Crafting Conundrums

Puzzles and Patterns for the Bead Crochet Artist

Ellie Baker
Susan Goldstine

CRC Press
Taylor & Francis Group

AN A K PETERS BOOK

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for the Bead Crochet Artist

Praise for *Crafting Conundrums*

"This beautifully illustrated book is a delight for both the bead crocheter and the student of mathematics. It can be enjoyed by the crafter without exploring the mathematics or the mathematician without trying the craft. But it will entice the crafter to do mathematics, and the mathematician to crochet tangible displays of her work.

Mathematics at its essence is the study of patterns and bead crochet is a beautiful medium for making tangible objects to display patterns. Here we have an extensive study of the possibilities for bead crochet by two knowledgeable mathematicians. Starting with a simple hexagonal grid, a wealth of possibilities emerge. The surprising complexity of this medium is explored in depth.

Readers can chose from clearly illustrated patterns or design their own while learning about the underlying mathematics. With ideas from topology, tilings, graph theory, knot theory, and group theory, an entire liberal arts mathematics course can be designed around this text. Whether you are looking to introduce mathematics topics into your arts curriculum, generate enthusiasm in your mathematics courses, or simply create stunning bead crochet, you will find great ideas for exploration."

—**Eve Torrence,** *Professor of Mathematics, Randolph-Macon College*

"This book is a collection of wonderful tools for mastering geeky and beautiful projects that in a tactile and creative way explore notions like universal covering space, four color theorem, wallpaper groups, and seven color tori that unfairly seem to be reserved for mathematicians only. Crafters, puzzle lovers, and pattern designers will be delighted to find clear instructions on how to do the projects. I hope that non-crafting mathematicians will also peek in the book to see how mathematical concepts can be expressed in amazingly visual ways. It is indeed written with experience and love of both math and craft."

—**Daina Taimina,** *Adjunct Associate Professor of Mathematics, Cornell University, and Author of* Crocheting Adventures with Hyperbolic Planes

"This delightful book will give readers a visual understanding of mathematically inspired designs in bead crochet ropes. It is a well-written book that straddles the fence between mathematics and craft. The theory, patterns, projects, and instructions are presented in a clear and concise manner. If the technical aspects don't interest you, then skip ahead to the pattern pages for a full library of designs. This book will keep the experienced bead crocheter busy for a long while."

—**Judith Bertoglio-Giffin,** *Bead Line Studios, www.beadline.com*

"This is a must-have book for anybody interested in bead crochet bracelets and cords. It provides a perfect balance between the design and construction of bead crochet, and the underlying mathematics that dictates what is and is not possible within this art form."

—**Gwen Fisher,** *beAd Infinitum, www.beadinfinitum.com*

"*Crafting Conundrums* is a wonderful book that shows the unity of art, craft, and mathematics. It is a feast for the eyes as well as the mind. The authors integrate accessible discussions of the mathematics of pattern and shape with design challenges and step-by-step instructions, so that readers can be as practical or as idea-oriented as the spirit moves them. *Crafting Conundrums* will join other special books, like Taimina's *Crocheting Adventures with Hyperbolic Planes* and belcastro and Yackel's *Making Mathematics with Needlework,* on my bookshelf. I need only to look at them to be reminded that art and math were 'separated at birth.' Baker and Goldstine have reunited them. I wish math could have been like this when I was in school!"

—**Sarah Kuhn,** *Professor of Psychology, University of Massachusetts Lowell*

"Baker and Goldstine offer a beautiful and precise mathematical introduction to the deep ideas of bead crochet, leading the reader to feel like a participant in the development. Myriad accompanying patterns afford readers an excellent venue for experiencing the mathematics themselves and for becoming severely addicted to bead crochet. The mathematical theory is followed by a generous invitation to further play with the mathematics through beading, theoretical exploration, or a combination of both. This book is a wonderful resource for people wanting to deepen their understanding of mathematics through crafting, people who love mathematics and crafting, and people who enjoy design. College libraries, parents of the mathematically gifted, and technically oriented people take note: You want this book!"

—**Carolyn Yackel,** *Associate Professor of Mathematics, Mercer University, and Coeditor of* Making Mathematics with Needlework *and* Crafting by Concepts

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For Olivia, Sophie, Mark, and Keiko

E.B.

For all of my teachers, both in and out of the classroom, and Kiko

S.G.

CONTENTS

Part Three: Instructions and Patterns

Preface by Ellie Baker

Long before I learned to bead crochet, I read a book about the search for the shape of the universe.* The book discussed many theories, but one that stuck in my mind was the notion that the universe might be shaped like a torus (which is the mathematical term for a donut shape; its plural is tori). Part of the idea was that the universe might actually be finite, despite having no apparent wall-like ends or edges, so that if you travelled in a straight line, you could eventually return to where you started. If you imagine yourself walking on the surface of a giant donut, you can probably convince yourself that, just like a sphere, a torus has this property. Later, when I first saw bead crochet bracelets, I fell in love with them—visually, tactilely, sculpturally…emotionally, really. At the time, I couldn't have explained what drew me to them. But eventually it hit me—they were tori! And, in my mind, it was a kind of mystical form—maybe even the shape of the universe!

When I first learned the basics of bead crochet, I taught the craft to my daughter Sophie, and together we became obsessed with creating our own bracelet designs. But producing new patterns turned out to be deceptively tricky, partly because of this quality the bracelets had of being continuous in any direction. That, combined with their underlying spiral structure from being crocheted in the round, seemed to be the source of much difficulty. Design elements had to fit exactly with no gaps and to flow seamlessly from left to right and top to bottom. Designing was most easily done on a flat surface with colored pencils and paper or their computer-aided analog. But visualizing and planning a finished bracelet from the traditional flat pattern layouts we found in books often turned out to be mind-boggling, at least for anything even slightly complex. Despite the difficulties, Sophie worked out a few simple but beautiful patterns. In one case, she tried crocheting a monochrome, patternless bracelet with white beads and then worked out her design directly on the bracelet by coloring the beads with markers. This approach had some merit, but was still confusing and wasteful.

* *How the Universe Got Its Spots* by Janna Levin.

I found myself frequently frustrated by my design attempts. On paper, I'd be convinced something would work, but once crocheted, the bracelets would come out with annoying pattern discontinuities. And seeing a problem did not mean it was easily correctable. A smooth, seamless quality, both in form and pattern, was integral to the beauty of each piece, and neither of us was willing to tolerate flaws. I found myself thinking more about how bracelets are topological tori (topology is a branch of mathematics that includes the study of objects such as tori) and wondered if perhaps there might be some useful mathematics out there that would help. I also noted that designing a valid bracelet pattern was related to tiling problems in mathematics. Think about tiling your kitchen floor with a limited set of ceramic tile shapes (say, triangles, squares, or hexagons of a uniform size), with the constraint that you must cover the entire floor with no gaps and without ever needing to cut or use a partial tile. And now imagine that your kitchen floor is a curved surface shaped like a donut. Could you design a set of tiles and a visually attractive way of assembling them that would be guaranteed to work? I tried googling "Torus Tiling," and, lo and behold, a link came up with a reference to a paper by the mathematician Marjorie Senechal called "Tiling the Torus." I figured the math was beyond my background, but nonetheless I gamely wrote to Dr. Senechal, sent her some pictures of our bracelets, and asked for a copy of her paper.

Dr. Senechal kindly replied, sending copies of two of her papers and a brief note, in which she mentioned a colleague who had once shown her "a beaded torus that illustrated a map-coloring theorem…," although she couldn't remember who made it. While the papers didn't appear directly applicable to bead crochet tilings, my interest was piqued, and I thought, surely, if there is a beaded torus out there that illustrates a math theorem, I ought to be able to find it. I started googling again, adding the term "map coloring" to my searches. I didn't find the beaded torus, but this time I hit the jackpot. I found an amazing source of ideas, inspiration, and mathematical expertise, my coauthor and collaborator, Susan Goldstine.

Susan, a math professor and long-time crafter, maintains a wonderful website with an abundance of links, pictures, and examples of fun mathematics, many of which are related to the arts. Her website includes a section on map coloring titled "Seven Color Tori," and it was this that I stumbled upon in my search. In mathematics, map coloring deals with a question a cartographer might ask, namely, "What is the minimum number of colors needed to paint any map so that no two bordering countries are painted the same color?" It turns out that on a globe or flat paper, the answer is four. No matter what map you can come up with, no matter how many countries and regardless of configuration, it can be painted with only four different colors such that each country is clearly distinguishable from the others.* Now, suppose the Earth is not a sphere, but is a torus instead. Setting aside all reasonable questions you might have about why anyone would consider such a supposition, how many colors would you need now? Surprisingly, it turns out that the answer is no longer four, but seven. And, if this is true, it should be possible to create a specific example of a map on a torus that absolutely can't be painted properly with any fewer than seven colors. This is the subject of Susan's webpage on seven-color tori. On it, she shows several lovely examples of maps on tori with exactly seven "countries" in which each country shares a border with all six others. Any such map is itself a visual proof that at least seven colors are needed on the torus. On her website, there were examples by multiple artists, including Susan herself, of seven-color tori in different media, including cloth, plaster, and ceramic. But at the time, there was no beaded bracelet.

So I set about trying to design one—in bead crochet, of course. Although there were flat diagrams of maps available online, none of these translated easily into a bracelet pattern. I tried for several days and couldn't get anywhere. I felt like I just couldn't wrap my head around it. Exasperated, I showed the problem to Sophie, who was rushing off to school. Interested, she grabbed some graph paper, beads, and a crochet hook on her way out the door. At the end of the school day, I got an excited call from her. "I did it!" she said. And then I had to wait impatiently until evening for her to return home and show me her solution. She had crocheted the tiny torus shown in the following photograph.

I checked carefully, and, sure enough, it seemed to work. I was still skeptical, however, because some of the touching borders between "countries" were only one bead long (such as the blue and brown in the photo above). Could these be places where the regions didn't really share a border, but had just a single point of intersection (like Colorado and Arizona in the United States)?† Sophie was pretty confident and proceeded to spend the evening crocheting up a larger, bracelet-sized version with elongated borders. Its correctness was now clearly indisputable: it had seven identically shaped blocks of color, each one bordering all of the other six.

Our bead crochet story started with Sophie's seven-color torus bracelet, but it didn't end there. Sophie and Susan corresponded and, collaborating, we went on to create a more refined version of the seven-color bracelet, whose pattern, along with Sophie's original, appears on pp. 134–135. Sophie, after a previous exasperating science fair project involving fruit flies, dared to wonder if there was any way she might do her next project on something completely different, preferably inanimate… perhaps bead crochet? It seemed crazy at first, but it happened, and she won an honorable mention at the Massachusetts state science fair. Once in college, Sophie

* As long as the countries have no noncontiguous areas.

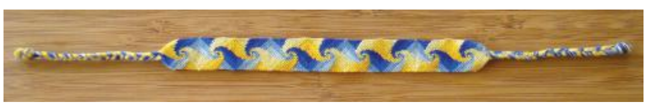
† Think about a pie top cut into eight pieces. All the pieces touch at a single point, the center, but they do not all share borders with each other. Each shares a border with only two adjacent pieces, not all eight. Thus it is possible to paint the countries (i.e., the slices) with just two alternating colors of frosting.

had much less time for bead crochet, although she did, to her own surprise, eventually decide to major in math. However, for the past five years Susan and I have continued to correspond, working on a variety of bead crochet design topics that expand on ideas and questions that began with Sophie's project. During this time, we've made tremendous progress. We've developed a deeper understanding and a new approach to creating bead crochet patterns and asked and answered an entertaining variety of mathematically inspired design questions. In the process, we've produced a multitude of novel and beautiful new designs unlike anything we'd seen before. With Susan and Sophie as travel partners, I've learned so much and covered exponentially more ground than I ever could have alone. This book is our effort to share with both the craft and recreational mathematics communities our newfound knowledge and passion for this fun and fascinating craft.

Ellie Baker

Preface by Susan Goldstine

As it happens, bead crochet is not my first foray into mathematical bracelet making. I didn't make the connection until some time after Ellie and I began our beading adventures, but in the summer after my sophomore year of college, Rachel Wells Hall, now a mathematics professor at Saint Joseph's University, introduced me to the world of friendship bracelets. Rachel and I were roommates at the Mills College Summer Mathematics Institute in 1991, and she designed and made friendship bracelets with more elaborate designs than I had ever seen before, mesmerizing patterns of interlocking waves and yin-yang signs. Armed with the techniques she taught me, I continued to design and make my own friendship bracelets well into graduate school. I still occasionally dust off and wear the bracelet shown here, made with extremely thin DMC flower thread, at a time when I had more manual dexterity and patience.



I suppose you could interpret this as a sign that I was predestined to join Ellie and Sophie's bead crochet project, but in fairness, the friendship bracelets are one in a long string of more or less mathematical crafts that I have taken up over the years. I have always been drawn to artistic applications and interpretations of mathematical ideas, no doubt influenced by my mother, who enjoys woodworking and gardening and photography and all manner of creative endeavors, and by my father, a computer science professor (now retired) with a Ph.D. in mathematics who never lost his fascination for tinkering with colorful math. My prebeading years were also filled with hexaflexagons, thanks to Martin Gardner's seminal recreational math books, and modular origami, thanks to the instigation of a college classmate, origami expert Tom Hull, and mobile making, thanks to the tutelage of another college classmate, board game designer and artist Alison Frane, and so on.

In the mathematical community, people who enjoy fusing mathematics with their creative endeavors tend to find each other. By the time I had settled into my current academic position, I was friends with sarah-marie belcastro and Carolyn Yackel, the cofounders of the Knitting Circle at the annual Joint Mathematics Meetings (JMM). I didn't attend the Knitting Circle for some years on the reasonable basis that I didn't actually knit, but when I showed sarah-marie a topological model I had sewn based on a passage in Lewis Carroll's *Sylvie and Bruno Concluded* in 2004,* she insisted I bring it to the circle and assured me that all crafts were welcome there. The following year, sarah-marie and Carolyn organized a special session at JMM on mathematics and the fiber arts, and that is where I was introduced to Daina Taimina's marvelous hyperbolic crochet.† I was so excited by her models of the hyperbolic plane and their potential use in teaching hyperbolic geometry that I prevailed on my close friend and colleague Katherine Socha to show me the rudiments of crochet.

This is where my crochet repertoire—beadless and restricted to the single crochet stitch—remained for the next few years. In the summer of 2008, having just survived my tenure review, the most nerve-wracking part of my academic career, I decided that it was time to branch out. I started visiting my local yarn shop and took up crochet in earnest. At the same time, I finally learned how to knit, and by the fall I had worked my way up to making patterned scarves.

And so it was in the middle of my renewed interest in fiber arts that I got an email in November out of the blue from a woman in Massachusetts I'd never met describing the bead crochet that she and her daughter were investigating. In addition to foreshadowing a number of the design challenges that are at the heart of this book, Ellie's message included a photograph of Sophie's first full-sized seven-color torus map bracelet. Delighted at this marvelous surprise in my inbox, I wrote back to ask if I could add Sophie's bracelet to my seven-color torus webpage and if Sophie could share some insight into how she had arrived at the pattern. Imagine my surprise when a few weeks

* The model, Fortunatus's Purse, is described in Chapter 7 of *Making Mathematics with Needlework: Ten Papers and Ten Projects*, edited by sarah-marie belcastro and Carolyn Yackel.

† Try an Internet search for "hyperbolic crochet." You won't regret it.

later, I found a padded envelope in my mailbox at work and opened it to find my very own seven-color map bracelet!

Thus began the correspondence that Ellie described in her preface and that ultimately led to this book. Ironically, it took me another three and a half years to learn to bead crochet, even though Sophie and Ellie gave me a short lesson when I met them for the first time in January 2009. In large part, their generosity is to blame, since they kept giving me bracelets based on our work together, so I was happy to keep knitting shawls and sweaters and working on bead crochet in the abstract. In the end, my crafting self overwhelmed my mathematical self—and I was a little sheepish when I gave math talks about bead crochet and had to admit that I couldn't bead crochet myself. In the spring of 2012, I picked up my own beads and hook, and I've been hooked ever since.

Were it not for the Internet and our many wonderful colleagues in mathematics and art, I would never have met Ellie Baker and would never have embarked on the longest and most rewarding project of my mathematical career. The fruits of our labors are here in this volume, and I hope that it brings you as much pleasure as it has brought me.

Susan Goldstine

Acknowledgments

We would like to thank Sophie Sommer for her work on the science fair project, "Mathematical Explorations in Bead Crochet," which was the starting point for this book. We also thank her for design contributions, for allowing us to publish several of her patterns, and for assistance with the section on how to bead crochet. Thank you to Alice Peters and Charlotte Byrnes for reading our early book proposal and supporting the idea of this book, and to Sunil Nair for seeing the book through to publication. Thank you to Charlotte Byrnes for all her extraordinary work and helpful advise on the production of the book.

Thanks to Norton Starr for his enthusiasm about seven-color torus bracelets, for inspiring our Ungar-Leech map design, and for kindly allowing us to reprint photos of his hydrostone torus. Kira Isaacs and Kate Magauran provided meticulous bead stringing work for many bracelet samples. Katie Sheena was a wealth of enthusiasm and crocheted many bracelet samples with unparalleled craftsmanship. Gwen Fisher was a great source of ideas and artistic inspiration. We particularly thank her for suggesting the idea of transforming tessellations in bead crochet. Thanks to Carolyn Yackel and sarah-marie belcastro for inspiring the bead crochet torus knot, K_7, and seven-color torus designs with their crocheted and knitted tori, and for allowing us to reprint a photograph of their work (and thanks also to Craig Coleman, who took the photograph, for his permission to reprint it). We would like to thank Danny Baker for ideas and for suggesting the patchwork torus knot construction method. Thanks to Saint Mary's College of Maryland and Sue Johnson for facilitating the visiting collaborator residency that enabled us to work together at SMCM. Thanks to Andrew Paul Keiper and the St. Mary's College Library Media Center for filming our bead crochet demonstration videos. Thanks to Robert Fathauer for his suggestions on constructing tessellated bracelet patterns. Florence Turnour graciously allowed us to reprint a photo of her beautiful artwork *Beaded Deformation: From Kepler's Star to the Night Sky*. Marian Goldstine spent many patient hours taking the bead crochet demonstration photographs for the chapter on how to bead crochet. Marcia Pronin, Rachel Holt, Sarah Gelson, Sarfraz Khan, and Joanna Knight helped guide us through the publication process at Taylor & Francis.

Although we've never met or spoken, we owe profound thanks to Judith Bertoglio-Giffin for her wonderful bead crochet books and patterns that started our journey, and to Linda Lehman for suggesting to the bead crochet yahoo group the use of size 20 crochet thread on larger circumferences. Robert Scharein's KnotPlot program was a great tool for producing illustrations of torus knots.

We are grateful to the Bridges Organization for all of their work in promoting mathematical art and artistic mathematics and for all the opportunities they have given us to share our bead crochet. We have displayed our bead crochet artworks, including a joint piece with Sophie Sommer, in five of the Exhibitions of Mathematical Art sponsored by Bridges at the national Joint Mathematics Meetings and at the international Bridges Conference between 2010 and 2014. We also ran a bead crochet workshop at the 2012 Bridges Conference at Towson University, an invaluable chance to put our bead crochet instruction into practice. Thanks to the staff and peer reviewers at the *Journal of Mathematics and the Arts*, especially editors Gary Greenfield and Craig Kaplan, and at *Math Horizons*, especially editor Bruce Torrence, for their assistance with our academic papers on bead crochet and mathematics.

Thank you to Allie Altman, Michael Klugerman, Charlotte Byrnes, Tabitha Vevers, Sandy Ganzell, and Jonathan Goldstine for reading portions of our early drafts and for their extremely helpful comments. Allie's sharp eye and careful reading clarified ideas and caught many typos and errors. Michael provided many illuminating discussions, helped with Adobe Illustrator and other technical challenges, and clarified thinking, illustrations, and writing in many ways. We would also like to thank Eve Torrence for her willingness to test some of the material in the manuscript with her students at Randolph-Macon College. Thank you to Sandy Spitzer whose presentation at Bridges 2012 inspired our use of star polygons as a conceptual aid.

Ellie would like to thank: Dan Margalit for allowing me to audit his enlightening knot theory class; Marjorie Senechal for her email mentioning "a beaded torus that illustrated a map coloring theorem" and for sending copies of her papers; Margo Seltzer for her encouragement, advice, and friendship; Linda Baker for her insight and for

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Introduction

Bead crochet bracelets have an allure that is hard to resist. In their most basic form, they are hollow, seamless tubes of delicate beads that are flexible enough to slip over the hand and onto the wrist. For the wearer and her companions, the appeal is both visual and tactile; the bracelet is firm but pliable, the beads packed into a sleek, snake-like skin. For the crafter, the technique, once mastered, is straightforward, and the choices endless; different colors, finishes, and shapes of beads can create effects that are simple or complex, subtle or bold, textured or smooth. But for the analytically minded, the greatest allure is in creating bracelet patterns. Behind the deceptively simple and uniform arrangement of beads is a subtle geometry that produces compelling design challenges and fascinating mathematical structures.

This book is for crafters, puzzle lovers, and pattern designers alike. Experienced bead crochet crafters looking for a project to curl up with may well choose to skip ahead to the pattern pages and begin crocheting from our abundance of unique, mathematically inspired designs. Those wishing to go beyond following others' patterns to designing their own will find many useful tools, template patterns, and a new methodology for understanding how to do so. Puzzle lovers without previous knowledge of bead crochet will find ample inspiration for learning the craft, or they may be sufficiently compelled by the design challenges alone and never even feel an urgent need to pick up a crochet hook. But for anyone who loves all of these things—bead crochet, pattern design, and puzzles—our methods, challenges, and patterns offer a cover-to-cover springboard for creative exploration.

Our presentation is structured in three parts. In Part One, A Design Framework, we describe the basic requirements and constraints of a valid bead crochet pattern and explain what makes designing in this medium so tricky. We then lay out our design framework and the ideas we've developed to facilitate easier creation of successful designs. We also point out a few design choices and issues unique to bead crochet patterns and offer our thoughts on how best to approach them. Much of the material in Part One was developed over several years as we puzzled through many design questions we posed to ourselves. A thorough understanding of this section will give readers a big leg up on approaching the design challenges in the next section.

Part Two, Design Challenges, presents a series of bead crochet design challenges informed by colorful bits of mathematics. Each chapter in this section begins with a design puzzle; most of the puzzles are posed in broad, nontechnical terms. In some chapters, we present multiple related challenges. Each challenge is presented with a brief introduction to the mathematical idea that inspired it written for those with an analytical bent but no particular advanced math background. Next, we discuss what made the challenge difficult, present some of our solutions, and describe the thinking and ideas behind our approach. We invite anyone who goes on to develop his or her own novel or interesting solutions to please let us know!*

Part Three, Instructions and Patterns, contains the bead crochet pattern pages. This is essentially the answer sheet at the back of the textbook. It contains nearly 100 original patterns, including solutions to all the design challenges. Readers who jump right to the patterns will want to be sure first to inspect the guide to reading the pattern pages (p. 133), which explains how to use and interpret the information presented. In addition, we outline the basics of bead crochet technique. We encourage anyone inspired to learn the craft (and we hope there will be many!) also to peruse the many videos and abundant additional teaching resources available online, including our demonstration videos and other supplementary materials posted on the publisher's website.† For technical basics, don't underestimate the value of interaction with a human teacher, and be sure to check out classes offered at your local bead store.‡

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* We can be contacted at craftingconundrums@gmail.com.

† http://www.crcpress.com/product/isbn/9781466588486 (under the "Downloads/Updates" tab).

‡ Or start a local bead-craft group—a lovely alternative to a book group!—to share ideas, difficulties, resources, patterns, expertise, and, of course, conversation and fun!

PART ONE

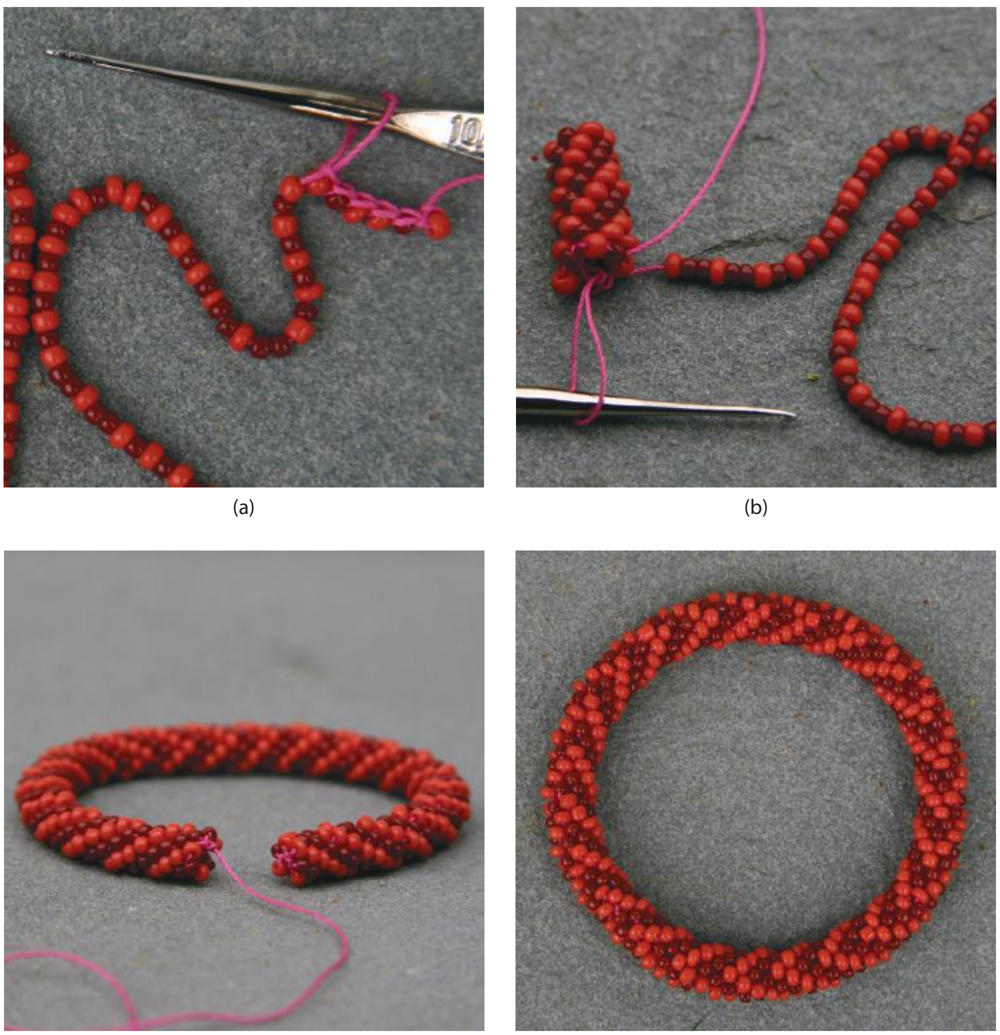
A Design Framework

CHAPTER 1

Exploring the Bead Plane

The process of making a bead crochet bracelet involves a sequence of five component activities: *pattern selection (or creation), stringing, starting, crocheting*, and *closing*. Step one, creating or choosing a pattern, is the primary focus of this book. With a pattern design in hand, step two is stringing the beads onto thread in the order specified by the design. After the beads are in place on the thread, step three involves creating an initial round of stitches, analogous to casting on in knitting (Figure 1.1(a)). In step four, the crafter crochets the beads in a continuous spiral, pulling one bead into each stitch (Figure 1.1(b)) until the rope reaches the desired length. In the final step, closing, the artist sews a seamless-looking connection between the two ends of the rope (Figures 1.1(c) and 1.1(d)). We provide a set of tutorials on techniques for starting, crocheting, and closing, as well as tips for simplifying the stringing process, in Chapter 8. Everything else is all about pattern design!

A pattern must provide the crafter with two essential pieces of information: which beads are strung onto the thread in what order, and how many beads are crocheted before the starting round is closed. The number of beads in the starting chain is the *circumference* of the bracelet. For instance, the bracelet in Figure 1.1 has circumference 6, as seen in the 6-bead chain in the first step shown. Alternately, we describe this as a *6-around bracelet*. As for the color sequence of the beads, most bracelets consist of a relatively short color sequence that is repeated multiple times to achieve the desired size. This segment of beads is the *repeat* of the pattern. The 6-around bracelet pictured here



(a) (b) (c) (d)

FIGURE 1.1 The poststringing stages of a bead crochet bracelet: starting, crocheting, closing, and the final bracelet. The pattern for this bracelet, Sophie's Herringbone, appears in the pattern pages on p. 175. Sophie Sommer designed and crocheted the bracelet photographed for our paper "Building a better bracelet: Wallpaper patterns in bead crochet" in the *Journal of Mathematics and the Arts*.

has a 26-bead repeat. There are various ways to provide the circumference and stringing order; most frequently, designs are presented as flat layout diagrams showing a single pattern repeat.

Torus Basics

From a mathematical perspective, a bead crochet bracelet is a torus, or donut shape, an object studied in the mathematical field of topology. Mathematicians will often represent a three-dimensional torus using a flat two-dimensional square diagram with the opposite edges *identified*. By this, we mean that the diagram indicates that the edges of the square are glued together in pairs (top and bottom, left and right) to construct a torus. On such a two-dimensional gluing diagram, an ant travelling across the square would appear to exit at one edge and re-enter at the opposite edge. Devotees of classic video games such as Asteroids have seen this in action, since moving the space ship across an edge of the screen causes it to reappear on the opposite side in just this way.

To understand how a torus is constructed from a square, imagine rolling the square sheet into a cylinder by gluing together the top and bottom edges. Then imagine gluing together the two circular ends of the cylinder, which is tantamount to gluing together the right and left edges of the original square. This thought experiment assumes the square sheet can be stretched or shrunk as needed. In the mathematical field of topology, this is a perfectly reasonable requirement. Topology studies the properties of objects that don't change as the objects are deformed by stretching, flexing, or shrinking, as long as they are not torn, punctured, or cut. Figure 1.2 shows how to construct a three-dimensional torus from a stretchy, flexible two-dimensional square.

It turns out that the rectangular flat layout diagrams used in bead crochet pattern books have a lot in common with the two-dimensional square models of tori used in mathematics! This is because the right and left edges are connected by the bead crochet stitch itself, while the top and bottom of the resulting cylinder are connected by sewing together the two ends. If the bracelet consists of multiple identical repeats, closing involves sewing the top of the last repeat to the bottom of the first repeat. However, the manner in which they connect is the same regardless of whether there is only one repeat or many. It's interesting to note that the stretchiness of the beaded fabric created by a crochet stitch on thread fits well with

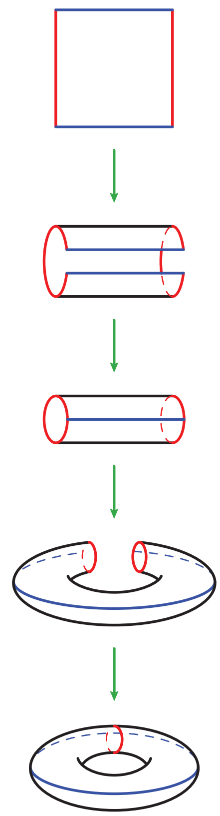


FIGURE 1.2 Turning a square into a torus.

the topological notion of objects that can be deformed by stretching—although bead crochet is admittedly limited in its stretching ability compared with the routine demands of a topologist. One aspect of this stretchiness is that a bead crochet bracelet can be rolled to move the beads from the inside of the bracelet (the part nearest the wrist) to the outside of the bracelet and vice versa. An actual donut is much less topological in this regard, and trying the same maneuver will only yield a broken donut and lots of crumbs. As we go along, we will point out other ways in which bead crochet bracelets help model ideas used in mathematics.

Flat Layouts

The first problem encountered by any bead crochet pattern designer is how to plan a design on a two-dimensional flat layout and visualize how it will look on a finished three-dimensional form. Although excellent software is currently available, such as jbead by Damian Brunold, which permits a simulated rolling of the rope from a pattern during the design process, this procedure has its own limitations. Our hope is that the ideas and design methods described in this book, if incorporated into programs such as jbead,

will lead to greatly improved software for supporting bead crochet pattern design.

The layouts we found in our early pattern book searches were left-leaning parallelograms like the ones shown in Figure 1.3. These particular layouts are for a 6-around bracelet, with stringing order left to right and bottom to top, as indicated by the numbering on the left-hand diagram. The number of beads in each row of a traditional diagonal layout is the circumference of the bracelet. Just like the flat torus diagrams in topology, the opposite edges of the parallelogram are identified. However, the identification from the right edge to the left shifts up one bead to account for the fact that the rightmost bead on one row is (once crocheted) adjacent to the leftmost bead on the row above it. This shift-up is a direct result of the bead crochet process of crocheting continuously in the round, creating the underlying spiral structure of bead crochet. This spiral structure can make it tricky to align motifs and is the source of much difficulty in designing bead crochet patterns.

The traditional bead crochet layout, which we call the *diagonal layout*, works reasonably well for some simple patterns, such as the blue diamonds on red shown on the right in Figure 1.3. However, it is by no means the only flat layout possible and, from a visualization standpoint, it turns out to be less than optimal for many designs.

For example, consider the 7-around design in Figure 1.4 consisting of blue, green, and pink hexagons, each composed of 19 beads. A single pattern repeat consists of 3 times 19, or 57, beads. There is only a single bead shown in the last row because the repeat length happens not to be a multiple of the 6-around circumference, a common phenomenon that we will discuss more soon. The green and blue hexagons are clear enough, but it's not immediately obvious that the pink beads also form a hexagon. To see that, you have to wrap your mind around exactly how all the edges will connect in a finished bracelet. And, unlike the simple two-dimensional torus models in

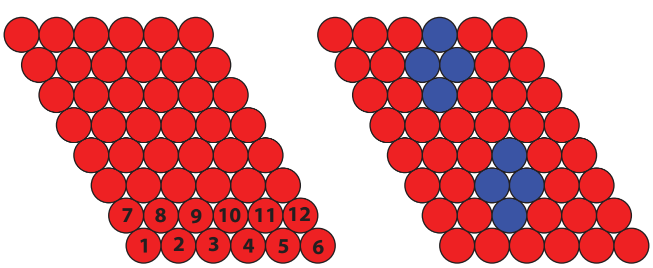


FIGURE 1.3 The traditional diagonal layout for a bead crochet chart. Charts are read from left to right and bottom to top, as indicated by the numbers on the leftmost chart.

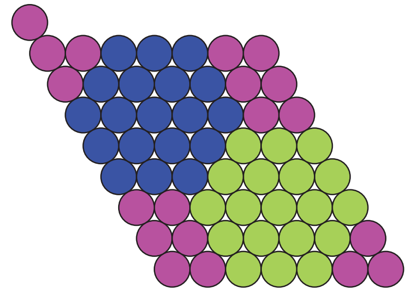


FIGURE 1.4 A pattern of hexagons in the diagonal layout. In this configuration, it is not clear that the pink beads form a hexagon.

mathematics, bead crochet's spiral structure introduces some funny extra constraints about exactly how the left and right and top and bottom edges are identified. Trying to visualize all this from a diagonal layout can be mind-boggling. Is there a better layout that would make it easier to visualize what is going on in the pattern?

If you imagine that you could peel a bracelet open like an orange and flatten it out, in one piece, cutting along any line running between beads, it is apparent that many different layouts are possible. For now, imagine that our bracelet is impractically tiny, consisting of just a single 57-bead repeat (or, if it's too hard to imagine such a tiny bracelet, imagine slicing out and laying flat a single pattern repeat from a larger bracelet). Figure 1.5 shows the same hexagonal pattern as in Figure 1.4, but in a variety of different possible flat layouts. Figure 1.5(a) shows the very simplest one, the uncrocheted strand (not shown here in its 57-bead entirety because of page width limitations), which is quite obviously not much help for visualization. Figure 1.5(b) shows the vertical layout used in some more recent bead crochet pattern books and websites, such as Judith Bertoglio-Giffin's pattern book, *Triangular Bead Crochet Ropes,* in which it is called the "zipper" layout, but it is no more helpful in this case than the diagonal layout. Figure 1.5(c) shows a variant of the traditional diagonal layout that uses a right-leaning diagonal instead of a left-leaning diagonal. Figures 1.5(d)–1.5(g) show four more possible valid flat layouts.

There are some numerical patterns in the vertical and diagonal layouts (Figures 1.4 and 1.5(b)–1.5(c)) that will pop up in various ways as we proceed. As noted before, in a left-leaning diagonal chart for an N-around bracelet, each row contains N beads—which is part of the reason why it is such a common chart. On the other hand, a right-leaning diagonal chart has N + 1 beads per row. For instance, in Figure 1.5(c), the chart for our 7-around bracelet has 8 beads per row. In the vertical layout (Figure 1.5(b)), the number of beads per row alternates

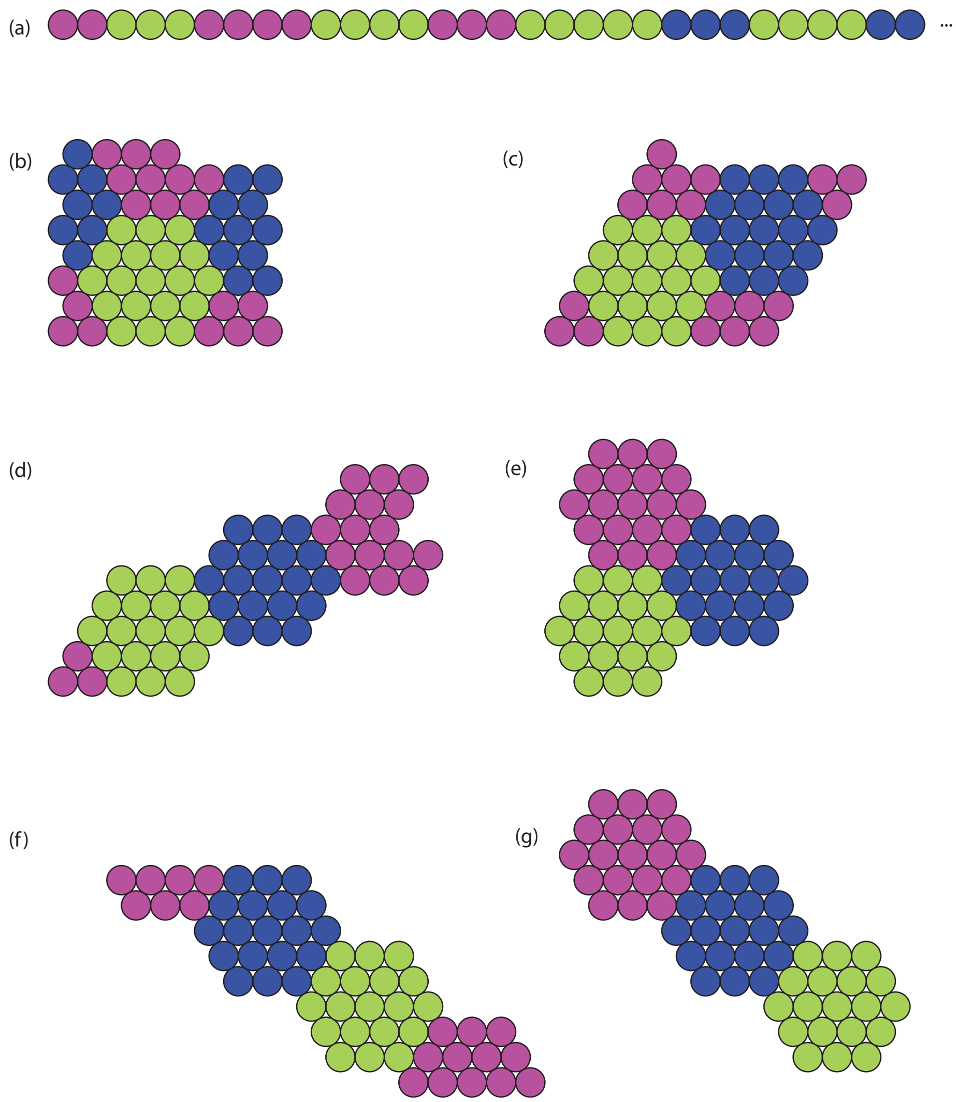


FIGURE 1.5 Different flat layouts of the hexagonal pattern. Layouts (a), (b), (c), and (f) provide the stringing order if read from left to right, bottom to top. Layouts (d), (e), and (g) do not.

between N and N + 1, creating the effect that produces its zipper-like edges.

Although the options in Figure 1.5 are all valid flat layouts for a 7-around bracelet,* you might notice that only some of them provide a straightforward indication of the correct stringing order. Ideally, a good layout will both add visual clarity *and* clearly indicate a way of mapping itself back into the single strand needed for stringing and crocheting. Unfortunately, for this particular pattern, the layouts that show the pattern most clearly (namely, (e) and (g), the two that keep the hexagons intact), don't provide an accurate stringing order (when read from bottom to top and left to right). But there's another problem that is even more serious. When you are constructing a design, you don't know ahead of time what it will look like; you want to be able to explore many possibilities. So you have no guidance in advance about what kind of layout to choose. Certain layouts turn out to be better for visualizing some types of designs than others. So what's a designer to do?

* That these are all valid layouts may not seem obvious at the moment, but we will explain how we determined this shortly (illustrated in Figures 1.11 and 1.12).

The Infinite Bead Plane

If you imagine you are a tiny explorer charting the colored surface of a bead crochet bracelet, you would find that, just as on a globe, the surface seems to continue forever

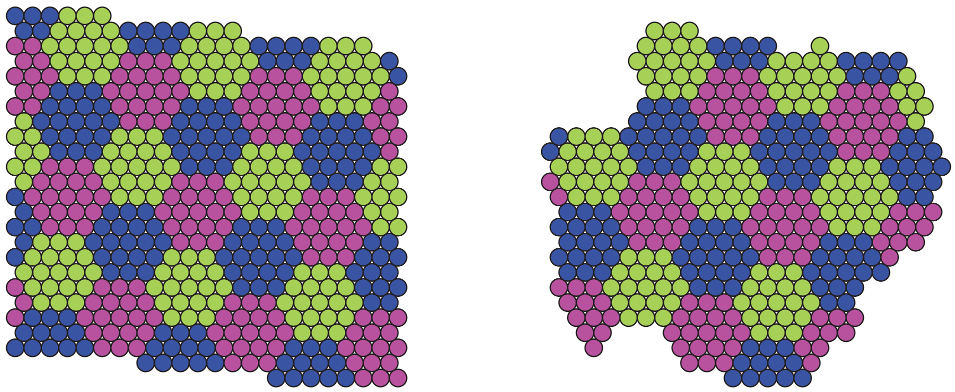


FIGURE 1.6 Two possible maps of the same bracelet. An intrepid ant crawling across the bracelet and taking notes on the colors along the way might create either of these charts.

regardless of the direction in which you choose to travel. From your perspective, you might even think, as some early explorers of the Earth did, that you were travelling on a flat plane extending infinitely in all directions. After all, you can always keep going without ever reaching an edge, but the distances are short, so you could also easily start looping around without realizing it. As you loop, you might notice that the terrain colors keep repeating themselves according to a predictable pattern, so you might begin to suspect that your theory about being on a flat planar surface was flawed. However, without any additional clues (like celestial bodies or a strong magnetic pull) to sway you in favor of one theory over the other, you could have trouble deciding whether you were on a curved surface, such as a donut, or on a flat plane with an infinitely extending repeated pattern. Either model works pretty well based on the evidence.

A key insight into bead crochet pattern design is that viewing a bracelet as just such an infinite plane is a powerful way to maximize a designer's sense of freedom while enabling a clear visualization of what is going on in a pattern. As long as you obey some simple rules about how the pattern repeats, you don't have to constrain yourself in advance by picking a particular flat layout shape. Designing directly in this conceptual space, which we call the *bead plane*, provides a unique way of capturing and interacting with all visual vantage points at once. Our tiny explorer charting a bracelet sporting the hexagonal pattern from Figure 1.5 might come up with either of the two equally valid "maps" shown in Figure 1.6 (or any of an infinite number of possible alternative peripheral shapes). Likewise, either of these (uncolored) would work fine as a flat layout with which to begin designing, mainly because either one is big enough to give a decent sense of what's going on in a 7-around bracelet pattern.

Almost all the patterns we could find in books or online prior to our own work are visibly based on the left-leaning diagonal layout or the vertical layout. Since we designed most of our patterns in the bead plane, they are not limited by any particular layout shape. The pattern section in this book contains an explosion of designs that, like the hexagonal example in Figure 1.6, are utterly confusing viewed solely on a diagonal or vertical chart. Designing in the bead plane lifts unnecessary constraints that have long plagued bead crochet designers.

There are two points that we need to establish to make this a workable design method: what are the repetition rules that give us a valid bracelet pattern for a given circumference, and how do you extract the stringing order from a pattern in the bead plane? We will address both issues shortly. The important point for now is that, as long as we follow appropriate rules, we could just as easily begin with a big—or essentially infinite—uncolored sheet of bead crochet graph paper (such as the ones provided on pp. 247–250) and start filling it in, much as our imaginary bead explorer might.

Hockey-Stick Translations

Imagine again that you are exploring the colored surface of a bead crochet bracelet. Because the surface is finite but has no edges for you to drop off, your travels may eventually bring you back to where you started. If you travel in a fixed direction and are mapping the colors encountered as you go, a return to where you started will cause the colors on the map to start repeating. What type of repetitions would you expect? Consider for instance a 7-around bracelet, and choose any bead on it at which to begin your travels. Because bracelets are crocheted in the round,

the crocheted beads form a spiral. So, if you were to travel in a constant direction from each bead to the next bead on the thread, you would be travelling in a continuous spiral moving up (or possibly down) the length of the rope, and you would eventually spiral right back to where you started. Whether you spiral up or down depends on whether you move right or left and on whether the bracelet was crocheted left- or right-handed. In general, left-handed crocheters use the same charts as right-handed crocheters but end up with mirror-reversed patterns. To make the remaining discussion more straightforward, we will describe everything in terms of right-handed crochet. For more information about the relationship between left-handed and right-handed bead crochet, see p. 111.

Alternatively, if you move exactly 7 beads horizontally to the right and then 1 bead diagonally down and to the right, that will bring you right back to where you started. More generally, for an N-around bracelet, moving to the right N beads and down-right 1 bead always brings you back to your starting point. Note also that the inverse form of the rule works as well: moving N beads to the left and up-left 1 bead likewise returns you to your starting position. We call this a *hockey-stick translation of length N* because if you were to map out this movement on a piece of bead crochet graph paper, you'd find it takes the shape of a hockey stick, as shown in Figure 1.7. If you have a bead crochet bracelet on hand, it's helpful to try actually tracking this movement with your finger to convince yourself that no matter where you begin, a hockey-stick translation of length N will always return you to your starting position. Figure 1.7 shows an example of what the hockey-stick translation looks like on one of our maps of the 7-around hexagonal pattern.

A quick check of the pattern should convince you that a hockey-stick translation of length 7 away from every bead is another bead of the exact same color. Thus this pattern consistently obeys the hockey-stick translation rule for hockey sticks of length 7 and is a valid 7-around pattern. In mathematics, a *translation* is the name for a motion that doesn't involve any rotations or flipping, such as sliding a penny from one position on a table to another without allowing it to rotate or flip over as you slide it. Since the hockey-stick translation rule applies to every possible starting point, taking the entire bead plane design (assuming we let it go on forever in all directions in the infinite bead plane) and sliding it along the hockey-stick translation will leave the color pattern entirely unchanged! As mathematicians would describe this property, the pattern is *preserved* by the hockey-stick translation.

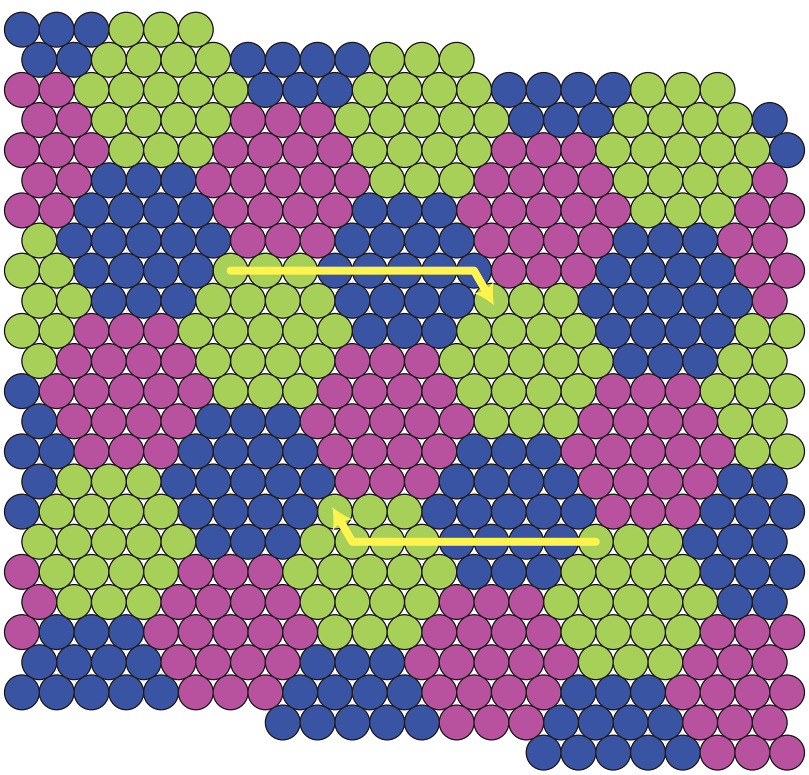


FIGURE 1.7 The hockey-stick translation of length 7. The two beads at either end of each yellow arrow represent the same bead in a 7-around bracelet.

The hockey-stick translation represents our first of two important rules about how bead crochet patterns must repeat. On a bead crochet graph for an N-around bracelet, any two beads that are related to one another by a hockey-stick translation of length N must be the same color, or equivalently the entire bead plane pattern is preserved by a hockey-stick translation of length N. Figure 1.8 shows a series of patterns that you can use to test your understanding of the hockey-stick translation rule. Can you identify which of the patterns consistently obey the hockey-stick translation rule and are thus valid bead crochet patterns, and which are not? For the valid bead crochet patterns, can you identify the circumference of the bracelet?

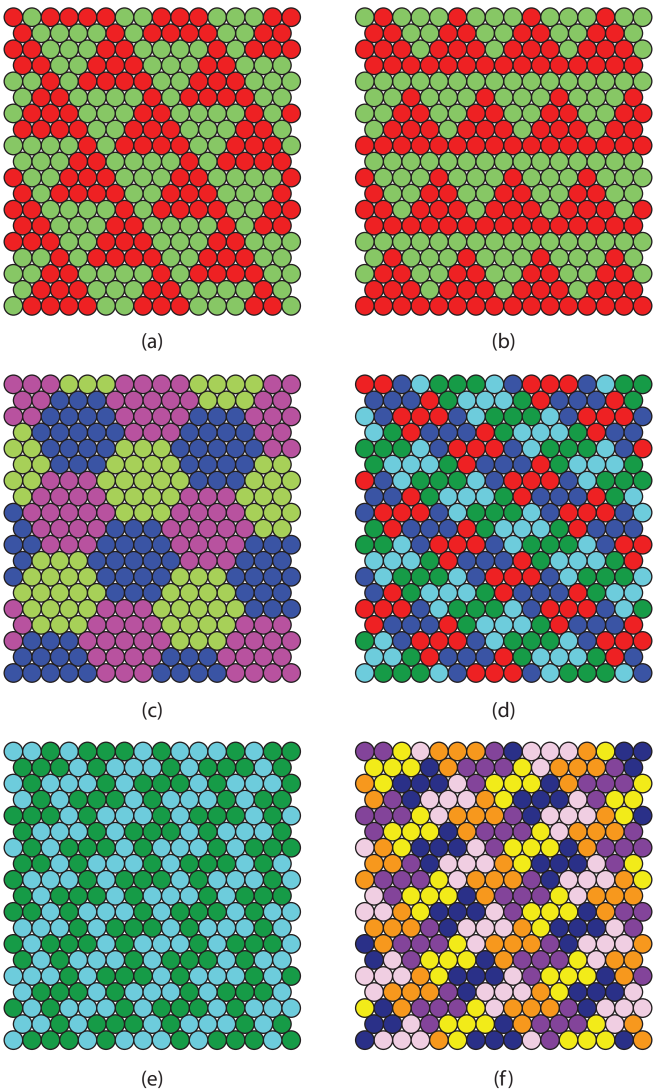


FIGURE 1.8 Some of these patterns are valid bead crochet patterns and some are not. Can you identify which are which? The valid patterns obey the hockey-stick translation rule for some N. This means that every bead in the pattern is the same color as the beads that are a hockey-stick translation of length N away from it. For the valid patterns shown here, N is 5, 6, or 7. (Solution: (a) is a valid 5-around pattern; (b) is not valid; (c) is a valid 7-around pattern; (d) is not valid; (e) is a valid 6-around pattern; (f) is a valid 6-around pattern.)

Note that patterns (d) and (f) are the same except for coloring—one of them is a valid bead crochet pattern and one is not. This is because only one of them is painted so that every bead is the same color as those beads related to it by a hockey-stick translation. Once you get used to it, identifying and constructing valid bead crochet patterns becomes second nature, but it can be confusing when you're just starting out.

Repeat Translations

The hockey-stick translation is the first of two rules about how bead crochet patterns repeat. It corresponds to how the left and right edges of a vertical layout are fastened together in a finished bracelet. The second rule about how patterns repeat corresponds to the way in which the top and bottom edges of a vertical layout are connected. If you again imagine yourself a tiny explorer navigating the surface of a bead crochet bracelet, the hockey-stick translation indicates how the pattern repeats itself as you move horizontally, circling the short way around the rope. The second rule, which we'll call the *repeat translation* rule, indicates how the pattern repeats as you move along the length of the rope. The repeat translation doesn't have a specific shape like the hockey stick—its shape will change from design to design depending on the length of the pattern repeat. It corresponds to how the first bead of a repeat connects to the first bead of the previous or following repeat.

In mathematical terms, the hockey-stick translation and the repeat translation together constitute a set of two *independent* translations, which creates what is called a *doubly periodic pattern in the plane*. All this means is that our bead crochet repeat in the flat layout of your choice can be used to fill (or in the technical parlance, *tile*) an infinite planar surface with no gaps or overlaps—or a finite one, like a piece of bead crochet graph paper. (Naturally, on a finite planar surface, some tiles may be cut off at the edges, like the tiles on the edge of a bathroom floor that are trimmed to fit the room.) What makes the translations independent is that they point in two nonparallel directions as shown on the left of Figure 1.9, which allows our repeat to fill the entire plane. Thus, as long as we create a repeating pattern on bead crochet graph paper that includes

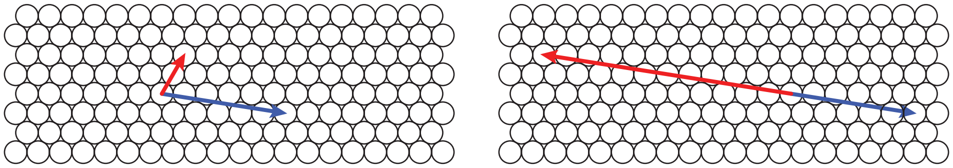


FIGURE 1.9 Independence of translations. The two translations on the left are independent because they move the plane in two nonparallel directions. By contrast, the translations on the right are not independent.

both a hockey-stick translation and another independent repeat translation, it qualifies as a valid bead crochet pattern, and it forms a doubly periodic tiling of the bead plane. Ensuring that the pattern you are developing has a hockey-stick translation can take some careful attention, but the repeat translation emerges naturally from the repetition of the color sequence that makes up the bracelet design. Figure 1.10 shows how the hockey-stick translation (in blue) and the repeat translation (in red) determine how layout tiles (in this case a vertical layout) are fitted together to tile the plane. Notice that in the bottom diagram we are drawing the blue hockey-stick translation as a straight line instead of the characteristic hockey-stick shape we have been using up to now.

In Figure 1.11, we explore different layout possibilities for a particular bracelet design, the hexagonal pattern from Figure 1.5. Figure 1.11(a) shows a sheet of bead crochet graph paper tiled with an uncolored vertical layout for a 7-around, 57-bead repeat design (the parameters of the hexagonal pattern). Starting at any tile, we can place adjacent tiles with either the hockey-stick translation of length 7 or a repeat translation of size 57, as marked by the blue and red arrows. Figure 1.11(b) shows the hexagonal pattern painted on the tiles constructed from the same vertical layout, and Figures 1.11(c) and 1.11(d) show it painted on two other layouts. The white arrows illustrate that the same two independent translations are used regardless of the choice of layout. We can draw two important conclusions from these diagrams: each valid layout will tile the bead plane with the same translations, and each layout can be painted so that identical copies of that layout fit together to produce the same bead plane design.

In fact, the reverse is also true: the hockey-stick and repeat translations tell us exactly what constitutes a valid bracelet layout. The reason we know that the unusual 3-hexagon layout in Figure 1.5(e) and Figure 1.11(d) is a valid layout for a 7-around bracelet with a 57-bead repeat is precisely because it tiles the bead plane with no overlaps when we apply our hockey-stick and repeat translations. All the other layouts in Figure 1.5 will tile the plane under the same translations.

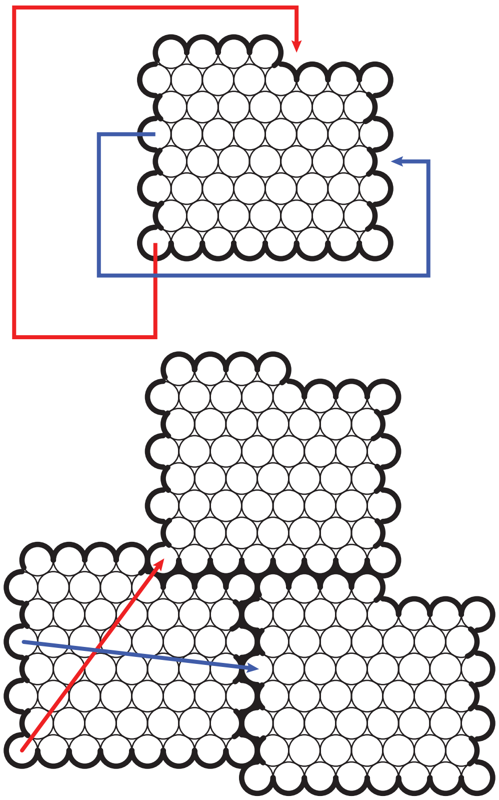


FIGURE 1.10 Two independent translations in the bead plane. The hockey-stick translation (in blue) and the repeat translation (in red) produce a periodic tiling of the plane.

Another way to think about what constitutes a valid layout is to consider a bead plane diagram with beads numbered according to a particular circumference and repeat length. Beads that are a hockey-stick or repeat translation away from one another are assigned the same number. Figure 1.12 (top) shows an example of a bead plane diagram of a 4-around, 21-bead repeat pattern with this numbering. The numbering would work for any pattern like this one, with N = 4 and R = 21. It's easy to construct the numbering by starting with a single horizontal row of R sequentially numbered beads (21 in this case, for example, the top row in Figure 1.12). We can think of this row as the uncrocheted strand of a single repeat, or the simplest flat layout. Then all the other numbers are filled in

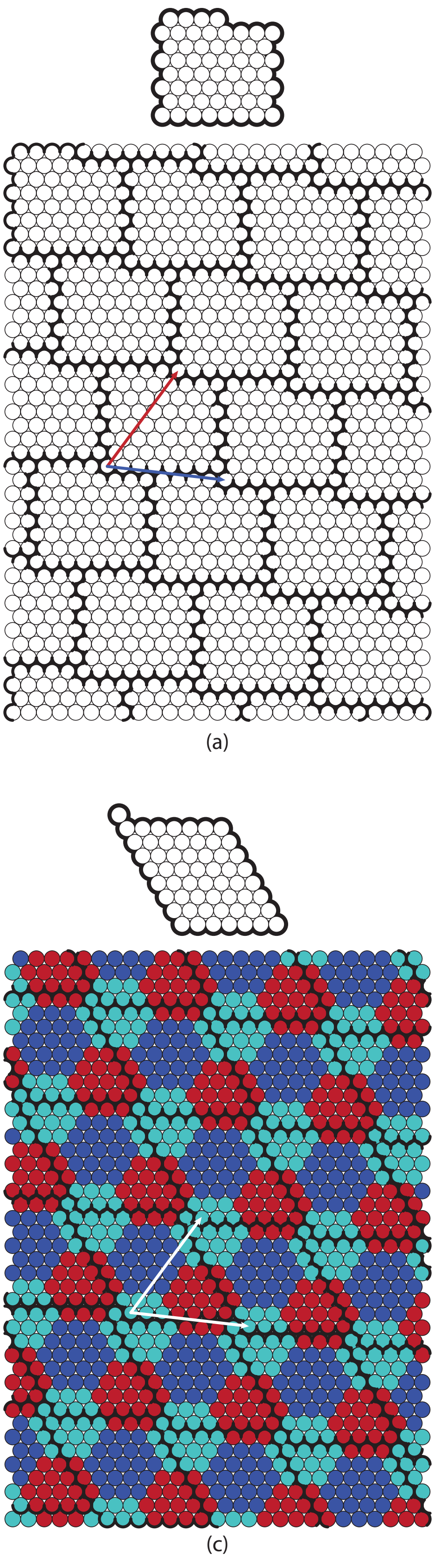


FIGURE 1.11 Tiling the bead plane with different layouts of the hexagon design from Figures 1.4 and 1.5. These are different ways of giving a pattern for the same bracelet design.

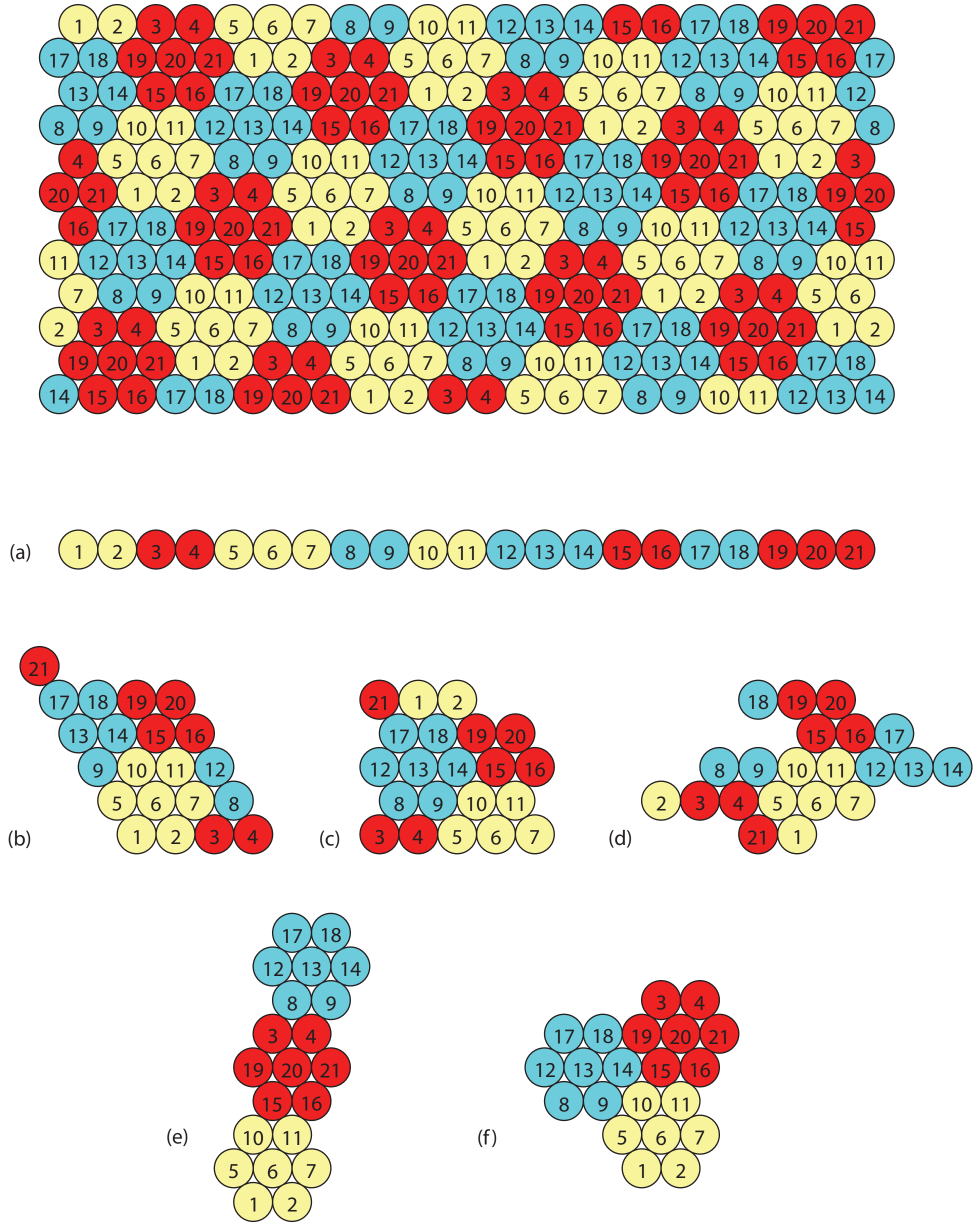


FIGURE 1.12 A numbered bead plane diagram (top) and example valid layouts (a–f) for a 4-around pattern with a 21-bead repeat.

according to the hockey-stick and repeat translation rules for a 4-around 21-bead repeat. Once we have this numbering, a valid layout is *any contiguous set of R beads that includes exactly one bead of each number*. Figure 1.12 shows a variety of such valid layouts. This process is analogous to carving out and laying flat a single tile representing one repeat from a bracelet. Limiting the tile to only one bead of each number on the bead plane ensures that we don't effectively grab the same bead twice (which is possible on a bead plane diagram, but not on a physical bracelet) and that every bead in the repeat is represented once and once only. This process also makes it easy to see how many choices there are for valid flat layouts. If the numbering on a layout can be read left to right, bottom to top, without a disruption in the number sequence (as it can in Figures 1.12(a)–1.12(d), but not 1.12(e)–1.12(f)), then that layout also provides an easy way of reading the stringing order. Another nice feature of the diagonal and vertical layouts (Figures 1.12(b) and 1.12(c)) is that they have a straightforward method for adjusting the values of R and N to get a comparable layout with a different circumference or repeat length.

Lattices and Tilings

Every repeating pattern in the bead plane has an associated *lattice* based on its two independent translations. As we will see, these lattices can aid in generating patterns. To construct a lattice for a bracelet, we need only two pieces of information: N, the bracelet circumference, and R, the pattern repeat length. Patterns with the same values of N and the same values of R will have the same underlying lattice. A lattice consists of a set of points (or beads) that are related to one another by combinations of the hockey-stick translation and the repeat translation. Given a layout tiling such as those shown in Figures 1.10 and 1.11, the lattice can be constructed by selecting the first bead in every layout tile (or the third, or the tenth, or the Nth). Figure 1.13 shows two different lattices. On the left is a lattice for the hexagonal pattern—or any other 7-around pattern with a repeat length of 57. On the right is a lattice for a 5-around pattern with a repeat length of 16. In each case, the lines in the grid between the lattice points show that you can travel from any bead in the lattice to any other bead by following some number of hockey-stick translations (or reversed hockey-stick translations) and some number of repeat translations (or reversed repeat translations). Since a valid bead plane bracelet pattern is preserved by both the hockey-stick translation and the repeat translation, any translation taking one lattice bead to another will preserve the bracelet pattern.

With a lattice in the bead plane, we can construct an associated *tiling* of the plane, a way of covering the bead plane with clusters of beads, or *tiles*, with no gaps or overlaps between tiles. Even better, we can choose a completely uniform tiling, in which each tile is exactly the same shape and contains exactly one bead from the lattice. Any such tile will contain exactly R beads and will be a possible flat layout for a single repeat of our bracelet.

In our terminology, the *lattice* refers only to the points (the gray beads) themselves, not the grid lines. However, drawing the grid lines to connect the lattice points can aid in producing a nicely shaped tile associated with the lattice, as shown in Figure 1.13. The tile (in red) also includes one lattice bead (in gray). The rule we used to determine each tile from the grid is that any bead on an interior section of grid is considered part of that section's tile, while beads intersecting the grid are chosen to be part of that tile or an adjacent tile using some consistent rule. For example, the rule we used here was that if more than half the bead was inside the grid boundary, that bead was included in the tile, and otherwise it was part of the adjacent section's tile. For beads exactly halfway

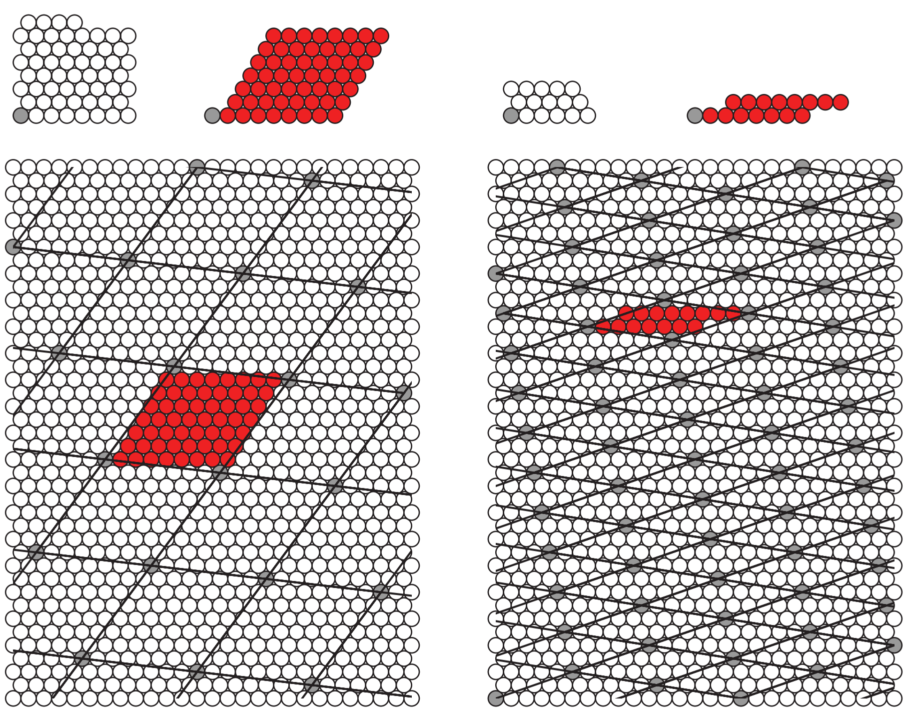


FIGURE 1.13 Lattices and possible associated tiles for a 7-around, 57-bead repeat (left), and a 5-around, 16-bead repeat (right). The vertical layout tile is also given in the upper left in each case.

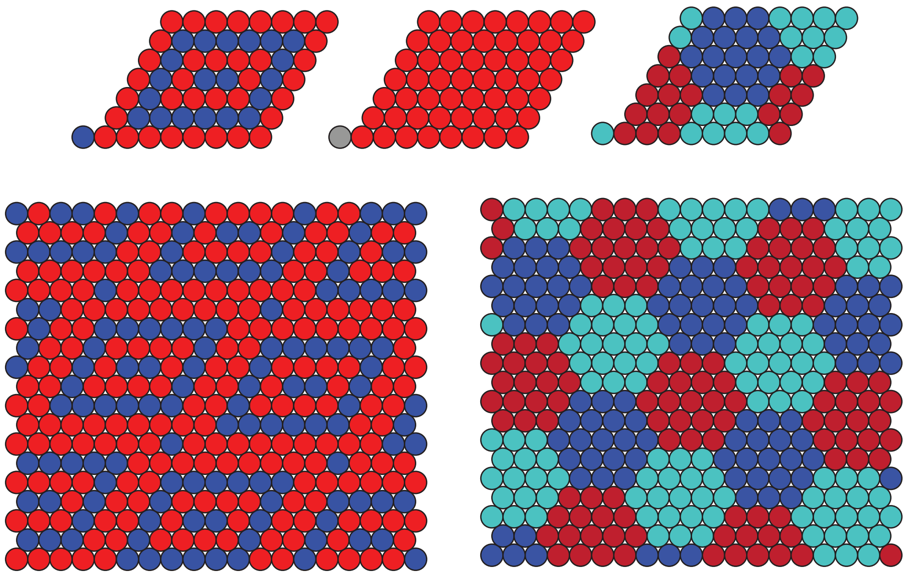


FIGURE 1.14 Two ways of coloring in a 57-bead tile for a 7-around bracelet to create a bracelet design.

inside the grid boundary, we simply made a consistent choice across all tiles.

Another important observation is that every tile can be moved into the position of any other tile with the translation that takes the lattice bead in the first tile to the lattice bead in the second tile. Since any translation that moves one lattice bead to another preserves a bracelet design, *the bead color choices within a single tile completely determine the entire pattern!* This makes perfect sense since each tile comprises a single repeat of the bracelet. Figure 1.14 shows how the tile produced from the lattice on the left of Figure 1.13 can be painted two different ways to produce two completely different looking patterns that nonetheless have the same lattice.

There are multiple ways to draw the grid lines of a lattice. The grid lines that we used in Figure 1.13 were determined by the hockey-stick translation and the repeat translation. In the interest of full disclosure, we should point out that while the hockey-stick translation is determined solely by the circumference of the bracelet, there are multiple choices for the repeat translation in a given bracelet pattern, which give rise to different grids, as demonstrated in Figure 1.15. A repeat translation moves the first bead in a repeat (marked in yellow) onto the first bead in the next repeat (marked in green). However, each bead in a bracelet appears over and over again in the bead plane pattern because of the hockey-stick translation rule, and each copy of the green bead gives a different repeat translation that generates the same lattice. Since a horizontal path in the bead plane always traces the beads in their stringing order, one possible repeat translation is simply the horizontal translation by R beads, where R is the repeat length, as shown at the bottom of Figure 1.15.

The flexibility in choosing grid lines for a lattice gives us more options for forming different tile shapes with the technique from Figure 1.14. In Chapters 6 and 7, we will see even more ways to use lattices and tiles to produce bracelet designs with intricately symmetric structures.

