\
\mathcal{H}(M,t) &= \mathcal{H}(x,y) e^{-\alpha z} e^{j(\omega t - \beta z)}
\end{align*}
$$

The amplitude decreases by the factor $e^{-\alpha z}$. Also, phase is constant if:

$$
\omega t - \beta z = ct
$$

This means that:

$$
\omega dt - \beta dz = 0
$$
And we have the phase speed:

$$v_p = \frac{dz}{dt} = \frac{\omega}{\beta}$$

Now we decompose the electromagnetic wave (Figure 1.1) with:

- a part parallel to the propagating way \( z \): \( E_z(x,y) \hat{u} \);
- a part perpendicular to the propagating way \( z \): \( \vec{E}_T(x,y) \).

\[ \text{Figure 1.1. Decomposition of the EM field} \]

In the case of transverse electromagnetic (TEM) fields, the waves are transversal; then:

\[
\begin{align*}
E_z &= 0 \\
H_z &= 0
\end{align*}
\]

which means that:

\[
\begin{align*}
\vec{u}.\vec{E} &= 0 \\
\vec{u}.\vec{H} &= 0
\end{align*}
\]
And the transversal fields satisfy the following group of equations:

\[
\begin{align*}
\text{div}_T \vec{E}_T &= 0 \\
\text{div}_T \vec{H}_T &= 0 \\
\text{rot}_T \vec{E}_T &= 0 \\
\text{rot}_T \vec{H}_T &= 0 \\
\gamma (\vec{u} \wedge \vec{E}_T) &= j \omega \mu \vec{H}_T \\
\gamma (\vec{u} \wedge \vec{H}_T) &= -j \omega \epsilon \vec{E}_T
\end{align*}
\]

To ensure the compatibility of these equations, we must have

\[
\gamma = \pm j \omega \sqrt{\mu \epsilon} = \pm j \beta
\]

And the TEM fields are lossless

\[
\alpha = 0
\]

From the quantity

\[
\alpha = 0 \text{ and } \beta = \omega \sqrt{\mu \epsilon}
\]

This gives

\[
v_p = \frac{\omega}{\beta} = \frac{\omega}{\omega \sqrt{\mu \epsilon}} = \frac{1}{\sqrt{\mu \epsilon}} = c
\]

The TEM fields are propagating at the same speed as the uniform plane waves \(c\).

From these equations, one can deduce the values of transversal electric and magnetic fields.

\[
\begin{align*}
\vec{H}_T &= \frac{1}{Z} (\vec{u} \wedge \vec{E}_T) \\
\vec{E}_T &= Z \left( \vec{H}_T \wedge \vec{u} \right)
\end{align*}
\]
where \( Z = \sqrt{\frac{\mu}{\varepsilon}} \) is the wave impedance.

TEM fields have the same properties as the uniform plane waves. The only difference is that the TEM fields \( \vec{E}_T \) and \( \vec{E}_T \) depend on \( x \) and \( y \). \( \vec{E}_T \), \( \vec{E}_T \) \( \vec{u} \) and a trirectangle and direct trihedral.

1.3. Solutions of the transverse electromagnetic waves

We have to satisfy:

\[
\begin{align*}
\text{rot} \vec{E}_T &= 0 \\
\text{div}_T \vec{E}_T &= 0
\end{align*}
\]

From the first equation, there is a scalar potential \( V \) that \( \vec{E}_T = -\text{grad}V \).

In the second formula, we obtain:

\[
\text{div}_T \left( \text{grad}V \right) = 0
\]

This means that the Laplacian of the potential \( V \) is zero:

\[ \Delta_T V = 0 \]

– If TEM waves exist, then there are at least two conductors.
– The conductors are at the same potential.
– The force lines (\( E \) field) are orthogonal to the equipotential.
If magnetic and electric fields are orthogonal, then the determination of the EM fields of TEM conductors is the same as a problem of electrostatic. We have to resolve:

\[
\begin{align*}
\vec{E}_T &= -\nabla \tilde{V} \\
\Delta \tilde{V} &= 0 \\
\vec{n} \times \vec{E}_r &= 0 \\
\vec{H}_T &= \frac{1}{Z} \left( \vec{u} \times \vec{E}_r \right) = -\frac{1}{Z} \vec{u} \times \nabla \tilde{V}
\end{align*}
\]

1.4. Characteristic parameters of the TEM lines

1.4.1. Capacitance per unit of length

The charge density of the conductors in Figure 1.3 is:

\[
\rho_S = D_n = \vec{n} \cdot \vec{D} = \varepsilon \vec{n}_v \vec{E} = -\varepsilon \nabla \tilde{V} \cdot \vec{n}_v = \varepsilon \nabla \tilde{V} \cdot \vec{n}
\]

\[
\rho_S = \varepsilon \left| \nabla \tilde{V} \right|
\]
On the surface $S_1$, the charge is $Q = \iiint_{S_1} \rho_s \, dS = \varepsilon \iiint_{S_1} \left| \nabla V \right| \, dS$.

On the surface $S_2$, the charge is $-Q$.

The potential between the two conductors $S_1$ and $S_2$ is:

$$V = \int_{S_1} E \, dl = -\int_{S_2} \nabla V \cdot dl = -\int_{S_1} dV = V_{S_1} - V_{S_2} = V_1 - V_2$$

Then, in the plane the capacity is

$$C = \frac{\varepsilon \iiint_{S_1} \left| \nabla V \right| \, dS}{\int_{S_1} \nabla V \cdot dl}$$
1.4.2. **Characteristic impedance**

![Characteristic impedance diagram](image-url)

**Figure 1.4. Characteristic impedance**

The superficial density of current is longitudinal because:

$$\bar{I}_S = \vec{n} \wedge \vec{H}_T = |H_T| \vec{u}$$

And the total current is:

$$I = \iiint_{S_2} \bar{I}_S \, d\vec{S} = \iiint_{S_2} (\vec{n} \wedge \vec{H}_T) \, \vec{u} \cdot d\vec{S} = \iiint_{S_2} |H_T| \, \vec{u} \cdot \vec{u} \, d\vec{S} = \iiint_{S_2} |H_T| \, d\vec{S}$$

Then:

$$I = \iiint_{S_2} |H_T| \, d\vec{S} = \frac{1}{Z} \iiint_{S_2} \left| \nabla \vec{V} \right| \, d\vec{S} = \frac{1}{Z} \frac{Q}{\varepsilon} = \frac{CV}{\varepsilon Z}$$

And by definition of the characteristic impedance

$$Z_c = \frac{V}{I} = \frac{\varepsilon Z}{C}$$

where $$Z = \sqrt{\frac{\mu}{\varepsilon}}$$ is the wave impedance of the plane waves.
We also have
\[ C = \varepsilon \frac{Z}{Z_c} \]

1.4.3. Conductance per unit of length

The lines of \( H \), are the equipotential. Let us calculate the flux \( \phi \) of the vector \( \vec{B} \).

\[ \mu \mu \varepsilon = \frac{G_T Z}{\nabla V} \]

We have:
\[ |H_T| dl = \frac{|E_T|}{Z} dl = \frac{\nabla V}{Z} dl = \frac{dV}{Z} \]

And the flux:
\[ |B| dl = \mu |H_T| dl = \mu \frac{dV}{Z} = \sqrt{\mu \varepsilon dV} \]
\[ \phi = \int_{S_2}^{S_1} \sqrt{\mu \varepsilon} dV = \sqrt{\mu \varepsilon} V = \frac{\mu V}{Z} \]

But:

\[ \phi = L I = \sqrt{\mu \varepsilon} V \]

\[ L = \frac{\sqrt{\mu \varepsilon} V}{I} = \sqrt{\eta \varepsilon} Z_c \]

This can be written as:

\[ L = \frac{\mu \varepsilon}{C} \]

Notice that the speed of the propagation of the TEM waves is the same as that of the classical theory of lines:

\[ v = \frac{1}{\sqrt{\mu \varepsilon}} = \frac{1}{\sqrt{LC}} \]

In the case of the TEM lines, we have, in fact, to remember only the expressions of the capacity, of the inductance and then of the characteristic impedance:

\[ C = \frac{\sqrt{\mu \varepsilon}}{Z_c} \]
\[ L = \sqrt{\mu \varepsilon} Z_c \]
\[ Z_c = \sqrt{\frac{L}{C}} \]

And we have the justification of the Kirchoff theory in the case of the TEM waves.
1.5. The power

1.5.1. Density

We define the density of magnetic power and of electric power by

\[
\begin{align*}
\bar{W}_M &= \frac{1}{4} \int \mu \tilde{H}_T \tilde{H}_T^* dS \\
\bar{W}_E &= \frac{1}{4} \int \varepsilon \tilde{E}_T \tilde{E}_T^* dS
\end{align*}
\]

1.5.2. Flux

If a phenomenon is periodic, its average value over a long time tends to be its average value in one period.

\[
\bar{P} = \Re \left\{ \frac{1}{2} \int \left( \tilde{E} \times \tilde{H}^* \right) \tilde{u} dS \right\} = \Re \left\{ \frac{1}{2} \int \left( \tilde{E} \times \tilde{H}^* \right) \tilde{u} dS \right\}
\]

where

\[
\tilde{E} = \tilde{E} e^{-j\beta z}
\]

\[
\tilde{H}^* = \frac{1}{Z} \tilde{u} \times \tilde{E}^* = \frac{1}{Z} \left\{ \left( \tilde{E} \tilde{E}^* \right) \tilde{u} - \left( \tilde{E}^* \tilde{u} \right) \tilde{E} \right\}
\]

Since \(\tilde{E}\) is perpendicular to \(\tilde{u}\), only the first part of the double vectorial product is different from zero, and using \(\tilde{E} = -\nabla \tilde{V}\), we get

\[
\bar{P} = \Re \left\{ \frac{1}{2Z} \int \tilde{E} \tilde{E}^* dS = \Re \int \left| \nabla \tilde{V} \right|^2 dS \right\}
\]
Now Green’s relation gives the integral with the \( \nabla \) of two functions \( \Phi, \Psi \) as:

\[
\int_S (\nabla \Phi \cdot \nabla \Psi + \Psi \Delta \Phi) \, dS = \oint_C \Phi \cdot \nabla \cdot \mathbf{n} \, dl
\]

where \( C \) is a close outline around surface \( S \).

**Figure 1.6. Green schemas**

To use Green’s formula, we take

\[ \Phi = \Psi = V \]

And using

\[ \Delta V = 0 \]
starting with an integral of order 2, we arrive at an integral of order 1 as:

\[
\iint_S \left| \mathbf{\nabla V} \right|^2 \, dS = \oint_c \mathbf{\nabla V} \cdot \mathbf{n} \, dl
\]

and

\[
\oint_c = \oint_{c_1} + \oint_{c_2}
\]

The integrals around \( C_1 \) and around \( C_2 \) are easy to calculate as:

\[
\oint_{c_1} = V_1 \oint_{c_1} \mathbf{\nabla V} \cdot \mathbf{n} \, dl = V_1 \oint_{c_1} \mathbf{\nabla V} \, dl = \frac{OV_1}{\varepsilon}
\]

In the same manner:

\[
\oint_{c_2} = -\frac{OV_2}{\varepsilon}
\]

And the total integral is then:

\[
\oint_c = \frac{Q}{\varepsilon} (V_1 - V_2)
\]

And we can obtain the average power flux:

\[
\bar{P} = \Re \left\{ \frac{1}{2Z} \frac{QV}{\varepsilon} \right\} = \Re \left\{ \frac{V^2}{2Z_c} \right\}
\]

In general, this is a real quantity

\[
\bar{P} = \frac{1}{2} \frac{V^2}{Z_c}
\]
1.6. Problems

1.6.1. The band-line

1) A band-line is made up of two parallel conductors (Figure 1.8). When we consider the simplified model (without the border effects), we can calculate the potential.

![Figure 1.8. The band-line and the waves](image)

2) Let us compute the total charge, the capacity by length, the inductance by length and the characteristic impedance.

SOLUTIONS.—

1) To simplify the problem, we neglect the border effects. It is the same if we have a magnetic wall at the two sides (Figure 1.9).

![Figure 1.9. The electric waves of the band-line when neglecting the border effects](image)
We have to verify

$$\Delta V = 0 \quad \text{or} \quad \frac{\partial^2 V}{\partial x^2} = 0$$

The problem is only a function of \(x\):

$$\frac{dV}{dx} = -E$$

$$V(x) = -E x + C^e$$

The conditions are:

$$\begin{cases} x = 0, V = 0 \\ x = a, V = a \end{cases} \Rightarrow E = -\frac{V}{a}$$

and:

$$V(x) = V \frac{x}{a}$$

2) The total charge is:

$$Q = \int_0^b \rho_s \, dl = -\varepsilon E \int_0^b \, dl = -\varepsilon E b = \varepsilon V \frac{b}{a}$$

The capacity by length is:

$$Q = CV; \quad C = \varepsilon \frac{b}{a}$$

The inductance by length is:

$$L = \frac{\mu \varepsilon}{C}; \quad L = \mu \frac{a}{b}$$
The characteristic impedance is:

\[ Z_c = \sqrt{\frac{L}{C}}; \quad Z_c = \sqrt{\frac{\mu}{\varepsilon \cdot b}} = \frac{a}{b} \]

We recover the same values by using the electrostatic theory.

1.6.2. *The coaxial cable*

1) Give the expressions of the potential of a coaxial cable.
2) Deduce the electric and magnetic waves.
3) What is the characteristic impedance, the capacity and the inductance by unit length and density power?

**SOLUTIONS.–**

1) Electric waves are radial and magnetic waves are perpendicular to electric waves (Figure 1.10).

![Figure 1.10. The waves of the coaxial cable](image)

We have to verify in circular cylindrical coordinates:

\[ \Delta V = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} = 0 \]
Symmetry revolution gives that $V$ is independent of $\theta$:

$$\frac{\partial^2 V}{\partial \theta^2} = 0$$

Then:

$$\frac{d}{dr} \left( r \frac{dV}{dr} \right) = 0 \quad \Rightarrow \quad r \frac{dV}{dr} = C_{te} \quad \Rightarrow \quad dV = C_{te} \frac{dr}{r}$$

A general solution is:

$$V(r) = C_{te} \log r + C_{1te}$$

And with the limiting conditions:

$$V(a) = V, \quad V(b) = 0$$

we have

$$V(r) = \frac{\log \left( \frac{b}{r} \right)}{\log \left( \frac{b}{a} \right)}$$

2) Computing the waves, the electric wave is purely longitudinal:

$$\vec{E} = \vec{E}_r, \quad E = -\frac{dV}{dr}$$

$$E = \frac{V}{r \log \left( \frac{b}{a} \right)}$$
The magnetic wave depends only on $\theta$:

$$\vec{H} = \vec{H}_0, \quad H = \frac{1}{Z} E$$

$$H = \frac{V}{r Z \log \left( \frac{b}{a} \right)}$$

3) The characteristic impedance is:

$$Z_c = \frac{Z}{2\pi \log \left( \frac{b}{a} \right)}$$

where $Z = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi$ in the air

The capacity and the inductance by unit length are, respectively:

$$C = \varepsilon \frac{Z}{Z_c} = \frac{2\pi \varepsilon}{\log \left( \frac{b}{a} \right)}$$

$$L = \frac{\mu \varepsilon}{C} = \frac{\mu}{2\pi} \log \left( \frac{b}{a} \right)$$

The density power is

$$\bar{p} = \frac{1}{2} \frac{V^2}{Z_c} = \frac{\pi}{Z} \frac{V^2}{Z \log \left( \frac{b}{a} \right)}$$
1.7. Bibliography


Losses in TEM Transmission Lines

2.1. Introduction

We consider dielectric losses and metallic losses. We say that the losses are so slight that the waves are not modified; this means that the waves are still TEM and we can use their TEM expressions.

2.2. Perturbation computing

If the losses are slight, the waves are only attenuated. The waves are multiplied by two factors:

\[ e^{-az} e^{j(\omega t - \beta z)} \]

The first factor characterizes the attenuation, while the second factor, which remains unchanged, characterizes the propagation.

The power flux becomes:

\[ \bar{P}(z) = \bar{P}_0(z) e^{-2az} \]
where $\bar{P}_0(z)$ is the power flux without losses, and:

$$\frac{-\partial \bar{P}(z)}{\partial z} = 2\alpha \bar{P}_0(z)e^{-2\alpha z} = 2\alpha \bar{P}(z)$$

$$\alpha = -\frac{1}{2} \frac{\partial \bar{P}(z)}{\partial z}$$

In this manner, we can characterize:
- dielectric losses $\alpha_D$;
- metallic losses $\alpha_M$.

### 2.3. Dielectric losses

#### 2.3.1. Determination from the dielectric constant

When losses are dielectric losses, then we assume that the dielectric constant has an imaginary part. If the losses angle $\phi$ is small, we have:

$$\bar{\varepsilon} = \varepsilon' - j\varepsilon'' = \varepsilon e^{-j\phi}$$

$$\begin{cases} 
\varepsilon' = \varepsilon \cos \phi \approx \varepsilon \\
\varepsilon'' = \varepsilon \sin \phi \approx \varepsilon \phi
\end{cases}$$

Without losses, we have:

$$(j\beta)^2 + \omega^2 \mu \varepsilon = 0$$

$$\alpha = 0 \quad \text{and} \quad \beta = \omega \sqrt{\mu \varepsilon}$$
With losses, we obtain:

\[
\gamma^2 + \omega^2 \mu \varepsilon = 0 \\
(\alpha + j \beta)^2 + \omega^2 \mu (\varepsilon' - j \varepsilon'') = 0 \\
\begin{cases}
\alpha^2 - \beta^2 + \omega^2 \mu \varepsilon' = 0 \\
2 \alpha \beta - \omega^2 \mu \varepsilon'' = 0
\end{cases}
\]

In the case of slight losses \(\alpha \ll \beta\), it seems that the propagation is not modified by the losses. From the first equation, we obtain:

\[
\beta \approx \omega \sqrt{\mu \varepsilon'} \\
v \approx \frac{1}{\sqrt{\mu \varepsilon'}}
\]

From the second equation, we obtain:

\[
\alpha = \frac{\omega^2 \mu \varepsilon''}{2 \beta} = \frac{\omega}{2} \sqrt{\mu \varepsilon'} \varepsilon'' = \frac{\beta}{2} \tan \phi
\]

\[
\alpha \approx \frac{\beta}{2} \phi = \frac{\pi}{\lambda} \phi
\]

where \(\phi\) is the loss angle.

2.3.2. **Determination from the Maxwell–Ampere relation**

Consider the second Maxwell–Faraday formulas from Chapter 1. This relation can be written as:

\[
\text{rot} \vec{H} = j \omega \varepsilon \vec{E} = \omega \varepsilon'' \vec{E} + j \omega \varepsilon' \vec{E}
\]
This relation is of the form:

\[ \text{rot} \vec{H} = \vec{J}_D + j \omega \varepsilon' \vec{E} \]

where \( \vec{J}_D \) is a conduction current, and from Ohm’s law its value is of the form:

\[ \vec{J}_D = \sigma_D \vec{E} \]

Then, we can say that there is a fictive conduction current of dielectric losses:

\[ \sigma_D = \omega \varepsilon'' \]

The losses in the dielectric are:

\[ \bar{P}_D = \frac{1}{2 \sigma_D} \iint_S \vec{J}_D \vec{J}_D^* dS \]

But the overall losses are:

\[ \bar{P} = \frac{1}{2} \Re \left( \frac{1}{Z} \iint_S \vec{E}\vec{E}^* dS \right) = \frac{1}{2Z} \iint_S \vec{E}\vec{E}^* dS \]

And:

\[ \alpha_D = \frac{1}{2} \frac{\bar{P}_D}{\bar{P}} = \frac{1}{2} Z \omega \varepsilon'' \]

That is:

\[ \alpha_D = \frac{\omega}{2} \varepsilon'' \sqrt{\frac{\mu}{\varepsilon'}} = \frac{\omega}{2} \sqrt{\frac{\mu}{\varepsilon'}} \varepsilon'' = \frac{\beta}{2} \tan \varphi = \frac{\beta}{2} \varphi \]
Then:

\[ \alpha_d = \frac{\pi \varepsilon''}{\lambda} \approx \frac{\pi}{\lambda} \phi \]

Then we obtain the same result as in the previous chapter. This is a general method used when we have small dielectric losses.

### 2.4. Metallic losses

When the metallic losses are small, we can compute them through perturbations.

![Figure 2.1. Frontier between two mediums](image)

In the case without losses, the electric field \( \vec{E} \) is perpendicular to the surface. Then, \( \vec{E}_{tg} = \vec{0} \) and the surface impedance \( Z_s \) is also zero because

\[
Z_s = \frac{E_{tg}}{H_{tg}} = \frac{E_{tg}}{H_T} = 0
\]
But in this case, we have a surface impedance:

\begin{align*}
Z_s &= (1 + j)R_s \\
R_s &= \frac{1}{\sigma \delta} \\
\delta &= \frac{2}{\sqrt{\mu \sigma \omega}}
\end{align*}

where \( \delta \) is the skin thickness of the metal. Then, there is certainly a tangential component \( \vec{E}_{tg} \) of the electric field \( \vec{E} \).

\[ |\vec{E}_{tg}| = Z_s |\vec{H}_T| \]

\[ \vec{E}_t \]

\[ \vec{E}_{ig} \]

Figure 2.2. Tangential component \( \vec{E}_{tg} \) of the electric field \( \vec{E} \)

The waves are not quite TEM. It shows that this component is

\[ \vec{E}_{tg} = \frac{\vec{J}_s}{\sigma} \]

where \( \vec{J}_s \) is a superficial density of current. The metallic losses are computed by the formula:

\[ \alpha_m = -\frac{1}{2} \frac{\partial \tilde{P}}{\partial z} \tilde{P}(z) \]
where \( -\frac{\partial P}{\partial z} \) are the losses by Joule’s effect \( \bar{P}_j \) in the conductor:

\[
\frac{\partial \bar{P}}{\partial z} = \bar{P}_j = \frac{1}{2} R_s \int_c \bar{J}_s \bar{J}_s^* \, dl = \frac{1}{2} R_s \int_c \bar{H} \bar{H}^* \, dl
\]

because

\[
\bar{H} = \bar{J}_s \wedge \bar{u}
\]

But we recall

\[
\bar{P}(z) = \Re \left\{ \frac{1}{2} \int \bar{E} \wedge \bar{H}^* \right\} \bar{u} \, dS
\]

And with the expression of \( \bar{E} \), we can find the expressions of \( \bar{P}(z) \):

\[
\bar{E} = -Z \left( \bar{u} \wedge \bar{H} \right)
\]

\[
\bar{P}(z) = \frac{1}{2} Z \int_s \bar{H} \bar{H}^* \, dS
\]

Then:

\[
\alpha_M = \frac{1}{2} \frac{R_s}{Z} \int_c \bar{H} \bar{H}^* \, dl
\]

With the wave impedance \( Z \), the skin effect \( \delta \) is:

\[
Z = \sqrt{\frac{\mu}{\varepsilon}}, \quad R_s = \frac{1}{\sigma \delta}, \quad \delta = \sqrt{\frac{2}{\mu \sigma \omega}}
\]
2.5. General case: dielectric losses and metallic losses

We still use the following perturbation method:

\[
\alpha = -\frac{1}{2} \frac{\partial \bar{P}}{\partial z} / \bar{P}(z)
\]

where the total average losses is:

\[
-\frac{\partial \bar{P}}{\partial z} = \bar{P}_M + \bar{P}_D
\]

These dielectric and metallic losses are additive and we have the overall attenuation:

\[
\alpha = \alpha_D + \alpha_M
\]

We recall the expressions of the dielectric and metallic powers in the line theory.

\[
\begin{aligned}
\bar{P}_D &= \frac{1}{2} GVV^* \\
\bar{P}_M &= \frac{1}{2} RII^*
\end{aligned}
\]

Figure 2.3. Equivalent circuit of the line over a very small $dx$ length.
\[2\alpha \bar{P} = \bar{P}_D + \bar{P}_M = \frac{1}{2} G V V^* + \frac{1}{2} R I I^*\]

But the power is:

\[\bar{P} = \frac{1}{2} V I^* \text{ with } Z_c = \frac{V}{I}\]

Then:

\[2\alpha \bar{P} = \frac{1}{2} G Z_c V I^* + \frac{1}{2} R V I^* = \frac{1}{2} \left( G Z_c + \frac{R}{Z_c} \right) V I^* = \left( G Z_c + \frac{R}{Z_c} \right) \bar{P}\]

and

\[\alpha = \frac{1}{2} \left( G Z_c + \frac{R}{Z_c} \right)\]

But we have shown in this chapter that the dielectric power is:

\[\bar{P}_D = \frac{\omega}{2} \varepsilon^* \int_S EE^* dS = \frac{\omega}{2} \varepsilon^* \tan \varphi \int_S EE^* dS = \frac{1}{2} G V V^*\]

But the density of electric power is

\[\bar{W}_E = \frac{1}{4} C V V^* = \frac{\varepsilon^*}{4} \int_S EE^* dS\]

Then, by combining these two equations, we have:

\[G = \omega C \tan \varphi\]
In the same manner, we can show that the power $R$ is of another form:

$$R = \varepsilon R_s \frac{\int \tilde{H}\tilde{H}^* dl}{C \int \tilde{H}\tilde{H}^* dS}$$

where $\frac{\int \tilde{H}\tilde{H}^* dl}{\int \tilde{H}\tilde{H}^* dS}$ is said to be the factor of geometry because this factor only depends on the form (the geometry) of the conductors of the line.