Readers with advanced mathematical training may have noticed that the process of extracting a bead plane diagram from a toroidal bracelet and constructing the associated lattice has a familiar ring. In fact, using the language of algebraic topology, when you imagine yourself as a tiny explorer mapping out a bracelet in the bead plane, you are constructing the *universal covering space* of the beaded torus. If you happen to be versed in algebraic topology, you might contemplate how the hockey-stick translation and the repeat translation are related to the fundamental group of the beaded torus.

Designing in the Bead Plane

How does all this help us create new designs? Designing directly in the bead plane allows an artist to visualize more immediately the relationships between shapes in a pattern. We can effectively see what's going on everywhere without having to choose a particular viewing angle or flat layout shape. With our knowledge of what constitutes a valid bead crochet pattern in the bead plane, we can

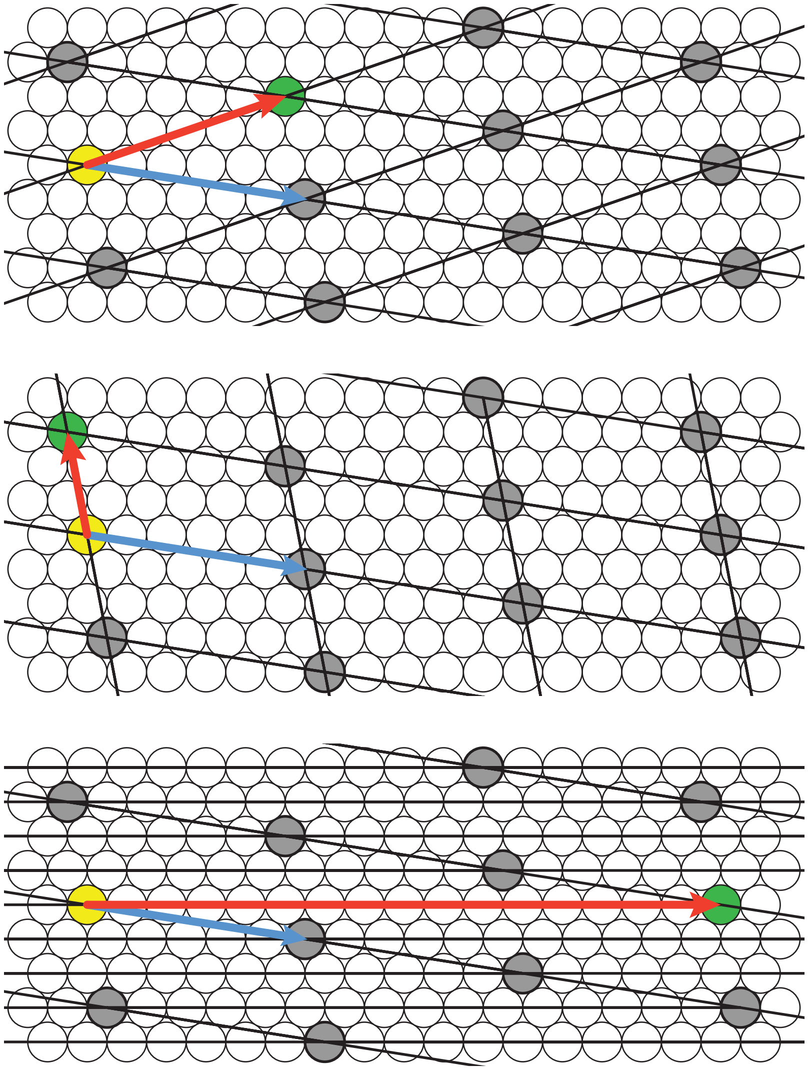


FIGURE 1.15 Different repeat translations for the lattice corresponding to a 5-around bracelet with a repeat length of 16. The first is the repeat translation read from the vertical layout, as used in the grid on the right of Figure 1.13. The second is the repeat translation read from the traditional diagonal layout. The third is a horizontal translation by 16 beads.

explore in a very immediate way how all the edges connect in a flat layout, and we can see quickly how a design change affects overall appearance as we adjust individual bead color choices. Although some supporting software could certainly be helpful (and we hope that our approach will inspire improvements in bead crochet design software), designing in the bead plane can still be accomplished the old-fashioned way, with bead crochet graph paper and erasable colored pencils.*

When you work in the bead plane, as you paint a bead blue, for example, you know that all beads that can be reached from it with hockey-stick translations must also be painted blue. You can begin with a desired circumference and adjust if it looks like a different circumference might work better. If you are willing to choose both the circumference and the repeat length in advance, the design process becomes quite straightforward, as illustrated in the sequence in Figure 1.16. It is usually sufficient to work with a grouping of tiles in which at least one tile is fully surrounded by others. This sequence shows, for a 7-around bracelet, how a decision to color one bead propagates (in this case via a hockey translation of length 7, and a repeat translation of size 57) to other beads in the graph. Any layout or lattice-derived tiling would work as a starting point, but, in this case, we chose to begin with a grouping of seven vertical layouts.

* If you have it, some sort of graphics software is certainly quite helpful. A vector graphics editor, such as Adobe Illustrator, although not essential, is a particularly wonderful tool, since it makes it easy to copy, paste, and move entire tiles or individual design elements.

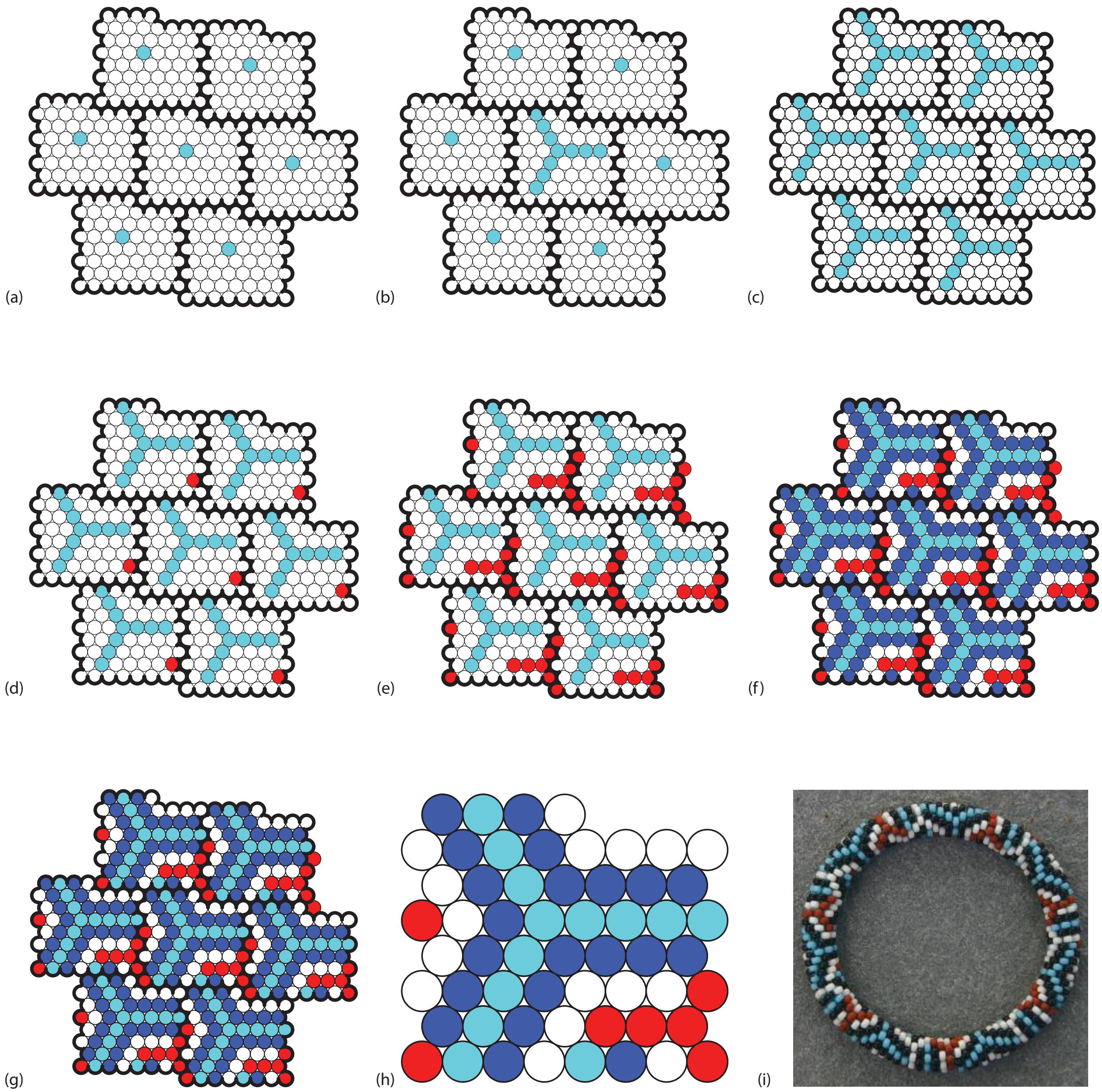


FIGURE 1.16 A design sequence in the bead plane (a–g) using a fixed repeat length of 57 and circumference of 7, and the resulting repeat (h) and bracelet (i). The bracelet appears in the pattern pages as Woven P3 (p. 226).

If you prefer a little more flexibility, you can simply choose a circumference and construct a design in several adjacent vertical layout strips, allowing the repeat length to be determined by the pattern (or even making a nonrepeating pattern). In fact, there are ways of designing bracelets without deciding on a repeat length *or* a circumference in advance! But for those techniques, you will have to wait until Part Two of this book.

Although a design in the bead plane gives a complete view of how the shapes and motifs in a pattern relate to one another, it can give a misleading impression of how the design will look on a finished bracelet. This is because the bead plane effectively simulates a view from all vantage points simultaneously, whereas a real bracelet exists in three-dimensional space, and not all of it can be seen at once. We will discuss strategies for addressing these issues shortly.

Extracting a Repeat from a Bead Plane Diagram

Once you've created a valid bead crochet design in the bead plane, you need the stringing order to create a bracelet from it, so you'll need to extract a flat layout from your design. If you've started with a clearly outlined tiling of vertical layouts, as in the design example in Figure 1.16, then extracting one of them is easy. But even if you haven't, reading a single repeat from a bead plane pattern is no problem at all. Moreover, this extraction can be in whatever flat layout you prefer. To use the pattern as a stringing guide, you should extract a layout that provides a clear indication of the correct stringing order. We typically use the vertical layout for this purpose because it is the most compact representation on a rectangular page, but, for short repeats, a diagonal layout works equally well. If your bead plane diagram is wide enough, even a single horizontal "strand" of R beads will do the trick!

Figure 1.17 shows the hexagonal pattern in the bead plane and, outlined in black, several different ways to extract a single repeat from it: two placements of a vertical layout, two placements of a diagonal layout, and two placements of the layout from Figure 1.5(e), one of which keeps the hexagonal motifs intact. This last layout does not provide an easy means of reading the stringing order, but it can nonetheless be used to tile the bead plane in this pattern, as can any of the layout shapes from Figures 1.4 and 1.5 for a 57-bead repeat in circumference 7. You can imagine that you are taking a cutout template of your preferred layout and that it is legitimate to translate it anywhere on the bead plane pattern.

To extract a repeat from any bead plane pattern, you need to know the bracelet circumference, N, and the repeat length, R, both of which are easy to work out. Take, for instance, the bead plane design in Figure 1.18, which is visibly doubly periodic. If you start with the upper left corner of one of the beige diamonds and scan the pattern for a hockey-stick translation connecting it to the upper left corner of another beige diamond, you find that a hockey-stick translation of length 6 preserves the entire design. (If you prefer, you can read this from any other recognizable

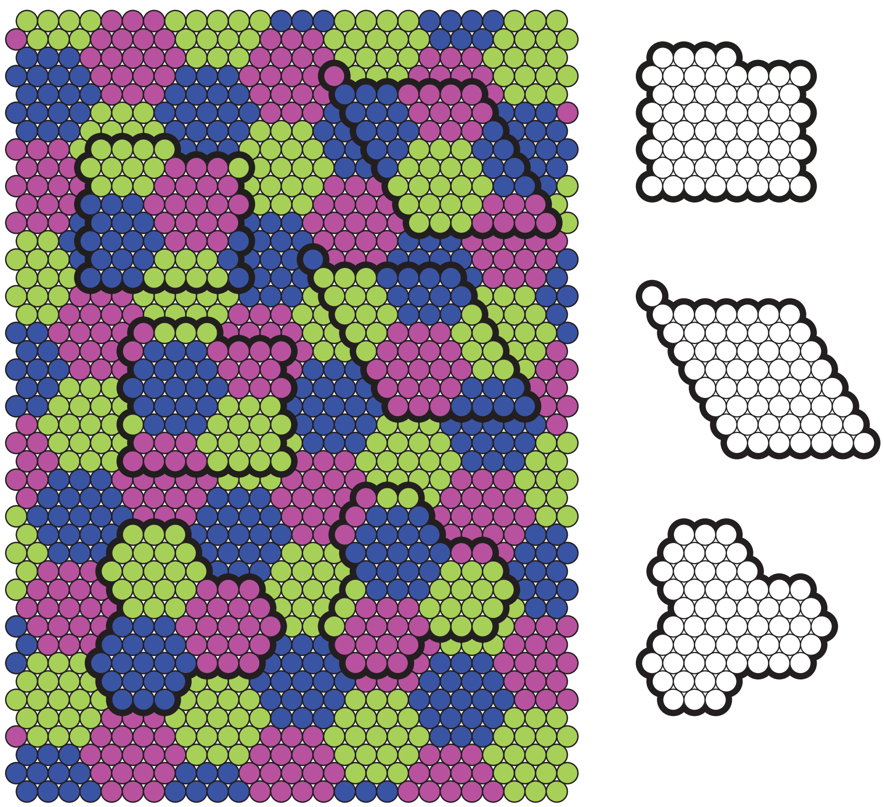


FIGURE 1.17 A 7-around pattern in the bead plane and examples of different ways to extract a flat layout from it. The vertical layout (top right) and diagonal layout (middle right) both provide a correct stringing order for the repeat when read from bottom to top and left to right. Since the stringing order is cyclical, it does not matter where the template is placed on the bead plane; the resulting pattern will be the same, although the rope's closing seam may wind up in a different spot.

pattern feature, such as the red dots in the middle of the beige parallelograms, or the lower left corners of the red diamonds.) In rare cases, more than one length of hockey stick works for a particular design, and you can choose your favorite. As we will see in Chapters 3 and 8, sometimes a minor change in a bead plane diagram will produce a pattern for a new circumference.

The design in Figure 1.18 creates a 6-around bracelet. By peeling out either a vertical layout strip or a diagonal layout strip of circumference 6 from the bead plane, as in Figure 1.19, you will find the stringing order for your bracelet, in which the repeat occurs over and over again. To determine the length of a single repeat, simply scan the stringing order for the point at which the color sequence repeats. Writing out the number of beads in each color makes it easier to confirm that the color sequence repeats through the entire length of the bracelet. In this case, the repeat is 24 beads long.

As we see from the two stringing patterns given in Figure 1.19, the stringing order you extract from a pattern varies depending on where you start your strip of the bead plane. Since each color sequence is repeated over and over again, the only difference between bracelets produced by

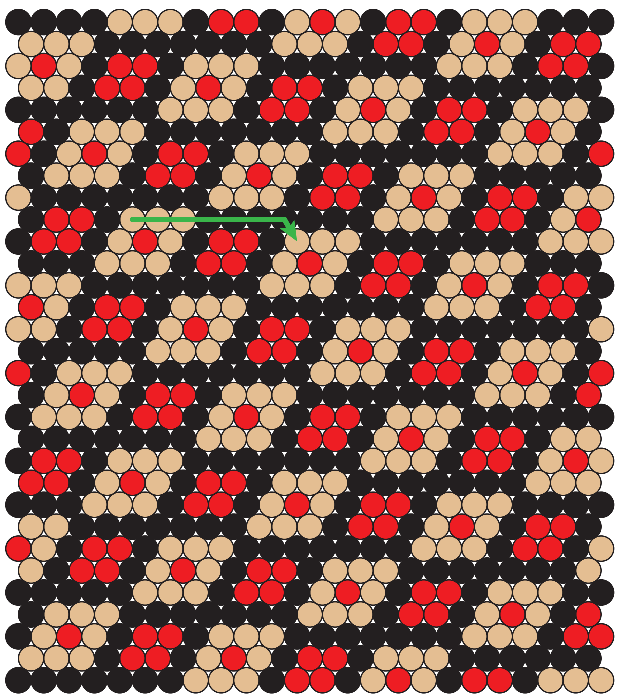


FIGURE 1.18 A bead plane pattern preserved by a hockey-stick translation of length 6.

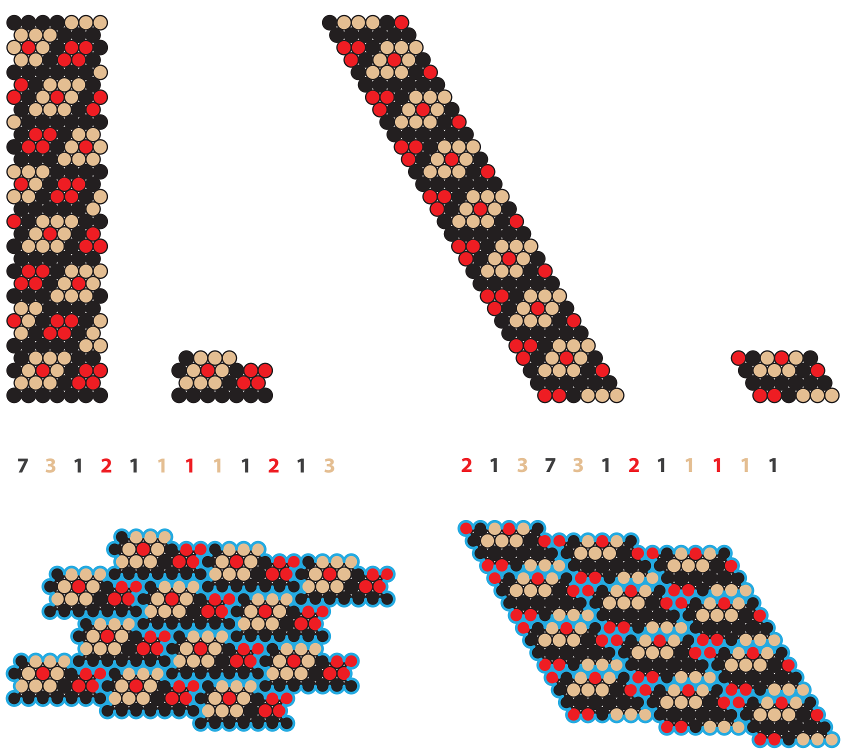


FIGURE 1.19 Extracting a single repeat of a pattern with unknown repeat length. The two color-coded numerical sequences are two possible stringing orders for a single repeat, each determined from the strip of the bead plane above it. This design appears in the pattern pages as Bon Bon Checkerboard (p. 223) using the first stringing order.

the two stringing orders is the position of the seam connecting the beginning of the bead crochet rope to the end. At the bottom of Figure 1.19, we confirm that we have extracted two correct stringing orders by tiling the bead plane with a single repeat and reconstructing our bead plane pattern.

FIGURE 1.20 Two pairs of bracelets with the same pattern closed with different physical twists. The patterns are Zadie's Wave (p. 185) and Diamond Zigzag (p. 153).

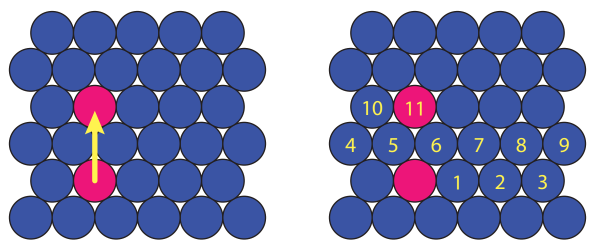


FIGURE 1.21 The vertical separation between beads. Two beads that are vertically "adjacent" are 2 rows apart and separated by an interval of 2N + 1 beads in an N-around bracelet. In this case, N = 5, so 2N + 1 = 11.

Twists

A surprising "twist" in our bead crochet design journey was the discovery that details of the final step, sewing the rope closed, can have an enormous impact on the appearance of the resulting bracelet. The story is all in the twist! It is a confusing story, however, because there are multiple ways to twist a bracelet before closing and each has a different visual impact.

This section has more analytic details than the previous sections, some of which are not essential for many aspects of bead crochet design. In making bracelets, you can use twists to great effect without understanding all of the theory behind them. If you are not numerically inclined and find parts of this discussion to be overly technical, we recommend that you glance over them and return to them as needed when you are actually designing or closing a bracelet. We particularly advise studying the photographs in Figures 1.20 and 1.27, which give nice illustrations of the effects of twisting.

As we've already noted, bead crochet rope has the wonderful quality of being flexible and a bit stretchy, a feature that contributes enormously to its tactile appeal. In fact, it is often so flexible that it's easy to introduce one or more full *physical twists* in one end of the rope before closing (sometimes inadvertently!), and doing so can dramatically alter the look of a design. Figure 1.20 shows two pairs of bracelets crocheted with the exact same pattern but closed with different physical twists, creating strikingly different looks. It is always a good idea to hold the two ends of a bracelet together in position and look carefully before closing to make sure the bracelet is twisted in the way that you intend.

In addition to full physical twists, it is also possible to introduce tiny partial *structural twists*. A structural twist is determined by the total number of beads in the bracelet, and many bracelets have just such a structural twist built into them, although you might have to inspect rather carefully to recognize one in a completed bracelet. The key to understanding whether or not a bracelet has a structural twist is to look more closely at how vertical layouts work. As we observed in Figure 1.5, the rows in a vertical layout for an N-around bracelet alternate between N and N + 1 beads. Two consecutive rows, which we call a *double row*, will therefore always contain exactly 2N + 1 beads. The offset between beads in consecutive rows means that the bead directly above a given bead will be two rows up—or equivalently, one double row up. This is illustrated for circumference 5 in Figure 1.21, which also shows that two vertically "adjacent" beads (because of the row in between, they do not actually touch) are exactly 2N + 1 beads apart.

Bracelets without a structural twist, which we call *structurally aligned*, are those whose total number of beads is an exact multiple of 2N + 1. All other bracelets have a

structural twist. It's easy to see whether or not a bracelet has a structural twist if you view it in its entirety in vertical layout form. If it is structurally aligned, there will be a complete double row at the top. Figure 1.22(a) shows a structurally aligned bracelet, and Figures 1.22(b)–1.22(e) show four others with structural twists. (These are too short to be real bracelets, but they suffice for compact examples.)

A structural twist forces a partial twist (less than 360°) of the bead crochet rope, which is necessary to line up the two ends of the bracelet for sewing. The reason we refer to this type of twist as structural (rather than physical) is because it changes the underlying structure of the bracelet in a way that can be detected with careful inspection. A structural twist can be combined with any number of full physical twists.

If the repeat of a design contains a multiple of 2N + 1 beads, it is *vertically aligned*, because each repeat stacks directly above the previous one on the crocheted rope. Vertically aligned designs guarantee a structurally aligned bracelet, as shown in the 7-around example in Figure 1.23. The yellow arrow in the bead plane diagram on the right shows the vertical stacking.

On the other hand, some designs in which consecutive repeats aren't vertically stacked will line up vertically every 3 repeats, or every 5 repeats. A pattern that stacks vertically at every Kth repeat is called *vertically aligned by K*. For example, consider the pattern in Figure 1.24 with a 35-bead repeat. This is a 7-around (N = 7) design, so 2N + 1 is 15, which does not divide evenly into the repeat size of 35. We can see this clearly in the jagged top of the single repeat shown on the left. Yet a set of three such repeats, shown in the middle, is 105 beads, which *is* evenly divisible by 15. Thus the triangle motif stacks vertically at every third repeat, as indicated by the yellow arrow in the bead plane diagram on the right. A bracelet constructed from this pattern can be made more symmetric by using a multiple of 3 repeats, which permits lining up every third triangle motif in a structurally aligned bracelet.

Taking a closer look at the numbers involved, we see that it is important that the double row length, 15, and the repeat length, 35, have a common factor of 5. Since 15 = 5 × 3, any design whose repeat length is a multiple of 5 that isn't already vertically aligned will be vertically aligned by 3. On the other hand, if we work

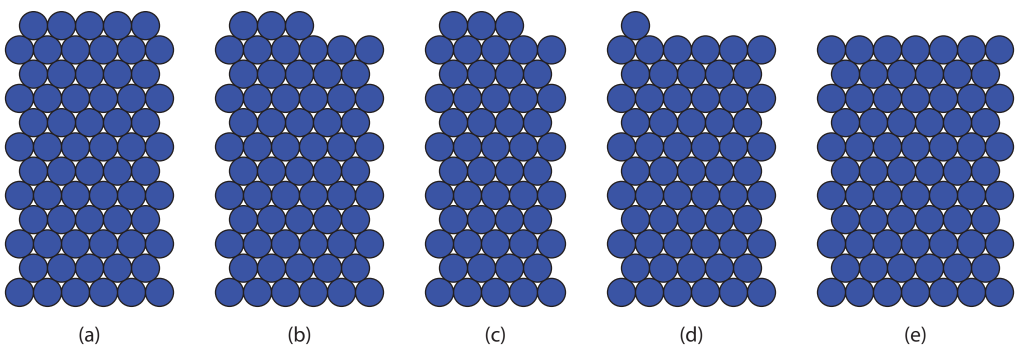


FIGURE 1.22 A structurally aligned bracelet (a) has a multiple of 2N + 1 beads, or a whole number of double rows in vertical layout form. All the other bracelets (b–e) are structurally twisted, which means they require the bead crochet rope to be twisted slightly to connect the first bead to the last bead at the closure.

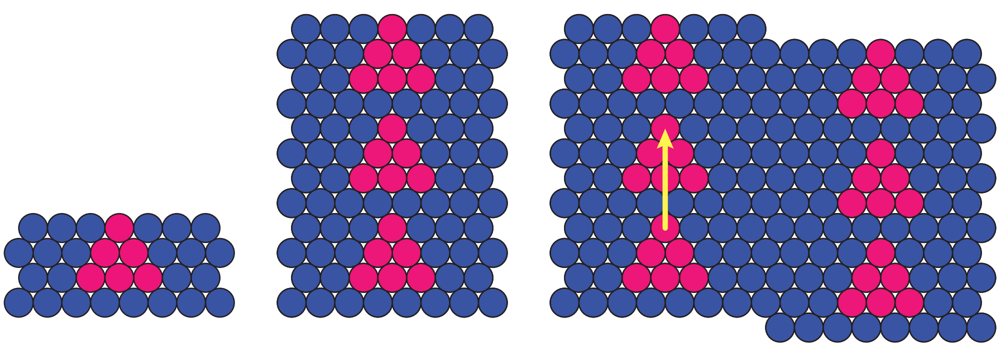


FIGURE 1.23 A repeat that is a multiple of 2N + 1 beads guarantees a structurally aligned bracelet and also a vertically aligned design.

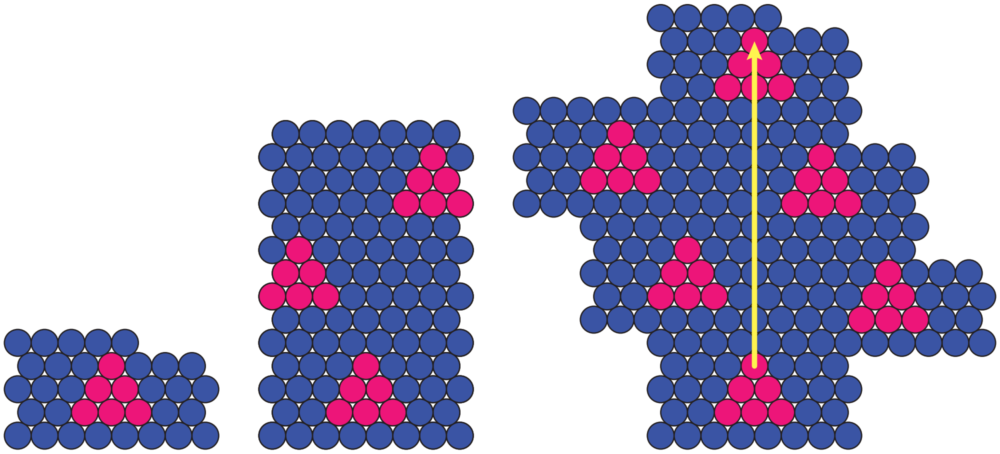


FIGURE 1.24 A pattern that is vertically aligned by three. In this case, the repeat size is 35 beads, and three such repeats is 105 beads, which is a multiple of 2N + 1 (shown stacked in vertical layout form in the middle). Looking at the pattern in the bead plane (right), the yellow arrow points out how the triangle in the fourth repeat will align vertically with the triangle in the first repeat.

in circumference 6, where the double row length is the prime number 13, whose only factors are 1 and 13, any pattern that is not vertically aligned can only be vertically aligned by 13, and motifs that are 13 repeats apart will be so far away from each other in a bracelet that the alignment will not be readily observable. In the following table, we give the possibilities for vertical alignment by K in common bracelet circumferences. In practice, only vertical alignments by 1, 3, and 5 will be apparent in a finished bracelet. As we see in the table, some circumferences, such as 7 and 10, allow more options for motif alignment because the double row length has more factors.

Circumference (N)	Double row (2N + 1)	Possible vertical alignments by K
4	9	1, 3, 9
5	11	1, 11
6	13	1, 13
7	15	1, 3, 5, 15
8	17	1, 17
9	19	1, 19
10	21	1, 3, 7, 21
11	23	1, 23
12	25	1, 5, 25

On the other hand, in many non–vertically aligned patterns we can line up every repeat or every Kth repeat with the right combination of structural and physical twists, like the bracelets in the upper left and lower right of Figure 1.20. In Chapter 3, Geometric Cross Sections, we will discuss the use of both structural and physical twists to line up motifs, even in cases when the bracelet itself is not structurally aligned. We will also see in Chapter 4 that sometimes it is handy to use physical twists to make elements that *are* vertically aligned appear slanted instead.

Twisting and alignment turns out to be a much subtler topic than one might expect! This is why we recommend holding the two ends of a bracelet together in position and inspecting your twist options carefully before closing. Sometimes adding or subtracting a repeat or two, as sizing constraints permit, can make a big difference in the available options. Twisting decisions significantly affect the symmetry and aesthetics of the finished bracelet, creating an extra layer of artistic choices for the designer. In the upcoming chapters, we will discuss much more about how to use both physical and structural twists to accomplish specific design goals.

Since the vocabulary introduced in this section can be confusing, here it is again in summarized review form:

Physical twist: a manual twist of one end of the rope before sewing closed; always a multiple of 360° clockwise or counterclockwise.

Double row: two consecutive rows in the vertical layout of a bracelet design. In circumference N, a double row has 2N + 1 beads.

Structurally twisted: describes a bracelet that does not have a multiple of 2N + 1 beads, where N is the circumference. A structural twist forces a twist of less than 360° in a closed bead crochet rope.

Structurally aligned: describes a bracelet that has an exact multiple of 2N + 1 beads, where N is the circumference. Note that a bracelet can be

structurally aligned and also be physically twisted; these are two independent properties.

Design that is vertically aligned: A pattern with a repeat size that is a multiple of 2N + 1 and thus lines up vertically at every single repeat.

Design that is vertically aligned by K: A pattern that stacks vertically at every Kth repeat; this occurs when the repeat length times K is a multiple of 2N + 1.

Cropping Visualization and the Limits of Physical Twists

Although the bead plane has many advantages as a design space, it has the drawback that it shows more of a bracelet pattern than is visible from a single viewing angle of the actual bead crochet rope, making it hard to judge how the design will look on the completed bracelet. Even the vertical layout can be misleading, since it shows the pattern all the way around the bead crochet rope. In Figure 1.25, we see the bead plane pattern and a vertical chart for a 9-around bracelet with a design of outlined hexagons, a pattern called Honeycomb. It is easy to make the mistake of imagining the bracelet will resemble the vertical strip on the right, which is 10 beads wide. Since the beads are actually crocheted into a tube, the amount of the pattern you can see on a bracelet from a single vantage point is closer to 5 beads wide.

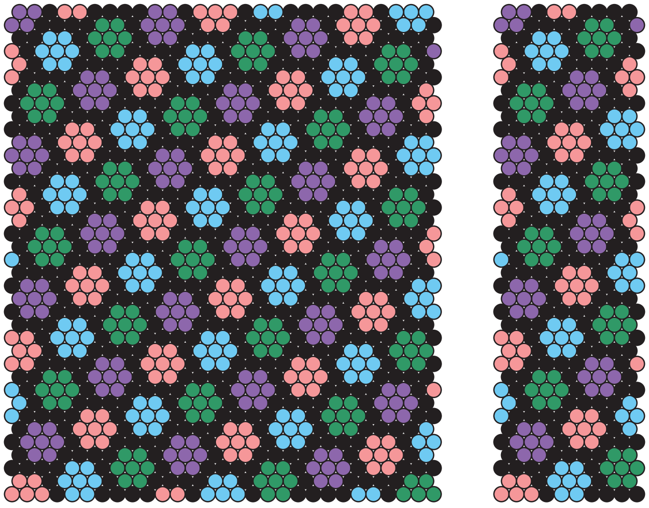


FIGURE 1.25 A bead plane pattern and the corresponding vertical chart. This is the pattern for Honeycomb, which appears on p. 154.

Existing bead crochet software has a nice solution to this problem. In jbead, the freeware application by Damian Brunold mentioned earlier, the appearance of the final bracelet is simulated with a strip that goes halfway across the vertical chart of a bracelet pattern. In addition, jbead simulates rolling the bead crochet rope by moving the viewing strip horizontally across the vertical layout—including moves that pass across the side seam in the chart. This gives a nice way to approximate views of the bead crochet rope from all sides.

Unfortunately, this method of simulating the bead crochet rope will only display a bracelet with no twists. Since physical and structural twists can have a strong impact on the visual effect of a pattern, getting a sense for how a pattern will appear when subjected to various twists is valuable. While we are unaware of any current software that allows this type of visualization, the bead plane offers a useful low-tech approach to the problem.

The technique described below is much easier to understand if you can manipulate an actual bead crochet rope and compare it to its bead plane pattern. If you are interested in designing your own patterns and visualizing twists using this method, we recommend referring back to this section after you have crocheted one of the patterns in this book and can experiment with twists yourself. While the nuances of twisting are subtle, they become more intuitive with practice and experience.

It is easy enough to isolate a 5-bead wide vertical strip of the bead plane by covering it with two pieces of paper, or even better, by a single piece of paper with a notch of the right width cut into it. As we show in Figure 1.26 for the Honeycomb pattern, rotating the paper gives us a glimpse of what we are likely to see in a finished bracelet with a twist. The upper left image is a completely vertical swath of the bead plane. In this strip, you can see that the green hexagons, which are two repeats apart in the bracelet, are almost lined up. We can simulate lining up every other repeat by slanting the paper about 3° to the left so that the edges are parallel to the line of green hexagons, as seen in the upper right. To make this alignment in the bracelet, you would twist the top of the bead crochet rope slightly to the *right* before closing to make the line of green hexagons stack up vertically.

Instead, you might choose to line up every fifth repeat. In the bead plane, we simulate this by rotating the paper about 8° to the right of vertical, as in the lower left image in Figure 1.26. The two pink hexagons at the bottom and top of the strip, which are 5 repeats apart, are now aligned in the strip, which you would accomplish by rotating the

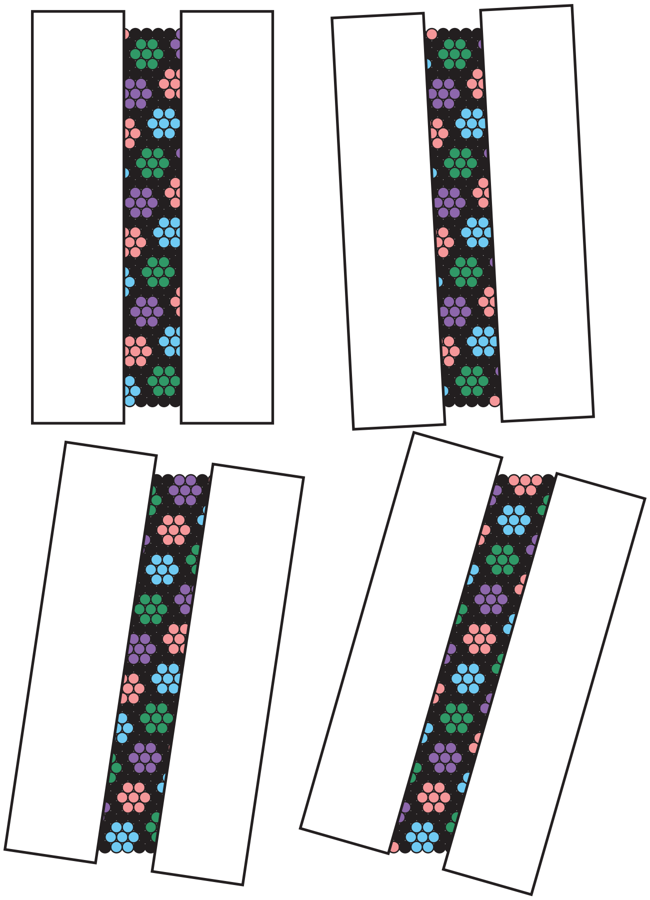


FIGURE 1.26 Using pieces of paper to visualize different twists of a bracelet.

top of the rope to the *left* until the pink hexagons are vertically stacked.

With a more extreme twist, you can line up every third repeat. This is shown in the lower right of Figure 1.26, in which the paper is rotated around 16° to the right of vertical. As you can see, this twist gives a very different visual effect than the other twists because it creates a vertical stack of adjacent hexagons, whereas the other twists cause the hexagons to spiral around the bracelet.

Honeycomb has a 52-bead repeat, and since 52 has no common factors with the length of a double row (19 beads), this design is not vertically aligned except by 19, which is almost the entire length of a bracelet. What the simulation in Figure 1.26 illustrates, and what experience closing bead crochet bracelets confirms, is that we can nonetheless line up every other repeat if we use an even number of repeats, or every third repeat if we use a multiple of 3 repeats, or every fifth repeat if we use a multiple of 5 repeats, as long as we perform the proper twist. In the pattern pages in Part Three, we will often give suggestions of possible alignments for patterns that aren't vertically aligned by indicating when every Kth repeat can be aligned with a twist. We will also describe some more precise ways of capturing twist information in Chapter 4. Keep in mind that twisting makes a lot more sense when you have an actual crocheted rope in your hands and can experiment with the available options before closing. Figure 1.27 shows a segment of Honeycomb twisted in each of the ways approximated in Figure 1.26. The first alignment is the segment in its untwisted position. We arranged the others by simply lining up the hexagons in every other repeat, every fifth repeat, and every third repeat, without stopping to think about whether we were twisting to the left or to the right or about exactly how far to twist.

One crucial question remains: how far can you actually twist a bracelet? After all, experimenting with different twists in the bead plane is only useful if the bead crochet is actually flexible enough to accommodate the simulated twist. As it happens, the final twist option in Figures 1.26 and 1.27 is close to the limit of what we have found feasible for bead crochet bracelets. This twist and the corresponding twist in the opposite direction are marked in orange on the bead plane in Figure 1.28. Deeper twists tend to push beads out of position and make the bead crochet too inflexible, though the actual bounds of twisting vary depending

FIGURE 1.27 A segment of bead crochet rope showing all of the twist options from Figure 1.26: untwisted, with every other repeat lined up, with every fifth repeat lined up, and with every third repeat lined up.

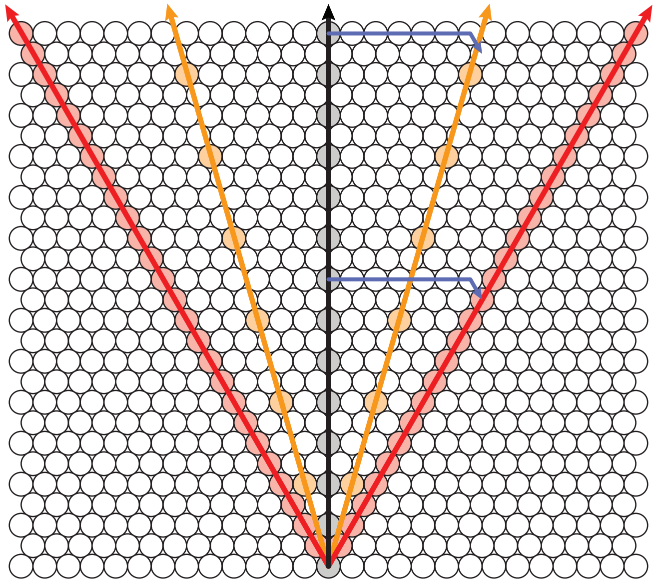


FIGURE 1.28 The limits of bracelet twists. The orange lines are about 16° from vertical, and twists less than or equal to the twists that make the orange beads line up vertically are feasible for most bead crochet bracelets. Twists between the orange and red lines are more problematic, and twists past the red lines, which are 30° from vertical, are not possible from a practical standpoint. The blue arrows are hockey-stick translations for a 6-around bracelet, which show how much of a twist results from rotating the top of a short 6-around bead crochet rope by 360° to the left.

on bead size, circumference, and looseness or tightness of the crochet stitches. More specific information about these effects is in Chapter 8. The red lines in Figure 1.28 mark the point beyond which twisting becomes physically impossible because the crochet stitches themselves prevent the beads from moving any farther.

Note that the angle seen in the bead plane is *not* the same as the angle through which you rotate the top of the bracelet to produce that alignment. Consider the two hockey-stick translations marked in blue in Figure 1.28, which are the translations for a 6-around bead crochet rope. Since a hockey-stick translation corresponds to one circuit around the circumference of the rope, rotating the top of a bead crochet rope that ends at the start of a blue arrow by 360° to the left will move the part of the bead plane at the other end of the blue arrow to the vertical axis marked in black. Consequently, in a bead crochet rope segment of 7 double rows, one full physical twist to the left will move the 30° right-leaning red axis to vertical, whereas in a segment of 13 double rows, one full physical twist will move the 16° right-leaning orange axis to vertical. A typical 6-around bracelet will have around 55 double rows, so performing one of the alignments marked in orange will take a little over 4 full twists. In practice, you don't need to know how many physical twists are required for the alignment you want; you simply twist the bead crochet rope until the pattern is lined up as you wish, as in Figure 1.27.

New Design Challenges

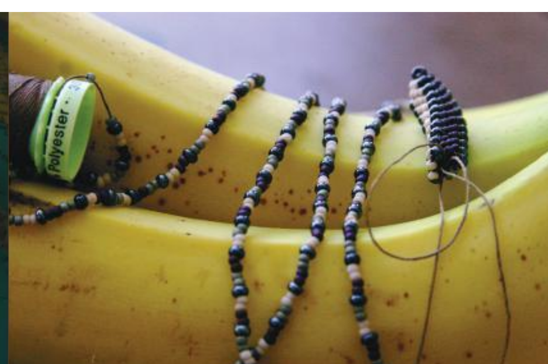
This completes our overview of the bead crochet design framework we have developed over the past few years. We discovered everything in this chapter—the infinite bead plane as a design space, the translation rules for bracelet patterns, using lattices and tilings to create bracelet designs, the effects of twists on bead crochet—in the process of solving some fascinating puzzles in bead crochet design. We were spurred on by intellectual curiosity, but also by our desire to make more beautiful and compelling beaded bracelets.

We are now ready to move on to Part Two, in which we present the various design challenges that caught our imagination and discuss how to solve them using the techniques from this chapter. But if you are so inclined, we invite you to copy one of the bead plane templates on pp. 247–250 (or download it from the book's website*), put down this book for a while, and start exploring.

* http://www.crcpress.com/product/isbn/9781466588486 (under the "Downloads/Updates" tab).

PART TWO

Design Challenges



CHAPTER 2

Seven-Color Tori

Mathematicians, like artists, are concerned with the essences of things. They can't resist taking commonplace objects and looking for hidden patterns that reveal their fundamental structure. Consider, for instance, the subject of maps. In our everyday lives, we use maps for specific, practical purposes: to navigate from place to place, to explore new terrain, to understand the relationships between faraway nations. It is the mathematician's impulse to peer beneath the surface and ask more elemental questions. What *is* a map? How does it convey information? How does the geometry of the world being mapped influence the underlying structure of the map itself?

The answers to these questions depend a great deal on what kind of map we're considering. Figure 2.1 shows two common types of maps that convey different types of information. On the left is a street map, charting an interconnected network of roads. From a mathematical standpoint, the map is a collection of intersecting lines and curves, and the names of the particular streets and where or whether these roads actually exist in the world are not a part of the essential structure of the map. On the right is a geopolitical map, showing a cluster of adjoining regions of land. For many applications, mathematics ignores certain features used in the practical interpretation of the map. How big are these regions? Are they countries, states, or territories of another type? Is the map oriented with north pointing up, down, or in an entirely different direction? In many mathematical endeavors, these questions are irrelevant: we might only care about the shapes and relationships between the regions on the map.

In 1852, a mathematics graduate student named Francis Guthrie posed a very basic question about geopolitical maps. He was pondering the common practice of coloring in the regions of a map (which we will call *countries* regardless of what territories they actually represent) to make it easier for the viewer to distinguish between neighboring countries. Since many maps have dozens of countries and it is hard to produce so many distinct colors, maps generally use the same color for different countries, provided that those countries don't share a common border. Guthrie wondered: what is the smallest number of colors that can be used to color any map, no matter how complicated, so that no countries with a common border are the same color? [*Explorations in Topology,* pp. 23–24].

There are a couple of ground rules that make Guthrie's question more precise. First of all, two countries that touch at a single point are not considered to share a border. This is why the two countries colored in blue in Figure 2.1 are allowed to have the same color. Second of all, each country must consist of a single contiguous region. For instance, in a map of the world, Alaska would be considered a separate country from the continental United States for coloring purposes. These rules clearly simplify the map-coloring question, but they also make it more interesting. If we abandon either rule, it turns out that no number of colors is enough to color every possible map.

The history of mathematics is full of questions that are simple to ask but unbelievably difficult to answer. Guthrie's map-coloring question is a prime example. It is not hard to establish that the magic number of colors is at least four by constructing a map like the one in Figure 2.2. In this map,

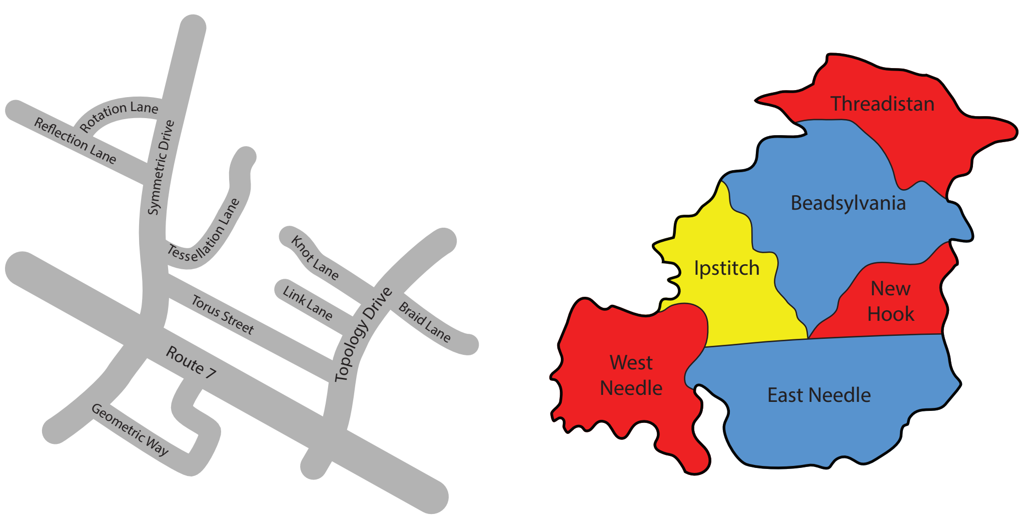


FIGURE 2.1 Two common types of maps: a road map, showing a network of interconnected roadways, and a geopolitical map, showing an area divided into separate countries.

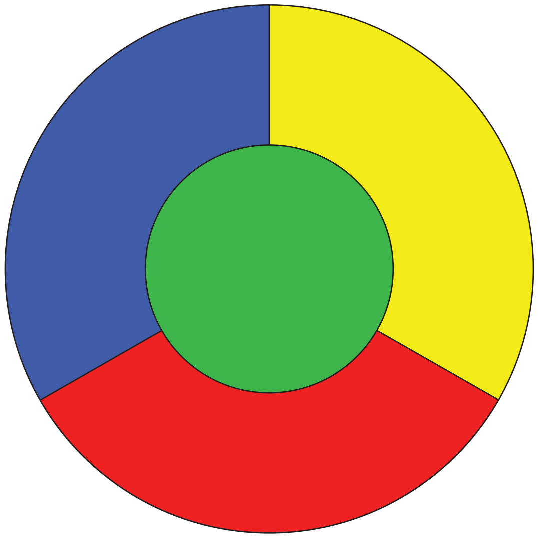


FIGURE 2.2 A map of four countries that all border one another. Because every pair of countries shares a border, this map requires four colors.

each of the four countries shares a border with every other country, so each country requires its own color. As more mathematicians grew interested in the map-coloring puzzle, they were incapable of producing a map that required five colors, and thus the Four-Color Conjecture—stating that four colors are sufficient to color any map—was established. However, there is a big difference between saying that no one has yet found a map that requires five colors and saying that no such map is possible. And coming up with a definitive proof that no matter how crazy a map you make, you can always color it with four colors, stymied the mathematical world for over a century. In an 1890 paper, P. J. Heawood proved that the magic number of colors is no greater than five,* and for 86 years the world's mathematicians were stuck in a map-coloring limbo: they knew that the number of colors needed was either four or five, but they couldn't prove which [*Explorations in Topology*, p. 99].

Heawood's 1890 paper includes another observation that should catch the attention of the curious bead crochet designer. As it happens, the number of colors required for an arbitrary map depends on the shape of the surface on which the map is drawn. Guthrie's original question, and the Four-Color Conjecture it produced, assumes that the map is drawn on a plane (such as a flat sheet of paper). But maps drawn on other surfaces may require more colors, and Heawood exhibited a map on a torus that has *seven* countries, each of which touches all the others. [*Explorations in Topology*, p. 267] This means that for a map on a torus, at least seven colors are needed!

This leads to our first bead crochet design challenge.

Challenge A *Can you design a bead crochet bracelet with seven identically shaped regions such that each region borders all the others?*

Having identically shaped countries is not necessary to prove that seven colors are required for torus maps, but it creates a more symmetric and hence more attractive bracelet.

Guthrie's original question was finally settled in 1976, when Kenneth Appel and Wolfgang Haken proved the Four-Color Conjecture, transforming it into the Four-Color Theorem. The event was somewhat anticlimactic, since the proof involves a computer verification of hundreds of pages of special cases and is not comprehensible to a human reader. Work continues to this day on finding a more streamlined and enlightening demonstration of why four colors suffice for all maps in the plane. In a curious twist, it happens that even though maps on the torus require three more colors than those on the plane, it is much easier to prove that *every* map on the torus can be colored with only seven colors—so much so that Heawood proved it in his original 1890 paper. Exhibiting a map that clearly required seven colors was one part of the proof—it established that *at least* seven colors were needed, thereby setting a lower bound on the correct answer. The remainder of the proof showed that one never needs more than seven, i.e., that seven is also an upper bound. Thus the general proof technique was to trap the correct answer tightly between an upper and a lower bound. The mathematics is too advanced to describe here, but we direct the interested reader to David Gay's delightful book, *Explorations in Topology: Map Colorings, Surfaces, and Knots,* to learn more about the math behind map coloring theorems.

Since the surface we live on is a sphere, it is natural to ask whether the map-coloring number for a sphere is also greater than four. In fact, even without knowing how many colors are required for planar maps, we can demonstrate that the number for spherical maps and planar maps is the same. The procedure is illustrated in Figure 2.3 and employs the same topological flexibility that we used in Chapter 1 when we made a torus out of a flat square. Given a map on the sphere, we puncture the sphere in the middle of one

* In fact, the full story is much more colorful—pardon the expression. Eleven years earlier, in 1879, Sir Alfred Kempe announced that he had proven the Four-Color Conjecture, and his proof was generally accepted by the mathematical community. Until Heawood's paper, no one noticed that Kempe's proof had a fatal flaw, but Heawood modified Kempe's argument to prove the Five-Color Theorem.

FIGURE 2.3 Translating a spherical map into a planar map for coloring. If we puncture the sphere in the middle of a country and stretch it into a flat disk, we get the flat map in the upper right. A coloring of the flat map can then be transferred back to the surface of the sphere.

of the countries and stretch the surface into a flat disk in the plane. The sphere and the map on it will be extremely distorted, but this will not change which countries are adjacent in the map. The flat map can now be colored in the plane with the minimum number of colors required, and if we undo the flattening we obtain a coloring of our spherical map with the same number of colors.

As we see from the stretching in Figure 2.3, the particular shapes of the countries in a map do not affect its coloring properties; the only thing that matters is which pairs of countries share a border. This adjacency information can also be represented in a mathematical construct called a *graph*. A graph is composed of *nodes* and *links,*[*] as shown in Figure 2.4, where the nodes are dots and the links are line segments or curves, smooth or wiggly. A link represents a connection between two nodes in the graph, and two nodes in a graph are considered *adjacent* if there is a link between them. Notice that the term "adjacent" is used for graphs in a different way than in common English: two nodes connected by a link are adjacent even if they are physically far apart. In a graph, all that matters is how the nodes are connected to each other by links, not where the nodes are placed in relation to each other or whether the links are straight or curved. Close inspection of the graphs in the lower center and lower right of Figure 2.4 reveals that they are in fact the same graph, since they both have five nodes with a link connecting each pair of nodes (or, in other words, every node is adjacent to every other node). If we choose, we may label the nodes or links in a graph with colors, numbers, or names. Mathematicians, scientists, and engineers use labeled graphs to answer questions about all manner of phenomena, from traffic regulation to online social networks to neurons and synapses in the brain.

Starting with a map, we can construct a graph in which each country is represented by a node and two nodes are adjacent precisely when the two corresponding countries share a border, as shown in Figure 2.5. We begin by placing a node in the middle of each country. We then draw a link connecting the nodes for each pair of bordering countries. If we are careful to draw each link so that it only passes through the two countries it connects, we can guarantee that none of the links cross between nodes. In fact, this process can also be run in reverse, and we invite the reader to ponder how a graph whose links never cross can be turned into a map with the same adjacencies.

Figure 2.6 shows this process applied to the map from Figure 2.2. The resulting graph is called a *complete graph on four nodes* because every node has links to every other node, just as every country in the map is adjacent to every other country. This graph is commonly denoted K_4. Similarly, the last two graphs in Figure 2.4 are complete graphs on five nodes, or K_5.

Much of the work on map coloring in mathematics was done using the theory of graphs, since graphs provide a

[*] In some discussions of graph theory, *nodes* are called *vertices,* and *links* are called *edges*.

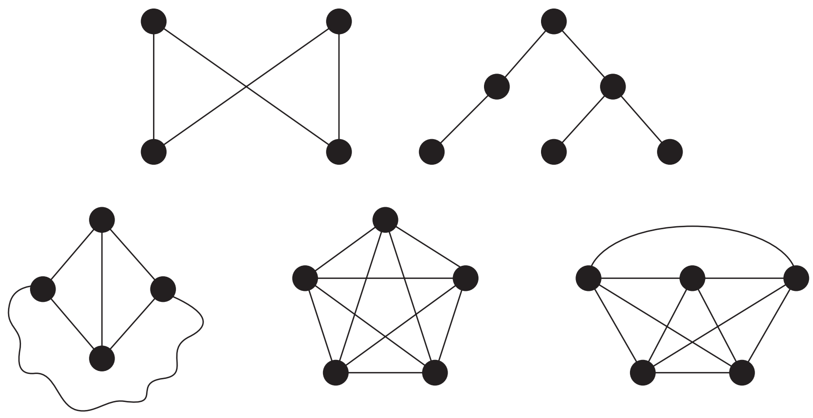


FIGURE 2.4 Examples of graphs. The dots are the nodes of the graphs and the lines and curves connecting them are the links. The information that specifies a graph is simply which nodes are connected by a link and does not depend on how the nodes are arranged or whether the links are straight or curved.

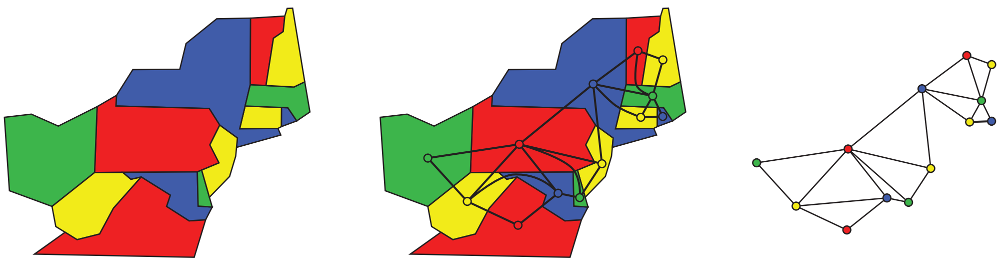


FIGURE 2.5 Extracting an adjacency graph from a map. The link representing each border between adjacent countries only crosses the two countries sharing that border, which guarantees that the links of the graph will not cross. In the image on the right, the map is removed and the links of the graph are straightened to simplify the diagram.

concise way of capturing all the critical adjacency information contained in a map while leaving out extraneous geometric details. For readers with some math background interested in learning more about the ins and outs of graph theory, *Topological Graph Theory* by Jonathan L. Gross and Thomas W. Tucker is a terrific resource. Representing a particular map as a graph with direct links between bordering countries translates the problem of assigning colors to countries into the problem of assigning colors to the nodes in a graph so that adjacent nodes are different colors. All maps with the same number of countries and same country adjacencies can be represented by a single graph, thus distilling infinite possibilities into a single representative form. A coloring solution to a graph works for all the maps it represents.

Using the procedure in Figure 2.5, we can represent any possible map on a surface by a graph on that surface whose links never cross one another. Depending on the map and choice of node placement, the links might need to curve around to avoid crossing one another. The lower left graph in Figure 2.4 is an example of such a graph.

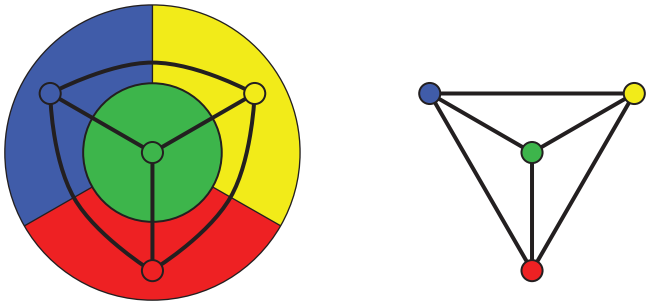


FIGURE 2.6 The adjacency graph for the four-color map from Figure 2.2 This graph (which appears in a different configuration in Figure 2.4) is the complete graph on four nodes, since there is an edge connecting every pair of nodes.

Note that although it appears different, it is actually the same graph as in Figure 2.6, the complete graph with four nodes, containing the same adjacency information. Both ways of drawing it are equally legitimate.

If it is possible to construct a map on a surface with N countries that all border one another, it is also possible to construct a complete graph on N nodes on that surface

with no link crossings. Thus, a complete four-node graph with no link crossings is possible on a sphere or plane, and a complete seven-node graph with no link crossings is possible on a torus.

From the Four-Color Theorem we know it is *not* possible to construct a planar map with more than four countries that all border one another. It is likewise impossible to construct a complete planar graph with no link crossings with more than four nodes since, if we could do so, we could then construct a map from the graph that would require more than four colors to paint. Similarly, it is impossible to construct a toroidal map with more than seven countries that all border one another, and thus it is impossible to construct a complete graph with more than seven nodes on a torus with no link crossings. To gain some intuition for the truth of these statements, see if you can draw a counterexample on paper, such as a complete graph on five nodes with no link crossings. In other words, take the lower right graph in Figure 2.4 and see if there is any way to move links and nodes around to get rid of the crossings. Then try it again on a bagel (permanent markers work great for drawing on bagels). You can draw the complete graph on seven nodes on a bagel without link crossings, though it is somewhat more challenging than drawing K_4 in the plane. Try going up to K_8 on a bagel with no crossings, however, and you will be stymied!

An Internet search reveals many fascinating seven-color torus maps in different media, but before our work on the problem we found none in bead crochet. A few particularly interesting maps in other media are shown in Figures 2.7–2.9.

Figure 2.7 shows a K_7 graph on a torus with no link crossings knitted by sarah-marie belcastro and a seven-color torus map crocheted by Carolyn Yackel. Figure 2.8 shows a patchwork cloth torus made by Susan Goldstine. Instructions for making your own are posted online at http://faculty.smcm.edu/sgoldstine/torus7.html. Figure 2.9 shows two views of a hydrostone model of a seven-color torus map created by Norton Starr, which was based on a map developed in 1953 called the Ungar-Leech map [Coxeter 1959, p. 288]. Following the suggestion of his student Geoffrey Wilson, Starr used string stretched between points on the torus to mark straight lines on the curved surface as a method of marking some of the country borders for painting. One of the interesting things about this model is that there is a single view (Figure 2.9, bottom) in which all seven countries can be seen touching one another, making it easier to verify the correctness of the proof. This gave rise to a sequel to Challenge A:

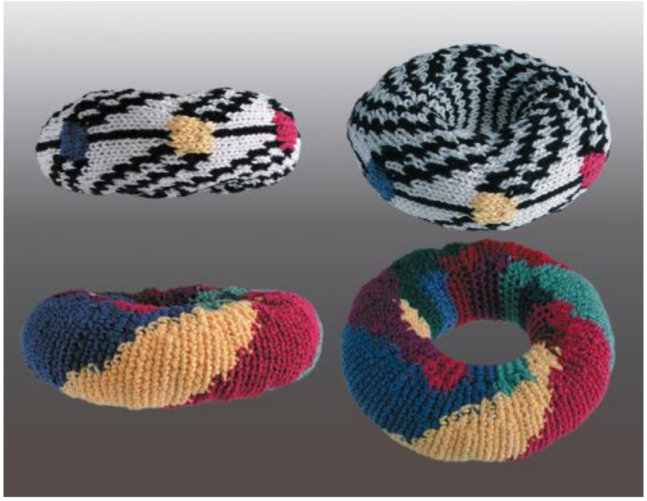


FIGURE 2.7 Seven-color tori by sarah-marie belcastro and Carolyn Yackel. The complete seven-node torus graph at the top was knitted by belcastro, and the seven-color torus map at the bottom was crocheted by Yackel. Photograph by Craig Coleman, Associate Professor of Art at Mercer University.

FIGURE 2.8 Fleece seven-color torus map by Susan Goldstine. The torus consists of seven congruent hexagons in fleece stitched together by hand.

Challenge B *Can you design the bracelet in Challenge A so that it is possible to see from a single viewpoint each of the seven regions "touching" each other?*

The K_7 graph that belcastro knit inspired a further bead crochet challenge:

Challenge C *Can you create a bracelet design that shows a complete graph on seven nodes with no link crossings?*

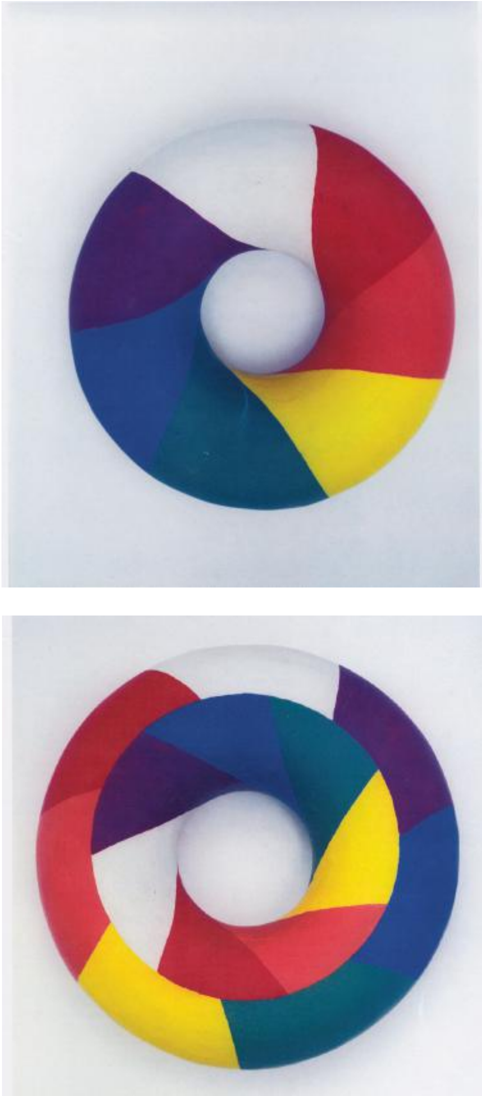


FIGURE 2.9 A hydrostone seven-color torus map created by Norton Starr, Brian E. Boyle Professor Emeritus of Mathematics and Computer Science at Amherst College (shown from both sides). From a single viewpoint (bottom), it is possible to see all seven countries touching one another.

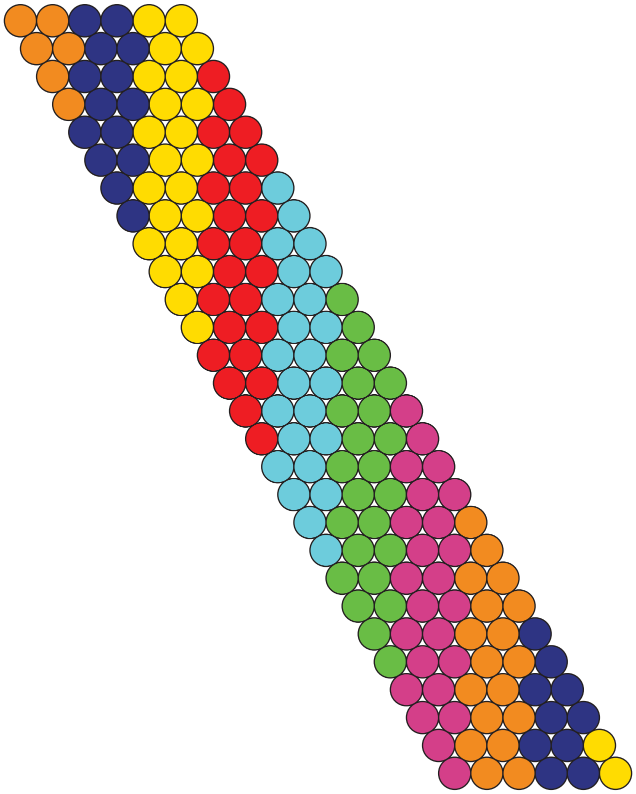


FIGURE 2.10 Sophie Sommer's first bead crochet seven-color torus.

To our knowledge, the first solution to challenge A was developed on a diagonal layout by Sophie Sommer and was presented as part of her twelfth grade science fair project, "Mathematical Explorations in Bead Crochet." Figure 2.10 shows her original pattern. This solution is too short for a bracelet, but Sommer had a resolution to this problem, which we'll discuss shortly.

When she created this design, Sommer didn't know about designing in the bead plane, so she had to think carefully about how all the edges would connect to confirm that all seven countries touched one another. To prove that it worked, she crocheted up the tiny version in Figure 2.10, from which it was possible to verify the map's adjacencies. This is also easily confirmed in the bead plane, as shown in Figure 2.11, and we can see more readily that some of the contacts are only 1 bead long. Astute observers might be

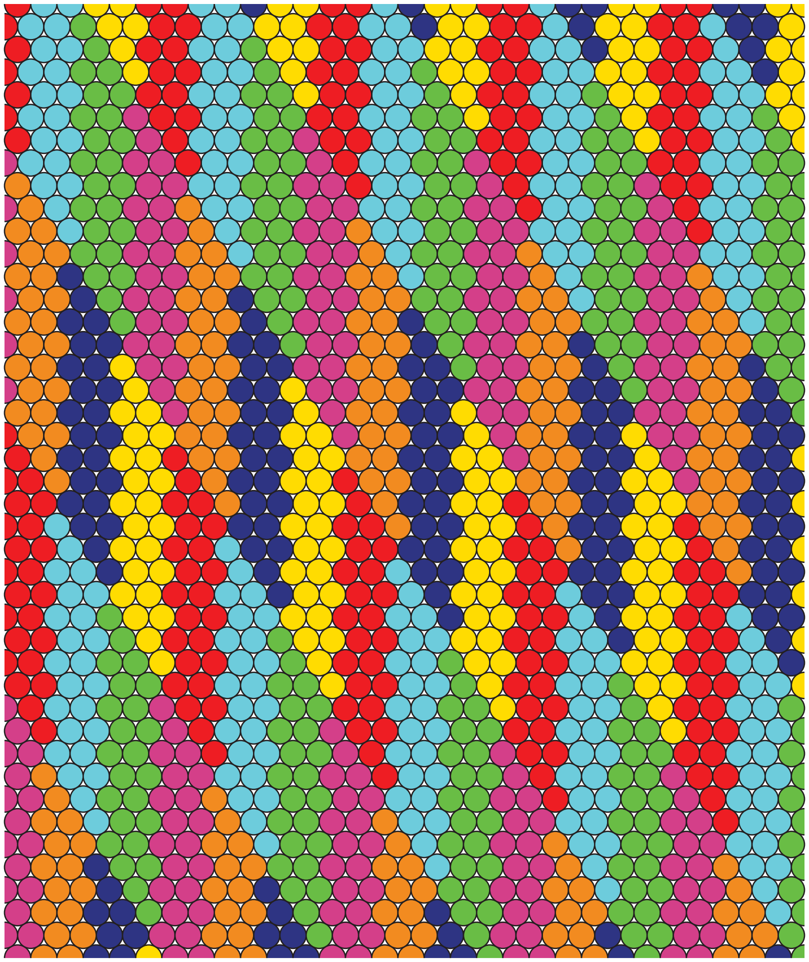


FIGURE 2.11 The bead plane chart for the miniature bracelet in Figure 2.10.

concerned that these 1-bead adjacencies are really point adjacencies, like Colorado and Arizona, and thus don't count as sharing a border. This concern is put to rest in Sommer's next version of the design.

To address the fact that her design was too short for a bracelet, Sommer noted that each row in the diagonal layout was repeated twice. She realized it was possible to lengthen the design without changing the adjacencies by simply repeating each row a few extra times, stretching it out lengthwise (diagonally) in a uniform fashion. This also had the effect of elongating the borders that were previously only 1 bead long. It turns out that using 8 or 9 repetitions of each row produces a nice size for a bracelet. Her finished product is shown at the top of Figure 2.12, and the full pattern, Sophie's Original Seven-Color Torus, appears on p. 134. A bead plane version elongated with 4 repetitions of each row is shown at the bottom of Figure 2.12 and gives a better sense for what happens as the pattern is stretched. Note that the 1-bead-long touching borders have lengthened to 3 beads long in this version, removing any doubt that these might read as single point adjacencies rather than truly shared borders. In the bracelet-sized version, these borders stretch to 4 beads in length.

Although lovely and mathematically correct, Sommer's original design has the drawback that it isn't easy to see all the countries touching one another in a single view. We set about trying to rectify this with a bracelet design that, like Starr's hydrostone model, would be more easily verifiable from a single viewing angle of the three-dimensional torus. Collaborating with Sommer, we first set out to work out a vertical layout for bead crochet because it was clear that Starr's design, rendered on a flat torus, was vertically based.* The final result of our collaboration is

* At the time, we had not seen the vertical layout in use elsewhere. As noted in Chapter 1, this layout is increasingly popular and appears in various sources, such as Judith Bertoglio-Giffin's book *Triangular Bead Crochet Ropes*, and in the freely available jbead software developed by Damian Brunold.

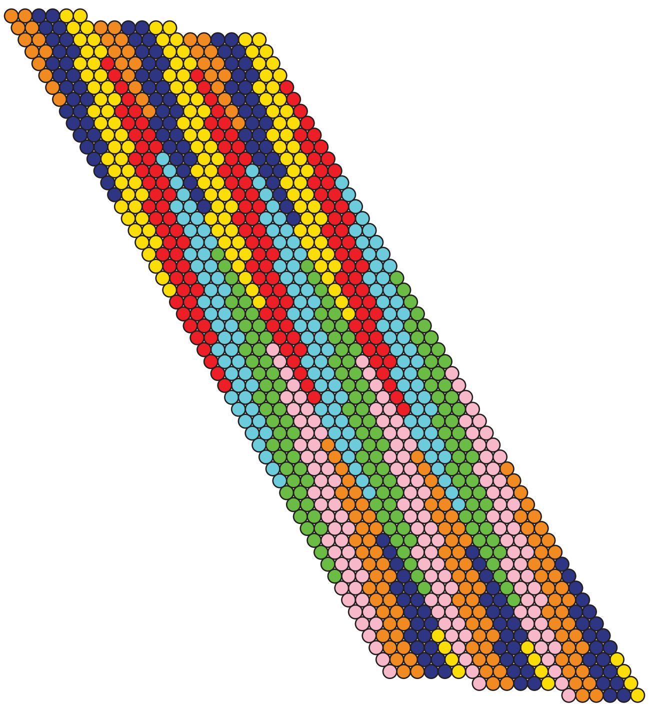


FIGURE 2.12 Expanded versions of Sophie Sommer's bracelet design from Figure 2.10. At the bottom is a bead plane diagram expanded 2 times, and at the top is the full-sized bracelet, which was expanded 4 times (to get 8 repetitions of each row).

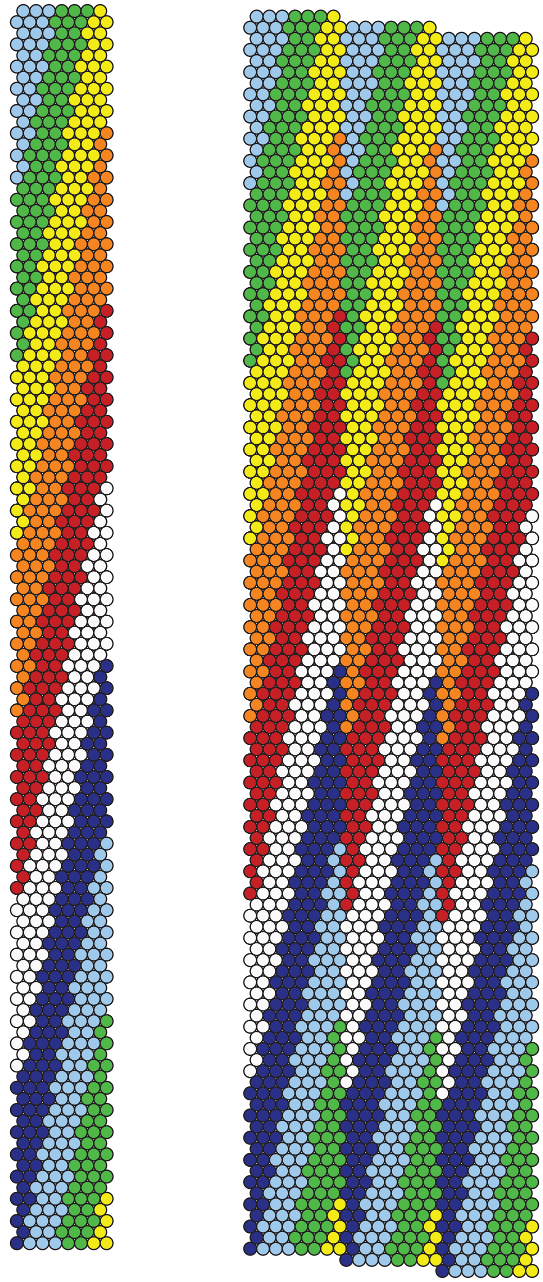


FIGURE 2.13 A vertical layout seven-color torus map fashioned after Norton Starr's hydrostone model of the Ungar-Leech map.

shown in vertical layout form at left in Figure 2.13, along with a bead plane rendition at right. The actual bracelet shown in Figure 2.14, front and back, mimics Starr's photos of his torus painted with the Ungar-Leech map. Our full pattern for this design, Seven-Color Torus (Ungar-Leech Map), appears on p. 135.

The basic structure of both seven-color designs is a parallelogram-shaped tile on a map with exactly seven tiles. Each tile touches one other tile on each of its long sides and two other tiles on both top and bottom. It is interesting to note how important the choice of layout was to the development of these designs. Consider cutting a diagonal layout from the bead plane version of the vertical design, or a vertical layout from the diagonal design. You can see the latter on the pattern page for Sophie's Original Seven-Color Torus on p. 134. Although you can still extract the correct stringing order, the resulting layouts look quite bewildering in isolation.

FIGURE 2.14 An Ungar-Leech map bracelet made from the pattern in Figure 2.13. This appears in the pattern pages as Seven-Color Torus (Ungar-Leech Map) on p. 135.

Our next challenge was to design a K_7 bracelet with no link crossings. Figure 2.15 shows how to extract a complete seven-node graph directly from the Ungar-Leech map design. Using the process from Figure 2.5, we place a node in the center of each country and connect the nodes with links. To make it easier to draw in the links, many of which cross the edge of a single vertical layout, we start with two copies of the Ungar-Leech pattern extracted from the bead plane diagram. Because of this duplication, each node appears twice in each diagram; nodes that are the same color represent the same node on the torus. By contrast, each link is drawn exactly once in each diagram. Note that each node has three links pointing downward and three links pointing upward, adding up to the required six links emanating from every node in a K_7 graph.

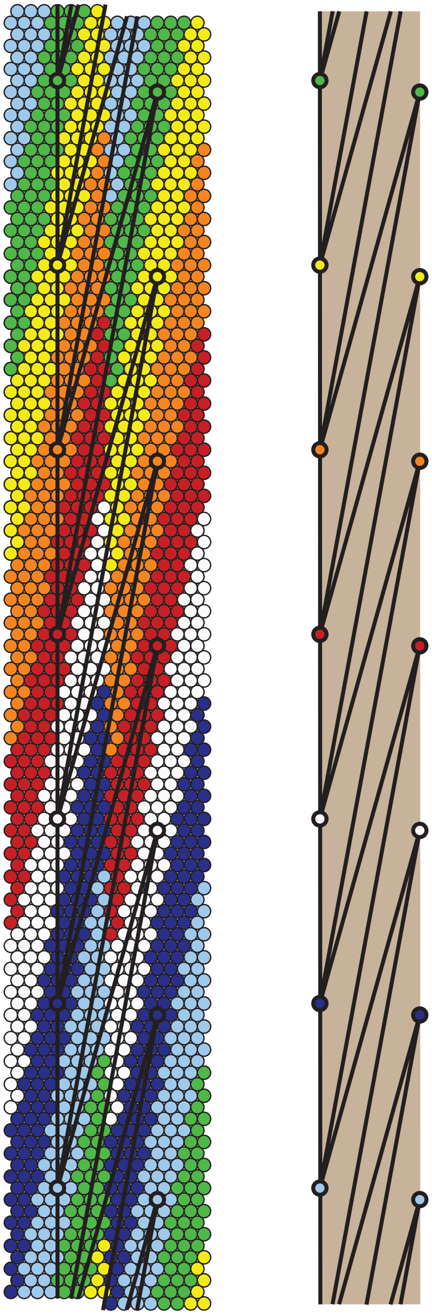
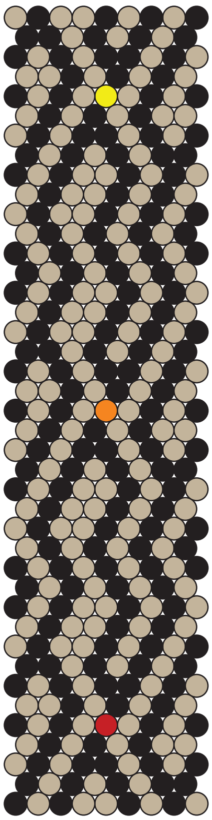


FIGURE 2.15 Converting a seven-color torus map into a complete seven-node graph on the torus with no link crossings. Each node appears twice on the flat diagram, but the hockey-stick translation glues each pair together into a single node. Note that each node has six links emerging from it, one connecting to each of the other six nodes.

To accommodate a more detailed representation of this K_7 graph on a bracelet, we expanded to circumference 8. Even with the extra design space, we were forced to allow some of the links to coincide as they emerge from the nodes and later fork into separate paths. Although this



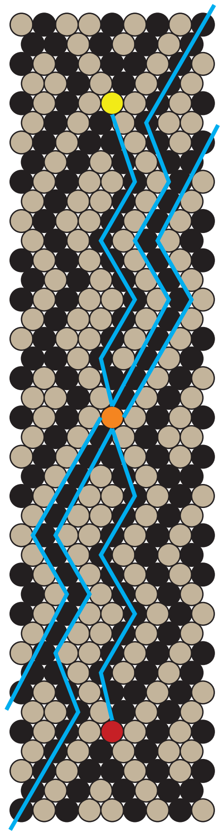


FIGURE 2.16 A segment of a pattern for a K_7 graph on a torus. Looking closely at the chart on the left, we see that only four lines of black beads emerge from the orange node because two pairs of links overlap. However, the sketch on the right shows that we can easily draw all six links connected to the orange node without any crossings.

FIGURE 2.17 A complete seven-node graph bracelet. This vertically aligned design appears in the pattern pages as Complete Seven-Node Graph (p. 136). This particular example uses size 11 Delica beads, which makes an extremely small bracelet, with 3mm Swarovski crystal bicones for the nodes.

FIGURE 2.18 Another complete seven-node graph bracelet. This design, which is more symmetric than the design in Figure 2.17, appears in the pattern pages as Symmetric Complete Seven-Node Graph (p. 137). This example uses size 11 Delicas in 24K gold-lined crystal and matte black with 3mm sterling rounds for the nodes.

technically causes the links to intersect, looking closely at the pattern snippet in Figure 2.16, we observe that the links can be separated into noncrossing paths by sketching in a little finer detail. One version of the resulting bracelet, with Swarovski crystal bicones in seven different colors representing the nodes, appears in Figure 2.17. The links connecting these seven nodes are depicted in black beads on a white background. The full pattern is provided on p. 136.

While the pattern in Figures 2.16 and 2.17 fits most people when worked in size 11 seed beads, we often like to use the slightly smaller Delica beads for 8-around bracelets. Since these designs are not flexible with respect to sizing, we also constructed a not-quite-vertical pattern for a slightly twisted bracelet in size 11 Delica beads in 8-around that would fit an average sized wrist. A bracelet made from this design using 3mm sterling rounds for the nodes, which are not colored in this rendition, is shown in Figure 2.18. The pattern, Symmetric Complete Seven-Node graph, appears on p. 137. We invite readers to try designing their own seven-color map or K_7 bracelets to meet their own particular bead size, bracelet size, and circumference wishes!

An interesting postscript to our seven-color map designs is that both can be painted with fewer colors if the number

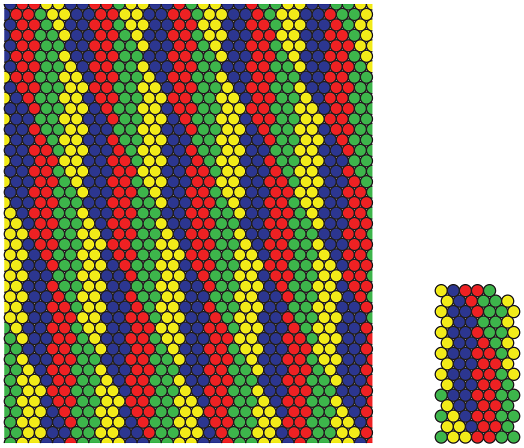


FIGURE 2.19 Sophie's original seven-color tiling in a four-color version. The repeat size reduces to four countries; two repeats, or eight countries, are four-colorable.

of countries in the bracelet exceeds seven. For example, Figure 2.19 shows a four-color version of the pattern with the exact same tile used in Sophie's original seven-color bracelet from Figure 2.10. The repeat size changes to 96, a multiple of four tiles rather than seven, and with two such repeats, or eight tiles, the pattern can be painted with only four colors. The Four-Color Theorem dictates that any planar map is four-colorable, so it's not surprising that a bead plane rendition of any pattern will be four-colorable. While it is not necessarily the case that any such four-color rendition will still be a valid bead crochet pattern—in other words, will obey the hockey-stick translation rule—in this case, it is. We will revisit the problem of determining a minimum coloring for a particular pattern in Chapter 6 on Escher designs.

CHAPTER 3

Geometric Cross Sections

Why are people drawn to symmetry? Are objects with symmetry inherently more beautiful? Or do we love symmetry because it makes things more orderly and thus more understandable? Whatever the reason, symmetry compels, especially for those with an artistic, scientific, or mathematical bent. Many of the design challenges in this book originated in a quest for greater symmetry, and that is especially true here in our treatment of the geometric cross-section designs.

A bead crochet design that fascinated us early in our explorations was the "caterpillar" from a pattern book by Judith Bertoglio-Giffin.[*] In this design, the standard bead crochet slip stitch is used throughout, but small and large beads are combined and carefully placed to give the appearance of a flattened rope. If done with large drop beads, which have the hole offset to one side, the effect is that of a caterpillar with legs jutting out—hence the name. Figure 3.1 shows a bracelet based on Bertoglio-Giffin's caterpillar design. Inspecting it carefully, we noticed that the design is not entirely symmetric: the caterpillar has more beads on its belly than on its back.

Riffing off the caterpillar design, Sophie Sommer came up with a related pattern that created the look of a triangular cross section, shown in Figure 3.2. Instead of drop beads at the edges (which we sometimes also call the "spines"), it used seed beads in a larger size. Although the design was very satisfying, generating a nicely tactile triangular cross section, we noticed that it, like the caterpillar, was not fully symmetric: not all sides of the triangle were the same. In fact, it was an isosceles triangle, in which only two sides were equal. Soon thereafter, Bertoglio-Giffin published a pattern book on triangular bead crochet ropes,[†] which focused on this same isosceles triangle pattern. Again we wondered why there were no equilateral triangles in her book. Did the spiraling structure of bead crochet create some sort of constraint that made more perfect symmetry impossible? Likewise, we also wondered if the caterpillar design could be made more symmetric, so that its belly and back would be the same width. These ponderings inspired the challenges in this chapter.

The isosceles triangle is what we refer to as a geometric cross-section design. If you imagine slicing the bead crochet rope in half, the two sliced ends seem to form a triangular shaped cross section. In mathematics, an equilateral triangle is the first of what are called the "regular polygons," polygons in which all sides and all angles are equal. The regular polygons are the most symmetric polygons. Figure 3.3 shows the regular polygons up to nine sides.

Although a caterpillar is not officially a polygonal cross section (because the sides are curved rather than straight), it is two sided and could be viewed as one more element

FIGURE 3.1 A bracelet based on Judith Bertoglio-Giffin's caterpillar design made by Sophie Sommer.

FIGURE 3.2 Sophie Sommer's isosceles triangle bracelet. The pink side is wider than the other two.

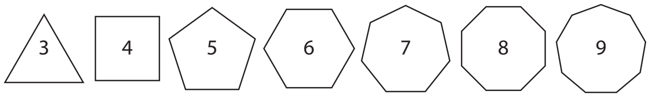


FIGURE 3.3 The regular polygons up to nine sides: triangle, square, pentagon, hexagon, heptagon, octagon, nonagon.

[*] Judith Bertoglio-Giffin, *Patterns and Graphing for Bead Crochet Ropes*, Glass Cat Books LLC, November 1, 2004.

[†] Judith Bertoglio-Giffin, *Triangular Bead Crochet Ropes: A Pattern Book of 3-Dimensional Bead Crochet Ropes*, Glass Cat Books, LLC, January 6, 2011.

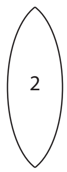


FIGURE 3.4 The digon, a two-sided figure.

all the way to the left in the chart in Figure 3.3. So let's also add the two-sided shape known as a digon (shown in Figure 3.4) to our list of regular objects.

Challenge *Can you construct an equilateral triangle design in bead crochet? How about the other regular geometric cross sections: square, pentagon, hexagon, heptagon, octagon, nonagon, etc.? How about a regular digon?*

Let's begin by examining a 9-around version of the isosceles triangle pattern, shown with five stacked repeats on the lower left in Figure 3.5. The dark blue beads represent the larger spine beads. Is there a simple way to convert it into an equilateral triangle?

The yellow section is just one column wider than the other two sections, so why not try lopping off a single vertical column of yellow to make the yellow section equal in size to the pink and green sections? Unfortunately, this then renders the pattern invalid as a vertical layout. It becomes quickly apparent that reducing the circumference (or N-around) of the pattern by one requires removing two vertical columns, but although lopping off two vertical columns, as shown in Figure 3.6, results in a valid vertical layout in 8-around, the yellow section is now too narrow instead of too wide. So neither strategy works.

The number of columns in a *valid* untwisted vertical layout is 2N + 1, so the number of vertical columns is always odd. Figure 3.7 shows the 2N + 1 = 19 vertical columns in an N = 9-around vertical layout. An easy observation to make is that 19 is not divisible by 3, so it's not too surprising that we were unable to create an equilateral triangle in 9-around with this type of layout. Likewise, in 8-around, we get 2N + 1 = 17 vertical columns, which is also not divisible by three.

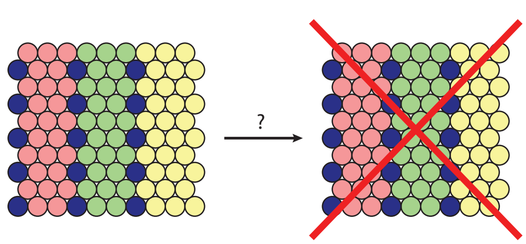


FIGURE 3.5 An isosceles triangle 9-around (left), in which the yellow stripe is larger than the green and pink. Can it be transformed into an equilateral triangle by simply lopping off one vertical column of yellow beads?

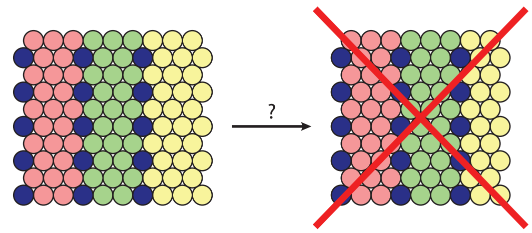


FIGURE 3.6 The 9-around isosceles triangle reduces to a valid 8-around isosceles triangle by removing two vertical columns of yellow. But now the yellow section is narrower than the pink and green sections, so it remains isosceles.

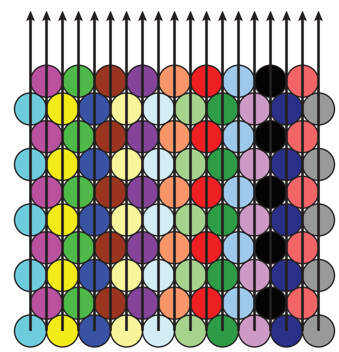


FIGURE 3.7 A 9-around vertical layout diagram showing its 19 vertical columns.

Are there any circumferences that have a number of vertical columns that *is* divisible by three? To answer that question, let's construct a quick table giving the value of 2N + 1 for different circumferences.

N-around	2N + 1	Divisible by 3?
3	7	No
4	9	Yes
5	11	No
6	13	No
7	15	Yes
8	17	No
9	19	No
10	21	Yes
11	23	No
12	25	No
13	27	Yes

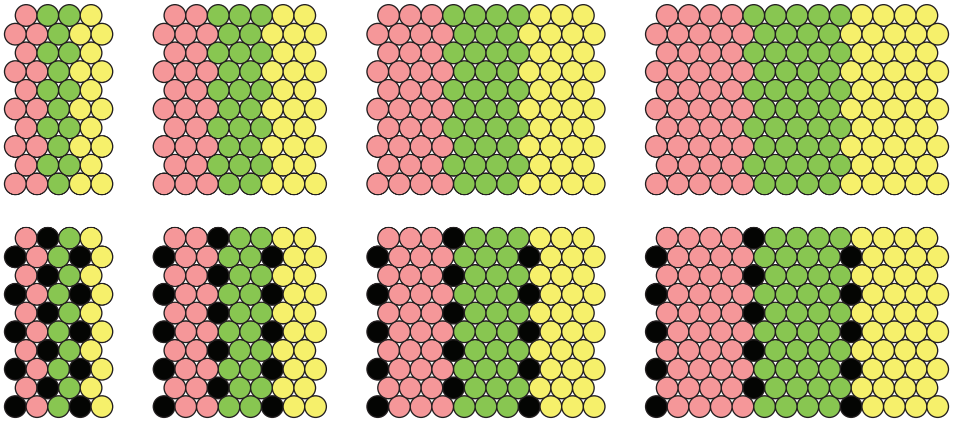


FIGURE 3.8 Equilateral triangle patterns in circumferences 4, 7, 10, and 13. The bottom row shows the spine beads in black in the leftmost column of each section.

As indicated in the table, $2N + 1$ for circumferences 4, 7, 10, 13,… are all divisible by three! Thus these are the circumferences that should permit an equilateral triangle in vertical layout format. In fact, it is no coincidence that these circumferences are all 3 apart. Each of these numbers leaves a remainder of 1 when you divide it by 3, or in other words, each can be written as $N = 3K + 1$ for some whole number K. This makes the number of columns equal to

$$2N + 1 = 2(3K + 1) + 1 = 6K + 2 + 1 = 6K + 3 = 3(2K + 1),$$

which is always a multiple of 3. Figure 3.8 shows what the even division of columns looks like for these circumferences (top row), as well as what the pattern looks like with the spine beads added in as the leftmost vertical column, shown in black, in each section (bottom row).

Looking carefully at the 7-around design in the larger bead plane layout shown in Figure 3.9, we can see more easily that the apparent shifting up and down of the spine-bead columns does not disrupt the overall symmetry of the three repeating stripes and that the pattern correctly obeys the required hockey-stick translation for $N = 7$. Furthermore, it is clear that expanding or shrinking by adding or removing two columns at a time to or from each vertical stripe maintains the validity of the pattern, so equilateral triangles in circumferences 4, 7, 10, 13, etc. are all possible.

Figure 3.10 shows a delightful equilateral triangular bracelet in 7-around made from the finished pattern, and Figures 3.11 and 3.12 show a 4-around version in a necklace and bracelet pair modeled by Sophie Sommer. Full pattern descriptions in 4-, 7-, and 10-around appear in the pattern pages section (pp. 138–140).

The value of $2N + 1$ clearly dictates which polygonal cross sections are possible using this type of untwisted vertical layout. Looking again at the table on p. 40, we can rule out some other possibilities right off the bat. Since $2N + 1$ is, by definition, an odd number, an untwisted

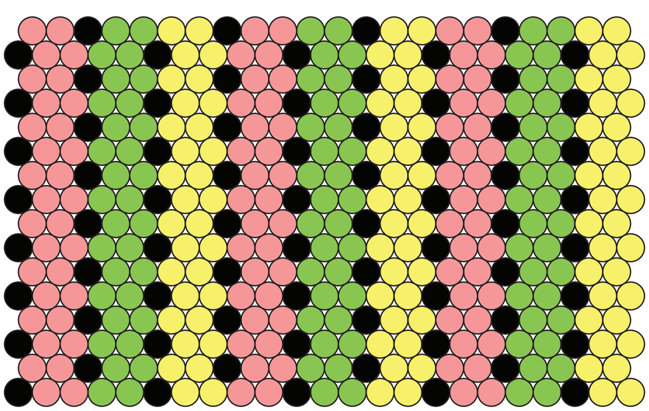


FIGURE 3.9 A bead plane diagram for the 7-around pattern in Figure 3.8.

FIGURE 3.10 Equilateral triangle in 7-around with size 8 and 11 seed beads. Pattern is on p. 139.

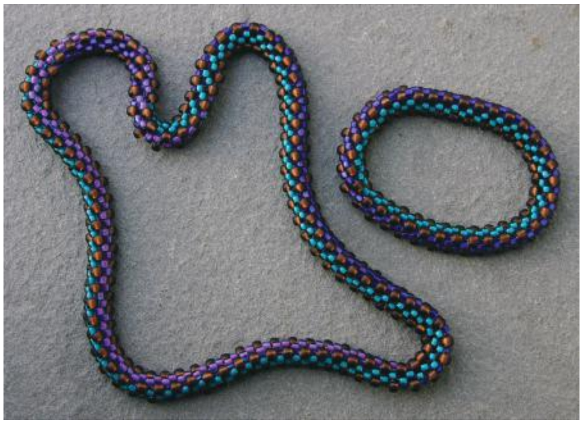


FIGURE 3.11 Equilateral triangle 4-around necklace and bracelet pair made with size 6 and 8 seed beads. The pattern is on p. 138.

FIGURE 3.12 Sophie Sommer wearing the equilateral triangle 4-around necklace and bracelet pair in Figure 3.11.

vertical layout design can never be divided into an even number of striped sections, so none of the polygonal cross sections with an even number of sides, such as the digon, square, hexagon, or octagon, are possible with this type of design. How about the other polygonal cross sections, those with an odd number of sides? A pentagon, heptagon, and nonagon are all possible, but only when 2N + 1 is divisible by 5, 7, and 9, respectively. As can be seen from the table, this happens at circumferences 7, 10, and 13, respectively. The associated designs, shown in Figures 3.13–3.15, fall out naturally. For those who want larger circumferences, wider stripes work for each of these as well.

Now that we have produced geometric cross-section designs with an odd number of sides, let's consider the more challenging problem of cross sections with an even number of sides, beginning with the two-sided digon. Figure 3.16 shows two possible patterns for an asymmetric caterpillar in 9-around. Both designs have asymmetric blue and pink (or belly and back) sections of slightly different widths.

As already noted, an untwisted vertical layout design cannot be transformed into a symmetric caterpillar with an equal number of beads on its belly and back because no matter what circumference we try, 2N + 1 is not going to divide evenly by 2. To divide by 2, we need an even repeat length, and for that we will have to consider nonvertical designs (i.e., those with a repeat length that is not a multiple of, or divisible by, 2N + 1). A simple diagonal layout with an even value of N, such as the 6-around pattern shown in Figure 3.17, might seem promising. It divides beautifully into two equal stripes, and perhaps the natural 30° slant could be untwisted before sewing closed to line up

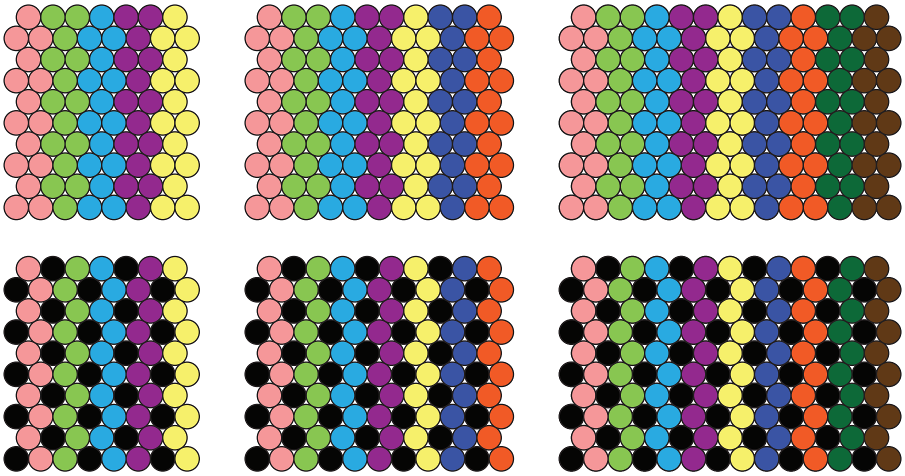


FIGURE 3.13 Designs for the remaining odd polygonal cross sections: pentagon, heptagon, and nonagon.

FIGURE 3.14 A pair of pentagonal cross-section bracelets, one in size 11 Delicas and size 11 seed beads, one in size 10 Delicas and size 8 seed beads (pattern on p. 144).

FIGURE 3.15 A heptagonal cross-section bracelet (pattern on p. 148).

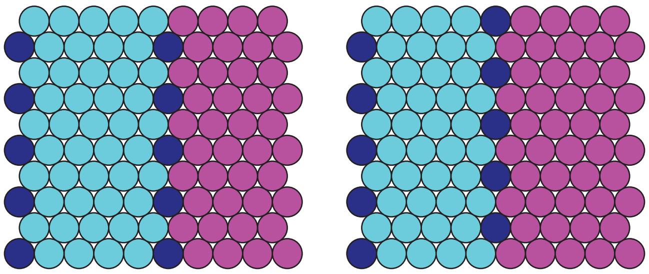


FIGURE 3.16 Vertical layout of two different caterpillar designs in 9-around. Neither one is symmetric: the blue and pink sections are not identical, dividing the vertical columns into sections of 10 and 9 or 9 and 10 (including the column of dark blue spine beads). No matter what the value of N, the required 2N + 1 columns can never be divided into two equal halves.

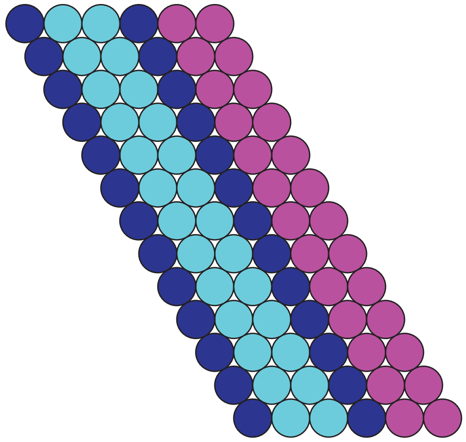


FIGURE 3.17 An unsuccessful strategy for a symmetric caterpillar. The diagonal spines slant too sharply to be rendered vertical by untwisting.

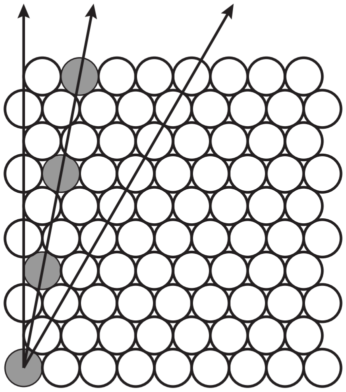


FIGURE 3.18 Choosing a diagonal between vertical and 30 degrees that minimizes the distance between spine beads.

the stripes and spines. Unfortunately, we know from our experiments with twists described in Chapter 1 that a 30° twist goes beyond the flexibility limit of bead crochet rope. It can be done, but the rope will be quite unpleasantly stiff and the bead packing will be contorted beyond anything that looks or feels like bead crochet.

A better solution is to use a diagonal that aims as close as possible to vertical (in order to achieve a more manageable twist), while also minimizing the vertical stacking distance between spine beads. The obvious intermediate choice in between vertical and 30° is shown in Figure 3.18. The spine beads stack at every third rather than every second horizontal row, but they are still close enough to read visually as connected points on a line. Note that the only other intermediate choices would result in a greater horizontal distance between the spine beads.

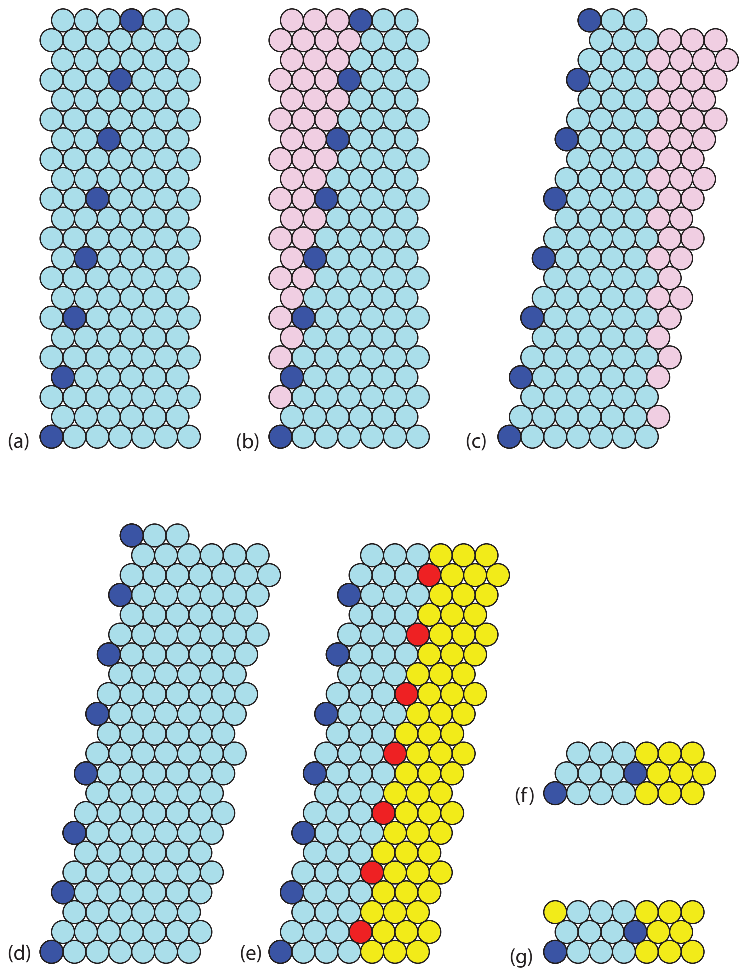


FIGURE 3.19 Constructing a symmetric caterpillar with the diagonal from Figure 3.18.

Having chosen a promising diagonal for the spines, we can play with it on a standard vertical layout (see Figure 3.19(a)) to see how we might use it to construct a symmetric two-sided digon. Shifting a few beads around with the hockey-stick translation, we can engineer a more personalized layout that makes it easier to see the diagonal stripes we're trying to construct. Figures 3.19(b) and 3.19(c) show how shifting beads from the left of the diagonal to the right edge of the layout accomplishes this. This shifting of beads is just a different, but equally valid, way of mapping the three-dimensional bracelet onto a two-dimensional plane. Note that the rearrangement conveniently preserves our ability to read the stringing order from bottom to top, left to right. Once we've constructed this new slanted layout, it is much easier to see how to place the second set of "caterpillar legs" (shown in red) in order to divide our bug evenly into two identically sized and shaped stripes (Figure 3.19(e)), thus creating a caterpillar with the same size back and belly. To make this chart neater, we have trimmed the incomplete repeat from the diagram. Figures 3.19(f) and 3.19(g) show a single repeat of the pattern in our new personalized diagonal layout and in standard vertical layout form, respectively.

Viewing the result in a larger bead plane diagram in Figure 3.20, we can confirm that it divides the plane into two-colored stripes while still obeying the hockey-stick translation rule for a 6-around bracelet. In this particular diagram, our caterpillar has dark blue legs on one side and red legs on the other, with a light blue belly and a yellow back. A gentle twist before closing the bracelet will line up the spine beads so they appear vertical and a symmetric digon results. By using a single color for the spine beads and belly/back beads, the design can be transformed into a fully symmetric caterpillar.

Exploring further, we can divide each of the 6-around symmetric caterpillar stripes into two and also create a 6-around square cross section, shown in Figure 3.21. Additional square designs derived from this same general

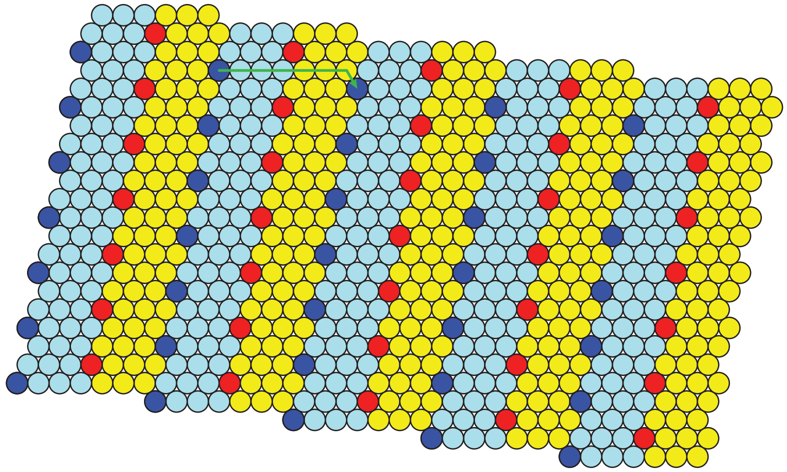


FIGURE 3.20 The bead plane diagram for the symmetric caterpillar from Figure 3.19, with belly and back the same width.

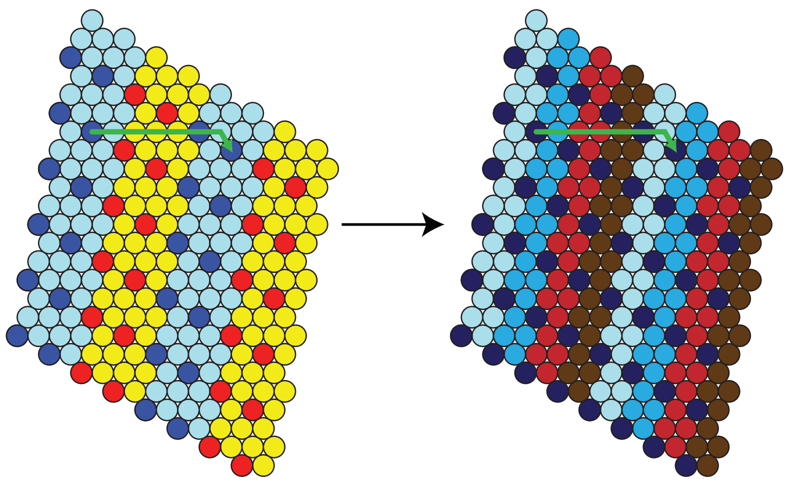


FIGURE 3.21 Developing a 6-around square design by dividing each of the symmetric caterpillar stripes in two (with hockey-stick translation shown by the green arrow).

structure are provided in the pattern section on pp. 141 and 143. The first two photographs in Figure 3.22 show how the natural slant of the pattern causes the spines to spiral around the rope. Untwisting this natural spiral before sewing the bracelet closed creates a square cross section, as shown in the third photograph of the same rope.

Closing this type of design can be a little tricky because the rope tends to spring back into its more natural position. However, because the slant is not too great, these bracelets can be closed quite neatly. Once sewn, the bracelet behaves as if it was always meant to be in its physically twisted (but visually untwisted) state. Small circumference bracelets like this 6-around tend to be fairly flexible, so, aside from the added difficulty in closing, twisting is not a problem. In some cases, however, larger circumference designs (such as the Square 10-around), depending on the beads and the amount of twisting needed, can end up stiffer and less supple, more bangle-like, than a usual rope. Using a looser crochet stitch helps reduce resistance to the twisting. In cases where flexibility is an issue, we've noted so on the pattern pages. The photos in Figure 3.22 are also a great example of how twisting options need to be considered by a designer—both the twisted and untwisted version of the bracelet are lovely in their own right, and the designer gets to choose from a variety of possibilities!

So what's still missing? The hexagon and octagon are yet to go. The hexagon, as we shall discuss shortly, turns out to be a strange beast with a somewhat more challenging personality. But the octagon is quite straightforward.

Looking again more carefully at the 6-around symmetric caterpillar design shown in Figure 3.20, we can see that

FIGURE 3.22 A square geometric cross-section bracelet design, based on the symmetric caterpillar, shown, left and top, prior to being untwisted and, bottom, after being untwisted and closed. This design appears in the pattern pages as Square 6-around (p. 142).

there is also a hockey-stick translation at 26-around, which will produce eight stripes instead of two. So the 6-around caterpillar could actually be crocheted in 26-around to produce an octagon! Of course, 26-around is rather too big a circumference to be practical, but, if we just squeeze a bit of the fat out by narrowing the stripes until it is a 2-around caterpillar, that problem is easily solved, as seen in Figure 3.23. The right-hand version shows it repainted in eight rainbow colors, but it could be crocheted in 10-around using either color scheme.

Figure 3.24 shows the final bracelet in two variations: with the natural spiral fully untwisted to create a proper octagon (back bracelet in photo), and a second version (front bracelet in photo) only partially untwisted, so that each stripe wraps once around the meridian (see Figure 4.1, p. 51). The design can also be done with larger spine beads to further accentuate the octagonal cross section.

This chapter's one remaining unsolved challenge is the hexagon—and you are no doubt wondering why we saved it for last. Unfortunately, this isn't a case of saving the best for last. Rather, it's because the hexagon turns out to be the middle schooler of the geometric cross sections: stubborn and intractable, although not entirely hopeless.

To ward off the inevitable disappointment at this news, we should quickly point out that a perfectly fine *irregular* hexagon is easy to implement in vertical layout form using three stripes each of two different widths. Shown in the pattern section on p. 146, Irregular Hexagon is akin to lining up three adjacent *asymmetric* caterpillars.

A *regular* hexagon, however, turns out to be possible only with spine beads that are quite far apart vertically,

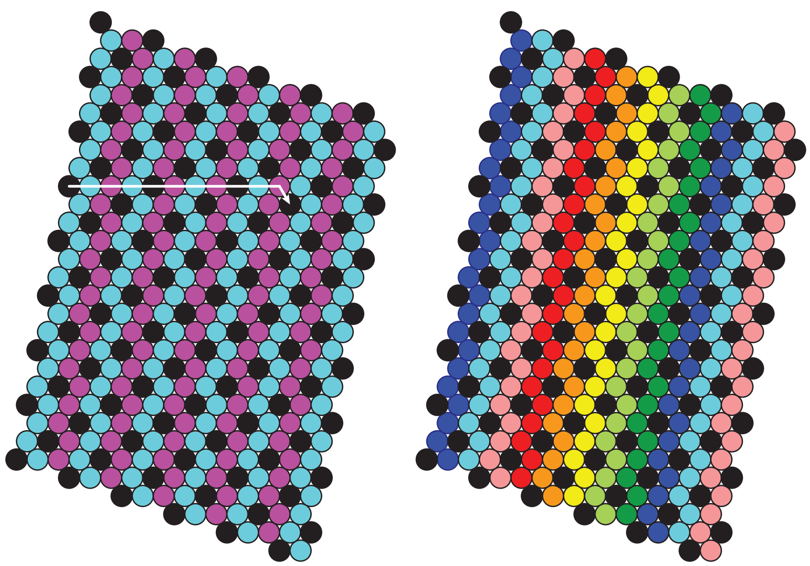


FIGURE 3.23 Creating a 10-around octagon from a skinny 2-around caterpillar.

FIGURE 3.24 A 10-around octagon in two different twists. The pattern, Octagon, appears on p. 149.

so although the bracelet can still be divided into six equal sections, the extra vertical distance between spine beads causes them to read as less connected than the beads in the next spine over (which are quite close together), and thus the visual and tactile sense of a hexagonal cross section is disrupted (see the pattern Hexagon with Spiral, p. 147).

We uncover the problem with a closer look at our diagonal options, shown in Figure 3.25. The gray bead and the red bead lie on the diagonal we used for our digon, square, and octagonal patterns. Keeping in mind that in circumference N, the long rows of our vertical chart have $N + 1$ beads and the short rows have N beads, we can see that the gray and red beads are $3N + 2$ beads apart. When $3N + 2$ is divisible by 2 (i.e., when N is even), we can put a bead midway between the gray and red to get a regular digon. Similarly, we obtain square and octagonal designs when $3N + 2$ is divisible by 4 and by 8, respectively. (You can check this against the patterns in this chapter: the square has $N = 6$, giving $3N + 2 = 20 = 4 \times 5$, and the octagon has $N = 10$, giving $3N + 2 = 32 = 8 \times 4$.)

Unfortunately, since $3N + 2$ has a remainder of 2 when we divide it by 3, it can never be a multiple of 3, much less a multiple of 6. This makes it impossible to divide a bracelet with this particular diagonal into 6 equal sides. So for a regular hexagon, we must resign ourselves to a diagonal running between the gray and blue beads in Figure 3.25, which puts our spines four rows apart. You might wonder if trying a left-leaning diagonal would help. Alas, it does not, and we invite numerically inclined readers to confirm this.

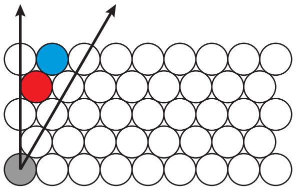


FIGURE 3.25 The hexagon diagonal problem. The gray-to-red diagonal, used for the digon, square, and octagon, does not work for a regular hexagon. The gray-to-blue diagonal does, but it leaves the spines too far apart to create a convincing polygonal cross section.

One More Challenge: Möbius Bands

Readers of a mathematical persuasion may be familiar with Randall Munroe's *xkcd*, the self-described "webcomic of romance, sarcasm, math, and language" that appears three times a week at xkcd.com. As it happens, Munroe tackled the subject of mathematical bracelets in one of his earlier comics.* The single-panel cartoon depicts a flat bracelet with a half-twist bearing the letters "WWED." Figure 3.26 shows a physical recreation of the drawing using a glued strip of paper. The caption of the comic reads, "What Would Escher Do?"

Along with the cultural phenomenon of "WWJD" bracelets, the comic references the artwork of M.C. Escher, who is famous for incorporating mathematical themes in his work. (We will explore Escher's work in more detail in Chapter 6.) The bracelet is in the shape of a mathematical object called a *Möbius band* (or *Möbius strip*), a flat strip whose ends are connected with a single half-twist. One of Escher's better-known woodcut prints, *Möbius Strip II*, depicts a line of ants crawling over the surface of a Möbius band and is doubtless the inspiration for the "Escher Bracelet" cartoon.

The Möbius band is a favorite object in popular mathematics because of its intriguing properties. If you have never played with one, we strongly recommend that you make one by cutting a thin strip of paper and gluing the ends together with a half twist. A particularly interesting exercise is to cut a Möbius band lengthwise down the

FIGURE 3.26 "What Would Escher Do?" A reenactment of the *xkcd* cartoon "Escher Bracelet," depicting a Möbius band.

* xkcd, *Escher Bracelet,* http://xkcd.com/88.

FIGURE 3.27 A cylinder and a Möbius band. While the cylinder has two sides (one with writing on it and one without) and two edges (the blue and the red), the Möbius band has one surface and one edge.

FIGURE 3.28 Mobius bracelets in 6- and 8-around. The patterns are provided on pp. 150 and 151.

middle of the strip; the result is rather surprising. (Hint: you do *not* end up with two thinner Möbius bands.) Some other curious properties of the Möbius band are illustrated in Figure 3.27, which shows two identical strips of paper with the ends glued together into a cylinder with no twist, on the left, and a Möbius band with a half twist, on the right.

The cylinder naturally has two sides: the outside, where the letters are printed, and the inside. On the other hand, the half twist in the Möbius band connects the front and back sides of the printed page, as seen in the glued join at the front where the unlettered side of the original paper connects to the lettered side. This means that the Möbius band has only one side! We also see by looking at the printing through the wrong side of the paper that there is a strange mirror reversal in effect: the "WWED" on the right of the join has travelled around the band and become "MMED" on the left of the join. As if being one-sided weren't strange enough, the Möbius band also has only one edge. Whereas the cylinder clearly has two separate edges, the blue edge and the red edge, following the colors on the right of Figure 3.27 reveals that the Möbius band has a single edge that is half blue and half red. If you ponder this carefully, it offers a clue to what happens if you cut the band lengthwise down the middle, as suggested previously.

Before we had devised any of the patterns in this chapter, we were inspired by the *xkcd* comic to attempt a beaded Möbius bracelet. In fact, our motivation for developing a symmetric caterpillar pattern stemmed from this desire to construct a Möbius design. In a regular caterpillar bracelet, which is joined like the strip on the left of Figure 3.27, it is not critical that the two sides of the caterpillar be the same width. However, to make a caterpillar into a Möbius bracelet, the half twist dictates that we sew the belly and the back of the caterpillar together, making lack of symmetry a serious design obstacle rather than an aesthetic quibble. Figure 3.28 shows two Möbius bracelets made using the techniques of this chapter.

Observant readers may notice that viewing these bracelets as Möbius bands involves a minor cheat. Since a bead crochet bracelet is actually a hollow tube, its surface always has two sides: the visible outside surface and the hidden interior surface. A bead crochet bracelet can only be considered a Möbius band if we view it as a flat strip and ignore the fact that it is actually hollow. In practice, this interpretation is natural, and the mathematicians we have shown these bracelets have immediately recognized them as Möbius bands. However, if you are interested in a more faithful fiber-art representation of a Möbius band and happen to knit, it is well worth seeking out the ingenious method of seamless Möbius knitting developed by designer Cat Bordhi*. At first glance, knitting a seamless object with only one edge will seem unlikely, since casting on a knitted piece creates one edge and binding off creates another. Bordhi's trick is to use a special Möbius cast on in

* Cat Bordhi, *A Treasury of Magical Knitting*, Passing Paws Press, 2004.

the middle of the band, from which you knit outward in a half-twisted surface until you bind off along the single edge of the finished Möbius band. Seamless knitted Möbius bands make lovely scarves and cowls, and their center-out construction gives more insight into their mathematical structure than the seamed, not-quite-flat bead crochet bracelet technique. Nonetheless, beaded Möbius bracelets have an undeniable visual and tactile appeal, and you don't have to wait for a cold day to wear one.

Faceted Bracelets: Another Design Springboard

Part of the appeal of the bracelets in this chapter is the textural effect produced by mixing beads of different sizes. For the geometric cross-section designs, the larger beads are arranged in straight spines that separate long flat strips of beads. If instead we cover our bracelet with simple geometric shapes, such as hexagons or parallelograms, and outline each shape in larger beads, we create a bracelet covered with small facets of color.

FIGURE 3.29 A collection of faceted bracelets. The patterns are Honeycomb (p. 154), Stained Glass Diamonds (p. 152), and Pressed Berries (p. 231).

You will find a number of faceted bracelet designs in the pattern pages, some of which are shown in Figure 3.29. If you are interested in designing your own faceted patterns, we recommend that you look at the techniques in Chapter 6 for creating tessellated bracelet designs.

CHAPTER 4

Torus Knots

One of our early design quests was to create bracelet patterns that illustrate torus knots. Like ballet dancers who make the difficult look easy, torus knots have a simple elegance that belies a precise choreography. For the bead crochet designer, they are a natural focus, since the various underlying spiral structures in all bracelets form some flavor of torus knot. We begin by explaining what they are and why we find them intriguing, starting with a few preliminary definitions.

Just as every point on a sphere or globe can be identified by its longitude and latitude, every point on a torus can be identified with two coordinates in a similar fashion. On a torus, we measure the two coordinates along longitude and meridian lines, as shown in Figure 4.1. In bead crochet terms, the length of a meridian line is the circumference or thickness of the rope itself, and the length of a longitude line is the circumference or size of the full bracelet. A [longitude, meridian] position on a torus also corresponds to an [X,Y] position on two perpendicular axes on a flat version of the torus. We will look more closely at the relationship between tori and their flat diagrams later.

A “knot” in this context is a length of string that has been twisted and looped in some way prior to having its two ends tied together (or attached somehow). In a torus knot, the string is looped through and around a torus before having its ends connected so that it can no longer be unlinked from the torus without untying or cutting it first (Figure 4.2).

More precisely, a *torus knot* is a knot that winds P times around a torus meridionally and Q times longitudinally [notated (P,Q)],* where P and Q are relatively prime. *Relatively prime* means that P and Q are two numbers with no common divisors other than 1, for example, 4 and 7, or 5 and 8. Figure 4.3 shows the torus knots from P = 3 to P = 9 and from Q = 2 to Q = 5. One famous knot you may have seen or heard about is the trefoil, or (3,2) torus knot, shown at the upper left in Figure 4.3. In torus knots, the torus does not have to be present; the string itself is the torus knot, so an imaginary torus is sufficient, as in the (11, 2) torus knot shown in Figure 4.4.

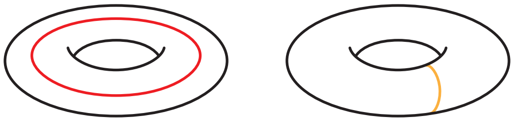


FIGURE 4.1 Longitude lines run along the circumference of the full bracelet. Meridian lines run along the circumference of the rope.

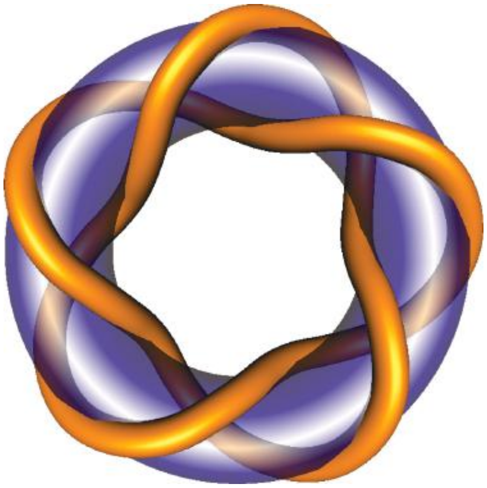


FIGURE 4.2 A length of string looped through and around a torus before having its ends connected (top), and a computer-generated diagram of the same knot (bottom). Notice that the bow in the upper picture is not a part of the mathematical knot; it is just a method of attaching the two ends of the string.

Elements missing from the chart in Figure 4.3 are those where P and Q are not relatively prime. You might immediately wonder what’s so special about relatively prime P and Q: why would we distinguish these cases from instances where P and Q have a common divisor? The answer to this question is related to a fun and mysterious fact that has interesting bead crochet design implications, namely, that the “string” in a torus knot can be embedded into or laid onto the torus surface without ever crossing over or under itself, a feat that is physically possible only for a knot with relatively prime P and Q. In fact, torus knots

* In some texts, the meaning of P and Q in this notation is reversed.

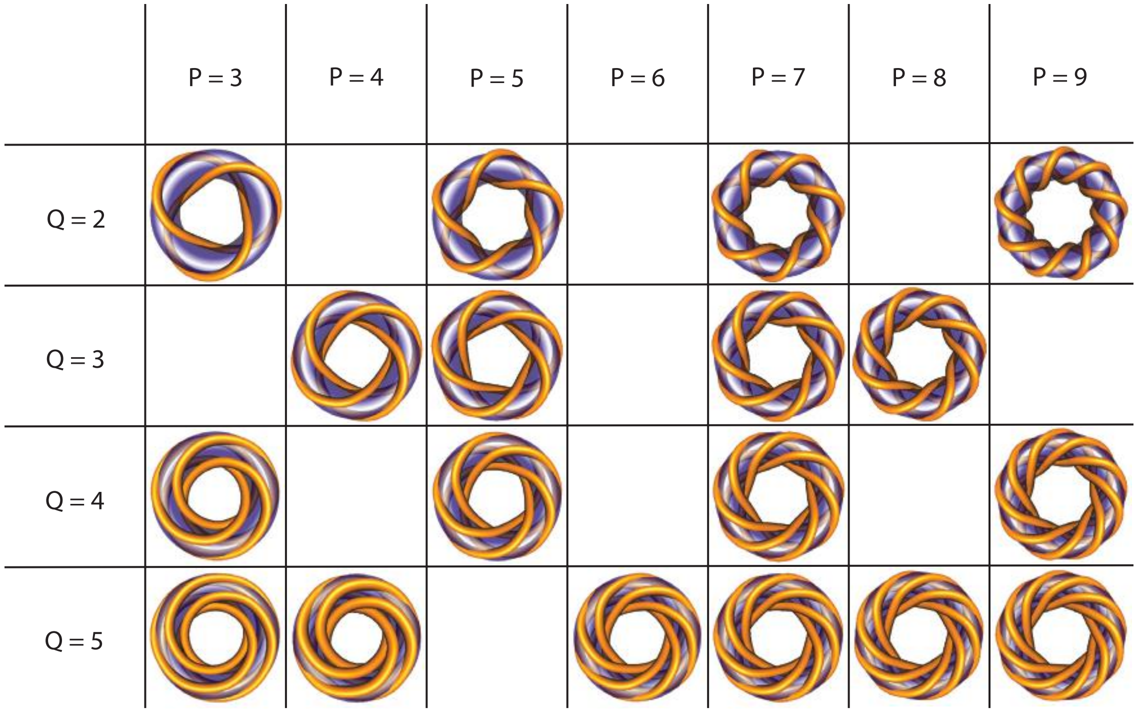


FIGURE 4.3 Examples of torus knots. These torus knot images were produced using the Knotplot software created by Rob Scharein.

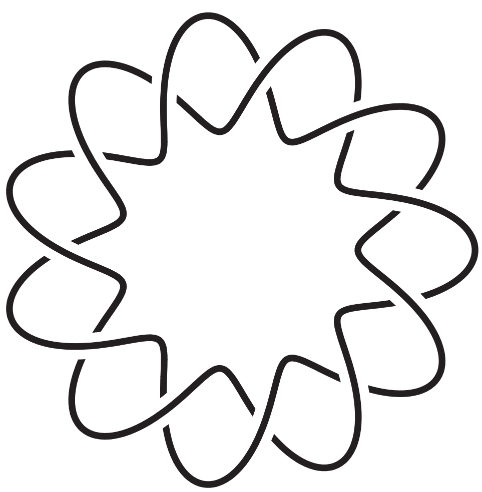


FIGURE 4.4 An (11, 2) torus knot without the torus. This image was produced using the Knotplot software created by Rob Scharein.

are precisely those knots that *can* be drawn on the surface of a torus without crossings. Furthermore, this ability to "draw" or embed the string on the torus surface without ever crossing itself holds true for any relatively prime P and Q, no matter how large! This might seem amazing and mind-bending, unless, of course, you are a mathematician who has spent time playing with such things. Try picking a couple of relatively prime numbers out of a hat, say, 52 and 27*. Now imagine trying to wrap a single piece of string around a bracelet so that it weaves 52 times through the hole (i.e., meridionally) and also 27 times around the full length of the bracelet (i.e., longitudinally) before connecting back to itself at the beginning, all without ever crossing over or under itself. Aside from the problem of avoiding crossings, you might also object that the string had better be thin enough in relationship to the size of the torus—a little tiny torus and a big fat string surely won't work. However, mathematicians think in theoretical terms and consider the string to be infinitely thin, the width of a single point. But even if we prefer our string to have a tangible width, we can always pick a torus size that is bigger in relationship to the string.

One way to think about a torus knot is to envision a spider traversing the torus and spinning a strand of silk to create the knot as it goes. If the spider travels in precisely the right unvarying direction, namely, at a slope of Q/P on a flat model of the torus, it can produce the desired knot. We shall see diagrams later that illustrate this more clearly, but a slope of Q/P is just the mathematical way of saying that the line changes by Q in the vertical direction as

* We know that 52 and 27 are relatively prime because 27 is 3^3 and thus its only prime factor is 3, which doesn't divide into 52.

it changes by P in the horizontal direction. This is exactly the direction needed to allow the spider to accomplish its goal of precisely Q longitudinal traversals and, simultaneously, P meridian traversals before returning to its starting position, where it can then glue the two ends of the strand together. The spider's special torus knot choreography is to spin (in both senses) around the torus the requisite number of times, both meridionally and longitudinally, without ever stepping on the same spot twice. While the spider's task might sound tricky, it is really just following a straight line with a precisely chosen slope, and if you've ever made a standard spiral pattern in bead crochet, you may already have created some interesting torus knots.

Challenge *Can you design a trefoil knot—or (3,2) torus knot—in bead crochet? Can you come up with a general method for designing any (P,Q) torus knot in bead crochet? As in a proper torus knot, the "string" component of the design (in this case some "linear" pattern of contrasting color beads) must never cross itself.*

Ideally, the gaps between the "string" should remain symmetric throughout, just as they are in the torus knots shown in Figure 4.3. However, unlike the torus knots in Figure 4.3, the lines need not be perfectly smooth. In fact, given the bead-sized "pixels" on our bracelet "canvas," some zigzagging is likely needed.

Construction Using Physical Twists

If you have a bagel or ceramic donut and simply want to draw a torus knot on it with a marker, it might take some thought to figure out how to do so, even for relatively small values of P and Q. If you have a laundry marker and a bagel to spare, this is a fun exercise to try before reading further. Colin Adams in *The Knot Book*[*] (a wonderful resource for those interested in learning more about mathematical knot theory) gives a method for drawing a (P,Q) torus knot that involves marking and connecting equidistant points along the inner and outer longitudinal "equators" of the torus. In bead crochet, we are "drawing" with beads and are thus constrained by bead pixelation, so we can't draw smooth curves or lines anywhere we want. Despite these added constraints, working in bead crochet we have a wonderful advantage over someone trying to draw lines on, say, a hard ceramic donut. Crocheted rope has flexibility and stretch to it, a feature that topologists dream about, and one that we will use to great advantage in our first set of constructions. As observed in Chapter 1, topologists study geometric properties that don't change with stretching and shrinking[†]— so the classic joke about them is that they don't know the difference between a donut and coffee cup … because either one can be deformed by stretching and shrinking into the other!

Overall, our approach is based upon either physical twists of the rope, natural spiraling of the pattern, or a combination of both. Before discussing bracelet design methods in detail, let's forget about bead crochet briefly and get oriented conceptually with a related pencil and paper "thought experiment" on how to construct a torus knot starting from a flat torus. This is only a thought experiment because our paper needs to be flexible and stretchy while still holding its shape, and most likely you don't have any such paper on hand (and neither do the topologists, except in their imaginations). After our experiment, we'll see how this method applies to bead crochet.

Consider a square flat torus with Q evenly spaced parallel vertical lines drawn on it, as shown upper left in Figure 4.5 for Q = 3. Now imagine gluing the identified edges together: first the right and left edges of the paper to produce a cylinder and then the top and bottom edges to produce the torus (using the method depicted in Figure 1.2). What happens to the vertical lines? The top row of Figure 4.5 illustrates this process. We can see that the lines on the flat paper form three separate, evenly spaced, longitudinal rings on the torus. You may notice this is related to the geometric cross-section designs described in Chapter 3. Now suppose we try it again, but give the cylinder a 1/3 twist before connecting the ends. What happens to the "vertical" lines now? Marvelously, we have connected the first line to the second, the second to the third, and the third back to the beginning again—to form a *single* loop that wraps around the torus three times longitudinally while at the same time wrapping just once meridionally, all without ever crossing itself—in other words, we have constructed a (1,3) torus knot, as shown in the bottom row of Figure 4.5!

This same method works for *any* values of relatively prime P and Q; a P/Q twist before closing on a cylinder with Q vertical lines will always produce a (P,Q) torus knot!

[*] *The Knot Book*, Colin Adams, American Mathematical Society, August 2004.

[†] As opposed to, for example, tearing or puncturing.

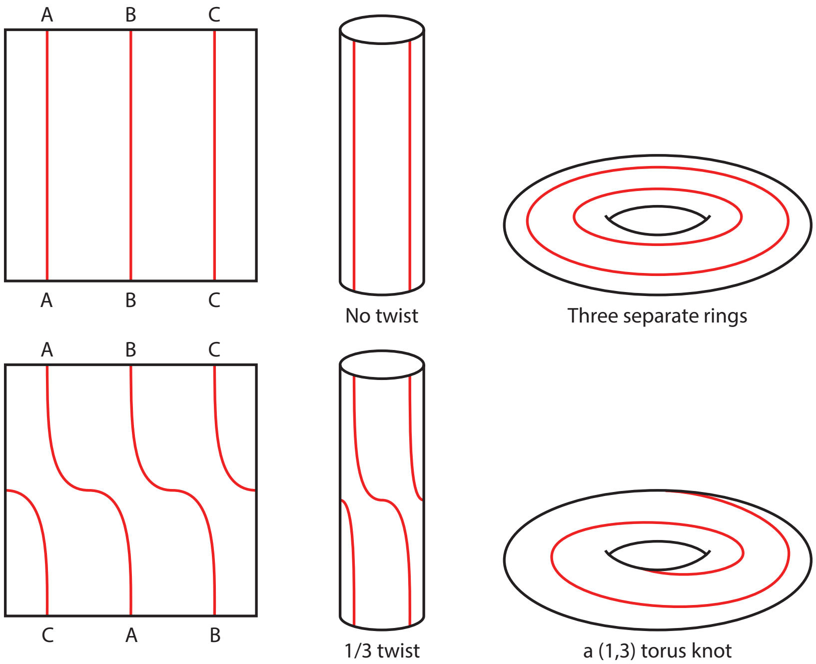


FIGURE 4.5 Thought experiment: starting with a flat torus with three evenly spaced vertical parallel lines, connect the left and right edges to roll it into a cylinder. Next, connect the top and bottom edges (now the circular ends of the cylinder) to form a torus. What happens to the three lines? Now try it again, but give the cylinder a 1/3 twist before closing. What happens to the lines now?

However, you may need some further convincing to see why this is true and to gain insight into what happens when P and Q are not relatively prime. For this, we'll turn to star polygons, adapting a lovely instructional idea we learned about from Sandy Spitzer at the 2012 Bridges Conference (a yearly interdisciplinary conference on mathematics and the arts).[*] Sandy presented a paper on using star polygons to provide students with intuition for understanding cyclic groups in mathematics. We noticed that they also provide a nice model for conceptualizing torus knot construction in bead crochet.

We construct a (P/Q) star polygon on a circle with Q equally spaced perimeter points by drawing lines connecting every Pth point. The construction process starts with any one of the perimeter points, and a link is drawn between it and the point P clockwise dots away.[†] The next link is drawn from that point to the following point P clockwise dots away, and so on, until we return to the starting point. When P and Q are relatively prime, this process is guaranteed to include all Q perimeter points, and the resulting figure is called a *regular star polygon*.[‡] Figures 4.6 and 4.7 show all possible star polygons produced on a circle with 8 points and 7 points, respectively. Note that different values of P sometimes produce the same star polygon. Testing star polygon construction using a variety of different values for P and Q should give you some valuable intuition for how the process behaves differently depending on the values of P and Q and, in particular, why relatively prime P and Q are needed to include all perimeter points.

For modeling purposes, we will think of the star polygon circle as the place in our torus knot construction where the two cylinder ends meet. The choice of Q represents the number of vertical lines on the cylinder, and the circle is a cross section (or overhead view) where all you can see is the circular cylinder end and dots at the endpoints of the vertical lines. The choice of P represents the amount of twisting prior to closing the two ends of the cylinder. Since the dots

[*] Sandy Spitzer, "Using Star Polygons to Understand Cyclic Group Structure," Bridges 2012 Conference Proceedings.

[†] Drawing in a counterclockwise direction is also fine, as long as the direction is consistent throughout.

[‡] If P and Q are not relatively prime, this process will *not* include all the perimeter points, in which case we obtain a star polygon for a smaller value of Q. For example, in Figure 4.6, the (2/8) star polygon is also a (1/4) star polygon. If the drawing process is begun again at one of the unused perimeter points, and this is repeated until all perimeter points have been used, the result is called a *star figure* (see http://mathworld.wolfram.com/StarFigure.html). Star figures have a connection to another construct from topology called a torus link. We do not discuss torus links in this book, but they are also interesting and can be represented in bead crochet using the techniques described here.

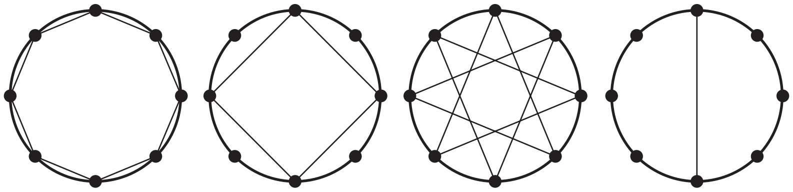


FIGURE 4.6 The star polygons for Q = 8. From left to right, P = 1 or 7, P = 2 or 6, P = 3 or 5, P = 4.

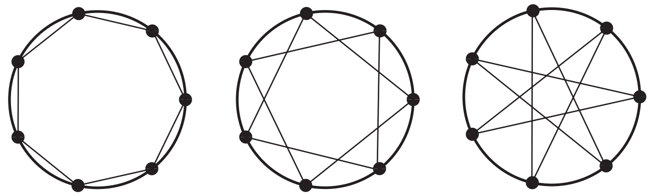


FIGURE 4.7 The star polygons for Q = 7. From left to right, P = 1 or 6, P = 2 or 5, P = 3 or 4. Since 7 is prime, for all P < 7, P and Q must be relatively prime, so all the points are connected.

divide the circle into Q evenly spaced sections, a connection between a given point and the point P dots clockwise away from it models a clockwise P/Q twist of the cylinder end. Thus, lines drawn between points show how the cylinder's vertical lines connect with one another at a closure sewn with a P/Q twist. For example, as shown middle right in Figure 4.6, a circle with eight evenly spaced perimeter points (Q = 8) linking every third point (P = 3) models the connections on a cylinder with eight vertical lines that is closed with a 3/8 twist at one end. Since for these values of P and Q, all eight of the perimeter points are included in the star, all eight of the original vertical lines would be connected in a single knot.

How many meridian traversals does a 3/8 twist produce? The physical twist of the rope causes each of the eight vertical lines to travel 3/8 of the way around the meridian. Summing all those partial traversals, we can calculate that the entire knot will travel around the meridian a total of $8 \times 3/8 = 24/8$ times for a total of three full meridian traversals. Thus the eight vertical lines of our cylinder will connect into a single knot that simultaneously travels around the torus eight times longitudinally and three times meridionally, which is precisely the definition of a (3,8) torus knot!

An alternate way of using a star polygon model to calculate the number of meridian traversals is to count how many times your hand has to travel around the circle during the drawing process. This also helps distinguish between identical star polygons with differing values for P. For example, during the drawing process of a (3/8) star polygon, as you count out dots to skip while moving clockwise in sequence, your pencil will travel exactly three times around the circle before returning to the starting point again. By contrast, if you try constructing the (5/8) star polygon, modeling a 5/8 twist, you will get the same regular star polygon, but this time your pencil will travel five times clockwise around the circle. So it is not just the final form of the star polygon that is informative, but also the process of creating it that helps model and visualize what is happening as we introduce different twists before closing the cylinder ends of the rolled flat torus.

What happens if P and Q have a common divisor, or in other words, are not relatively prime? For example, let's try P = 2, which divides evenly into 8 and represents a 2/8 (or 1/4) twist. In this case we get the star polygon shown middle left in Figure 4.6, in which only half the points on the circle wind up interconnected. What kind of knot is constructed now? By playing with star polygons you should be able to make several observations about their behavior. The first and most important is that only when P and Q are relatively prime are all Q points in the circle included in the polygon. Hence only in this case is a *single* torus knot produced that uses all Q vertical lines.

From a star polygon perspective, a clockwise twist of P/Q is functionally equivalent to a counterclockwise twist of 1 – (P/Q), e.g., 1/8 clockwise and 7/8 counterclockwise produce the same star polygon. In fact, so do all twists of 1/8 and 7/8, regardless of direction. Similarly, twists of 1/8, 9/8, 17/8, etc., all produce the same star polygon. This does not mean that the torus knots produced with these twists are identical, only that the final star polygons are. The torus knots are all identifiably different, depending on the direction of twist and number of meridian traversals.

Now that we know how to construct a specific (P,Q) torus knot with our hypothetical stretchy paper by drawing

vertical lines and applying the appropriate P/Q twist before closing the cylinder, we can use this idea to create torus knots in bead crochet. As mentioned earlier, our thought experiment suggests a method based on the geometric cross-section designs in Chapter 3. Using these designs, for odd values of Q we can construct perfectly vertical dotted lines equally spaced on bead crochet rope. Using drop beads for the spines can give the lines a more solid appearance. For example, the Equilateral Triangle 7-around design shown on p. 41 creates Q = 3 vertical lines, and the Pentagon design on p. 42 creates Q = 5 vertical lines.

But can we achieve a twist in the rope as desired before closing to create a torus knot? For the types of twists we want, the answer is yes. A single full twist, although insufficient by itself for our purposes, is generally no problem for a bracelet length rope—in fact, if you have made a lot of bead crochet ropes, you've probably inadvertently closed bracelets with a full twist already. Multiple full twists can tax the flexibility of the rope and might be more easily noticed, but if you do not limit yourself to bracelets—if long necklaces will do—you can generally achieve multiple full twists without reaching the physical torque limit of bead crochet.

However, by themselves, single or multiple full twists are not useful because they always connect each vertical line directly back to itself, so the ratio represented by the twist can never have relatively prime P and Q. For example, two full twists with a Q of 3 would give a ratio of 6/3, and 6 and 3 are not relatively prime. Thus, for our purposes, we also need smaller partial twists like 1/3, 2/3, or 3/5. How can we accomplish those? Fortunately, the regular geometric cross-section designs are constructed symmetrically so that each repeat is composed of Q smaller identical sections, each effectively adding 1/Qth of the rope's circumference. By using a single color for all sides of the polygon instead of a different color for each side, we can make these smaller sections of the repeat identical to each other. This leaves us with a single background color and a contrasting spine color that represents the "string" for our knot. Then, due to the symmetry of the cross-section designs, it is no problem to leave off (or add on) one or several of the small identical sections, and doing so will induce various fractional twists with Q in the denominator. If we need a twist greater than one, we can accomplish this by combining full physical twists of the rope with these smaller twists induced by partial repeats. For example, a 5/3 twist is the same as a 1 and 2/3 twist, and it can be accomplished by combining a full twist plus a 2/3 twist induced by omitting 2/3 of the final repeat.

FIGURE 4.8 A (5,3) torus knot bracelet produced from the equilateral triangle cross-section design on p. 41, using drop beads and size 11 seed beads in 7-around. The final repeat on the bracelet uses only 7 of the 21 beads in a repeat to induce a partial twist of 2/3. This is added to a full twist to produce the requisite 5/3 twist before closing.

The bracelet shown in Figure 4.8 is an example of exactly this approach based on the equilateral triangle 7-around design. In this case, the 21-bead repeat of the original triangle pattern is portioned into three identical 7-bead sections, and the final repeat is reduced by two sections to only 7 beads, thereby enabling a 2/3 counterclockwise twist. This 2/3 twist was added to a full physical counterclockwise twist to produce the 5/3 twist needed to get a (5,3) torus knot.*

Using partial repeats to produce fractional twists and possibly one or more full physical twists, we can achieve a P/Q twist of a geometric cross-section design with Q faces. Keep in mind, however, that, depending on stitch gauge, circumference, and bead type, multiple full twists can create too much torque in a bracelet-sized rope, so a longer necklace-length piece could be needed.

There is nothing like playing with the bracelets themselves to clarify these ideas. If the math seems confusing, try experimenting with different twists and partial repeats on one of the regular cross-section designs. Figure 4.9 shows the 4-around Equilateral Triangle bracelet (pattern on p. 138) and *all* the (P,3) torus knots that it can produce up to P = 7. We made all the bracelets shown from the same crocheted piece, twisted, closed, reopened, retwisted, and

* Note that reducing the last repeat to only 7 beads could alternatively have been used to induce a 1/3 twist clockwise, so the same exact bracelet could also be closed into a (1,3), (2,3), or (4,3) torus knot, depending on the direction chosen for the induced twist.

(a)

(b)

(c)

(d)

(e)

FIGURE 4.9 All the (P,3) torus knots up to P = 7. These examples were constructed using the exact same triangular cross-section pattern with different closing twists: (a) (1,3), (b) (2,3), (c) (4,3), (d) (5,3), (e) (7,3). What would be the next knot in the series and how would you produce it?

closed again to try out all the various twist and partial repeat options (which is why the thread tail is visible in the photos). This particular bracelet ends with only two-thirds of the final repeat (using 6 of 9 beads in the last repeat), forcing either an N + 1/3 twist in the counterclockwise direction or an N + 2/3 twist in the clockwise direction. A 1/3 twist produces a (1,3) knot, a 2/3 twist a (2,3) knot, a 4/3 twist a (4,3) knot, etc.

So far, our examples have used only the odd geometric cross sections (triangle, pentagon, etc.) because those are the designs that produce perfectly vertical lines. However, we can also use the even cross sections (digon, square, etc.),

FIGURE 4.10 A pair of mirror image (3,4) torus knots in 6-around. Can you figure out the difference in the construction? Hint: the bracelets have a slightly different number of total beads.

in which the pattern has a natural slant to it. The trick is first to straighten out the natural slant of the spines by untwisting them (as we might normally do for these designs), and only then to apply the desired P/Q twist. Figure 4.10 shows two (3,4) torus knot bracelets created using this approach, based on the Square 6-around pattern on p. 142. With this trick in mind, it should also be possible, for example, to figure out how to achieve a (3,2) torus knot using as a basis the Möbius band designs in Chapter 3. In fact, the Möbius band bracelet itself is already a (1,2) torus knot design!