In conclusion, the dielectric power $\overline{P}_D$ and the conductance $G$ depend directly on the frequency $\omega$. But the metallic power $\overline{P}_M$ and the resistance depend on $\sqrt{\omega}$ because $R_s = \frac{1}{\sigma \delta}$ and $\delta = \sqrt{\frac{2}{\mu \sigma \omega}}$.

2.6. Problems

2.6.1. The transmission line with low losses

A transmission line is considered as a cascade of elementary ladders of length $l$. The system is an infinite ladder $Z_D, Z_S$.

1) Give the characteristic impedance of the line (with $l$ small):

i) in the lossless case;

ii) in the case of small loss: $R << Ls, G << Cs$. 
2) For the elementary cell load by $Z_c$, give the value of $v_e/v_s$.

We write $v_e/v_s = e^{\gamma i}$. Also give the attenuation constant $\alpha$, the propagation constant $\beta$, and the speed of the phase $v = \omega/\beta$.

- the attenuation constant $\alpha$;
- the propagation constant $\beta$;
- the speed of the phase $v = \omega/\beta$.

**Solutions.**

1) Characteristic impedance.

Consider an elementary cell of length (Figure 2.5), where:

$$Z_s = (R + sL)l$$

$$Z_D = \frac{1}{(G + sC)l}$$
If the fourpoles is closed on a load which is the characteristic impedance $Z_c$, then we see $Z_c$ at the input:

$$Z_E = Z_C = Z_S + \frac{Z_D Z_C}{Z_D + Z_C}$$

$$Z_C^2 - Z_S Z_C - Z_S Z_D = 0$$

On solving this equation, we find that if $Z_S \approx l$ and $Z_D \approx \frac{1}{l}$, then the product $Z_S Z_D$ is independent of $l$. If $l$ is small, we can neglect the quantity $Z_S Z_C$ which is proportional to $l$.

$$Z_C = \sqrt{Z_S Z_D} = \frac{\sqrt{R + sL}}{\sqrt{G + sC}}$$

When the line is lossless, i.e. $R = 0$ and $G = 0$, we have:

$$Z_C = \frac{L}{\sqrt{C}} = R_C$$

An infinite ladder is a pure resistance.
In the case of losses, we write the characteristic impedance as:

\[
Z_c = \sqrt{\frac{L}{C}} \left(1 + \frac{R}{sL} \left(1 + \frac{G}{sC}\right)\right)^{\frac{1}{2}}
\]

And in the case of small losses:

\[
R \ll sL, \quad G \ll sC
\]

The characteristic impedance is of the form:

\[
\left(\frac{1 + \epsilon_1}{1 + \epsilon_2}\right)^{\frac{1}{2}} \approx \left(1 + \frac{\epsilon_1}{2}\right)\left(1 - \frac{\epsilon_2}{2}\right) \approx 1 + \frac{1}{2}(\epsilon_1 - \epsilon_2)
\]

and

\[
Z_c \approx \sqrt{\frac{L}{C}} \left[1 + \frac{1}{2} \left(\frac{R}{sL} - \frac{G}{sC}\right)\right]
\]

In the sinusoidal case, \(s = j\omega\)

\[
Z_c = \frac{R + jL\omega}{\sqrt{G + jC\omega}} \text{ with losses}
\]

\[
Z_c \approx \sqrt{\frac{L}{C}} \left[1 + \frac{j}{2\omega} \left(\frac{G}{C} - \frac{R}{L}\right)\right] \text{ with small losses}
\]

where \(Z_c\) is a characteristic of the line at a given frequency.
2) Phase and propagation constant.

From Figure 2.5, we have:

\[
\begin{align*}
V_E &= (Z_S + Z_D \parallel Z_C) I_E \\
V_E &= Z_S I_E + V_S
\end{align*}
\]

From these two equations, we can define the relationship:

\[
\frac{V_E}{V_S} = 1 + \frac{Z_S}{Z_D} + \frac{Z_S}{Z_C}
\]

But:

\[
\begin{align*}
\frac{Z_S}{Z_D} &= (R + sL)(G + sC) I^2 \\
\frac{Z_S}{Z_C} &= \sqrt{(R + sL)(G + sC)} I
\end{align*}
\]

And if \( l \) is very small, then we can neglect the first term and

\[
\frac{V_E}{V_S} = 1 + \frac{Z_S}{Z_C}
\]

\[
\frac{V_E}{V_S} = 1 + l \sqrt{(R + sL)(G + sC)}
\]

that is

\[
\frac{V_E}{V_S} = 1 + s l \sqrt{LC} \left(1 + \frac{R}{sL}\right)^{\frac{1}{2}} \left(1 + \frac{G}{sC}\right)^{\frac{1}{2}}
\]
The propagation phenomenon is such that:

\[
\frac{V_e}{V_s} = e^{\gamma l} \quad \text{with} \quad \gamma_i = \alpha l + \beta l
\]

\[
e^{\gamma l} = 1 + s l \sqrt{LC} \left( 1 + \frac{R}{sL} \right)^{\frac{1}{2}} \left( 1 + \frac{G}{sC} \right)^{\frac{1}{2}}
\]

If \( l \) is small, then the second part of the expression is somewhat smaller than 1.

\[
\gamma_i \approx \log \left[ 1 + s l \sqrt{LC} \left( 1 + \frac{R}{sL} \right)^{\frac{1}{2}} \left( 1 + \frac{G}{sC} \right)^{\frac{1}{2}} \right]
\]

\[
\gamma_i \approx s l \sqrt{LC} \left( 1 + \frac{R}{sL} \right)^{\frac{1}{2}} \left( 1 + \frac{G}{sC} \right)^{\frac{1}{2}}
\]

Now we define the propagation constant (unit of length):

\[
\gamma = \alpha + j \beta = \frac{\gamma_i}{l}
\]

\[
\gamma = s \sqrt{LC} \left[ 1 + \frac{R}{sL} + \frac{G}{sC} + \frac{RG}{s^2 LC} \right]^{\frac{1}{2}}
\]

Losses are small:

\[
\frac{R}{sL} \text{ and } \frac{G}{sC} << 1
\]

\[
(1 + u)^{\frac{1}{2}} \approx 1 + \frac{u}{2} - \frac{u^2}{8}
\]

with \( u = \frac{R}{sL} + \frac{G}{sC} + \frac{RG}{s^2 LC} \)
By limiting ourselves to the terms in \( \frac{1}{s} \), we obtain:

\[
(1+u)^{\frac{1}{2}} \approx 1 + \frac{1}{2s} \left( \frac{R}{L} + \frac{G}{C} \right) + \frac{1}{2s^2} \frac{RG}{LC} - \frac{1}{8s^3} \left( \frac{R}{L} + \frac{G}{C} \right)^2
\]

\[
(1+u)^{\frac{1}{2}} \approx 1 - \frac{1}{8s^2} \left( \frac{R}{L} - \frac{G}{C} \right)^2 + \frac{1}{2s} \left( \frac{R}{L} + \frac{G}{C} \right)
\]

and by ordering, we obtain:

\[
\gamma = \frac{1}{2} \sqrt{LC} \left( \frac{R}{L} + \frac{G}{C} \right) + s \sqrt{LC} \left[ 1 - \frac{1}{8s^2} \left( \frac{R}{L} - \frac{G}{C} \right)^2 \right]
\]

and with \( R_c = \sqrt{\frac{L}{C}}, G_c = \sqrt{\frac{C}{L}} \)

We obtain

\[
\gamma = \frac{1}{2} \left( \frac{R}{R_c} + \frac{G}{G_c} \right) + s \sqrt{LC} \left[ 1 - \frac{1}{8s^2} \left( \frac{R}{L} - \frac{G}{C} \right)^2 \right]
\]

In sinusoidal state \( s = j\omega, \gamma = \alpha + j\beta \), we obtain the loss term \( \alpha \) and the term of propagation \( \beta \):

\[
\alpha = \frac{1}{2} \left( \frac{R}{R_c} + \frac{G}{G_c} \right)
\]

\[
\beta = \omega \sqrt{LC} \left[ 1 + \frac{1}{8\omega^2} \left( \frac{R}{L} - \frac{G}{C} \right)^2 \right]
\]

Considering only the first order, we obtain:

\[
\alpha = \frac{1}{2} \left( \frac{R}{R_c} + \frac{G}{G_c} \right)
\]

\[
\beta = \omega \sqrt{LC} = \frac{\omega}{v} \quad \text{where} \quad v = \frac{1}{\sqrt{LC}}
\]
where \( v \) is the speed of the phase \( v = \omega / \beta \).

We can verify that small losses (first order) do not modify the speed of the phase. In the second order, this speed is a function of the frequency:

\[
v(\omega) = \frac{1}{\sqrt{LC} \left[ 1 + \frac{1}{8\omega^2} \left( \frac{R}{L} - \frac{G}{C} \right)^2 \right]}
\]

In these conditions, the propagation of impulsions will be dispersive. It means that all the Fourier components of an impulsion will propagate at different speeds. This induces a deformation of the impulsions (spectrum display in Figure 2.6).

![Input and Output](image-url)

**Figure 2.6. Spectrum display of an impulsion**

### 2.6.2. Coaxial cable with losses

Let us consider a coaxial cable with a geometry shown in Figure 2.7. The space \( a \leq r \leq b \) is full of dielectric \( \epsilon = \epsilon_0 \epsilon_r \) (\( \epsilon_0 \) is the empty constant, while \( \epsilon_r \) is the relative constant).
1) From Gauss and Ampere theorems, give the capacity \( C \) and the inductance \( L \) by unit of length.

2) Give the resistance \( R \) if \( \sigma \) is the skin effect.

3) What is the conductance \( G \) in a function of \( \varphi \)?

4) From the preceding results, give the characteristic impedance \( Z_c \) and the propagation constant \( \gamma \) in the cases:

   i) where the losses are negligible,

   ii) where the losses are slight \((L\omega \gg R, C\omega \gg G)\). We will show that \( \gamma \) is of the form:

   \[
   \gamma = \alpha_1 + \alpha_2 + j\beta
   \]

5) Give the expression of \( \alpha_1 \) and \( \alpha_2 \) as a function of \( \varepsilon_r \), of the frequency \( f \) and of the dimensions \( a \) and \( b \).

   If \( \tan \varphi \) is independent of the frequency, at what frequency are the dielectric losses equal to the metallic losses?

   We give \( \tan \varphi = 510^{-4}, \sigma = 5.810^7 \text{\Omega}^{-1}\text{m}^{-1}, a = 1.29 \text{mm}, b = 4.7 \text{mm} \).

**Solutions.**

1) Length capacity and length inductance. Gauss and Ampere theorems.

   For \( C \) and \( L \), we have to solve a static problem.
Gauss theorem: The flux of the electric vector \( \vec{E} \) through a closed surface is equal to \( \frac{q}{\varepsilon} \), where \( q \) are the inside charges to this surface and \( \varepsilon \) is the dielectric constant of the medium.

\[
\int \int_{\Sigma} \vec{E} \cdot d\vec{S} = \frac{q}{\varepsilon}
\]

In Figure 2.8:

\( d\vec{S} = dS \hat{n} \)

\( \vec{E} \) is a radial vector

\( \vec{E} \cdot d\vec{S} = 0 \) on the extremities

\( \vec{E} \cdot \hat{n}dS = Eds \) on the surface (\( \Sigma \))

\( E \) is a constant vector \( E = E(r) \) on the surface (\( \Sigma \)):

\[
E(r) \cdot 2\pi r = \frac{q}{\varepsilon}
\]
then \( E(r) = \frac{q}{2\pi r \varepsilon} \)

and the potential between the two metallic conductors \( a \) and \( b \) is

\[
V = |V_a - V_b| = \int_a^b E(r) \, dr = \frac{q}{2\pi \varepsilon} \int_a^b \frac{dr}{r} = \frac{q}{2\pi \varepsilon} \log \frac{b}{a}
\]

where

\[
C = \frac{q}{V} = \frac{2\pi \varepsilon}{\log \frac{b}{a}} \, \text{F/m}
\]

but

\[
\varepsilon = \varepsilon_0 \varepsilon_r
\]

\[
\varepsilon_0 = \frac{10^{-9}}{36\pi}
\]

\[
C = \frac{24.1 \varepsilon_r}{\log_{10} \frac{b}{a}} \, \text{pF/m}
\]

Ampere’s theorem: the circulation of \( \vec{H} \) on a closed outline is equal to the summary of the currents which go through the surface made by this outline.

\[
\oint_{(C)} \vec{H} \, d\vec{l} = I
\]
We consider the closed outline \((C)\) as:

\[
\oint \vec{H} \cdot d\vec{l} = 2\pi r \cdot H(r) = I
\]

\[
H(r) = \frac{I}{2\pi r} \quad \text{and} \quad B(r) = \frac{\mu I}{2\pi r}
\]

where \(\mu\) is the permeability. Now the flux \(\phi\) of the vector \(\vec{B}\) through the surface between the conductors is:

\[
\phi = LI = \iint_{S} \vec{B} \cdot d\vec{S}
\]

Considering that the element of surface \(dS\) comprises an elementary flux (on a unit length \(l=1\)) between the two conductors:
\[ d\phi = B(r)dr.l = \frac{\mu I}{2\pi r}dr.l \]

\[ \phi = \frac{\mu I}{2\pi} \int_{a}^{b} \frac{dr}{r} = \frac{\mu I}{2\pi} l \log \frac{b}{a} \]

\[ L = \frac{\phi}{ll} = \frac{\mu}{2\pi} \log \frac{b}{a} \text{ Henry} / m \]

Another method is to consider the magnetic energy and the element of volume \( dv = 2\pi rldr \):

\[ W = \frac{1}{2} L(l)I^2 = \frac{1}{2} \iiint_v \mu H^2 dv \]

\[ L = \frac{1}{lI^2} \iiint_v \mu H^2 dv = \frac{1}{lI^2} \iiint_v \mu \frac{l^2}{4\pi r^2} 2\pi r dr \]

that is:

\[ L = \frac{\mu}{2\pi l} \int_{a}^{b} \frac{dr}{r} = \frac{\mu}{2\pi} \log \frac{b}{a} \]

But \( \mu_0 = 4\pi 10^{-7} \) and \( L = 2.10^{-7} \log \frac{b}{a} \) Henry / m or

\[ L = 0.46 \log_{10} \frac{b}{a} \mu H / m \]

2) In this case, we consider that the current circulates in a shell of \( \delta \) thickness (Figure 2.10).
But we have two conductors of conductivity $\sigma = 1/\rho$ (\(\rho\) is the resistivity). The thickness of the skin effect is $\delta = \sqrt{\frac{2}{\mu \tau \omega}}$.

Then, for the exterior conductor (b):

$$R_E = \rho \frac{l}{S_E} = \frac{l}{\sigma S_E} \text{ with } S_E = 2\pi b \delta \text{ and } l = 1$$

$$R_E = \frac{1}{2\pi b \sigma \delta}$$

And for the interior conductor (a), we only have to change (b) to (a):

$$R_I = \frac{1}{2\pi a \sigma \delta}$$

The total resistance (per unit length) is

$$R = R_E + R_I = \frac{1}{2\pi \sigma \delta \left[ \frac{1}{a} + \frac{1}{b} \right]} \text{ } \Omega/m$$
Application: in the case of copper

\[ \sigma = 5.810^7 \ (\Omega.m)^{-1} \]
\[ \mu = \mu_0 \]
\[ \delta = \frac{6.610^{-2}}{\sqrt{f}} \]

and as an example:

\[ \begin{align*}
  f &= 1 MHz \quad \delta = 66 \mu m \\
  f &= 1 GHz \quad \delta = 2.1 \mu m 
\end{align*} \]

The thickness of the skin effect is greater at low frequencies.

The resistance due to the skin effect is:

\[ R = 4.10^{-8} \left( \frac{1}{a} + \frac{1}{b} \right) \sqrt{f} \]

To make a comparison, we take:

\[ b = 4a \text{ and } a = 5.10^{-3} m \]

\[ R = 10^{-5} \sqrt{f} \]

and we give examples:

\[ \begin{align*}
  f &= 1 MHz = 10^6 H \quad \sqrt{f} = 10^3 \quad R = 10^{-2} \Omega / m \\
  f &= 10 GHz = 10^{10} H \quad \sqrt{f} = 10^5 \quad R = 1 \Omega / m 
\end{align*} \]

3) Conductance per unit length \( G \).
A complex permittivity is a characteristic of the losses:

$$\tilde{\varepsilon} = \varepsilon(1 - jtg\varphi) = \varepsilon_0\varepsilon_r(1 - jtg\varphi)$$

And we have a conductance:

$$G = \omega C \text{tg}\varphi \quad \text{with} \quad C = \frac{2\pi \varepsilon}{\log \frac{b}{a}}$$

Then:

$$G = \frac{2\pi \varepsilon_0\varepsilon_r}{\log \frac{b}{a}} \omega \text{tg}\varphi \quad (\Omega m)^{-1}$$

$$G = \frac{151\varepsilon_r f \text{tg}\varphi}{\log \frac{b}{a}} \quad p(\Omega m)^{-1}$$

and the conductance increases with the frequency.

4) When losses are negligible, the characteristic impedance and the propagation speed are

$$Z_c = R_c = \sqrt{\frac{L}{C}} = \frac{1}{2\pi} \sqrt{\frac{\mu}{\varepsilon}} \log \frac{b}{a}$$

$$\gamma = j\omega \sqrt{LC} = j\omega \sqrt{\varepsilon_0\varepsilon_r\mu}$$

From the preceding exercise, we have:

$$\gamma = \alpha + j\beta$$
where:

\[
\alpha = \frac{R}{2R_c} + \frac{G}{2G_c} = \alpha_1 + \alpha_2
\]

\[\begin{align*}
\alpha_1 &= \frac{R}{2R_c} = \frac{R}{2} \sqrt{\frac{C}{L}} \quad \text{metallic losses} \\
\alpha_2 &= \frac{G}{2G_c} = \frac{G}{2} \sqrt{\frac{L}{C}} \quad \text{dielectric losses}
\end{align*}\]

5) Expressions of \(\alpha_1\) and \(\alpha_2\).

\[
\alpha_1 = \frac{1}{2\log \frac{b}{a}} \sqrt{\frac{\pi \varepsilon_0 \varepsilon_r}{\sigma}} \left[ \frac{1}{a} + \frac{1}{b} \right] \sqrt{f}
\]

\[
\alpha_2 = \pi \sqrt{\frac{\mu_0 \varepsilon_0 \varepsilon_r}{\sigma}} \tan \phi \, f
\]

The dielectric and metallic losses are equal when

\[
\sqrt{f} = \frac{1}{\sqrt{\pi \mu_0 \sigma}} \cdot \frac{1}{\tan \phi} \cdot \frac{1}{2\log \frac{b}{a}} \left[ \frac{1}{a} + \frac{1}{b} \right]
\]

A numerical application gives:

\[
\sqrt{f} = 5.0510^4
\]

that is

\[
f = 2550 \text{ MHz}
\]

\[
f = 2.55 \text{ GHz}
\]
2.7. Bibliography


3

Determination of the Characteristics of TEM Lines

3.1. Introduction

In this chapter, we present two methods for the determination of the characteristics of TEM lines. These methods are:

– conform transformations;
– finite differences.

3.2. Conform transformations

3.2.1. Determination of the capacity

The capacity $C$ allows us to find the characteristic impedance $Z_c$ and the self-inductance $L$ using the formulas.

$$
\begin{align*}
Z_c &= \frac{\varepsilon Z}{C} \\
L &= \frac{\mu \varepsilon}{C}
\end{align*}
$$
Knowing $C$, we can also characterize the losses:

\[
\begin{align*}
G &= \omega C \text{tg} \varphi \\
R &= \frac{\varepsilon R_s}{C} \int \frac{\vec{H} \vec{H}^* \cdot dl}{S} \\
\end{align*}
\]

where \( \frac{\int \vec{H} \vec{H}^* \cdot dl}{\iint S} \) is the factor of geometry and depends only on the form of the conductors of the line.

Then, we have to solve:

\[
\begin{align*}
\Delta V(x,y) &= 0 \\
V &= C^x \text{ on the conductors}
\end{align*}
\]

### 3.2.2. Transformation in the complex plane

Let us consider two complex planes \((x, jy)\) and \((u, jv)\), and a transformation \(W(z)\) that allows movement from one plane to another (Figure 3.1).

![Figure 3.1. Conform transformation](image)
If the function $W(z)$ is analytic, then it admits a unique derivative:

\[
\begin{align*}
\frac{\partial W}{\partial x} &= \frac{\partial u}{\partial x} + j \frac{\partial v}{\partial x} \\
\frac{\partial W}{\partial y} &= -j \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}
\end{align*}
\]

or

\[
\begin{align*}
\frac{\partial W}{\partial x} &= \frac{\partial u}{\partial x} + j \frac{\partial v}{\partial x} \\
\frac{\partial W}{\partial y} &= -j \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}
\end{align*}
\]

This leads to the well-known Cauchy–Riemann conditions.

\[
\begin{align*}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}
\end{align*}
\]

Cauchy–Riemann

Starting from the first relation and using the second, we get:

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2}
\]

We can do the same with the $y$ variable and this means that:

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0 \\
\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= 0
\end{align*}
\]

\[
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\]

The two functions $u$ and $v$ are said to be conjugate harmonic functions. This means that when you know the first function, you also know the second function (and vice versa).
The problem is finding the transformation that simplifies the problem of the determination of the capacity:

\[ W = u(x, y) + jv(x, y) \]

First we have to know some properties of this conform transformation.

### 3.2.3. Orthogonality

![Figure 3.2. The two families of curves](image)

Let us compute the following quantities:

\[
\begin{align*}
\text{grad} u &= \nabla u = \bar{a} \frac{\partial u}{\partial x} + \bar{b} \frac{\partial u}{\partial y} \\
\text{grad} v &= \nabla v = \bar{a} \frac{\partial v}{\partial x} + \bar{b} \frac{\partial v}{\partial y} = -\bar{a} \frac{\partial u}{\partial y} + \bar{b} \frac{\partial u}{\partial x}
\end{align*}
\]

from Cauchy – Rieman

and:

\[
\nabla u \cdot \nabla v = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x} = 0
\]
\[ \nabla u \cdot \nabla v = 0 \]

Then, the two families of curves \( u = C^u \) and \( v = C^v \) are orthogonal.

### 3.2.4. Position of \( \nabla u \) versus \( \nabla v \)

We compute the vector \( \vec{c} \wedge \nabla u \) from Figure 3.3.

**Figure 3.3. Unity vectors**

\[
\vec{c} \wedge \nabla u = \frac{\partial u}{\partial x} (\vec{c} \wedge \vec{a}) + \frac{\partial u}{\partial y} (\vec{c} \wedge \vec{b})
\]

But

\[
\vec{c} \wedge \vec{a} = \vec{b} \text{ and } \vec{c} \wedge \vec{b} = -\vec{a}
\]

\[
\vec{c} \wedge \nabla u = -\frac{\partial u}{\partial y} \vec{a} + \frac{\partial u}{\partial x} \vec{b}
\]

From Cauchy–Riemann, we get

\[
\vec{c} \wedge \nabla u = \frac{\partial v}{\partial x} \vec{a} + \frac{\partial v}{\partial y} \vec{b} = \nabla v
\]
\( \vec{c} \wedge \nabla u = \nabla v \)

It verifies that \( \nabla u \) and \( \nabla v \) are orthogonal. Then, starting from \( \nabla u \) and turning from \( \pi/2 \), we arrive at \( \nabla v \).

### 3.2.5. Recapitulation

In Figure 3.4 we summarize the properties of the functions \( u(x, y) \) and \( v(x, y) \). The last property is given without demonstration.

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]

\( \Delta u = \Delta v = 0; \quad \nabla u \cdot \nabla v = 0 \)

\( \vec{c} \wedge \nabla u = \nabla v; \quad |\nabla u|, |\nabla v| = \frac{1}{h} \)

**Figure 3.4. Properties of the transformation function \( W(x, y) \)**

### 3.2.6. Example of computation

We choose in the plane \( (u, v) \) a capacity which is easy to calculate. The problem is to find the frequency
transformation $W(z)$. For example, in the case of the coaxial cable, the transformation is

$$W = \log z = \log (\rho e^{j\theta}) = \log \rho + j\theta$$

![Figure 3.6. Transformation for the coaxial cable](image)

And the capacity is

$$C = \frac{\varepsilon v_{ll} - v_{ll}}{u_{ll} - u_{ll}} = \frac{2\pi \varepsilon}{\log b - \log a}$$

$$C = \frac{2\pi \varepsilon}{\log \frac{b}{a}} \text{ per unit length.}$$

### 3.3. Finite differences method

Finite differences method can be used for lines with complicated geometry (Figure 3.7), but it takes a lot of computation.