Although both bracelets in Figure 4.10 are (3,4) torus knots, you may notice that they have an interesting difference: their knots spiral in opposite directions, one clockwise, the other counterclockwise. In fact, they are mirror images of one another. You may also have noticed that the spirals in the Figure 4.9 set of bracelets alternate direction as P increases. These observations are related to another tidbit from knot theory, namely, that there are two distinct variants of a (P,Q) knot: a knot and its mirror image.*

* The exception to this rule is when P or Q is equal to one, in which case the mirror versions are actually the same knot, an unknotted loop.

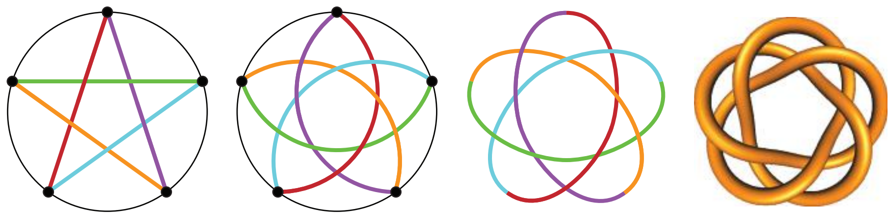


FIGURE 4.11 Star polygons can be constructed with P and Q reversed in meaning. This figure shows a sequence of steps relating a (5/3) star polygon—constructed with five perimeter points on a circle with links connecting every third point—to a (5,3) torus knot. The image on the right was generated by the Knotplot software created by Rob Scharein.

The mirror image bracelets in Figure 4.10 are both (3,4) torus knots done in 6-around with drop beads (in white) and size 11 seed beads (in brown), based on the Square 6-around design on p. 45. You can think of them as two related species of the same torus knot.

When we first considered the problem of constructing mirror image but otherwise identical torus knots in bead crochet, we originally believed the only possible construction method was to use the same pattern and crochet one left handed and the other right handed, which forces the spirals to run in opposite directions. If you are ambidextrous, or can team up with an opposite-handed partner, right- and left-handed crocheting is perhaps the ideal solution to this problem. Teaching yourself to crochet both ways has the added benefit of enabling you to teach both left- and right-handed friends to bead crochet. Eventually, however, we realized a less taxing solution was possible, as demonstrated by the bracelet pair in Figure 4.10, which avoided resorting to left- and right-handed crocheting. As an extra credit challenge, try figuring out exactly how this mirror image pair was constructed. A hint is that one bracelet has 10 more beads in it than the other (and both end with fewer than the entire 20-bead repeat in the pattern). If you are picky, you might therefore complain that the two bracelets are not true identical mirror images. However, the casual observer would never know, so we're quite pleased with them. We'll have a chance to play a bit more with mirror image knots in Chapter 5 on knotted and linked bracelets.

A final interesting side note on star polygons is that it is also informative to construct them with P and Q reversed, i.e., using P perimeter points on a circle with links connecting every Qth point. Our original interpretation is more useful for modeling bead crochet torus knot construction using physical twists, but the reversed meaning is more helpful for modeling the appearance of the knot itself, offering yet another way to gain insight into torus knot structure. With some modifications, a "reversed" star polygon looks like a flattened, torus-less version of the actual knot (called a two-dimensional projection), at least when $P < Q$. Figure 4.11 shows a sequence of images relating a "reversed" (5/3) star polygon to a (5,3) torus knot. Notice that the leftmost illustration in Figure 4.11 is also a (5,2) star polygon. It is interesting to consider how the curving of the lines in the first step of the progression in Figure 4.11 would differ in a similar sequence relating the star polygon to a (5,2) torus knot instead (see Figure 4.2 for a good illustration of a (5,2) torus knot).

Construction Using Natural Twists

So far so good—we've proposed a successful approach for solving the torus knot challenge in this chapter. However, these designs rely heavily on physical twisting, which has some annoying limitations in bead crochet. Is there another construction method that doesn't depend so much on physical twists, but instead develops a pattern that spirals naturally at just the right slope and can thus be closed without any significant torquing of the rope? One benefit of untwisted bracelets is full preservation of the rope's natural flexibility, which is one of bead crochet's most appealing tactile qualities.

How can we understand and control the natural spiral of a pattern to create torus knot designs that wrap longitudinally and meridionally according to a specific desired (P,Q) plan? This is exactly the question we address next, using a different construction method that eliminates the need for any significant physical twisting. Here again, before describing how to do it in bead crochet, we return to a flat torus thought experiment to see how it might be done in principle. The approach might take a little mind-bending conceptually, but it makes good intuitive sense once you understand the ideas.

For this thought experiment, imagine constructing a patchwork of flat tori that is P tori wide and Q tori high, as shown on the left in Figure 4.12 for $P = 3$ and $Q = 2$.

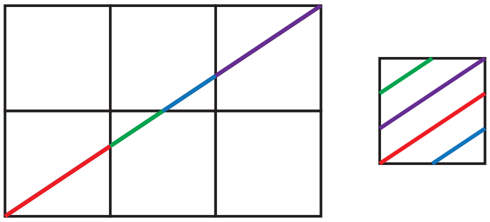


FIGURE 4.12 Constructing a (3,2) torus knot using a three-by-two patchwork of flat tori. On the left is the patchwork of transparent flat tori three wide and two high with a line drawn from the lower left corner to the upper right. On the right is what it would look like if the individual tori are stacked together like a deck of cards, thus effectively gluing together the associated interior points on all tori in the patchwork. If we next also identify the exterior points (i.e., the left and right and top and bottom edges), the result is a (3,2) torus knot drawn on the torus surface.

Then draw a straight line from the lower left corner of the patchwork to the upper right. By design, since each square in the patchwork is one torus, this line spans P tori in the horizontal direction while simultaneously spanning Q tori in the vertical direction. Thus it has exactly the slope of Q/P required for the desired (P,Q) torus knot. With some imagination, this may conjure up the image of a two-dimensional map produced by an explorer charting the surface of a torus, looping three times through the hole while simultaneously looping two times around the long way. Keep that image in mind as we continue.

On a standard flat torus, the left and right edges correspond, as do the top and bottom edges, so you can visualize rolling it up into a torus in three dimensions by attaching the corresponding edge points (as shown in Figure 1.2). Our patchwork of flat tori is likewise a two-dimensional map of a single torus. However, in our patchwork, there are multiple correspondences of both interior and edge points. Here, too, one can visualize the same kind of correspondence operation: rolling the whole patchwork up so that *all* corresponding points, both edge and interior, are connected. An alternate way to think about this is to imagine it as two sequential operations: a first step that connects all the interior points by breaking apart the patchwork into its constituent flat tori and stacking them up like a deck of cards, and a second step that connects all their edge points in the usual manner. The result will be a single torus with interior layers, like a set of nested Russian dolls, or layers of the earth's crust.

One more leap of imagination is needed to complete our thought experiment. Suppose the patchwork fabric layers are actually transparent and infinitely thin. Because of their transparency, even after stacking and rolling, we can still see the entire diagonal line we've traced on the patchwork. No matter what layer of fabric it was originally drawn upon, it shows through on the torus surface. Furthermore, since the diagonal line has been deliberately chosen with a slope that will traverse three torus horizontal lengths while at the same time traversing two torus vertical lengths, and since 3 and 2 are relatively prime, the result is a (3,2) torus knot drawn on the torus surface.* On the right, Figure 4.12 shows the transparent, infinitely thin, stacked tori resulting from this process. Figure 4.13 shows, bagel-rendered, the final result of our imaginary process applied to the patchwork in Figure 4.12.

So, how do we extend this idea from thought experiments and bagels to a practical torus knot construction in bead crochet? Figure 4.14 illustrates a construction for a (4,3) torus knot design using vertical bracelet layouts as the patchwork elements. To simplify things, we've used an untwisted bracelet (i.e., a vertical layout with an integral number of double rows),† but as we'll see later, this is not required. In this example, the layout length is short, only 8 double rows, which is clearly not long enough (a bracelet made with size 11 seed beads typically requires at least 50 double rows), but it suffices to demonstrate the idea.

A decision is clearly required about which beads to "paint" to represent the straight line. Given the coarse pixelation of bead crochet, which precludes painting partial beads, the line typically needs to zigzag to stay on course. For symmetry and aesthetics, we want a short zigzag that closely tracks the slope of the diagonal and a rope length that provides space for an integral number of these zigzags, as shown in the example in Figure 4.14.

Although the design in Figure 4.14 is not long enough for an actual bracelet, we can use it anyway, repeating it multiple times to get a longer rope. To get a good size for a bracelet, 52 double rows work nicely. Since the original patchwork design creates a traversal of the meridian every 2 double rows, 52 double rows result in 26 meridian traversals or a (26,3) torus knot. In this particular bracelet, however, we chose to play with the twist as well by untwisting some of those traversals to generate the interesting looking (20,3) torus knot shown at the bottom of Figure 4.15.

* For additional insight into the importance of having relatively prime P and Q, try drawing some patchworks with diagonal lines using P and Q that are not relatively prime, such as P = 6 and Q = 4, and see how those patchworks behave.

† Recall from Chapter 1 that a vertical layout is composed of alternating rows of N + 1 and N beads, where N is the rope circumference. A double row is defined as a chunk of two consecutive rows in a vertical layout, i.e., a row of N + 1 and a row of N beads.

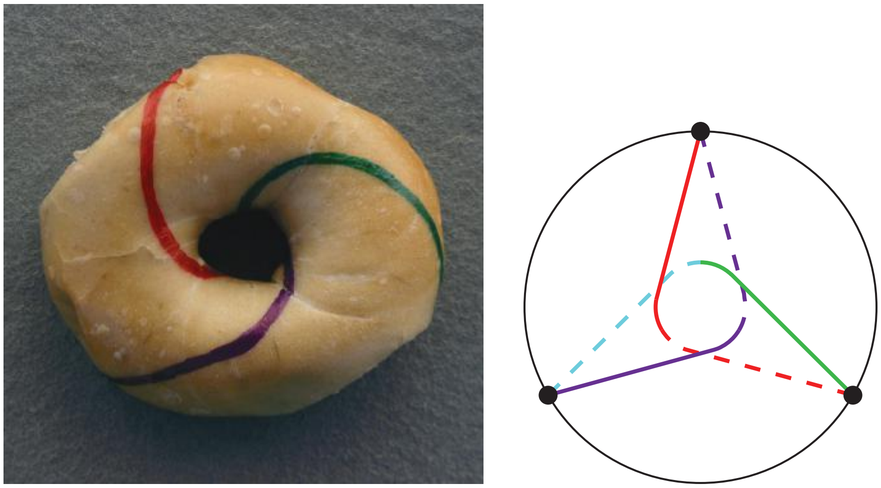


FIGURE 4.13 Left: A bagel rendition of the (3,2) torus knot from the patchwork in Figure 4.12. The blue portion of the line is there, too, but can be seen on the underside only. Right: a modified "reversed" star polygon diagram (similar to those in Figure 4.11) of the same knot with the underside portions shown as dotted lines.

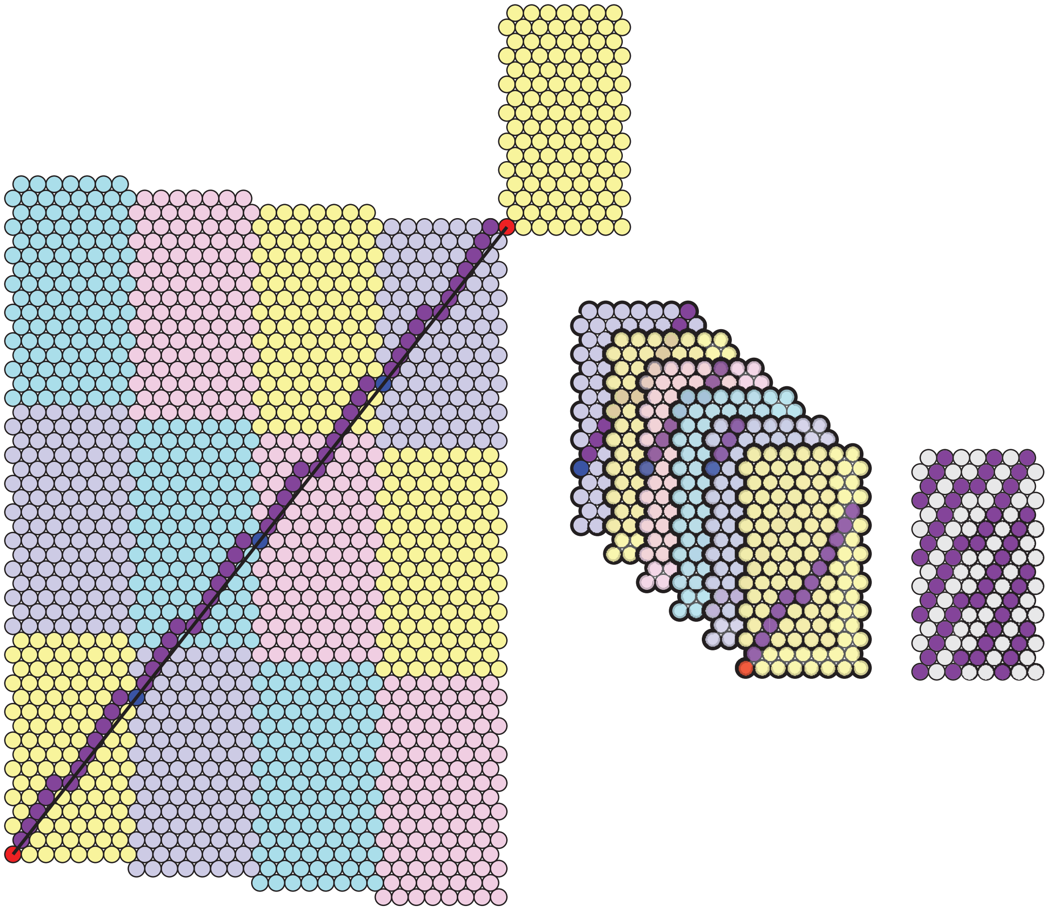


FIGURE 4.14 A (4,3) torus knot construction using a four by three patchwork of "full" bracelets in vertical layout form. The bracelet of 8 double rows here suffices for an example of the technique, but is too short for a wearable bracelet. For the construction, imagine the patchwork material is infinitely thin and transparent. To connect the corresponding interior points, patchwork components are stacked up like a deck of transparent cards to display the complete pattern in a single vertical layout, shown at right.

FIGURE 4.15 A full-size bracelet made using the construction and design shown in Figure 4.14. With 52 double rows and a twist, the design produced a (20,3) torus knot (at the bottom). This pattern, Long Zag (P,3) Torus Knot, appears on p. 158.

Figure 4.16 shows the patchwork method applied to a (3,2) torus knot design on a full bracelet-sized patchwork. This design is also provided in the pattern pages as the Zigzag (3,2) Torus Knot on p. 156. Although our choice of zigzag here resulted in the "line" stepping out of the patchwork bounds at the top end, this is not a problem, because there really are no patchwork bounds. If you visualize an additional patchwork element on the upper right you can see where that snippet of the zigzag belongs on the full single layout. As long as the line begins at the lower left corner of the original patchwork and ends at the upper right corner (which is another way of saying it must have a slope of Q/P on the bead crochet grid), and the zigzagging of the line does not cause it to intersect itself, the design will work. Too narrow a bracelet circumference or too wide a zigzag could cause problems with self-intersection that would not arise with a straight, thin line, but in this case, the design works quite well.

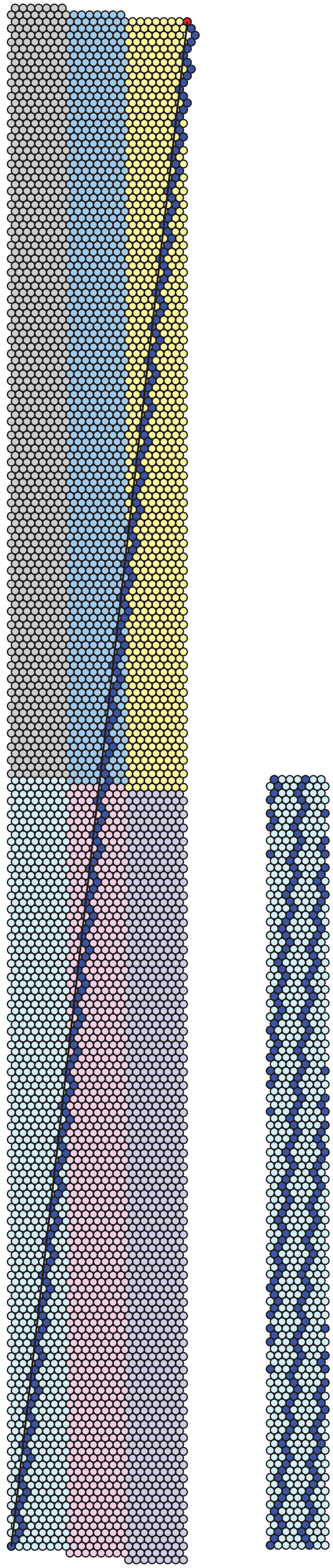


FIGURE 4.16 The patchwork and final bracelet design for the Zigzag (3,2) Torus Knot pattern on p. 156 and shown bottom on the pear of the chapter header photo, p. 50.

We have glossed over several technical difficulties that can arise with this type of construction method. One is that creating a large full-sized patchwork like the one in Figure 4.16 can be unwieldy, possibly requiring poster-sized bead crochet graph paper. A second difficulty is that to achieve a symmetric design we likely need to tinker with the bracelet circumference and length parameters to find a patchwork whose diagonal has room for an integral number of the small repeating zigzag elements. Unfortunately, the unwieldiness of patchwork construction makes this type of trial and error unappealing. While the patchwork idea is useful for conceptual understanding, it is possible to determine the slope and length of a patchwork diagonal line using simple numerical calculations and thereby avoid patchwork construction altogether. Once we have chosen a repeating zigzag design that tracks a particular slope, its exit point on one side of a vertical layout determines (via a hockey-stick translation) where it enters again on the opposite side. So the patchwork is not needed for this task either since we have an alternate method to determine the complete course of a specific zigzag pattern on a full bracelet layout.

On a square flat torus, determining the slope of the diagonal is straightforward because the slope needed for a (P,Q) torus knot is simply Q/P (i.e., a *rise* of Q over a *run* of P). Unfortunately, a bracelet vertical layout is not square and, complicating matters further, the relevant units on the horizontal and vertical axes are different. So our slope calculation is a bit more complex. For readers interested in trying this "slope calculation approach" who don't mind a few gory technical details, the remainder of this section describes a method of calculating both the length and slope of the diagonal based on the patchwork input values of P, Q, the bracelet length L, and the circumference N.

Our first task is to define a coordinate system on a bead plane grid that measures columns of beads on the horizontal axis and rows of beads on the vertical axis. Figure 4.17 shows such a coordinate system. Note that, in bead crochet, the columns are separated by *half* the width of a bead and the rows by the height of a bead, which accounts for the different units on the two axes. The adjacent vertical layout elements in a patchwork are placed a hockey-stick translation away from one another, as marked by the blue and green beads in Figure 4.17. For bracelets of circumference N on this coordinate system, the hockey-stick translation adds 2N + 1 in the horizontal direction and subtracts 1 in the vertical direction.

To construct a (P,Q) knot, consider a P by Q patchwork where

L = bracelet length in double rows, and
N = bracelet circumference.

Using the bead plane coordinate system defined in Figure 4.17, each of the Q layout elements in a patchwork adds 2L in the vertical direction. However, due to the hockey-stick translation, the Y-coordinate of the upper right point on a patchwork diagonal is shortened by one for each of the P elements in the horizontal direction. Thus the patchwork rise is (2LQ) – P. Each of the P elements in the patchwork adds 2N + 1 in the horizontal direction. Thus the patchwork run is P(2N + 1). So the slope of the

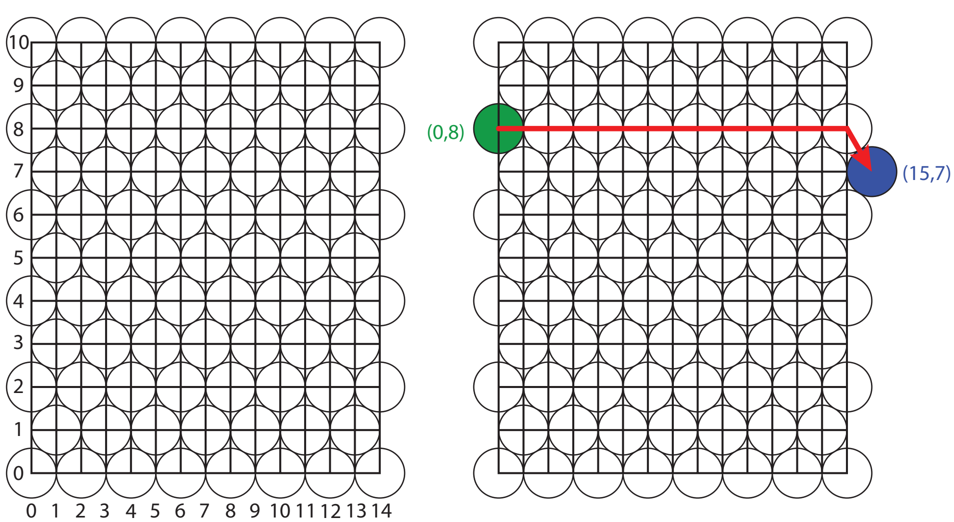


FIGURE 4.17 A bead crochet coordinate system that measures columns (half beads on the X-axis) and rows (whole beads on the Y-axis). On this coordinate system, for bracelets of circumference N, a hockey-stick translation adds 2N + 1 horizontally and subtracts one vertically. The hockey-stick translation in the diagram on the right is for N = 7.

diagonal line running from the lower left to the upper right corners of a patchwork (i.e., rise/run) is equal to

$$\frac{(2LQ) - P}{P(2N+1)}$$

For a symmetric zigzag that closely tracks this slope, we need a rise and run with a common divisor such that rise/run is reducible to a ratio with relatively small whole numbers in the numerator and denominator. We can attempt to find such a ratio by adjusting our choices for the values of N and L. Tinkering with the length of the full bracelet, L, is feasible, since there is always some limited flexibility in this parameter. Tinkering with the circumference N is also an option, and in this regard, it's useful to note that 2N + 1 is prime for 5-, 6-, 8-, 9-, and 11-arounds and factorable for 7-, 10-, and 12-arounds. So these latter circumferences are more likely to produce a reducible slope.

Let's try the method on an example with P = 4 and Q = 3. If we choose L = 49 and N = 10, we get 2LQ – P = (2)(49)(3) – 4 = 290 and P(2N + 1) = 4(21) = 84. So rise/run = 290/84 = 145/42, which doesn't reduce any further. However, if we increase L to 52, we suddenly get a rise/run of 308/84 = 11/3, which has much smaller numbers in the numerator and denominator. This permits zigzag segments separated by 11 beads vertically and 3 half beads horizontally that fit perfectly in the 52 double rows of a full bracelet. Using a single full 52-double-row vertical layout as our canvas, we can place these grid beads (the yellow beads in Figure 4.18) accordingly. Once the grid is laid out, we can experiment with different zigzag options on it, taking care to choose one that does not overlap itself. Four possibilities are shown in gray in the upper right. We can then experiment further with widening or otherwise modifying the design as shown in the progression in Figure 4.18. Figure 4.19 shows two resulting bracelets, the expected (4,3) torus knot done on the planned 52 double rows in size 11 seed beads and a (5,3) torus knot that resulted from using smaller size 11 Delica beads on a longer rope (i.e., with a larger L). These bracelets, which have an identical foreground and background pattern around the accent beads, also offer a taste of the Escher designs coming up in Chapter 6.

Combining Patchworks and Twists

All the patchwork examples above use "untwisted" bracelet components (i.e., using the terminology of Chapter 1, they are structurally aligned). However, as shown in Figure 4.20, this is not necessary. On the left is an example with untwisted components. On the right is an example in which the bracelet components are three beads short of a full double row at the top, forcing a slight twist at the bracelet closure. In this latter case, we can still create a patchwork diagonal, and the small twist at the closure makes up for the lost bit of span in this diagonal. In this particular example, the diagonal happens now to line up perfectly with a natural diagonal of the bead plane, so no zigzagging is even needed (although this won't be true in most cases). One drawback to using twisted patchwork components is that it can further complicate the slope calculations described above.

In general, it's quite useful to combine both physical twists and natural twists to achieve a desired knot. For example, consider the Zigzag (3,2) Torus Knot on p. 61. For sizing purposes, you might want to adjust the length of the bracelet to be a bit shorter by using fewer than 45 repeats. Unfortunately, doing so removes a portion of the final meridian traversal. However, strategically leaving off a few repeats permits a tiny physical twist to replace the missing bit of natural pattern twist. In this case, *as long as the total number of repeats is odd*, you can omit a few repeats and still obtain the desired (3,2) knot with the correct choice of twist. Since physical twists run into problems with torquing the rope and natural twists run into sizing constraints, combining both techniques is often the best approach to get exactly the knot you want.

Sometimes, even armed with the slope calculations from the previous section, it becomes clear that a nicely reducible slope is simply not achievable on a bead crochet grid for a workable rope length and circumference. In this case, the only practical solution is to choose a slope that is as close as possible to the required one, and then rely more heavily on twisting to achieve the desired knot.

A final point worth mentioning is that we can use the natural slant of many existing patterns to produce torus knots designs. However, we might need a reverse engineering* of the slope calculations described above to figure out in advance exactly what torus knot(s) would result. Alternatively, we can avoid the math and try simply making the bracelet to find out! Consider, for example, the hexagonal pattern shown in Figure 4.21, which you may recognize from its frequent appearances in Chapter 1. It turns out that it generates the Zigzag (3,2) Torus Knot trefoil design on p. 61, as shown in the bottom example in Figure 4.21.

* Reverse engineering would start with the known slope, L, and N and calculate P and Q from them.

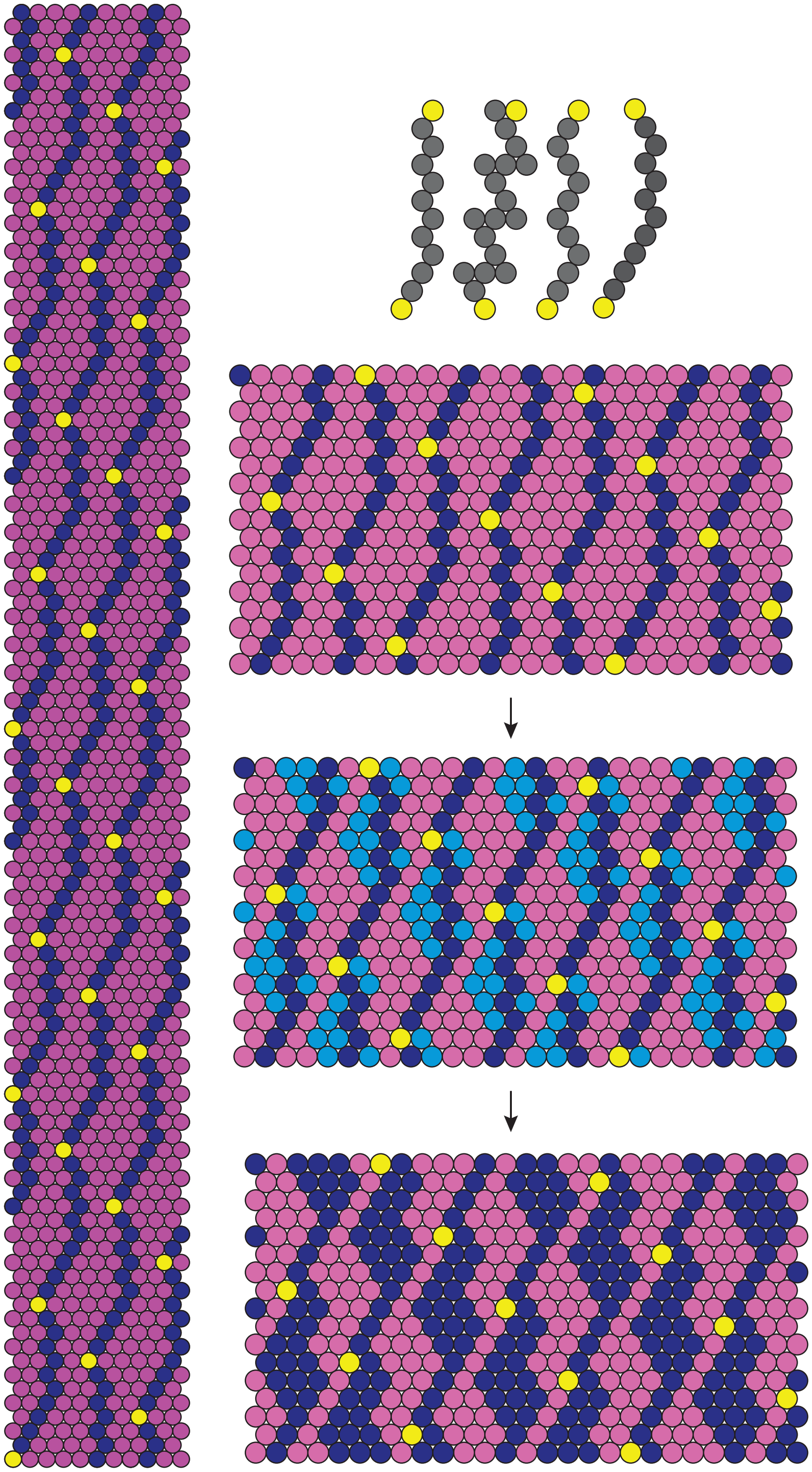


FIGURE 4.18 (4,3) Torus Knot design and transformation. On the left is one possible complete bracelet design for P = 4, Q = 3, L = 52, N = 10 with a calculated slope of 11/3 on the bead crochet coordinate system. Note that this slope is easily seen in the yellow accent beads, which are separated from one another by 11 beads vertically and 3 half beads horizontally. The basic design can then be transformed as desired while still maintaining this same slope, as shown on the right. In this case, it has been transformed into two identical interlocking (4,3) torus knots in each color with the yellow accent beads in between.

FIGURE 4.19 (5,3) (left) and (4,3) (right) torus knot bracelets created from the design developed in Figure 4.18. The pattern, Knotted Snakes, appears on p. 159.

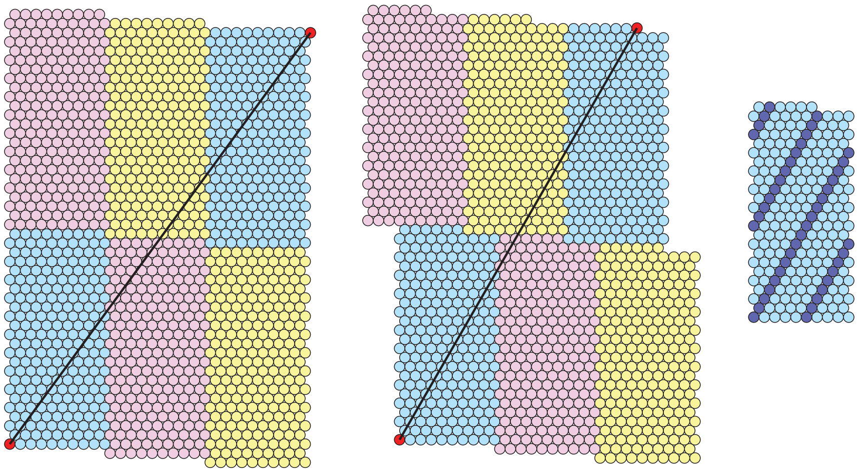


FIGURE 4.20 It is possible to create a patchwork diagonal even when the patchwork elements are not composed of an integral number of double rows. In this case, a small amount of physical twist makes up for the smaller run of the diagonal line.

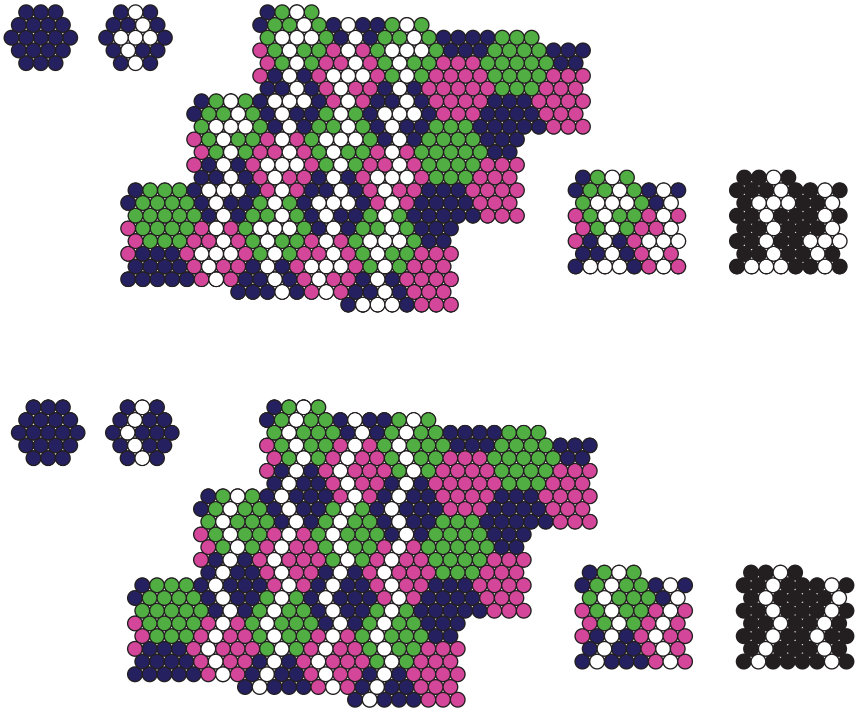


FIGURE 4.21 Generating torus knot patterns from the natural slant of existing tilings/patterns.

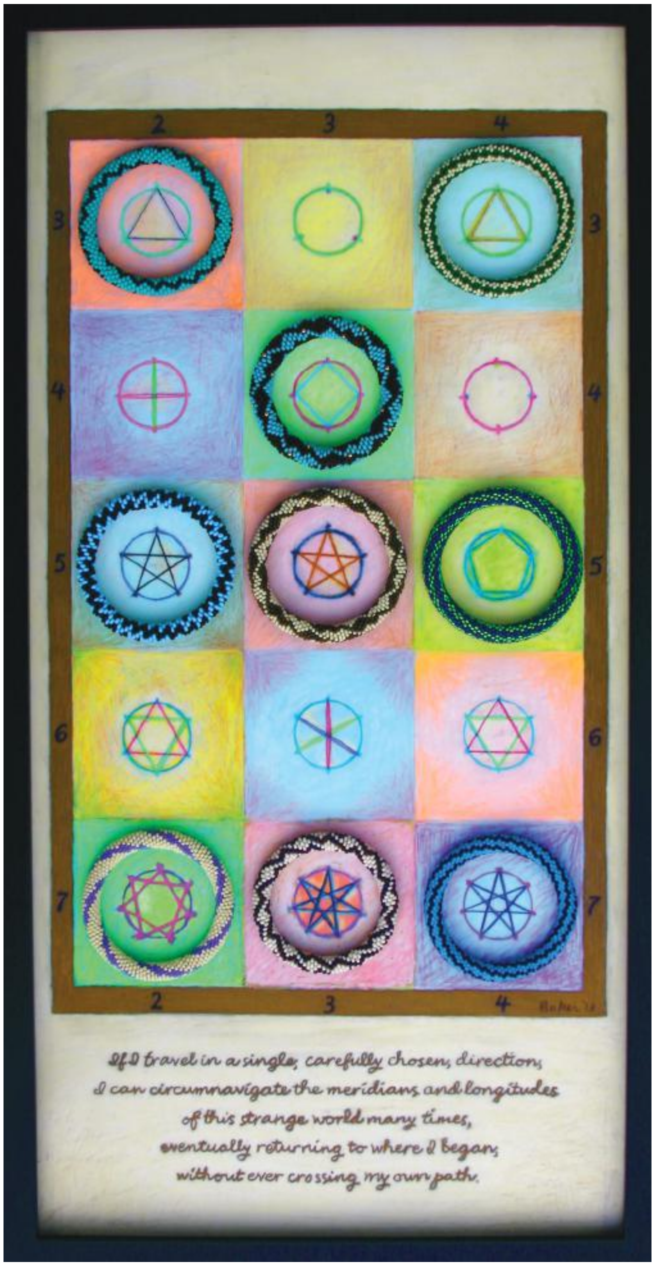


FIGURE 4.22 *The Torus Traveler's Journey*, an art piece included in the Joint Mathematics Meetings Exhibition of Mathematical Art in January 2014. All but two of the patterns used are provided in our pattern pages: the (3,2) on p. 156, the (5,2) on p. 161, the (4,3) and (5,3) on p. 159, and the (P,4) knots on p. 162.

However, we can use the same pattern to generate other (3,2) torus knot designs with a different zigzag, such as the one in the top example.

You should now have plenty of tricks up your sleeve for producing almost any sort of bead crochet torus knot you desire! Using the techniques we've described, Figure 4.22 circles back to where we began. Recreating in bead crochet a variation of Figure 4.3 from the start of the chapter, it displays a window into the same conceptual infinite graph, although this time the axes are swapped, with P on the vertical axis and Q on the horizontal axis.* For P between 3 and 7 and Q between 2 and 4, wherever the (P,Q) pair is relatively prime, it shows a bead crochet (P,Q) torus knot bracelet and also a hand-drawn, "reversed-meaning," (P/Q) star polygon. In the remaining spots, where the (P,Q) pair is not relatively prime and thus neither a (P,Q) torus knot nor (P/Q) star polygon is possible, it shows a compound star figure with component star polygons in separate colors.

Titled *The Torus Traveler's Journey,* this wall art piece is intended to invite the viewer to ponder connections between torus knots and star polygons.

* In addition to swapping axes, it displays a smaller window than the one in Figure 4.3. Figure 4.22 can be mapped onto Figure 4.3 by rotating it 90° counterclockwise and then flipping the Q axis so the smaller numbers are at the top.

CHAPTER 5

Knotted and Linked Bracelets

In the last chapter, we began by considering what would happen if we wrapped a string around a torus and fastened the ends together to obtain a torus knot. If we dispense with the torus and explore what happens when we take a length of string, rearrange it any way we like, and fasten the ends together, then we enter the more general realm of *knot theory*, a fascinating branch of topology. The field originated in the 1800s, inspired by a beautifully imaginative but utterly incorrect theory about the nature of the chemical elements.* The study of knots continued long after scientists came to a proper understanding of the elements, and it blossomed into its own field of inquiry. After all, many mathematicians don't care whether a subject has obvious practical applications. They are spurred on by the challenge and intrigue of discovery for its own sake, and they understand that a field of mathematics that seems completely useless when it is developed is often unexpectedly useful later on. In recent decades, the story has come full circle with the discovery of exciting new connections between knot theory, genetics, and molecular chemistry.

For a student of knot theory, bead crochet bracelets offer a new avenue of fun exploration because they are, in many ways, the perfect medium for modeling knots. And, of course, they also offer yet another way to adorn yourself with a mathematical model! The primary focus of knot theory is to understand, categorize, and identify different knots. A knot is a strand that is (perhaps) tangled in some way and then connected at the ends. A bead crochet invisible join provides the perfect way of creating such a connection with a smooth seamlessness that helps capture the essence of a mathematical knot. Unlike a piece of tied string, bead crochet bracelets have no obstacle at the join to interfere with manipulating and exploring the knot in its varied forms. Additionally, the slipperiness of glass or metal beads facilitates easy manipulation between the different forms of a knot. A mathematical knot can be rearranged in any way possible and still be considered the same knot, as long as it is never cut. For example, you may find it surprising that the purple knots and the blue knots in Figure 5.1 are all different incarnations of the same knot, the *trefoil* (which you may recall was introduced in Chapter 4 as a type of torus knot). Careful inspection also reveals that the purple and blue knots are subtly different versions of the trefoil. In each configuration, whenever the strand of each knot crosses itself, the part of the strand that is on the top in the purple knot is on the bottom in the blue knot, creating a kind of left- and right-handed trait. Color aside, no matter how you manipulate them, you will never be able to make the purple knot look exactly like the blue one! They are fundamentally different creatures. These kinds of fundamental traits are what knot theorists seek to identify.

One useful way of identifying a knot is to manipulate it into a form with the least possible number of *crossings*, or places where the strand crosses over or under itself. This number is the *minimum crossing number* of the knot, and it is the sort of fundamental property that helps topologists distinguish one knot from another. The simplest knot is a plain ring (such as a normal bead crochet bracelet), which has zero crossings and is called the *unknot*. There is no knot with a minimum crossing number of one or two; you might attempt drawing one to convince yourself of this. The trefoil, with three crossings, turns out to be the simplest after the unknot, and it is also the only three-crossing knot, although as discussed before, it has left- and right-handed forms. Figure 5.2 shows all the distinct knots (disregarding handedness) up to a minimum crossing number of seven.

Bead crochet knots, depending on length, can be worn as either necklaces or bracelets and, since there are multiple choices for which hole the head or hand might go through, they provide additional fun design choices. Although this does not involve the pattern design techniques we discuss in the other chapters, we couldn't resist including this additional lovely connection between mathematics and bead crochet, a connection that inspired this set of challenges.

Challenge *Can you make a knotted trefoil bracelet and use it to prove that all the bracelets depicted in Figure 5.1 are also trefoils? Can you create a set of bead crochet bracelets or necklaces representing each of the knots up to seven crossings, as shown in Figure 5.2?*

You may notice that some of the knots in Figure 5.2 are tied (no pun intended) to the designs from Chapter 4. In fact, three of the knots with seven or fewer crossings (not counting the unknot) are torus knots. Of course, one of them is the trefoil, which we have already identified as a (3,2) torus knot. Can you spot the other two torus knots in Figure 5.2?

Another interesting characteristic of every knot is its *braid representation*, which can be useful to consider when designing knotted bracelets. In mathematics, a braid is a set of N strands connected to two horizontal bars, one at the top and one at the bottom; the strands may cross over or under one another while moving from the top bar to the bottom one. As illustrated in the braid on the left in Figure 5.3, each strand must always travel downward from

* Colin Adams, *The Knot Book*, American Mathematical Society, 2004.

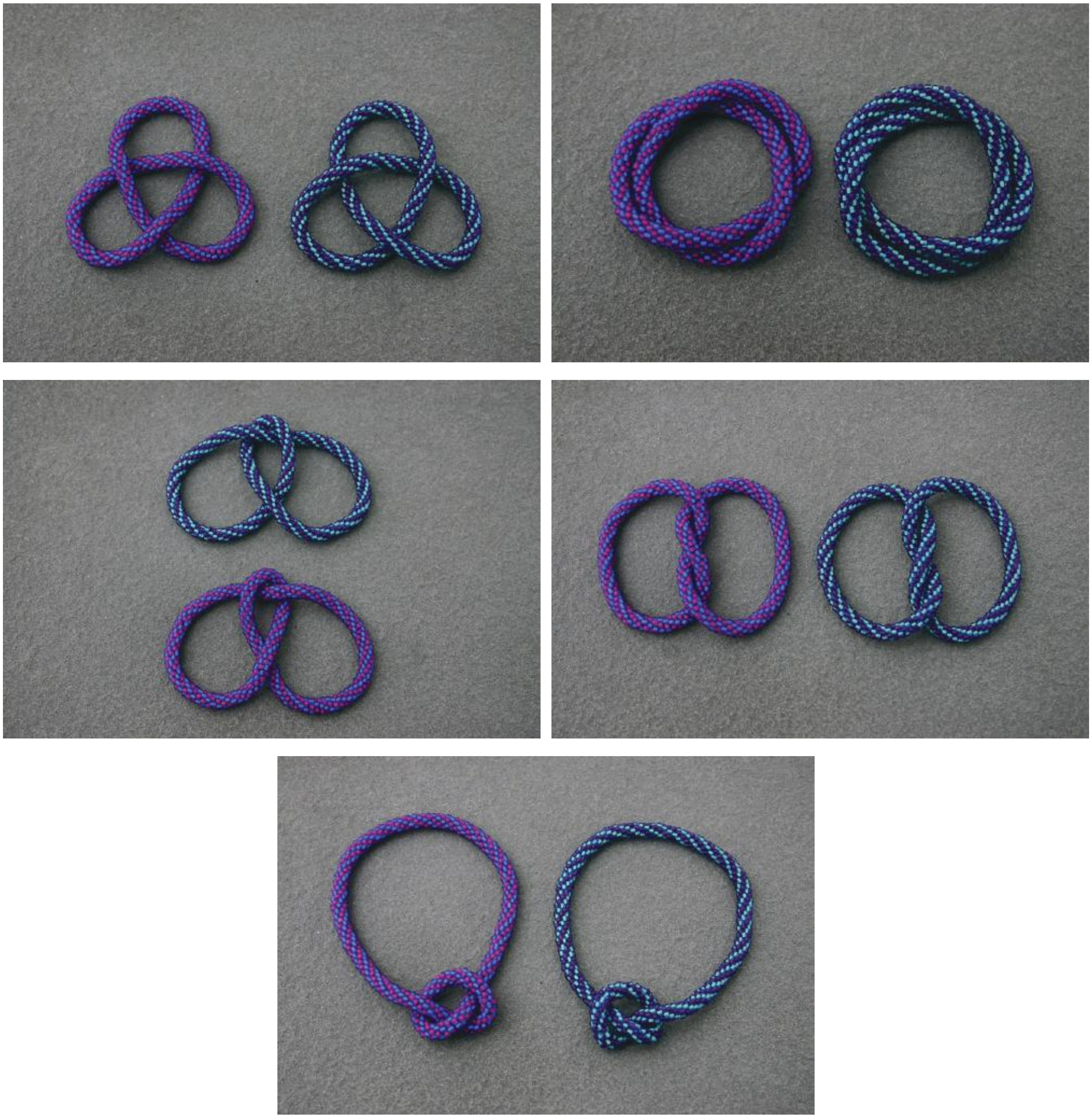


FIGURE 5.1 Left- and right-handed trefoil knots and some of their many forms. These bracelets are made with size 8 seed beads in circumference 5-around (4-around works well too!) with a simple alternating bead color pattern.

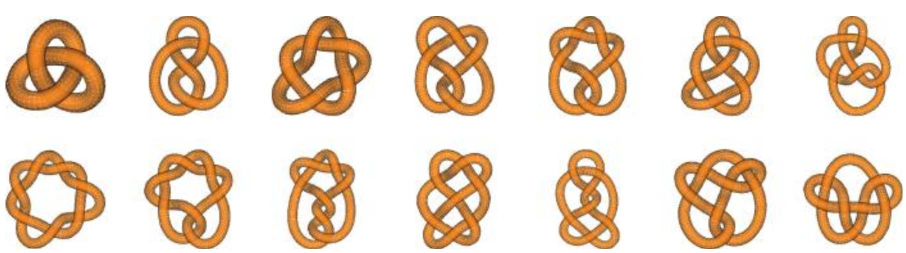


FIGURE 5.2 The knots up to seven crossings. These images were produced using the KnotPlot software written by Robert Sharein.

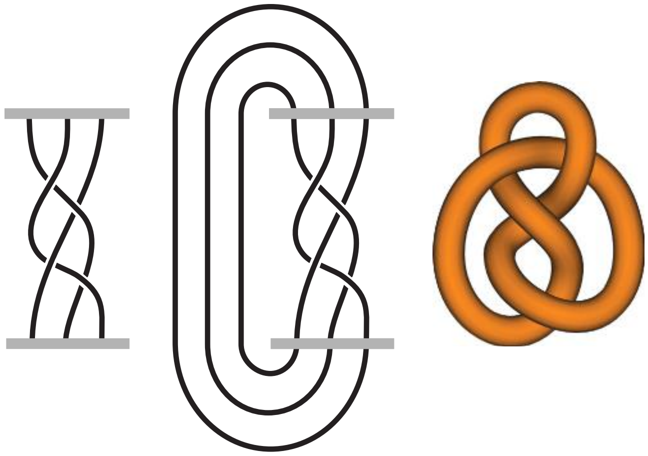


FIGURE 5.3 The figure-eight knot (right), in a braid representation (left), and in a closed braid representation (middle).

FIGURE 5.4 Two ways of wearing the figure-eight knot from bottom left of the chapter header page. On the top is the three-strand closed braid representation shown in Figure 5.3 (middle), which works well for a bracelet. On the bottom, Teddy is modeling a nice form for a necklace.

the top (so no doubling back upward). A strand may land in a new position on the bottom bar by the time it reaches the bottom. Every knot has at least one braid representation. For example, Figure 5.3 shows a knot called the *fig re-eight knot* (on the right, and also shown bottom left in the chapter header photograph) and a braid representation of it (on the left) in a three-strand braid. If we extend the strands beyond the top bar and connect them to the strands at the bottom bar as in the middle illustration of Figure 5.3, we find a *closed braid representation* of the figure-eight knot. A closed braid representation is simply another configuration of the original knot.

A closed braid representation for a knot presents a nice way to wear it, allowing the knotted bracelet to wrap around your wrist in a single direction (always clockwise or always counterclockwise). The number of strands in the braid representation is the number of times the bracelet winds around in that configuration, which helps determine how long to make a bracelet if you wish to wear it this way. For instance, to wear a figure-eight knotted bracelet in the closed braid representation depicted in Figure 5.3, you would make a bracelet a little more than three times the length of a regular bead crochet bracelet; you need a bit of extra length to accommodate the over and under crossings in the braid. Figure 5.4 shows two arrangements of a figure-eight knotted bracelet: the three-strand closed braid representation on the top, and a nonbraid necklace form on the bottom.

We can further generalize the ideas of knot theory by considering what happens when we use more than one strand to make a *link*. The name is logical, since links are simply knots that are linked together in some way. Knot theorists also seek ways of understanding, identifying, and categorizing links. One obvious feature of a link is the number of component knots it contains, and another is minimum crossing number—just as for individual knots. Figure 5.5 shows links with up to six crossings. The link on the bottom left is a famous three-component link called the Borromean Rings that has the amusing property that if any one of the three components is removed, the other two components are unlinked. Links offer another fascinating approach to jewelry design with bead crochet. The bracelet shown in Figure 5.6 is composed of a trefoil linked with the unknot, forming a blue (3,2) torus knot wrapped around an actual torus, with the brown and green bracelet playing the role of the torus. Depending on the lengths of the different components, linked bracelets can be worn in an endless variety of ways. Conveniently, every link also has a closed braid representation.

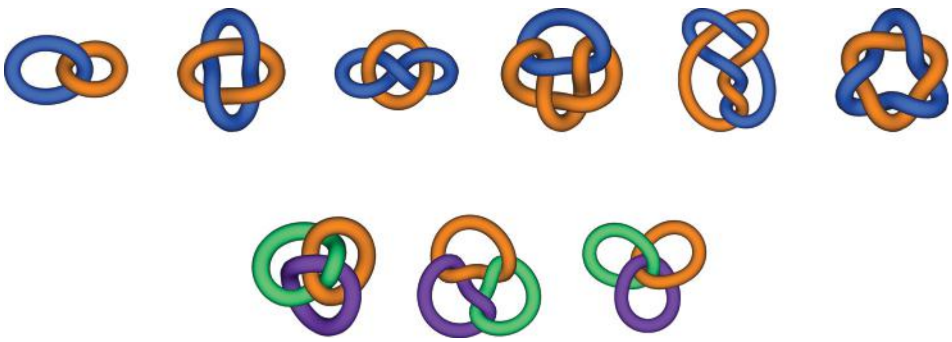


FIGURE 5.5 The links up to six crossings. These images were produced using the KnotPlot software written by Robert Sharein.

FIGURE 5.6 A linked bracelet.

As another challenge, we suggest investigating the possibilities for creating linked bead crochet bracelets to form interesting jewelry. Readers interested in seeking inspiration for other possible knotted and linked bracelet ideas, or who simply wish to learn more about knot theory, are enthusiastically urged to consult one of our favorite resources: *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots* by Colin C. Adams. Knotted and linked bracelets make a wonderful teaching aid for a class in knot theory. If you happen to be a knot theory student, we can't imagine a more appropriate class contribution than a set of bead crochet knots and links, which can be used to adorn a wall as easily as a wrist, and which will likely be employed for years to come as treasured classroom manipulatives.

FIGURE 5.7 Celtic knot designs. The patterns for these bracelets are provided on pp. 163–164.

Celtic Knots: Another Design Springboard

One more interesting knot design motif is the Celtic knot, as explored in the two bracelet designs shown in Figure 5.7. These designs simply paint a picture of a knot or link using color or blank space to indicate the "string" passing over or under itself. Celtic knots have a rich history in art throughout the ages, and a little research will reveal many beautiful knot designs that can be used as the basis for bead crochet patterns. We hope our readers will be inspired to explore the use of these motifs further.

CHAPTER 6

Escher Designs

When the subject of mathematics and art arises, one of the first names to enter the conversation is M.C. Escher. A 20th-century Dutch artist, Escher is famous for his incorporation of mathematical themes in his artwork, which includes woodcut prints, lithographs, wood carvings, and many other forms of visual art.

One of Escher's favorite artistic themes involves what he termed "regular divisions of the plane." In mathematical language, these are known as *tessellations* or *tilings* of the plane, and they are patterns that divide the plane into copies of one or several shapes with no gaps between them. Tessellations are a common feature of the traditional decorative arts and are certainly familiar to many fiber artists, especially quilters. What fascinated Escher and filled pages and pages of his sketchbooks were tessellations in which each shape was a recognizable form from nature. Figure 6.1 is an example of a tessellation of the plane by identical fish in the style of Escher (though less sophisticated than his sketches and prints).

The marvelous volume *M.C. Escher: Visions of Symmetry* by Doris Schattschneider contains a complete collection of Escher's tessellation sketches as well as the various artworks they inspired. Escher often used his regular divisions of the plane as launching points for more complex works in which the distinction between foreground and background is blurred in provocative ways. *Visions of Symmetry* also shows the earlier decorative arts traditions from around the world that inspired Escher's experiments.

Always keen to explore new geometric challenges, Escher modified some of his regular divisions of the plane to other surfaces, creating carvings of tessellations of a sphere and, most interestingly for our purposes, a set of tiled columns in The Hague displaying tessellations of a cylinder. This offers an intriguing invitation to the bead crochet artist. After all, what is a cylinder but a bead crochet bracelet that hasn't been sewn closed yet?

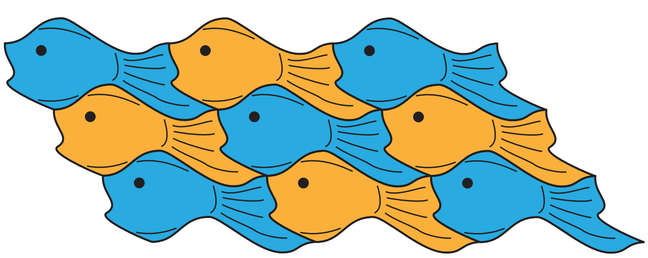


FIGURE 6.1 A tessellation of the plane by congruent fish. In this tessellation, we can use two colors to ensure that neighboring fish are different colors, but other tessellations may require more colors.

With this motivation, we call a bead crochet design an *Escher design* if it consists of copies of a single shape in multiple colors that cover the bracelet with no gaps, making a regular division of the beaded torus. The photograph at the head of this chapter shows two examples of such bracelets, one tiled with fish and one with lizards. Our terminology is admittedly somewhat arbitrary, as many of Escher's regular divisions of the plane use two or three interlocking shapes instead of just one. Moreover, since Escher incorporated Möbius bands in his artwork as noted in Chapter 3, you could make a convincing case for naming the Möbius bracelet after Escher. However, since the challenge that led us to the patterns in this chapter is the one most directly inspired by Escher's artwork, these are the designs we choose to honor with his name.

Challenge *Can you design an Escher bracelet—a bracelet tiled by a single shape in multiple colors? Can you design one with two colors of beads? Three colors? As many colors as you want? Can you design an Escher bracelet with aesthetically appealing tiles? Symmetric tiles? Tiles in recognizable shapes?*

Deforming Polygonal Tilings

Designing Escher bracelets is an undertaking in which the bead plane approach to pattern creation really shines. After all, as we saw in Chapter 1, a bracelet pattern in the bead plane is a tiling of the plane by pattern repeats, and we can group the beads in a single repeat into different shapes. In the bead plane, an Escher bracelet pattern is no more nor less than a tessellation of the bead plane by a single shape that incorporates a hockey-stick translation. For instance, Figure 6.2 shows the bead plane pattern for the bracelet of fish in the chapter header photograph. The plane is tiled by black and blue fish, and a single repeat of the bracelet consists of one fish of each color. Our challenge for this chapter boils down to this: can you produce an aesthetically pleasing arrangement of interlocking, congruent shapes in the bead plane that is preserved by a hockey-stick translation?

A good place to start is to follow Escher's lead. From his sketches and his descriptions of his artistic process, we know that he began his regular divisions of the plane into animal forms with simple tilings by polygons. The sketches collected in Schattschneider's *Visions of Symmetry* often reveal Escher's penciled scaffoldings of triangles, squares, parallelograms, pentagons, or hexagons.

The general process is shown in Figure 6.3, in which we start with a tiling by squares. To make more interesting tiles,

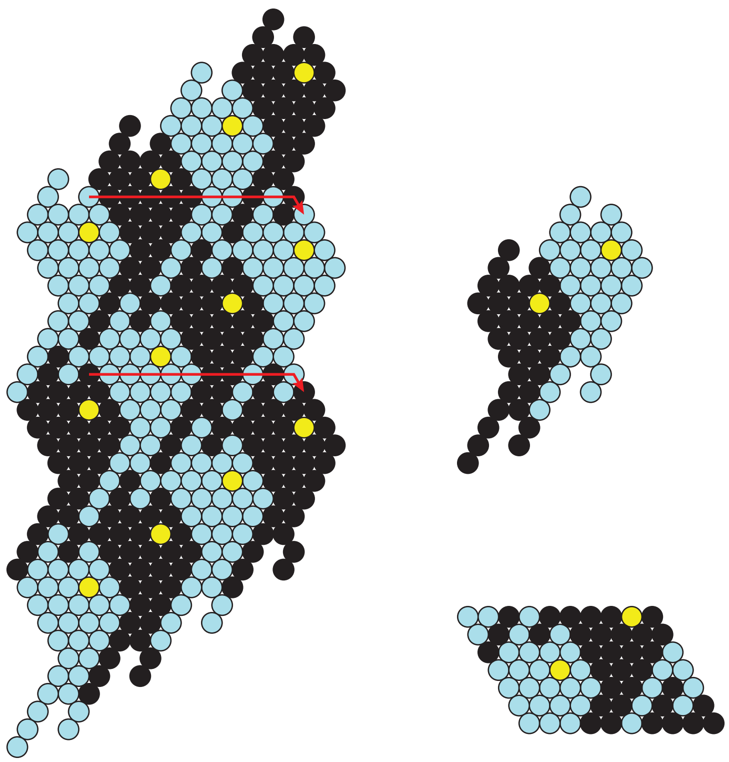


FIGURE 6.2 The pattern for the fish bracelet in the chapter header. In the bead plane, the pattern is a regular division of the plane into fish. The red arrows in the bead plane diagram on the left show that the tiles are related by the hockey-stick translation for a 10-around bracelet. On the right are two representations of a single bracelet repeat, one preserving the fish shapes and the other preserving the order of bead stringing. The pattern for this bracelet, Escher Fish, appears on p. 204.

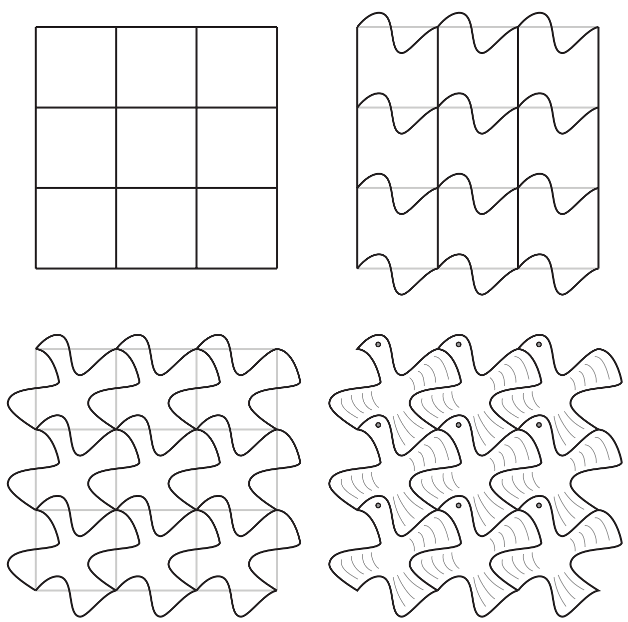


FIGURE 6.3 A common method for producing tessellations of the plane. Starting with the square tiling in the upper left, we deform the horizontal edges into a curve, then deform the vertical edges into a different curve. Decorating the resulting curved tiles gives a pattern of interlocking birds.

we replace the sides of the squares with curves that strike our fancy. If you try this, you will probably find yourself making numerous adjustments to the curves to get exactly the shape that you want, and this sort of tinkering with shapes is a large component of designing Escher bracelets in the bead plane. In Figures 6.1 and 6.3, the curved tiles are transformed into fish and birds by drawing in fins and feathers. It is much harder to render fine details in beads, and indeed most of our tessellated bracelet patterns are coarser and more abstract than Escher's regular divisions of the plane. The larger the bracelet circumference, the greater the opportunity for detail.

To transfer this technique of modifying polygonal tilings to bracelet designs, we first need some polygonal bead tilings. Unlike Escher, we cannot start with just any polygonal tiling of the plane. We have an extra constraint: we must also incorporate a hockey-stick translation. One simple starting point arises from considering the two diagonal layouts of a bracelet chart. As we saw in Chapter 1, the conventional left-leaning diagonal chart for an N-around bracelet has a right-leaning counterpart, and while the left-leaning chart has N beads per row, the right-leaning chart has N + 1 beads per row. Overlaying these two charts gives a parallelogram of N by N + 1 beads that tiles the plane

with a checkerboard of diamonds, as shown in Figure 6.4 for N = 6. Because these diamonds fit into both the left- and right-leaning diagonal charts for an N-around bracelet, their edges line up in both diagonal directions to create this simple two-color tiling.

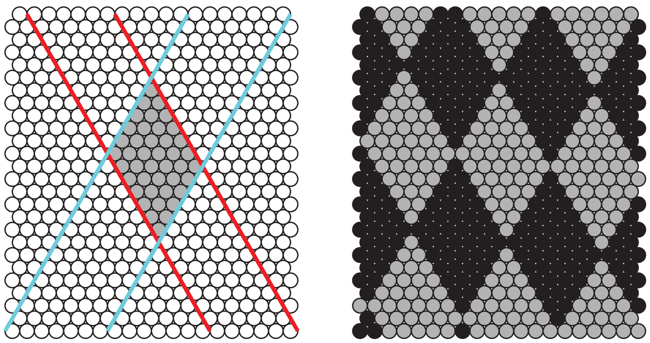


FIGURE 6.4 A simple polygonal tiling of a 6-around bracelet. In the diagram on the left, the red lines mark the traditional left-leaning diagonal bracelet chart with 6-bead rows, while the blue lines mark the corresponding right-leaning chart with 7-bead rows. The six-by-seven bead parallelogram in their intersection yields the diamond tiling on the right. This pattern appears in the pattern pages as Harlequin (p. 165).

With a little experimentation, we can create a variety of more complicated parallelogram tilings starting from one of the standard bracelet layouts. For instance, Figure 6.5 shows the formation of two different tilings by parallelograms in circumference N = 7. Each begins with the vertical layout for a 7-around bracelet. Since this layout contains 2N + 1 = 15 vertical columns of beads, we can divide it into three strips of five columns each. These strips, marked in shades of gray, alternate between 2 and 3 beads per row, with 5 beads in each double row. We then cut the strips into parallelograms with right-leaning top and bottom edges. Depending on the height that we choose for the parallelograms, we might need to make adjustments to account for the effect of the hockey-stick translation. In Figures 6.5(a)–6.5(c), the parallelograms are composed of 5 columns with 4 beads in each column. As luck would have it, with tiles of this height the hockey-stick translation between vertical layouts lines up the top and bottom parallelogram edges uniformly, as marked by the black diagonal in Figure 6.5(b). Figure 6.5(c) shows this tiling rendered with four colors for a simpler design.

If we instead take parallelograms with 5 beads per column, as in Figures 6.5(d)–6.5(g), we must take a little more care in how we position the tiles. If we simply slice up each vertical layout with right-leaning diagonals as

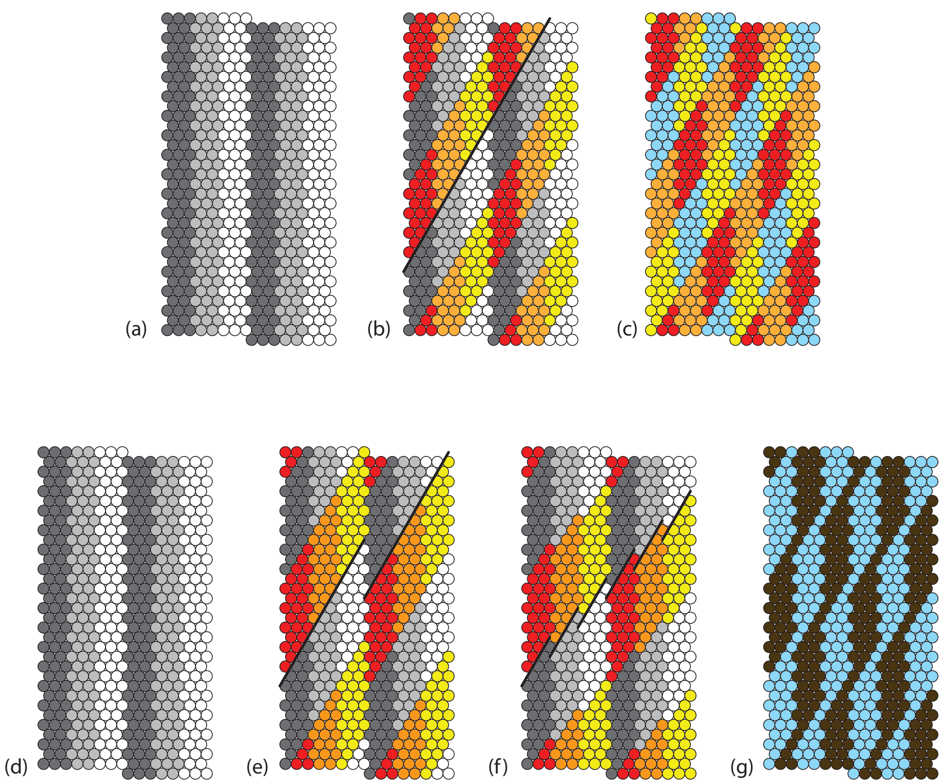


FIGURE 6.5 Parallelogram tilings of a 7-around bracelet. The starting point for each tiling is a division of the vertical layout for a 7-around bracelet into three equal-sized vertical strips.

in Figure 6.5(e), the bottom edges and top edges of the parallelograms are aligned differently between vertical layouts than within vertical layouts, as marked by the broken black diagonal. This poses a problem for adapting the geometric deformation technique of Figure 6.3 to this tiling in the bead plane. For instance, if we adjust the beads on the right edges of the yellow parallelograms, the effect on the neighboring tiles will be different than if we adjust the corresponding beads on the right edges of the orange parallelograms. On the other hand, if we realign the three vertical strips in each vertical layout by sliding the middle strip down 2 rows and the right strip down 4 rows, we find the modified tiling in Figure 6.5(f), in which the relative spacing of the parallelograms is uniform throughout the tiling. Coloring this tiling in two colors yields the simplified pattern in Figure 6.5(g). As a bonus exercise, see if you can figure out which of the Escher designs at the end of the book is based on this tiling!

We can also begin this process with diagonal strips instead of vertical strips to get many more parallelogram tilings. Figure 6.6 shows a few of the possibilities. The top two tessellations are cut from left-leaning strips, and the bottom two tessellations are cut from right-leaning strips. In general, figuring out how to arrange the parallelograms within each strip to get a uniform arrangement of tiles involves some tinkering to find an alignment that works.

Since the bead plane forms a hexagonal grid, with each bead surrounded by six touching beads, it should not come as a surprise that there are also many simple hexagonal tilings that make nice starting points for tessellations. A little experimentation with hexagonal clusters of beads produces tessellations like those in Figure 6.7, both of which are excellent springboards for more complex tessellations. You may recognize the upper pattern from Chapter 1, in which we used it to illustrate different bracelet charts in the bead plane. Notice that while the top arrangement of regular hexagons in a 7-around bracelet can be colored with three colors, the bottom arrangement of irregular hexagons in an 8-around bracelet needs four colors.*

So, how do you find this sort of polygonal pattern? The simplest way is to take a promising shape (in general, hexagons and parallelograms work well), tile the bead plane with it, and then search for hockey-stick translations for reasonable circumferences. In the case of the irregular hexagon

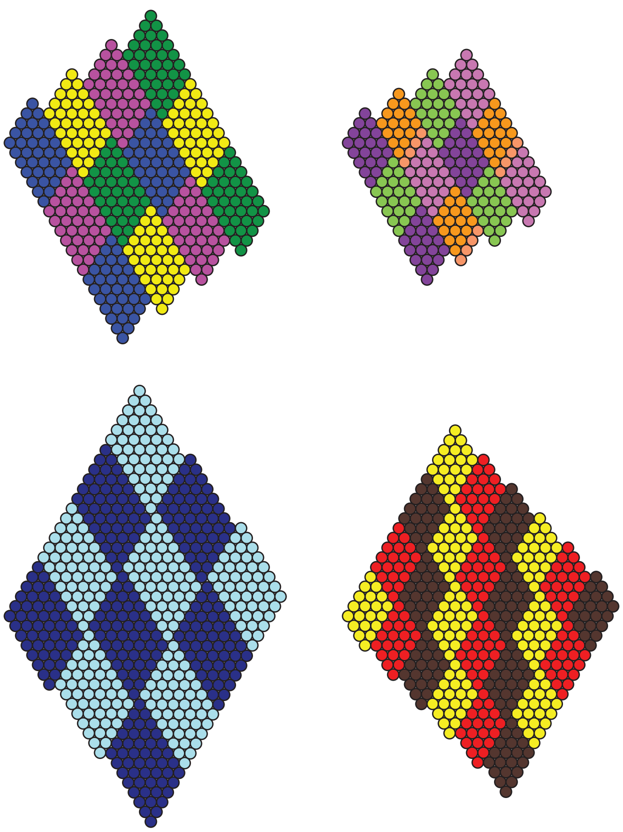


FIGURE 6.6 Further examples of parallelogram tilings. From left to right and top to bottom, these tilings are in circumference 10, 8, 8, and 7.

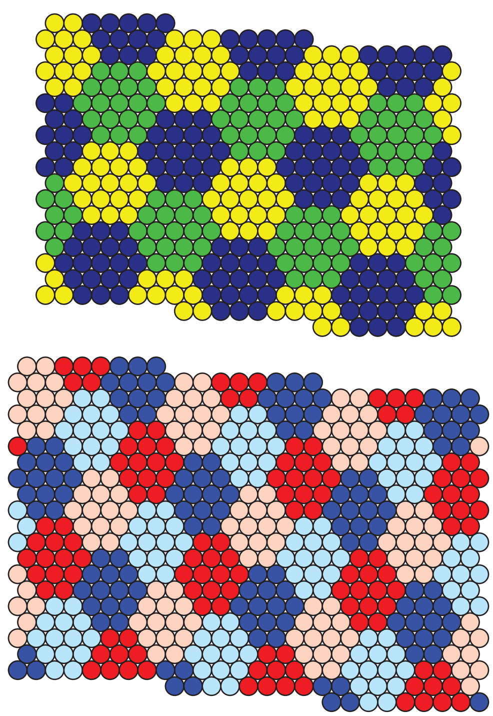


FIGURE 6.7 Tessellations with hexagons. The top is a 7-around pattern of regular hexagons, and the bottom is an 8-around pattern of irregular hexagons.

* Recall from Chapter 3 that a polygon is *regular* if all of its side lengths are the same and all of its angles are the same, and it is *irregular* otherwise.

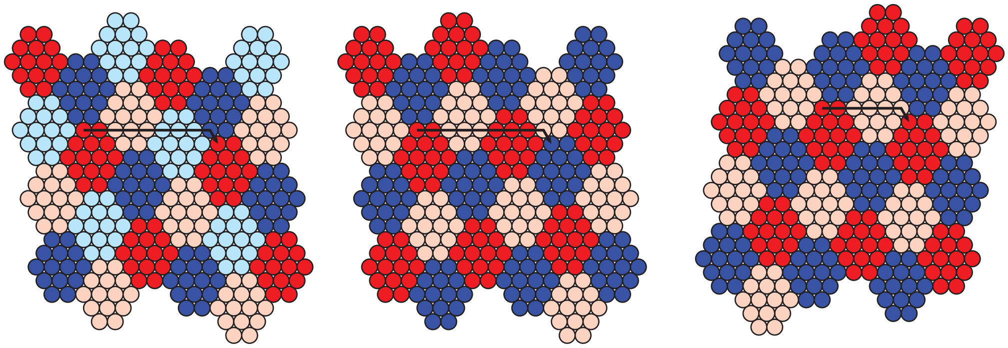


FIGURE 6.8 Colorings of hexagonal tessellations in the bead plane. On the left is the 8-around hexagonal pattern from Figure 6.7 with the hockey-stick translation marked. In the center, we see that the same tiling colored with three colors is not a valid bracelet pattern. However, if we mirror reverse the three-colored tiling, we find a shorter hockey-stick translation that gives a valid 5-around bracelet pattern.

FIGURE 6.9 Circumference 8 and 5 bracelets tiled with hexagons. For a more interesting aesthetic effect, the four center beads in each hexagon are black. The 8-around pattern on the left appears in the pattern pages as Diamond Glow (p. 168).

from the second tiling in Figure 6.7, there is a hockey-stick translation of length 8 between certain pairs of hexagons.

As shown in Figure 6.8, it is the relationship between this hockey-stick translation and the arrangement of hexagons that forces us to use four colors. If we color the hexagons with only three colors, hexagons at either end of the hockey stick are different colors, and we no longer have a valid bracelet pattern. However, if we reflect this pattern through a vertical axis (in other words, take its mirror image), we stumble upon a new hockey-stick translation of length 5. The reversed pattern, shown on the right of Figure 6.8, actually *can* be colored with only three colors. As we will see shortly, reflecting a tessellation through a vertical axis can sometimes take a bracelet pattern where the circumference is too small or too large and turn it into a viable bracelet design. Figure 6.9 shows two bracelets based on the two valid patterns in Figure 6.8.

With a collection of polygonal tilings at our disposal, we are ready to experiment with deforming polygons. While the process is analogous to the deformation of squares in Figure 6.3, the pixilated quality of the bead plane gives the technique a different feel in practice. Instead of bending straight edges into curves, the bead designer is adjusting the colors of beads in each tile to gradually change the tile shapes a few beads at a time. Figure 6.10 shows the process in stages applied to the second hexagon tiling from Figure 6.7. At each step, one or two beads from each tile are colored to put them in an adjacent tile. These changes are applied uniformly so that within each pattern in the progression, all the tiles have the same shape.

In most of the transitions in Figure 6.10, the color changes occur in the same position in each tile. However, if you look closely at the center design in the bottom row, you will see that the blue tiles are altered at the top, while the red and cream tiles are altered at the bottom, resulting in two different orientations of the tile shape: the blue tiles are oriented as 2's, and the red and cream tiles as upside down 2's. In the final pattern, a similar adjustment gives Z-shaped tiles that are once again all oriented the same way. Figure 6.11 shows bracelets made with these final two patterns.

Assembling Shaped Tiles

There is nothing to stop us from applying the same trial-and-error process we used to form hexagonal tilings directly to nonpolygonal shapes. For instance, consider the 7-bead snowflake tile in Figure 6.12. The shape is simple enough that with a little experimenting, you can fill the bead plane with snowflakes. In fact, since the arrangement is not mirror symmetric, you can do this in two ways. The first way, with each snowflake slightly to the left of the snowflake below it, only has a hockey-stick translation of length 3. Even if this weren't too short to be practical, it would force neighboring snowflakes to be the same color. (Technically, this tiling also has a hockey-stick translation of length 16, but that circumference is too large for any but the most intrepid bead crocheter.)

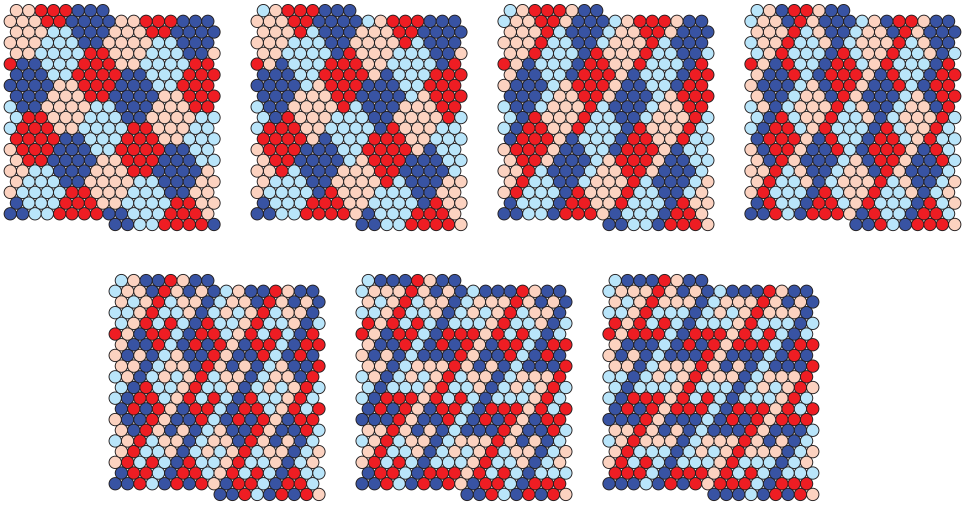


FIGURE 6.10 Deforming a polygonal tiling into a more elaborate Escher bracelet design. In each successive chart, at most two beads from each tile change color.

FIGURE 6.11 Bracelets based on the last two patterns from Figure 6.10. The patterns are sections U and W of Tessellation Evolution Necklace, which is on p. 214 in the pattern pages.

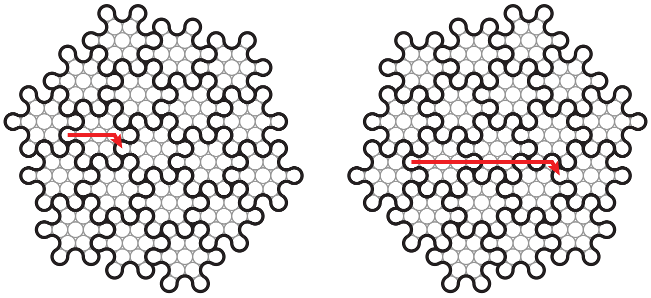


FIGURE 6.12 Creating a tessellation directly from a shaped tile. The snowflake tile tessellates the plane in two different ways that are mirror reflections of each other. The tiling on the left has a 3-around hockey-stick translation, making it unsuitable for a bracelet. Fortunately, the tiling on the right has a 9-around hockey-stick translation.

However, if we reflect the tiling so that each snowflake is slightly to the *right* of the one below it, we find a hockey-stick translation for a 9-around bracelet, and we're in business!

Now, we need to decide how to color the snowflakes. One way is to use the smallest number of colors possible, and we can determine this number by following the process in Figure 6.13. Each time we color in a snowflake, we need to apply the same color to any snowflake that is a hockey-stick translation away. First, we color a snowflake and its translations light blue, then we color one of its neighbors and its translations dark blue. Next, we observe that the purple snowflakes touch both light blue and dark blue snowflakes, so we are forced to use a third color. Similarly, the green snowflakes touch snowflakes of all three colors so far, so we add a fourth color. From this point on, we can fill the remaining snowflakes with the four colors we have already used. It is not difficult to fill in the rest of the color pattern, since we can always find an uncolored snowflake that touches three different colors, which tells us that we must apply the fourth. The collection of bracelets in Figure 6.25 contains a four-color snowflake bracelet.

Just because we only need four colors doesn't mean we have to use four colors. For aesthetic reasons, we might want to incorporate more, and once we know the minimum number of colors, it is easy to add more in a systematic way. In Figure 6.13, we can see that if we follow a line of snowflakes by going to the right and slightly up, we cycle through all four colors. If instead we put a cycle of five colors in this line, as in Figure 6.14, then applying the hockey-stick translation repeatedly generates a pattern of snowflakes in five colors. By the same process we can add even more colors,

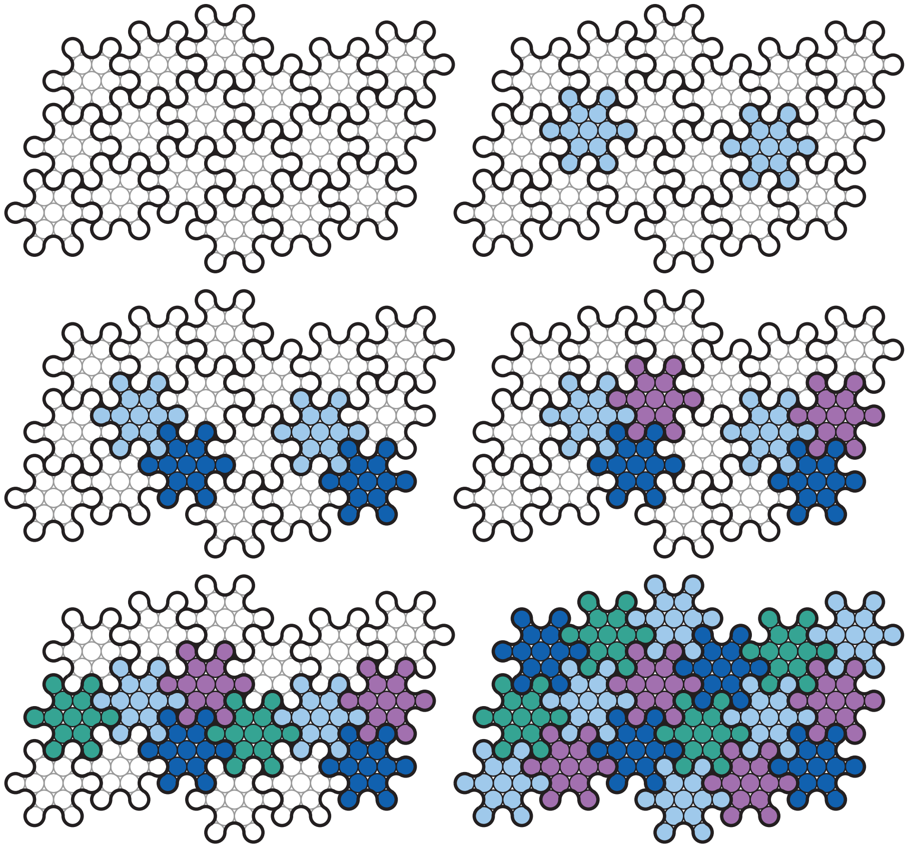


FIGURE 6.13 Coloring the snowflake tessellation. To color the tiling, we go snowflake by snowflake, giving snowflakes separated by a hockey-stick translation the same color. For this tiling, we find that at least four colors are required. The four-colored pattern appears in the pattern section as Snowflakes (p. 197).

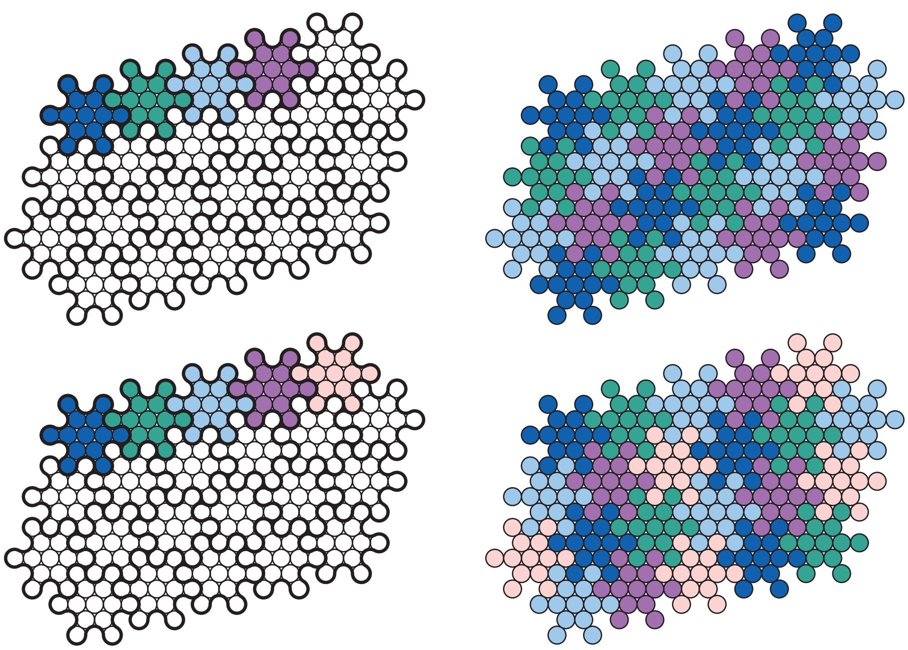


FIGURE 6.14 Four-color and five-color snowflake patterns. In each case, once we pick a color order in the strips shown on the left, the remaining colors are determined by the hockey-stick translation.

but the more colors we add, the longer the repeat length of the pattern, which can make bracelet sizing more difficult.

So far, we have seen Escher bracelet patterns that require three colors (Figures 6.7 and 6.8) or four colors (Figures 6.10 and 6.13) to ensure that neighboring tiles have different colors. This raises the natural question of how many colors are required for tessellated bracelet patterns in general. Not surprisingly, this issue is closely related to our very first design challenge from Chapter 2, that of designing seven-color torus map bracelets. In fact, each of the seven-color torus map bracelets (pp. 134–135) also fits our definition of an Escher bracelet, since all seven countries in our bracelet map have exactly the same shape. However, these patterns have a different flavor than the tessellated bracelets in this chapter, since there is only one tile of each color. As it happens, for many of the tilings we have shown in this chapter so far, if it were physically possible to make a bracelet with only seven tiles, the bracelet would also be a seven-color torus map; two examples are shown in Figure 6.15. It is not as clear how many colors are required in a pattern with multiple tiles in each color, though the seven-color map theorem on the torus guarantees that we will need no more than seven.

The Four-Color Theorem for maps on a plane also applies to maps on a cylinder. If you haven't seen this result before, it's worth pondering how to take a map on a cylinder and stretch it into a map on the plane; the process is analogous to stretching a spherical map into the plane as described on p. 29, with the bonus that you don't need to poke a hole in your map first. This means that any Escher tiling of a bracelet can be colored with four colors—until we sew the ends together to form a torus. This leads us to suspect that most tessellated bracelets with relatively small tiles can be colored with four colors.

On the other hand, by choosing the number of tiles in your bracelet carefully, it is no great feat to force a fifth color into the picture. For instance, suppose you make a bracelet

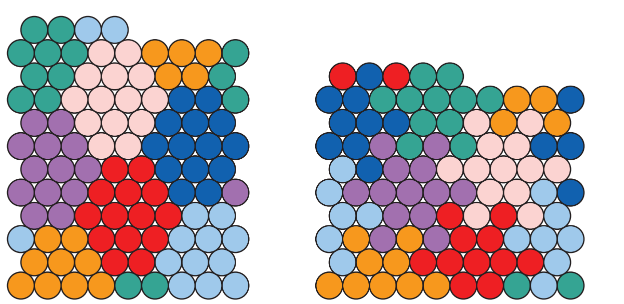


FIGURE 6.15 Escher designs as seven-color torus maps. These charts have too few beads in them to be shaped into a torus and sewn together, but if they could be, each of the seven tiles would touch all of the others.

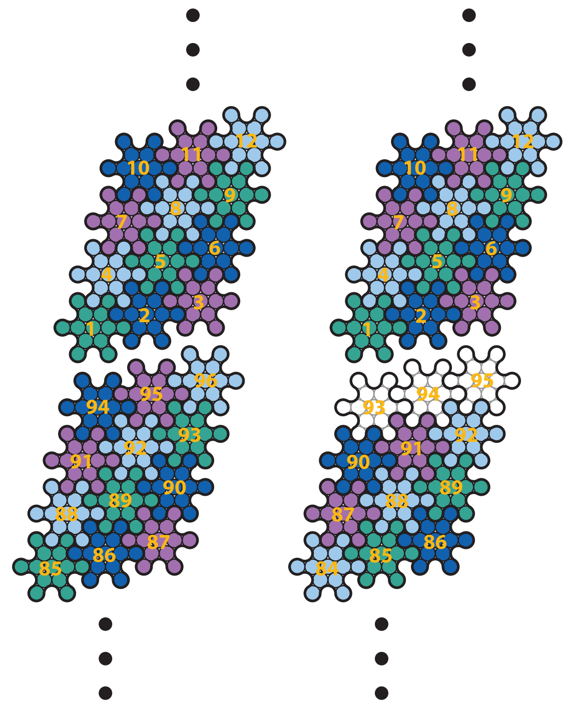


FIGURE 6.16 A four-color impasse. Here are two hypothetical snowflake bracelets that fasten together across the gap in the middle of each bead strip. If the total number of snowflakes is a multiple of four, then the colors line up properly at the seam as on the left. If the total number of snowflakes is not a multiple of four, then the last few tiles touch all four colors and cannot be colored at all.

with the snowflake tiling using exactly 95 tiles. As we saw in Figures 6.13 and 6.14, if you use only four colors, once you have colored in the first four tiles, all the other colors are determined. Since the total number of tiles is not a multiple of four, an attempt to color a 95-tile bracelet with four colors will reach an impasse at the last few tiles. Figure 6.16 illustrates the problem with two bracelets, sliced along the edges of the snowflake tiles and laid flat in the plane. However, since 95 *is* a multiple of five, the five-color pattern from Figure 6.14 works perfectly here.

What is not clear to us is whether there is a tessellated bracelet pattern with multiple tiles of each color that requires more than five colors. We invite you to ponder this question—perhaps you will come up with the answer that has thus far eluded us!

Generating Tiles from Lattices

The hexagon and snowflake tessellations began with a promising tile shape. Full of hope, we pushed copies of our shape around the bead plane to find a configuration that covered the plane and then searched for a hockey-stick translation.

This gave us very little control over the bracelet circumference and how the tiles were arranged in the final pattern. Is there a technique that allows us to have more influence over the size of our bracelet and the arrangement of tiles?

Let's take a closer look at the 5-around hexagon tiling from Figure 6.8. There are 14 beads in each hexagon and three colors of hexagons that are repeated to cover the bracelet. Consequently, the repeat for this bracelet pattern is $3 \times 14 = 42$. Because the tiles are all the same shape and the same orientation and the colors are arranged symmetrically, the pattern breaks down nicely into three segments of 14 beads each that are identical except for a shuffle of the colors we use. For instance, if we start at the leftmost bead in a hexagon, the stringing pattern is

4 2 3 3 2
4 2 3 3 2
4 2 3 3 2

Since moving horizontally in the bead plane gives us a readout of the beads as they are strung on the thread, this tells us that if we move 14 beads to the right (or left) of the leftmost bead in a hexagon, we will land on the leftmost bead in another hexagon. In Figure 6.17, the leftmost beads of each hexagon are marked in black to illustrate this relationship. Furthermore, we know that applying a hockey-stick translation of length 5 will also take us from a black bead to another black bead. The horizontal and hockey-stick translations point in two independent directions, so by the mathematics of lattices explained in Chapter 1, we can travel from any black bead to any other with some combination of these two translations.

Armed with this understanding of the lattice connected to a known tiling, we can now run the process in reverse. Let's say we want to make a 5-around Escher bracelet, but instead of having 14 beads in each tile, we want 13 beads in each tile. In this case, our lattice should be generated by the hockey-stick translation of length 5 and a horizontal translation by 13 beads. The resulting lattice appears in Figure 6.18.

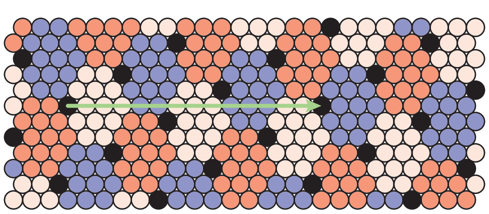


FIGURE 6.17 The lattice for the 5-around hexagonal tiling from Figure 6.8. Each hexagon contains 14 beads, which produces a 14-bead horizontal translation in the lattice.

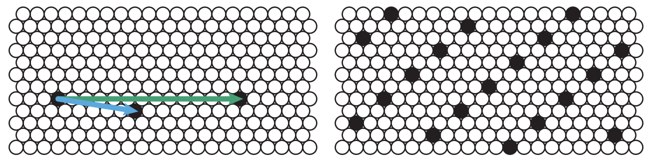


FIGURE 6.18 Creating a lattice for a specific circumference and tile size. On the left, the blue arrow indicates the hockey-stick translation for a 5-around bracelet and the green arrow is a horizontal translation by 13 bead lengths. Applying all possible combinations of these translations produces the lattice on the right, which marks the relative positions of 13-bead congruent tiles in a pattern of circumference 5.

With our lattice in hand, we can fill in tile shapes by coloring beads around the lattice as shown in Figure 6.19. The tiles are grown incrementally from the top of the figure to the bottom in three different patterns (shown in columns left, middle, and right). Lattice points that are separated by a hockey-stick translation represent the same bead in the bracelet, so we always color them the same color. As long as we keep each tile the same shape with respect to the lattice point as we color in beads and make sure not to overlap our colors, once each tile has 13 beads in it we will have an Escher bracelet design. Figure 6.20 shows the middle design from Figure 6.19 in bead crochet.

Growing Symmetric Tilings

In the previous design method, we used the regularity of our starting lattice to produce tiles that were all the same shape. Since each tile grew from its lattice point in the same way, translating the entire bead plane to move one lattice point into the position of a second lattice point will cause the tile around the first lattice point to fill exactly the same space as is vacated by the tile around the second. A more subtle way to evolve an Escher design in the bead plane is to use symmetry to create tiles that have the same shape.

Take the design that emerges in Figure 6.21, for example. In this case, we have chosen a circumference of 6 and a two-color palette of light and dark blue. To force the light and dark tiles to have the same shape, we gradually color them in while maintaining symmetry around a center point, marked with a red dot that is midway between a light bead and a dark bead. Since we are making a 6-around design, the red dot is repeated at hockey-stick intervals of length 6 just like the rest of the pattern. Every time we color a bead in light blue, we color the bead on the opposite side of the center in dark blue, and vice versa. By following this procedure, we ensure that the light and dark tiles have exactly the same shape, because

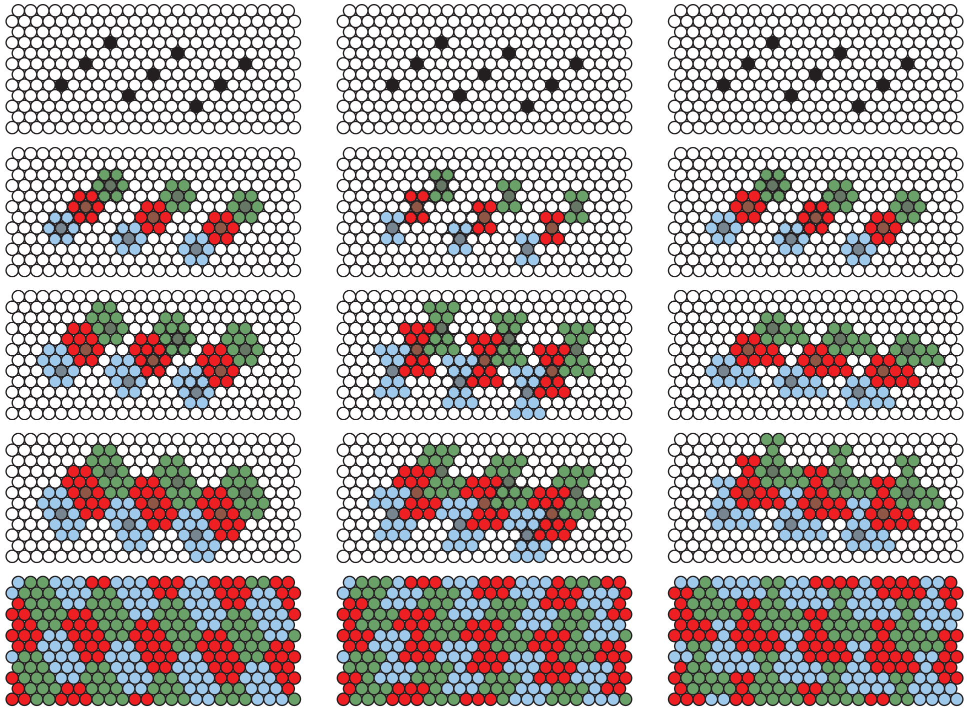


FIGURE 6.19 Three tessellations produced with the lattice from Figure 6.18. Since the lattice accommodates a 13-bead tile, we add beads incrementally to all the tiles in a given pattern until every tile has 13 beads.

FIGURE 6.20 A bracelet made with the design in the bottom center of Figure 6.19. The pattern appears in the pattern pages as Naptime (p. 171).

if we rotate the pattern of light beads by 180° around the center point, it will exactly cover the pattern of dark beads.

In the pattern section, you will find many two-color Escher designs with a similar 180° symmetry. Some of them were made with this method of growing tiles, while others began with symmetric polygonal tilings of the type discussed earlier in the chapter. The frequency of this design element in our patterns reflects an inherent symmetry in the bead crochet process: if you rotate any bead crochet chart by 180°, thereby reversing the order in which you string the beads, there will be no detectable difference in the final bracelet. This symmetry also appears in the numerical stringing pattern for the design in Figure 6.21. Starting at a point midway between two red dots, the pattern of beads on the strand is

1 1 1 1 1 2 1 1 2 3 1 6 1 3 2 1 1 2 1 1 1 1 1
1 1 1 1 1 2 1 1 2 3 1 6 1 3 2 1 1 2 1 1 1 1 1

Notice that if you read this pattern backward, you get the identical pattern but with the two colors swapped, demonstrating the symmetric design process.

By using the bead plane more adventurously, we can generate more complex symmetries between tiles. Consider the two regular division drawings in Figure 6.22, in which the tiles appear in three different orientations, one for each color. In each design, the black tiles are rotated from the blue tiles by 120°, which are rotated from the red tiles by 120°, which in turn are rotated from the black tiles by 120°. Patterns with this type of symmetry appear frequently in Escher's regular division sketches and are also popular in quilting. In fact, readers who quilt will recognize the design on the left as the traditional tumbling blocks quilt pattern.

We can emulate this type of symmetry in the bead plane with an analogous process to the one from Figure 6.21 by

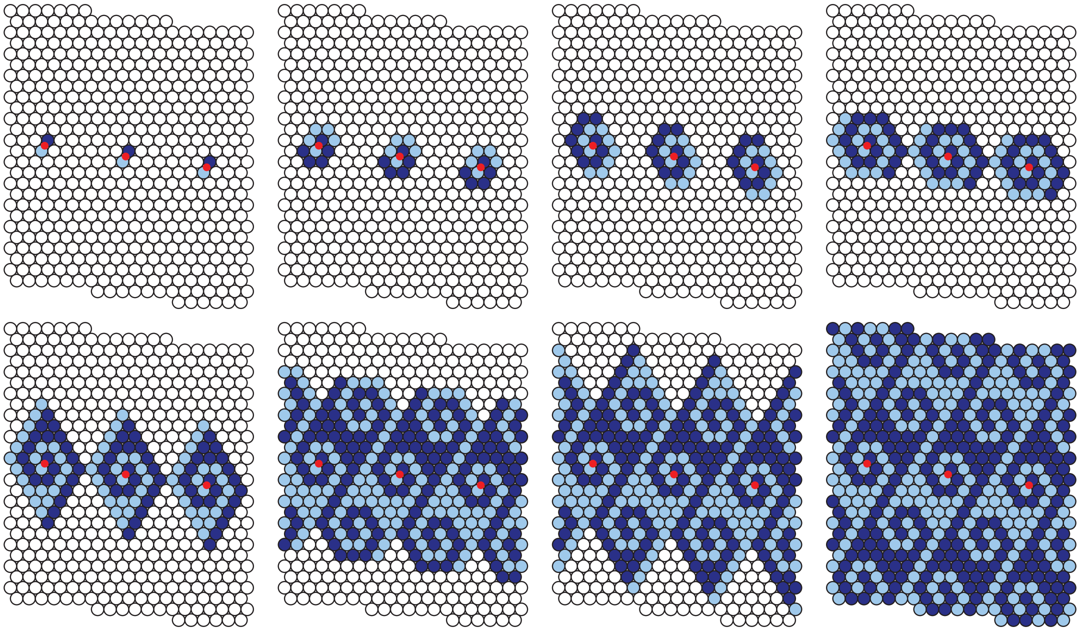


FIGURE 6.21 Growing a two-color Escher pattern using symmetry. We start with a seed pattern of the two colors, then add beads symmetrically around the center point, marked in red, so that a 180° rotation interchanges the colors. This design is the pattern for the Crashing Waves bracelet (p. 189).

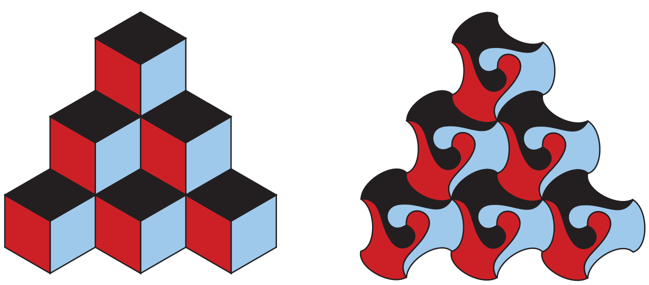


FIGURE 6.22 Three-color tessellations of the plane by congruent tiles. The tiling on the left is the classic tumbling blocks pattern used in quilting, and the tiling on the right is a deformation of it. In each tessellation, tiles of different colors are the same shape but rotated by 120°.

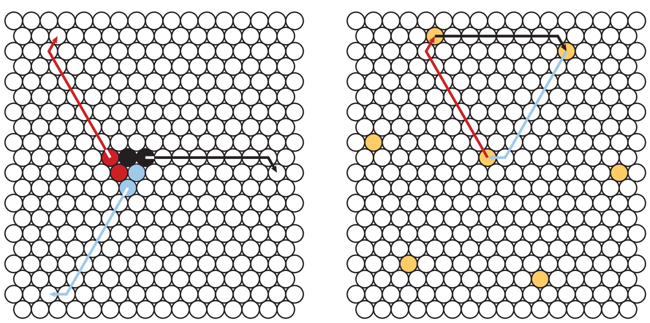


FIGURE 6.23 Translations for a three-color tessellation with tiles rotated by 120°. Since the black tile is translated by the hockey-stick translation, the red and blue tiles must be translated by rotations of the hockey-stick translation, as marked in the left diagram. On the right, we see the lattice formed by our three translations.

starting with a symmetric three-color seed pattern and maintaining the 120° relationships between tiles as we color in the plane. However, there is a significant difference introduced by the 120° angle. While a hockey-stick translation rotated by 180° is simply a reversed hockey-stick translation, a hockey-stick translation rotated by 120° is an entirely different type of translation, as shown in Figure 6.23 for a 7-around bracelet. In our seed pattern, the black tiles must have a hockey-stick translation of length 7. Since rotating by 120° counterclockwise around the center of the pattern will place the black tiles on top of the red tiles, the red tiles—and thus the entire Escher design—must have the translation marked in red that we get by rotating the hockey stick by 120° around the center. The relationship between the red and blue tiles gives us a translation in a third direction marked in blue.

As shown in the diagram on the right of Figure 6.23, if we apply all three translations in turn, we end up back where we started. This means that we really only have two *independent* translations; for instance, we can achieve the red translation by performing the reverse of the blue translation and then the reverse of the black translation (or vice versa). Since our entire Escher pattern will be preserved by

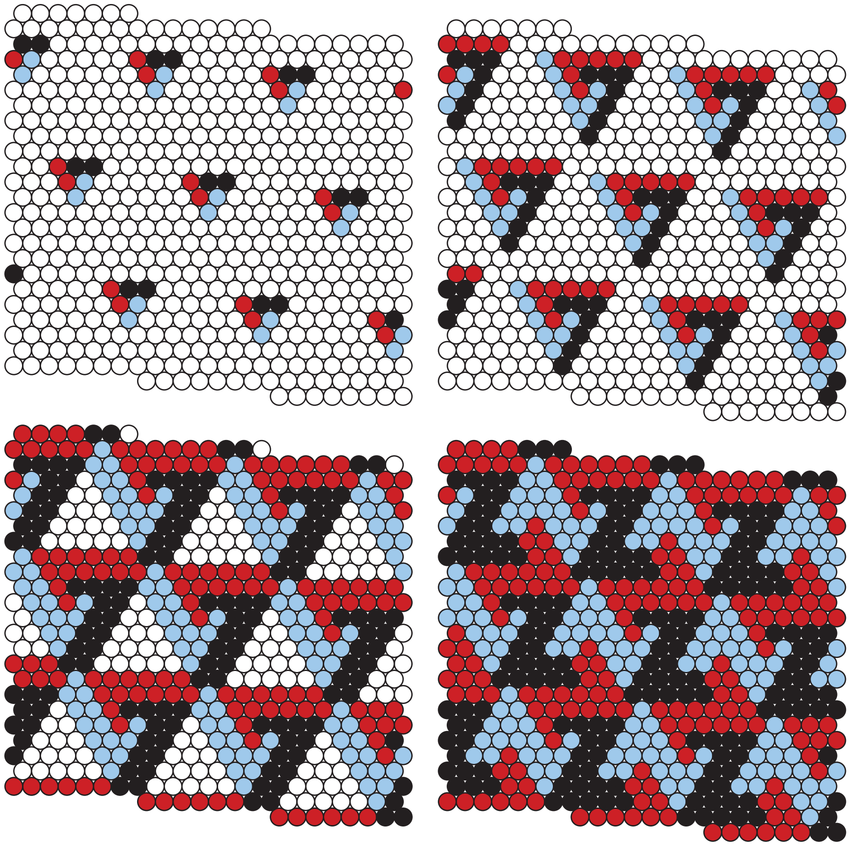


FIGURE 6.24 Growing a three-color Escher pattern using symmetry. At each stage, we apply the three translations shown in Figure 6.23.

all three translations, the final pattern will correspond to the lattice marked in yellow. In particular, the repeat length, as determined by the lattice, must be 57 beads.* Since we are forming congruent tiles in three colors, we know in advance that when we are done, each tile will contain 57/3 = 19 beads.

Notice what has happened here; we picked a circumference of 7 and added a symmetry condition on our three colors of tiles, and that information alone completely determined the *size* of our tiles! We will look more closely at different types of symmetry in the next chapter, and we will see that imposing symmetry forces all sorts of interesting constraints on bracelet designs. Upon reflection, you might be struck by how lucky we were that the repeat length forced by our choice of symmetry was evenly divisible by 3, and you might suspect that the game was rigged. In fact, it was. If you would like a little geometric puzzle, try applying the same 120° rotation condition in circumference 6; you will find that the repeat length is *not* divisible by three and if you try to form an Escher design with this method, you will always have beads left over. For a bigger challenge, see if you can figure out which circumferences *do* work!

Fortunately for us, this process *does* work for a 7-around bracelet, as confirmed in Figure 6.24. Here, we start with our seed pattern, translated by the lattice from Figure 6.23. At each stage, we add more beads of each color while preserving the 120° rotations that take one tile color to another. Since we only have 19 beads of each color to fill in, it is not long before we have colored the entire plane. What is particularly interesting about this pattern, and what makes it unique among the Escher designs we've seen here, is that the tiles are only truly congruent in the bead plane. In an actual bracelet, which you can see at the top of the assortment of Escher bracelets in Figure 6.25, the tiles are curved around the meridian of the bracelet in different orientations, making the three-dimensional shape of the black tiles quite different than the three-dimensional shape of either the blue tiles or the red tiles.

* You can read the repeat length from the lattice using the principles outlined in Chapter 1. In this case, it is easiest to use the traditional left-leaning diagonal layout: starting at one of the lattice beads, it takes 8 rows of 7 beads each plus one extra bead to reach the next lattice bead.

FIGURE 6.25 An assortment of Escher bracelets. The bracelet at the top is from the pattern in Figure 6.24, the bracelet on the lower left is from the snowflake tiling in Figure 6.13, and the center and lower right bracelets are the same patterns as in the chapter header. The center bracelet is Escher Fish (p. 204), and the remaining bracelets, clockwise from the top, are Flying Z's (p. 172), Gliding Vines (p. 218), Escher Lizards (p. 201), Four-Color Flowers (p. 198), Snowflakes (p. 197), and Interlocking Vines (p. 192).

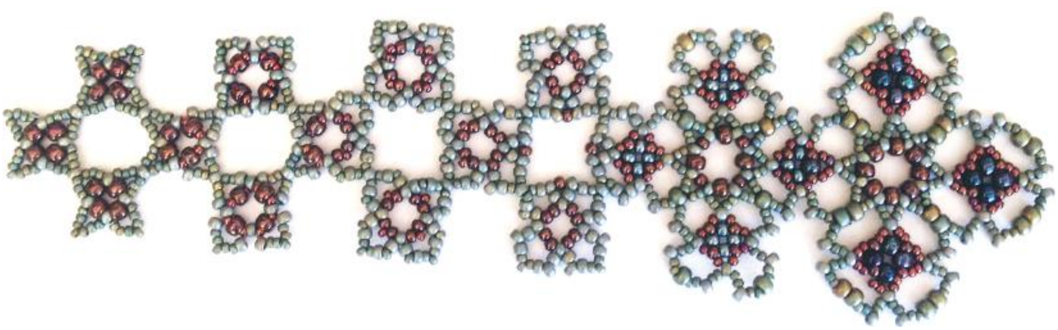


FIGURE 6.26 *Beaded Deformation: From Kepler's Star to the Night Sky*, by Florence Turnour. In this piece, the pattern gradually evolves from the Kepler's Star weave, developed by Gwen Fisher, on the left, to the Night Sky Weave, also by Fisher, on the right.

Escher Transformations: Another Design Springboard

Through our bead crochet adventures, we've gotten to know the extraordinary bead weaver and mathematician Gwen Fisher. She and Florence Tumour, also a mathematician and bead artist, are the women behind BeAd Infinitum, which produces intricate bead weaving patterns and tutorials and jewelry. After reading a blog post about a bead crochet sampler lariat, Fisher emailed us with a design challenge of her own. The lariat she had seen online contained over a dozen patterns in succession with abrupt transitions between consecutive patterns. Fisher wondered if we could use our design techniques to create bead crochet ropes in which each design transitions smoothly into the next to create a pattern that subtly evolves along the length of the rope. In her message, she included a link to an astonishing piece of Turnour's bead weaving, shown in Figure 6.26.

FIGURE 6.27 An Escher transformation necklace. This necklace appears in the pattern pages as Waves and Diamonds Transformation Necklace (p. 208). For a chunkier effect, the necklace is worked in larger beads (size 8 seed beads) than we typically use for 6-around patterns.

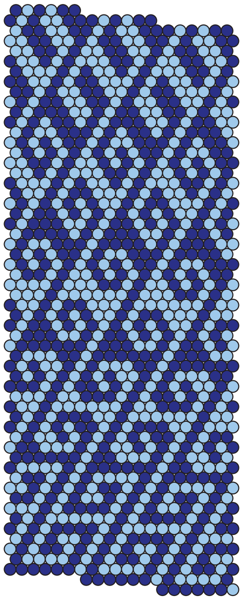


FIGURE 6.28 A chart for the upper section of the necklace in Figure 6.27. The design was created by deforming the pattern from Figure 6.21 in both directions. In this chart, each motif appears only once, but in the necklace each is repeated twice.

FIGURE 6.29 Marian Goldstine modeling her Waves and Diamonds necklace with a matching set of bracelets. These pieces are made with size 11 black seed beads and 1.8mm sterling silver beads.

In *Beaded Deformation: From Kepler's Star to the Night Sky*, Turnour interpolates between two of Fisher's bead weaving patterns in a slowly transforming sequence of beaded tilings. She took her inspiration from the work of the mathematician and artist Craig Kaplan, and also from the gradually transforming designs of another artist … M.C. Escher.

Escher bracelet patterns lend themselves naturally to this type of subtly shifting design. After all, one of the principal methods for making elaborate tessellations in bead crochet is to deform simpler patterns a few beads at a time, as we saw in Figure 6.10. To make a transforming Escher design, we simply place these deformations in sequence on a single bead crochet rope. In practice, this often involves tinkering to create smooth transitions between the deformations, a process that incorporates a great deal of trial and error, a large design space (a computer graphics package that allows you to copy and rearrange designs is extremely helpful), and a great deal of patience. The necklace pattern shown in Figures 6.27–6.29 is an example of this technique in action, and there are several more transformation designs at the end of the book.

CHAPTER 7

Wallpaper Groups

In our bead explorations so far, we have frequently been guided by a search for symmetry. When we constructed seven-color torus maps on bracelets, we wanted all of the countries to be the same shape. When we made bracelets with geometric cross sections, we were particularly interested in cross sections that were regular polygons. When we tiled our bracelets with Escher designs, we sometimes used symmetry as a tool to create pleasing tessellations. In all of these endeavors, we have danced around a central question, one of great significance to mathematics, the sciences, and the arts: what *is* symmetry?

Different disciplines have different answers to this question. Not surprisingly, mathematics takes a precisely targeted approach. Rather than thinking of symmetry in its entirety as a concept, mathematicians prefer to discuss a specific symmetry of a particular object. To keep things simple, let's consider symmetry in the context of patterns in a flat plane. A *symmetry* of a pattern is a way of moving the plane that repositions the pattern but does not change its appearance. As it happens, our design techniques have been forged by symmetries from the very beginning. Every bracelet design in the bead plane has a hockey-stick translation and a repeat translation that preserve it; if you apply these translations to the bead plane, the bracelet design remains unchanged. In other words, the hockey-stick translation and the repeat translation are symmetries of the bead crochet design!

Consider the patterns in Figure 7.1, each excerpted from a bracelet pattern in this book. The markings in orange and red indicate the symmetries of each pattern. The line of blue waves at the top, which we view as repeating forever to the left and to the right, has a *translational symmetry*: if we perform a translation of the plane along the orange arrow, each wave will move into the position of the next wave in the line, and we won't notice that the pattern has changed in any way. In fact, this is a hockey-stick translation, though the bead plane is rotated to make the translation horizontal. The yellow and black star has a *rotational symmetry*: if we rotate the plane by 60° around the green bead in the center, the star will look exactly the same as before the rotation. Notice that since we can rotate the star repeatedly, the star is also preserved by rotations of 120°, 180°, 240°, and 300° around the same center. A rotation of the plane by 360° doesn't count as a symmetry,[*] since all points in the plane return to their original positions.

The symmetries of the remaining patterns involve reflections, which are a little harder to visualize because they require moving the plane in three dimensions. For instance, the pattern of white flowers and green leaves in Figure 7.1 has a *refl ction symmetry* across the red line in the middle of the pattern, the *axis of refl ction*. If we imagine that the page is detached from the book and that the ink has soaked through to the other side of the page, then spinning the page by 180° around the red line would not change the appearance of the design of flowers and leaves. In theory, we can achieve the same effect by holding the page up to a mirror, but we would have to be very careful about where we held the page and the mirror to align the pattern properly. By contrast, the black vine at the bottom of Figure 7.1, which we view as continuing forever in a line like the blue waves, almost has a reflection symmetry across its red axis, except that the fronds on the top and the fronds on the bottom aren't lined up properly. However, if we also translate—or *glide*—the vine to the right by half the distance between fronds, the vine will fill the same space in the plane that it did before. This combined movement, a reflection and a translation parallel to the axis of reflection, is a *glide refl ction*, and so the vine has a *glide refl ction symmetry*.

As it happens, the only ways of rearranging the plane without stretching, compressing, folding, or tearing it—commonly known as the *rigid motions of the plane*—are translations, rotations, reflections, and glide reflections.[†] It is not immediately obvious that there are no other types of rigid motions. For instance, since a reflection followed by a translation produces a new type of motion called a glide reflection, we might imagine that a rotation followed by a reflection would produce a new type of motion, a "rotation reflection." In fact, it turns out that a rotation followed by a reflection is merely another reflection across a different axis.

To understand the full symmetry structure of a shape or pattern, mathematicians look at the collection of its symmetries and how they interact with each other. This way of viewing symmetry allows us to recognize the common structure of designs that superficially look quite different. For instance, Figure 7.2 shows three different repeating designs in the plane with the same essential symmetry structure. The top row shows the three unadorned designs, while the middle and bottom rows are labeled to illustrate the symmetries of each pattern: axes of reflection and glide reflection are marked with colored lines, and centers of rotation are marked with colored dots. Since each design

[*] From a technical viewpoint it does, but it is the same symmetry as the one that doesn't move the plane at all. Since every possible pattern has this symmetry (if you don't move it, it looks the same), mathematicians consider it a *trivial* symmetry—important for mathematical bookkeeping, but not a significant feature of a pattern.

[†] For an accessible proof of this fact, see Dorothy K. Washburn and Donald Crowe, *Symmetries of Culture: Theory and Practice of Plane Pattern Analysis*, University of Washington Press, 1998, p. 271.

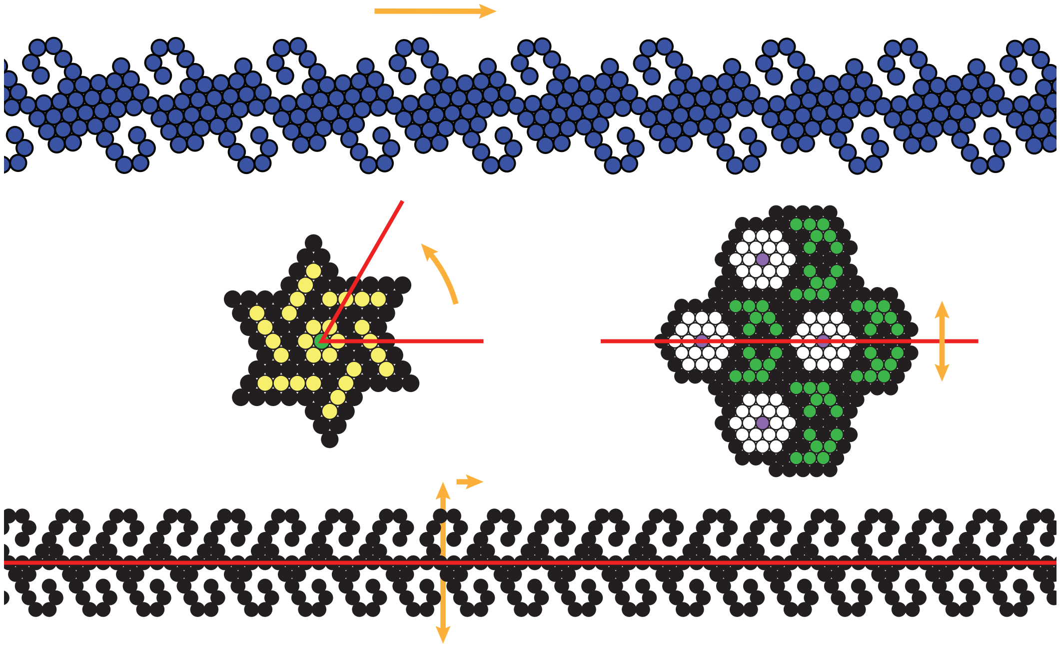


FIGURE 7.1 The four types of planar symmetries. The top design is preserved by a horizontal translation, the middle left design by a rotation by 60°, the middle right design by a reflection across a horizontal axis, and the bottom design by a glide reflection with a horizontal axis.

fills the infinite plane and repeats over and over again, the axes and centers of symmetry also repeat throughout the plane, and marking them all would make the symmetry types hard to discern. However, the bottom diagram in each column shows representatives of all the types of nontranslational symmetry, including a center of rotation for each possible angle and axes of reflection and glide reflection in each possible direction. Each axis of reflection or glide reflection can be reached from the marked axes by a translational symmetry of the design, and each center of rotation can be reached from the marked centers by a combination of reflections and translations of the design. To make it easier to inspect the individual reflections and glide reflections of the patterns, the middle row marks only two reflections and two glide reflections per pattern.

We describe the specific symmetries in Figure 7.2 and their relationships to each other in more detail below. Some of these symmetries are more readily apparent than others, and it is not essential to fully understand all of them. What is important is that all three patterns have exactly the same types of symmetries.

In each pattern in Figure 7.2, we have rotations by 60° (centers marked in red), rotations by 120° (centers marked in blue), and rotations by 180° (centers marked in green). Notice that while we can also rotate each pattern through a red center by 120° or 180° because we can repeat the 60° rotation, we *cannot* rotate around the green or blue centers by 60° without visibly changing the pattern. Through each red center, there are six axes of reflection (marked in orange) arranged 30° apart. There are glide reflections (axes marked in purple) in six directions. And finally, each design has translation symmetries along each of the reflection axes.

By contrast, while the two designs in Figure 7.3 are both based on the same triangular grid, the symmetry structures of the designs are different. The uncolored triangular pattern on the left is the middle pattern from Figure 7.2 and has the symmetries described before. However, in the pattern on the right, the upward-pointing triangles are gray and the downward-pointing triangles are white. Since any rotation by 60° or 180° turns the upward-pointing triangles downward and vice versa, the design on the right has no symmetries of this type. Similarly, half of the reflections and glide reflections of the design on the left do not preserve the design on the right. Notice that although all three centers of rotation marked in the design on the right are for rotations by 120°, they are fundamentally different in their relationship to the design itself. For instance, no rigid motion of the gray and white triangular grid could move the center of a gray triangle to a point where six triangles meet without changing the appearance of the pattern.

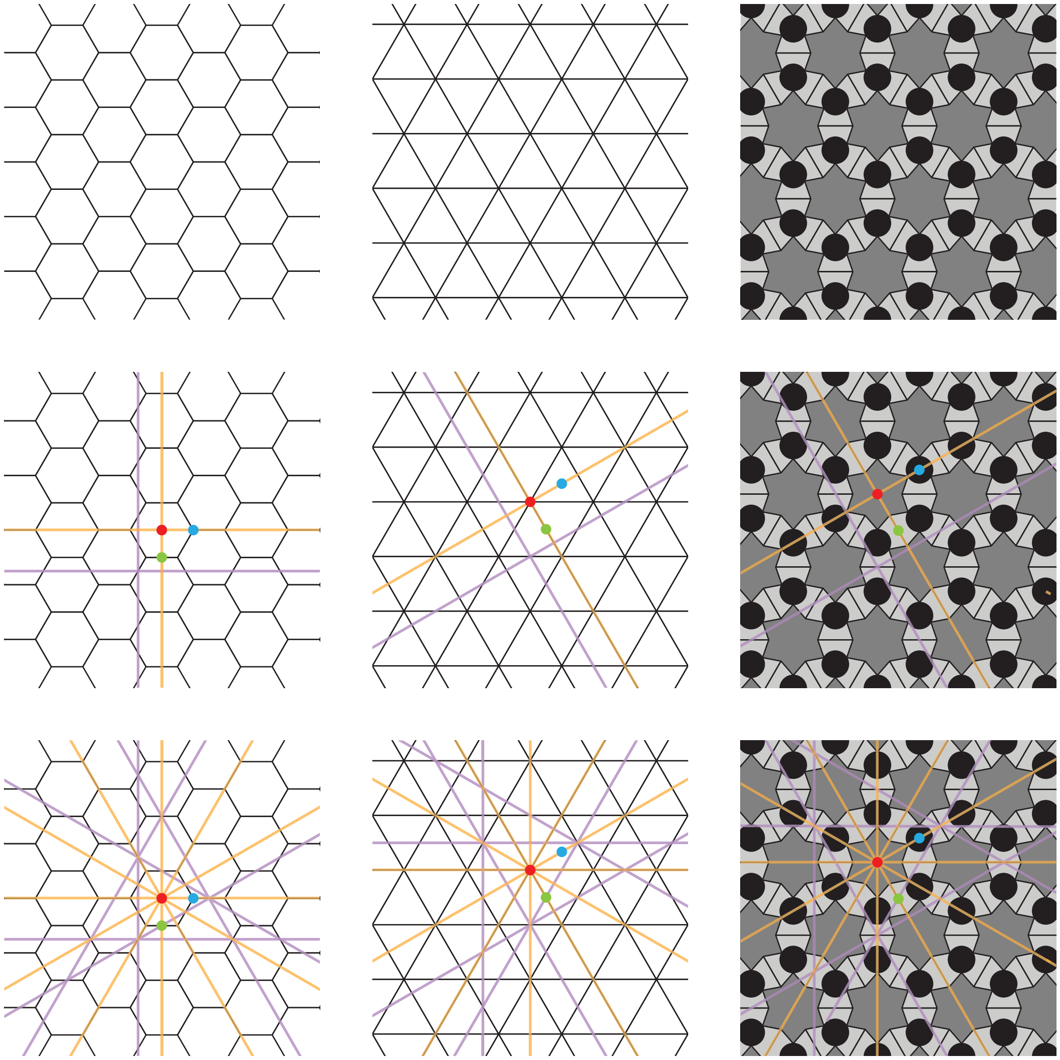


FIGURE 7.2 Designs with the same symmetry structure. Even though these three patterns are quite different in appearance, they all have the same symmetry structure. In the bottom two rows, representatives of each kind of nontranslational symmetry are marked as follows: centers of 60° rotations are red, centers of 120° rotations are blue, centers of 180° rotations are green, axes of reflection are orange, and glide reflection axes are purple. The bottom row shows reflection axes and glide reflection axes in *every* direction that gives a symmetry for the pattern, while the middle row shows fewer axes to make it easier to study the individual symmetries.

The collection of symmetries of a pattern is called the *symmetry group* of the pattern, and it is an example of a much more general type of mathematical object called a *group*. The theory of groups is rather abstract and intricate, and we will not describe it in any detail here, since the information that we can glean from the figures in this chapter will suffice for our purposes. If you are interested in learning more about group theory, Keith Devlin's *Mathematics: The Science of Patterns: The Search for Order in Life, Mind, and the Universe* has a nice chapter on the subject, and Nathan Carter's *Visual Group Theory* is a lovely introduction to the field that goes into more technical detail. An excellent undergraduate text on the subject is Dan Saracino's *Abstract Algebra: A First Course*.

Each of the designs in Figures 7.2 and 7.3 has translational symmetries in two independent directions, just like bracelet patterns in the bead plane. This means that we can reproduce each design by taking a finite piece of the design and filling the plane with copies of that piece, in the same way that we can tile the bead plane with a single repeat of a bracelet pattern. For example, we can build the design on the right of Figure 7.2 by translations of any one of the three tiles shown on the right of Figure 7.4. As you may have noticed if you spend time in wallpapered

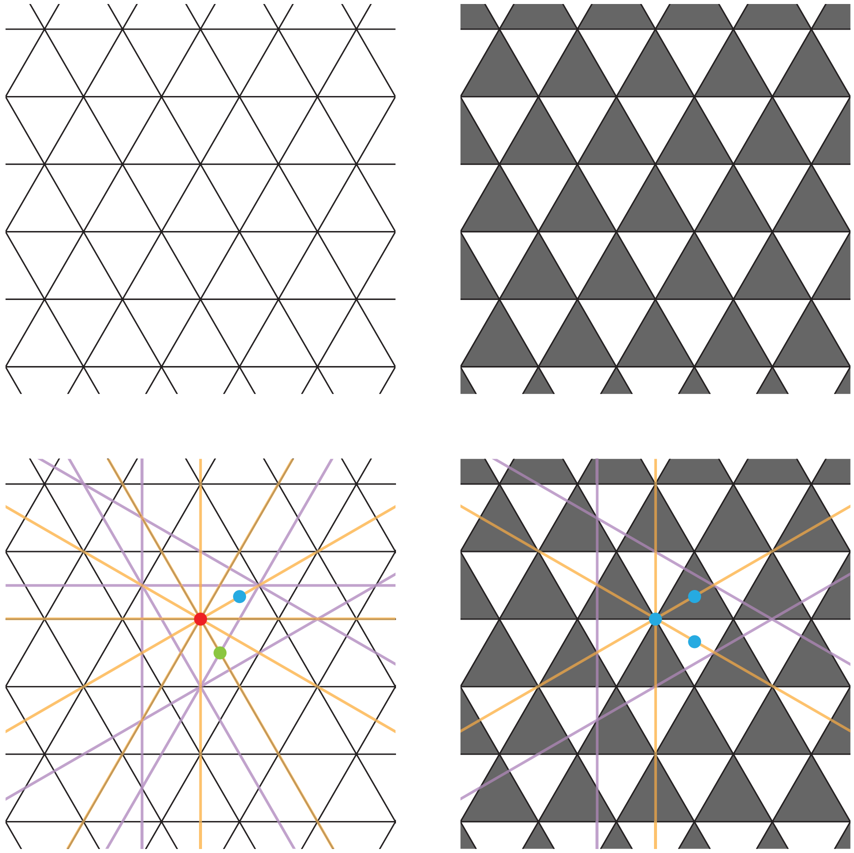


FIGURE 7.3 Designs with different symmetry structures. In the design on the right, all of the gray triangles point upward. Consequently, the rotation by 60° centered at the red dot in the left pattern does not preserve the pattern on the right, since it would make all of the gray triangles point downward.

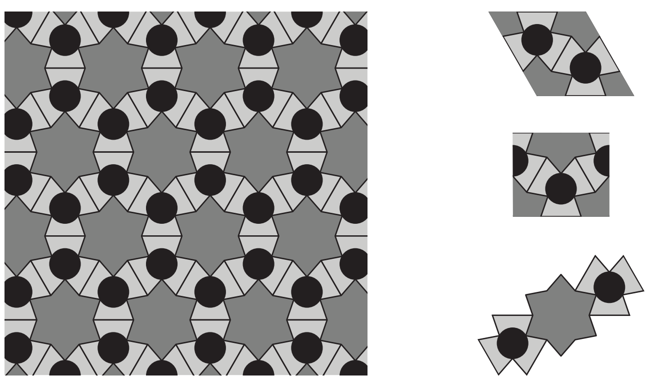


FIGURE 7.4 Building an infinite plane pattern from a finite pattern. We can form the design on the left by fitting together translations of any of the tiles on the right.

rooms, the designs on wallpaper fall into this category for practical reasons, since repeating patterns allow efficient printing of strips of wallpaper that cover a wall of any size easily. For this reason, any design in the plane with two independent translations is called a *wallpaper pattern*, and its group of symmetries is a *wallpaper group*. Since symmetry groups of repeating patterns are used in crystallography, wallpaper groups are also known as *plane crystallographic groups*, and in fact the notation that we will use to describe the different wallpaper groups in this chapter, one of several in common use, is the notation adopted by the International Union of Crystallography [Washburn and Crowe, *Symmetries of Culture*]. For instance, the common symmetry group of the patterns in Figure 7.2 is denoted P6M, while the symmetry group of the pattern on the right of Figure 7.3 is denoted P3M1. We will say more about this notation and its meaning shortly.

Extrapolating from the variations in Figure 7.2, it is not hard to see that there is an infinite variety of planar designs for the wallpaper group P6M. In the rightmost design, for example, we can expand or contract the dark circles, or make the stars more or less pointy, or make countless other changes as long as we make them symmetrically. Thus, there are infinitely many patterns that share the same symmetry structure. This raises a natural question: if there

are infinitely many patterns corresponding to each wallpaper group, are there also infinitely many possible wallpaper groups?

Surprisingly, the answer is no. An amazing theorem, proved independently by Evgraf Fedorov in 1891 and by George Pólya in 1924,* tells us that there are exactly 17 different wallpaper groups. This means that out of the endless variety of wallpaper patterns, there are only 17 possible symmetry structures! In other words, all the patterns that can be used to tile the plane belong to one of only 17 basic categories.

There are lovely systems for recognizing which wallpaper group a particular design exhibits, and we will describe one of them in detail presently. But first, we present our final design challenge.

Challenge *Every bead plane design for a bracelet is a wallpaper pattern. Which of the 17 wallpaper groups are possible in a bracelet design? Can you create a bracelet pattern demonstrating each?*

Identifying Wallpaper Groups

While symmetry groups were first studied by mathematicians and crystallographers for purely scientific reasons, they have important applications in the humanities as well. One of the most intriguing uses of symmetry classification is in the work pioneered by anthropologist Dorothy K. Washburn and mathematician Donald W. Crowe in their seminal 1988 book, *Symmetries of Culture: Theory and Practice of Plane Pattern Analysis*. Washburn and Crowe studied the decorative art traditions of various ancient and contemporary cultures, and they found that different cultural groups favor different symmetry structures. These preferences for certain symmetries in pottery, fabrics, and other decorated artifacts are pronounced enough to shed light on trade routes in antiquity through the spread of various symmetric design schemes. To help others perform similar pattern analysis, *Symmetries of Culture* includes flowcharts for recognizing the symmetry groups of the wallpaper patterns, as well as the symmetry groups of *frieze patterns*, which repeat in only one direction (like ornamental stitching on a garment hem), and other types of symmetries.

In Figure 7.8, we present a similar flowchart based on the wallpaper-pattern recognition algorithm of mathematician Brian Sanderson.† Proving that this classification gives all the possible wallpaper groups is beyond the scope of this book, but interested readers will find proofs of this and several other symmetry classification theorems in the excellent volume *The Symmetries of Things* by Conway, Burgiel, and Goodman-Strauss. Even without a rigorous proof, the material that follows is fairly technical. If you prefer not to delve into the details of wallpaper group structure but would like to incorporate some interesting rotations or reflections into your bead crochet designs, we suggest that you skip ahead to the section on designing wallpaper bracelets on p. 104. Whether or not you skip ahead, if you happen to own an iPad or iPhone, we strongly recommend that you acquire Jürgen Richter-Gebert's phenomenal (and rather addictive) app iOrnament, which allows you to draw designs for any of the wallpaper groups with a wide array of artistic effects. In addition, iOrnament features explanations of the theory of wallpaper groups with interactive illustrations that you can manipulate on your touch screen.

The first step in our pattern recognition scheme is to search for rotational symmetries. Any rotational symmetry of a wallpaper pattern has the property that if we repeat it enough times, the plane returns to its original position. The number of times we must apply the rotation to return to our starting point is called the *order* of the rotation. For instance, the star on the left of Figure 7.5 has a 60° rotational symmetry, and 60° is the smallest angle of rotation that preserves the design. Since performing six 60° rotations gives us a full 360° rotation, returning the plane to its starting position, a rotation of 60° has order 6. We can also describe this by saying that the star has

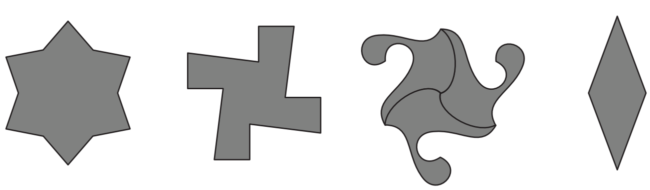


FIGURE 7.5 Designs with rotation order 6, 4, 3, and 2, from left to right. These are the only rotation orders in any of the wallpaper groups, in which the only possible angles of rotation are multiples of 60° and 90°.

* Dorothy K. Washburn and Donald W. Crowe, *Symmetries of Culture: Theory and Practice of Plane Pattern Analysis*, University of Washington Press, 1988.

† Dror Bar Natan's Image Gallery: Symmetry: Tilings: Sanderson's Algorithm, http://www.math.toronto.edu/~drorbn/Gallery/Symmetry/Tilings/Sanderson/index.html.

sixfold rotational symmetry. This reflects the fact that the star has six equally spaced points, each of which is carried to the next by rotation by 60°. The remaining shapes in Figure 7.5 have rotational symmetries with order 4, 3, and 2, respectively, corresponding to rotations by 90°, 120°, and 180°.

The only rotations that occur in the wallpaper groups are rotations with order 6, 4, 3, or 2. There are also wallpaper groups with no rotational symmetries at all.* Notice that since a rotation by 60° can be repeated to achieve rotations by 120° and 180°, any wallpaper group with rotations of order 6 also has rotations of orders 3 and 2. Likewise, since a rotation by 90° can be repeated to achieve a rotation by 180°, any wallpaper group with rotations of order 4 also has rotations of order 2. In the flowchart in Figure 7.8, our recognition algorithm first divides the wallpaper patterns into clusters based on the largest order of rotational symmetry. Within each cluster, the patterns are distinguished by the types of reflections and glide reflections that preserve it.

At this point, you may be wondering why there are no rotations of order 5, or order 7, or any higher order. While this is easier to prove than the full classification of the wallpaper groups, the math is still more technical than appropriate in this discussion. However, it is related to the analogous and less cryptic phenomenon, illustrated in Figure 7.6, that the only regular polygons that can tile the infinite plane are the equilateral triangle, the square, and the regular hexagon. Each of these polygons fits neatly in a tiling, with six triangles to a corner, four squares to a corner, and three hexagons to a corner. By contrast, any attempt to tile the plane with regular pentagons is doomed to failure: three pentagons to a corner leaves a gap, and four pentagons don't fit around a corner without overlapping.

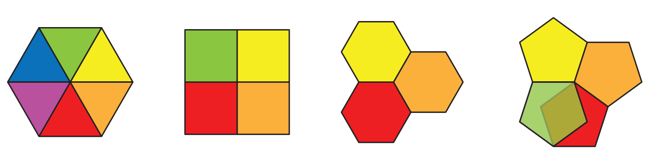


FIGURE 7.6 Why the only regular polygons that can tile the plane are the triangle, the square, and the hexagon. If we attempt to fill a plane with regular pentagons, we reach an impasse and are forced to either leave gaps or overlap the pentagons.

* Except the trivial symmetry, which is a rotation by 0°. Since performing this symmetry a single time returns the plane to its original position, this is considered a rotation of order 1.

Regular polygons with more than six sides, and correspondingly larger angles, fare even worse. Notice that between them, the uncolored tilings of regular triangles, squares, and hexagons have rotational symmetries of orders 6, 4, 3, and 2—precisely the orders that appear in the wallpaper groups.

In order to complete the wallpaper pattern analysis, we need to take a closer look at glide reflections. Any pattern with a reflection and a translation parallel to its reflection axis, such as the pattern at the top of Figure 7.7, has glide reflection symmetry in an incidental way, since we can combine its reflection and its translation. Because this glide reflection breaks down into a reflection that preserves the pattern and a translation that also preserves the pattern, it is a *decomposable* glide reflection. On the other hand, the vine pattern at the bottom of Figure 7.7 has an *indecomposable* glide reflection. As we observed while investigating the same design in Figure 7.1, the reflection through the red axis changes the alignment of the top and bottom fronds, so this glide reflection cannot be separated into a reflection and a translation each of which preserve the pattern. In spite of this, there is a translation by twice the marked distance that does preserve the design, but without a reflection symmetry, the glide reflection is nonetheless indecomposable. Among the wallpaper groups, reflections and indecomposable glide reflections are present or absent in all possible combinations: there are groups with reflections but no indecomposable glide reflections, groups with indecomposable glide reflections but no reflections, groups with both symmetry types, and groups with neither.

The Figure 7.8 flowchart identifies each wallpaper pattern by considering several aspects of the pattern: what is its maximum order of rotational symmetry (or equivalently, what is its smallest angle of rotational symmetry), does it have reflection symmetries, does it have indecomposable glide reflection symmetries, and what is the relationship between its reflection axes and its centers of rotation? Some of these symmetries are tricky to spot without practice. To make them easier to detect, Figures 7.9–7.15 show each of the representative patterns from the flowchart with markings to illustrate the rotations, reflections, and glide reflections of each pattern. Many of the symmetries marked are not needed for the pattern recognition algorithm; for instance, none of the indecomposable glide reflections in Figures 7.9–7.11 are relevant in distinguishing the groups with maximum rotation order 6, 4, and 3.

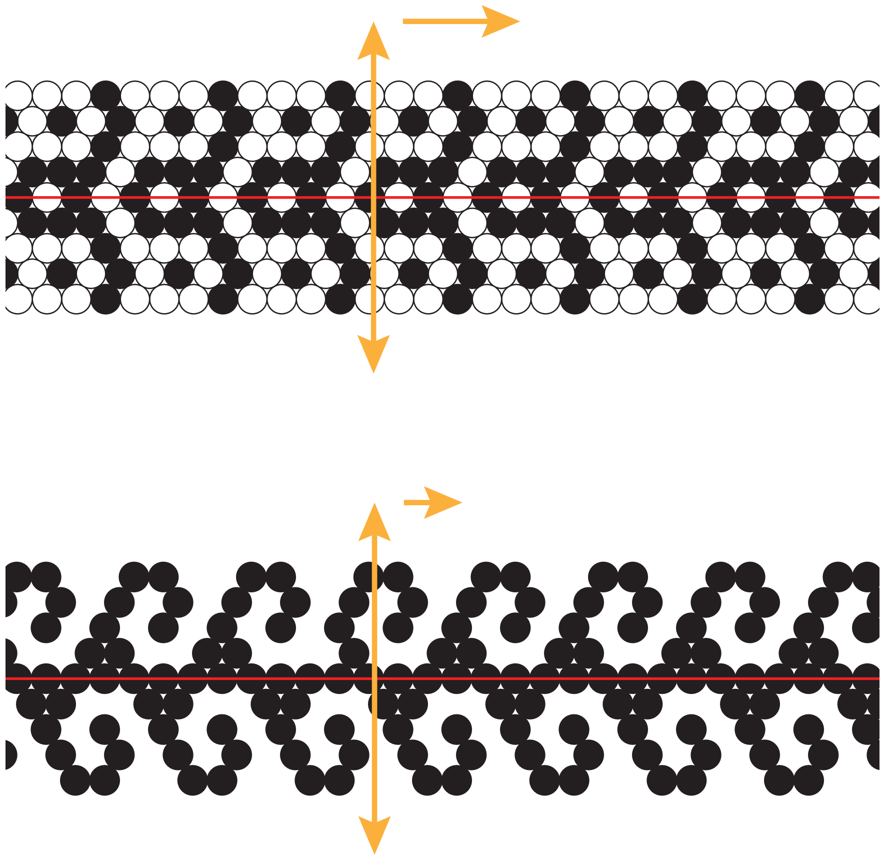


FIGURE 7.7 Decomposable vs. indecomposable glide reflections. The top design has a glide reflection, but it is a combination of a reflection that preserves the design and a translation that also preserves the design and is thus decomposable. By contrast, only performing *both* the marked reflection and the marked translation preserves the bottom design; either symmetry by itself does not preserve the pattern because the hooks will not line up properly. The bottom design therefore has an indecomposable glide reflection symmetry.

The crystallographic notation that identifies the 17 wallpaper groups is closely related to the questions that drive the flowchart. Explaining the notation in full detail is rather complicated (for instance, the rationale behind CMM vs. PMM and P3M1 vs. P31M is tricky), and readers who wish to see a complete account of its intricacies should consult Washburn and Crowe. However, a few salient points are easy to glean from the symbol for a given group. In a symbol with numbers, the largest (often only) number is the maximum rotation order. The letter M, which stands for "mirror" to avoid confusion with R for "rotation," indicates a reflection symmetry, and the letter G indicates an indecomposable glide reflection symmetry. So, for example, P6M has a rotation of order 6 and a reflection, while P2 has a rotation of order 2 but no rotations of higher order, no reflections, and no glide reflections. As you can see by inspecting Figures 7.9–7.15, some of the groups have symmetries that are not included in their symbols; examples include P4M, which contains a glide reflection, P4G, which contains a reflection, and CMM, which contains a rotation of order 2.

Wallpaper Groups in the Bead Plane

Now that we have an algorithm for recognizing the 17 wallpaper groups, we can consider which of these groups are symmetry groups for bead crochet patterns in the bead plane. To begin, let's take a closer look at the geometry of the bead plane itself.

As we have seen since our introduction to this design tool in Chapter 1, the bead plane is a conceptual space—an infinite plane that is extrapolated from the design of a bead crochet bracelet. The conceptual nature of the bead plane extends to our idealized representation of the beads. Crafters who have worked with seed beads before know

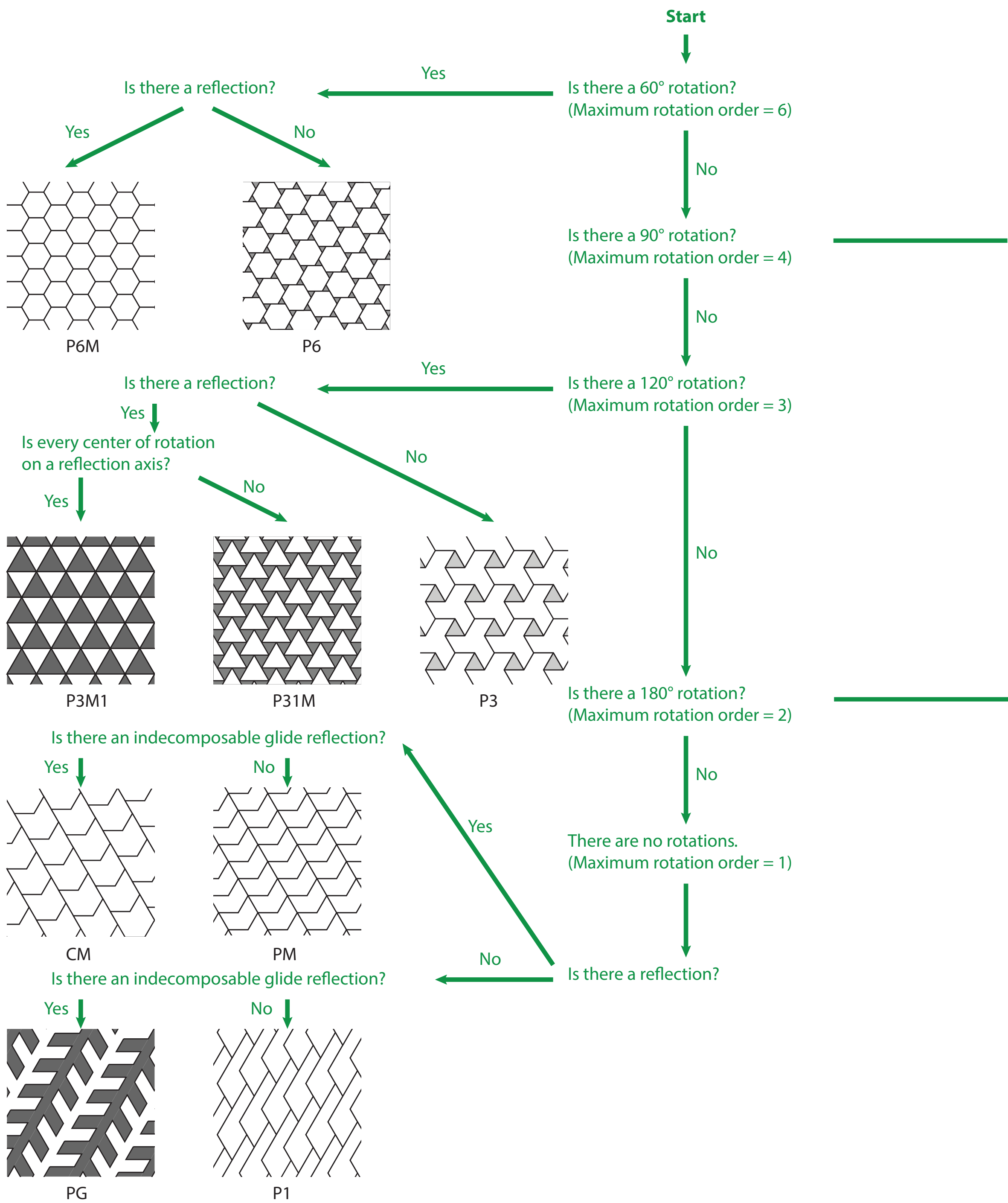
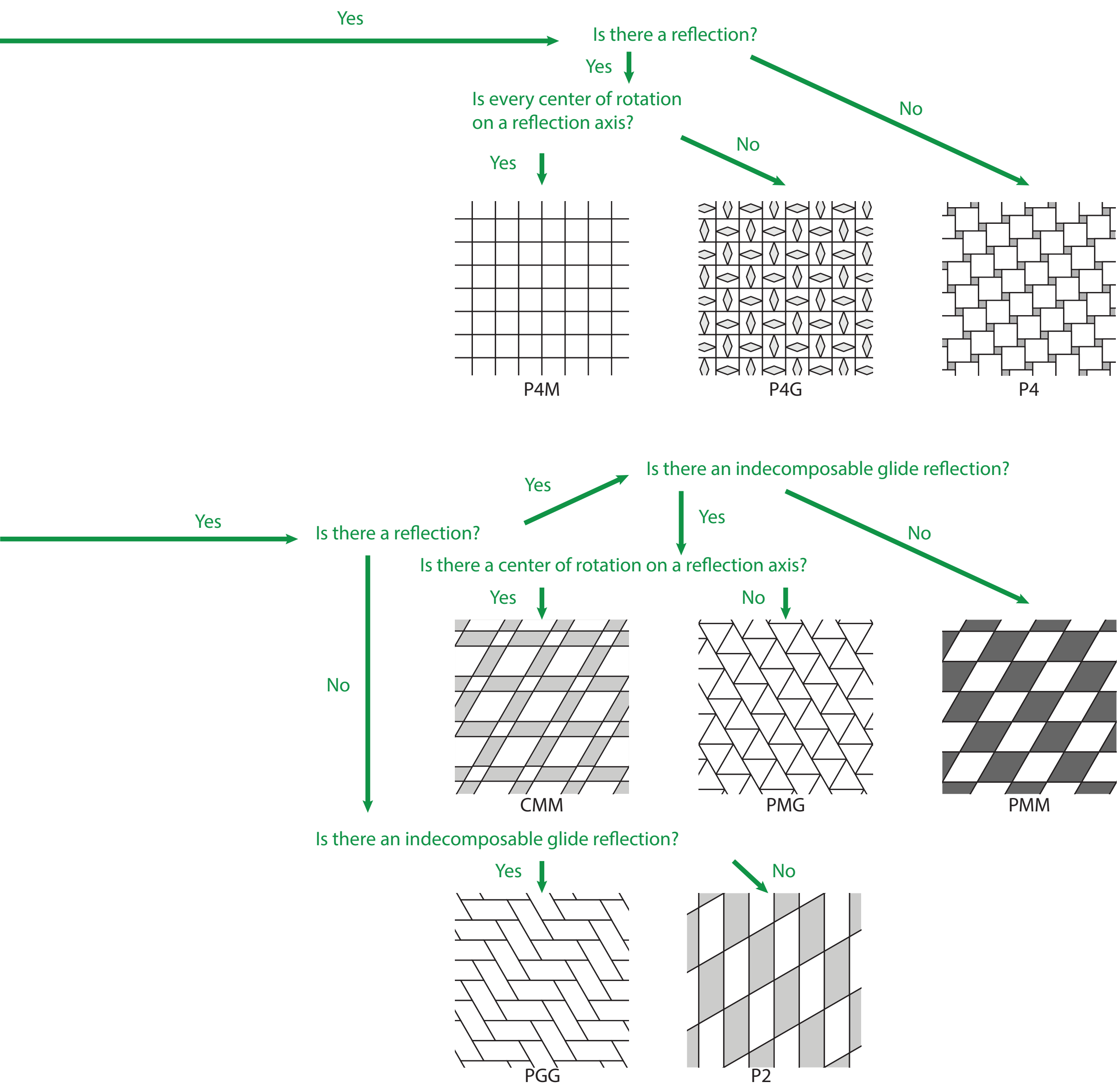


FIGURE 7.8 A flowchart for identifying the symmetry structure of a wallpaper pattern.