We have to solve

$$\Delta V = 0$$

with

$$V = e^{j\omega} \text{ on the conductors}$$

We square the element (Figure 3.7).
We compute the voltages at finite elements, for example:

\[
\begin{align*}
V(x+h, y) &= V(x, y) + h \frac{\partial V}{\partial x} + \frac{h^2}{2!} \frac{\partial^2 V}{\partial x^2} + \ldots \\
V(x-h, y) &= V(x, y) - h \frac{\partial V}{\partial x} + \frac{h^2}{2!} \frac{\partial^2 V}{\partial x^2} + \ldots
\end{align*}
\]

The addition of these two quantities gives us the second derivative versus \( x \), we do the same versus \( y \).

\[
\begin{align*}
\frac{\partial^2 V}{\partial x^2} &= \frac{1}{h^2} \left\{ V(x+h, y) + V(x-h, y) - 2V(x, y) \right\} \\
\frac{\partial^2 V}{\partial y^2} &= \frac{1}{h^2} \left\{ V(x, y+h) + V(x, y-h) - 2V(x, y) \right\}
\end{align*}
\]

However, we must have

\[
\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0
\]

\[
\Delta V = \frac{1}{h^2} \left[ V(x+h, y) + V(x-h, y) + V(x, y+h) + V(x, y+h) \right]
\]

\[
-\frac{4}{h^2} V(x, y) = 0
\]

at all points of the line.
The \((x,y)\) potential is the average of the four neighbor potentials \(A, B, C, D\) (Figure 3.8).

\[
V(x,y) = \frac{1}{4} \left[ V(x + h, y) + V(x - h, y) + V(x, y + h) + V(x, y + h) \right]
\]

**Figure 3.8. Average potential \((x, y)\)**

### 3.3.1. Example of the finite differences method

Suppose we have to determine the potentials of the very simple square line shown in Figure 3.9. The potentials of this line are, respectively, 100, 20, 60 and 80 volts.

**Figure 3.9. A simple rectangular line**
The new values of the potentials are given in the function of the old values. Then, we create an iteration, i.e.

\[
\begin{align*}
V_A &= \frac{1}{4} [V_B + V_C + 100 + 80] \\
V_B &= \frac{1}{4} [V_A + V_D + 100 + 20] \\
V_C &= \frac{1}{4} [V_A + V_D + 80 + 60] \\
V_D &= \frac{1}{4} [V_B + V_C + 60 + 20]
\end{align*}
\]

To begin the iteration, we suppose that \( V_B = V_C = V_D = 0 \) for the first iteration.

\[
\begin{align*}
V_A &= \frac{180}{4} = 45 \text{ volts} \\
V_B &= \frac{1}{4} [45 + 120] = \frac{165}{4} = 41.3 \text{ volts} \\
V_C &= \frac{1}{4} [45 + 140] = \frac{185}{4} = 46.3 \text{ volts} \\
V_D &= \frac{1}{4} [41.3 + 46.3 + 80] = 41.9 \text{ volts}
\end{align*}
\]

Then, we obtain the first three iterations.

<table>
<thead>
<tr>
<th>iteration1</th>
<th>iteration2</th>
<th>iteration3</th>
<th>true value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_A )</td>
<td>45</td>
<td>66.9</td>
<td>74.9</td>
</tr>
<tr>
<td>( V_B )</td>
<td>41.3</td>
<td>57.2</td>
<td>61.2</td>
</tr>
<tr>
<td>( V_C )</td>
<td>46.3</td>
<td>62.2</td>
<td>66.2</td>
</tr>
<tr>
<td>( V_D )</td>
<td>41.9</td>
<td>49.9</td>
<td>51.9</td>
</tr>
</tbody>
</table>
With only three iterations, we can calculate the true values.

3.4. Problems

3.4.1. Conform transformations

We consider the transformations:

\[ W = k z^2 \text{ and } W = k \sqrt{z} \]

1) We write \( W = u + jv \) and \( z = x + jy \). Show that the two functions are analytic.

2) Give the curves described by a point \( M(x, y) \) in the complex plane \( (x, y) \) when \( u = e^{i\theta} \) and when \( v = e^{i\theta} \).

Solutions.–

1) \( W = k z^2 \) is analytic.

\[ u + jv = k (x + jy)^2 = k \left( x^2 - y^2 + 2jxy \right) \]

Then

\[
\begin{align*}
  u &= k \left( x^2 - y^2 \right) \\
  v &= 2kxy \\
  \frac{\partial u}{\partial x} &= 2kx = \frac{\partial v}{\partial y} \\
  \frac{\partial u}{\partial y} &= -2ky = -\frac{\partial v}{\partial x}
\end{align*}
\]

It means that \( W = k z^2 \) is analytic.
2) The curves $u = c^t e$ with $x^2 - y^2 = \frac{u}{k}$ are a family of hyperbolas asymptotic to the bisector of the angle $x\hat{0} y$.

The curves $v = c^t e$ with $xy = \frac{v}{2k}$ are also a family of hyperbolas perpendicular to the first family (Figure 3.10).

![Perpendicular families of hyperbolas](image)

**Figure 3.10.** *Perpendicular families of hyperbolas*

This conform transformation permits us to determine the potential and the wave lines of an electric dihedral (Figure 3.11) or the characteristics of a hyperbolas line (Figure 3.12).

![Electric dihedral](image)

**Figure 3.11.** *Electric dihedral*
1a) \( W = k \sqrt{z} \) is analytic:

\[(u + jv)^2 = k^2(x + jy)\]

Then

\[
\begin{align*}
    u^2 - v^2 &= k^2x \\
    2uv &= k^2y
\end{align*}
\]

By comparing with the previous exercise, there is a change \((u,v) \leftrightarrow (x,y)\) and we have

\[
\begin{align*}
    \frac{\partial x}{\partial u} &= \frac{\partial y}{\partial v} \\
    \frac{\partial x}{\partial v} &= -\frac{\partial y}{\partial u}
\end{align*}
\]

It means that \( W = k \sqrt{z} \) is analytic.

1b) When \( u = e^{\alpha} \), the value of \( v \) is \( v = \frac{k^2y}{2u} \)

and

\[
u^2 - \frac{k^4y^2}{4u^2} = kx^2\]
\[ x = \frac{u^2}{k^2} - \frac{k^2 y^2}{4u^2} \] parabolas

When \( v = c^e \), we have \( x = \frac{k^2 y^2}{4v^2} - \frac{v^2}{k^2} \) perpendicular parabolas.

The curves \( u = c^e \) and \( v = c^e \) are given in Figure 3.13.

![Figure 3.13. Curves \( u = c^e \) and \( v = c^e \)](image)

We can determine, for example, the characteristics of a parabolic line (Figure 3.14).

![Figure 3.14. A parabolic line](image)
3.4.2. *Eccentric coaxial line using conform transformations*

1) We consider the eccentric coaxial line and the conform transformation:

\[ W = \text{Log} \left( \frac{z + jh}{z - jh} \right) \]

![Figure 3.15. An eccentric coaxial line](image)

**Figure 3.15. An eccentric coaxial line**

Calculate the coordinates of a point \( M(x, y) \) in the complex plane \( (x, y) \) as functions of the parameters \( u \) and \( v \).

\[
M \begin{cases} 
  x = x(u, v) \\
  y = y(u, v)
\end{cases}
\]

2) Show that \( W = W(z) \) is an analytic function.

3) Give the curve equations

\[ u = e^{ic} \quad \text{and} \quad v = e^{ic} \] in the complex plane \( (x, y) \)

4) Using the preceding results, calculate the capacity \( C \) (per unit length) and the self-inductance \( L \) (per unit length) of the eccentric coaxial line. Also calculate the characteristic
impedance $Z_c$ as a function of the geometric parameters of the line $a, b, d$.

5) What happens to $Z_c$ when the two conductors are centered? Calculate $Z_c$ when $d \ll a$ and $d \ll b$.

**Solutions.**–

1) We calculate the inverse of the transformation:

$$ W = \log\frac{z + jh}{z - jh} \iff z = jh\frac{e^w + 1}{e^w - 1} $$

or

$$ z = jh \coth \frac{W}{2} = -h \cot \frac{W}{2} $$

From the first equation, and using $W = u + jv$, we have:

$$ z = jh \frac{e^u + e^{-jv}}{e^u - e^{-jv}} $$

To get a purely real denominator, we multiply it by its conjugate complex:

$$ z = jh \frac{e^u + e^{-jv}}{e^u - e^{-jv}} \frac{e^u - e^{-jv}}{e^u - e^{-jv}} $$

After calculations, we get:

$$ z = x + jy = h \frac{\sin v + jshu}{chu - \cos v} $$

$$ \begin{cases} x = h \frac{\sin v}{chu - \cos v} \\ y = h \frac{shv}{chu - \cos v} \end{cases} $$
2) Analytic function.

\[
\begin{align*}
\frac{\partial x}{\partial u} &= -h\frac{\sin v\sin h}{(\sin - \cos v)^2} \\
\frac{\partial y}{\partial v} &= -h\frac{\sin v\cos h}{(\sin - \cos v)^2}
\end{align*}
\]

then \( \frac{\partial u}{\partial x} = \frac{\partial y}{\partial y} \)

and:

\[
\begin{align*}
\frac{\partial x}{\partial v} &= h\frac{\sin(v\sin h - \cos v)}{(\sin - \cos v)^2} = h\frac{\sin v\cos h - 1}{(\sin - \cos v)^2} \\
\frac{\partial y}{\partial u} &= h\frac{\sin(v\cos h - \sin v)}{(\sin - \cos v)^2} = h\frac{1 - \sin v\cos h}{(\sin - \cos v)^2}
\end{align*}
\]

then \( \frac{\partial u}{\partial y} = -\frac{\partial y}{\partial x} \)

Using the fact that \( \cos^2 u + \sin^2 u = 1, \ ch^2 u - sh^2 u = 1, \)

\( W = W(z) \) is then an analytic function.

3) Curves \( u = e^x \) and \( v = e^x \).

We start from the expressions of \( x \) and \( y \):

\[
\begin{align*}
\begin{cases}
x = h\frac{\sin v}{\sin - \cos v} \\
y = h\frac{\sin v}{\sin - \cos v}
\end{cases} \quad [3.1] \\
\end{align*}
\]

Taking \( u = e^x \), we eliminate the variable \( v \) using

\( \cos^2 v + \sin^2 v = 1 \)

\[
\begin{align*}
(1) \Rightarrow y\sin v &= x\sin h \\
(2) \Rightarrow y\cos v &= y\sin u - h\sin u
\end{align*}
\]
and by adding the two square equations, we have:

\[
x^2 - 2hv \coth u + y^2 + h^2 = 0
\]

\[
x^2 + (y - h \coth u)^2 = \frac{h^2}{\sinh^2 u}
\]

This is the equation of circles

of centers \( \begin{cases} x_u = 0 \\ y_u = h \coth u \end{cases} \) and of radius \( R_u = \frac{h}{\sinh u} \)

In particular, when \( u = 0 \), we have \( (R_0 = \infty, x_0 = 0, y_0 = \infty) \) the axis \( 0\overline{x} \).

And when \( u = \infty \), we have \( (R_\infty = 0, x_\infty = 0, y_\infty = h) \) the point \( h \).

![Figure 3.16. Curves \( u = c^v \)](image)

Taking \( v = e^u \), we eliminate the variable \( u \) using \( ch^2 u - sh^2 u = 1 \)

\[
\begin{align*}
(1) \Rightarrow xshu &= y \sin v \\
(2) \Rightarrow xchu &= x \cos v + h \sin v
\end{align*}
\]
Using the same method as above, we obtain:

\[ x^2 - 2hx \cot g v + y^2 - h^2 = 0 \]

\[ (x^2 - h \cot g v)^2 + y^2 = \frac{h^2}{\sin^2 v} \]

This is the equation of circles of centers \( \begin{cases} x_v = h \cot g v \\ y_v = 0 \end{cases} \) and of radius \( R_v = \frac{h}{\sin v} \)

When \( x = 0 \), we have \( y^2 = h^2 \left[ \frac{1}{\sin^2 v} - \cot g^2 v \right] = h^2 \); then \( y = \pm h \) and all the circles pass through the points \( \pm h \).

In addition, when \( v = 0 \), we have \( (R_0 = \infty, x_0 = \infty, y_0 = 0) \) the axis \( 0\bar{y} \).

When \( v = \pi/2 \), we have \( (R_{\pi/2} = h, x_{\pi/2} = 0, y_{\pi/2} = 0) \) a symmetric circle.

When \( v = \pi \), we have \( (R_\pi = \infty, x_\pi = -\infty, y_\pi = 0) \) axis \( -0\bar{y} \).

\[ \text{Figure 3.17. Curves } v = c^{\text{le}} \]
The two families of circles are orthogonal, as we can see in Figure 3.18.

4) Study of the eccentric coaxial.

Recall that if $u$ is the potential, then $\nu = -\frac{\phi}{\varepsilon}$, where $\phi$ is the flux of the displacement vector $\bar{D}$. We can construct an eccentric coaxial as shown in Figure 3.19.
We have to solve

\[
\begin{align*}
    d &= h \left[ \coth u_B - \coth u_A \right] \\
    b &= \frac{h}{shu_B} \\
    a &= \frac{h}{shu_A}
\end{align*}
\]

A solution of these equations gives:

\[
\begin{align*}
    C &= -\varepsilon \frac{\Delta v}{\Delta u} \\
    L &= -\mu \frac{\Delta u}{\Delta v} \\
    Z_C &= \sqrt{\frac{L}{C}}
\end{align*}
\]

With the variations:

\[
\begin{align*}
    v: 0 &\rightarrow 2\pi \\
    u: u_A &\rightarrow u_B
\end{align*}
\]

we obtain:

\[
\begin{align*}
    C &= \frac{2\pi \varepsilon}{u_A - u_B} \\
    L &= \mu \frac{u_A - u_B}{2\pi}
\end{align*}
\]

We have to calculate the difference \( u_A - u_B \) with the three equations that give \( a \), \( b \) and \( d \).

Compute \( \text{ch}(x - y) \) with \( x = u_A \) and \( y = u_B \).
\[
ch(x - y) = chx chy - shx shy
\]

\[
ch(x - y) = shx shy \left[ \coth x \coth y - 1 \right]
\]

But:

\[
2 \coth x \coth y = \coth^2 x + \coth^2 y - (\coth x - \coth y)^2 = \frac{1 + sh^2 x}{sh^2 x} + \frac{1 + sh^2 y}{sh^2 y} - (\coth x - \coth y)^2
\]

This means that:

\[
2 \coth x \coth y = 2 + \frac{1}{sh^2 x} + \frac{1}{sh^2 y} - (\coth x - \coth y)^2
\]

And with the two surrounded expressions:

\[
ch(x - y) = \frac{1}{2} shx shy \left[ \frac{1}{sh^2 x} + \frac{1}{sh^2 y} - (\coth x - \coth y)^2 \right]
\]

But \( x \) is \( u_A \) and \( y \) is \( u_B \) and with the expressions of \( a \), \( b \) and \( d \) we have a very simple expression:

\[
ch(u_A - u_B) = \frac{a^2 + b^2 - d^2}{2ab}
\]

and:

\[
C = \frac{2\pi\varepsilon}{\text{Argch} \left( \frac{a^2 + b^2 - d^2}{2ab} \right)}
\]

\[
L = \frac{\mu}{2\pi} \text{Argch} \left( \frac{a^2 + b^2 - d^2}{2ab} \right)
\]
The characteristic impedance is:

$$Z_c = \sqrt{\frac{L}{C}} = \frac{1}{2\pi} \sqrt{\frac{\mu}{\epsilon}} \text{Argch} \left( \frac{a^2 + b^2 - d^2}{2ab} \right)$$

and with

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi = 377\,\Omega$$

we have

$$Z_c = 60 \sqrt{\frac{\mu_r}{\epsilon_r}} \text{Argch} \left( \frac{a^2 + b^2 - d^2}{2ab} \right)$$

5) $Z_c$, when the two conductors are centered and when $d \ll a$ and $d \ll b$.

Recall that

$$\text{Argch} x = \log \left[ x + \sqrt{x^2 - 1} \right]$$

In the case where the two conductors are centered, we have:

$$d = 0 \Rightarrow x + \sqrt{x^2 - 1} = \frac{b}{a}$$

$$Z_c = 60 \sqrt{\frac{\mu_r}{\epsilon_r}} \log \frac{b}{a}$$

We recall the well-known characteristic impedance of the centered coaxial.

In the case where the deviation is small $d \ll a$ and $d \ll b$, we use
\[ x = \frac{a^2 + b^2 - d^2}{2ab} = \frac{a^2 + b^2}{2ab} \left\{ 1 - \frac{d^2}{a^2 + b^2} \right\} = \frac{a^2 + b^2}{2ab} (1 - \varepsilon) \]

And the computation of the following quantity is of the form:

\[ x + \sqrt{x^2 - 1} \approx \frac{b}{a} \left\{ 1 - \frac{d^2}{b^2 - a^2} \right\} \]

and

\[ \log[x + \sqrt{x^2 - 1}] \approx \log \left( \frac{b}{a} \right) - \frac{d^2}{b^2 - a^2} \]

Then, we have:

\[ Z_c \approx \frac{1}{2\pi} \sqrt{\frac{\mu}{\varepsilon}} \left\{ \log \left( \frac{b}{a} \right) - \frac{d^2}{b^2 - a^2} \right\} \]

In the case of a small correction, the first part of the expression corresponds to a centered coaxial, while the second part corresponds to the correction.

3.5. Bibliography


Part 2
Guides
4.1. Introduction

We start from Maxwell’s equations:

\[
\begin{align*}
\text{rot} \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\text{rot} \vec{H} &= \vec{i} + \frac{\partial \vec{D}}{\partial t} \\
\text{div} \vec{E} &= \rho \\
\text{div} \vec{H} &= 0
\end{align*}
\]

where \( \vec{i} \) is the conduction vector, \( \vec{D} \) is the displacement vector and \( \rho \) is the cubic density of charges.

If \( \sigma \) is the conductivity, from Ohm’s law we have:

\[\vec{i} = \sigma \vec{E}\]
The waves are homogeneous because they are independent of their amplitudes. They are also isotropic because they are independent of their direction.

4.2. Equations for a lossless medium

We try to find a solution of the form:

\[
\begin{align*}
\vec{E} &= \vec{E}(x, y)e^{(j\alpha - \gamma z)} \\
\vec{H} &= \vec{H}(x, y)e^{(j\alpha - \gamma z)}
\end{align*}
\]

\[\gamma = \alpha + j\beta\]

where \(\beta\) is the propagation constant and \(\alpha\) characterizes the losses.

The waves are a sum of transversal waves \((\vec{E}_T, \vec{H}_T)\) and longitudinal waves \((E_z, H_z)\). These two parts (transversal and longitudinal) depend only on the coordinates \((x, y)\).

\[
\begin{align*}
\vec{E}(x, y) &= \vec{E}_T(x, y) + E_z(x, y)\vec{u} \\
\vec{H}(x, y) &= \vec{H}_T(x, y) + H_z(x, y)\vec{u}
\end{align*}
\]

In the lossless case, \(\alpha = 0\) and \(\gamma = j\beta\). The time and space differential operators are, respectively:

\[
\begin{align*}
\frac{\partial}{\partial t} &= j\omega \\
\frac{\partial}{\partial z} &= -j\beta
\end{align*}
\]
The Maxwell’s equations without current and charges are now:

\[
\begin{align*}
\nabla \times \vec{E} &= -j \omega \mu \vec{H} \\
\nabla \times \vec{H} &= j \omega \varepsilon \vec{E} \\
\n\text{div} \vec{E} &= 0 \\
\n\text{div} \vec{H} &= 0
\end{align*}
\]

Simplifying by \( e^{j\omega t} \), we obtain:

\[
\begin{align*}
\nabla \times \left[ (\vec{E}_T + E_z \vec{u}) \ e^{-j\beta z} \right] &= -j \omega \mu \left[ (\vec{H}_T + H_z \vec{u}) \ e^{-j\beta z} \right] \quad [4.1] \\
\nabla \times \left[ (\vec{H}_T + H_z \vec{u}) \ e^{-j\beta z} \right] &= j \omega \varepsilon \left[ (\vec{E}_T + E_z \vec{u}) \ e^{-j\beta z} \right] \quad [4.2] \\
\n\text{div} \left[ (\vec{E}_T + E_z \vec{u}) \ e^{-j\beta z} \right] &= 0 \quad [4.3] \\
\n\text{div} \left[ (\vec{H}_T + H_z \vec{u}) \ e^{-j\beta z} \right] &= 0 \quad [4.4]
\end{align*}
\]
Using the following properties with \( a(z) = e^{-j\beta z} \) and \( \vec{A} = (\vec{A}_r + A_z \vec{u}) \):

\[
\text{rot}\, a(z) \vec{A} = a(z) \text{rot}\, \vec{A} + \text{grad} \, a(z) \land \vec{A}
\]

\[
\text{grad} \, a(z) = a'(z) \text{grad} z = a'(z) \vec{u}
\]

We obtain for the first relation [4.1]

\[
\text{rot}\, \vec{E}_r - \vec{u} \land \left[ j\beta \vec{E}_r + \text{grad}_r E_z \right] = - \omega \mu \left( \vec{H}_r + H_z \vec{u} \right)
\]

But \( j\beta \vec{E}_r + \text{grad}_r E_z \) is a transversal wave while \( \text{rot}\, \vec{E}_r \) is a longitudinal wave. Then we obtain two more equations [4.5] and [4.6] to be satisfied:

\[
\begin{align*}
\left\{ \begin{array}{l}
\text{rot}\, \vec{E}_r = - j\omega \mu \, H_z \vec{u} \\
\vec{u} \land \left[ j\beta \vec{E}_r + \text{grad}_r E_z \right] = j\omega \mu \, \vec{H}_r
\end{array} \right. \\
[4.5] \\
[4.6]
\end{align*}
\]

In the same manner, we have:

\[
\begin{align*}
\left\{ \begin{array}{l}
\text{rot}\, \vec{H}_r = j\omega \varepsilon \, E_z \vec{u} \\
\vec{u} \land \left[ j\beta \vec{H}_r + \text{grad}_r H_z \right] = - j\omega \varepsilon \, \vec{E}_r
\end{array} \right. \\
[4.7] \\
[4.8]
\end{align*}
\]

and the first two equations [4.3] and [4.4]:

\[
\begin{align*}
\left\{ \begin{array}{l}
\text{div}\, \vec{E}_r = j\beta \, E_z \\
\text{div}\, \vec{H}_r = j\beta \, H_z
\end{array} \right. \\
[4.9] \\
[4.10]
\end{align*}
\]

For example, calculating the double vectorial product \( \vec{u} \land [4.6] \) we have an expression of the form:

\[
\vec{A} \land (\vec{B} \land \vec{C}) = \vec{B}.(\vec{A} \land \vec{C}) - \vec{C}.(\vec{A} \land \vec{B})
\]
We do the same with [4.8] and obtain:

\[
\begin{align*}
\text{grad}_T E_z + j\beta \cdot \bar{E}_T + j\omega \mu \cdot (\bar{\mu} \wedge \bar{H}_T) &= 0 \quad [4.11] \\
\text{grad}_T H_z + j\beta \cdot \bar{H}_T - j\omega \varepsilon \cdot (\bar{\varepsilon} \wedge \bar{E}_T) &= 0 \quad [4.12]
\end{align*}
\]

Now we compute \( \text{div} [4.11] \) or \( \text{div} [4.12] \) and using the properties [4.7] and [4.9] and the divergence of a vectorial product:

\[
\text{div}(\bar{A} \wedge \bar{B}) = \bar{B} \cdot \text{rot}\bar{A} - \bar{A} \cdot \text{rot}\bar{B}
\]

\[
\begin{align*}
\Delta_T E_z + (k^2 - \beta^2) E_z &= 0 \quad [4.13] \\
\Delta_T H_z + (k^2 - \beta^2) H_z &= 0 \quad [4.14]
\end{align*}
\]

\[
k = \omega \sqrt{\mu \varepsilon}
\]

Now we calculate \( \bar{\mu} \wedge [4.11] \) and \( \bar{\mu} \wedge [4.12] \) and obtain the transversal vectors \( \bar{E}_T \) and \( \bar{H}_T \) as a function of the longitudinal components of the electric field \( E_z \) and the magnetic field \( H_z \).

\[
\begin{align*}
(k^2 - \beta^2) \bar{E}_T &= j\omega \mu \left( \bar{\mu} \wedge \text{grad}_T H_z \right) - j\beta \cdot \text{grad}_T E_z \quad [4.15] \\
(k^2 - \beta^2) \bar{H}_T &= -j\omega \varepsilon \left( \bar{\varepsilon} \wedge \text{grad}_T E_z \right) - j\beta \cdot \text{grad}_T H_z \quad [4.16]
\end{align*}
\]

Equations [4.13]–[4.15] are fundamental. But it is quite difficult and time-consuming to recover these equations. This is why we are giving equations [4.13]–[4.15] as results.
4.3. Limiting conditions

\( \vec{\tau}, \vec{u}, \vec{n} \) are the unit direct vectors on the conductor and they satisfy:

\[
\vec{n} = \vec{\tau} \wedge \vec{u}
\]

We have to verify on the perfect conductor:

\[
\begin{align*}
\vec{E}_{tg} &= \vec{0} \\
\vec{H}_{normal} &= \vec{0}
\end{align*}
\]

This means that we have on the conductor:

\[
\begin{align*}
\vec{E}.(\vec{\tau} + \vec{u}) &= 0 \\
\vec{H}.\vec{n} &= 0
\end{align*}
\]
We consider the first equation to satisfy. The electric vector \( \vec{E} \) has a transversal component and a longitudinal component:

\[
\vec{E} = \vec{E}_r + E_z \vec{u}
\]

Then, we have on the conductor:

\((\vec{E}_r + E_z \vec{u}).(\vec{r} + \vec{u}) = 0\)

which gives:

\(\vec{E}_r.\vec{r} + E_z = 0\)