Yes
Is there a reflection?
Yes
Is every center of rotation on a reflection axis?
No
No
Yes
P4M
P4G
P4
Is there an indecomposable glide reflection?
Yes
Yes
Yes
Is there a reflection?
No
Is there a center of rotation on a reflection axis?
Yes
No
No
CMM
PMG
PMM
Is there an indecomposable glide reflection?
Yes
No
PGG
P2

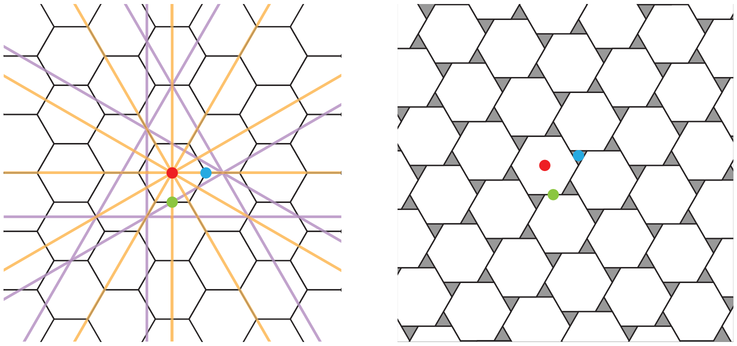


FIGURE 7.9 Wallpaper patterns with maximum rotation order 6: P6M and P6. Centers of 60° rotations are marked in red, centers of 120° rotations are marked in blue, and centers of 180° rotations are marked in green. The orange lines are reflection axes, and the purple lines are glide reflection axes.

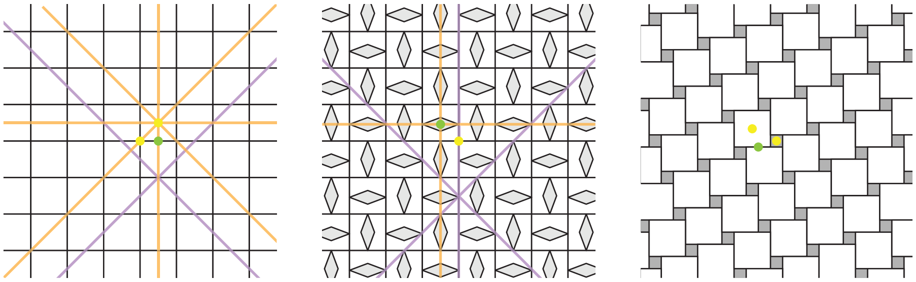


FIGURE 7.10 Wallpaper patterns with maximum rotation order 4: P4M, P4G, and P4. Centers of 90° rotations are marked in yellow, and centers of 180° rotations are marked in green. The orange lines are reflection axes, and the purple lines are glide reflection axes. What distinguishes P4G (middle) from P4M (left) in the flowchart in Figure 7.8 is that the center of 90° rotation does not lie on a reflection axis.

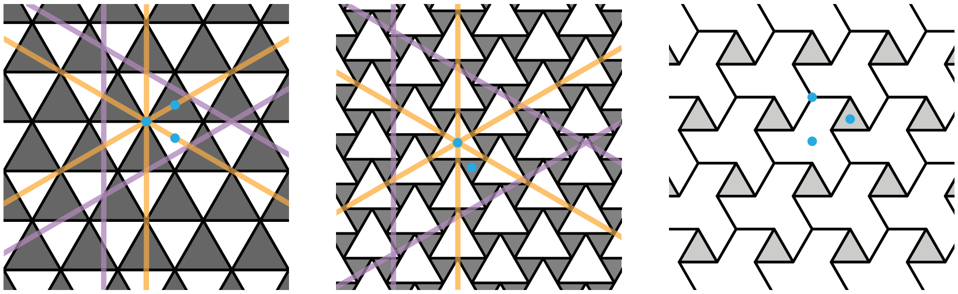


FIGURE 7.11 Wallpaper patterns with maximum rotation order 3: P3M1, P31M, and P3. Centers of 120° rotations are marked in blue, the orange lines are reflection axes, and the purple lines are glide reflection axes. What distinguishes P31M (middle) from P3M1 (left) in the flowchart in Figure 7.8 is the presence of a center of a 120° rotation that does not lie on a reflection axis.

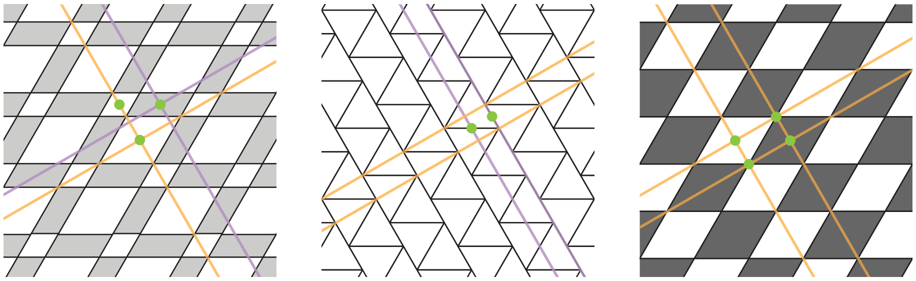
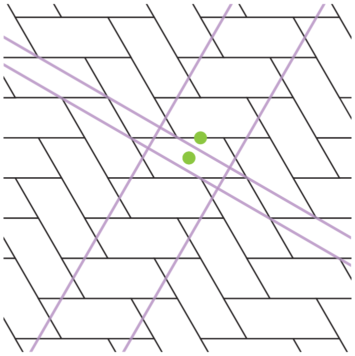


FIGURE 7.12 Wallpaper patterns with maximum rotation order 2 and reflections: CMM, PMG, and PMM. Centers of 180° rotations are marked in green. The orange lines are reflection axes, and the purple lines are glide reflection axes. What distinguishes PMG (middle) from CMM (left) in the flowchart in Figure 7.8 is that none of its centers of rotation lie on a reflection axis (as opposed to a glide reflection axis).



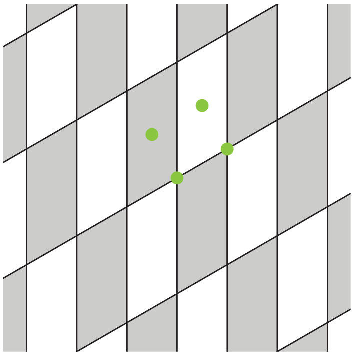
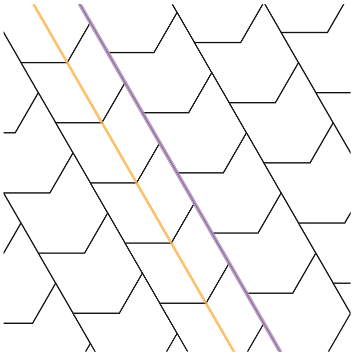


FIGURE 7.13 Wallpaper patterns with maximum rotation order 2 and without reflections: PGG and P2. Centers of 180° rotations are marked in green, and the purple lines are glide reflection axes.



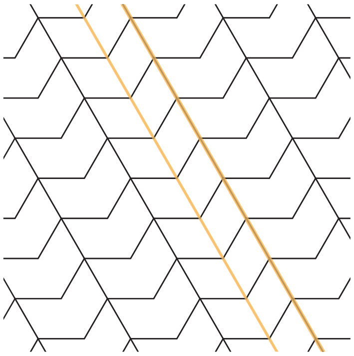


FIGURE 7.14 Wallpaper patterns with reflections and without rotations: CM and PM. The orange lines are reflection axes, and the purple lines are glide reflection axes. The difference between the symmetry groups is that CM has an indecomposable glide reflection and PM does not.

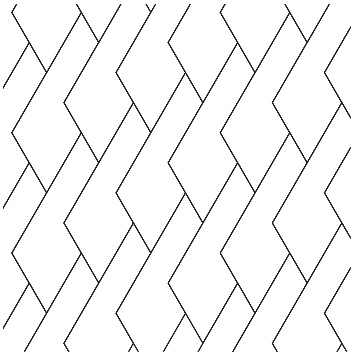


FIGURE 7.15 Wallpaper patterns with neither reflections nor rotations: PG and P1. The purple lines are glide reflection axes. Notice that P1, the simplest of the wallpaper groups, has no symmetries at all except translations.

that the perfect circles we use in our bead plane diagrams are not an accurate representation of the most common shape of seed beads, which are actually slightly flattened on the top and bottom. The middle diagram in Figure 7.16 gives a closer approximation of the profile of a typical seed bead, also known as a rocaille bead. Many of the rotations and reflections of the idealized bead plane, such as the reflection shown on the left of Figure 7.16, are not quite symmetries of this more realistic bead plane because they change the orientation of the beads, as shown on the right of Figure 7.16.

On the other hand, most of the symmetries in the bead plane will not translate into symmetries of an actual bracelet no matter what shape of bead we use, since the bead plane pattern is wrapped up into a tube and then sewn

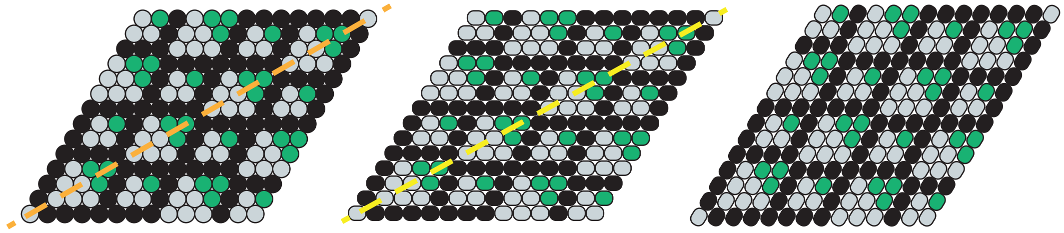


FIGURE 7.16 Idealized bead shapes compared to actual bead shapes. In the grid of perfect circles on the left, this pattern from Shadow Boxes (p. 220) has a reflection symmetry along the orange axis. However, most seed beads have a flattened profile closer to that in the middle diagram. Reflecting the entire diagram around the yellow axis produces the pattern on the right, which is not quite the same as the pattern in the middle because of the bead shape.

FIGURE 7.17 Bracelets from the pattern in Figure 7.16 made from beads of different shapes. The bracelet on the left contains gemstone and metal round beads, which are essentially spherical and have a circular profile. By contrast, the bracelet on the right contains standard glass seed beads with a flatter profile.

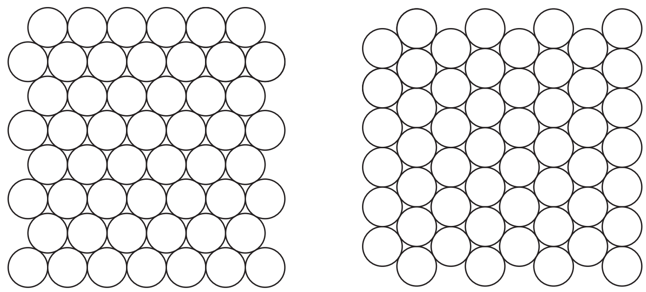


FIGURE 7.18 Why order 4 rotations are impossible in the bead plane. Rotating the bead plane by 90° changes the arrangement of the beads, so no matter how we color in the bead plane, the design cannot be preserved by a 90° rotation.

into a toroidal bracelet. Figure 7.17 shows two bracelets made from the pattern in Figure 7.16, Shadow Boxes: one with spherical beads and the other with rocaille beads. In both bracelets, the reflection axes in the bead plane translate into two spirals that wrap around the bead tube, one through the larger green and silver diamonds in the pattern and one through the smaller diamonds. The remnants of the planar symmetry appear in the reflection symmetry of the individual diamonds in the pattern, which is somewhat more apparent in the bracelet with spherical beads, and in the uniformity of the black gaps between the diamonds.

In answering this chapter's challenge for ourselves, we have chosen to use the idealized bead plane with circular beads in a perfect hexagonal packing. We have done so partly because it is a much more interesting mathematical exercise, and partly because we have found that, as with Shadow Boxes, bead plane patterns with symmetries of the plane that are not symmetries of the bracelet, such as reflections through oblique axes and rotations by 60°, often make compelling bracelets.

Looking over the 17 wallpaper groups, we find that we can cross three of them off the bead crochet list immediately. Without any coloring, the bead plane itself has no symmetries of order 4, because rotating it by 90° changes the arrangement of the beads, as shown in Figure 7.18. Therefore, we cannot possibly make bead crochet patterns for the groups P4M, P4G, and P4.

In an interesting counterpoint, some needle crafts work on a square grid instead of a hexagonal grid, which yields an entirely different set of symmetry constraints. In her chapter, "Symmetry Patterns in Cross Stitch," of the book *Making Mathematics with Needlework: Ten Papers and Ten Projects*, Mary D. Shepherd demonstrates that counted cross stitch can produce all of the wallpaper groups with no rotations of order 6 or 3. There are 12 such groups, and Shepherd includes cross-stitch designs for all 12 groups in photographs taken from her artwork, "Wallpapers in cross stitch."

While rotations by 90° are excluded in bead crochet, all the other types of symmetries in the wallpaper groups—rotations by multiples of 60°, reflections, and glide

reflections—are symmetries of the uncolored bead plane. To determine which of the remaining groups are actually possible in a bracelet, we need to think more carefully about how the bead plane is colored. It should come as no surprise that the major constraint on symmetries in bead crochet patterns is imposed by the hockey-stick translation.

Although it may not be immediately obvious why, the hockey-stick translation forces any bead plane pattern with either a vertical or horizontal reflection axis to be vertically striped: all the beads in each vertical column must be the same color. The cause of this restriction is illustrated in Figure 7.19, which shows the consequences of a vertical reflection on the left and the consequences of a horizontal reflection on the right. At the top, we have a section of the bead plane with a single reflection axis marked and one bead colored in light blue. The hockey-stick translation of the bead plane must preserve the entire pattern, so the translations of the reflection axes in the middle diagrams must also be reflection axes. (For the purposes of this illustration, the circumference is 5, but a similar phenomenon arises in any circumference.) At the bottom, applying the hockey-stick translation and the various reflections (following the arrows from top to bottom in each case) forces the entire column of the light blue bead to be light blue. The same analysis will show that every other column has to have a single color.

With this fact and our wallpaper flowchart in hand, it is not difficult to show that any bead plane bracelet pattern with both a reflection and a 120° rotation is *severely* constrained. In fact, it turns out that the only possible patterns for these groups are those shown in Figure 7.20. A complete proof of this fact appears in our article "Building a better bracelet: Wallpaper patterns in bead crochet" in the *Journal of Mathematics and the Arts*, and if you are interested in the details, we invite you to peruse the paper or, better yet, formulate your own proof.

The groups represented by the patterns in Figure 7.20 are P6M and P3M1. However, there is one other group that contains both a reflection and a 120° rotation, P31M, which leads us to the inescapable conclusion that just like P4M, P4G, and P4, P31M is impossible in bead crochet. We are left with 13 of the 17 wallpaper groups that are hypothetically realizable in bead crochet, though two of them, P6M and P3M1, offer only the very limited design possibilities of Figure 7.20. Figure 7.21 shows sample bracelets for both groups.

Fortunately, the remaining 11 wallpaper groups are all attainable in bead crochet with much greater design flexibility, as demonstrated in Figure 7.22. The artwork photographed here is *Crystallographic Bracelet Series*, a framed piece we made for the Exhibition of Mathematical Art at the national Joint Mathematics Meetings. For each of the 11 wallpaper groups P1, P2, P3, P6, PG, PMG, PGG, PM,

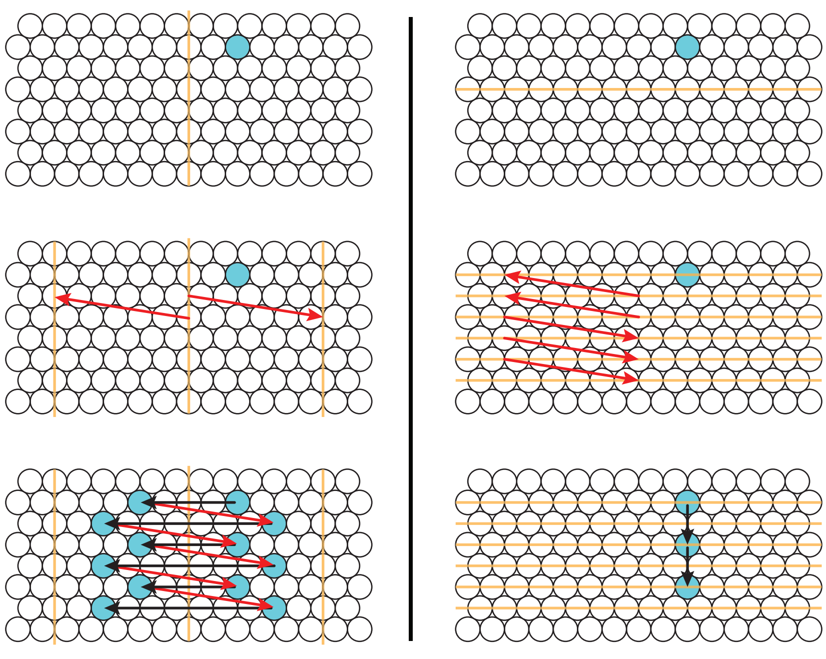


FIGURE 7.19 Constraints caused by vertical or horizontal reflections. In the sequence on the left, we consider the effects of a vertical reflection axis, and in the sequence on the right, we consider a horizontal reflection axis. In each case, the hockey-stick translations (in red) and the reflections (in black) force all the beads in a vertical column to be the same color.

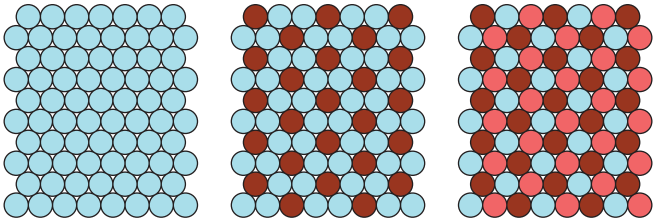


FIGURE 7.20 The only possible bracelet designs whose wallpaper group has a 120° rotation and a reflection. Left to right, these patterns have symmetry groups P6M, P6M, and P3M1, respectively. The second and third patterns are only valid in circumferences of the form 3K + 1, such as 4, 7, and 10.

FIGURE 7.21 Bracelets for wallpaper groups P6M and P3M1. The bracelet on the left appears in the pattern pages as Porcupine Pentagonal (p. 145). It is based on the two-color pattern in the middle of Figure 7.20, but the isolated beads are protruding drop beads in purple, and the background beads are smaller size 11 seed beads in black, making the bracelet appear different than the charted design. The bracelet on the right is a 4-around bracelet in 3mm round beads in three colors following the chart on the right of Figure 7.20.

FIGURE 7.22 *Crystallographic Bracelet Series*, an art piece by the authors included in the Joint Mathematics Meetings Exhibition of Mathematical Art in January 2012. This framed artwork includes a bracelet for each of the bead crochet wallpaper groups except for P6M and P3M1 and swatches of the corresponding bead plane patterns. Nine of these patterns appear in the pattern section (pp. 200–233).

CMM, CM, and PMM (appearing from left to right and top to bottom), there is a segment of a bead plane pattern with that group of symmetries together with the corresponding bead crochet bracelet. We invite you to analyze each bead plane pattern using the flowchart in Figure 7.8 to verify the identities of the different groups. All of the patterns are shown in more detail in the *Journal of Mathematics and the Arts* paper mentioned above, and all of them except those for PM and CMM in the bottom left (which we have replaced with improved bracelet designs) appear in the pattern pages for this book.

Designing Wallpaper Bracelets

The most powerful method for designing bracelets for a given wallpaper group in a given circumference is to evolve a design using symmetries, much as we grew symmetric Escher designs at the end of the previous chapter. Figure 7.23 shows this process in action for a P6 bracelet in circumference 6. In order to have a P6 design, we must have a rotation by 60°. We choose a center for our rotation and color it and its hockey-stick translations green, as in the first diagram. But since the green bead in the middle is a center of rotations by 60° and 120°, rotating this line of green beads around the middle bead shows that all seven of the beads in the second diagram must be green and must also be centers of rotation by 60°. From here on, we gradually fill in new colors, making sure that each time we color a bead we color in all the beads that are symmetrically placed around each of the green centers. After seven steps, we have filled in all of the beads inside the original hexagon of green beads and created a pattern that can tile the entire bead plane. Since the design has a 60° rotational symmetry

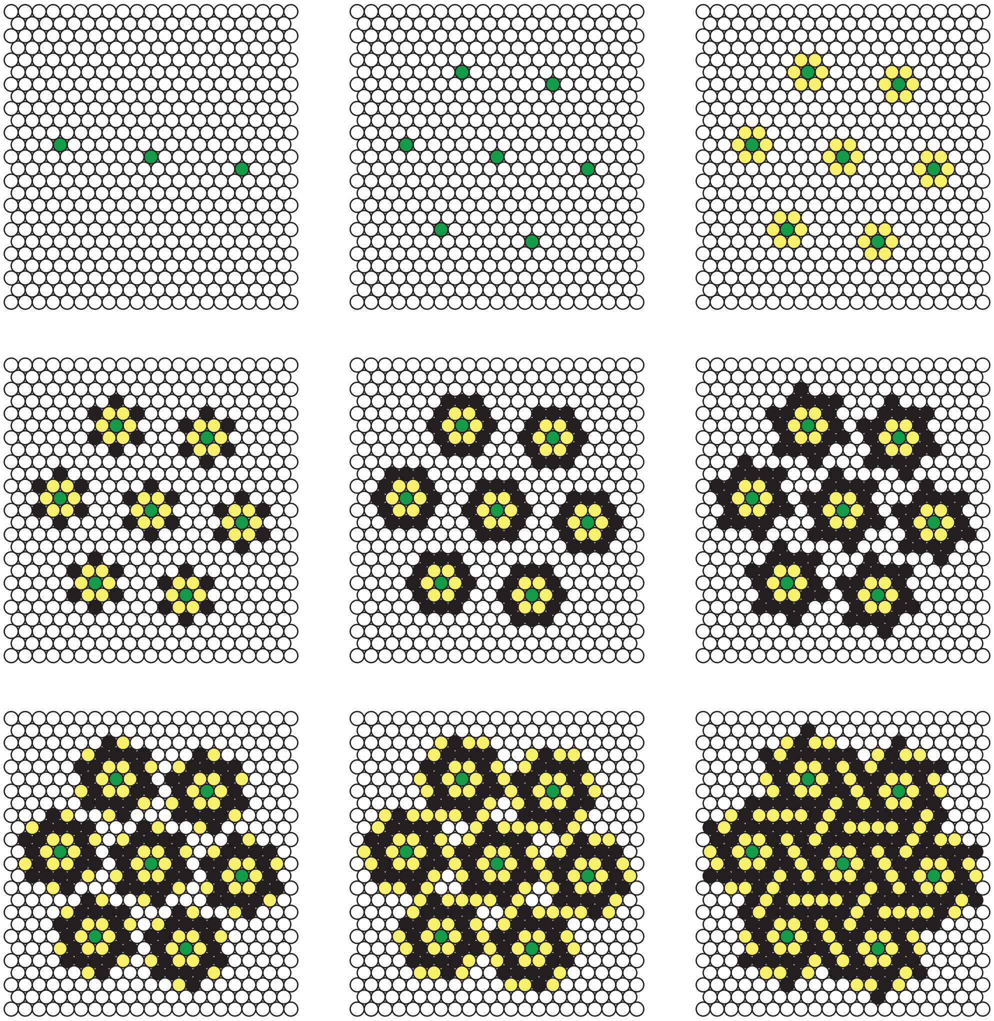


FIGURE 7.23 Evolving a P6 design. At the top left, we begin with a green bead marking the center of a 60° rotation along with its hockey-stick translations for a 6-around bracelet. In the top middle, we apply rotations by 60° and 120° around the middle green bead to determine four more beads that must be colored green. From this point onward, we color in the beads around the green beads using two constraints: the pattern must be the same around each green bead, and rotating by 60° around each green bead must preserve the pattern. This pattern is a variation of Rattan 6-around (p. 234).

and is not one of the P6M patterns from Figure 7.20, its wallpaper group must be P6.

In the diagrams in Figure 7.23, we created our pattern at the same time that we uncovered the constraints forced by a 60° rotation, choosing colors as we went according to our aesthetic judgment. Instead, we could take a more systematic approach, first determining all of the constraints of a given symmetry and making our bead color choices afterward. While the second approach requires us to delay the gratification of seeing a beautiful pattern emerge, it can uncover fascinating geometric and numerical patterns.

For instance, suppose we want to make a bracelet whose bead plane pattern contains a reflection. Since a vertical or horizontal reflection inevitably leads to a vertically striped bracelet, we choose an oblique reflection axis that is 30° counterclockwise from vertical, as shown in Figure 7.24. Once again, we set a circumference of 6 and mark the hockey-stick translations of our reflection axis. We now start with a bead in the bead plane and trace all of the beads that must be the same color under the various reflections and the hockey-stick translation, as in the middle diagram. If we repeat this process and use a new color for each starting bead and its reflections and translations, we get the diagram on the right of Figure 7.24, which serves as a universal template for a bead crochet pattern in circumference 6 with the chosen reflection axis. Beads that are the same color in the universal template must be

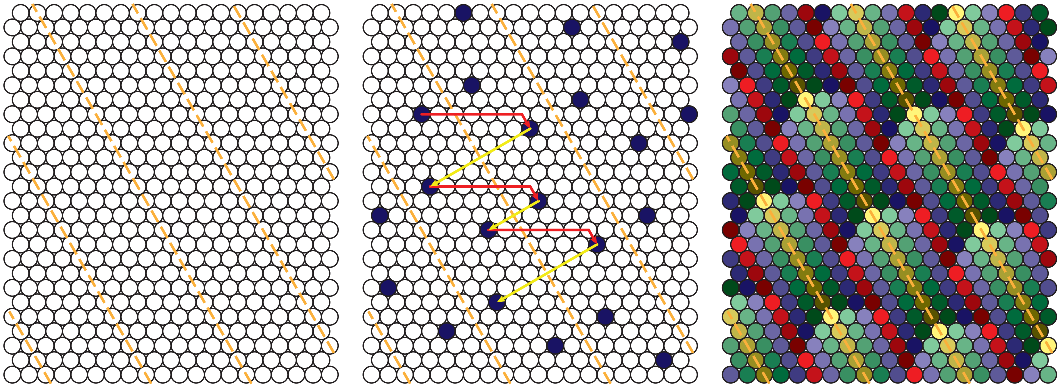


FIGURE 7.24 A universal template for a 6-around bracelet with a left-leaning reflection axis. On the left is the reflection axis along with all of its hockey-stick translations. In the middle, we take a single bead and fill in all of the beads that must be the same color because of the hockey-stick translation (red) and the reflection (yellow). On the right, we have repeated this process with all the other beads to give a color-coded diagram of which beads must be the same color.

the same color in *any* pattern that has the marked reflection! Conversely, if we take the 28 different colors in the universal template and replace them with the colors of our choosing, we will produce a pattern with the marked reflection.

Notice that the universal template in Figure 7.24 has a 48-bead repeat. This is not a repeat length we chose; rather, it was forced upon us by our circumference and the symmetry we desired. Similarly, the repeat length of 43 in Figure 7.23 is completely determined by our choice of circumference and rotational symmetry. This is an indication that symmetry comes at a cost. If we pursue a particular symmetric structure, we lose control over the repeat length for our circumference of choice.

The universal template in Figure 7.24 has no rotations, has a reflection of our choosing, and has an indecomposable glide reflection (with an axis along a line of red beads) that emerges from the bead crochet structure. Consequently, its wallpaper group is CM, and most of the patterns we can make by coloring in the template to our liking will also have wallpaper group CM. However, if we prefer to make a pattern with no indecomposable glide reflections, we can simply match the colors on either side of each red axis to produce a *refl ction* there instead. In fact, as Figure 7.25 demonstrates, we can make a universal template for the corresponding symmetry structure, which gives us a pattern with symmetry group PM. The middle diagram in Figure 7.25 is this universal PM template, and on

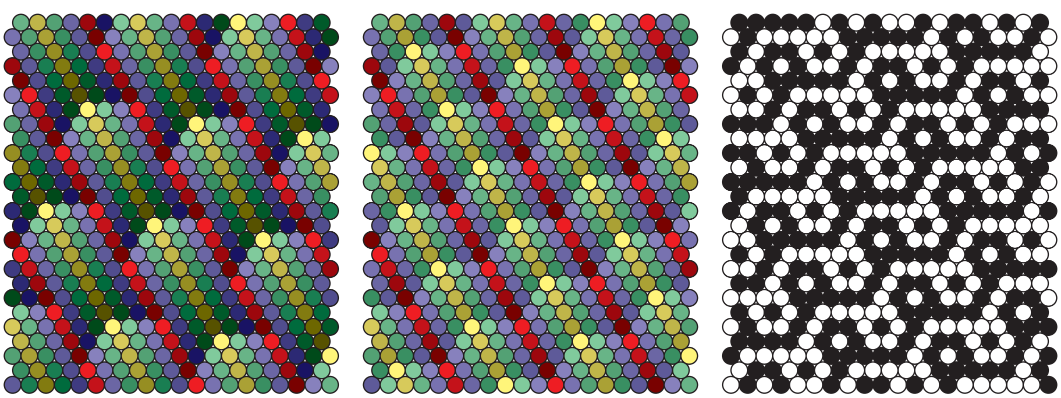


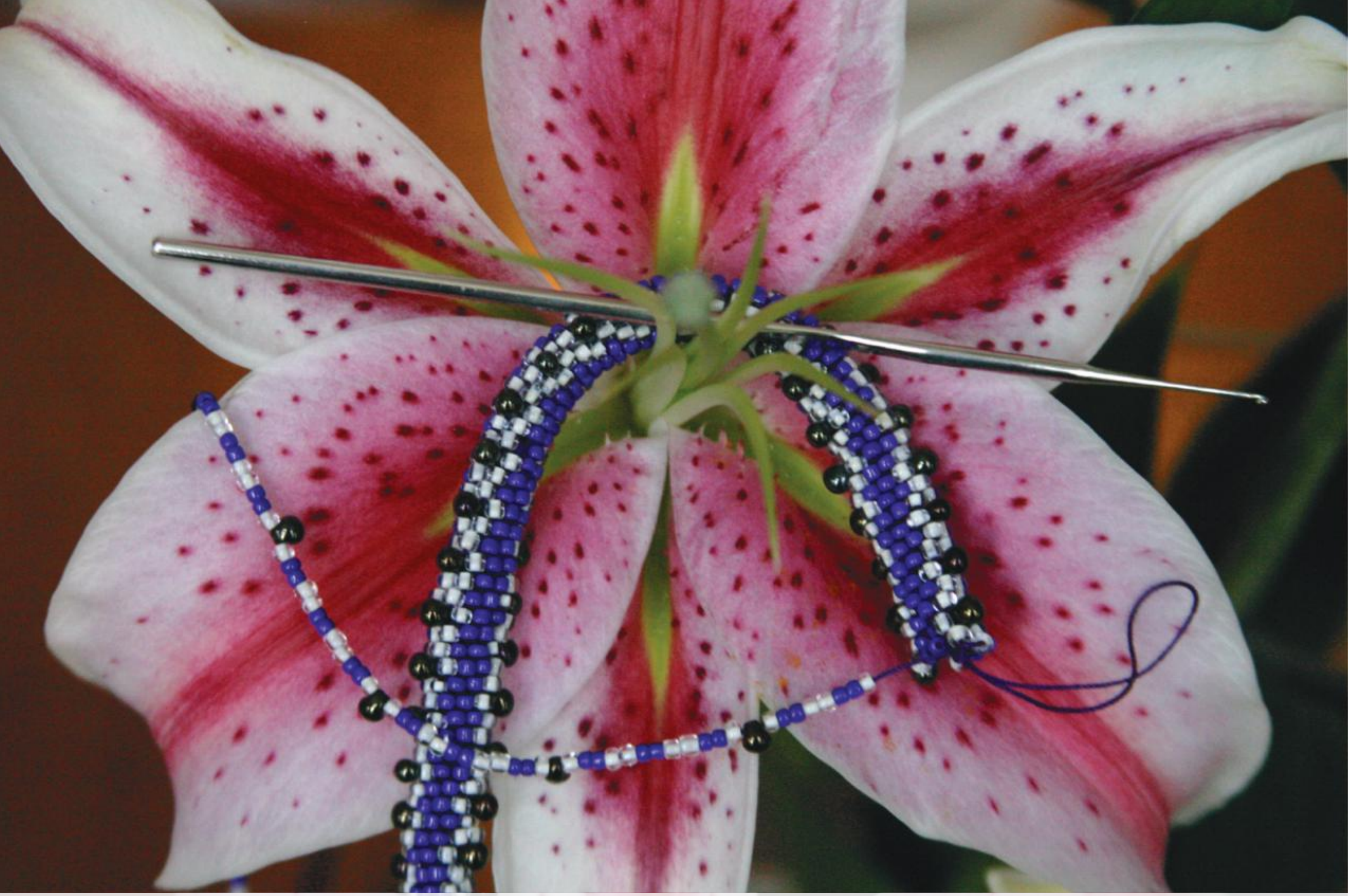
FIGURE 7.25 Constructing a PM bracelet pattern. On the left is the reflection template from Figure 7.24, which corresponds to the wallpaper group CM because it contains an indecomposable glide reflection. A minor modification gives the middle template, which no longer contains an indecomposable glide reflection and thus has symmetry group PM. On the right is the pattern for the Yin-Yang Reflection bracelet (p. 169), derived from the PM template.

the right is a bracelet pattern, Yin-Yang Reflection, created from the template. As a bonus, Yin-Yang Reflection is also an Escher design.

As you might imagine, this is only the beginning of the wallpaper bracelet story. There are all sorts of little tricks that go into making bracelets for groups like PG and CMM. There are fascinating numerical patterns in the repeat lengths of bracelets with different circumferences and reflection axes. There is the question of which wallpaper groups can support Escher designs. We encourage you to explore these ideas by making your own wallpaper bracelet designs and templates and asking your own questions, and we hope that you find the journey as intriguing as ours has been.

PART THREE

Instructions and Patterns



CHAPTER 8

How to Bead Crochet

The process of making a bead crochet bracelet involves five main steps: pattern selection, stringing, starting, crocheting, and closing. In this chapter we will go through each of these steps in turn to give you the skills you need to use the patterns in this book—or better yet, to use the principles from the previous chapters to design and make your own bracelets! Even if you already bead crochet, we hope you will find some of the tips and techniques here a useful supplement to your crafting skills.

Bead crochet is a challenging skill to teach yourself, and the best way to learn is to find a class at a local bead shop or persuade a friend who already bead crochets to show you the ropes. Fortunately, even if you don't have access to these opportunities, there are many resources besides this book to help you pick it up on your own. There are numerous demonstration videos online, including a set of supplementary videos for *Crafting Conundrums* that you can find through the publisher's website (http://www.crcpress.com/product/isbn/9781466588486). There is also a lovely way to practice bead crochet at a larger scale before attempting to make a bracelet with seed beads (which generally range from 1 to 4 millimeters in diameter) and a tiny crochet hook. We strongly recommend that if you are teaching yourself to bead crochet, you acquire some large plastic craft beads, called *pony beads*, which are available at most major craft stores, along with the other supplies listed below. In fact, because these larger beads are also much easier to photograph clearly, most of the demonstration photos in this chapter show bead crochet with large pony beads. Marian Goldstine was kind enough to take all of the pony-bead crochet photographs for us.

Before we delve into the bracelet-making process, here is a quick checklist of the supplies that you will need no matter what type of beads you are using:

- Beading needle for stringing
- Sewing or tapestry needle for closing
- Crochet hook
- Thread
- Beads
- Scissors

As we discuss the bead crochet process, we will talk about how to choose the size and type of the beads, needles, string, and crochet hook. Different-sized tools and materials affect the ultimate appearance of the bracelet, and selecting the thread, beads, and crochet hook size are an important part of the creative process.

If you are attempting your first bead crochet piece and would like to make the sample in large plastic beads that is demonstrated here (and whose pattern is given at the end of the section on pattern selection), you will need the following supplies:

- 9mm plastic pony beads in five colors, about 40 beads of each color.
- Thick yarn (labeled as Aran weight, or as using a crochet hook of size I, J, or K), preferably in white or a light color. You only need about 12 yards of yarn, so any ball or skein you can buy will have enough.
- A size I, J, or K crochet hook.
- A yarn needle with an eye small enough to fit through the beads. You will use this for both stringing and closing.
- Scissors.

Because both of the authors and the majority of readers are right handed, the photographs here demonstrate standard right-handed crochet, in which the right hand holds the hook and makes all of the stitches while the left hand holds the bracelet and thread. However, the exact same method can be performed in left-handed crochet. Here is how the instructions in this chapter are modified for left-handed crocheters:

- Stringing is *precisely the same* as for right-handed crochet. If you are reading the beads off of a chart, you read them from left to right and bottom to top *no matter which hand you crochet with*.
- Starting, crocheting, and closing are exactly mirror reversed from the techniques shown in the photographs. As far as possible, we have written the instructions in a hand-neutral way, but the images will all be reversed from left to right. (Front, back, top, and bottom stay the same.) While right-handed crochet will add beads counterclockwise (as viewed from the top), left-handed crochet will add beads clockwise, and the final bracelet will be the mirror reverse of its right-handed counterpart.

Pattern Selection

The most important feature of a design from a practical viewpoint is its circumference. Not only will the circumference determine how many beads you need to string and

crochet—larger circumference, more beads—but it will also determine what kind of beads you use and impact the flexibility of the final bracelet. We have more precise information about which beads work for which circumferences in the stringing section, but the general principle is that larger circumferences require smaller beads. As a general rule, larger circumferences also create stiffer bracelets, so if you are looking for a more supple form you will do best with circumferences 7-around and under until you have a sense of how tightly you crochet with different hooks and thread types.

After circumference, the next most important aspect of a bracelet pattern is the repeat length. Bracelets with shorter repeats are generally easier to string and easier to size, since you have more options for the total bracelet length with a short repeat than with a long one. While you consider repeat length, it is also worth looking at the complexity of the color sequence in a repeat; the more complicated the numerical pattern and the more colors appear, the more you will have to concentrate to make sure you string the beads in the right order. (There are tricks for checking your stringing even for long, complex repeats, as described in the stringing section.)

The pattern pages in this book are designed to make it easy to find the circumference and repeat length of each pattern along with suggestions about bead choices, twist choices, and sizing. On p. 133, we have a guide to how this information is presented on the pattern pages.

Our information on bracelet sizing is always approximate, since there are many factors that make size information vary sharply from crafter to crafter and project to project. First of all, bead sizes are far from precise. While there is a standard sizing system that is used across bead manufacturers, the actual dimensions of beads of a given size fluctuate between bead makers. As if that weren't enough of a confounding factor, how loosely or tightly you crochet also affects the length of your overall bracelet, as well as the flexibility of the bead crochet rope. The tightness or looseness of your crochet and the corresponding size of your stitches is known as the crochet *gauge*. Finally, the sweet spot of sizing, a bracelet small enough to hug the wrist without slipping off, but big enough to roll over the hand with gentle stretching, is actually fairly broad for most people. This means that there is a range of lengths that each person can comfortably wear.

As an illustration of why bracelet sizing is finicky, consider the three bracelets in Figure 8.1, which were all made by the same person from the same pattern in beads that

FIGURE 8.1 The reason sizing information is imprecise. Here are three bracelets made from the same pattern by the same person with size 8 seed beads and 3mm silver rounds. Although the bracelets are the same size, they have 18, 17, and 16 pattern repeats, respectively, because of differences in bead measurements and crochet gauge. The pattern is a simple variation of the 4-around hexagonal tiling on p. 12, with a silver bead in the middle of each hexagon.

are officially the same size. As you can see in the photograph, these bracelets are all roughly the same length. However, each was made with a different number of repeats. The bracelet on the left and the bracelet in the middle were both made with a size 8 crochet hook on heavy-duty topstitching thread, but because the first bracelet contains smaller beads than the second, the first bracelet uses 18 repeats while the second bracelet only uses 17 repeats. The bracelet in the middle and the bracelet on the right use exactly the same beads, but because the bracelet on the right uses thicker crochet thread and a larger size 6 hook to get a larger gauge, it uses only 16 repeats to achieve the same bracelet size.

The best way to deal with this uncertainty is to try some patterns with short repeat lengths and see how your gauge compares to ours. Then you can either adjust our sizing notes to account for how much smaller or larger your stitches are than ours, or you can try to adjust your gauge by holding your thread more loosely or tightly or by changing to a different hook size. In general, the larger the hook, the looser the stitches. However, some people find that a smaller hook eases the struggle to push the hook into each stitch, allowing them to keep their hands more relaxed and loosen their stitches. If you find that your bead crochet is especially tight, then another way to loosen it up that works nicely for smaller circumferences is to crochet

around a toothpick, as described later in the section on starting a bracelet.

If you are just beginning to bead crochet, or if you are trying our patterns for the first time and are unsure of how your gauge compares to ours, then we suggest you begin with one of the patterns listed below. The first three are particularly good for novice bead crocheters, because they are 5-around bracelets and use larger beads that are easier to see and manipulate. The remaining patterns are circumference 6 or 7 and all have short, simple repeats for easy stringing and flexible sizing.

- Trefoil Dissection, p. 160
- Naptime, p. 171
- Herringbone Reflection in 5-around, p. 176
- Escher (P,2) Torus Knot, p. 161
- Tricolor Zigzag, p. 173
- Four-Color Sawtooth, p. 174
- Sophie's Herringbone, p. 175
- Aztec Wave, p. 184
- Googly Eyes, p. 217
- Shadow Boxes, p. 220
- Brick Walkway, p. 222
- Bon Bon Checkerboard, p. 223
- Caged Zigzag, p. 224
- Triangle Twist, p. 227

For your very first bead crochet project (or your first project in small beads), we strongly recommend that you string a 5-around bracelet in five colors by stringing the colors in cyclic order: ABCDE, ABCDE, ABCDE, etc. Crocheting this pattern will create five colored spirals around your crochet rope and is particularly easy to work with because the color of the bead you are stitching onto the rope will always be the same as the color of the bead where you insert the crochet hook, so it's always clear where your next stitch goes. If you are making the five-spiral pattern in seed beads, try using size 8 seed beads, which have the virtue of being larger than the size 11 seed beads we most commonly use, and string around 90 beads of each color or 450 beads in total. As mentioned at the beginning of the chapter, the pony-bead bracelet shown in the photographic demonstration that follows takes 40 beads of each color or 200 beads in total.

Stringing

Most of the beads we use for bead crochet bracelets are glass *seed beads*, which have the virtue of coming in a variety of colors, finishes, and sizes. The sizes of seed beads are given numerically—the larger the number, the smaller the size. For instance, the seed beads we most commonly use are size 11 seed beads, which are roughly 1.5mm in diameter.* This size is also commonly denoted as size 11/0 or size 11^0, but we will stick with the simpler numerical notation, which is prevalent among retail bead suppliers. We also use size 8 seed beads, which are about 2.5mm, and size 6 seed beads, which are about 3mm. The other sizes of seed beads that are readily available are size 15, which is smaller than warranted for the bead crochet patterns in this book but can be used for designs with very large circumferences, and size 10, which is less common and which we use only occasionally. The beads in the top row of Figure 8.2 are sizes 11, 11, 8, and 6, respectively.

The bead sizes and measurements above are for standard glass seed beads in the rocaille form, which looks more or less like a bagel. However, there is a separate class of seed beads called precision cylinder beads that have completely flat sides with no rounding at the top and bottom. The most common type of precision cylinder bead is the *Delica*, and since this is the name that you are most likely to see when you are bead shopping, we use Delica as the generic term for precision cylinder beads. However, Toho also makes a precision cylinder bead called Aiko, and Toho Aiko beads can be used interchangeably with Delicas and are exactly the same size. Confusingly, although Delicas use the same size numbers as standard rocaille seed beads, they are substantially smaller. In the bottom left of Figure 8.2 are Delicas in sizes 11 and 10, and you can see that the size 10 Delicas are closer in size to the size 11 rocaille beads above them. As the term "precision cylinder" implies, Delicas are extremely uniform in size, and this regularity together with their smaller size make them an excellent choice for bracelets in circumference 8 or higher.

In addition to seed beads and Delicas, there are also small round beads in metals and gemstones. Metal round beads are relatively easy to string, combine nicely with seed beads, and give bracelets an extra bit of luxury. We often use sterling silver or gold-filled beads in our designs, and we find that 2mm round metal beads mix well with size 11 seed beads and size 11 Delicas, as do 1.8mm and 2.5mm rounds,

* The bead measurements we give here are from the website of Fusion Beads, a great all-purpose online bead store, at http://www.fusionbeads.com/Seed-Bead-Q-and-A.

FIGURE 8.2 Beads of various sizes, types, and finishes. Left to right and top to bottom, the beads are size 11 color-lined glass seed beads, size 11 transparent glass seed beads with a matte finish, size 8 opaque glass seed beads with a matte finish, size 6 opaque glass seed beads, size 11 Delicas with a luster finish, size 10 semi-matte silver-lined Delicas, 3.4mm drop beads with an AB (iridescent) finish, 2mm sterling silver rounds, and 2mm black onyx rounds. Above the beads are two stringing needles, a collapsing-eye needle at the top and a big-eye needle beneath it.

with different textural effects. We use gemstone beads more sparingly, not only because they are harder to find in small sizes and more expensive, but also because their sizing is less reliable and their holes are often so small that it is very hard to string them on thread thick enough for bead crochet. In the bottom right of Figure 8.2 are 2mm sterling silver rounds and 2mm black onyx rounds.

Finally, some of our designs, like the Möbius band bracelets on pp. 150–151, use drop beads (also called fringe beads), which are shaped like teardrops and jut outward from the core of a bead crochet rope. The most common size of drop bead, and the one used in our designs, is the 3.4mm drop bead, shown in the bottom center of Figure 8.2.

In addition to size and color, it is important to pay attention to the finish on your beads. Many beads have shiny finishes, and the glittering highlights can obscure intricate color patterns. If you are crocheting a pattern such as a multicolor Escher bracelet where the shapes of the colored regions are an important feature of the pattern, it is best to ensure that all or most of your beads have a matte finish. Unfortunately, since sparkly beads are flashy and sell better, bead shops tend to carry more glossy beads than matte beads, so we often supplement our bead store purchases, where we can inspect the beads in person and be sure what we're getting, with online shopping, where it is harder to tell what the beads look like but the selection is much wider. We are also guided by bead weaver Gwen Fisher's observation that bright highlights on dark beads are especially confusing to the viewer's eye, so when we do incorporate shiny beads, we usually choose them for the lighter colors in our designs. Glittery beads can also confuse the eye while crocheting, so matte colors are a good choice for the novice bead crocheter as well.

Naturally, since this step of the process is stringing, you will also need some string. For most of our bracelets, we use heavy-duty topstitching thread, which is thin enough to pass through all of the beads described above (with the possible exception of some gemstone rounds) but strong enough hold up to the wear and tear of bead crochet. Along with many other bead crochet practitioners, we favor Gutermann topstitch thread, a high-quality thread that is available at many major fabric stores. The thread is not cut from the spool until after the entire bracelet is crocheted, and it is important to make sure your spool has enough thread on it before you begin. A spool of Gutermann's thread is generally enough to make two bracelets in 6- or 7-around with size 11 seed beads, but a higher circumference bracelet in Delica size 11 beads can use up more than half the spool. For higher circumference bracelets, topstitching thread is too thin to support larger beads. Here we follow bead crochet expert Linda Lehman's lead and turn to size 20 crochet cotton, which is thicker than topstitching thread but still thin enough to pass through seed beads of sizes 11 and larger as well as metal round beads and drop beads. Size 20 crochet

cotton is harder to find than topstitching thread, so unless you have a local store that specializes in crochet, you will probably need to order it online.

To help you choose the appropriate bead and thread size for your pattern, here is a table showing which circumferences we have found most feasible for the different types of beads and thread. Since different people crochet at different gauges, your results may vary, so please experiment to see what works best for you. We should also point out that other bead crochet crafters make bracelets and necklaces in much larger circumferences than we do. Some of them use an alternate method of bead crochet that replaces slip stitches, as described in the method outlined below, with a different stitch known as single crochet that produces a more solid fabric tube inside the beads. All of our patterns and design techniques work with both types of stitch.

Bead Size and Type	Thread	Circumferences
Size 6 seed beads	Size 10 or 20 crochet cotton	4–6
Size 8 seed beads	Size 20 crochet cotton or topstitching thread	4–6
Size 11 seed beads	Size 20 crochet cotton	8–10
Size 11 seed beads	Topstitching thread	4–7
Size 11 Delicas	Topstitching thread	8–11

While the thread is mostly hidden inside the bead crochet tube, it is visible around the edges of rocaille and round beads and can have an effect on the final appearance of a crochet piece. In most cases, we pick either a neutral thread color or a thread color that coordinates with our beads. It is much harder to see what you are doing if you work with dark thread, so it is best to stick with light colors, especially when you are just starting out.

The last thing you will need to string your beads is a stringing needle. Most seed beads are small enough to require a specialty needle that can accommodate the thread while still being thin enough to pass through the beads. The two most common types of stringing needles are shown at the top of Figure 8.2. For most purposes, we recommend a big-eye needle, which is a long, straight strip of metal with a slit down the middle that can be opened to insert the thread and then released. Big-eye needles will pass through all but the small-holed gemstone beads and are as easy to manipulate as a conventional sewing needle. If you have beads with smaller holes, then you will need to switch to a collapsing-eye needle, which is thinner and more flexible—and consequently somewhat more challenging to handle while you string.

Once you are ready to string, you will need a clear surface, good lighting, and your pattern all at hand. A piece of felt, foam, or other soft fabric is useful so that you can arrange your beads on a flat surface to pick them up easily without having them roll away. A small toaster-oven size tray lined with soft fabric that you can hold on your lap is also useful. You may find it helpful to rewrite the numerical stringing pattern using colors that match the beads you are actually stringing to make the pattern easier to follow. It is exceedingly important that you string the beads in the correct order to produce the design you want, so just as wood workers measure twice and cut once, you want to check your stringing very carefully before you begin to crochet. Placing removable markers such as bits of paper, string, or pipe cleaner between repeats helps keep track of the stringing order and how many repeats you have already strung. Markers can also help you directly compare the color sequence in different repeats as shown in Figure 8.3, which is an easy way to check for errors in complicated patterns. For some relatively short repeats, creating a row of small bead piles that exactly matches your pattern sequence can simplify the stringing process by enabling you to pick up beads in order without having to think about the pattern.

FIGURE 8.3 Proofreading patterns with long repeats. In this strand, segments of supermarket twist ties separate the repeats. Lining one repeat up against another makes it easy to check that they have the same bead color sequence without counting beads.

If the unthinkable happens and you discover a stringing error as you are crocheting a bracelet, there are a couple of options for correcting the mistake without tearing out your crochet and starting over. Sometimes, the error can be corrected by removing a small number of beads. If none of these beads are metal, then you may be able to break off the offending beads with a pair of pliers. When trying this approach, be careful not to damage the thread as you crack the beads you are discarding. Wrapping the bead in a soft cloth before applying the pliers can help avoid thread damage, as can practicing your cracking technique on spare beads and thread first. Alternately, you can stop crocheting, cut your thread (being sure to leave a tail a few inches long), correct the stringing error, and then join the newly severed thread to your bracelet using the method on p. 127. While this does make crocheting more complicated during the joining process, this correction technique allows you to repair more convoluted stringing errors and only requires the standard tools for bead crochet.

Unless your pattern has a particularly long repeat, it is worth stringing an extra repeat or two, since it is easier to stop crocheting before you reach the end of your beads than to add more beads to a strand in progress. If you reach the end of your beads and you find that your bracelet is shorter than you'd like, you can cut your thread and string the beads from the other end. In this situation, you need to be careful of two things: unwind a generous amount of thread before you cut so that you have enough to finish crocheting, and be sure to string your beads *in the reverse order from the pattern*, since you are now stringing them from the opposite end of the thread.

Starting

Now that your beads are strung, you are ready to begin the crochet process. This means that it is time to decide which size hook you will use. As noted above, for bead crochet with large plastic beads, you will use a size I, J, or K hook, but the crochet hooks for bracelets with seed beads and thread are much smaller and measured on a different size scale, using numbers instead of letters. As with seed beads, the larger the size number, the smaller the hook. We recommend that new bead crocheters start with a size 11 (1.1mm) hook for topstitching thread or a size 9 (1.5mm) hook for size 20 crochet thread, then adjust the hook size once you have a sense of your gauge. Optimal hook size varies a lot from person to person; one of us uses a size 11 hook and the other uses a size 8 hook to get roughly the same gauge.

The fine details of hand positioning can vary a lot from person to person, and the illustrations here represent just one approach. What is important to focus on in these instructions is how the beads and thread are formed into stitches. With practice, you will discover precise hand motions and positioning that are most comfortable and efficient for you. In the beginning, however, getting the thread and beads to do what you want can feel quite awkward.

Throughout the instructions that follow, we describe the string supporting the beads as "thread," even though the photographs depict yarn, since thread is what you will use for most of your bead crochet projects.

Here is how you make your initial crochet chain (Figures 8.4–8.27). While we show the process for large plastic beads, at the end of this section we give a tip for transitioning to smaller beads.

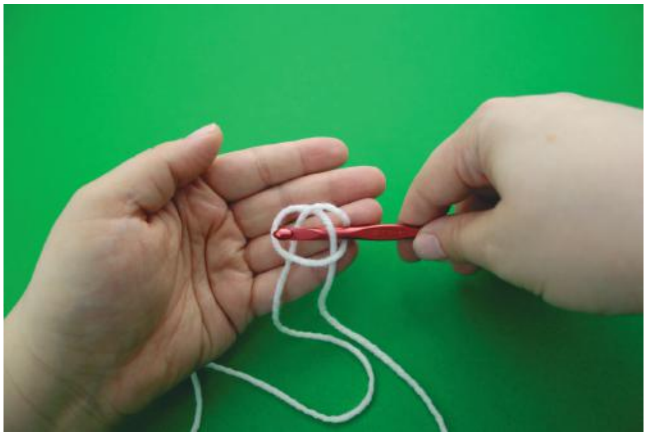


FIGURE 8.4 Begin a slip knot by making a loop in the thread and laying the tail end of the thread (on the right) under the loop. Stick the crochet hook under part of the tail of thread that is underneath the loop.

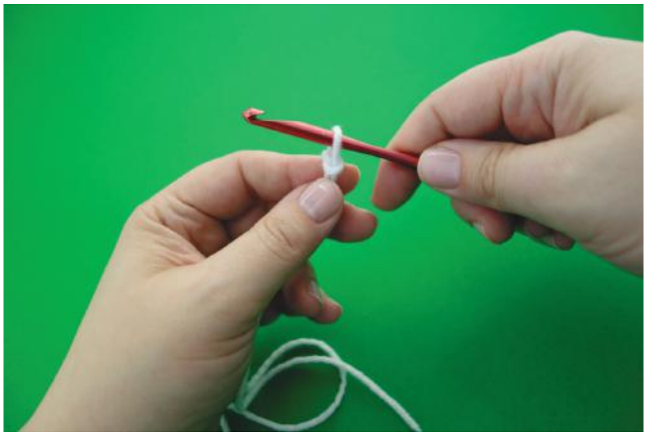


FIGURE 8.5 Pull both ends of the rope to tighten the slip knot.

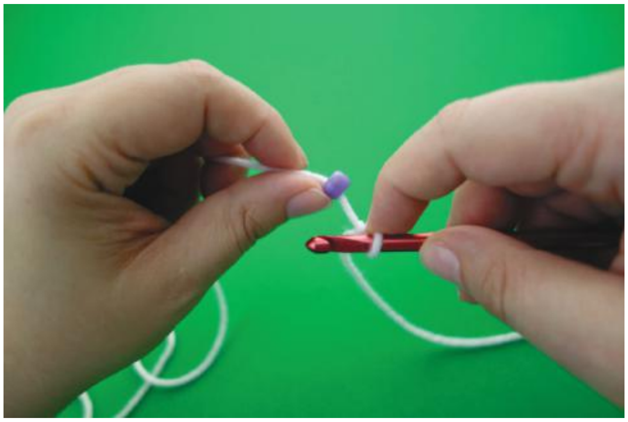


FIGURE 8.6 Start the initial bead chain by pulling the first bead all the way down to the slip knot.

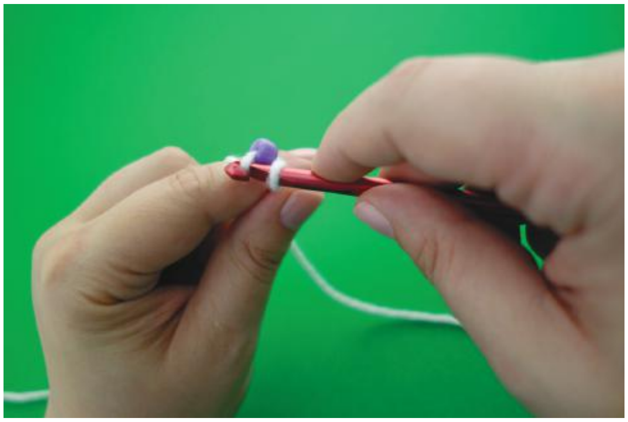


FIGURE 8.7 Keeping the bead close to the slip knot, wrap the thread from the back of the crochet hook to the front, trapping the first bead between the thread in the hook and the slip knot. You will always wrap your thread around the hook from back to front.

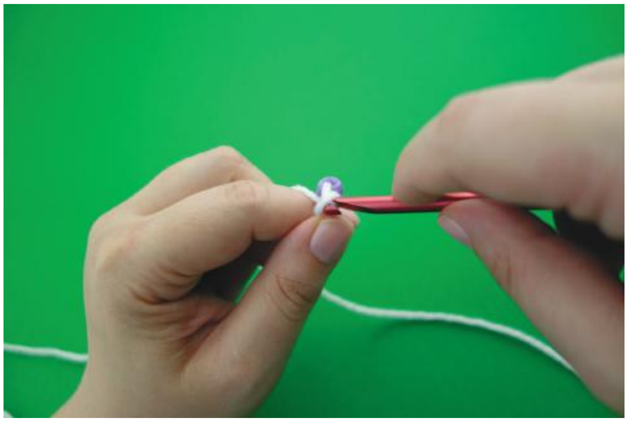


FIGURE 8.8 Pull the thread through the loop on the crochet hook.

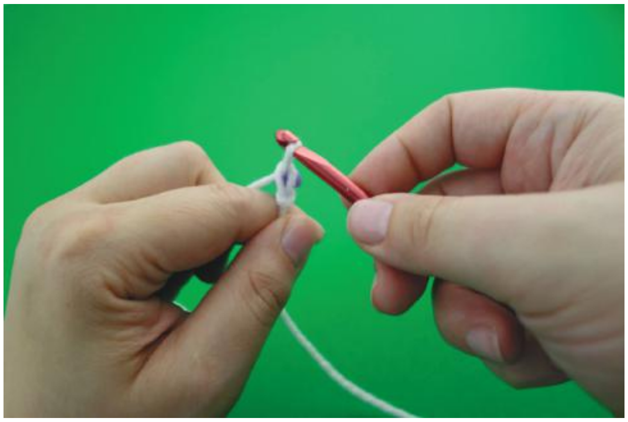


FIGURE 8.9 You have completed your first chain stitch. You should have a single loop of thread on your crochet hook, and the yarn will form a V shape coming out of the slip knot.

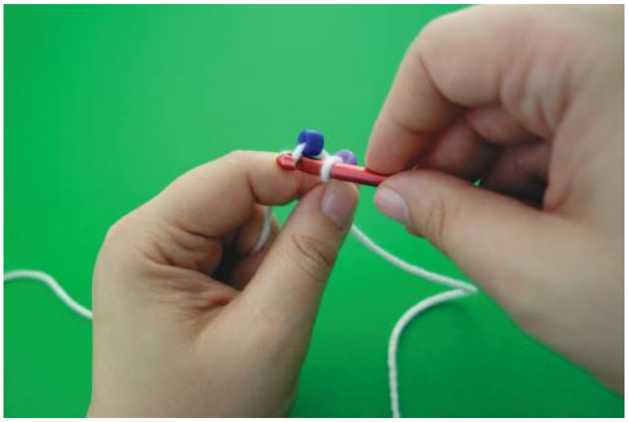


FIGURE 8.10 Repeating the steps from Figures 8.6 and 8.7, pull the second bead down next to the crochet hook and wrap the thread around the crochet hook.

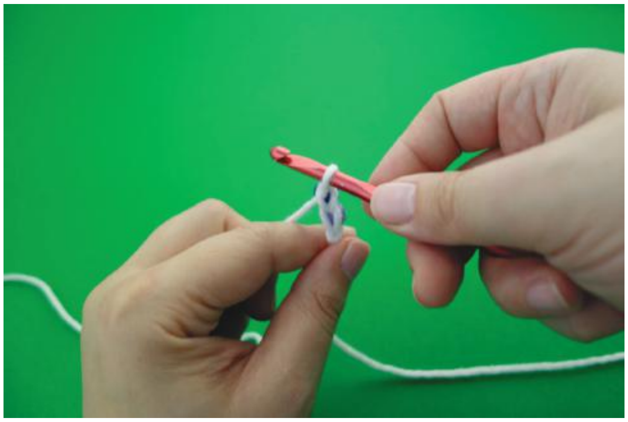


FIGURE 8.11 Repeating the step from Figure 8.8, pull the thread through the loop on the crochet hook to create the second chain stitch. Continue with the remaining beads in the initial chain, pulling each down, wrapping the thread around the hook, and pulling the thread through the loop that is on the hook.

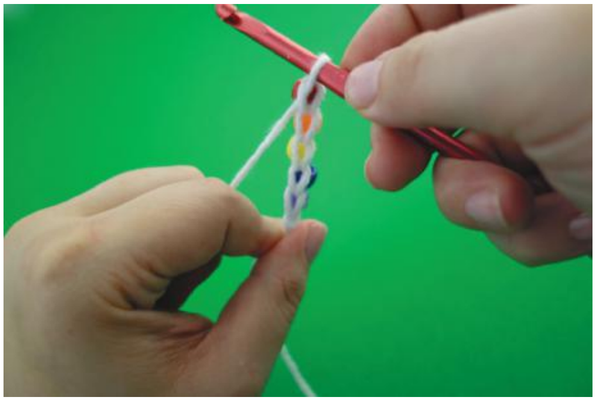


FIGURE 8.12 When you have as many beads on your chain as the circumference of your bracelet (which is 5-around in this example), you have completed the initial chain. The thread should form a line of V shapes as shown here, with all the beads on the same side of the V's.

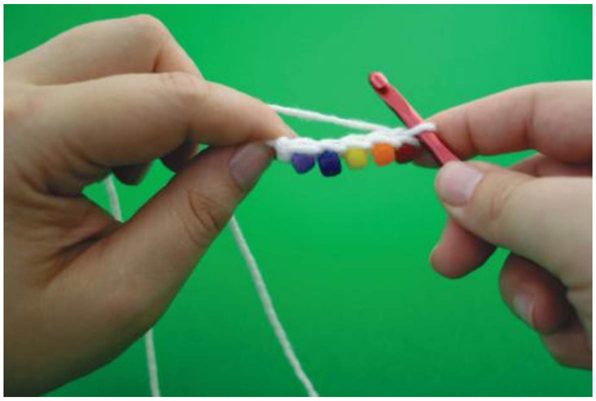


FIGURE 8.13 In preparation for closing the chain, hold the initial chain as shown in the photograph. Keep the hook in your dominant hand and grasp the tail with your other hand. Let the beads hang down below the V-shaped chain stitches.

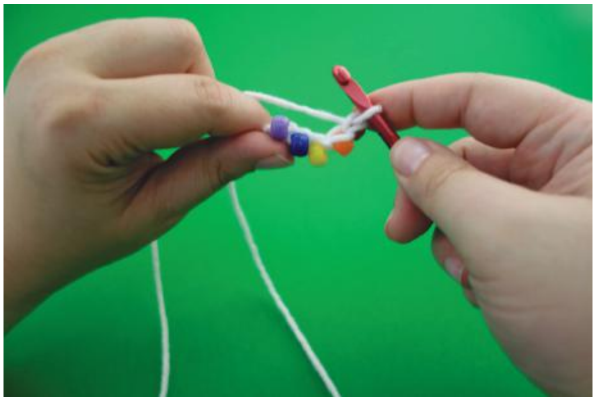


FIGURE 8.14 With the hand that is grasping the tail of the thread, rotate the first bead *forward* and upward. It is very important to rotate the bead *toward* you to make sure that the chain stitches stay on the inside of the bead crochet rope.

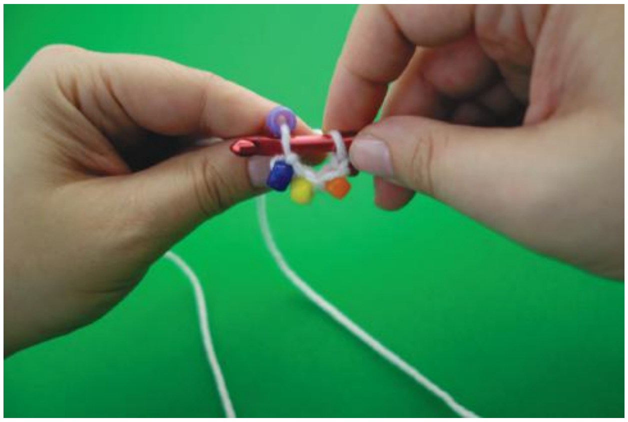


FIGURE 8.15 Insert the crochet hook from the inside to outside through the loop of thread holding the first bead.

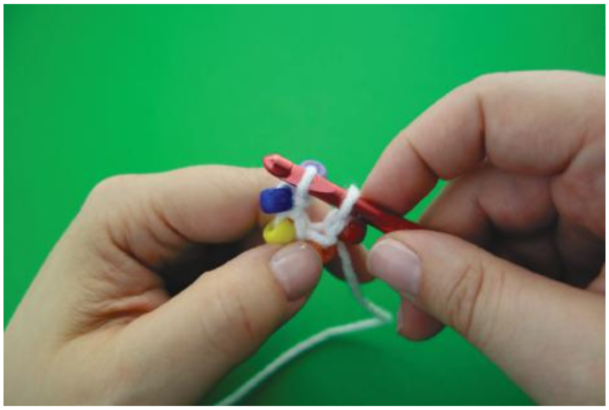


FIGURE 8.16 Push the first bead back behind the crochet hook. Looking closely at the photograph, you can see the first bead sitting behind the hook with the hole pointing up and down.

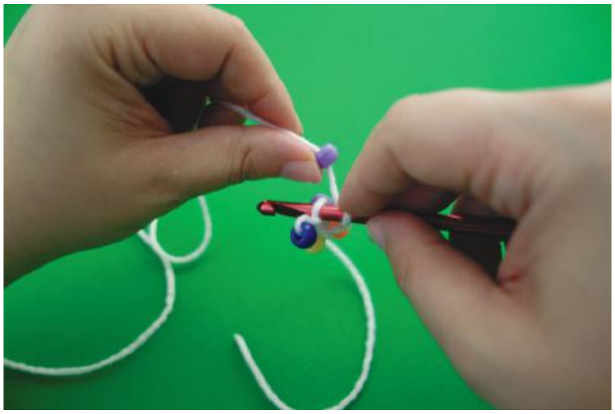


FIGURE 8.17 Pull the next bead on the strand of thread down toward the crochet hook. Hold the bead close to the previous stitch, making sure the thread is pulled taut so there is no loose thread between the bead and the stitch on the hook.

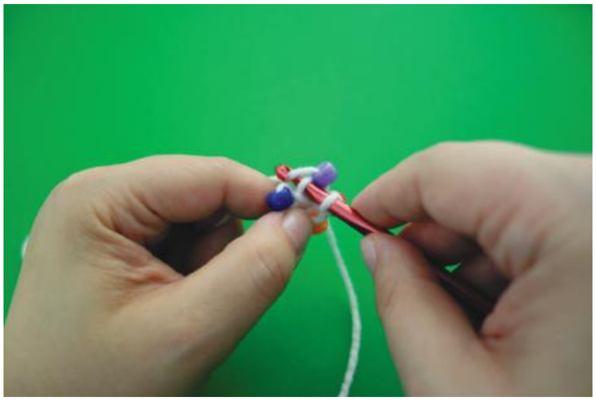


FIGURE 8.18 Wrap the thread around the crochet hook.

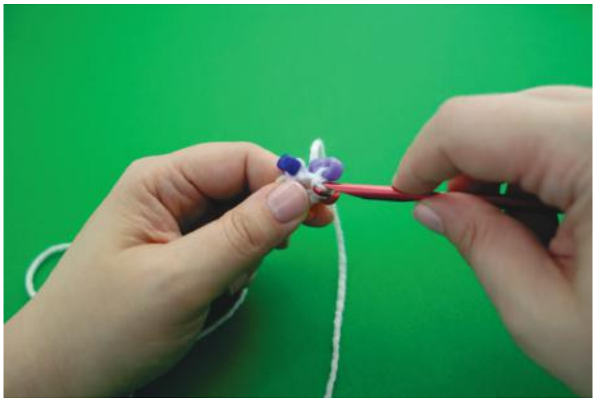


FIGURE 8.19 Pull the thread through *both* loops on the crochet hook. It is helpful to twist the hook so that the notch points downward when you are pulling the thread through. There will be a single loop of thread remaining on your hook at the end of the stitch. If you find it easier, you can pull the thread through the two loops in two separate steps, one loop at a time. You have now completed your first bead crochet slip stitch and closed the round.

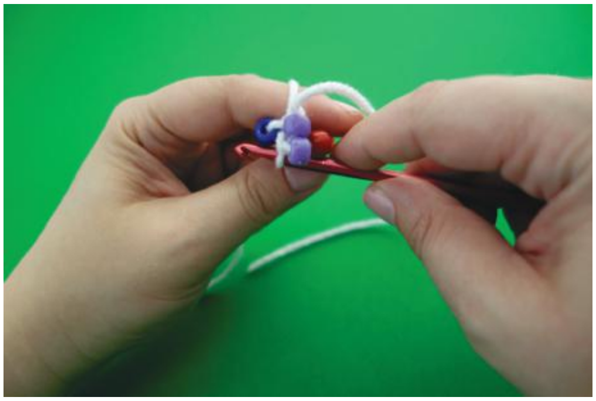


FIGURE 8.20 Here is a closer look at the beads after the first slip stitch. Viewing the crochet from the side, with the new stitch rotated downward, you can see that the first bead is lying flat with its hole pointing up and down, while the bead you have just stitched on is sitting on its side with its hole pointing sideways.

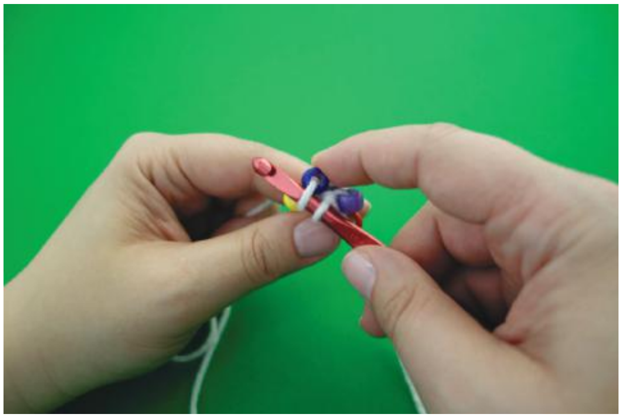


FIGURE 8.21 Insert the crochet hook from the inside of the ring of beads to the outside through the loop of thread holding the next bead in the initial chain. Push the bead back behind the crochet hook.

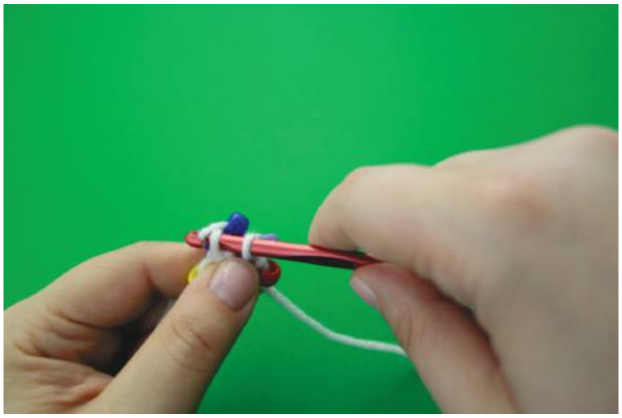


FIGURE 8.22 Repeating the steps from Figures 8.17 and 8.18, pull the next bead on the strand of thread all the way down to the crochet hook and wrap the thread around the crochet hook.

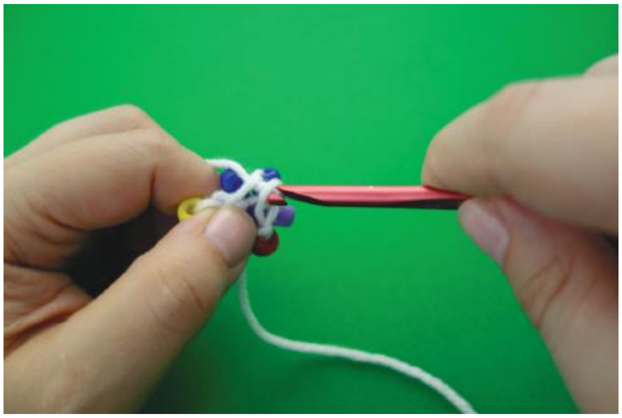


FIGURE 8.23 Repeating the step from Figure 8.19, pull the thread through *both* loops on the crochet hook. You will have one loop of thread remaining on your hook. This completes your second bead crochet slip stitch.

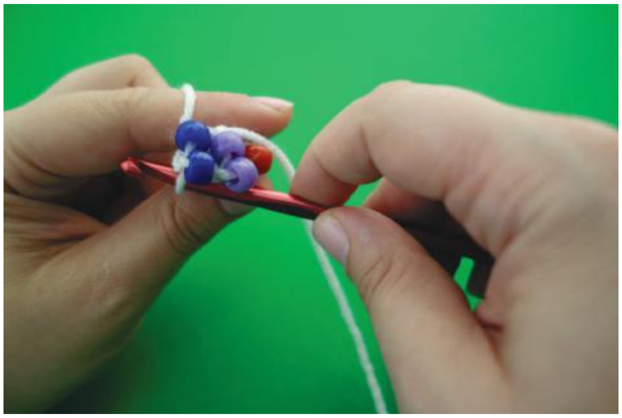


FIGURE 8.24 Here is a closer look at the beads after the second slip stitch, viewed from the side. You can see how the two initial chain beads (rotated to the top) are lying flat side by side, and the two newly stitched beads are sitting on their sides with their holes pointing sideways.

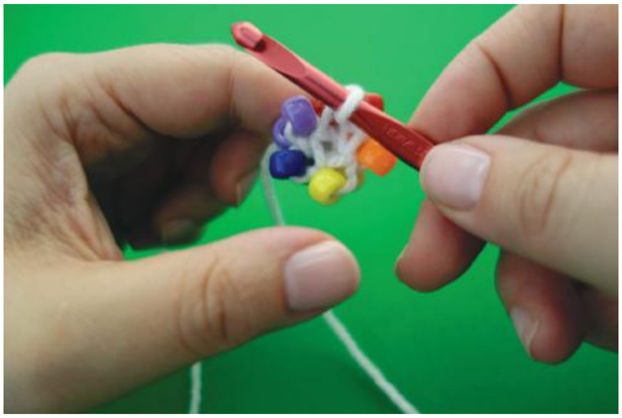


FIGURE 8.25 Continue to crochet on new beads until you have stitched into each of the initial chain stitches. When you are done, you will have a round of beads all sitting on their sides on the top of the bead crochet rope as shown here.

FIGURE 8.26 Here is the bottom of the rope after a few rounds. Note that all the initial chain beads are lying flat.

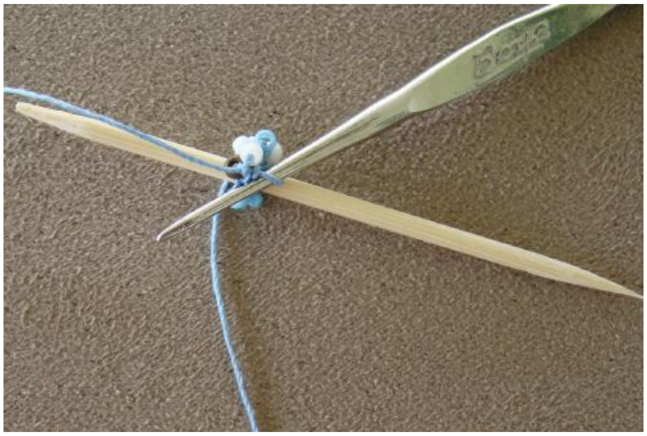


FIGURE 8.27 Crocheting a seed bead bracelet around a toothpick. Putting a toothpick into the center of the round in the early stages of crochet can help ensure that you insert your hook into the correct bead for each new stitch. If you have trouble crocheting loosely enough to get a nice, flexible bead tube, leaving the toothpick inside the bracelet may keep your stitch work more open. Pull the toothpick forward when it is more than halfway covered by the crochet.

The first time you close up an initial chain (as in Figures 8.13–8.25) using seed beads in a pattern where bead colors are often repeated, it is easy to mix up the order of the chain beads. To keep the chain untangled and make sure you don't skip any beads, you may find it helpful to start crocheting around a toothpick, as shown in Figure 8.27. As mentioned earlier, keeping the toothpick in the middle of your tube as you work can also help you keep your stitches loose, especially for size 11 seed beads in circumferences 6 and 7. However, working around a toothpick is a bit cumbersome, so in the long run switching to a larger hook may be a better strategy.

Crocheting

Once the first round of stitches is complete, you will keep using the same bead crochet slip stitch until you reach the end of your rope (literally, not metaphorically). In particular, the only difference between the stitch shown in the following photographs (Figures 8.28–8.35) and the last stitch from the previous section is that you will insert your hook into a slip stitch instead of a chain stitch. Crocheting the middle of a bead crochet rope is much easier than crocheting the first few rounds, when the rope is still a bit floppy and you don't have an established tube to grip. If you have a friend who already knows how to bead crochet, see if you can prevail on him or her to start a bead crochet rope for you so that you can begin learning at this stage.

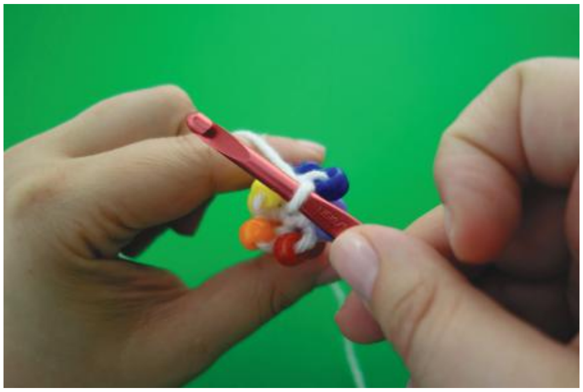


FIGURE 8.28 Here is a bead crochet rope in progress. There will always be as many beads as the circumference of the bracelet sitting on their sides in a circle at the top of the rope. In this photograph, the blue bead is the last bead that was stitched onto the rope. The last bead will be on the side of the hook toward your dominant hand, and the bead directly on the other side of the hook (in this case, yellow) is the bead you will stitch into next.

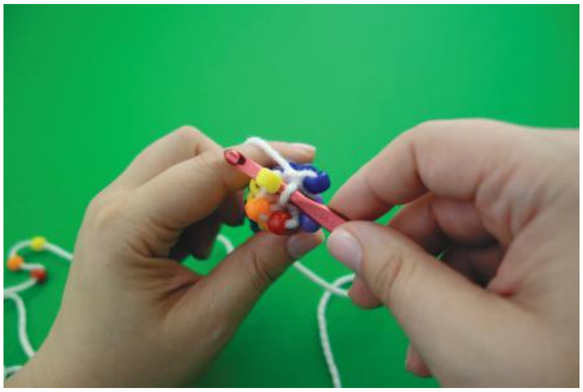


FIGURE 8.29 Insert the hook from the inside of the rope to the outside through the loop of thread holding the next (yellow) bead.

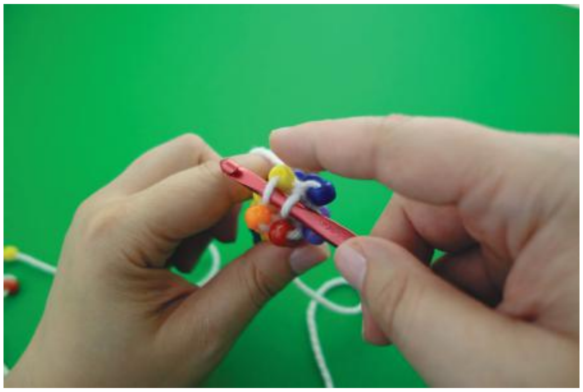


FIGURE 8.30 Push the bead on top of the hook back behind the crochet hook. With some experience, you may find that you can insert the hook and push the bead back into place at the same time. The yellow bead is now lying flat, just like all the beads below the top of the rope.

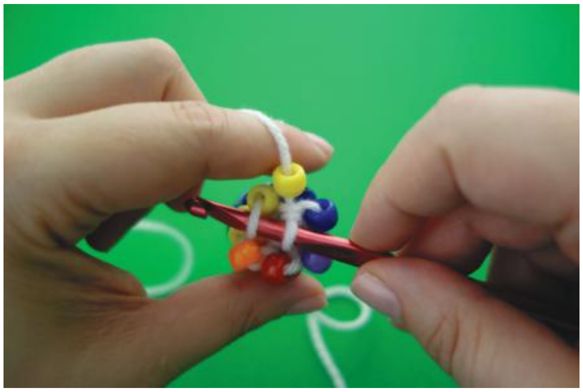


FIGURE 8.31 Pull the next bead on the strand of thread down toward the crochet hook. Hold the bead close to the previous stitch, making sure the thread is pulled taut so there is no loose thread between the bead and the stitch on the hook. Notice that the working thread is coming up from the previous stitch between the blue bead and the yellow bead. Be sure that the thread doesn't accidentally wrap itself around any of the beads already in the rope.

FIGURE 8.32 Wrap the thread around the crochet hook. As always, wrap the hook from back to front.

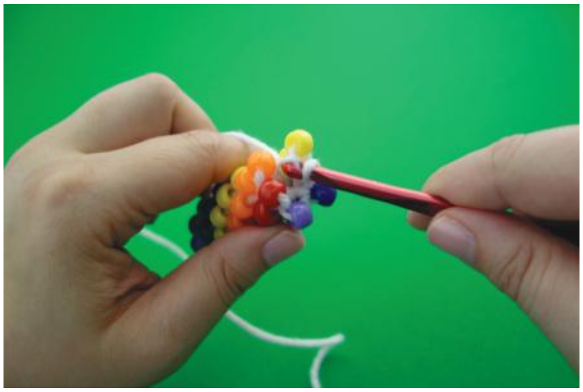


FIGURE 8.33 Pull the thread through *both* loops on the crochet hook. You can do this in one smooth step or in two steps, one per loop, if you find that easier.

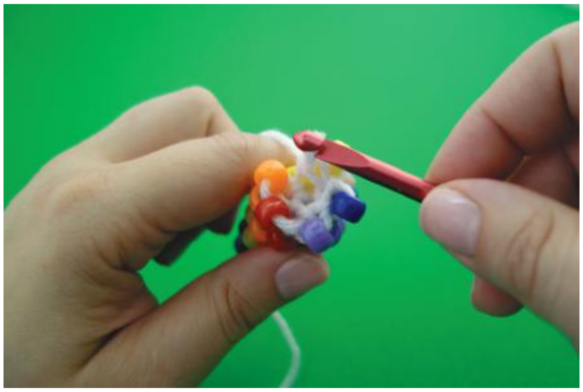


FIGURE 8.34 The bead crochet slip stitch is complete. There is one loop of yarn left on the crochet hook. In this photograph, the next stitch goes into the orange bead. To make the photographs here easier to follow, we only show one bead at a time on the working thread. In practice, pulling the beads down one at a time is absurdly inefficient. When you have enough experience that the bead crochet slip stitch feels somewhat natural, try holding a stack of beads in your nondominant hand and feeding them one at a time as in Figure 8.35.

As you bead crochet, you will use up your thread in the stitches until you find that there is virtually no thread left between your crochet hook and the new beads waiting to be fed into the stitches. When this happens, stop crocheting and slide all of the remaining beads down the thread until you have a good length of thread between the beads waiting to be crocheted and the bead crochet rope. As you do this, be careful not to tangle the thread. (Some people drape the thread and beads loosely around a finger in such a way that it feeds continuously as they crochet and are thus able to avoid having to periodically stop and slide beads—but this is a fine point for very experienced bead crocheters only!)

FIGURE 8.35 Holding a stack of beads ready to feed into the crochet stitches. Once you are comfortable with the basic bead crochet stitch, sliding multiple beads down at a time will make the bead crochet process more streamlined and efficient.

No matter how much bead crochet experience you have, mistakes happen. Maybe you forgot to slide a new bead down or to flip the old bead over before making your slip stitch. Maybe you were holding the yarn too loosely and the last few beads look wobbly. Maybe you don't know if you made a mistake but something just doesn't look right. The good news is that you can always pull out stitches (or in crochet parlance, *rip* them out) until you are back to a point where everything looks okay and you can get a fresh start (Figures 8.36–8.38).

It usually takes at least a few hours to crochet a bracelet, so you might not want to do it all in one sitting. If you are going to put a bead crochet rope aside for a while, it is a good idea to enlarge the final loop so that the stitches you have already made are less likely to come undone (Figure 8.39). You can also secure it by fastening a safety pin through the loop.

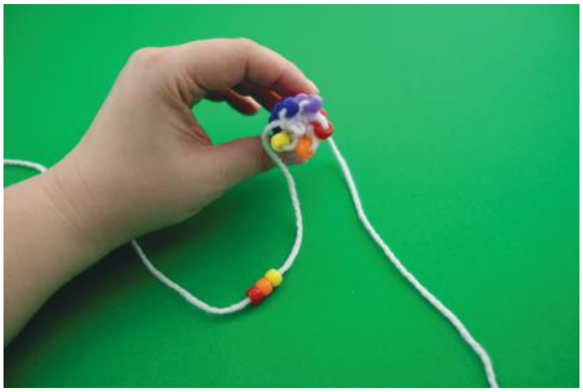


FIGURE 8.36 Here is a bead crochet rope after a few stitches were ripped out. In order to start again, you need to find the loop of the last stitch on the rope and insert your hook into it. Here, the loop is coming out of the blue bead and pointing down and to the right.

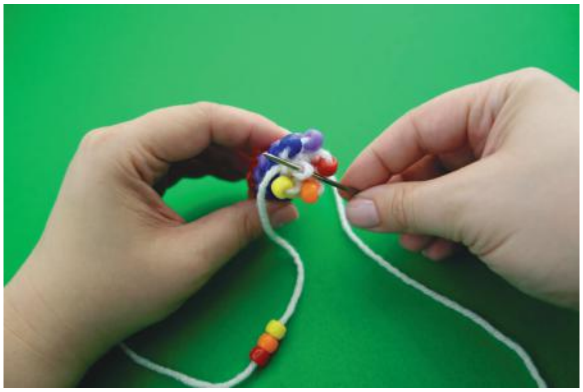


FIGURE 8.37 If the loop is too tight to insert the crochet hook, loosen it with something narrower. A tapestry or sewing needle works well. If you really can't force the loop open, gently rip out another stitch and try again with the previous loop.

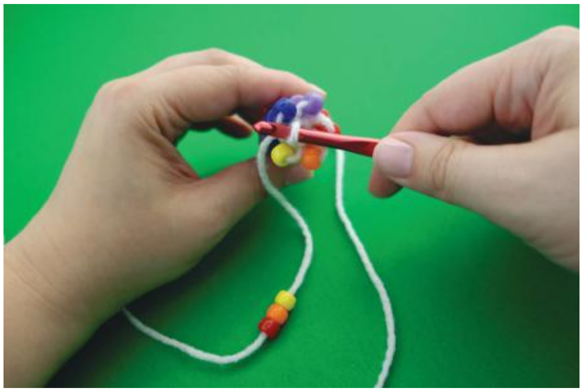


FIGURE 8.38 Insert the crochet hook through the last stitch from the inside of the bead crochet rope to the outside, and you're ready to start crocheting again.

FIGURE 8.39 Securing a bead crochet rope in progress. Pull on the crochet hook to expand the final loop and make the last stitch harder to pull out by accident. When you are ready to begin crocheting again, put the crochet hook into the enlarged loop and pull it taut. If you find that the last bead you crocheted onto the rope has come loose, rip out the last stitch and reinsert the hook as shown previously.

Closing

When you have crocheted your final bead, you are ready to begin the closing process by fastening off the end of your bead crochet rope (Figure 8.40).

You are now ready to sew the ends of the rope closed in an invisible join using a tapestry or large sewing needle. The sewing can be done with either the thread tail at the initial chain or the thread tail at the end of the rope, and different bead crocheters prefer to use different ends. In these instructions, we show how to sew with the initial thread tail, but closing with the end tail is very similar. In both cases, you alternate between the start and end of your rope, sewing the beads on each side in the order they were crocheted. When sewing the join, the tapestry needle *never* passes through a bead hole, which is why you can use a larger needle.

Each circular end of the rope has a step or notch in it where it drops off after the final bead. These two notches

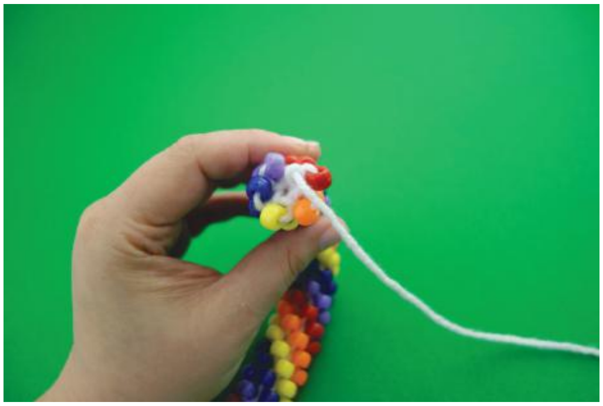


FIGURE 8.40 Fastening off the end of a bead crochet rope. After your final bead crochet slip stitch, cut off the thread leaving a tail of at least several inches. Wrap the thread around the hook and pull the tail of the thread all the way through the final stitch, tugging the end to tighten the final loop. The final stitch is now secure, and the crochet will not unravel.

will lock together like puzzle pieces at the join. Although the rope can be twisted one or more times before joining, there is only one correct way to align the beads at the top. Before sewing, hold the beads aligned in position and inspect how the pattern connects at the two ends. If the pattern does not continue smoothly at the join, there is a stringing error at the join, which you must correct before proceeding. The more you twist a bracelet, the harder it is to hold the ends together as you close, so novices should stick with gentle twists or untwisted bracelets.

The diagram in Figure 8.41 shows a conceptual overview of the closing process, in which you use the thread tail to sew the ends of the bracelet together for an apparently seamless join. The process is shown step by step in Figures 8.42–8.53.

The last step is to secure the thread ends into the center of your bead crochet rope. In order to make sure that they don't slip out of the crochet, do the following.

- Tie a knot in your thread as close as possible to your crochet rope. With your closing needle, pull the thread into the body of the bracelet until you feel the knot pop through the crochet stitches. (Sometimes, you can actually hear the little pop.) You can skip this step as long as you secure the thread end thoroughly in the next step.
- With the needle, insert the thread between stitches into the hollow core of the bracelet, pulling the thread so that it travels along the length of the bead crochet rope, and then pull the needle out between stitches. Repeat this several times to secure a long segment of the tail in the interior of the bracelet. Be careful not to stick your needle into the hole of a bead.

When your thread ends are secure, hide them as shown in Figure 8.54.

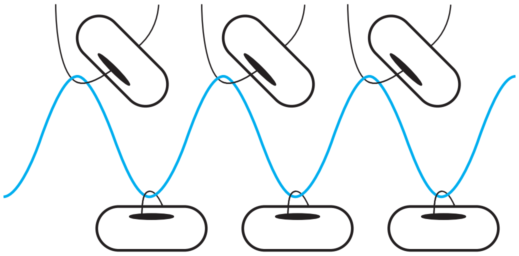


FIGURE 8.41 A schematic of the closing process. You sew with the blue thread, alternating between the initial chain beads (shown at the bottom) and the final beads (shown at the top), using the closing thread to push the final beads flat.

FIGURE 8.42 To begin closing the bracelet, thread the tail of your initial chain into a tapestry or sewing needle and bring the two ends of the bracelet together. This is the point at which you will decide what twist to use in closing your bracelet, so inspect the pattern carefully to make sure the design elements are arranged as you would like them.

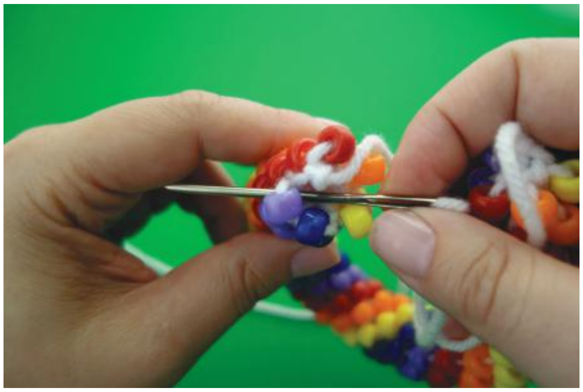


FIGURE 8.43 The final bead that you crocheted onto the rope (in this image, the red bead) is raised above the rest of the round. This is the last bead you will sew through. Stick your needle under the bead below and to its left (or to its right if you are crocheting left handed), from the inside of the rope toward the outside.

FIGURE 8.44 Push the bead on the needle back behind the needle. You are pushing the bead the same way you push it for each crochet stitch, after which the bead will lie flat like all the beads below it.

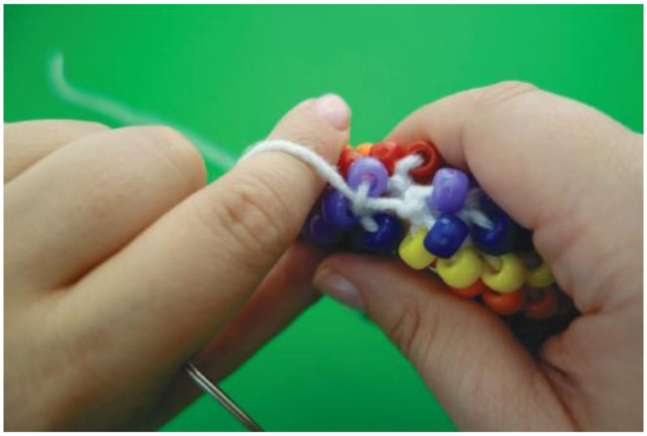


FIGURE 8.45 Pull the thread through, being careful not to knock the bead out of position.

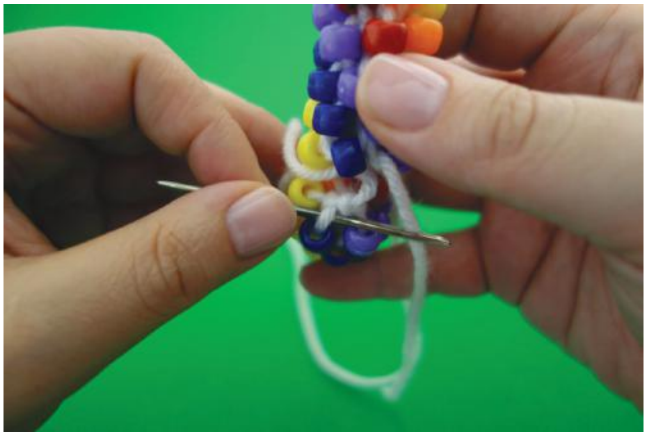


FIGURE 8.46 Pass the needle through the yarn above the second bead in the initial chain. Keep the needle pointing in the direction that you are sewing the seam (right to left, if you are working right handed).

FIGURE 8.47 Pull the thread through.

FIGURE 8.48 Stick the needle from the inside of the bead crochet rope to the outside under the next bead at the end of the rope.

FIGURE 8.49 Push the bead on the needle back behind the needle and pull the thread through.

FIGURE 8.50 Here is what the closing seam looks like so far. At this point, it is a good idea to check and make sure that your bracelet has the twist that you intended. If not, pull out the stitches you have sewn, correct the twist, and start over. If so, tighten your stitches and keep sewing, alternating between the beads in the initial chain and the beads at the end of the rope.

FIGURE 8.51 At this stage, there are two beads (both red) left to sew into place. Pull your needle through the thread over the last bead in the initial chain.

FIGURE 8.52 Stick your needle under the last bead in the bead crochet rope, push the bead back behind the needle, and pull the thread through.

FIGURE 8.53 You have now sewn all of the beads of the final seam together. Take a look at your seam and decide whether you are satisfied with it before securing the ends of the thread. If you like, you can continue to sew a few more beads together to help stabilize the join.

FIGURE 8.54 Hiding the ends of the thread. After you have woven in the end of the thread to your satisfaction, give the thread a firm tug in the direction that you are sewing and cut it off about 1/4 inch from the bracelet. Tug the bracelet on either side of the thread end until the end retracts into the middle of the bead crochet rope.

Joining New Thread

In most cases, a bead crochet rope is made with a single spool of thread. However, there are some cases in which you need to join a new piece of thread onto your work. For instance, if you are making a long bead crochet necklace, the beaded strand will be so long that even if it can hypothetically fit onto a single spool of thread, it will be such a nuisance to manage all the beads in a single strand that it is easier to divide the beads among several spools. Alternately, if you string your beads onto a partially used spool of thread and find that you don't have enough thread to finish, you may have to restring the remaining beads on another spool and join the new thread to the old. Finally, if you are in the middle of a rope and find an egregious stringing error that cannot be fixed by breaking a few beads off of the strand, you can always cut the thread, fix the stringing error, and then join the remainder of the bead strand to your rope.

Here, we demonstrate a simple method for incorporating a new beaded strand into a rope in progress. For clarity, we show two different colors of thread in the photographs (Figures 8.55–8.64), but in practice both strands would be the same color.

FIGURE 8.55 When you reach the second-to-last bead on the old thread (shown in white), lay your new thread (shown in blue) against it so that the working ends of the two strands of yarn are pointing in the same direction. Here you can see the tail of the blue thread pointing away from the working yarn, and the orange bead, ready to be stitched on, is on the white thread. Wrap both threads around the crochet hook.

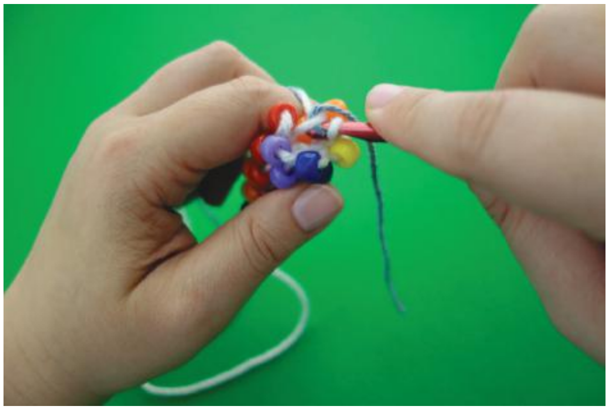


FIGURE 8.56 Pull both threads through both loops on the crochet hook.

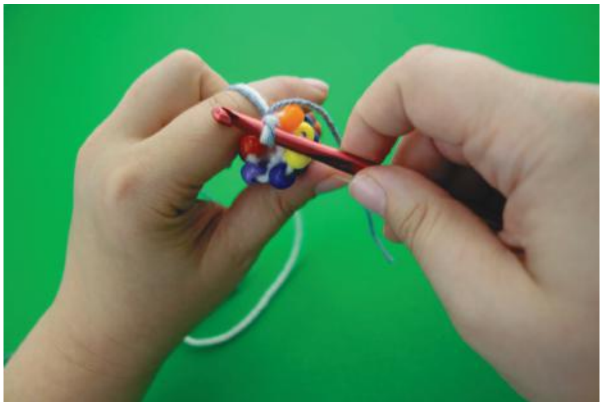


FIGURE 8.57 The second to last bead from the first thread is now stitched onto the rope. Notice that both threads are looped around the crochet hook, and the new thread is now attached to the rope. For the next couple of stitches, you will hold both threads together.

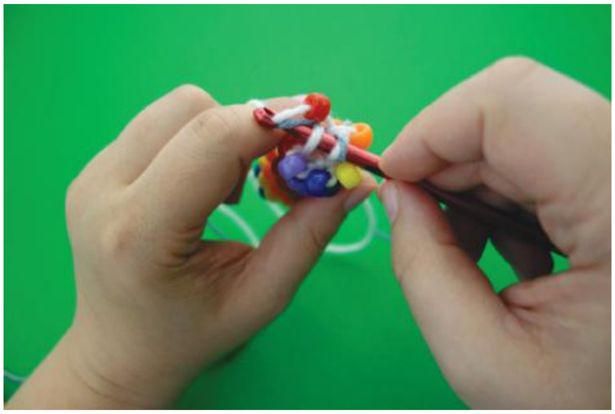


FIGURE 8.58 Make another bead crochet slip stitch using both threads. Here, after inserting the hook into the red bead on the rope and pushing it back (notice that the hook only passed through the old thread), pull the final bead down the first thread and wrap both threads around the crochet hook. Pull both threads through both loops.

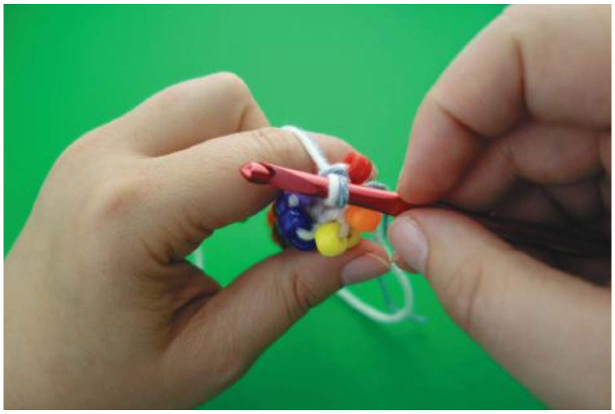


FIGURE 8.59 The stitch is complete, and there are no more beads left on the first thread.

FIGURE 8.60 In the next stitch, pull the first bead from the second thread down toward the crochet hook, still holding both threads together. Pull both threads through both loops.

FIGURE 8.61 When this stitch is complete, drop the first thread. If you haven't cut the first thread from its spool, do so now, leaving a tail a few inches long. Pull the tail of the first thread under the crochet hook and into the middle of the ring of beads at the top of the rope. From here on, you will crochet around that tail to hide it in the interior of the crochet rope.

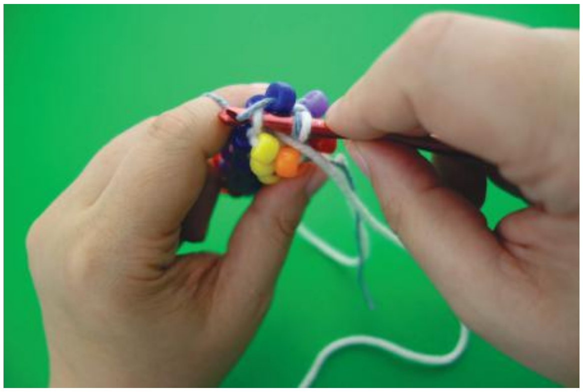


FIGURE 8.62 Make the next stitch using only the new thread. Pull the second bead on the new thread down to the hook, wrap the thread around the crochet hook, and pull it through both loops.

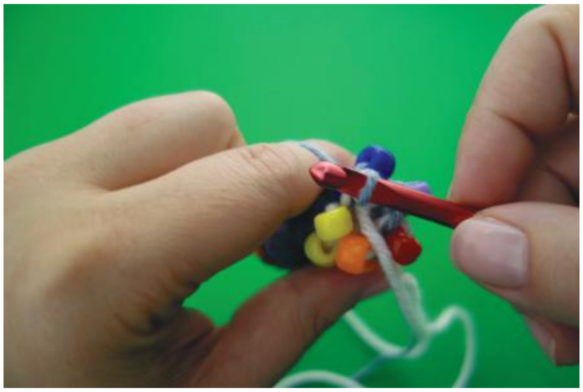


FIGURE 8.63 Now you are back to crocheting with a single thread. Continue crocheting as usual, pulling the tails of the old and new thread into the center of the bead crochet rope and stitching around them. You may need to give both tails a firm tug to make sure that you don't have a loop of thread poking out of the bead crochet rope.

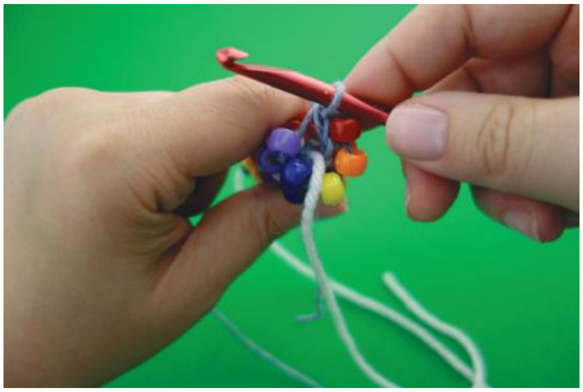


FIGURE 8.64 This is what the rope looks like a full round after the join. Notice that all of the visible stitches are in the new blue thread, and that the white and blue tails are protruding from the center of the bead crochet rope. Once you have a couple of inches of the tails secured in the middle of the rope, you can snip off the ends, being careful not to sever the working thread.

FIGURE 8.65 Stringing a transformation necklace. This is the Waves and Diamonds Transformation necklace (p. 208), strung with size 11 black seed beads and 1.8mm sterling silver rounds, with a paper tag marking the end of each segment of the pattern. The necklace itself is in Figure 6.29.

Tips for Transformation Designs

Most of the patterns in the pages that follow are for bracelets and use a single pattern repeat throughout the whole design. By contrast, the Escher transformation patterns on pp. 207–216 are mostly necklaces and contain a number of different color sequences, so stringing them is a little more challenging. For the three necklace designs (and any other particularly long piece of bead crochet), it is best to string the beads on several different spools of thread and use the join technique described above to crochet them together. Even if a single spool has enough thread to crochet your entire necklace, managing that long a string of beads—especially when you need to slide them down the thread as you crochet—is much more cumbersome than joining strands together as you crochet.

Since you are stringing different color sequences as you progress through the pattern, it is helpful not only to place a tag at the end of each repeat but also to label the repeats so that you know which part of the pattern is on each section of your bead strands. Figure 8.65 shows the bead strands for a transformation necklace with bits of paper marking each segment of beads. A stringing needle is generally not sharp enough to pierce paper, so use a sewing needle or straight pin to poke a hole before you thread the paper onto your strand. We recommend leaving the tags on the thread and tearing them off as you crochet so that you can keep track of how far you are in the design and correct any unfortunate stringing errors as they arise.

We hope you find these instructions and tips useful as you explore our patterns or your own. Happy crocheting!

Resources

Books and Papers

Adams, Colin Conrad. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*. Providence: American Mathematical Society, 2004.

Baker, Ellie and Susan Goldstine. "Bead Crochet Bracelets: What Would Escher Do?" In *Bridges Towson: Mathematics, Music, Art, Architecture, Culture: Conference Proceedings*, edited by Robert Bosch, Douglas McKenna, and Reza Sarhangi, pages 567–572. Phoenix: Tessellations Publishing, 2012.

belcastro, sarah-marie and Carolyn Yackel, ed. *Making Mathematics with Needlework: Ten Papers and Ten Projects*. Wellesley, MA: A K Peters, Ltd., 2007.

Bertoglio-Giffin, Judith. *Bead Crochet Ropes*. Manchester, NH: Glass Cat Books LLC, 2002.

Bertoglio-Giffin, Judith. *Patterns & Graphing for Bead Crochet Ropes*. Manchester, NH: Glass Cat Books LLC, 2004.

Bertoglio-Giffin, Judith. *Triangular Bead Crochet Ropes: A Pattern Book of 3-Dimensional Bead Crochet Ropes*. Printed by CreateSpace, 2011.

Bordhi, Cat. *A Treasury of Magical Knitting*. Friday Harbor, WA: Passing Paws Press, 2004.

Carter, Nathan. *Visual Group Theory*. Washington, DC: The Mathematical Association of America, 2009.

Conway, John H., Heidi Burgiel and Chaim Goodman-Strauss. *The Symmetries of Things*. Wellesley, MA: A K Peters, Ltd., 2008.

Coxeter, H. S. M. "The Four-Color Map Problem, 1840–1890." *The Mathematics Teacher* 52 (1959): 283–289.

Devlin, Keith. *Mathematics: The Science of Patterns: The Search for Order in Life, Mind, and the Universe*. New York: W. H. Freeman and Co, 1997.

Gay, David. *Explorations in Topology*. Burlington, MA: Academic Press, 2006.

Goldstine, Susan and Ellie Baker. "Building a Better Bracelet: Wallpaper Patterns in Bead Crochet." *Journal of Mathematics and the Arts* 6 (2012): 5–17. (DOI:10.1080/17513472.2012.654423)

Goldstine, Susan, Sophie Sommer, and Ellie Baker. "Beading the Seven Color Map Theorem." *Math Horizons* (September 2013): 22–24.

Gross, Jonathan L. and Thomas W. Tucker. *Topological Graph Theory*. New York: Dover Publications, 2001.

Saracino, Dan. *Abstract Algebra: A First Course*. Long Grove, IL: Waveland Press, Inc., 2008.

Schattschneider, Doris. *M.C. Escher: Visions of Symmetry*. New York: Harry N. Abrams, 2004.

Senechal, Marjorie. "Tiling the Torus and Other Space Forms." *Discrete and Computational Geometry* 3 (1988): 55–72.

Washburn, Dorothy K. and Donald W. Crowe. *Symmetries of Culture: Theory and Practice of Plane Pattern Analysis*. Seattle: University of Washington Press, 1988.

Software

iOrnament by Jürgen Richter-Gebert, http://www.science-to-touch.com/en/iOrnament.html.

jBead by Damien Brunold, http://www.jbead.ch.

KnotPlot by Robert G. Scharein, http://knotplot.com.

Online Bead and Crochet Suppliers

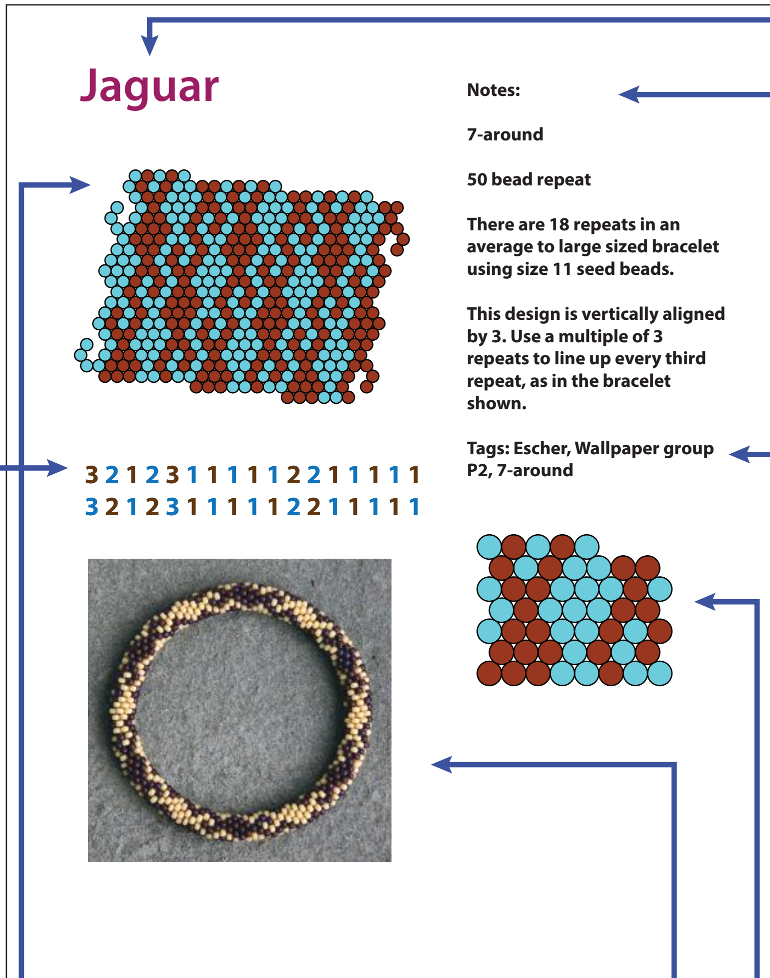
Fire Mountain Gems, http://www.firemountaingems.com.

Fusion Beads, http://www.fusionbeads.com.

Jo-Ann Fabric and Craft Stores, http://www.joann.com.

Pattern Pages

Anatomy of a Pattern Page



Pattern Name

Pattern Notes
The notes include the circumference of the bracelet, the length of the repeat, and information about bead types, thread type, bracelet sizing, and twist options.

As noted in the how-to section, seed bead sizes vary from brand to brand, which can affect bracelet sizing. Delicas (cylindrical beads that are smaller and more precisely formed than regular seed beads) can be used interchangeably with Toho Aiko Cylinder Beads.

When no thread is specified, we recommend heavy duty topstitching thread.

Tags
These indicate the major features of the pattern including geometric content and circumference to make it easier to find patterns with various attributes. The pattern index (p. 239) includes page references to where certain tags are discussed.

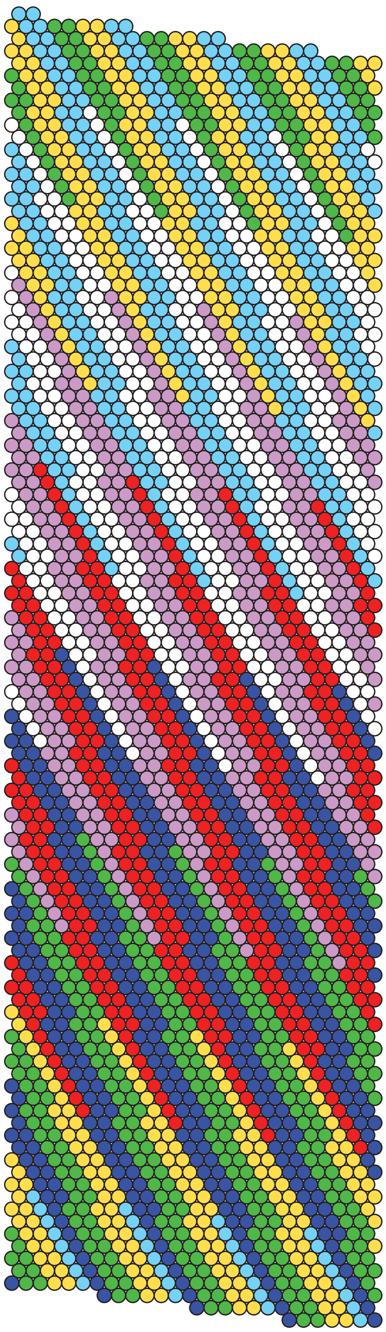
Repeat Chart
This chart duplicates the information given in the numerical pattern in a vertical layout. In a few cases, the numerical pattern begins in a different place than the repeat chart. The beads are strung in order from left to right and *bottom to top*.

Bracelet Photograph

Bead Plane Pattern
This is a patch of the infinite bead plane pattern for the bracelet. The borders of the patch are chosen to highlight features of the design and have no bearing on making a bracelet.

Numerical Pattern
For a single repeat, the number of beads of each color to be strung in order from left to right, top to bottom. In some patterns, we use shorthand for repeated segments as follows:
[1 1 × 4] means **1 1 1 1 1 1 1 1**

Sophie's Original Seven-Color Torus



Notes:

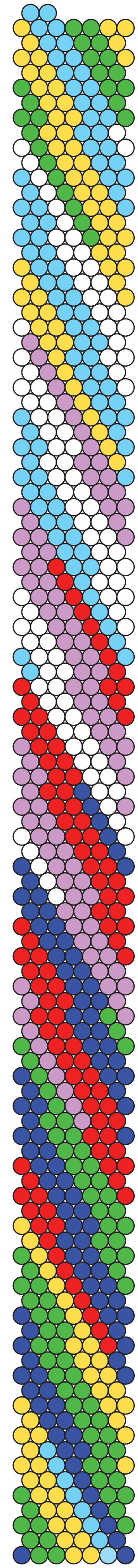
6-around

No repeat (672 beads)

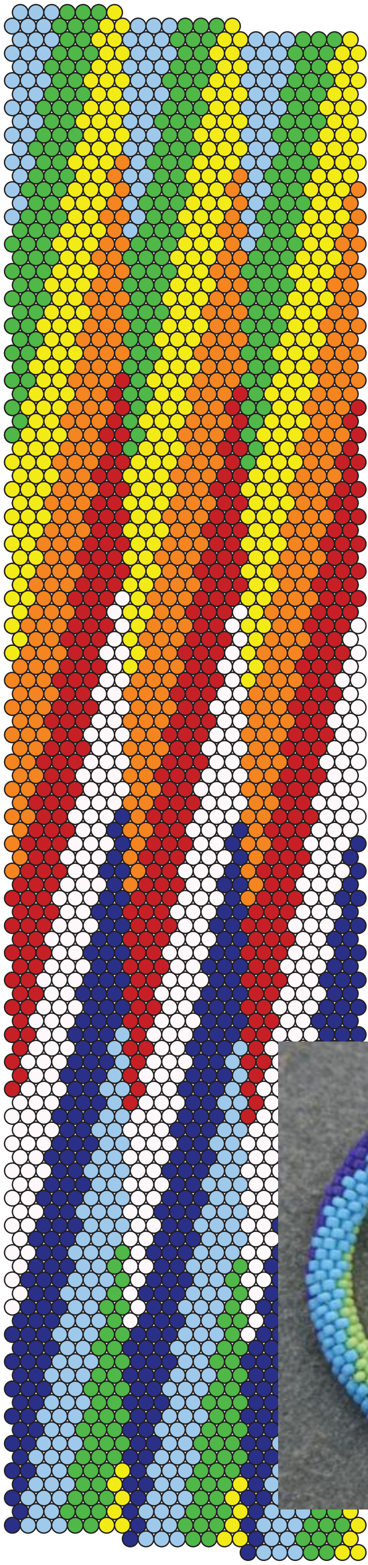
Uses size 11 seed beads.
This pattern is easier to read (but harder to fit on a page) in a traditional diagonal chart. For a longer bracelet, repeat each segment 9 times instead of 8.

Tags: Seven-Color, Escher, 6-around

[1 2 2 1 × 8] [2 2 2 × 8]
[1 2 2 1 × 8] [2 2 2 × 8]
[1 2 2 1 × 8] [2 2 2 × 8]
[1 2 2 1 × 8] [2 2 2 × 8]
[1 2 2 1 × 8] [2 2 2 × 8]
[1 2 2 1 × 8] [2 2 2 × 8]
[1 2 2 1 × 8] [2 2 2 × 8]



Seven-Color Torus (Ungar-Leech Map)



Notes:

7-around

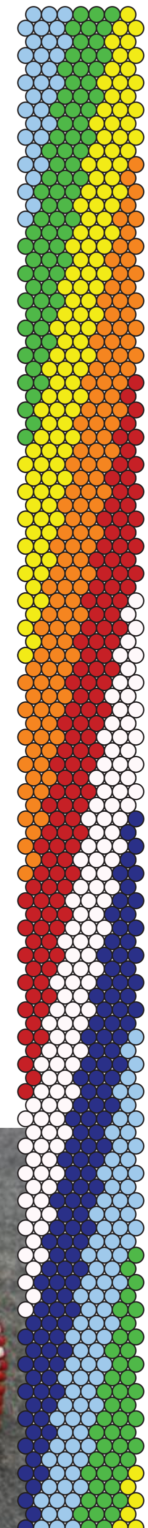
No repeat. The entire bracelet of 840 beads is shown at right, and we omit the numerical pattern for lack of space. There are 120 beads of each color.

There is no sizing flexibility except by changing tension or bead size. The bracelet shown is done in size 11 seed beads and fits a small to average size hand.

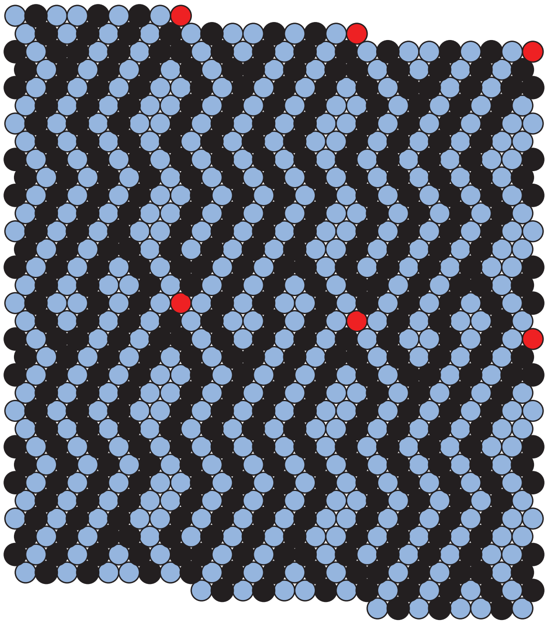
This seven-color torus design is based on the Ungar-Leech map, in which each of seven counries touches all the others.

Use care to choose colors that all contrast one another well for the best effect. Be careful not to twist when closing.

Tags: Seven-Color, Escher, 7-around



Complete Seven-Node Graph



1 1 1 1 2 1 1 1 1 1 1 1 1 1 1 2 1
1 1 2 1 1 1 1 1 1 1 1 2 1 1 1 1 1
1 1 2 1 1 1 1 1 1 1 2 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 1 2
1 1 1 1 1 1 1 2 1 1 1 1 1 1 1 1 1
1 2 1 1 1 2 1 1 1 1 1 1 1 1 1 1 2
1 1 1 1 1

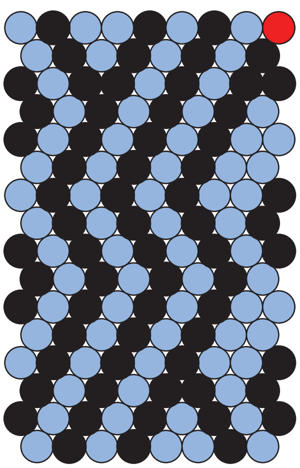
Notes:

8-around

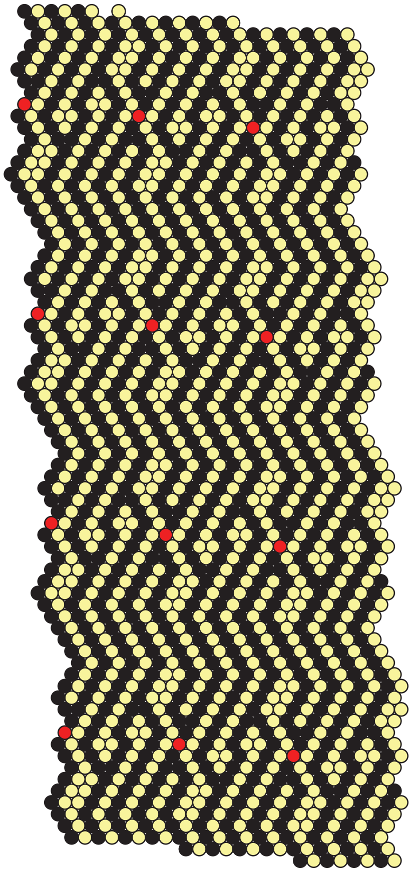
136 bead repeat

Use exactly 7 repeats with size 11 seed beads and seven larger beads for an average sized bracelet that shows a complete graph on seven nodes with no link crossings. Use seven different color Swarovski crystal bicones in place of the red bead on the graph (photo on left) or try all 3mm sterling silver rounds (photo on right). Use size 20 crochet thread to avoid pattern shift problems with a loose stitch.

Tags: Seven-Color, 8-around



Symmetric Complete Seven-Node Graph



Notes:

8-around

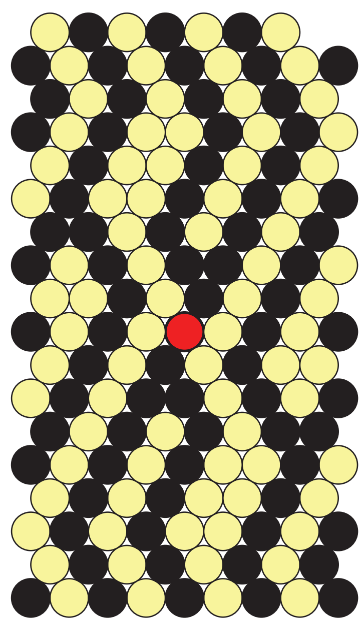
152 bead repeat

Use exactly 7 repeats to create a K_7 bracelet, fully connected graph on seven nodes with no link crossings.

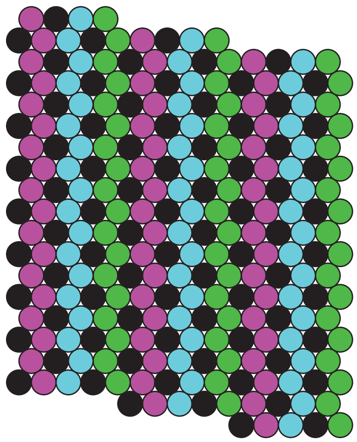
This pattern is designed for size 11 Delicas and produces an average-sized bracelet. It has a natural twist that must be untwisted before closing. The node beads (3mm sterling silver rounds in the bracelet shown, red on the chart) should line up when closed.

Tags: Seven-Color, Wallpaper Group P2, 8-around

[1 1 × 10] 1 2 [1 1 × 3] 1 2 [1 1 × 3] 1 2 [1 1 × 4] 2 1 1 1 2 [1 1 × 5] 2 [1 1 × 2] 1 [1 1 × 2] 2 [1 1 × 5] 2 1 1 1 2 [1 1 × 4] 2 [1 1 × 3] 1 2 [1 1 × 3] 1 2 [1 1 × 14]



Equilateral Triangle 4-around



1 1 1 1 1 1 1 1 1

Notes:

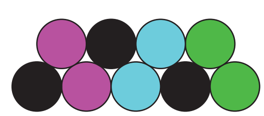
4-around

9 bead repeat

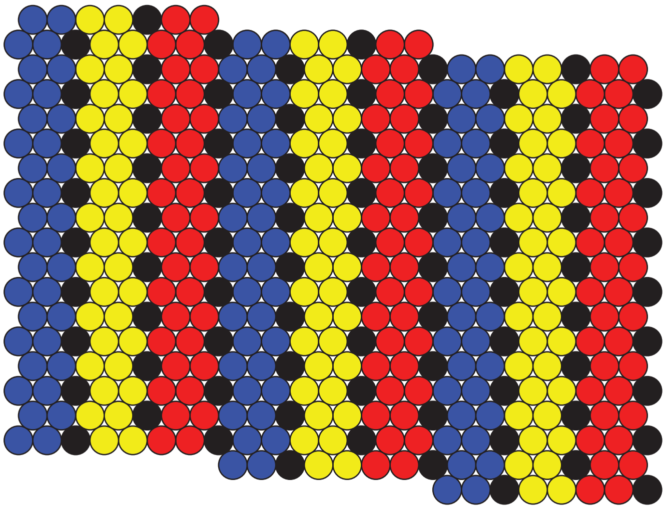
Use 34 repeats in an average sized bracelet using size 8 seed beads and size 6 seed beads for the spines (shown in black on bead plane diagram) as in the bracelet on the right.

A great quick bracelet using large beads! Try about 86 repeats for a matching slip-on necklace (check sizing for your head). Use care to avoid introducing a twist when closing. This pattern is vertically aligned, and the short repeat length makes sizing totally flexible. For a bolder effect, try 4mm semiprecious stones for the spines as in the bracelet on the left.

Tags: Geometric Cross Section, 4-around



Equilateral Triangle 7-around



2 1 2 2 1 2 2 1 2

Notes:

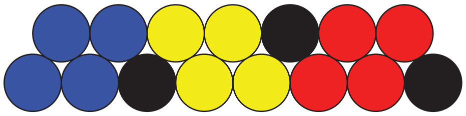
7-around

15 bead repeat

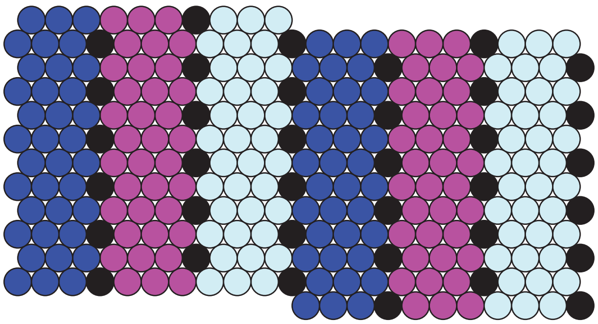
Use 54 repeats in an average sized bracelet using size 11 seed beads and size 8 seed beads for the spines (shown in black).

This pattern is vertically aligned and has extremely flexible sizing.

Tags: Geometric Cross Section, 7-around



Equilateral Triangle 10-around



Notes:

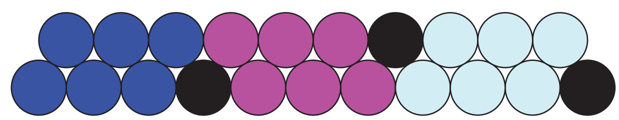
10-around

21 bead repeat

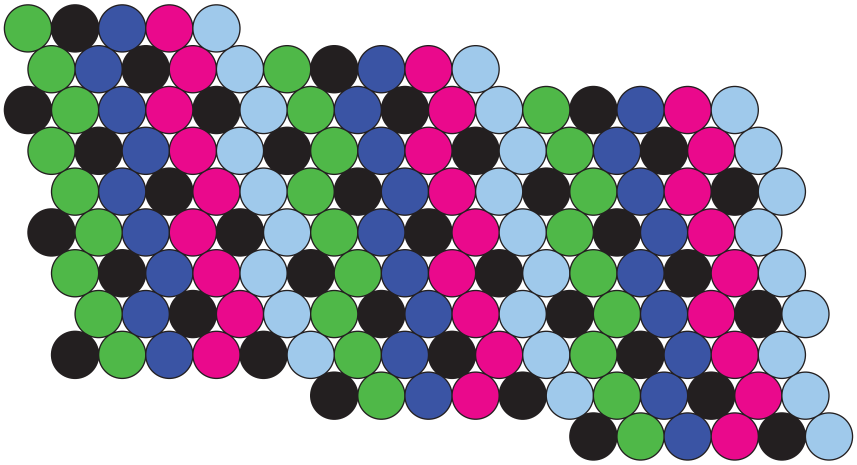
Use 58 repeats in an average sized bracelet using size 11 Delicas and size 10 Delicas (shown in black) for the spines.

Tags: Geometric Cross Section, 10-around

3 1 3 3 1 3 3 1 3



Square 5-around



1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

Notes:

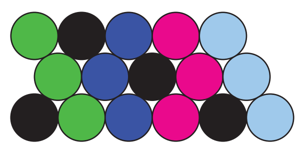
5-around

16 bead repeat

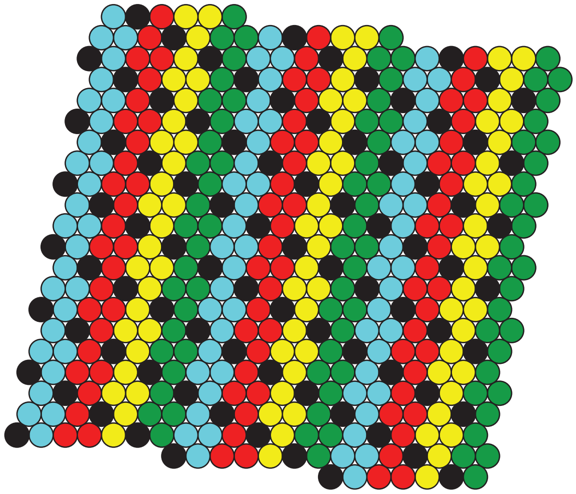
Use 24 repeats in an average sized bracelet using size 8 seed beads and size 6 seed beads (shown in black) for the spines.

Carefully untwist the natural spiral before joining to get the square cross section. A fun, fast bracelet (although untwisting at the join takes a little extra care). This makes a nice chunky pair with the Equilateral Triangle 4-around (p. 138).

Tags: Geometric Cross Section, 5-around



Square 6-around



1 1 2 1 1 1 2 1 1 1 2 1 1 1 2 1

Notes:

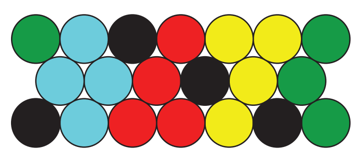
6-around

20 bead repeat

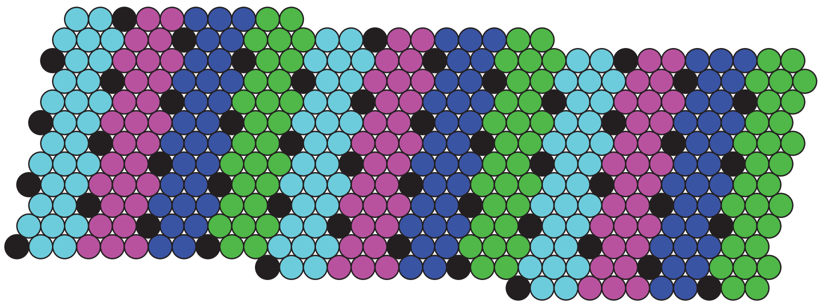
There are 34 repeats in an average sized bracelet using size 11 seed beads and size 8 seed beads for the spines (shown in black). Or use 22 repeats with size 6 and 8 seed beads and size 20 crochet thread. Both bracelets are shown.

This bracelet uses a twist of more than 360° to create the appearance of an untwisted square cross section. Alternate twists in which each color forms either a (3,1) or (2,1) torus knot are also possible (and are less tricky to close).

Tags: Geometric Cross Section, 6-around



Square 10-around



1 2 3 2
1 2 3 2
1 2 3 2
1 2 3 2

Notes:

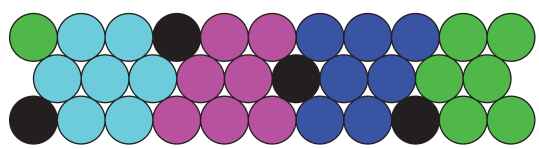
10-around

32 bead repeat

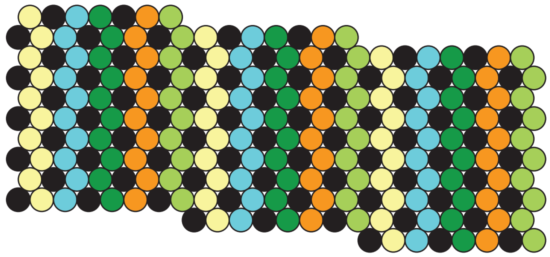
There are 42 repeats in an average sized bracelet using size 11 Delicas and size 11 seed beads for the spines (shown in black).

The bracelet in the photograph uses a twist to create the appearance of an untwisted square cross section. Alternate twists and/or fewer colors are also possible. The twist shown here tends to make a stiffer bangle-like bracelet.

Tags: Geometric Cross Section, 10-around



Pentagon



1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

Notes:

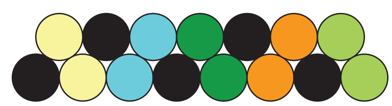
7-around

15 bead repeat

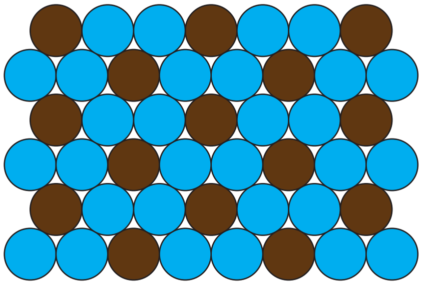
The upper bracelet uses 61 repeats and size 11 Delicas with size 11 seed beads for the spines (shown in black). Using the slightly larger beads for the spines creates the pentagonal (five-sided) cross section. The lower bracelet uses 45 repeats with size 10 Delicas and size 8 seed beads for the spines.

This pattern is vertically aligned and has extremely flexible sizing.

Tags: Geometric Cross Section, 7-around



Porcupine Pentagonal



Notes:

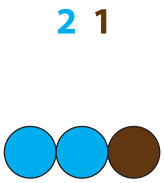
7-around

3 bead repeat

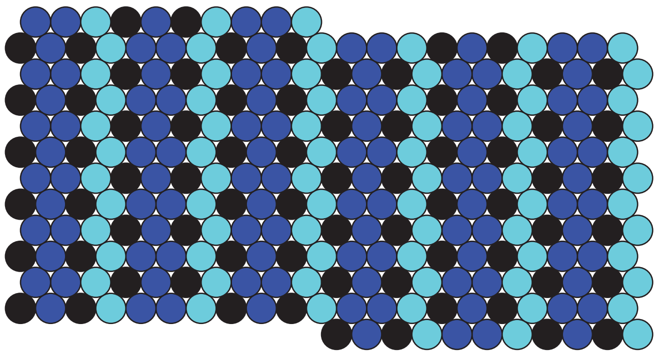
There are 240 repeats in an average sized bracelet, using size 11 seed beads and 3.4mm drop beads (in brown). The bracelet requires a multiple of 5 repeats for the spines to line up. The bracelets shown are closed with no twist (top photo) and closed with four full twists (bottom photo).

Because of the short repeat, sizing is extremely flexible.

Tags: Geometric Cross Section, Wallpaper Group P6M, 7-around



Irregular Hexagon



1 1 1 1 1 2

Notes:

10-around

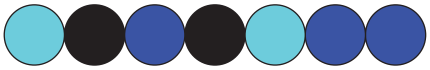
7 bead repeat

There are 144 repeats in an average sized bracelet using size 11 seed beads and size 8 seed beads (shown in black) for the spines with size 20 crochet thread.

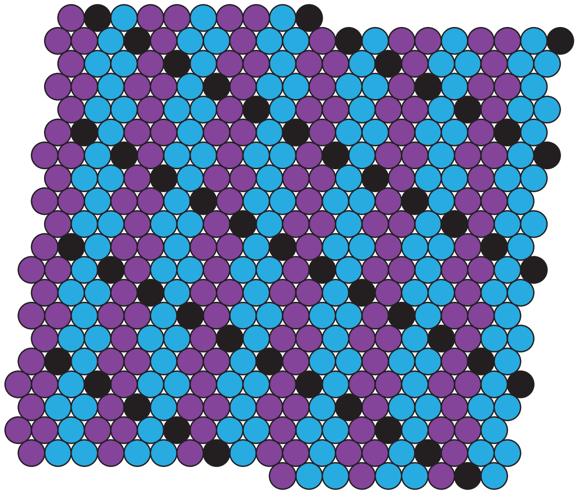
Use a multiple of 3 repeats and close without a twist to get the hexagonal cross section. Alternatively, use a number of repeats that is not a multiple of 3 to construct torus knot designs.

Because of the short repeat, sizing is very flexible.

Tags: Geometric Cross Section, Wallpaper Group P2, 10-around



Hexagon with Spiral



1 2 1 2 1 1 1 2 1 2 1 1

Notes:

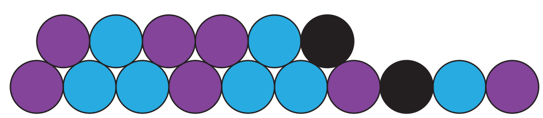
9-around

16 bead repeat

There are 72 repeats in an average sized bracelet using size 11 Delicas and size 11 seed beads (shown in black) for the spines. Use a multiple of 3 repeats to form the hexagonal shape. This bracelet has a less distinct hexagonal feel than other geometric cross-section designs due to the greater vertical distance between the spine beads.

Carefully untwist the natural spiral of the six sides before joining to achieve the hexagonal cross section.

Tags: Geometric Cross Section, Wallpaper Group P2, 9-around



Heptagon

Notes:

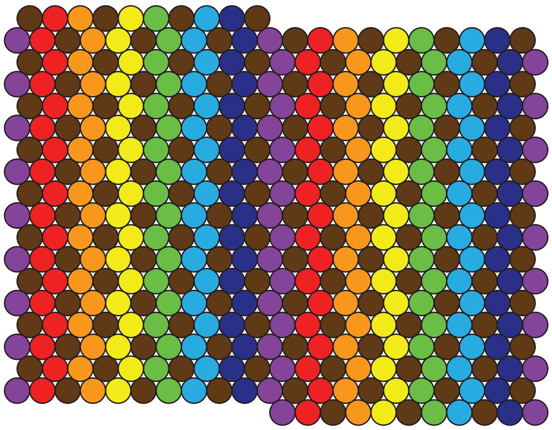
10-around

21 bead repeat

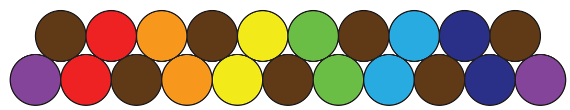
There are 60 repeats in an average sized bracelet using size 11 Delicas and size 11 seed beads (shown in brown) for the spines.

This pattern is vertically aligned and has extremely flexible sizing.

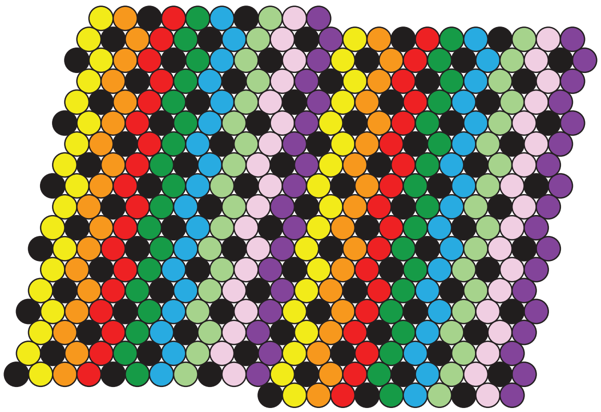
Tags: Geometric Cross Section, 10-around



1 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1



Octagon



Notes:

10-around

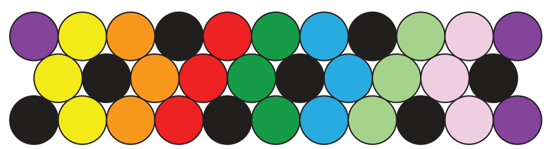
32 bead repeat

There are 40 repeats in an average sized bracelet using size 11 Delicas and size 11 seed beads (shown in black) for the spines. Because the repeat is short compared to the circumference, sizing is very flexible.

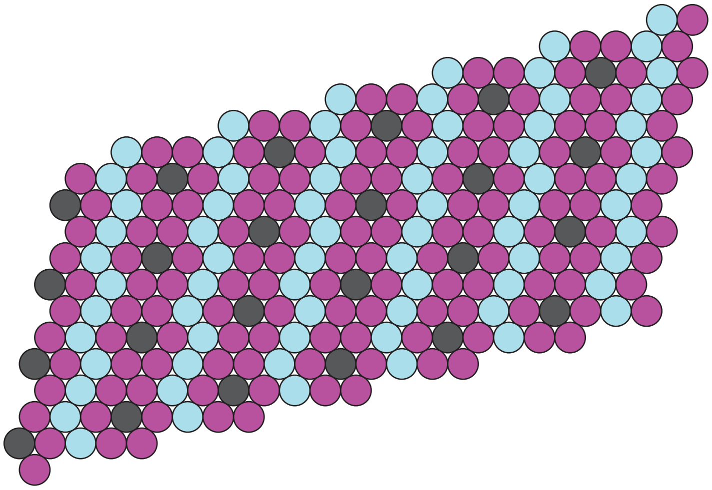
The two bracelets shown below use two different twist variations.

Tags: Geometric Cross Section, 10-around

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1



Möbius Band 6-around



1 1 1 2 1 2 1 1

or try an unstriped Mobius:

1 9

Notes:

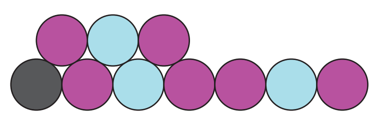
6-around

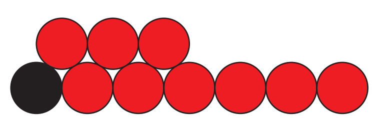
10 bead repeat

There are 75 repeats in an average sized Möbius bracelet using size 11 seed beads with 3.4mm drop beads (shown in gray). Use an odd number of repeats for a Möbius band.