We know \( \vec{E}_r \) as a function of \( E_z \), and using equation \([4.15]\) we obtain:

\[
\left(k^2 - \beta^2\right) (\vec{E}_r.\vec{r} + E_z) = j\omega\mu \left[ \vec{u} \wedge \nabla \vec{H}_z \right].\vec{r}
\]

\[-j\beta \nabla \vec{E}_z.\vec{r} + (k^2 - \beta^2)E_z = 0\]

On the walls of the guide, we have:

\[
\begin{cases}
E_z = 0 \\
\nabla \vec{E}_z = \vec{0}
\end{cases}
\]

and it is:

\(\vec{r}.(\vec{u} \wedge \nabla \vec{H}_z) = 0\)

that is

\((\vec{r} \wedge \vec{u}).\nabla \vec{H}_z = \vec{n}.\nabla \vec{H}_z = 0\)

This is the normal derivative which is zero:

\(\vec{n}.\nabla \vec{H}_z = \frac{\partial H_z}{\partial n} = 0\)
The second limiting condition is computed by using the second formula [4.16], which gives $\tilde{H}_f$ as a function of $H_z$. Then, we have to consider on the conductor:

$$\tilde{H} \cdot \hat{n} = \tilde{H}_f \cdot \hat{n} = 0$$

that is

$$\left(k^2 - \beta^2\right) \tilde{H}_f \cdot \hat{n} = -j \omega \varepsilon \left[\hat{u} \wedge \nabla \times \tilde{E}_z\right] \cdot \hat{n} - j \beta \nabla \times \tilde{H}_z \cdot \hat{n} = 0$$

$$(\hat{n} \wedge \hat{u}) \nabla \times \tilde{E}_z + \frac{\beta}{\omega \varepsilon} \nabla \times \tilde{H}_z \cdot \hat{n} = 0$$

$$-\hat{r} \cdot \nabla \times \tilde{E}_z + \frac{\beta}{\omega \varepsilon} \frac{\partial H_z}{\partial n} = 0$$

The second part is zero on the conductor; then $E_z$ has to be also zero on the conductor.

$$E_z = 0$$

Then the limiting conditions on the conductor are as follows:

$$\begin{cases} 
\hat{n} \cdot \nabla \times \tilde{H}_z = \frac{\partial H_z}{\partial n} = 0 \\
E_z = 0 
\end{cases}$$

### 4.4. Progressive and evanescent waves

In the case of progressive waves, we define the propagation speed $v_p$ with:

$$j \beta = \Gamma = j \frac{\omega}{v_p}$$
We have evanescent waves if:

$$\alpha = \Gamma$$

Then we have to find a solution $\Psi$ of the Helmholtz equation satisfying the limiting conditions.

$$\Psi = \begin{cases} 
E_z(\xi, \eta) \\
H_z(\xi, \eta) 
\end{cases}$$

$$\Delta \Psi(\xi, \eta) + \left(k^2 + \Gamma^2 \right) \Psi(\xi, \eta) = 0$$

$$E_z = 0, \quad \frac{\partial H_z}{\partial n} = 0$$

In a solution, the constant $k_c$ depends only on the geometry of the guide and on the medium.

$$k^2 + \Gamma^2 = k_c^2 = ct$$

In the case of dispersive waves, $\Gamma = j\beta$ and we arrive at the relation of dispersion:

$$k^2 - \beta^2 = k_c^2$$

where we have:

$$\begin{cases} 
k(\omega) = \omega \sqrt{\mu \varepsilon} = \frac{\omega}{v} \\
\beta(\omega) = \frac{\omega}{v_p(\omega)}
\end{cases}$$

where $v$ is the medium speed of the uniform plane waves. In the empty medium, this speed is

$$v = C = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}.$$
Then

\[ k^2 = \frac{\omega^2}{v^2} = k_c^2 + \beta^2 \]

which means that:

\[ \omega = \pm \sqrt{k_c^2 + \beta^2} \]

At the cutoff, we have:

\[ \beta = 0 \quad \text{and} \quad \omega = \omega_c = vk_c \]

This is true in the general case and also for transverse electric (TE) and transverse magnetic (TM) waves.

Now we return to the propagating constant:

\[ \beta^2 = k^2 - k_c^2 \]
and:

\[ \beta = \pm \sqrt{k^2 - k_C^2} \]

This propagating constant exists if

\[ k \geq k_C \quad \text{or} \quad \frac{\omega}{\nu} \geq \frac{\omega_c}{\nu} \]

i.e.

\[ \omega \geq \omega_c \]

Waves are propagating (\( \beta \) is real) if we are situated above the cutoff (\( \omega \geq \omega_c \)).

Waves are evanescent (\( \beta \) is imaginary) if we are situated under the cutoff (\( \omega \leq \omega_c \)).

4.5. Propagating waves

Propagation is due to the factor:

\[ e^{j(\omega t - \beta z)} \]

Propagation is direct (i.e. it goes toward the positive \( z \)) if \( \beta \geq 0 \).

On the contrary, propagation is inverse (i.e. it goes toward the negative \( z \)) if \( \beta \leq 0 \).
Then:

$$\beta = \pm k \sqrt{1 - \left( \frac{k_c}{k} \right)^2} = \pm \frac{\omega}{v} \sqrt{1 - \left( \frac{\omega_c}{\omega} \right)^2}$$

With \( \beta = \frac{\omega}{v_p} \), we obtain the speed of the waves:

$$v_p = \frac{v}{\sqrt{1 - \left( \frac{\omega_c}{\omega} \right)^2}}$$

The speed of the waves is always greater than the speed of the plane waves in the same medium.

$$v_p \geq v$$

We have the same formulas for the length waves:

$$\lambda_g \geq \lambda$$
where $\lambda_g = v_p T = \frac{2\pi v_p}{\omega}$ is the length wave in the guide and $\lambda = v T = \frac{2\pi v}{\omega}$ is the length in the same medium without guide.

We also define the cutoff wave $\lambda_c$ as:

$$k_c = \frac{\omega_c}{v} = \frac{2\pi}{v T_c} = \frac{2\pi}{\lambda_c}$$

which means that a cutoff frequency $f_c$ with:

$$\lambda_c = \frac{v}{f_c}$$

Recalling that $\lambda = \frac{v}{f}$ and $\lambda_g = \frac{v_p}{f}$, we obtain the well-known formulas:

$$\frac{1}{\lambda^2} = \frac{1}{\lambda_g^2} + \frac{1}{\lambda_c^2}$$

or:

$$\lambda_g = \frac{\lambda}{\sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2}}$$

where

$\lambda_g$ is the wave guide length;

$\lambda_c$ is the cutoff length (this depends on the guide geometry);

$\lambda$ is the length of work.
These relations are true for all types of guides and for all types of waves \((TE, TM, TEM, \ldots)\).

### 4.6. Group speed

This is the propagation speed of energy:

\[
v_G = \frac{d\omega}{d\beta}
\]

We define the speed phase \(v_p\) and the speed of the uniform waves \(v\) as:

\[
v_p = \frac{\omega}{\beta}
\]

\[
v = \frac{\omega}{k}
\]

In the \((\omega, \beta)\) diagram, we have:

\[
\begin{align*}
\tan \gamma &= v_G \\
\tan \theta &= v_p
\end{align*}
\]

\[\text{Figure 4.5. } (\omega, \beta) \text{ diagram and the speeds of energy and phase}\]
From:

\[ k^2 - \beta^2 = k_c^2 = \text{cte} \]

We obtain, by differentiation:

\[ k \, dk - \beta \, d\beta = 0 \]

Using different definitions of \( v, v_p \) and \( v_G \), we obtain:

\[ v^2 = v_p \cdot v_G \]

We know that \( v_p \geq v \), then we always have \( v_G \leq v \). In the air \( v = c \), then:

\[ c^2 = v_p \cdot v_G \]

4.7. **Average power flux**

The average power flux is by definition:

\[
\bar{P} = \frac{1}{2} \iint_{S} \left( \tilde{E} \wedge \tilde{\mathcal{H}}^* \right) \overline{\tilde{u}} \, ds
\]

where:

\[
\begin{align*}
\tilde{E} & = (\tilde{E}_T + E_z \overline{u}) \, e^{i(\omega t - \beta z)} \\
\tilde{\mathcal{H}} & = (\tilde{H}_T + H_z \overline{u}) \, e^{i(\omega t - \beta z)}
\end{align*}
\]

Then we have to consider the product:

\[
[(\tilde{E}_T + E_z \overline{u}) \wedge (\tilde{H}_T^* + H_z^* \overline{u})] \overline{u}
\]

which only has one non-zero term:

\[
(\tilde{E}_T \wedge \tilde{H}_T^*) \overline{u}
\]
and:

\[
\bar{P} = \frac{1}{2} \Re \oint_{S} \left( \vec{E}_T \land \vec{H}_T^* \right) \cdot \vec{u} \, ds
\]

We know \( \vec{E}_T \) and \( \vec{H}_T^* \) as a function of the components \( E_Z \) and \( H_Z \):

\[
\begin{align*}
\vec{E}_T &= j \frac{kZ}{k_c^2} \left( \vec{u} \land \text{grad}_T H_Z \right) - \frac{j \beta}{k_c^2} \text{grad}_T E_Z \\
\vec{H}_T^* &= j \frac{k}{Zk_c^2} \left( \vec{u} \land \text{grad}_T^* E_Z \right) + \frac{j \beta}{k_c^2} \text{grad}_T^* H_Z
\end{align*}
\]

The components \( E_Z \) and \( H_Z \) are real (except in the helix case) and we have to consider three products of the forms:

\[
\begin{align*}
\left( \vec{u} \land \text{grad}_T H_Z \right) \land \left( \vec{u} \land \text{grad}_T E_Z \right) &= \text{grad}_T H_Z \land \text{grad}_T E_Z \\
\text{grad}_T E_Z \land \left( \vec{u} \land \text{grad}_T E_Z \right) &= \left| \text{grad}_T E_Z \right|^2 \vec{u} \\
\left( \vec{u} \land \text{grad}_T H_Z \right) \land \text{grad}_T H_Z &= -\left| \text{grad}_T H_Z \right|^2 \vec{u}
\end{align*}
\]

Then the average power flux can be divided into two terms:

\[
\bar{P} = \frac{1}{2} \oint_{S} \frac{k^2 + \beta^2}{k_c^4} \left( \text{grad}_T E_Z \land \text{grad}_T H_Z \right) \cdot \vec{u} \, ds
\]
\[
+ \frac{1}{2} \oint_{S} \frac{\beta k}{k_c^4} \left( \frac{1}{Z} \left| \text{grad}_T E_Z \right|^2 + Z \left| \text{grad}_T H_Z \right|^2 \right) \, ds
\]
4.7.1. Stokes’ theorem

The value of the first term of the average power flux is determined by using Stokes’ theorem (or the rotational theorem) where, from Figure 4.6, (C) is a closed contour of a surface (S).

\[ \oint_S \text{rot} \vec{A} \cdot d\vec{s} = \oint_C \vec{A} \cdot d\vec{l} \]

![Figure 4.6. Surface (S) bounded by a closed contour (C)](image)

We start from the identity:

\[ \text{rot} (a \vec{A}) = a \text{rot} \vec{A} + \text{grad} a \wedge \vec{A} \]

With \( a = E_z \) and \( \vec{A} = \text{grad}_r H_z \), we obtain:

\[ \text{rot} \left( E_z \text{grad}_r H_z \right) = E_z \text{rot} \left( \text{grad}_r H_z \right) + \text{grad} E_z \wedge \text{grad} H_z \]

Then a \( \text{rot} \) of a \( \text{grad} \) is zero and the first part of \( \bar{P} \) is:

\[ \oint_S \left( \text{grad}_r E_z \wedge \text{grad}_r H_z \right) \bar{u} ds = \oint_S \text{rot} \left( E_z \text{grad}_r H_z \right) \bar{u} ds \]
From Stokes’ theorem:

\[ \oint_S (\nabla_r E_Z \wedge \nabla_r H_Z) \cdot \mathbf{u} ds = \oint_S \nabla \times (E_Z \cdot \nabla_r H_Z) \cdot \mathbf{u} ds \]

\[ = \oint_C E_Z \cdot \nabla_r H_Z \mathbf{u} d\mathbf{l} \]

This quantity is zero because on the contour (C) we have:

\[ \begin{cases} E_Z = 0 \\ \nabla_r \vec{H} \cdot \mathbf{n} = \frac{\partial H_Z}{\partial n} = 0 \end{cases} \]

The \( P \) remainder is then:

\[ \bar{P} = \frac{1}{2} k_0^2 \oint_S \left( \frac{1}{Z} \left| \nabla_r E_Z \right|^2 + Z \left| \nabla_r H_Z \right|^2 \right) ds \]

4.7.2. Ostrogradsky’s theorem

The second term of the average power flux will be determined by using Ostrogradsky’s theorem (or divergence theorem).

\[ \oint_S \nabla \cdot \mathbf{A} ds = \oint_C \mathbf{A} \cdot \mathbf{u} d\mathbf{l} \]

We will consider this theorem using:

\[ \mathbf{A} = \psi \cdot \nabla \psi \text{ with } \psi = E_Z \text{ or } \psi = H_Z \]

But first recall:

\[ \begin{cases} \nabla \cdot (b \cdot \vec{B}) = b \cdot \nabla \vec{B} + \nabla b \cdot \vec{B} \\ \nabla \cdot (\psi \cdot \nabla \psi) = \psi \cdot \nabla \left( \nabla \psi \cdot \nabla \psi \right) + \left| \nabla \psi \right|^2 \end{cases} \]
From the property:
\[ \text{div}(\text{grad } \psi) = \Delta \psi = -k_c^2 \psi \]
we obtain the integral:
\[ \iint_S \text{div}(\psi \text{grad } \bar{\psi}) \, ds = -k_c^2 \iint_S \psi^2 \, ds + \iint_S \left| \text{grad } \psi \right|^2 \, ds \]
which is zero because from Ostrogradsky’s theorem we have:
\[ \iint_S \text{div}(\psi \text{grad } \bar{\psi}) \, ds = \oint_C \psi \text{grad } \bar{\psi} \cdot \bar{n} \, dl = 0 \]
if \( \psi = E_z \)
or \( \psi = H_z \)
By recalling that on contour of the conductor we have:
\[ \begin{cases} E_z = 0 \\ \text{or grad } \bar{H}_z \bar{n} = \frac{\partial H_z}{\partial n} = 0 \end{cases} \]
the power is then:
\[ \bar{P} = \frac{1}{2} \frac{k \beta}{k_c^2} \iint_S \left( \frac{E_z^2}{Z} + Z H_z^2 \right) \, ds \]
We have two terms in \( E_z^2 \) and \( H_z^2 \). The powers are decoupling and additives (this is true in the case of a homogenous guide). We can decompose them on the TE and TM waves.

4.8. Power density

By definition, the power density is:
\[ \bar{W} = \bar{W}_E + \bar{W}_M = \frac{1}{4} \iint_S \bar{E} \bar{E}^* + \frac{1}{4} \iint_S \mu \bar{H} \bar{H}^* \]
with:
\[
\begin{align*}
\vec{E} &= \vec{E}_T + E_z \vec{u} \\
\vec{H} &= \vec{H}_T + H_z \vec{u}
\end{align*}
\]
as in the case of \( \vec{P} \), we obtain:
\[
\vec{W}_E = \vec{W}_M = \frac{1}{4 k_c^2} \iint_S \left( \varepsilon E_z^2 + \mu H_z^2 \right) ds
\]
and a total of:
\[
\vec{W} = \vec{W}_E + \vec{W}_M = \frac{1}{2 k_c^2} \iint_S \left( \varepsilon E_z^2 + \mu H_z^2 \right) ds
\]

4.9. Energy speed

By definition, the speed of the energy is:
\[
v_E = \frac{\vec{P}}{\vec{W}}
\]
But using \( Z = \sqrt{\frac{\mu}{\varepsilon}} \):
\[
\vec{W} = \frac{1}{2 k_c^2} \sqrt{\varepsilon \mu} \iint_S \left( \sqrt{\frac{\varepsilon}{\mu}} E_z^2 + \sqrt{\frac{\mu}{\varepsilon}} H_z^2 \right) ds = \frac{k^2}{2 k_c^2} \sqrt{\varepsilon \mu} \frac{2 k_c^2}{\beta k} \vec{P}
\]
\[
\vec{W} = \frac{k}{\beta} \sqrt{\varepsilon \mu} \vec{P} = \frac{k}{\beta v} \vec{P} = \frac{v_p}{v^2} \vec{P}
\]
and:
\[
\vec{P} = \frac{v^2}{v_p} \vec{W} = v_G \vec{W}
\]
Then:

\[ v_E = v_G = \frac{v^2}{v_p} = \frac{d\omega}{d\beta} \]

The speed of the energy is \( v_G \leq v \).

We will give the properties of the \( TE(E_Z = 0) \) waves and the \( TM(H_Z = 0) \) waves.

### 4.10. First example of \( TE \) waves

We can say that the \( TE(E_Z = 0) \) waves are also the \( HH(H_Z \neq 0) \) waves.

The problem is finding the different waves starting from \( H_Z \) given, respectively, by the equation and the limiting condition:

\[
\begin{align*}
\Delta H_Z + \left( k^2 - \beta^2 \right) H_Z &= 0; \quad Z = \frac{\mu}{\varepsilon}; \quad k = \sqrt{\frac{\mu}{\varepsilon}} \\
\frac{\partial H_Z}{\partial n} &= 0, \text{ on the conductors}
\end{align*}
\]

Knowing \( H_Z \), we can determine the other waves.

\[
\begin{align*}
\vec{E}_{TE} &= j \frac{k Z}{k_C^2} \left( \vec{u} \wedge \text{grad} \vec{H}_Z \right) \\
\vec{H}_{TE} &= -j \frac{\beta}{k_C^2} \text{grad} \vec{H}_Z
\end{align*}
\]
With:

\[ \vec{H}_{TE} = \frac{\beta}{kZ}(\vec{u} \wedge \vec{E}_{TE}) \]

The impedance for the TE waves is:

\[ Z_{TE} = \left| \frac{\vec{E}_T}{\vec{H}_T} \right| = Z \frac{k}{\beta} = Z \frac{v_p}{v} \]

But \( v_p \geq v \), then \( Z_{TE} \geq Z = \sqrt{\frac{\mu}{\varepsilon}} \)

We also have:

\[ Z_{TE} (or Z_H) = \mu v_p \]

The power flux is:

\[ P_{TE} = \frac{1}{2} \frac{\beta k}{k_c^2} Z \iint H_z^2 ds \]

and the energy:

\[ W_{TE} = \frac{1}{2} \frac{\mu k^2}{k_c^2} \iint H_z^2 ds \]

We define the speed of the energy as:

\[ \overline{P} = v_E \overline{W} \quad \Rightarrow \quad v_E = v_G = \frac{v^2}{v_p} \]
4.11. Second example of TM waves

We can say that the $TM (H_z = 0)$ waves are also the $E (E_z \neq 0)$ waves.

The problem is finding the different waves starting from $E_z$ given, respectively, by the equation and the limiting condition:

$$\Delta E_z + \left(k^2 - \beta^2\right)E_z = 0; \quad Z = \sqrt{\frac{\mu}{\varepsilon}}; \quad k = \sqrt{\mu \varepsilon}$$

$$E_z = 0, \text{ on the conductors}$$

Knowing $E_z$, we can determine the other waves.

$$E_{TM} = -j \frac{\beta}{k_c^2} \text{grad}E_z$$

$$H_{TM} = -j \frac{k}{Z k_c^2} (\bar{u} \wedge \text{grad}E_z)$$

With:

$$H_{TM} = \frac{k}{\beta Z} (\bar{u} \wedge \vec{E}_{TM})$$

The impedance for the TM waves is:

$$Z_{TM} = \frac{|\vec{E}_T|}{|\vec{H}_T|} = Z \frac{\beta}{k} = Z \frac{\gamma}{\nu_p}$$

But $\nu_p \geq \nu$, then $Z_{TM} \leq Z = \sqrt{\frac{\mu}{\varepsilon}}$
We also have:

\[ Z_{TM} (or \ Z_E) = \frac{1}{\varepsilon v_p} \]

The power flux is:

\[ P_{TM} = \frac{1}{2} \frac{\beta k}{Z k_C^2} \iint E_Z^2 dS \]

and the energy:

\[ W_{TM} = \frac{1}{2} \frac{\varepsilon k^2}{k_C^2} \iint E_Z^2 dS \]

We define the speed of the energy as:

\[ \overline{P} = v_E \overline{W} \Rightarrow v_E = v_G = \frac{v^2}{v_p} \]

### 4.12. Inverse waves

For direct waves \((\beta \geq 0)\) and in the two cases (TE and TM), we gave the vectors \(\vec{H}_{TE}\):

\[
\begin{align*}
\vec{H}_{TE} &= \frac{\beta}{kZ} (\vec{u} \wedge \vec{E}_{TE}) \quad \text{direct TE} \\
\vec{H}_{TM} &= \frac{k}{\beta Z} (\vec{u} \wedge \vec{E}_{TM}) \quad \text{direct TM}
\end{align*}
\]

In the case of inverse waves \((\beta \leq 0)\) and for TE and TM, the vectors are changed so as to be \(-\vec{H}_{TE}\) and \(-\vec{H}_{TM}\) (Figure 4.7).
4.13. Behavior of the $TE$ and $TM$ waves versus the position of frequency in connection with the cutoff

From:

$$k_c^2 = k^2 - \beta^2$$

we obtain:

$$\beta^2 = k^2 - k_c^2 = \frac{\omega^2 - \omega_c^2}{v}$$

4.13.1. Above the cutoff $\omega > \omega_c$

Above the cutoff, we have: $\omega > \omega_c$; then $\beta$ is real and there is propagation. The waves are in:

$$e^{i\omega t} e^{-i\beta z}$$
4.13.2. At the cutoff $\omega = \omega_c$

At the cutoff, we have $\omega = \omega_c$; then there is no propagation ($\beta = 0$) and the phase velocity $v_p = \frac{\omega}{\beta}$ is infinite. The waves are only in:

$e^{j\omega t}$

4.13.3. Under the cutoff $\omega < \omega_c$

Under the cutoff, $\beta$ is imaginary and we can write $\alpha = j\beta$; then there is no propagation and:

$$\alpha^2 = k_c^2 - k^2 = \frac{\omega_c^2 - \omega^2}{v}$$

and:

$$\alpha = k_c \sqrt{1 - \left( \frac{k}{k_c} \right)^2} = k_c \sqrt{1 - \left( \frac{\omega}{\omega_c} \right)^2}$$

The waves are multiplied by the factor:

$e^{j\omega t} e^{-\alpha z}$

This looks like attenuation and the attenuation length is:

$$L = \alpha^{-1} = \frac{1}{k_c \sqrt{1 - \left( \frac{\omega}{\omega_c} \right)^2}}$$
### 4.13.4. Summary

<table>
<thead>
<tr>
<th>Multiplication</th>
<th>TE</th>
<th>TM</th>
</tr>
</thead>
<tbody>
<tr>
<td>propagating $\omega &gt; \omega_c$</td>
<td>$e^{-j\beta z}$</td>
<td>$E_z = \frac{Z}{k} Z$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$H_z = \frac{k}{\beta} Z \left( \hat{u} \times \nabla \times E_z \right)$</td>
</tr>
<tr>
<td>cut off $\omega = \omega_c$</td>
<td>1</td>
<td>$H_z = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$Z_{1z} = \infty$</td>
</tr>
<tr>
<td>evanescent $\omega &lt; \omega_c$</td>
<td>$e^{-\alpha z}$</td>
<td>$H_z = -\frac{\alpha}{k^2 + \alpha^2} \nabla \times \nabla \times E_z$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$E_z = \frac{Z}{k} Z \left( \hat{u} \times \nabla \times H_z \right)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$Z_{1z} = \frac{k}{\alpha}$ low if $\omega \ll \omega_c$</td>
</tr>
</tbody>
</table>

### 4.14. Bibliography


5

Losses in Guides

5.1. Introduction

First, we have to evaluate the currents in the sides of the conductors. Recall that the lossy conductors are equivalent to perfect conductors with a skin effect of thickness:

\[ \delta = \sqrt{\frac{2}{\mu \sigma \omega}} \]

which gives a superficial resistance:

\[ R_s = \frac{1}{\sigma \delta} \]

This induces a superficial density of current:

\[ \mathbf{J}_s = \mathbf{n} \wedge \mathbf{H}_g \]

Joule’s power is then

\[ P_J = \frac{1}{2} R_s \int_S \mathbf{J}_s \mathbf{J}_s^* dS \]

Or, with (C) being a closed outline of the surface (S)
\[ P_J = \frac{1}{2} R_S \oint \mathbf{J}_S \mathbf{J}_S^* dS \]

**Figure 5.1.** Superficial density of current \( \mathbf{J}_s \) and direct dihedral \((\mathbf{n}, \mathbf{u}, \mathbf{\tau})\)

### 5.2. TE waves

The magnetic wave parallel to the conductors is:

\[ \mathbf{H}_tg = \mathbf{H}_T + H_z \mathbf{u} \]

Then with

\[ \mathbf{H}_T = -j \frac{\beta}{k_c^2} \text{grad} H_z \]

we have:

\[ \mathbf{J}_s = -j \frac{\beta}{k_c^2} (\mathbf{n} \wedge \text{grad} H_z) + (\mathbf{n} \wedge \mathbf{u}) H_z \]

\((\mathbf{n} \wedge \mathbf{u})\) is the vector \(\mathbf{\tau}\), and in the first part of the equation we have also: \(\mathbf{n} = \mathbf{u} \wedge \mathbf{\tau}\):

\[ \mathbf{J}_s = H_z \mathbf{\tau} - j \frac{\beta}{k_c^2} [ (\mathbf{u} \wedge \mathbf{\tau}) \wedge \text{grad} H_z ] \]
The double vectorial product gives:

\[ J_s = H_z \vec{t} - j \frac{\beta}{k_C^2}[(\vec{u}\cdot \nabla H_z)\vec{t} - (\nabla H_z, \vec{t})\vec{u}] \]

The first part of this double vectorial product is zero and 
\[ \nabla \tilde{H}_z, \vec{t} = \frac{dH_z}{d\tau} : \]

\[ J_s = H_z \vec{t} + j \frac{\beta}{k_C^2} \left( \frac{dH_z}{d\tau} \right) \vec{u} \]

The first part is tangential \( J_t = H_z \vec{t} \), while the second part is longitudinal \( J_{\text{long}} = j \frac{\beta}{k_C^2} \left( \frac{dH_z}{d\tau} \right) \vec{u} \)

5.3. **TM waves**

We only have

\[ \tilde{H}_g = \tilde{H}_f \]

and:

\[ \tilde{H}_t = -j \frac{k}{Z k_C^2} \left( \vec{u} \times \nabla \tilde{E}_z \right) \]

\[ J_s = -j \frac{k}{Z k_C^2} \left[ \vec{u} \times \left( \vec{u} \times \nabla \tilde{E}_z \right) \right] \]

which is

\[ J_s = -j \frac{k}{Z k_C^2} \left( \frac{dE_z}{dn} \right) \vec{u} \]
We only have a longitudinal part:

\[ \vec{J}_{\text{long}} = -j \frac{k}{Z k_c} \left( \frac{dE_z}{dn} \right) \vec{u} \]

### 5.4. Attenuation in the cases of TM and TM waves

We suppose that the losses are slight and do not change the waves:

\[ \vec{P}(z) = \vec{P}_0(z) e^{-2az} \]

This means that

\[ \alpha = -\frac{1}{2} \frac{d \vec{P}(z)}{dz} \]

With losses by Joule’s effect:

\[ -\frac{d \vec{P}(z)}{dz} = P_j = \frac{1}{2} R_s \oint_C J_s J_s^* dS \quad \text{with} \quad R_s = \frac{1}{\sigma \delta} \]

we obtain TE waves.

After the tedious computation we obtain:

\[ P_j = \frac{1}{2 \sigma \delta} \oint_C \left( H_{Z}^2 + \frac{\beta^2}{k_c^4} \left( \frac{dH_z}{d\tau} \right)^2 \right) dl \]

\[ \alpha_{TE} = \frac{1}{2} \frac{1}{Z \sigma \delta} \frac{k^2 - \beta^2}{\beta k} \oint_S H_z^2 ds \]
$TM$ waves.

$$P_j = \frac{1}{2\sigma \delta Z k^2} \frac{1}{C} \left( \frac{dE_z}{dn} \right)^2 dl$$

$$\alpha_{TM} = \frac{1}{2} \frac{1}{Z \sigma \delta \beta (k^2 - \beta^2)} \frac{k}{C} \left( \frac{dE_z}{dn} \right)^2 dl$$

5.5. Problem

5.5.1. Waves between two parallel metallic and lossy planes

1) A wave is propagating between two metallic, infinite and parallel conductors. The losses are supposed to be slight and the place between the two conductors ($\varepsilon_\infty, \mu_\infty, \sigma_\infty$) is full of dielectric ($\varepsilon_d, \mu_d, \sigma_d$).

![Diagram of two parallel plates](image-url)

**Figure 5.2. Two parallel plates**
In the plane \( y = C^{te} \), give the expressions of the longitudinal components \( E_{zd} \) and \( E_{zc} \) in the dielectric and in the conductor. Let us choose the origin at \( x = a/2 \), then the problem is symmetrical versus \( 0z \) and \( E_{zd} \) is an odd function of the variable \( x \).

2) Show that with the limiting conditions the report is always a constant \( \frac{E_z}{H_y} = C^{te} \). Then, deduce a method to compute the propagation constant \( \gamma \).

3) To resolve this last equation, we suppose that the dielectric losses and the displacement current are negligible. Then, compute \( \alpha \) and \( \beta \) as a function of \( a \), \( R_{sc} \) (superficial resistance of the conductor), \( \eta_d \) (wave impedance in the dielectric) and \( k_d \).

4) Show that the electric field is practically transversal in the dielectric and longitudinal in the conductor.

SOLUTIONS.–

1) We use the perturbation method which states that there are no geometrical modifications of the waves. We use the ideal lossless expressions of the waves multiplied only by the attenuation factor \( e^{-az} \). There is only a change:

\[
j\beta \rightarrow \gamma = \alpha + j\beta
\]

and the propagation equation:

\[
\Delta_r E_z + (k^2 - \beta^2)E_z = 0
\]

becomes

\[
\Delta_r E_z + (k^2 + \gamma^2)E_z = 0
\]
or

\[ \Delta_r E_z + K^2 E_z = 0 \text{ with } K^2 = k^2 + \gamma^2 \]

In the plane \( y = C \), we have:

\[ \frac{d^2 E_{zd}}{dx^2} + K^2 E_{zd} = 0 \text{ with } K^2 = \gamma^2 + \omega^2 \mu_d \varepsilon_d \]

in the dielectric and

\[ \frac{d^2 E_{zc}}{dx^2} + K^2 E_{zc} = 0 \text{ with } K^2 = \gamma^2 + \omega^2 \mu_c \varepsilon_c \]

in the conductor.

Now we have to give the fields in the two elements (dielectric and conductor).

In the dielectric, the general solution is:

\[ E_{zd} = A_i e^{jk_d x} + B_i e^{-jk_d x} \]

But we have symmetry versus \( 0z \):

\[ E_{zd}(-x) = -E_{zd}(x) \]

This gives:

\[ A_i = -B_i \]

and

\[ E_{zd} = 2jA_i \sin K_d x = C_i \sin K_d x \]
In the dielectric, there is a small component \( E_{Zd} \) (Figure 5.3) to fit the component of the conductor \( E_{Zc} \).

![Figure 5.3. The waves in the dielectric](image)

Now in the conductor, the general solution is

\[
E_{Zc} = A_2 e^{iK_c x} + B_2 e^{-iK_c x}
\]

And physically in the conductor, the field \( E_{Zc} \) has to decrease when the variable \( x \) increases. Then, the constant \( A_2 = 0 \) and we also have to choose a solution of

\[
K_c^2 = \frac{\omega^2}{\mu_c} \frac{1}{\varepsilon_c}
\]

so that \( jK_c \) has a positive real part:

\[
\Re\left(jK_c\right) \geq 0
\]

and

\[
E_{Zc} = C_2 e^{-jK_c x}
\]

2) The limiting conditions at the separation of two mediums is the equality of the tangential electric and magnetic components:

\[
E_{t1} = E_{t2}
\]

\[
H_{t1} = H_{t2}
\]
The first condition gives:

\[
\begin{align*}
\left[ E_{Zd} \right]_{x=\frac{a}{2}} &= \left[ E_{Zc} \right]_{x=\frac{a}{2}} \\
\left[ E_{Yd} \right]_{x=\frac{a}{2}} &= \left[ E_{Yc} \right]_{x=\frac{a}{2}}
\end{align*}
\]

[5.1]

[5.2]

The second condition gives:

\[
\begin{align*}
\left[ H_{Zd} \right]_{x=\frac{a}{2}} &= \left[ H_{Zc} \right]_{x=\frac{a}{2}} \\
\left[ H_{Yd} \right]_{x=\frac{a}{2}} &= \left[ H_{Yc} \right]_{x=\frac{a}{2}}
\end{align*}
\]

[5.3]

[5.4]

For example, if we know the expression of the report \( \frac{(1)}{(4)} \), then it is possible to know:

\[
\begin{align*}
\left[ \frac{E_{Zd}}{H_{Yd}} \right]_{x=\frac{a}{2}} &= \left[ \frac{E_{Zc}}{H_{Yc}} \right]_{x=\frac{a}{2}}
\end{align*}
\]

We know \( E_{Zd} \) and \( E_{Zc} \) and using the well-known formula gives the transversals as a function of the longitudinal ones.

We recall:

\[
(k^2 - \beta^2) \tilde{H}_T = -j \omega \epsilon \left[ \bar{u} \wedge \text{grad}_t E_Z \right] - j \beta \text{grad}_t H_Z
\]

and with the change:

\( j \beta \rightarrow \gamma \)

\[
(k^2 + \gamma^2) \tilde{H}_T = -j \omega \epsilon \left[ \bar{u} \wedge \text{grad}_t E_Z \right] - \gamma \text{grad}_t H_Z
\]

we obtain:

\[
\tilde{H}_T = -j \frac{\omega \epsilon}{K^2} \left[ \bar{u} \wedge \text{grad}_t E_Z \right] - \frac{\gamma}{K^2} \text{grad}_t H_Z
\]
which gives:

\[ H_y = -j \frac{\omega \varepsilon_y}{K^2} \frac{\partial E_Z}{\partial y} - \gamma \frac{\partial H_z}{\partial y} \]

But \( H_z \) does not depend on the variable \( y \), and we have:

\[
\begin{aligned}
H_{yd} &= -j \frac{\omega \varepsilon_y}{K_d} C_1 \cos(K_d x) \\
H_{yc} &= -j \frac{\omega \varepsilon_y}{K_c} C_2 e^{-jK_c x}
\end{aligned}
\]

and the condition:

\[
\frac{C_1 \sin(K_d \frac{a}{2})}{-j \frac{\omega \varepsilon_y}{K_d} C_1 \cos(K_d \frac{a}{2})} = \frac{C_2 e^{-jK_c \frac{a}{2}}}{-\frac{\omega \varepsilon_y}{K_c} C_2 e^{-jK_c \frac{a}{2}}}
\]

This gives with the simplifications:

\[
tg(K_d \frac{a}{2}) = j \frac{\varepsilon_d}{\varepsilon_c} \frac{K_c}{K_d}
\]

3) Dielectric losses can be neglected:

\[ \gamma_d = \alpha_d + j \beta_d \approx j \beta_d \]

and

\[ K_d^2 = k_d^2 + \gamma_d^2 \approx k_d^2 - \beta_d^2 \]

This means that \( K_d \) is slight and the component \( E_{zd} = C_1 \sin K_d x \) too:

\[
tg(K_d \frac{a}{2}) \approx K_d \frac{a}{2}
\]
Then:

\[ K_d \approx \frac{a}{2} \frac{\varepsilon_d K_c}{\varepsilon_c K_d} \]

and

\[ K_d^2 = j \frac{2 \varepsilon_d K_c}{a \varepsilon_c} \]

In the conductor, this is the displacement current which is neglected.

Recall that we have:

- \( \text{rot} \vec{H} = \sigma \vec{E} \), in a perfect conductor;

- \( \text{rot} \vec{H} = \frac{\partial \vec{D}}{\partial t} = j \omega \varepsilon \vec{E} \) in a perfect dielectric.

Then, it is possible to go from the dielectric to the conductor by using a fictitious permittivity:

\[ \varepsilon_c = \frac{\sigma_c}{j \omega} \]

and

\[ K_d^2 = j \omega \frac{2 \varepsilon_d}{a} (jK_c) \]

In the conductor, we see that the electric \( z \) wave can be written as:

\[ E_{z_c} = C_z e^{-jk_c z} \text{ with } \Re (jK_c) \geq 0 \]

In the conductor, the losses are due to the skin effect:

\[ jK_c = \frac{1 + j}{\delta_c} \text{ with } \Re (jK_c) = \frac{1}{\delta_c} \geq 0 \]
Then

\[ K_d^2 = j \omega \frac{2 \varepsilon_d}{a \sigma_c} \frac{1+j}{\delta_c} \]

With the skin thickness:

\[ \delta_c = \sqrt{\frac{2}{\mu_c \sigma_c \omega}} \]

Using the expression of the surface resistance:

\[ R_{Sc} = \frac{1}{\sigma_c \delta_c} \]

We have a slight value of \( K_d \):

\[ K_d^2 = \gamma_d^2 + k_d^2 = j \omega \frac{2}{a} \varepsilon_d (1+j) R_{Sc} \]

In dielectric and in metal we have the same permeability:

\[ \mu = \mu_d = \mu_c \]

Then

\[ \gamma_d^2 = -k_d^2 + K_d^2 = -\omega^2 \mu \varepsilon_d + j \omega \frac{2}{a} \varepsilon_d (1+j) R_{Sc} \]

\[ \gamma_d^2 = -k_d^2 \left[ 1 - \frac{2 j (1+j) R_{Sc}}{ak_d \eta_d} \right] \]

where the impedance wave is:

\[ \eta_d = \sqrt{\frac{\mu}{\varepsilon_d}} \]
In the expression of $K_d^2$, the second part is slight and we can write:

$$
\gamma_d = jk_d \left[ 1 - j(1+j) \frac{R_{Sc}}{ak_d \eta_d} \right]
$$

or

$$
\gamma_d = \frac{R_{Sc}}{a \eta_d} + j \left[ k_d + \frac{R_{Sc}}{a \eta_d} \right]
$$

which is of the form:

$$
\gamma_d = \alpha_d + j \beta_d
$$

The expression of $\gamma_d$ gives us the values of the propagation and the attenuation:

$$
\begin{align*}
\alpha_d &= \frac{R_{Sc}}{a \eta_d} \\
\beta_d &= k_d + \frac{R_{Sc}}{a \eta_d}
\end{align*}
$$

4) We have:

$$
E_{z_d} = C_1 \sin K_d x
$$

From the other general formula:

$$
(k^2 - \beta^2) \tilde{E}_T = j \omega \mu \left[ \bar{u} \wedge \nabla_T \bar{E}_z \right] - j \beta \nabla_T \bar{E}_z
$$

we show that:

$$
E_{x_d} = -\frac{\gamma_d}{K_d} C_1 \sin K_d x
$$
Then:
\[
\frac{E_{zd}}{E_{xd}} = \frac{C_1 \sin K_d x}{\frac{\gamma_d}{K_d} C_1 \sin K_d x} = -\frac{K_d}{\gamma_d} \tan K_d x \approx \frac{x}{\gamma_d} (K_d)^2 \leq 1
\]

Then from Figure 5.4 the electric waves are quasi-transversal in the dielectric:

\[E_{zd} \leq E_{xd}\]

\[\text{Figure 5.4. Electric field in the dielectric}\]

Also using the computation of \(E_{ZC}\) and \(E_{XC}\):

\[
\begin{align*}
E_{Zc} &= C_2 e^{-jK_c x} \\
E_{Xc} &= j \frac{\gamma_c}{K_C} C_2 e^{-jK_c x}
\end{align*}
\]

we have:

\[
\left| \frac{E_{Xc}}{E_{Zc}} \right| = \left| \frac{j \frac{\gamma_c}{K_C} C_2 e^{-jK_c x}}{\frac{\gamma_c}{K_C} C_2 e^{-jK_c x}} \right| = \left| j \frac{\gamma_c}{K_C} \right|
\]
Using \( jK_C = \frac{1 + j}{\delta_C} \), we obtain:

\[
\left| \frac{E_{xC}}{E_{zc}} \right| = \frac{\gamma_C \delta_C}{\sqrt{2}} \leq 1 \text{ because } \delta_C \text{ is slight.}
\]

Then from Figure 5.5 the electric waves are quasi-longitudinal in the conductor:

\( E_{xC} \leq E_{zc} \)

We have the next configuration for the electric fields in the depth \( \delta \) of the conductor and in the dielectric (Figure 5.5).

\[\text{Figure 5.5. Electric fields in the dielectric and in the conductor (\( \delta \))}\]

5.6. Bibliography


6.1. Introduction

In this chapter, we will study the rectangular guide successively with $TM$ and $TE$ waves (Figure 6.1). In general, we find in other studies that the $TE$ is explained first and then the $TM$. We decided to start with the $TM$ because the $TM$ waves are easier to study and we will see at the end that only the $TE_{0n}$ (and the $TE_{m0}$) are not degenerated. A particular study (on the problem) is given for the fundamental $TE_{01}$.

![Figure 6.1. The rectangular guide](image)
6.2. **TM rectangular guide**

6.2.1. **The fields**

In the case of the *TM* mode, the component is $H_z = 0$, and we have to determine $E_z(x,y)$ which is a solution of the propagation equation:

$$\Delta_x E_z(x,y) + (k^2 - \beta^2) E_z(x,y) = 0$$

that is

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + k_c^2 E_z = 0$$

with $k_c^2 = k^2 - \beta^2$. We look for a solution of the next form by separating the two variables $x$ and $y$:

$$E_z(x,y) = F(x)G(y)$$

Then

$$\frac{F'(x)}{F(x)} + \frac{G'(y)}{G(y)} + k_c^2 = 0$$

We put

$$k_c^2 = k_x^2 + k_y^2$$

where $k_x^2$ and $k_y^2$ are functions of $x$ and $y$, respectively. We now have to find the solutions of two differential equations:

$$\begin{align*}
F'(x) + k_x^2 F(x) &= 0 \\
G'(y) + k_y^2 G(y) &= 0
\end{align*}$$
which give two general solutions:

\[
\begin{align*}
F(x) &= A \cos k_x x + B \sin k_x x \\
G(y) &= C \cos k_y y + D \sin k_y y
\end{align*}
\]

Then, the field \( E_z(x, y) \) is to be of the form:

\[
E_z(x, y) = [A \cos k_x x + B \sin k_x x][C \cos k_y y + D \sin k_y y]
\]

Now we have to impose the limiting conditions which must be (Figure 6.1):

\[
\vec{E}_g = 0 \text{ on the sides}
\]

These limiting conditions impose

\[
\begin{align*}
E_z(x, y) &= 0; \quad x = 0 \text{ and } x = a \quad \forall y \\
E_z(x, y) &= 0; \quad y = 0 \text{ and } y = b \quad \forall x
\end{align*}
\]

To verify these limiting conditions, we must have

\[
A = C = 0
\]

\[
k_x = \frac{m\pi}{a}; \quad k_y = \frac{n\pi}{b}
\]

We can say that we have a \( TM_{mn} \) propagation.

And with the propagation factor \( e^{j(\omega t - \beta_{mn} z)} \):

\[
E_z(x, y) = E_0 \sin\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) e^{j(\omega t - \beta_{mn} z)}
\]

The other electric fields are given with the formula:

\[
\vec{E}_r = -\frac{j\beta}{k_C} \text{grad}_r E_z
\]
\[
\begin{align*}
E_x &= -j\beta \frac{\partial E_z}{\partial x} = -j\beta \frac{E_0}{k_c^2} \frac{m\pi}{a} \left[ \cos\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) \right] e^{j(\alpha - \beta z)} \\
E_y &= -j\beta \frac{\partial E_z}{\partial y} = -j\beta \frac{E_0}{k_c^2} \frac{n\pi}{b} \left[ \sin\left(\frac{m\pi}{a} x\right) \cos\left(\frac{n\pi}{b} y\right) \right] e^{j(\alpha - \beta z)}
\end{align*}
\]

and using

\[
\tilde{H}_T = -\frac{jk}{k_c^2 Z} \left[ \tilde{u} \wedge \text{grad}_T E_z \right] \text{ with } \tilde{u} \wedge \text{grad}_T E_z
\]

\[
\begin{align*}
0 \wedge \begin{vmatrix}
\frac{\partial E_z}{\partial x} & -\frac{\partial E_z}{\partial y} \\
\frac{\partial E_z}{\partial y} & \frac{\partial E_z}{\partial x}
\end{vmatrix}
= 0 \\
1 \\
\end{align*}
\]

\[
\begin{align*}
H_x &= \frac{jk}{k_c^2 Z} \frac{\partial E_z}{\partial y} = \frac{jk}{k_c^2 Z} E_0 \frac{n\pi}{b} \left[ \sin\left(\frac{m\pi}{a} x\right) \cos\left(\frac{n\pi}{b} y\right) \right] e^{j(\alpha - \beta z)} \\
H_y &= -\frac{jk}{k_c^2 Z} \frac{\partial E_z}{\partial x} = -\frac{jk}{k_c^2 Z} E_0 \frac{m\pi}{a} \left[ \cos\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) \right] e^{j(\alpha - \beta z)} \\
H_z &= 0
\end{align*}
\]

\textbf{6.2.2. The dispersive relation}

The constant $k_c$ depends only on the integer values of the couple $(m,n)$.

\[
k_{c_{mn}}^2 = k^2 - \beta^2 = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2
\]
We will see that we have the same relation of dispersion as for the $TE_{mn}$ fields. In the rectangular guides, we find that the $TE_{mn}$ and the $TE_{nm}$ are degenerated.

The fields are multiplied by the factor:

$$e^{j(\omega t - \beta_{mn} z)}$$

And $\beta_{mn}$ is positive and real. Then to assume propagation, we must have

$$\beta_{mn}^2 = k^2 - k_{Cmn}^2 > 0$$

This induces

$$k^2 > k_{Cmn}^2$$

or

$$\frac{2\pi}{\lambda} > \frac{2\pi}{\lambda_{Cmn}}$$

And the length wave is maximum because

$$\lambda < \lambda_{Cmn}$$

Now let us come back to the integer value of $\beta$ and we obtain the relation of dispersion:

$$\beta_{mn} = \sqrt{\omega \mu \epsilon - \frac{\pi^2}{2} \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)}$$

There is no propagation if $\beta_{mn} = 0$. Then, we say that we are at the cutoff and

$$\omega_{Cmn} = \frac{\pi}{\sqrt{\mu \epsilon}} \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$
or

\[ f_{C_{mn}} = \frac{1}{2\sqrt{\mu \varepsilon}} \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \]

The fundamental mode corresponds to the lower frequency of cutoff. But \( m = 0 \) or \( n = 0 \) is impossible because in these cases \( E_z \) should be zero and there is no propagation (\( E_z \) is a product of \( \sin(\frac{m\pi}{a}x) \) and \( \sin(\frac{n\pi}{b}y) \)).

The fundamental corresponds to \( m = 1 \) and \( n = 1 \). The fundamental is the TM\(_{11}\):

\[ f_{C_{11}} = \frac{1}{2\sqrt{\mu \varepsilon}} \sqrt{\frac{1}{a^2} + \frac{1}{b^2}} \]

We can observe that for the TE\(_{mn}\), we will see that the wave exists even if \( m = 0 \) or \( n = 0 \) (\( H_z \) will be a product of \( \cos(\frac{m\pi}{a}x) \) and \( \cos(\frac{n\pi}{b}y) \)). Only the TE\(_{0n}\) and the TE\(_{m0}\) are not degenerated.

6.2.3. The power flux

Taking the results of Chapter 5, the power flux is with \( H_z = 0 \):

\[ \bar{P} = \frac{k \beta}{2 Z k_c^2} \int_{S} E_z^2 \, ds \]

Let us consider, for example, the second equation we have:

\[ \bar{P} = \frac{k \beta}{2 Z k_c^2} E_0^2 \int_{0}^{b} \int_{0}^{a} \sin^2(\frac{m\pi}{a}x) \sin^2(\frac{n\pi}{b}y) \, ds \]
And with:

$$\sin^2 X = \frac{1}{2}(1 - \cos 2X)$$

we obtain

$$\overline{P} = \frac{1}{8} \frac{a b k \beta_{mn}}{Z k^2 c_{mn}} E_0^2$$

### 6.2.4. Attenuation

$$\alpha_{TM} = \frac{1}{2} \frac{1}{Z \sigma \delta \beta (k^2 - \beta^2)} \int \int \left( \frac{dE_Z}{dn} \right)^2 dl$$

We know $$\int \int E_Z^2 ds$$ and we have to calculate the integral:

$$\int \left( \frac{dE_Z}{dn} \right)^2 dl$$

But:

$$\frac{dE_Z}{dn} = \mathbf{n} \cdot \nabla_E E_Z = \frac{\partial E_Z}{\partial x} + \frac{\partial E_Z}{\partial y}$$

and:

$$\left( \frac{dE_Z}{dn} \right)^2 = E_0^2 \left[ \frac{m\pi}{a} \cos\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) + \frac{n\pi}{b} \sin\left(\frac{m\pi}{a} x\right) \cos\left(\frac{n\pi}{b} y\right) \right]^2$$
Now consider the four currents in the four sides (Figure 6.2).

\[ x = 0 \text{ and with } y : 0 \rightarrow b \quad \left( \frac{dE_z}{dn} \right)_{x=0}^2 = E_0^2 \left( \frac{m\pi}{a} \right)^2 \sin^2 \left( \frac{n\pi}{b} y \right) \]

\[ x = a \text{ and with } y : b \rightarrow 0 \quad \left( \frac{dE_z}{dn} \right)_{x=a}^2 = E_0^2 \left( \frac{m\pi}{a} \right)^2 \sin^2 \left( \frac{n\pi}{b} y \right) \]

\[ y = 0 \text{ and with } x : a \rightarrow 0 \quad \left( \frac{dE_z}{dn} \right)_{y=0}^2 = E_0^2 \left( \frac{n\pi}{b} \right)^2 \sin^2 \left( \frac{m\pi}{a} x \right) \]

\[ y = b \text{ and with } x : 0 \rightarrow a \quad \left( \frac{dE_z}{dn} \right)_{y=b}^2 = E_0^2 \left( \frac{n\pi}{b} \right)^2 \sin^2 \left( \frac{m\pi}{a} y \right) \]

Then, we have

\[
\oint_C \left( \frac{dE_z}{dn} \right)^2 dl = \int_0^b \left( \frac{dE_z}{dn} \right)_{x=0}^2 dy + \int_a^0 \left( \frac{dE_z}{dn} \right)_{x=a}^2 dy + \int_0^a \left( \frac{dE_z}{dn} \right)_{y=0}^2 dx + \int_b^a \left( \frac{dE_z}{dn} \right)_{y=b}^2 dx
\]
And after computation, we obtain:

\[ \oint \left( \frac{dE_Z}{dn} \right)^2 dl = E_0^2 \left[ \left( \frac{m\pi}{a} \right)^2 b + \left( \frac{n\pi}{b} \right)^2 a \right] \]

Then, we obtain the attenuation coefficient:

\[ \alpha_{TM_{mn}} = \frac{2R_s k}{ab\beta Z} \frac{m^2 b^3 + n^2 a^3}{m^2 b^2 + n^2 a^2} \]

or

\[ \alpha_{TM_{mn}} = \frac{2R_s}{Z} \sqrt{\frac{k_{C_{mn}}^2}{k^2}} \frac{m^2 b^3 + n^2 a^3}{m^2 a b + n^2 a^3 b} \]

It is important to know the attenuation of the fundamental mode \( m = n = 1 \):

\[ \alpha_{TM_{11}} = \frac{2R_s}{Z} \sqrt{\frac{k_{C_{11}}^2}{k^2}} \frac{b^3 + a^3}{a b^3 + a^3 b} \]

**6.2.5. Field lines**

The differential equation of the field line \( C \) is given by

\[ \vec{E} \wedge \vec{dC} = 0 \]

Then

\[
\begin{align*}
|E_x| & |dx| & E_y dz - E_z dy = 0 \\
|E_y| & |dy| & E_z dx - E_x dz = 0 \\
|E_z| & |dz| & E_x dy - E_y dx = 0
\end{align*}
\]
And we find the differential equation for the electric wave:

\[
\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z}
\]

**Figure 6.3. Field line for the wave \( \vec{E} \)**

And we can do the same in the case of the magnetic wave.

6.2.5.1. Magnetic field lines along a plane \( x = cte \)

We consider the plane \((y0z)\). In this plane, we only have one field which is not zero. If we consider the propagation term:

\[
H_z = 0 \\
H_y = -j \frac{k}{Zk_C} E_0 \frac{m\pi}{a} \left[ \cos \left( \frac{m\pi}{a} x \right) \sin \left( \frac{n\pi}{n} y \right) \right] e^{-j\beta z}
\]

there is only one term and its real part is:

\[
\Re (H_y) = -\frac{k}{Zk_C} E_0 \frac{m\pi}{a} \cos \left( \frac{m\pi}{a} x \right) \sin \left( \frac{n\pi}{n} y \right) \sin(\beta z)
\]
We have to fit:

\[
\frac{dy}{\Re_e(H_y)} = \frac{dz}{\Re_e(H_z)} \quad \text{where } \Re_e(H_z) = 0 \text{ in the cases of } TM \text{ waves}
\]

Then:

\[
\Re_e(H_y) = 0
\]

If we consider the fundamental \( m = 1, n = 1 \):

\[
-\frac{k}{\beta \varepsilon_0} E_0 \frac{\pi}{a} \cos \left( \frac{\pi}{a} x \right) \sin \left( \frac{\pi}{n} y \right) \sin (\beta z) dz = 0
\]

By integration:

\[
\frac{k}{\beta \varepsilon_0} E_0 \frac{\pi}{a} \cos \left( \frac{\pi}{a} x \right) \sin \left( \frac{\pi}{n} y \right) \cos (\beta z) = C^e
\]

But \( x \) is a constant and we have to fit:

\[
\sin \left( \frac{\pi}{n} y \right) \cos (\beta z) = C^e = A^2
\]

We can study this curve, remarking that \( A < 1 \), by using another variable \( u \):

\[
\begin{align*}
\sin \left( \frac{\pi}{n} y \right) &= A u \\
\cos (\beta z) &= \frac{A}{u}
\end{align*}
\]
This gives the magnetic field lines shown in Figure 6.4.

\[ E_z = E_0 \left[ \sin \left( \frac{m \pi}{a} x \right) \sin \left( \frac{n \pi}{b} y \right) \right] e^{-j \beta z} \]

\[ E_y = -j \frac{\beta}{k_c^2} E_0 \frac{n \pi}{b} \left[ \sin \left( \frac{m \pi}{a} x \right) \cos \left( \frac{n \pi}{b} y \right) \right] e^{-j \beta z} \]

which gives real parts in the case of the fundamental mode, \( m = 1, n = 1 \):

\[ \Re_e(E_z) = E_0 \sin \left( \frac{\pi}{a} x \right) \sin \left( \frac{\pi}{b} y \right) \cos(\beta z) \]

\[ \Re_e(E_y) = -\frac{\beta}{k_c^2} E_0 \frac{\pi}{b} \sin \left( \frac{\pi}{a} x \right) \cos \left( \frac{\pi}{b} y \right) \sin(\beta z) \]

In this case, we have to fit:

\[ \frac{dy}{\Re_e(E_y)} = \frac{dz}{\Re_e(E_z)} \]
\[ k_c^2 \frac{b}{\pi} \frac{\sin \left( \frac{\pi}{b} y \right)}{\cos \left( \frac{\pi}{b} y \right)} dy = -\beta \frac{\sin(\beta z)}{\cos(\beta z)} dz \]

We have an integration of the form: \( \frac{u'}{u} \)

which gives \( \log(u) \):

\[ k_c^2 \frac{b^2}{\pi^2} \log \left( \cos \frac{\pi}{b} y \right) = -\log(\cos \beta z) + \log A^2 \]

where \( \log A^2 \) is the constant of integration. We consider the fundamental then:

\[ k_c^2 = k_{c11}^2 = \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} \]

and:

\[ \frac{b^2}{\pi^2} k_{c11}^2 = 1 + \frac{b^2}{a^2} \]

Then, we obtain:

\[ (1 + \frac{b^2}{a^2}) \log \left( \cos \frac{\pi}{b} y \right) = -\log(\cos \beta z) + \log A^2 \]

And the integration gives:

\[ \left( \cos \frac{\pi}{b} y \right)^{1 + \frac{b^2}{a^2}} (\cos \beta z) = Cte = B^2 \]
This curve is described by using:

\[
\begin{cases}
\left(\cos \frac{\pi y}{b}\right)^2 = \frac{B}{u} \\
\cos \beta z = Bu
\end{cases}
\]

And we obtain a family of electric curves perpendicular to the precedent which was the family of magnetic curves (Figure 6.5).

\[\text{Figure 6.5. Electric and magnetic field lines for the TM}_{11}\text{ mode in a rectangular guide}\]

We can do the same in the planes \(y = C^{te}\) and \(z = C^{te}\).

6.3. **TE rectangular guide**

6.3.1. **The fields**

Now in the case of the **TE** mode, the component is \(E_Z = 0\) and we have to determine \(H_Z(x, y)\), which is a solution of the same propagation equation as for **TE** waves:

\[
\Delta_r H_Z(x, y) + (k^2 - \beta^2) H_Z(x, y) = 0
\]
that is
\[ \frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + k_c^2 H_z = 0 \]

with \( k_c^2 = k^2 - \beta^2 \). As in the case of \( TM \), we look for a solution by separating the two variables \( x \) and \( y \):

\[ H_z(x, y) = F(x)G(y) \]

Then
\[ \frac{F'}{F}(x) + \frac{G'}{G}(y) + k_c^2 = 0 \]

We put
\[ k_c^2 = k_x^2 + k_y^2 \]

where \( k_x^2 \) and \( k_y^2 \) are functions of \( x \) and \( y \), respectively. We now have to find the solutions of two differential equations:

\[
\begin{align*}
F'(x) + k_x^2 F(x) &= 0 \\
G'(y) + k_y^2 G(y) &= 0
\end{align*}
\]

which give two general solutions:

\[
\begin{align*}
F(x) &= A \cos k_x x + B \sin k_x x \\
G(y) &= C \cos k_y y + D \sin k_y y
\end{align*}
\]

Then, the field \( H_z(x, y) \) is to be of the form:

\[ H_z(x, y) = [A \cos k_x x + B \sin k_x x][C \cos k_y y + D \sin k_y y] \]
Now we have to impose the limiting conditions which must be (Figure 6.1):

\[ \vec{n}. \text{grad} \vec{H}_z = \frac{\partial H_z}{\partial n} = 0 \quad \text{on the sides} \]

This means that on the sides we must have

\[ \begin{align*}
\frac{\partial H_z}{\partial x} &= 0; \quad x = 0 \text{ and } x = a \quad \forall y \\
\frac{\partial H_z}{\partial y} &= 0; \quad y = 0 \text{ and } y = b \quad \forall x
\end{align*} \]

To verify these limiting conditions, we must have:

\[ B = D = 0 \]

\[ k_x = \frac{m\pi}{a}; \quad k_y = \frac{n\pi}{b} \]

We can say that we have a TE\(_{mn}\) propagation.

With the propagation factor \( e^{j(\omega t - \beta z)} \):

\[ H_z(x, y) = H_0 \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{j(\omega t - \beta z)} \]

The other electric fields are given with the formula:

\[ \vec{E}_r = \frac{j \beta}{k_c} \text{grad}_r \vec{H}_z \]

and

\[ \vec{E}_r = \frac{j k Z}{k_c} [\vec{u} \wedge \text{grad}_r \vec{H}_z] \]
6.3.2. The dispersive relation

The constant $k_c$ depends only on the integer values of the couple $(m,n)$. We have the same relation of dispersion as for the $TM_{mn}$ fields. In the rectangular guides, the $TE_{mn}$ and the $TM_{mn}$ are degenerated.

$$k_{cmn}^2 = k^2 - \beta^2 = k_r^2 + k_i^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

The fields are multiplied by the factor:

$$e^{j(\omega \tau - \beta_{mn} \tau)}$$

And $\beta_{mn}$ is positive and real. Then to assume propagation, we must have

$$\beta_{mn}^2 = k^2 - k_{cmn}^2 > 0$$

This induces

$$k^2 > k_{cmn}^2, \quad \frac{2\pi}{\lambda} > \frac{2\pi}{\lambda_{cmn}}$$

As for $TM$ waves, the length wave has a maximum

$$\lambda < \lambda_{cmn}$$

Now let us return to the integer value of $\beta$ and we obtain the relation of dispersion:

$$\beta_{mn} = \sqrt{\omega^2 \mu \varepsilon - \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)}$$
There is no propagation if $\beta_{mn} = 0$. Then we say that we are at the cutoff and

$$\omega_{C_{mn}} = \frac{\pi}{\sqrt{\mu \varepsilon}} \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

or

$$f_{C_{mn}} = \frac{1}{2\sqrt{\mu \varepsilon}} \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

The fundamental mode corresponds to the lower frequency of cutoff. If $a < b$, this corresponds to $m=0$ and $n = 1$. The fundamental is the $TE_{01}$:

$$f_{C_{01}} = \frac{1}{2b\sqrt{\mu \varepsilon}}$$

On the contrary, if $a > b$, the fundamental will be the $TE_{10}$ ($m = 1$ and $n = 0$).

$$f_{C_{10}} = \frac{1}{2a\sqrt{\mu \varepsilon}}$$

We observe that the $TE_{mn}$ exist even if $m = 0$ or $n = 0$. The $TE_{0n}$ and the $TE_{m0}$ are not degenerated.

### 6.3.3. The power flux

Taking the results of Chapter 5, the power flux is with $E_Z = 0$:

$$\bar{p} = \frac{1}{2} \frac{Z k \beta}{k_c^2} \int \int_{S} H_Z^2 ds$$
Let us consider, for example, the second equation we have:

\[
P = \frac{1}{2} \frac{Z k \beta}{k_C^2} H_0^2 \int_0^a \int_0^b \cos^2 \left( \frac{m\pi x}{a} \right) \cos^2 \left( \frac{n\pi y}{b} \right) ds
\]

And with:

\[
\cos^2 X = \frac{1}{2} (1 + \cos 2X)
\]

we obtain:

\[
P = \frac{1}{8} \frac{a b Z k \beta_{mn}}{k_{Cmn}^2} H_0^2
\]

### 6.3.4. Attenuation of the fundamental \( m = 0 \) and \( n = 1 \)

The result of the attenuation of the \( TE_{mn} \) is more complicated because we have to determine the integral

\[
\Phi \left( H_Z^2 + \frac{\beta^2}{k_C^4} \left( \frac{dH_Z}{dn} \right)^2 \right) dl.
\]

\[
\alpha_{TE_{mn}} = \frac{1}{2} \frac{1}{Z \sigma \delta} \frac{k^2 - \beta^2}{k_C^4} \frac{\Phi \left( H_Z^2 + \frac{\beta^2}{k_C^4} \left( \frac{dH_Z}{dn} \right)^2 \right)}{\int_S H_Z^2 ds}
\]

We will give only the simple attenuation of the fundamental:

\[
\alpha_{TE_{01}} = \frac{R_S}{ab \beta_{10} k Z} \left[ 2b k_C^2 + ak^2 \right]
\]
6.4. Problems

6.4.1. The fundamental TE\textsubscript{01} mode of the rectangular guide

1) Give the electric and magnetic waves for a rectangular guide of dimensions \(a\) and \(b\) \((a < b)\) in the case of the TE\textsubscript{0n} modes. Consider the particular case of the fundamental TE\textsubscript{01}

2) Find and give the magnetic and electric lines in the case of the TE\textsubscript{0n} modes (particular case of the TE\textsubscript{01}).

Solutions.–

1) The TE\textsubscript{0n} modes correspond to:

\[
\begin{align*}
E_Z &= 0 \\
H_Z &= H_0 \cos \left( \frac{n\pi}{b} y \right)
\end{align*}
\]

with two perpendicular fields:

\[
\begin{align*}
\vec{E}_T &= j \frac{kZ}{k_c^2} \left( \uvec{u} \times \text{grad}_z H_z \right) \\
\vec{H}_T &= -j \frac{\beta}{k_c^2} \text{grad}_t H_z
\end{align*}
\]

and with dispersion:

\[
k_{c0n}^2 = k^2 - \beta_{0n}^2 = \frac{n\pi}{b}
\]

Then from the vectors \(\vec{H}_T\) and \(\vec{E}_T\), we have, respectively:

\[
\begin{align*}
H_x &= 0 \\
H_y &= j \beta \frac{b}{n\pi} H_0 \sin \left( \frac{n\pi}{b} y \right)
\end{align*}
\]
\[
\begin{align*}
E_x &= jkZ \frac{b}{n\pi} H_0 \sin\left(\frac{n\pi}{b} y\right) \\
E_y &= 0
\end{align*}
\]

with \( kZ = \omega \sqrt{\frac{\mu}{\varepsilon}} \sqrt{\frac{\mu}{\varepsilon}} = \omega \mu \)

We only have three components: \( E_x, H_y, H_z \).

In the case of the fundamental \( TE_{01} \) \((a < b)\), we have:

\[
f_{c_{01}} = \frac{c}{2b}; \quad \lambda_{c_{01}} = 2b
\]

2) Electric and magnetic lines in the case of the \( TE_{0n} \).

We recall that the equations of the magnetic (electric) lines are given by the formulas:

\[
\frac{dx}{\Re_e(H_x)} = \frac{dy}{\Re_e(H_y)} = \frac{dz}{\Re_e(H_z)}
\]

In our case, we have only three components \( E_x, H_y, H_z \). This leads us to determine the magnetic lines in the plane \((yz)\).

*Magnetic lines in the plane \((yz)\)*

We have to fit:

\[
\frac{dy}{\Re_e(H_y)} = \frac{dz}{\Re_e(H_z)}
\]
with:

\[
\begin{align*}
H_y &= j \beta \frac{b}{n\pi} H_0 \sin \left( \frac{n\pi}{b} y \right) e^{j(\omega t - \beta z)} \\
\Re_e(H_y) &= -\beta \frac{b}{n\pi} H_0 \sin \left( \frac{n\pi}{b} y \right) \sin(\omega t - \beta z) \\
H_z &= H_0 \cos \left( \frac{n\pi}{b} y \right) e^{j(\omega t - \beta z)} \\
\Re_e(H_z) &= H_0 \cos \left( \frac{n\pi}{b} y \right) \cos(\omega t - \beta z)
\end{align*}
\]

This means that

\[
\frac{n\pi}{b} \cos \left( \frac{n\pi}{b} y \right) dy = -\beta \sin \left( \frac{n\pi}{b} y \right) \sin(\omega t - \beta z) dz
\]

Integrating now

\[
\log \left[ \sin \left( \frac{n\pi}{b} y \right) \right] = -\log \left[ \cos(\omega t - \beta z) \right] \pm B^2
\]

and at the beginning \( t = 0 \):

\[
\sin \left( \frac{n\pi}{b} y \right) \cos(\beta z) = \pm A^2
\]

In the case of the \( TE_{01} \) mode, we have to give \( y \) in function of \( \beta z \) (the constant \( A \) must be <1):

\[
\sin \left( \frac{\pi}{b} y \right) \cos(\beta z) = \pm A^2
\]
The curve is studied by using an auxiliary variable $u$ as:

$$\begin{align*}
\sin \left( \frac{\pi}{b} y \right) &= A u \\
\cos (\beta z) &= \pm \frac{A}{u}
\end{align*}$$

Figure 6.6. Magnetic field lines for the dominant TE$_{01}$ mode in the rectangular guide

The signs $-$ and $+$ correspond to the domains $\left( \frac{\lambda_g}{4}, \frac{3\lambda_g}{4} \right)$ and $\left( 0, \frac{\lambda_g}{4} \right)$ with $\left( \frac{\lambda_g}{4}, \lambda_g \right)$ respectively.

Electric lines in the planes (x0y) and (x0z)

There is only one electric field and with the propagation constant, we have:

$$E_x = jkZ \frac{b}{n\pi} H_0 \sin \left( \frac{n\pi}{b} y \right) e^{j(\omega t - \beta z)}$$
Then:

\[
\Re_e(E_x) = -jkZ \frac{b}{n\pi} H_0 \sin\left(\frac{n\pi}{b} y\right) \sin(\omega t - \beta z)
\]

\(E_x\) varies in:

\[
\begin{cases}
\sin\left(\frac{\pi}{b} y\right) \text{ order } y \text{ if } n=1.
\sin\left(\frac{2\pi}{\lambda_x} z\right) \text{ order } z
\end{cases}
\]

The first variation is described by the curve shown in Figure 6.7, while the second variation is described by the curve shown in Figure 6.8.

**Figure 6.7. Variation of \(E_x\) versus \(y\)**

**6.4.2. Rectangular TE\(_{01}\) guide with dielectric**

To force the propagation in a rectangular guide of dimensions:

\[
a = 1.016 \text{ cm and } b = 2.286 \text{ cm}
\]
of the fundamental electromagnetic mode $TE_{01}$ at 5 GHz, we introduce a dielectric $\varepsilon_r$ into it.

1) What are the variations of $\varepsilon_r$ to only have in $TE_{01}$ mode?

2) We take $\varepsilon_r = 2.25$. What are the length waves in the guide?

Figure 6.8. Variation of $E_x$ versus $Z$

**Solutions.**

1) The equation of propagation is:

$$k_C^2 = k^2 - \beta^2$$

$$\left(\frac{2\pi}{\lambda_C}\right)^2 = \left(\frac{2\pi}{\lambda}\right)^2 - \left(\frac{2\pi}{\lambda_g}\right)^2$$

$$\frac{1}{\lambda_C^2} = \frac{1}{\lambda^2} - \frac{1}{\lambda_g^2}$$

Then:

$$\lambda_g = \frac{\lambda_C \lambda}{\sqrt{\lambda_C^2 - \lambda^2}}$$
We have propagation if $\lambda_c > \lambda$ with $\lambda = \frac{v}{f}$ and

$$v = \frac{1}{\sqrt{\varepsilon \mu_0}} = \frac{1}{\sqrt{\varepsilon_0 \varepsilon_r \mu_0}}.$$ 

In the case of the $TE_{mn}$ modes, we have:

$$\frac{1}{\lambda_c^2} = \left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2$$

where $\lambda_c$ is only a function of the guide dimensions and it is a constant for a given mode $TE_{mn}$.

In the case of the fundamental $TE_{01}$:

$$\frac{1}{\lambda_c^2} = \left(\frac{1}{2b}\right)^2$$

This gives

$$\lambda_c = 2b$$

We must have

$$4b^2 > \frac{1}{f^2 \varepsilon_0 \varepsilon_r \mu_0}$$

or

$$\varepsilon_r > \frac{1}{4b^2 f^2 \varepsilon_0 \mu_0}$$

that is:

$$\varepsilon_r > 1.73$$
2) We have to determine the next propagating mode.

We have for the first modes:

\[ TE_{01} : f_c^2 = \frac{v^2}{4b^2} \]
\[ TE_{10} : f_c^2 = \frac{v^2}{4a^2} \]
\[ TE_{02} : f_c^2 = \frac{v^2}{b^2} \]

.........etc

But

\[ b > 2a \Rightarrow b^2 > 4a^2 \Rightarrow f_c(TE_{02}) < f_c(TE_{10}) \]

And \( f_c(TE_{02}) \) corresponds to an \( \varepsilon_r = 4(1.73) = 6.92 \)

Then we must have:

\[ 1.73 < \varepsilon_r < 6.92 \]

This corresponds to the repartition:

\[
\begin{array}{c|c|c}
& TE_{01} & TE_{02} & TE_{10} \\
\varepsilon_r = & 1.73 & 6.92 & \\
\end{array}
\]

**Figure 6.9. Repartition of the modes**
6.5. Bibliography


7

Circular TM and TE Guides

7.1. Introduction

In this chapter, we will study the circular guide successively with TE and TM waves (Figure 7.1). Then we will give the expressions of the different fields. Due to the property of the Bessel functions and in contrast to the rectangular guide, we will see that only the TE$_{0n}$ and the TM$_{1n}$ are degenerated. Also, the TE$_{11}$ (which is not degenerated) is the fundamental mode.

Figure 7.1. The circular waveguide

7.2. Properties of the TE and TM circular waveguide

We have to give a solution of the propagation equation:

$$\Delta_r \psi(r, \theta) + \left(k^2 - \beta^2 \right) \psi(r, \theta) = 0$$
where $\psi(r, \theta)$ is the $TE$ or the $TM$ wave. Then using the Laplacian expression in cylindrical coordinates, we obtain:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \left( k^2 - \beta^2 \right) \psi = 0$$

With the limiting conditions:

$$\begin{align*}
\text{if } \psi(r, \theta) = E_z & \quad \text{then } \psi(a, \theta) = 0 \quad TM \\
\text{if } \psi(r, \theta) = H_z & \quad \text{then } \frac{\partial \psi(a, \theta)}{\partial r} = 0 \quad TE
\end{align*}$$

These limiting conditions state that the electric field is perpendicular to the conductor or the magnetic field is tangent to this same conductor.

To find a solution, we separate the two variables:

$$\psi(r, \theta) = F(r) \cdot G(\theta)$$

and by using $k_c^2 = k^2 - \beta^2$, we obtain:

$$\frac{r}{F(r)} \frac{d}{dr} \left( r \frac{dF(r)}{dr} \right) + k_c^2 r^2 + \frac{1}{G(\theta)} \frac{d^2 G(\theta)}{d\theta^2} = 0$$

The first two terms are only a function of the variable $r$, and if they are equal to a constant $v^2$, then the last term is only a function of $\theta$ and is equal to the constant $-v^2$.

The solution is:

$$\begin{align*}
\frac{r}{F(r)} \frac{d}{dr} \left( r \frac{dF(r)}{dr} \right) + k_c^2 r^2 &= v^2 \\
\frac{1}{G(\theta)} \frac{d^2 G(\theta)}{d\theta^2} &= -v^2
\end{align*}$$
The last equation in $\theta$ is with the solution:

\[
\begin{align*}
G''(\theta) + \nu^2 G(\theta) &= 0 \\
G(\theta) &= A \cos \nu \theta + B \sin \nu \theta
\end{align*}
\]

We must have a symmetry revolution:

\[G(\theta + 2\pi) = G(\theta)\]

then $2\pi \nu = 2\pi m$ with $\nu = m \in \mathbb{N}$. We choose the solution:

\[G(\theta) = \cos m \theta\]

We observe that if we turn the axes of an angle of $\frac{\pi}{2}$, we have a solution:

\[G(\theta) = \sin m \theta\]

Then we can say that the polarization is different from $\frac{\pi}{2}$.

Now, the first equation in $r$ with the transformation of the variable is:

\[
\begin{align*}
\rho &= k_c r \\
\frac{d^2 F}{d \rho^2} + \frac{1}{\rho} \frac{d F}{d \rho} + \left(1 - \frac{m^2}{\rho^2}\right) F(\rho) &= 0
\end{align*}
\]

We recover the Bessel equation. The solution is a combination of the Bessel functions of the first kind $J_m(\rho)$ and the second kind $N_m(\rho)$. But $N_m(\rho)$ is infinite when $\rho = 0$, then the solution is $J_m(\rho)$.
And the general solution is:

\[
\psi (r, \theta) = \begin{cases} 
E_Z, & \cos \psi \theta = \theta_m C_m E_r J_k r, \\
0, & \end{cases} 
\]

\[
H_Z = \psi_0 J_m (k_c r) \cos m\theta 
\]

7.3. **TM circular waveguide**

The TM circular waves are:

\[
E_Z (r, \theta) = E_0 J_m (k_c r) \cos m\theta 
\]

The limiting conditions impose on the conductor:

\[
E_Z (a, \theta) = 0 \quad \forall \theta 
\]

Then:

\[
J_m (k_c a) = 0 
\]

This means:

\[
k_c a = x_{mn} 
\]

where \(x_{mn}\) is the \(n\)th root of \(J_m\). The roots \(x_{mn}\) are given in tabular format; for example, we give the first roots of the first Bessel functions in Figure 7.2.

\[
\begin{array}{|c|c|c|c|}
\hline
& x_{m1} & x_{m2} & x_{m3} \\
\hline
J_0 & 2.405 & 5.520 & 8.654 \\
J_1 & 3.832 & 7.016 & 10.173 \\
J_2 & 5.135 & 8.417 & 11.620 \\
\hline
\end{array}
\]

**Figure 7.2. Zeros \(x_{mn}\) of the first Bessel functions \(J_m\)**
The wavelength and the frequency at the cutoff are:

\[ \lambda_{\text{cmn}} = \frac{2\pi a}{x_{\text{cmn}}} \quad \text{and} \quad f_{\text{cmn}} = \frac{v_{\text{cmn}}}{2\pi a} \]

Then we have the mode:

\[ TM_{mn} \]

With a dispersive relation:

\[ \omega = \sqrt{\beta^2 + \frac{x_{\text{cmn}}^2}{a^2}} \]

Now we give the expression of the different waves that are given from:

\[
\begin{cases}
\vec{E}_{Tmn} = -j\frac{\beta}{k^2_{\text{cmn}}} \text{grad} \vec{E}_{Zmn} \\
\vec{H}_{Tmn} = -j\frac{k}{Zk^2_{\text{cmn}}} (\vec{u} \wedge \text{grad} \vec{E}_{Zmn})
\end{cases}
\]

This gives:

\[
\begin{cases}
E_{rmn} = -j\frac{\beta}{k^2_{\text{cmn}}} E_0 J_m \left(k_{\text{cmn}} r\right) \cos m\theta \\
E_{\theta mn} = j\frac{\beta}{k^2_{\text{cmn}}} E_0 \frac{m}{r} J_m \left(k_{\text{cmn}} r\right) \sin m\theta \\
E_{Zmn} = E_0 J_m \left(k_{\text{cmn}} r\right) \cos m\theta
\end{cases}
\]
\[
H_{r_{mn}} = -j \frac{k}{k_{cmn}^2} E_0 m \frac{Z}{r} J_m(k_{cmn} r) \sin m\theta \\
H_{\theta_{mn}} = -j \frac{k}{k_{cmn}} E_0 m \frac{Z}{r} J'_m(k_{cmn} r) \cos m\theta \\
H_{Z_{mn}} = 0
\]

and the \textit{TM} impedance:

\[
Z_E = \frac{E_r}{H_\theta} = -\frac{E_\theta}{H_r} = Z \frac{\beta}{k}
\]

\section*{7.4. \textit{TE} circular waveguide}

The \textit{TE} circular waves are given from the values of the \(H_z(r,\theta)\):

\[
H_z(r,\theta) = H_0 J_m(k_c r) \cos m\theta
\]

In this case, the limiting conditions impose:

\[
\left. \frac{\partial H_z}{\partial r} \right|_{r=a} = 0
\]

This means:

\[
J'_m(k_c a) = 0
\]

or:

\[
k_c a = x'_{mn}
\]
where $x_{mn}$ is the $n$th root of $J'_m$.

<table>
<thead>
<tr>
<th></th>
<th>$x'_{m1}$</th>
<th>$x'_{m2}$</th>
<th>$x'_{m3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J'_0$</td>
<td>3.832</td>
<td>7.016</td>
<td>10.173</td>
</tr>
<tr>
<td>$J'_1$</td>
<td>[1.841]</td>
<td>5.331</td>
<td>8.536</td>
</tr>
<tr>
<td>$J'_2$</td>
<td>3.054</td>
<td>6.706</td>
<td>9.969</td>
</tr>
</tbody>
</table>

**Figure 7.3.** Zeros $x'_{mn}$ of the first derivatives of the Bessel functions $J'_m$

The wavelength and the frequency at the cutoff are:

$$\lambda_{c mn} = \frac{2\pi a}{x'_{mn}} \quad \text{and} \quad f_{c mn} = \frac{v x'_{mn}}{2\pi a}$$

Then we have the mode:

$$TE_{mn}$$

with a dispersive relation:

$$\omega = \sqrt{v^2 \beta^2 + \frac{x'^2_{mn}}{a^2}}$$

As in the case of TM waves, we give the expression of the different waves that are given from:

$$\begin{cases}
\vec{E}_{Tmn} = j \frac{kZ}{k_{Cmn}^2} (\vec{u} \wedge \nabla \vec{d}H_{Zmn}) \\
\vec{H}_{Tmn} = -j \frac{\beta_{mn}}{k_{Cmn}^2} \nabla \vec{d}H_{Zmn}
\end{cases}$$
This gives:

\[
\begin{align*}
E_{r,mn} &= j \frac{kZ}{k_{C,mn}^2} H_0 \frac{m}{r} J_m(k_{C,mn}'r) \sin m \theta \\
E_{\theta,mn} &= j \frac{kZ}{k_{C,mn}'} H_0 J_m(k_{C,mn}'r) \cos m \theta \\
E_{Z,mn} &= 0
\end{align*}
\]

\[
\begin{align*}
H_{r,mn} &= -j \frac{\beta}{k_{C,mn}'} H_0 J_m(k_{C,mn}'r) \cos m \theta \\
H_{\theta,mn} &= j \frac{\beta}{k_{C,mn}'} H_0 \frac{m}{r} J_m(k_{C,mn}'r) \sin m \theta \\
H_{Z,mn} &= H_0 J_m(k_{C,mn}'r) \cos m \theta
\end{align*}
\]

and the TE impedance:

\[
Z_H = \frac{E_r}{H_\theta} = -\frac{E_\theta}{H_r} = Z \frac{k}{\beta}
\]

7.5. Fundamental mode and classification of the modes

From Figures 7.2 and 7.3, we see that the fundamental mode is the TE_{11} because:

\[x_{11}' < x_{01}\]

We also see that the TE_{0m} are degenerated with the TM_{1m} and the first modes are given in Figure 7.4.
The modes $TE_{0m}$ and $TM_{0m}$ have a symmetry revolution because they do not depend on $\theta (m=0)$. We also give the Bessel function aspect of $m$ order with $m \neq 0$ (Figure 7.5).

$$m \neq 0$$

**Figure 7.4. Classification of the first modes**

**Figure 7.5. Bessel function of $m$ order**
We can see that:

\[ x'_{mn} < x_{mn} \]

Then:

\[ \omega_c \left( TE_{mn} \right) < \omega_c \left( TM_{mn} \right) \]

which is a generalization of Figure 7.4.

\[ m = 0 \]

The aspect of \( J_0 \) and of \( J_1 \) gives properties of the case \( m = 0 \).

Figure 7.6. Aspect of \( J_0 \) and \( J_1 \)

We have:

\[ x'_{0n} = x_{1n} \]

because:

\[ J'_0(x) = -J_1(x) \]
and

\[ x_{0n} < x_{1n}' = x_{1n} \]

This gives:

\[ \omega_c (TE_{0n}) = \omega_c (TM_{1n}) > \omega_c (TM_{0n}) \]

### 7.6. Utilization band of the fundamental mode \( TE_{11} \)

The cutoff frequency of the fundamental is in \( Hz \) if \( a \) is given in \( m \):

\[ f_c (TE_{11}) = \frac{1.84}{2\pi \sqrt{\mu \varepsilon}} \frac{Hz}{a(m)} \]

A more appropriate dimension is in \( GHz \) with \( a \) in \( cm \). We have in the air:

\[ f_c (TE_{11}) = \frac{8.791}{a(cm)} GHz \]

We do the same for the next \( TM_{01} \) mode:

\[ f_c (TM_{01}) = \frac{11.483}{a(cm)} GHz \]

This gives the utilization band in the case of the fundamental mode:

\[ \frac{8.791}{a(cm)} < f_c (GHz) < \frac{11.483}{a(cm)} \]

In fact, the practical band is weak because we have important losses near the cutoff.
7.7. Field lines of the first modes

7.7.1. The fundamental $TE_{11}$

7.7.2. $TM_{01}$ Symmetry of revolution

There is a symmetry of revolution. Also the field $E$ is zero on the axis of the cylindrical because $J'(0) = 0$.

7.7.3. $TE_{21}$ Quadrupolar mode

We have a quadrupolar mode (the $TE_{21}$ has four arches). In the case of the $TE_{31}$ we have six arches and the fields are sixpolar.
7.7.4. **TE\(_{01}\) Degenerated mode with the TM\(_{11}\)**

This is the only mode without an electric field on the conductors. Then there are no superficial currents on these conductors and the current only depends on \(\theta\).

We will see that the attenuation of the \(TE_{01}\) is zero at high frequencies (60–100 GHz) and that is why these are used to simultaneously transport a lot of communications (500,000 or more).

7.7.5. **TM\(_{11}\) Degenerated mode with the TE\(_{01}\)**

7.8. **Power flux and attenuations**

We give without demonstration the results of the two power fluxes (\(TE_{mn}\) and \(TM_{mn}\)) and the two attenuations (\(TE_{mn}\) and \(TM_{mn}\)).
In fact, the demonstrations use the waves given in [7.3].

\[
\begin{align*}
\mathcal{P}_{TE_{mn}} &= \frac{1}{4} k \frac{\beta_{mn}}{\sigma_{mn}^2} S Z H_0^2 \left(1 - \frac{m^2}{\sigma_{mn}^2}\right) J_m^2(x_{mn}) \\
\mathcal{P}_{TM_{mn}} &= \frac{1}{4} k \frac{\beta_{mn}}{\sigma_{mn}^2} S \frac{E_0^2}{Z} J_m(x_{mn})
\end{align*}
\]

\[
\begin{align*}
\alpha_{TE_{mn}} &= \frac{R_S}{a Z} \left[1 - \left(\frac{x_{mn}'}{ka}\right)^2\right] \frac{1}{2} \left[\frac{x_{mn}^2}{(ka)^2} + \frac{m^2}{x_{mn}^2 - m^2}\right] \\
\alpha_{TM_{mn}} &= \frac{R_S}{a Z} \sqrt{1 - \left(\frac{x_{mn}}{ka}\right)^2}
\end{align*}
\]

In the two cases (power flux and attenuation), the expressions of the \(TM_{mn}\) are simpler but the fundamental is a \(TE_{mn}\) mode.

We can also note that the behavior of the attenuation of the \(TE_{mn}\) when we have large values of the frequency \((\omega \to \infty)\) is of the form:

\[
\alpha_{TE_{mn}} \approx \sqrt[3]{\omega} \left[\frac{a_1}{\omega^2} + a_2 m^2\right]
\]

where \(a_1\) and \(a_2\) are two constants.

- If \(m \neq 0\), the behavior of \(\alpha_{TE_{mn}}\) is \(\alpha_{TE_{mn}} \approx \sqrt[3]{\omega}\) and the attenuation grows as \(\omega^{\frac{1}{2}}\).

- If \(m = 0\), the behavior of \(\alpha_{TE_{0n}}\) is \(\alpha_{TE_{0n}} \approx \frac{1}{\omega^{\frac{7}{3}}}\) and the attenuation decreases as \(\omega^{-\frac{3}{2}}\). This is an important property.
of the $TE_{0n}$ and they are used at high frequencies to transport a lot of communications.

7.9. Problems

7.9.1. Semi-circular and quadrant guide

From the spectrum ($TE$ and $TM$) of the circular guide, find the modes that propagate:

1) in the semi-circular guide;
2) in the quadrant guide.

Do not forget that in the circular guide the $TE_{mn}$ and the $TM_{mn}$ can have two polarizations. Then the variation versus $\theta$ takes the general form:

$$A \cos m \theta + B \sin m \theta$$

SOLUTION.--

1) Semi-circular guide

In the case of the TM modes, we have:

$$E_z = E_0 J_m (k_c r) [A \cos m \theta + B \sin m \theta]$$
or:

\[ E_Z = \hat{E} \left[ A \cos m\theta + B \sin m\theta \right] \]

But the electric behavior must be zero on the conductors:

\[ E_Z = 0 \text{ for } \theta = \frac{\pi}{2} \]

\[
\begin{cases}
  \text{m even: } m = 2p \quad E_Z \left(\frac{\pi}{2}\right) = \hat{E} \left[ A \cos p\pi + B \sin p\pi \right] \Rightarrow A = 0 \\
  \text{m odd: } m = 2p + 1 \quad E_Z \left(\frac{\pi}{2}\right) = \hat{E} \left[ A \cos \left(p\pi + \frac{\pi}{2}\right) + B \sin \left(p\pi + \frac{\pi}{2}\right) \right] \\
  \Rightarrow B = 0
\end{cases}
\]

Then in the TM mode with even case \( m = 2p \):

\[ E_Z = E_0 J_{2p} (k_c r) \sin 2p\theta \]

and in the TM mode with odd case \( m = 2p+1 \):

\[ E_Z = E_0 J_{2p+1} (k_c r) \cos (2p+1)\theta \]

All the values of \( p \) are possible unlike in the case of \( p = 0 \) because at this time \( E_z \) will be zero and the mode does not exist.

\textit{In the case of the TE modes, we have:}

\[ H_Z = H_0 J_m (k_c r) \left[ A \cos m\theta + B \sin m\theta \right] \]

or

\[ H_Z = \hat{H} \left[ A \cos m\theta + B \sin m\theta \right] \]
The limiting conditions are different:

\[ \frac{\partial H_Z}{\partial \theta} = 0 \quad \text{when } \theta = \frac{\pi}{2} \]

\[ \frac{\partial H_Z}{\partial \theta} = m \hat{H} [-A \sin m\theta + B \cos m\theta] \]

\[ \begin{cases} m \text{ even: } m = 2p & \Rightarrow B = 0 \\ m \text{ odd: } m = 2p + 1 & \Rightarrow A = 0 \end{cases} \]

Then in the TE mode with the even case \( m = 2p \):

\[ H_Z = H_0 J_{2p}(k_c r) \cos 2p\theta \]

and in the TE mode with the odd case \( m = 2p+1 \):

\[ H_Z = H_0 J_{2p+1}(k_c r) \sin(2p+1)\theta \]

All the values of \( p \) are possible.

2) Quadrantal guide

**TM modes**

\[ E_z = \hat{E} [A \cos m\theta + B \sin m\theta] \]

And we have to satisfy the limiting conditions:

\[ E_z = 0 \text{ for } \theta = 0, \frac{\pi}{2} \]

It gives:

\[ A = 0, m = 2p \]
With a solution:

\[ E_Z = E_0 J_2 (k_c r) \sin 2p \theta \]

We have the modes \( TM_{2p,n} \) with \( p \neq 0 \).

**TE modes**

Using the same method, we find:

\[ H_Z = H_0 J_2 (k_c r) \cos 2p \theta \]

We have the modes \( TE_{2p,n} \) with all values of \( p \).

### 7.9.2. Angle \( \alpha \) Guide

We consider a guide with a section of angle \( \alpha \). Give the corresponding \( TM \) and \( TE \) waves.

\[
\psi = \psi_0 J_\nu(k_c r) \begin{bmatrix} \cos \nu \theta \\ \sin \nu \theta \end{bmatrix}
\]

**SOLUTION.**

In general, we have fields of the form:
**TM fields**

In this we have electric fields $\psi = E_z$ with:

\[
E_z = E_0 J_\nu (k_c r) \sin \nu \theta \quad \text{because} \quad E_z = 0 \quad \text{for} \quad \theta = 0
\]

We must have:

\[
\sin \nu \theta = 0
\]

that is:

\[
\nu = p \frac{\pi}{\alpha}
\]

Then we have for the TM modes:

\[
\begin{cases}
\text{TM} & \frac{\pi n}{2}, \quad \text{without} \quad p = 0 \\
E_z = E_0 J_\nu \left( k_c r \right) \sin \left( p \frac{\pi}{\alpha} \theta \right) & \text{for} \quad p \neq 0
\end{cases}
\]

**TE fields**

In this case, we have magnetic fields $\psi = H_z$ with:

\[
H_z = H_0 J_\nu (k_c r) \cos \nu \theta \quad \text{because} \quad \frac{\partial H_z(0)}{\partial \theta} = 0
\]

and we must have:

\[
\frac{\partial H_z}{\partial \theta} = -\nu H_0 J_\nu (k_c r) \sin \nu \theta = 0
\]
Then:
\[
\nu = p \frac{\pi}{\alpha}
\]

and we have for the TE modes:
\[
\begin{cases}
  TE & p = 0 \text{ is permitted} \\
  H_z = H_0 J_k \frac{p^2}{r} (k_r r) \cos(p \frac{\pi}{\alpha} \theta)
\end{cases}
\]

If \( \alpha = \frac{\pi}{2} \) and if \( \alpha = \pi \), we recover the precedent results.

And when \( \alpha = 2\pi \), we have the classic results of the circular guide.

### 7.9.3. Computing the power flux and the attenuations for TM and TE fields

Compute this power flux and attenuation by using the result (Lommel’s integral):
\[
\int_0^a J_m^2(k r) r dr = a^2 \left[ J_m^2(k a) + \left( 1 - \frac{m^2}{k^2 a^2} \right) J_m^2(k a) \right]
\]

### 7.10. Bibliography


PART 3
Cavities
8.1. Introduction

Without loss of generality and to understand the phenomena, we consider the simple case of the fundamental incident wave $TE_{01}$. It will be easier to consider after any sort of $TE_{m,n,p}$ or $TM_{m,n,p}$ wave.

8.2. The fundamental waves

We recall that the fundamental wave $TE_{01}$ is formed by two magnetic fields $H_z$ and $H_y$ and one electric field $E_x$:

\[
\begin{align*}
H_z &= H_0 \cos \left( \frac{\pi}{b} y \right) e^{-j\beta z} \\
H_y &= j \beta \frac{b}{\pi} H_0 \sin \left( \frac{\pi}{b} y \right) e^{-j\beta z} \\
E_x &= jkZ \frac{b}{\pi} H_0 \sin \left( \frac{\pi}{b} y \right) e^{-j\beta z}
\end{align*}
\]
8.3. Construction of the cavity

To simplify, we first consider a magnetic plane at \( z = 0 \) (Figure 8.1).

![The first magnetic plane](image)

Figure 8.1. The first magnetic plane

In the case of \( TE_{01} \), we have an incident and a reflected wave for \( H_z \):

\[
\begin{align*}
H_{zr} &= H_0^- \cos \left( \frac{\pi}{b} y \right) e^{-j\beta z} \\
H_{zi} &= H_0^+ \cos \left( \frac{\pi}{b} y \right) e^{j\beta z}
\end{align*}
\]

The total magnetic wave \( H_z = H_{zr} + H_{zi} \) is then:

\[
H_z = \left[ H_0^+ e^{j\beta z} + H_0^- e^{-j\beta z} \right] \cos \left( \frac{\pi}{b} y \right)
\]

The limited condition at the origin \( z = 0 \) imposes \( H_z = 0 \) and we have \( H_0^+ = -H_0^- = H_0 \).

This gives:

\[
H_z = -H_0 \left[ e^{j\beta z} - e^{-j\beta z} \right] \cos \left( \frac{\pi}{b} y \right)
\]
and:

\[ H_z = -2 j H_0 \cos \left( \frac{\pi}{b} y \right) \sin \beta z \]

We do the same for the magnetic wave \( H_y \) and the electric wave \( E_x \). For example, in the case of the magnetic wave and remembering that going from the incident to reflected wave is equivalent to changing the sign of the propagation constant \( \beta \) into \(-\beta\):

\[
\begin{align*}
H_{yr} &= j \beta \frac{b}{\pi} H_0^- \sin \left( \frac{\pi}{b} y \right) e^{-j\beta z} \\
H_{yi} &= -j \beta \frac{b}{\pi} H_0^+ \sin \left( \frac{\pi}{b} y \right) e^{j\beta z}
\end{align*}
\]

The same limited condition imposes at \( z = 0 \) and gives a total magnetic wave \( H_y = H_{yi} + H_{yr} \):

\[ H_y = 2 j \beta \frac{b}{\pi} H_0 \sin \left( \frac{\pi}{b} y \right) \cos \beta z \]

Using the same, we also obtain:

\[ E_x = E_{xr} + E_{xi} = 2kZ \frac{b}{\pi} H_0 \sin \left( \frac{\pi}{b} y \right) \sin \beta z \]

The different waves are a product of \( \sin \beta z \) or \( \cos \beta z \) and the limiting conditions are satisfied at the origin \( z = 0 \):

\[
\begin{align*}
H_z &= 0 \\
H_y \text{ is maximum at } z = 0 \\
E_x &= 0
\end{align*}
\]
These limiting conditions are also verified for $\beta z = p\pi$ recalling that $\beta = 2\pi/\lambda_g$:

$$z = p\frac{\lambda_g}{2}$$

We can place a perfect conductor without perturbing the planes if

$$z = \frac{\lambda_g}{2}, 2\frac{\lambda_g}{2}, \ldots; p\frac{\lambda_g}{2}$$

### 8.4. The cavity

We place a perfect conductor at $z = d$. Then, we have a condition on propagation constant $\beta$:

$$H_z(z = d) = 0$$

that is

$$\sin \beta d = 0$$

and:

$$\beta = p\frac{\pi}{d}$$

Recall that in the general case $m \neq 0$ and $n \neq 0$, we have

$$k_c^2 = k^2 - \beta^2 = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}$$

Then:

$$k_{m,n,p}^2 = k_c^2 + \beta^2 = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} + \frac{p^2\pi^2}{d^2}$$
And the resonance frequency of the constructed cavity with \( k^2 = \omega^2 \mu \varepsilon \) is:

\[
\omega_{m,n,p} = \frac{\pi}{\sqrt{\mu \varepsilon}} \left[ \frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{p^2}{d^2} \right]^{1/2}
\]

And in our case, considering the \( TE_{0,1,p} \), it is:

\[
\omega_{0,1,p} = \frac{\pi}{\sqrt{\mu \varepsilon}} \left[ \frac{1}{b^2} + \frac{p^2}{d^2} \right]^{1/2}
\]

### 8.5. The waves in the cavity

By considering the time factor \( e^{j \omega t} \), the waves in the cavity are then

\[
\begin{align*}
H_Z &= -2jH_0 \cos \left( \frac{\pi}{b} y \right) \sin \left( \frac{p\pi}{d} z \right) e^{j \omega t} \\
H_y &= 2jp \frac{b}{d} H_0 \sin \left( \frac{\pi}{b} y \right) \cos \left( \frac{p\pi}{d} z \right) e^{j \omega t} \\
E_x &= 2\omega_{0,1,p} \frac{b}{\pi} \mu H_0 \sin \left( \frac{\pi}{b} y \right) \sin \left( \frac{p\pi}{d} z \right) e^{j \omega t}
\end{align*}
\]

The magnetic and the electric field lines are computed by using the real parts of the wave. We have in the case of the fundamental \( TE_{0,1,1} \):

\[
\begin{align*}
\Re_e (H_Z) &= 2H_0 \cos \left( \frac{\pi}{b} y \right) \sin \left( \frac{\pi}{d} z \right) \sin \omega t \\
\Re_e (H_y) &= -2\frac{b}{d} H_0 \sin \left( \frac{\pi}{b} y \right) \cos \left( \frac{\pi}{d} z \right) \sin \omega t \\
\Re_e (E_X) &= 2\omega_{0,1,1} \frac{b}{\pi} \mu H_0 \sin \left( \frac{\pi}{b} y \right) \sin \left( \frac{\pi}{d} z \right) \cos \omega t
\end{align*}
\]
In the case of the magnetic fields, we have to fit

\[ \frac{dy}{\mathcal{R}_e(H_y)} = \frac{dz}{\mathcal{R}_e(H_z)} \]

This means that:

\[ \frac{dz}{\cos\left(\frac{\pi}{b}y\right)\sin\left(\frac{\pi}{d}z\right)} = -\frac{dy}{b\sin\left(\frac{\pi}{b}y\right)\cos\left(\frac{\pi}{d}z\right)} \]

or:

\[ \frac{1}{d} \cos\left(\frac{\pi}{d}z\right)\frac{dz}{\sin\left(\frac{\pi}{d}z\right)} = -\frac{1}{b} \cos\left(\frac{\pi}{b}y\right)\frac{dy}{\sin\left(\frac{\pi}{b}y\right)} \]

This means that:

\[ d\left(\log\left(\sin\frac{\pi}{a}z\right)\right) = -d\left(\log\left(\sin\frac{\pi}{b}y\right)\right) \]

that is

\[ \sin\left(\frac{\pi}{d}z\right)\sin\left(\frac{\pi}{b}y\right) = \text{Cons} \tan t = \pm A^2 \]

There is only one electric field \( E_x \) and its real part is given by the next field:

\[ \mathcal{R}_e(E_x) = 2\omega_0,1,1 \mu b H_0 \sin\left(\frac{\pi}{b}y\right)\sin\left(\frac{\pi}{d}z\right)\cos\omega t \]
And we give the field lines in the two planes \((x0y)\) and \((x0z)\):

![Field lines in the two planes](image)

**Figure 8.2.** The magnetic lines from \(0\) to \(d = \frac{\lambda_g}{2}\)

![Electric lines versus y and z](image)

**Figure 8.3.** The electric lines versus \(y\) and \(z\)

### 8.6. Electric and magnetic energies in the cavity

#### 8.6.1. Electric energy

We have by definition:

\[
\bar{W}_E = \frac{1}{4} \iiint_V \varepsilon \tilde{E} \tilde{E}^* dV
\]

Within the case of the \(TE_{0,1,1}\):

\[
E = E_x = 2\omega_{0,1,1} \mu \frac{b}{\pi} H_0 \sin\left(\frac{\pi}{b} y\right) \sin\left(\frac{\pi}{d} z\right) e^{j\omega t}
\]
Then:

$$
\bar{\mathcal{W}}_E = \frac{\varepsilon}{4} \int_{0}^{b} \int_{0}^{d} \int_{0}^{a} 4 \alpha_{0,1,1}^2 \mu^2 \frac{b^2}{\pi^2} H_0^2 \sin^2 \left( \frac{\pi}{b} y \right) \sin^2 \left( \frac{\pi}{d} z \right) \ dx \ dy \ dz
$$

or:

$$
\bar{\mathcal{W}}_E = \alpha_{0,1,1}^2 \mu^2 \frac{b^2}{\pi^2} H_0^2 \int_{0}^{a} \int_{0}^{b} \int_{0}^{d} \sin^2 \left( \frac{\pi}{b} y \right) dy \int_{0}^{d} \sin^2 \left( \frac{\pi}{d} z \right) dz
$$

and using:

$$
\int_{0}^{a} \ dx = a
$$

$$
\int_{0}^{b} \sin^2 \left( \frac{\pi}{b} y \right) dy = \frac{b}{2}
$$

$$
\int_{0}^{d} \sin^2 \left( \frac{\pi}{d} z \right) dz = \frac{d}{2}
$$

we obtain, with $V = abd$ being the volume of the cavity:

$$
\bar{\mathcal{W}}_E = \frac{1}{4} \alpha_{0,1,1}^2 \mu^2 \varepsilon \frac{b^2}{\pi^2} H_0^2 abd = \frac{1}{4} \alpha_{0,1,1}^2 \mu^2 \varepsilon \frac{b^2}{\pi^2} H_0^2 V
$$

8.6.2. Magnetic energy

$$
\bar{\mathcal{W}}_M = \frac{1}{4} \iiint_{V} \mu \tilde{H} \tilde{H}^* \ dV
$$

Magnetic $TE_{0,1,1}$ vector has two components:

$$
\tilde{H} = H_y \tilde{j} + H_z \tilde{u}
$$
and

$$\tilde{H} \tilde{H}^* = H_y H_y^* + H_z H_z^*$$

So:

$$\overline{W}_M = \frac{\mu}{4} \int_0^a \int_0^d \int_0^b \left[ 4 \frac{b^2}{d^2} H_0^2 \sin^2 \left( \frac{\pi}{b} y \right) \cos^2 \left( \frac{\pi}{d} z \right) + 4 H_0^2 \cos^2 \left( \frac{\pi}{b} y \right) \sin^2 \left( \frac{\pi}{d} z \right) \right] dx dy dz$$

$$\overline{W}_M = \mu \frac{b^2}{d^2} H_0^2 \int_0^a dx \int_0^b \sin^2 \left( \frac{\pi}{b} y \right) dy \int_0^d \cos^2 \left( \frac{\pi}{d} z \right) dz$$

$$+ \mu H_0^2 \int_0^a dx \int_0^b \cos^2 \left( \frac{\pi}{b} y \right) dy \int_0^d \sin^2 \left( \frac{\pi}{d} z \right) dz$$

and using the fact that:

$$\int_0^b \sin^2 \left( \frac{\pi}{b} y \right) dy = \int_0^b \cos^2 \left( \frac{\pi}{b} y \right) dy = \frac{b}{2}$$

$$\int_0^d \cos^2 \left( \frac{\pi}{d} z \right) dz = \int_0^d \sin^2 \left( \frac{\pi}{d} z \right) dz = \frac{d}{2}$$

we have a very simple result:

$$\overline{W}_M = \frac{1}{4} \mu H_0^2 \left[ 1 + \frac{b^2}{a^2} \right] V$$

Now we will show that the two energies (electric and magnetic) are the same. The resonant frequency $\omega_{0,1,1}$ is given by

$$\omega_{0,1,1}^2 \mu \varepsilon = \pi^2 \left[ \frac{1}{b^2} + \frac{1}{d^2} \right]$$
Then:

\[ \bar{W}_E = \frac{1}{4} \mu H_0^2 \left[ 1 + \frac{b^2}{a^2} \right] V = \bar{W}_M \]

And electric plus magnetic energies in the cavity are:

\[ \bar{W} = \bar{W}_E + \bar{W}_M = 2\bar{W}_E = 2\bar{W}_M = \frac{1}{2} \mu H_0^2 \left[ 1 + \frac{b^2}{a^2} \right] V \]

8.7. Quality factor \( Q \) of the cavity

The quality factor \( Q \) of the cavity is a function of the total energy \( \bar{W} \) and the dissipating power \( p \).

\[ Q = \omega_{n,m,p} \frac{\bar{W}}{P} \]

![Figure 8.4. The cavity](image)

When we consider that there is no dielectric, then the only one dissipating power is by Joule’s effect:

\[ P = P_j = \frac{1}{2} R_s \int_S \bar{H}_g \bar{H}^*_q dS \]
where \( R_s = \frac{1}{\sigma \delta} \) is the surface impedance and \( \delta = \sqrt{\frac{2}{\mu \sigma \omega}} \) is the skin thickness of the metal.

We have to compute Joule’s dissipating powers in the three cases:

- at \( z = 0; \) \( P_j \) \( z = 0; \)

- at \( z = d; \) \( P_j \) \( z = d; \)

- the contribution of the lateral surfaces; \( P_j \) \( Z \text{lateral}; \) \( x = 0, \) \( x = a, \) \( y = 0 \) and \( y = b. \)

In the case of the \( TE_{011}, \) we have only two magnetic waves:

\[
\begin{align*}
H_z &= -2 j H_0 \cos \left( \frac{\pi}{b} y \right) \sin \left( \frac{\pi}{d} z \right) e^{j \omega t} \\
H_y &= 2 j \frac{b}{d} H_0 \sin \left( \frac{\pi}{b} y \right) \cos \left( \frac{\pi}{d} z \right) e^{j \omega t}
\end{align*}
\]

\( P_j \) \( z = 0 \) is computed using the values of the magnetic wave for \( z = 0: \)

\[
\begin{align*}
H_z &= 0 \\
H_y &= 2 j \frac{b}{a} H_0 \sin \left( \frac{\pi}{b} y \right) e^{j \omega t}
\end{align*}
\]

Then:

\[
\left[ P_j \right]_{z = 0} = R_s H_0^2 \frac{ab^3}{d^2}
\]
is computed with the values of the magnetic wave for \( z = d \):

\[
\begin{align*}
H_z &= 0 \\
H_y &= -2j \frac{b}{d} H_0 \sin\left(\frac{\pi}{b} y\right) e^{j\omega t}
\end{align*}
\]

The square values of the magnetic wave for \( z = d \) are the same as for \( z = 0 \); then:

\[
\left[ P_j \right]_{z=d} = \left[ P_j \right]_{z=0} = R_S H_0^2 \frac{ab^3}{d^2}
\]

In the case of \( x = 0 \), there are two components of the magnetic waves for the computation of \( \left[ P_j \right]_{x=0} \):

\[
\tilde{H}_{bg} = H_y \tilde{j} + H_z \tilde{u}
\]

and

\[
|H_{bg}|^2 = |H_y|^2 + |H_z|^2
\]

This gives:

\[
\left[ P_j \right]_{x=0} = 2R_S H_0^2 \int_0^d \left[ \cos^2\left(\frac{\pi}{b} y\right) \sin^2\left(\frac{\pi}{d} z\right) + \frac{b^2}{d^2} \sin^2\left(\frac{\pi}{b} y\right) \cos^2\left(\frac{\pi}{d} z\right) \right] dy \, dz
\]

or

\[
\left[ P_j \right]_{x=0} = \frac{1}{2} R_S H_0^2 \left[ bd + \frac{b^3}{d} \right]
\]

It is the same in the case of \( x = a \):

\[
\left[ P_j \right]_{x=a} = \left[ P_j \right]_{x=0} = \frac{1}{2} R_S H_0^2 \left[ bd + \frac{b^3}{d} \right]
\]
We do the same in the cases of $y = 0$ and $y = b$:

\[
\begin{bmatrix}
P_j \end{bmatrix}_{y=b} = \begin{bmatrix}
P_j \end{bmatrix}_{y=0} = R_0 H_0^2 ad
\]

In total, we have

\[
P_j = 2 \begin{bmatrix}
P_j \end{bmatrix}_{z=0} + 2 \begin{bmatrix}
P_j \end{bmatrix}_{y=0} + 2 \begin{bmatrix}
P_j \end{bmatrix}_{z=0}
\]

This gives the power:

\[
P_j = R_0 \frac{H_0^2}{d^2} \left[ 2ab^3 + bd^3 + b^3d + 2ad^3 \right]
\]

Now it is possible to compute the quality factor:

\[
Q = \frac{\omega a b d \left[ b^2 + d^2 \right]}{2R_0 \left[ 2ab^3 + bd^3 + b^3d + 2ad^3 \right]}
\]

where:

\[
\omega_{0,1} = \frac{\pi}{\sqrt{\mu \epsilon}} \left[ \frac{1}{b^2} + \frac{1}{d^2} \right]^{1/2} \quad R_0 = \frac{1}{\sigma \delta} \quad \text{and} \quad \delta = \sqrt{\frac{2}{\mu \sigma \omega_{0,1} \omega}}
\]

It is possible to determine this quality factor by the measurement of the power of the cavity as a function of frequency. We have in Figure 8.5:

\[
Q = \frac{\omega_0}{\Delta \omega} = \frac{\omega_0}{\omega_2 - \omega_1}
\]

To have a value of the quality factor, we consider the case that the cavity is perfectly symmetrical and the dielectric is the air:

\[
a = b = d \quad \text{and} \quad \mu = \mu_0, \quad \epsilon = \epsilon_0
\]
Then:

\[
Q = \frac{\sqrt{2} \pi}{6} \frac{\mu_0}{R_s \sqrt{\varepsilon_0}} \]

If we consider a cavity with dimensions and properties:

\[
a = b = d = 3 \text{ cm} \\
\sigma = 5.81 \times 10^7 \Omega /\text{m} \\
R_s = 0.022 \ \Omega
\]

then we have:

\[
f_{0,1,1} = 7.07 \ \text{GHz} \\
Q = 12700
\]

It is important to compare this result with the classical quality factors of low-frequency circuits \( L, C \) which are approximately several hundred.
8.8. Bibliography


9.1. Introduction

Circular resonant cavities are the most used microwave domain. That is why we give the results of the general $TE_{m,n,p}$ and $TM_{m,n,p}$ waves.

9.2. The fundamental propagative $TM_{m,n}$ and $TE_{m,n}$ waves

We have to solve:

$$\Delta \psi + (k^2 - \beta^2) \psi = 0$$

With the limited conditions on the conductors:

$$\psi = \begin{cases} 
TM \text{ waves if } E_z \neq 0 \ (H_z = 0) \\
\text{or} \\
TE \text{ waves if } H_z \neq 0 \ (E_z = 0) 
\end{cases}$$

To satisfy the limited conditions, we can only have discrete values of $E_z$ or $H_z$. We define the modes of propagation $TM_{m,n}$ or $TE_{m,n}$:
Where $J_m$ is the Bessel functions, and the transversal vectors are determined from the following formulas:

\[
\begin{align*}
\bar{E}_{Tmn} &= -j \frac{\beta_{mn}}{k_{Cmn}^2} \text{grad} \bar{E}_{Zmn} + j \frac{kZ}{k_{Cmn}^2} (\bar{u} \wedge \text{grad} \bar{H}_{Zmn}) \\
\bar{H}_{Tmn} &= -j \frac{\beta_{mn}}{k_{Cmn}^2} \text{grad} \bar{H}_{Zmn} - j \frac{k}{Zk_{Cmn}^2} (\bar{u} \wedge \text{grad} \bar{E}_{Zmn})
\end{align*}
\]

### 9.3. TE and TM stationary waves

As for $TE_{01}$ rectangular waves, the different fields are multiplied by a constant of propagation $e^{j(\omega t - \beta z)}$. We consider a first magnetic plane at $z = 0$ (see Figure 9.1).

![Figure 9.1. The first magnetic plane at z = 0](image-url)
In the case of $TE_{mn}$, we have an incident and a reflected wave for $H_z$:

\[
\begin{align*}
H_{zr} &= H_0^* J_n \left( x_{mn} \frac{r}{a} \right) \cos n\theta e^{-j\beta z} \\
H_{zi} &= H_0 J_n \left( x_{mn} \frac{r}{a} \right) \cos n\theta e^{j\beta z}
\end{align*}
\]

The total magnetic wave $H_z = H_{zr} + H_{zi}$ is then:

\[
H_z = \left[ H_0^* e^{j\beta z} + H_0 e^{-j\beta z} \right] J_n \left( x_{mn} \frac{r}{a} \right) \cos n\theta
\]

The limited condition at the origin $z = 0$ imposes $H_z = 0$ and we have $H_0^* = -H_0 = H_0$.

This gives:

\[
H_z = -2jH_0^* J_n \left( x_{mn} \frac{r}{a} \right) \cos n\theta \sin \beta z
\]

In the case of $TM_{mn}$ waves, we must obtain $E_z$ proportional to $\cos \beta z$:

\[
E_z = 2E_0 J_n \left( x_{mn} \frac{r}{a} \right) \cos n\theta \cos \beta z
\]

### 9.4. Realization of a cavity

The different waves are a product of $\sin \beta z$ or $\cos \beta z$ and the limiting conditions are satisfied at the origin $z = 0$:

\[
\begin{align*}
TE_{mn} \rightarrow H_z (z = 0) &= 0 \\
TM_{mn} \rightarrow E_z (z = 0) &= \text{max}
\end{align*}
\]
These limiting conditions are also verified for $\beta H = p\pi$ by recalling that $\beta = 2\pi/\lambda_g$:

$$H = p\frac{\lambda_g}{2}$$

We can place a perfect conductor without perturbing the planes if:

$$H = \frac{\lambda_g}{2}; \frac{2\lambda_g}{2}; \ldots; p\frac{\lambda_g}{2}$$

### 9.5. The cavity

We place a perfect conductor at $z = H$ (Figure 9.2).

![Figure 9.2. Realization of the circular cavity](image)

Then we have a condition on propagation constant $\beta$:

$$\begin{cases}
    TE_{mn} \rightarrow H_z (z = H) = 0 \\
    TM_{mn} \rightarrow E_z (z = H)_{\text{max}} \quad \text{i.e. } \sin \beta H = 0
\end{cases}$$

and

$$\beta = p\frac{\pi}{H}$$
Recall that if $m \neq 0$ and $n \neq 0$, we have:

$$k^2_c = k^2 - \beta^2 = \left(\frac{x_{mn}}{a}\right)^2$$  \hspace{1cm} TE case

or

$$k^2_c = k^2 - \beta^2 = \left(\frac{x_{mn}}{a}\right)^2$$  \hspace{1cm} TM case

Then:

$$k^2_{m,n,p} = k^2_{Cmn} + \beta^2 = k^2_{Cmn} + \frac{p^2 \pi^2}{H^2}$$

And the resonance pulsations of the constructed cavities with $k^2 = \omega^2 \mu \varepsilon$:

$$\omega_{TE\,m,n,p} = \sqrt{\left(\frac{x_{mn}}{a}\right)^2 + \frac{p^2 \pi^2}{H^2}} \hspace{1cm} TE$$

$$\omega_{TM\,m,n,p} = \sqrt{\left(\frac{x_{mn}}{a}\right)^2 + \frac{p^2 \pi^2}{H^2}} \hspace{1cm} TM$$

9.6. Curve representations

Now the resonant frequencies are:

$$\begin{align*}
 f_{TE\,m,n,p} &= \frac{\nu}{2} \sqrt{\left(\frac{x_{mn}}{\pi a}\right)^2 + \left(\frac{p}{H}\right)^2} \\
 f_{TM\,m,n,p} &= \frac{\nu}{2} \sqrt{\left(\frac{x_{mn}}{\pi a}\right)^2 + \left(\frac{p}{H}\right)^2}
\end{align*}$$

On the diagram $\omega = \omega(\beta)$, we see that the $\omega_{m,n,p}$ are a set of discrete frequencies (Figure 9.3).
We can observe that the two last equations can be written as:

\[
(fD)^2 = \frac{4}{\pi} \left( \frac{x_{mn}^2}{\pi} \right)^2 + \left( \frac{p\nu}{2} \right)^2 \left( \frac{D}{H} \right)^2
\]

where \( D \) is the diameter of the cavity and we use to give the quantity \((fD)^2\) as a function of the relative value of \((D/H)^2\).
If the value \( (D/H)^2 \) is given, the vertical in Figure 9.4 gives the order of the modes present in the cavity with their resonant frequencies.

### 9.7. Frequent and particular examples of modes

- If \( \frac{D}{H} \leq 1 \), we have the configuration \( \text{cylinder} \) and the fundamental is \( TE_{11} \) and after \( TE_{111}, TM_{010}, TM_{011}, \ldots \)
- If \( \frac{D}{H} \leq 1 \), we have the configuration \( \text{cylinder} \) with the modes \( TE_{110}, TE_{111}, TE_{112}, TM_{012}, \ldots \)
- If \( \frac{D}{H} \geq 1 \), we have the configuration \( \text{cylinder} \) with the modes \( TM_{010}, TE_{111}, TM_{011}, \ldots \)
- If \( \frac{D}{H} \geq 1 \), we have the configuration as a drop \( \text{drop} \) with the modes \( TM_{010}, TM_{110}, TE_{111}, \ldots \)

We can note that:
- First, we have \( x_{0n} = x_{1n} \), then \( TE_{0np} = TM_{1np} \), and these two modes degenerate.
- The modes \( TM_{mn0} \) are independent of \( H \).
- The most used mode is the \( TE_{011} \), but it is coupled with some other mode.
- The \( TM_{011} \) has lower losses.
9.8. Examples of the fields of current modes

– Case of the $TE_{111}$;

![Figure 9.5. Electric and magnetic field lines for the $TE_{111}$ mode in the cylindrical cavity](image)

– Case of the $TM_{010}$: none is dependent on $\theta$. The electric field is maximum on the axis.

![Figure 9.6. Electric and magnetic field lines for the $TM_{010}$ mode in the cylindrical cavity](image)

– $TE_{011}$ is the mode of greatest interest and the most-used mode in microwaves because its attenuation is zero at high frequencies. But the $TE_{011}$ is degenerated with the $TM_{111}$. Then, we can add wedges to eliminate this mode.
Figure 9.7. Electric and magnetic field lines for the $TE_{011}$ and elimination of the $TM_{111}$

9.9. Bibliography


Index

A, B
ampere, 25, 40, 42
amplitude, 4
analytic function, 65, 67
attenuation, 4, 23, 30, 33, 104, 110, 112, 119, 129, 131, 141, 163, 164, 170, 198
attenuation length, 104
average power flux, 15, 93, 94, 95, 96
band-line, 16
Bessel, 151, 153, 154, 157, 159, 192
border effects, 16

C
capacitance, 8, 9
capacity, 9, 12, 16, 17, 18, 20, 40, 51, 54, 56, 57, 65
Cauchy-Rieman, 53, 55
cavity, 175, 176, 178, 179, 181, 182, 184, 187, 188, 193, 194, 196–198
characteristic impedance,
  10, 12, 16, 18, 20, 32–35, 40, 47, 51, 73
circular, 18, 151, 154, 156, 165, 170, 191, 194
closed surface, 41
coaxial, 18, 39, 40, 57, 73, 74
communications, 163, 165
complex
  permittivity, 47
  plane, 52, 61, 65
conductance, 11, 32, 40, 46, 47
conduction, 26, 79
conductivity, 45, 79
conform transformation, 51, 52, 54, 56, 61, 62, 65
copper, 46
correction, 74
cubic density of charges, 79
current, 10, 26, 44, 130, 163, 198
cutoff, 127, 128, 140, 155, 157, 161, 196
above the, 89, 103
at the, 88, 104, 127, 140, 155, 157
under the, 89, 104
cylindrical coordinates, 152

D
decoupling powers, 97
degenerated, 123, 127, 128, 139, 140, 151, 158, 163, 198
density, 8, 13, 18, 20, 31, 97
dielectric, 23, 24, 26, 27, 30, 31, 32, 39, 40, 41, 48, 111–114, 116–118, 120, 121, 146, 147, 184, 187
differential equation, 124, 131, 132, 137
direct wave, 102
direction, 80, 103
dispersive, 39, 87, 126, 139, 155, 157, 155, 157
displacement, 70, 79, 112, 117
divergence, 83, 96
double vectorial product, 13, 82, 109

E, F, G, J
eccentric coaxial, 65, 70
electric, 3, 4, 6, 8, 13, 16, 18, 19, 27, 28, 31, 41, 62, 83, 85, 88, 112, 114, 117, 120, 121, 125, 132, 134, 136, 138, 142, 143, 145, 152, 163, 166, 169, 175, 177, 179–181, 183, 184, 198, 199
line, 142, 145, 181
elementary cell, 33, 34
energy, 44, 92, 98, 99, 100, 102, 181, 182, 184
speed, 98
equipotential, 7, 11
evanescent, 86, 87, 88, 89
fictitious permittivity, 177
field line, 131, 132, 134, 136, 145, 162, 179, 181, 198, 199
finite difference, 51, 57, 59
flux, 11, 13, 23, 24, 41, 43, 70, 100, 102, 128, 140, 163, 164, 170
fundamental, 83, 123, 128, 131, 133–135, 140–143, 147, 148, 151, 158, 161, 162, 164, 175, 179, 191, 197
Gauss, 40, 41
geometric parameter, 66
geometry factor, 32, 52
Green, 13, 14
group speed, 92
joule’s, 29

H, I, K, L
harmonic function, 53
Helmhotz, 87
hyperbolas, 62, 63
homogeneous, 79, 80
impedance, 7, 10, 12, 16, 18, 20, 29, 32–35, 40, 47, 51, 66, 73, 100, 101, 112, 118, 156, 158
impulsion, 39
inductance, 12, 16, 17, 18, 20, 40, 51
integration, 133, 135
inverse wave, 102, 103
isotropic, 4, 79, 80
iteration, 60
Kirchoff, 12
Laplacian, 152
limiting condition, 19, 84, 86, 87, 99, 101, 112, 114, 125, 138, 152, 154, 156, 167, 177, 178, 193, 194
linear, 3, 4, 79
longitudinal wave, 80, 82
lossless, 6, 32, 34, 79, 80, 112
lower frequency, 128, 140

M, O, P
line, 143, 181
Maxwell, 3, 25, 79, 81
metallic, 23, 24, 27, 28, 30, 32, 40, 42, 48, 111
microwave, 191, 198, 84
Ohm’s law, 26
orthogonality, 54
parabolic line, 64
perfect conductor, 84, 107, 117, 178, 194
permeability, 4, 43, 118
perturbation, 23, 30, 112
phase, 4, 5, 35, 92, 104
polarization, 153
potential, 7, 9, 16, 18, 42, 59, 62, 70
power, 13, 18, 20, 23, 24, 31, 32, 97, 100, 102, 107, 128, 140, 163, 164, 170, 184, 187
problem, 8, 16, 17, 40, 54, 56, 99, 101, 111, 112, 123
progressive, 86, 88
propagating constant, 4, 88, 89
speed, 47, 86, 92

Q, R, S
quadrantal, 165, 167
quadrupolar mode, 162
quality factor Q, 184
quasi-longitudinal, 121
quasi-transversal, 120
rectangular, 59, 123, 127, 134, 136, 139, 142, 145, 146, 151, 175
relation of dispersion, 87, 127, 139
repartition (classification) of the modes, 149
resistance, 32, 34, 40, 45, 46, 118
resistivity, 45
rotational, 95
self, 51, 65
semi-circular, 165
sinusoidal, 35, 38
sixpolar mode, 162
skin
  effect, 29, 40, 45, 46, 107, 117
  thickness, 28, 118, 185
small losses, 35, 39
speed of the phase, 33, 39
wave, 90
Stokes, 95, 96
superficial
density of current, 10, 28, 107, 108
resistance, 107, 112
surface bounded by a closed contour, 95
surface impedance, 27, 185
symmetry revolution, 19, 153, 159

T, U, V

TE, 88, 92, 97, 99, 102, 103, 108, 110, 123, 136, 151, 152, 156, 158, 165, 166–170, 179, 181, 182, 192, 195

TEM, 3, 6, 7, 8, 12, 23, 51, 92
thickness, 44, 45, 46, 107
TM, 88, 92, 97, 101–103, 109–111, 123, 124, 133, 137, 139, 151, 152, 154, 156, 157, 165–170, 192, 195
transformation in the complex plane, 52
transmission line, 32
transversal wave, 80, 82
two parallel metallic planes, 111
unit direct vector, 84
vectorial product, 83
variable separation, 124, 137, 152