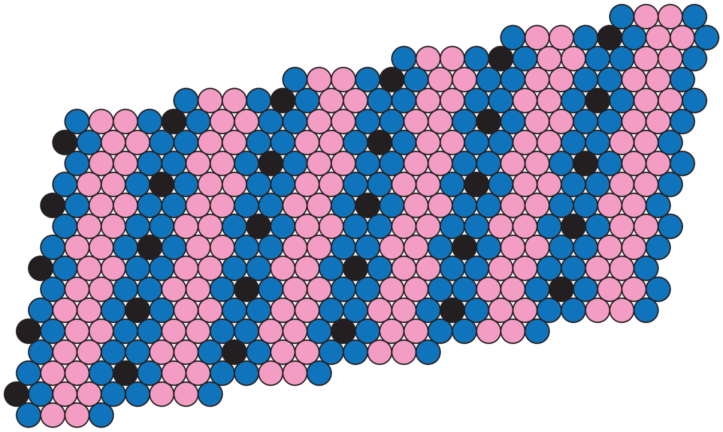
The pattern will naturally spiral once around the meridian about every 26 repeats. Thus the natural spiral of a 75 repeat bracelet will include about three wraps of the drop beads and make a (3,2) torus knot. The bracelets shown were made by partially untwisting before sewing to create a (1,2) torus knot. This pattern can also be made with an even number of repeats to create a two-sided symmetric caterpillar bracelet.

Tags: Möbius, Torus Knot, Wallpaper Group P2, 6-around





Möbius Band 8-around



1 1 2 2 2 2 2 1

Notes:

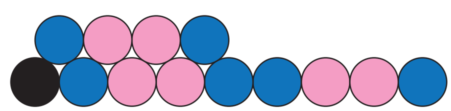
8-around

13 bead repeat

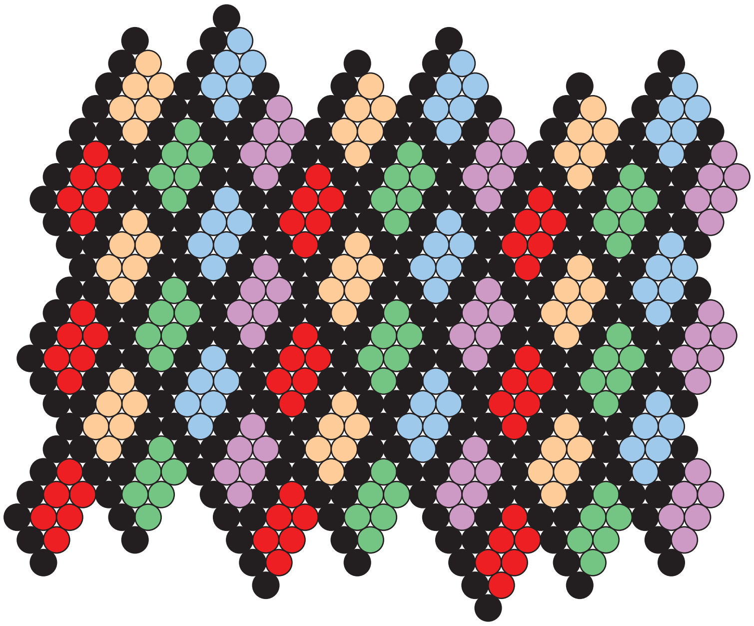
There are 83 repeats in an average sized bracelet using size 11 Delicas and 3.4mm drop beads (shown in black).

Use an odd number of repeats for a Möbius band. To create the Möbius half-twist, untwist the pattern's natural spiral until the spines only wrap once around the meridian of the bracelet before sewing closed. For a solid color version, replace the pink and blue with a single color. For a chunkier effect, try this pattern with size 11 seed beads instead of the Delicas, and use size 20 crochet thread.

Tags: Möbius, Torus Knot, Wallpaper Group P2, 8-around



Stained-Glass Diamonds



1	1	1	2	2	1	2	2
1	1	1	2	2	1	2	2
1	1	1	2	2	1	2	2
1	1	1	2	2	1	2	2
1	1	1	2	2	1	2	2

Notes:

8-around

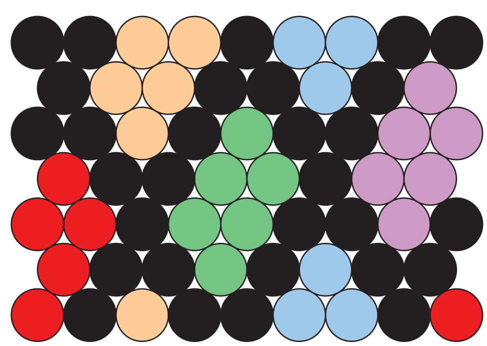
60 bead repeat

For the faceted effect, use larger beads for the diagonals than for the diamonds. The average sized bracelets photographed here have 17 repeats in size 10 and size 11 Delicas (in front) and 14 repeats in size 8 and size 11 seed beads (behind). Use size 20 crochet thread for the seed bead version.

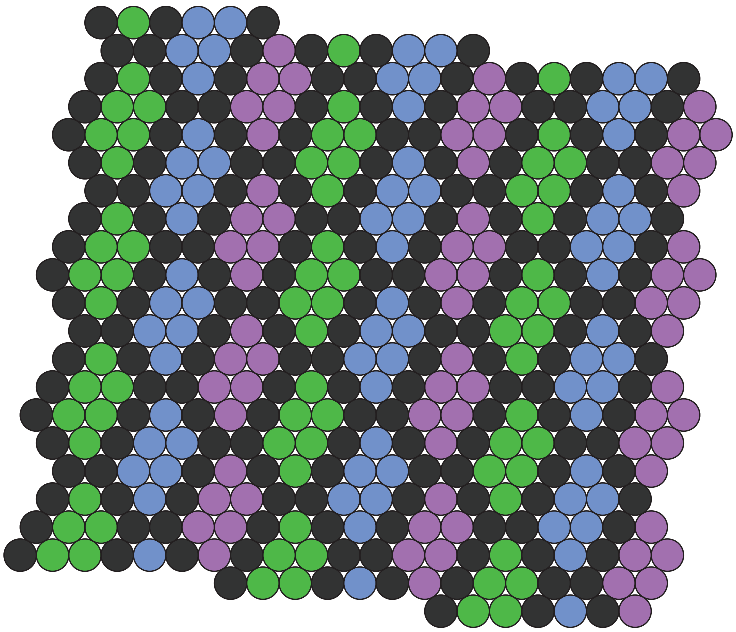
A minor twist lines up every repeat as in the bracelets shown.

To create the appearance of stained glass, use foil-lined or color-lined transparent beads for the colored diamonds and outline them in dark, matte beads.

Tags: Faceted, 8-around



Diamond Zigzag



2 2 1 1 1 1 1 2
2 2 1 1 1 1 1 2
2 2 1 1 1 1 1 2

Notes:

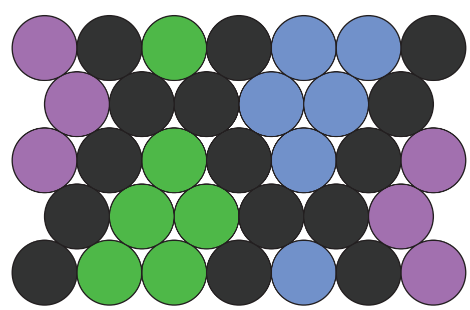
6-around

33 bead repeat

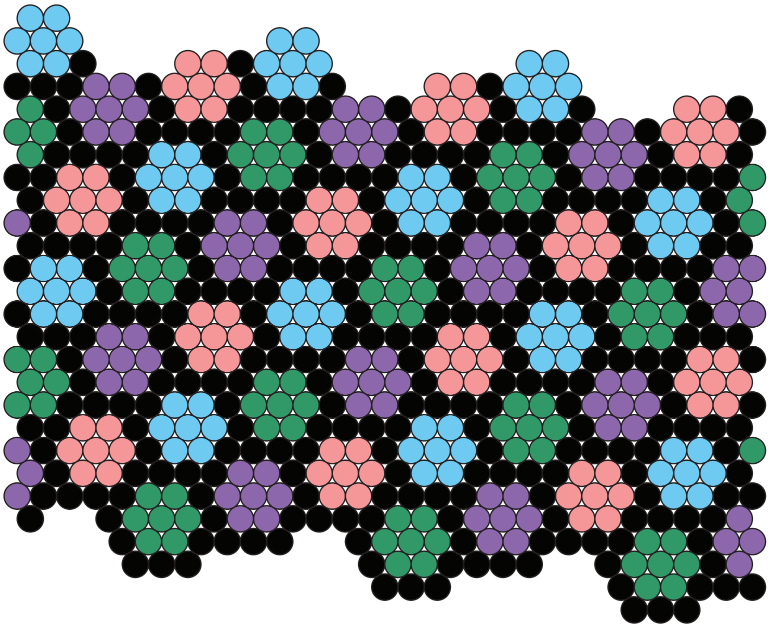
There are 19 to 20 repeats in an average sized bracelet using size 11 seed beads for the diamonds and size 8 seed beads for the outlines. The different bead sizes create the bracelet's faceted texture.

Every repeat can be aligned with a minor twist, as shown in the bottom right of Figure 1.20 on p. 19. The bracelet in the photograph, which is closer to the natural alignment of the design, allows each color to wrap twice around the bracelet.

Tags: Faceted, 6-around



Honeycomb



4 2 1 3 1 2
4 2 1 3 1 2
4 2 1 3 1 2
4 2 1 3 1 2

Notes:

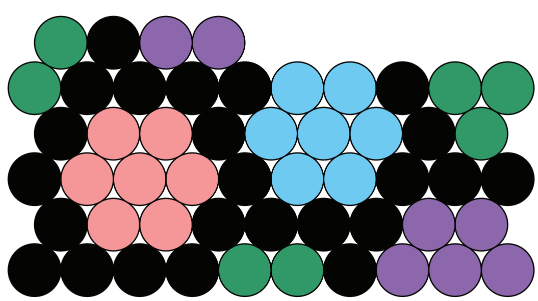
9-around

52 bead repeat

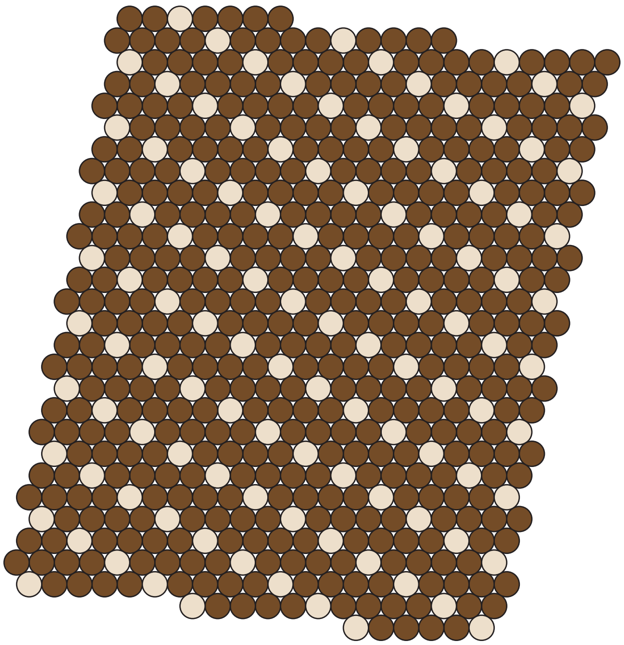
With an even number of repeats (20 or 22 in an average sized bracelet), every other repeat can be aligned. The textured effect comes from using size 11 Delicas for the hexagons and size 10 Delicas for the outlines. Outlining with size 11 seed beads produces a similar effect. The larger beads in circumference 9 give this bracelet a bangle-like stiffness.

While the pattern diagrammed here uses four colors for the hexagons (shown in the left of the photograph), it can be modified to incorporate more colors. The upper bracelet on the right has five colors; the lower bracelet, seven.

Tags: Faceted, 9-around



(P,4) Torus Knot 6-around



1 4

Notes:

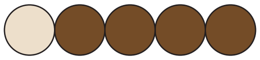
6-around

5 bead repeat

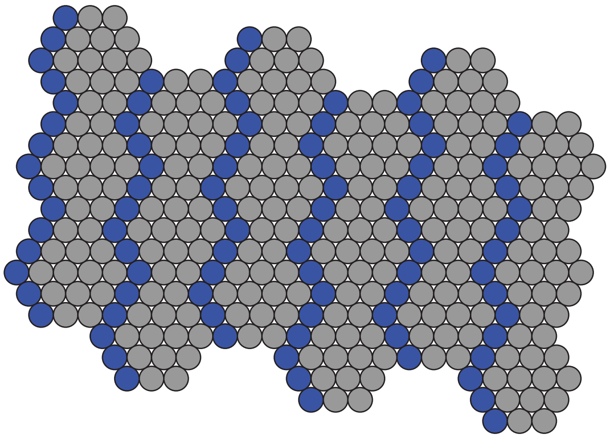
There are 135 repeats in an average sized bracelet using size 11 seed beads and 2.8mm drop beads for the spines (shown in off-white). You can make a (3,4) torus knot like the one photographed here with an odd number of repeats and an appropriate twist before closing to achieve the desired P = 3 meridian wraps. Note that a multiple of 4 repeats gives a square cross-section bracelet.

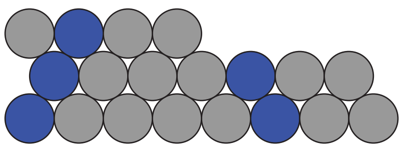
Because of the short repeat, sizing is very flexible.

Tags: Torus Knot, Geometric Cross Section, Wallpaper Group P2, 6-around



Zigzag (3,2) Torus Knot





1 4 1 2 1 3 1 3 1 2

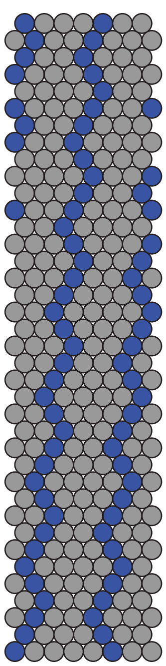
Notes:

7-around

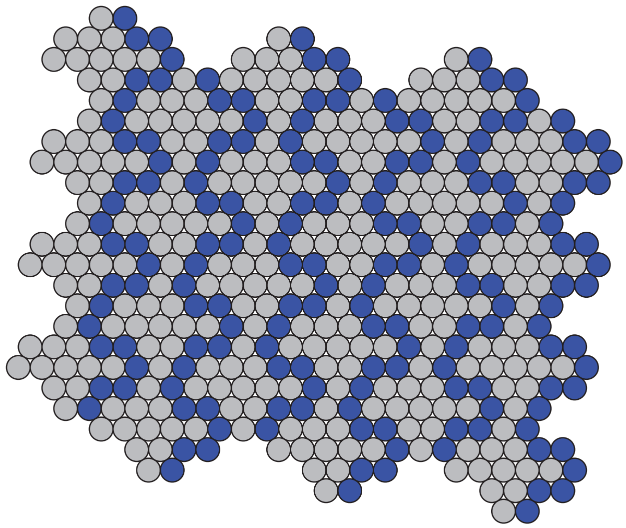
19 bead repeat

There are 45 repeats in a large sized bracelet using 2mm sterling silver rounds and size 10 seed beads, making a (3,2) torus knot with no twist. For a smaller bracelet, leave off a few repeats, but make sure the total number of repeats is odd. Take care to ensure the twist you use creates the torus knot you want. This pattern is based on the Hexagonal Grid pattern (p. 166). In addition to a single repeat, we show below a vertical segment of 15 repeats that completes a single meridian wrap.

Tags: Torus Knot, Wallpaper Group P2, 7-around



Squiggly (3,2) Torus Knot



1 3 2 2 2 1 1 5 1 1

or

1 2 2 2 1 4 1 1 1 4

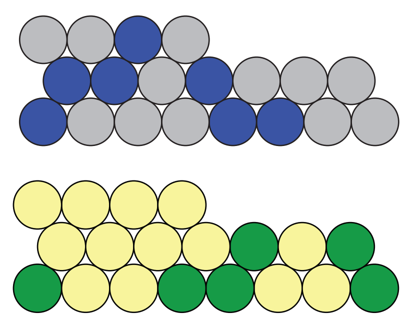
Notes:

7-around

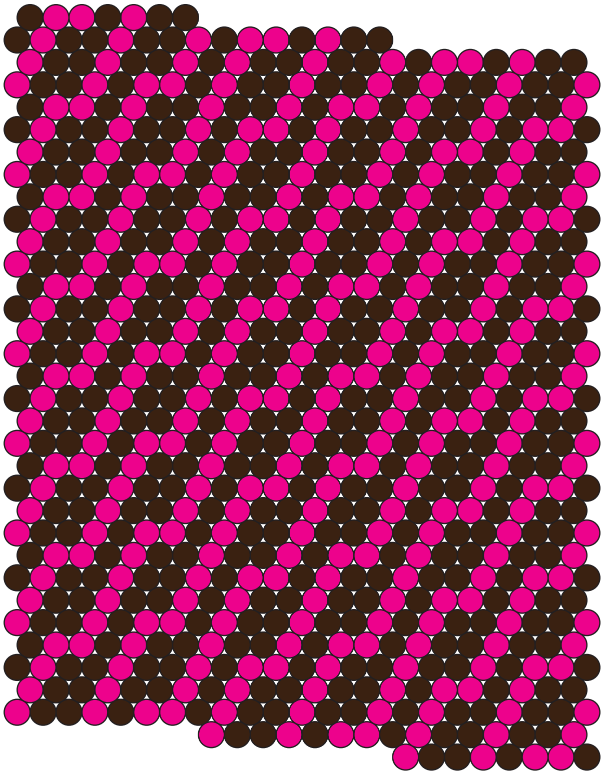
19 bead repeat

There are 41 to 45 repeats in an average sized bracelet using size 11 seed beads and size 10 seed beads for the "string." A bracelet with exactly 45 repeats forms a (3,2) torus knot with no twist, whereas one with 43 or 41 repeats will have a small structural twist.
A torus knot requires an odd number of repeats. This pattern is based on the Hexagonal Grid pattern (p. 166). An alternate variation is given in green and cream.

Tags: Torus Knot, 7-around



Long Zag (P,3) Torus Knot



1 2 1 1 2 1 1 2
1 2 1 1 1 2
1 2 1 1 2 1 1 2

Notes:

7-around

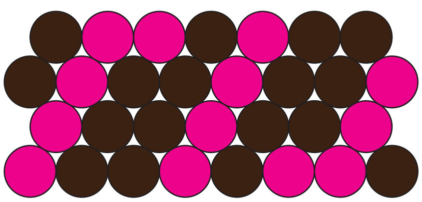
30 bead repeat

There are 26 repeats in an average sized bracelet using size 11 seed beads.

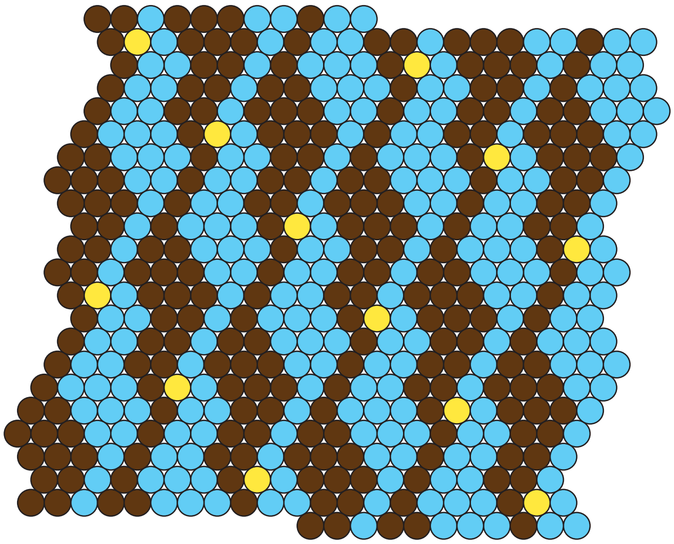
This pattern is vertically aligned, and each repeat produces a meridian wrap, so P repeats with no twist forms a (P,3) torus knot or link. If P is a multiple of 3, a torus link results; otherwise, a torus knot results.

Use a physical twist before closing if you want to change the alignment and the number of meridian wraps. The bracelet shown uses 26 repeats and two full twists to create a (20,3) torus knot instead of the (26,3) torus knot that would have resulted without the twist.

Tags: Torus Knot, Wallpaper Group P2, 7-around



Knotted Snakes



1 1 3 1 1 2 2 1 3 2 1 2
2 1 2 3 1 2 2 1 1 3 1

Notes:

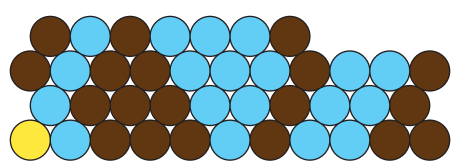
10-around

39 bead repeat

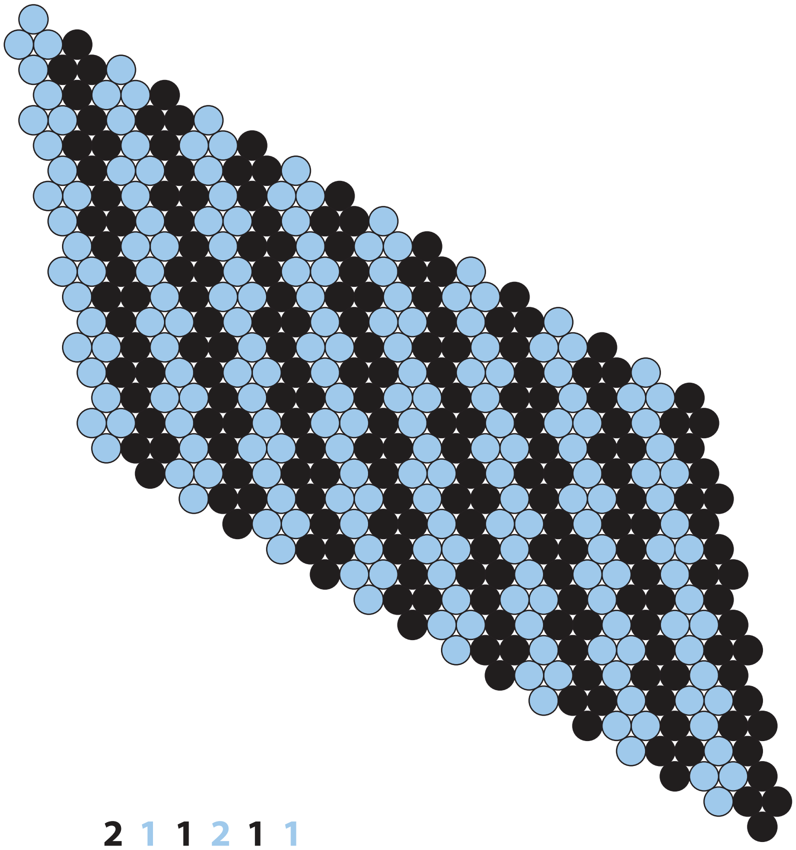
There are 28 repeats in a small sized bracelet with no twist using size 11 seed beads and size 20 crochet thread. You can also use size 11 Delicas with additional repeats and a twist.

This design is vertically aligned by 7 and was constructed to produce interlocking (4,3) torus knots with no twist using the patchwork design method described in Chapter 4. As long as the number of repeats is not a multiple of 3, you can obtain (4,3) torus knots with a suitable twist. The interlocking knots are rotated 180° from one another around the gold accent beads.

Tags: Torus Knot, 10-around



Trefoil Dissection



2 1 1 2 1 1

Notes:

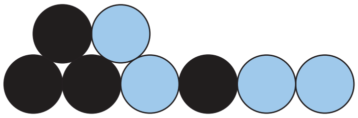
5-around

8 bead repeat

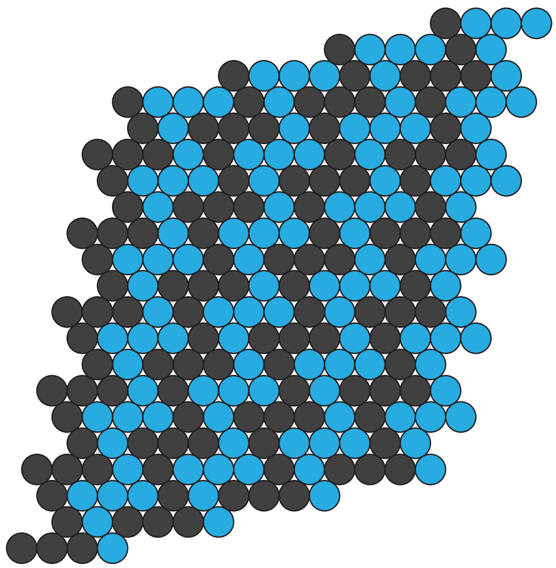
There are 57 repeats in an average sized bracelet using size 8 seed beads. The short repeat makes sizing very flexible.

Use an odd number of repeats to make a torus knot. The natural spiral of the bracelet forms an interlocking pair of (5,2) torus knots. To obtain a pair of trefoil knots as in the photograph, untwist the spiral until the beads of each color only cross the meridian of the bracelet three times instead of five.

Tags: Torus Knot, Escher, Wallpaper Group P2, 5-around



Escher (P,2) Torus Knot



3 1 1 3 1 1

Notes:

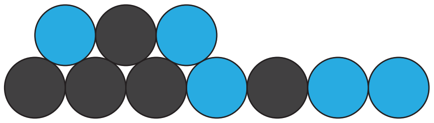
6-around

10 bead repeat

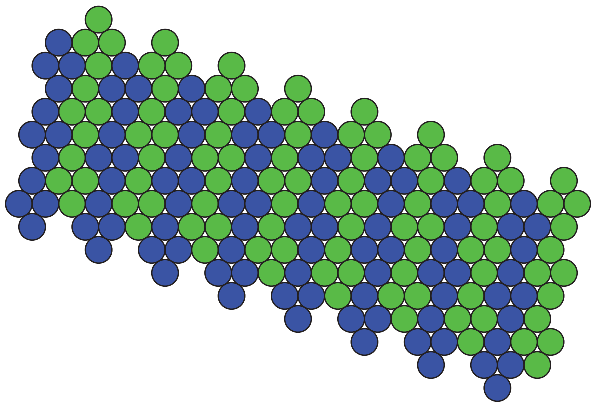
The average sized bracelet shown here has 71 repeats using size 11 seed beads. The short repeat makes sizing very flexible.

Use an odd number of repeats with an appropriate twist to construct interlocking torus knots in the two colors. The amount of twist determines the value of P. In the bracelet photographed here, there are two interlocking (5,2) torus knots, so P = 5. If the pattern is instead twisted so that P = 3, the result is two interlocking trefoil knots.

Tags: Torus Knot, Escher, Wallpaper Group P2, 6-around



Escher (P,4) Torus Knot 10-around



Notes:

10-around

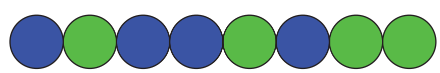
8 bead repeat

Use a multiple of 4 repeats to get two sets of four identical stripes. Use an odd number of repeats to get a (P,4) torus knot. For example, the bracelet in the larger photo has 157 repeats in size 11 Delicas and is twisted to create a (5,4) torus knot. The smaller photo shows an alternate non-Escher variation with a different twist used to create a (7,4) torus knot.

Because of the short repeat, sizing is very flexible.

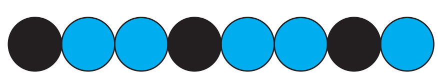
Tags: Torus Knot, Escher, Wallpaper Group P2, 10-around

1 1 2 1 1 2

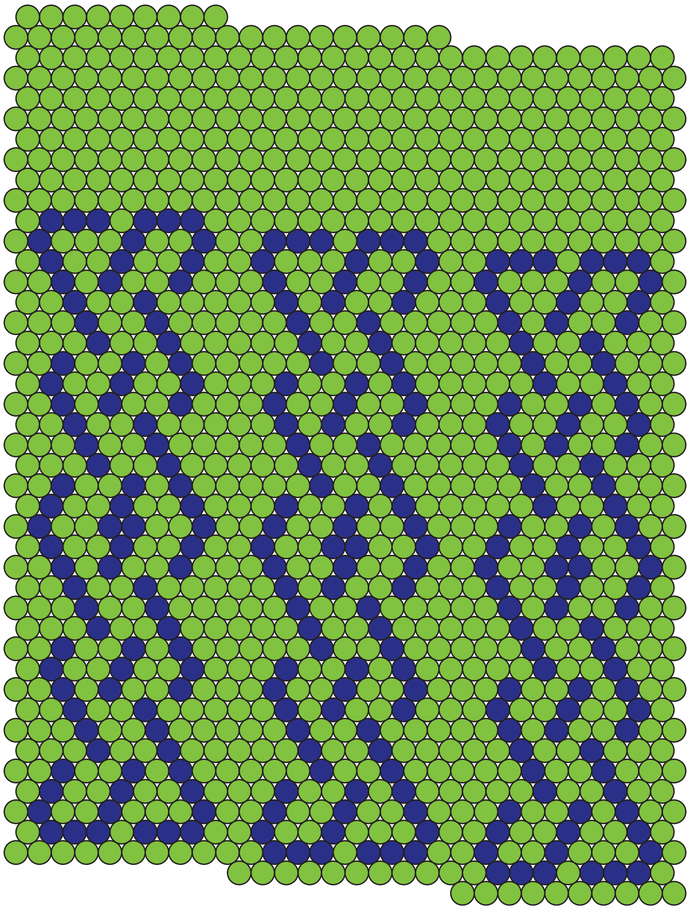


or try the non-Escher variation at left:

1 2 1 2 1 1



Single Strand Celtic Knot



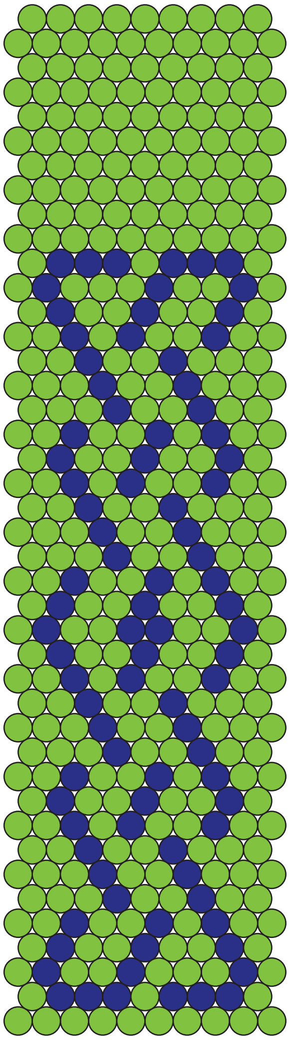
Notes:

9-around

399 bead repeat

There are 3 repeats in an average sized bracelet using size 11 Delicas. Because of the extreme length of the repeat, we omit the numerical pattern.

Tags: Celtic Knot, Wallpaper Group P2, 9-around



Three-Strand Celtic Knot

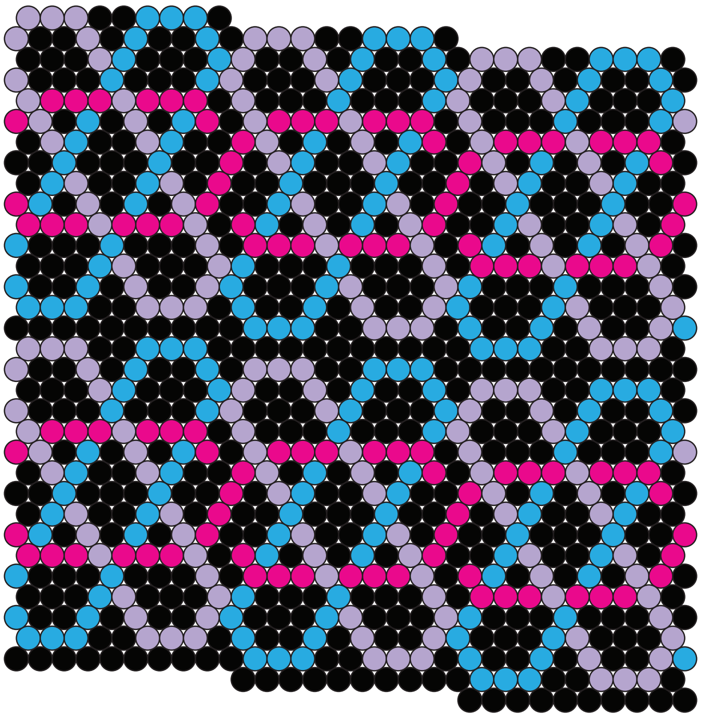
Notes:

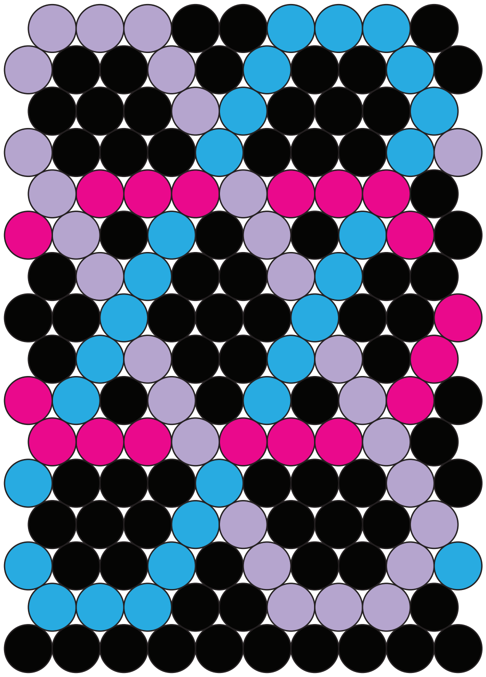
9-around

152 bead repeat

There are 8 repeats in an average sized bracelet using size 11 Delicas. This pattern is vertically aligned.

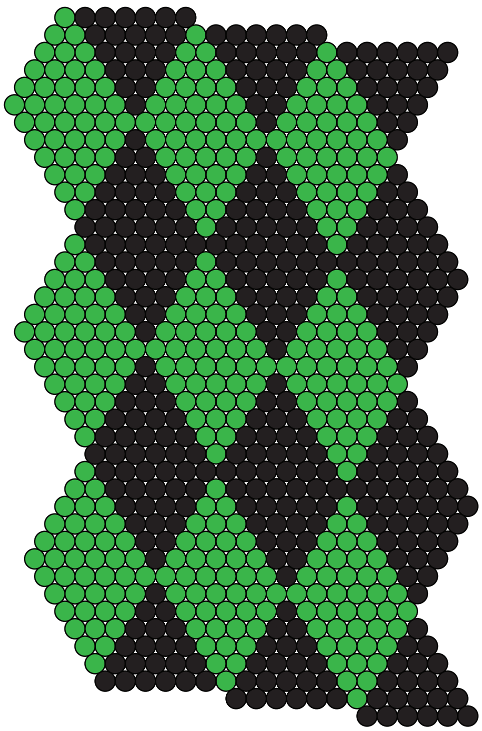
Tags: Celtic Knot, Wallpaper Group P2, 9-around





[1 x 11] 3 2 3 1 1 2 1 1 1 2 1 1 3 1
1 3 1 1 3 1 3 1 1 3 1 3 1 1 1
1 1 1 1 1 1 1 1 2 1 1 2
1 1 1 1 2 1 3 1 2 1 1 1 1
2 1 1 2 1 1 1 1 1 1 1 1
1 1 1 3 1 3 1 1 3 1 3 1 1 3 1
1 3 1 1 2 1 1 1 2 1 1 3 2 3

Harlequin



Notes:

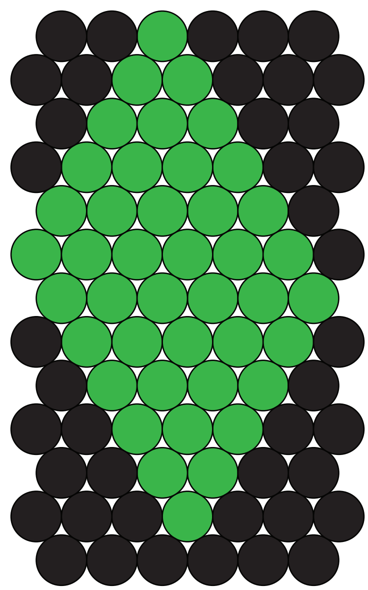
6-around

84 bead repeat

There are 8 repeats in an average sized bracelet using size 11 seed beads.
A small twist lines up every repeat as in the bracelet shown.

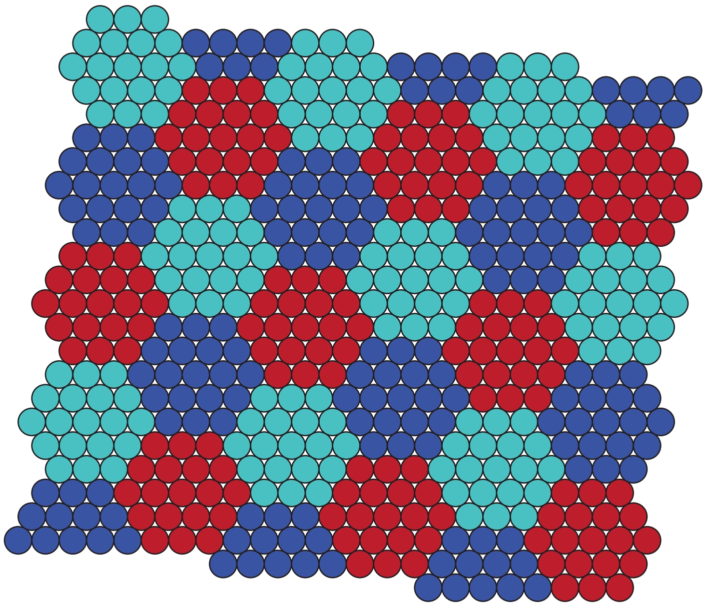
This design is useful as a template for creating other Escher designs.

Tags: Escher, Wallpaper Group P2, 6-around



12 1 5 2 4 3 3 4 2 5 1
12 1 5 2 4 3 3 4 2 5 1

Hexagonal Grid



5 3 4 4 3
5 3 4 4 3
5 3 4 4 3

Notes:

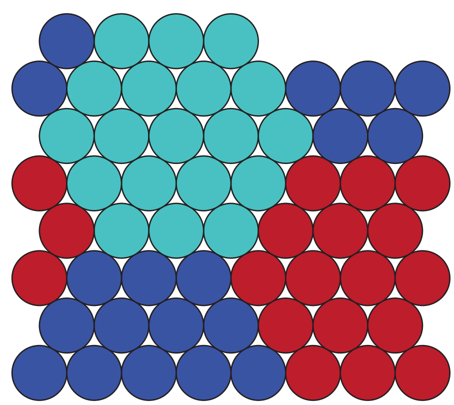
7-around

57 bead repeat

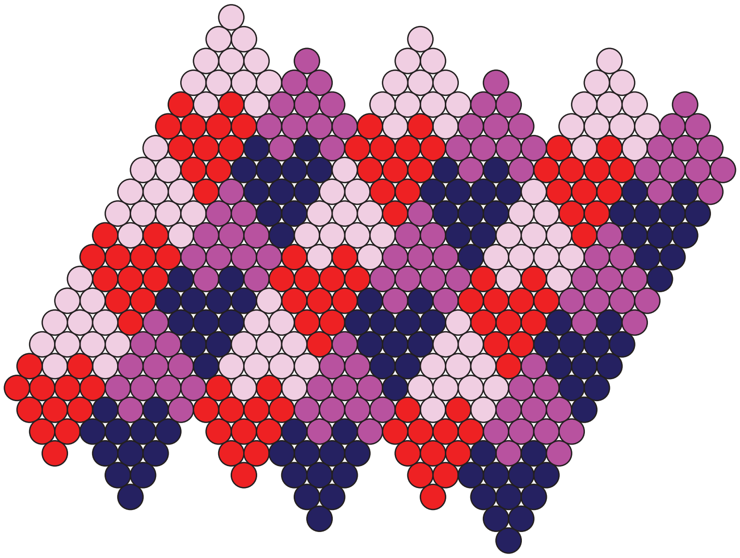
There are 14 repeats in an average sized bracelet using size 11 seed beads.

Use a multiple of 2 repeats and a twist to line up every other repeat as in the bracelet shown. This pattern is a useful template for other designs.

Tags: Escher (tricolor), Wallpaper Group P3, 7-around



Four-Color Valentine



Notes:

7-around

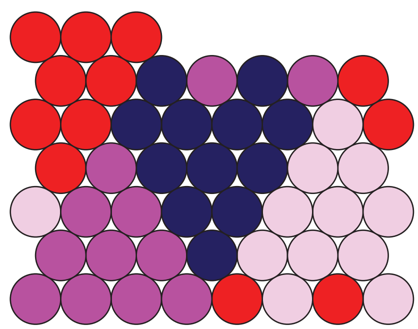
48 bead repeat

There are 18 repeats in an average sized bracelet using size 11 seed beads.

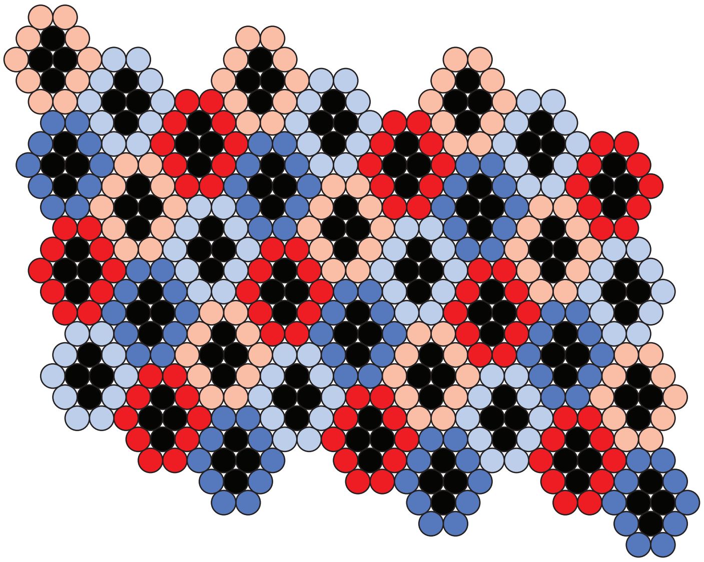
Use a multiple of 3 repeats and a small twist to line up every third repeat as in the bracelet shown.

Tags: Escher (four-color), 7-around

4 1 1 1 1 3 1 4 2 2 3 1
1 3 2 2 4 1 3 1 1 1 1 4



Diamond Glow



Notes:

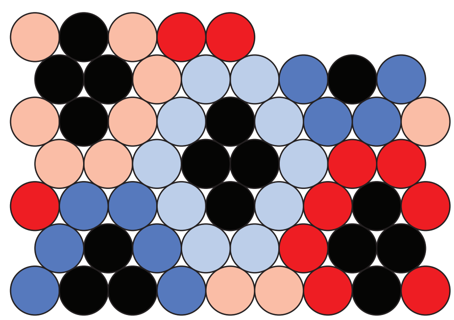
8-around

56 bead repeat

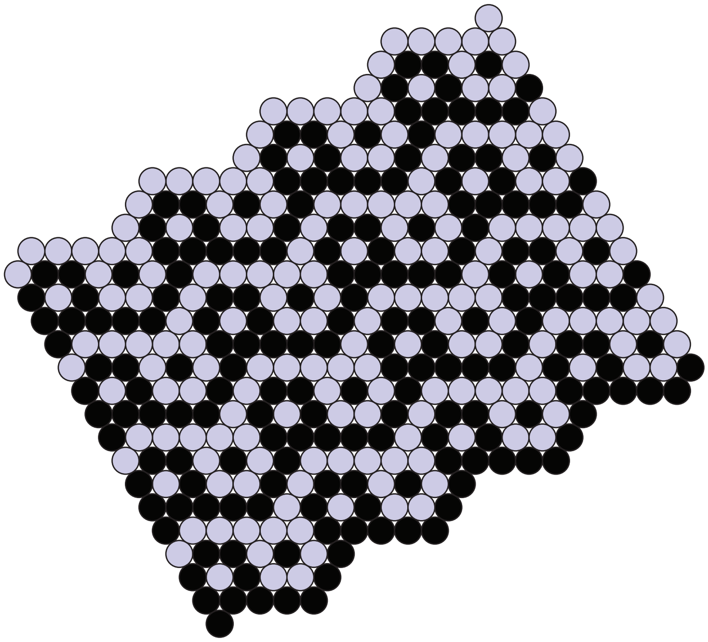
There are 20 repeats in an average sized bracelet using size 11 Delicas. Be sure to choose four colors that contrast well for the outlines of the diamonds.

Tags: Escher (four-color), 8-around

1	2	1	2	1	1	1	1	1	1	2
1	2	1	2	1	1	1	1	1	1	2
1	2	1	2	1	1	1	1	1	1	2
1	2	1	2	1	1	1	1	1	1	2



Yin-Yang Reflection



5 1 1 1 1 2 1
1 2 1 1 1 1 5

Notes:

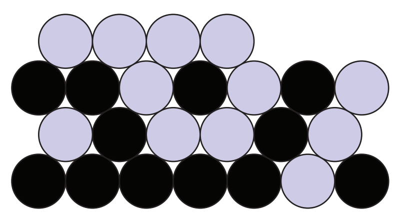
6-around

24 bead repeat

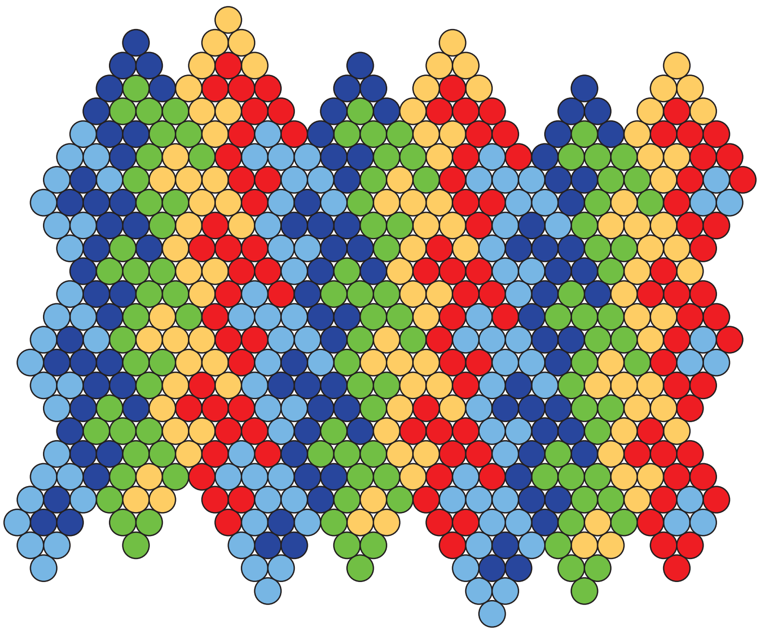
There are 33 repeats in an average sized bracelet using size 11 Delicas, or 30 repeats using size 11 seed beads.

Use a multiple of 3 repeats and a slight twist before sewing to line up every third repeat, as in the bracelet shown.

Tags: Escher, Wallpaper Group PM, 6-around



Five-Color Jigsaw



Notes:

8-around

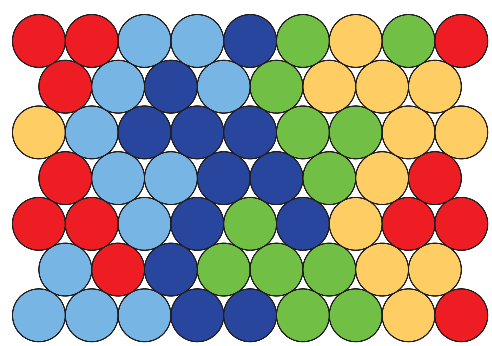
60 bead repeat

Use 18 repeats in size 11 Delicas for an average sized bracelet. The number of repeats does not affect the alignment, so sizing is flexible.

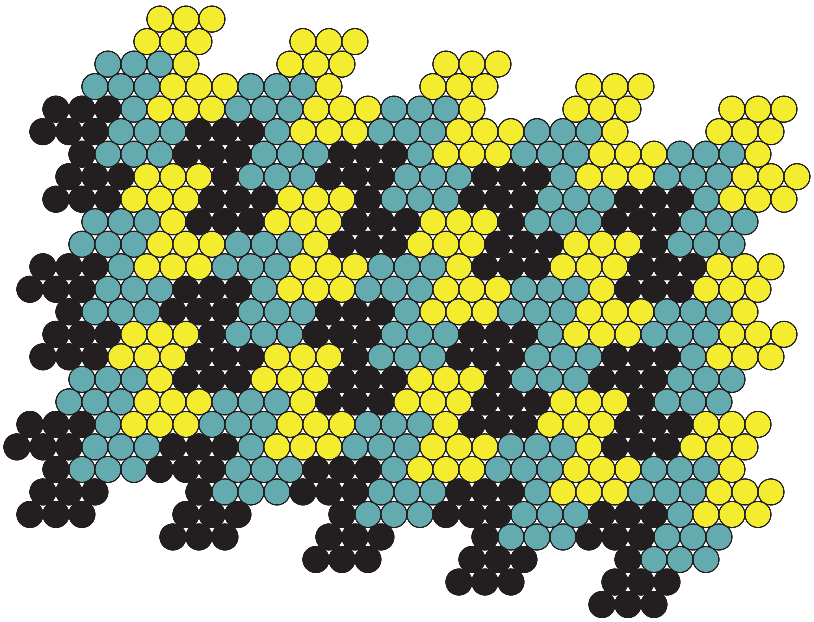
A minuscule twist lines up every repeat, as in the bracelet on the right. The bracelet on the left is twisted in the opposite direction nearly two full twists.

Tags: Escher (five-color), 8-around

3	2	2	1	1	1	1	1
3	2	2	1	1	1	1	1
3	2	2	1	1	1	1	1
3	2	2	1	1	1	1	1
3	2	2	1	1	1	1	1



Naptime



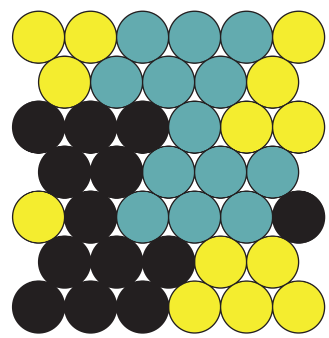
Notes:

5-around

39 bead repeat

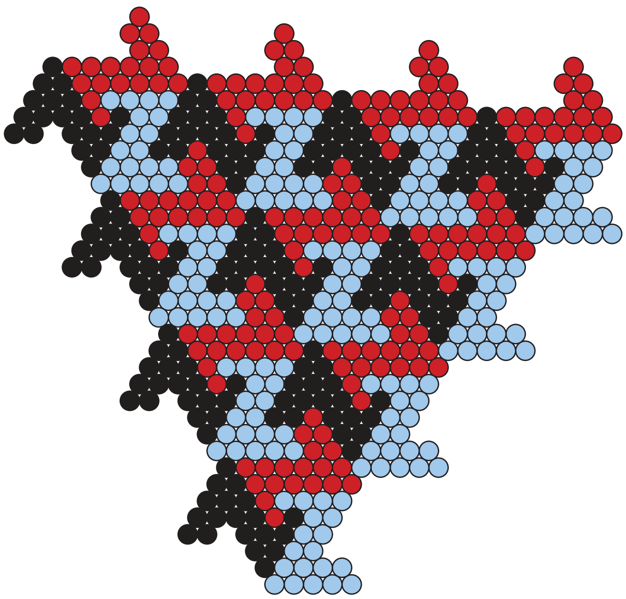
There are 11 repeats in an average sized bracelet using size 8 seed beads. Use a small twist to line up every repeat as in the bracelet shown.

Tags: Escher (tricolor), 5-around



3 3 3 3 1
3 3 3 3 1
3 3 3 3 1

Flying Z's



1 6 5 2 1 4 2 2 2 2 1
3 2 4 1 1 2 3 1 4 2 6

Notes:

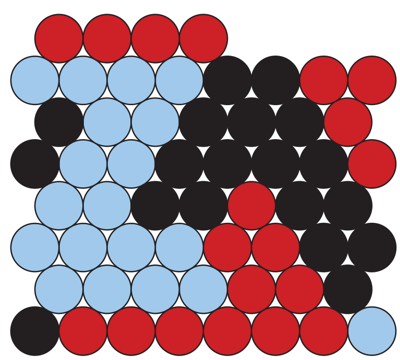
7-around

57 bead repeat

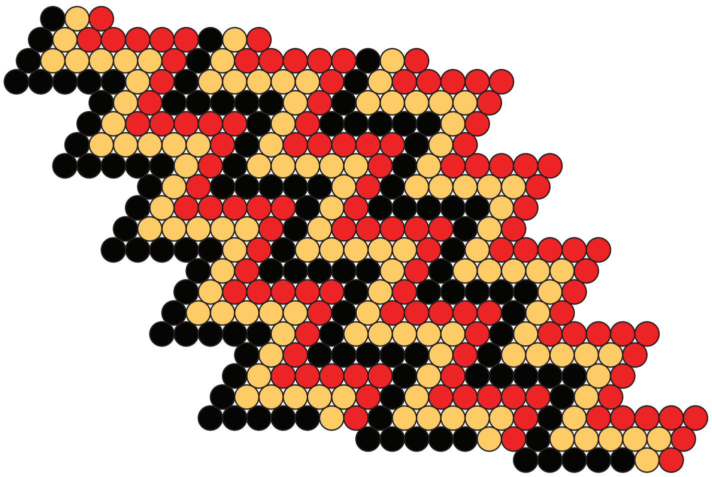
There are 14 or 15 repeats in an average sized bracelet using size 11 seed beads.

The bracelet pictured below has 15 repeats and no twist, which lines up every fifth repeat. A small twist will align every third repeat instead. Alternately, if you use a multiple of 2 repeats, you can line up every other repeat with a small twist in the opposite direction.

Tags: Escher (tricolor), 7-around



Tricolor Zigzag



Notes:

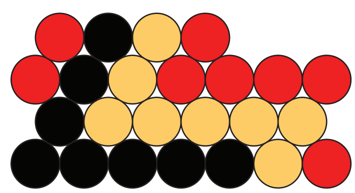
6-around

24 bead repeat

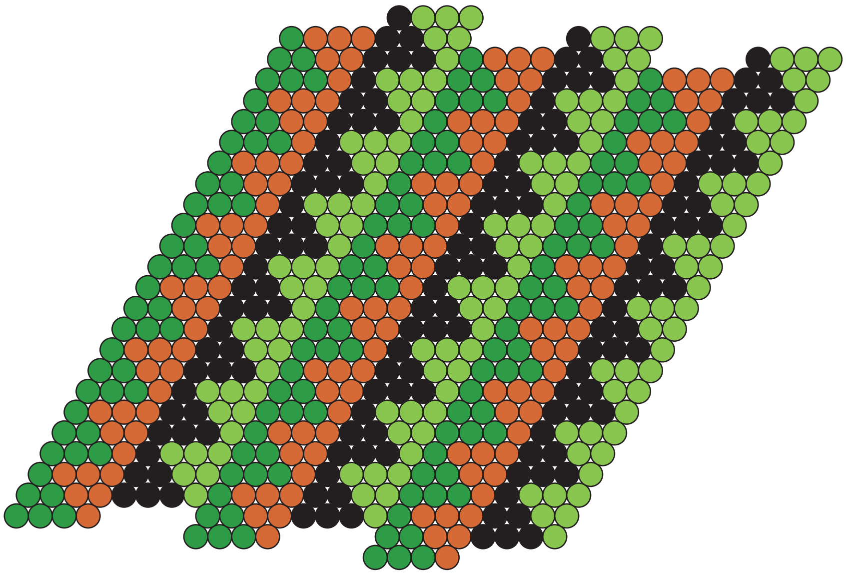
There are 30 repeats in an average sized bracelet using size 11 seed beads. Since the repeat is short and alignment does not have a significant effect on the appearance of a bracelet, sizing is very flexible.

Tags: Escher (tricolor), Wallpaper Group PM, 6-around

5 1 1 1
5 1 1 1
5 1 1 1



Four-Color Sawtooth



Notes:

7-around

24 bead repeat

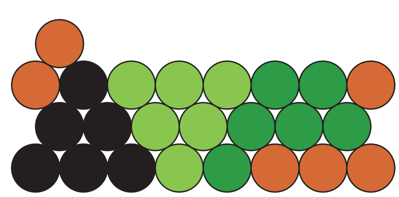
There are 33 to 35 repeats in an average sized bracelet using size 11 seed beads. Sizing is very flexible.

This pattern is related to a common left-leaning sawtooth design in two colors. The smaller teeth in this design allow two parallel tracks of two colors each to spiral around the bracelet.

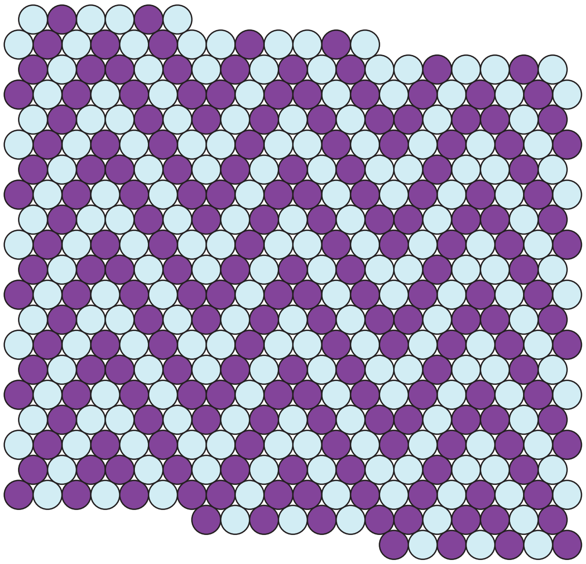
Using two colors instead of four can produce either of the designs in the right of the photograph.

Tags: Escher (four-color), Wallpaper Group PM, 7-around

3 1 1 3
2 2 3 1
1 3 2 2



Sophie's Herringbone



Notes:

6-around

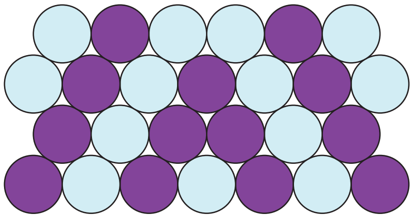
26 bead repeat

There are 28 repeats in an average sized bracelet using size 11 seed beads. Since the pattern is vertically aligned and the repeat is short, sizing is extremely flexible.

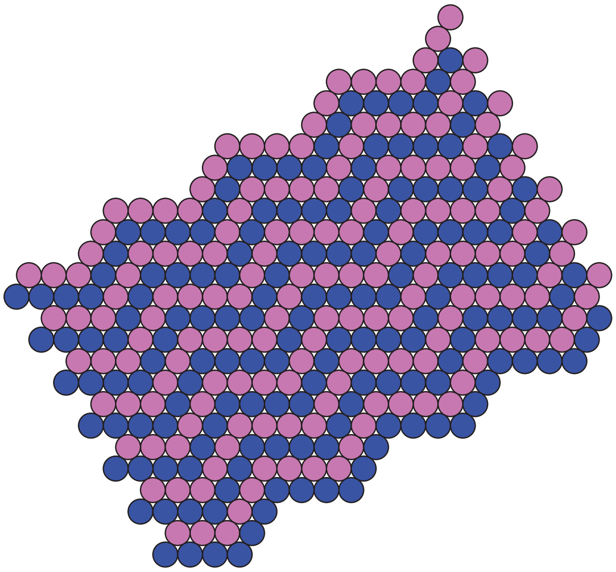
The bracelet shown has 26 repeats in 2mm sterling silver rounds and size 11 seed beads. Use care when closing to avoid a twist, unless you want one!

Tags: Escher, Wallpaper Group P2, 6-around

1 1 1 1 1 1 1 1 2 1 2
1 1 1 1 1 1 1 1 2 1 2



Herringbone Reflection



Notes:

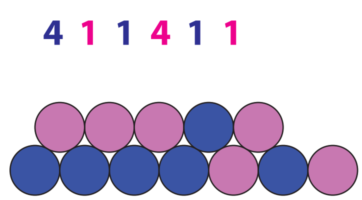
6-around or 5-around

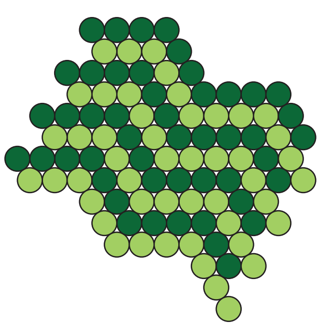
12 bead repeat

There are 62 repeats in an average sized 6-around bracelet using size 11 seed beads, as shown in the photograph.

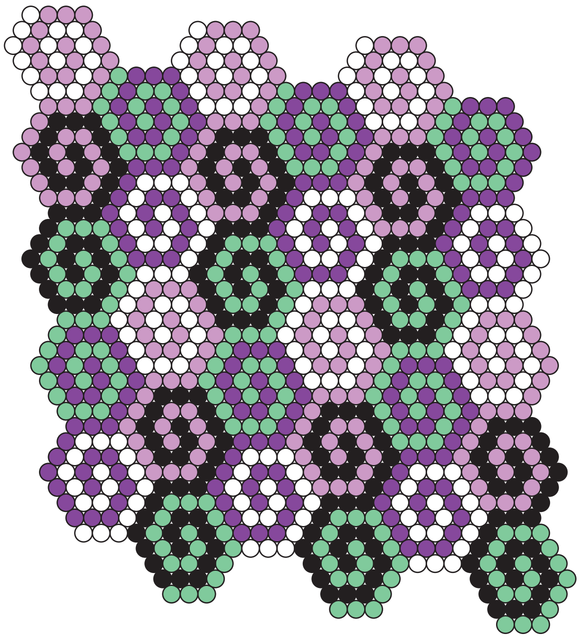
This bracelet can also be done as a 5-around in size 8 beads with the same numerical repeat, as shown in the green chart below. In the 5-around version, which takes around 35 repeats, the reflection axis leans right instead of left. Sizing is extremely flexible in both circumferences.

Tags: Escher, Wallpaper group CMM, 6-around, 5-around





Ultimate Swirl



Notes:

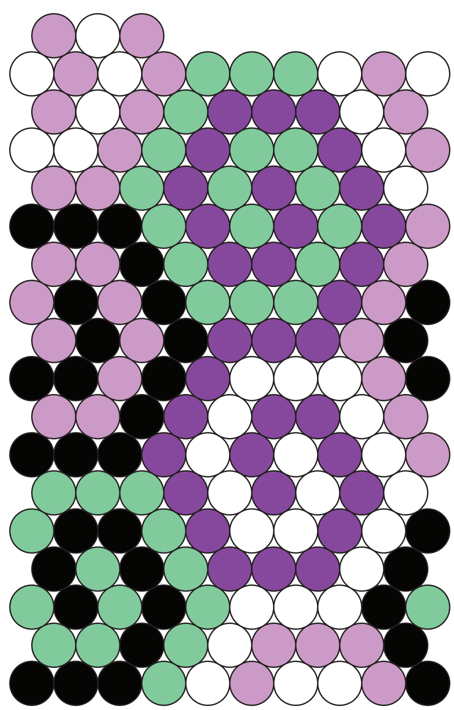
9-around

165 bead repeat

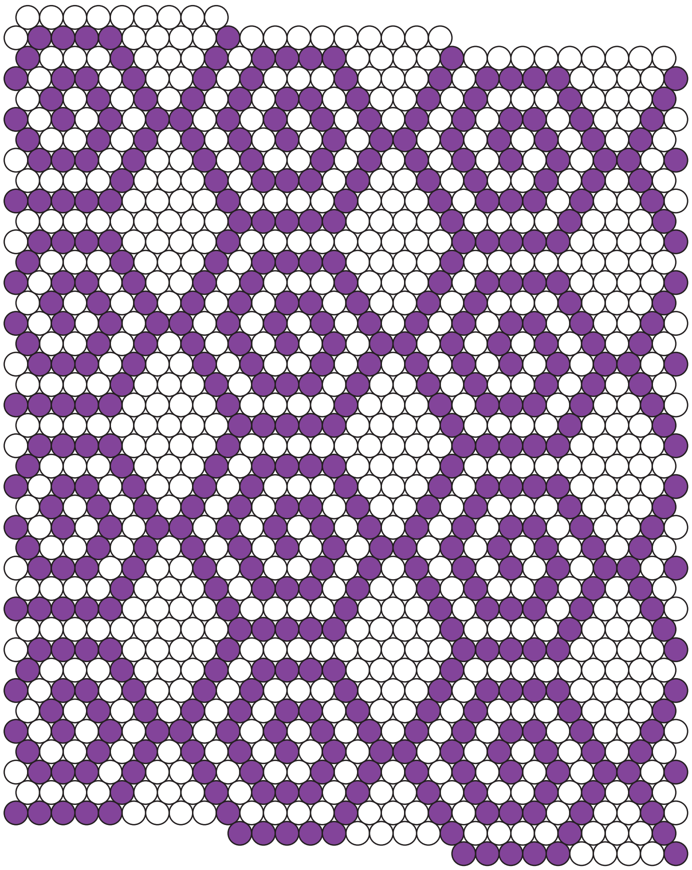
There are 8 repeats in a large sized bracelet using size 11 Delicas. Given the large repeat, sizing is not very flexible with this design. Choose five colors that all contrast well with one another.

Tags: Escher (five-color), 9-around

3 1 1 1 2 1 1 2 1 1 1 3 1 1 1 1 1 1 3 1 1 1 1 1 1
3 1 1 1 2 1 1 2 1 1 1 3 1 1 1 1 1 1 3 1 1 1 1 1 1
3 1 1 1 2 1 1 2 1 1 1 3 1 1 1 1 1 1 3 1 1 1 1 1 1
3 1 1 1 2 1 1 2 1 1 1 3 1 1 1 1 1 1 3 1 1 1 1 1 1
3 1 1 1 2 1 1 2 1 1 1 3 1 1 1 1 1 1 3 1 1 1 1 1 1



Coils and Diamonds



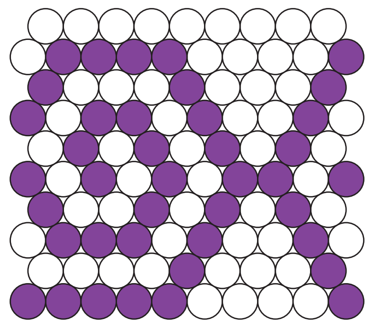
Notes:

9-around

95 bead repeat

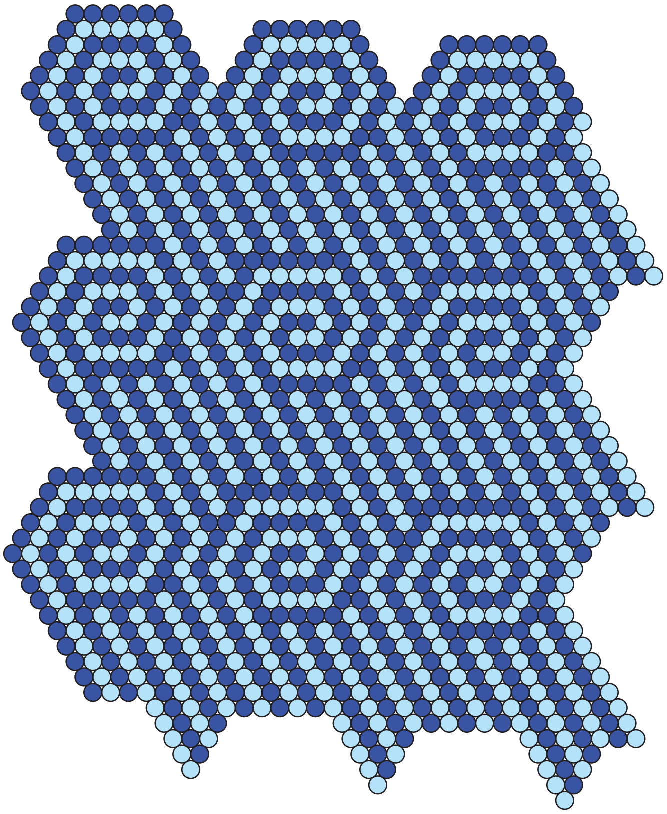
There are 13 repeats in an average sized bracelet using size 11 Delicas, as in the bracelet shown (front and back). This design is vertically aligned.

Tags: 9-around



5 4 **1** 4 **1** 3 **1** 1 **3** 1 **1** 2 **1** 1 **1** 2
1 1 **1** 1 **1** 1 **1** 1 **1** 1 **1** 1 **2**
1 **1** 1 **1** 1 **1** 1 **1** 1 **1** 1 **1** 1 **2**
1 **1** 2 **1** 1 **1** 3 **1** 3 **1** 1 **4** 4 **1** 9

Foamy Wave



Notes:

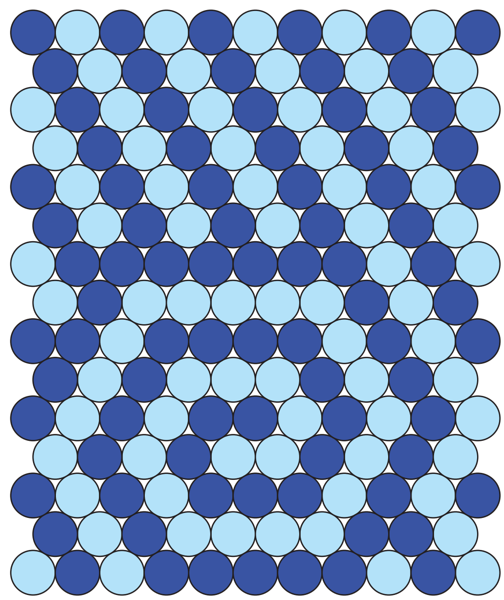
10-around

158 bead repeat

There are 7 repeats in an average sized bracelet using size 11 seed beads and size 20 crochet thread. Or use 8 repeats with size 11 Delicas.

Twist when closing to line up every repeat as in the bracelet shown (front and back).

Tags: 10-around



1 1 1 5 [1 1 × 3] 4 2 [1 1 × 2] 1
3 [1 1 × 4] 2 [1 1 × 4]
2 [1 1 × 4] 3 [1 1 × 2] 2 1 4
[1 1 × 3] 5 [1 1 × 2] 7 [1 1 × 33]

Escher Foamy Wave

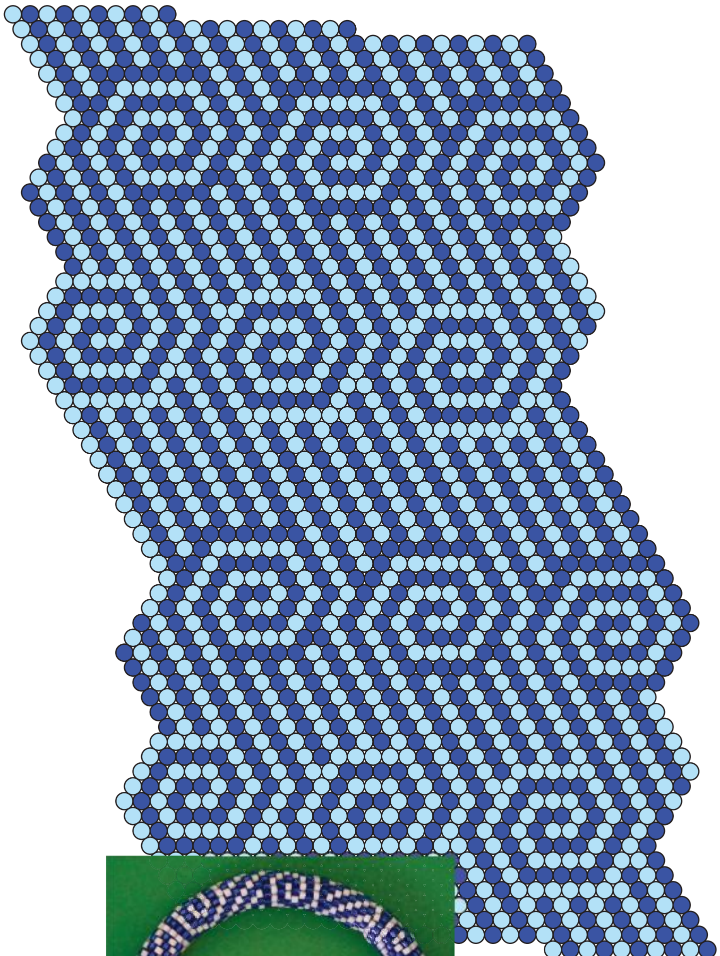
Notes:

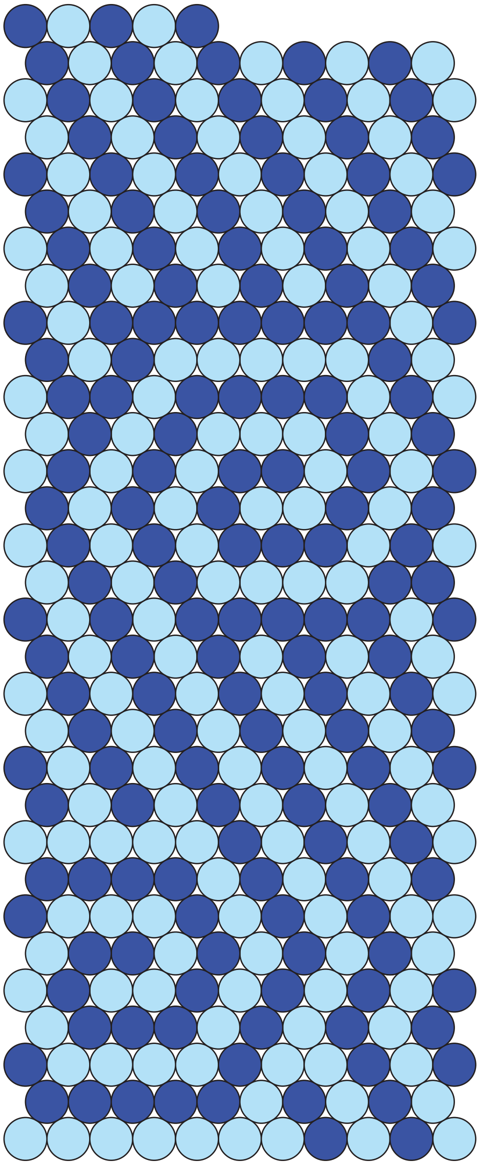
10-around

320 bead repeat

There are 4 repeats in an average sized bracelet using size 11 Delicas. Twist when closing to line up the repeats as in the bracelet shown. This design has no sizing flexibility.

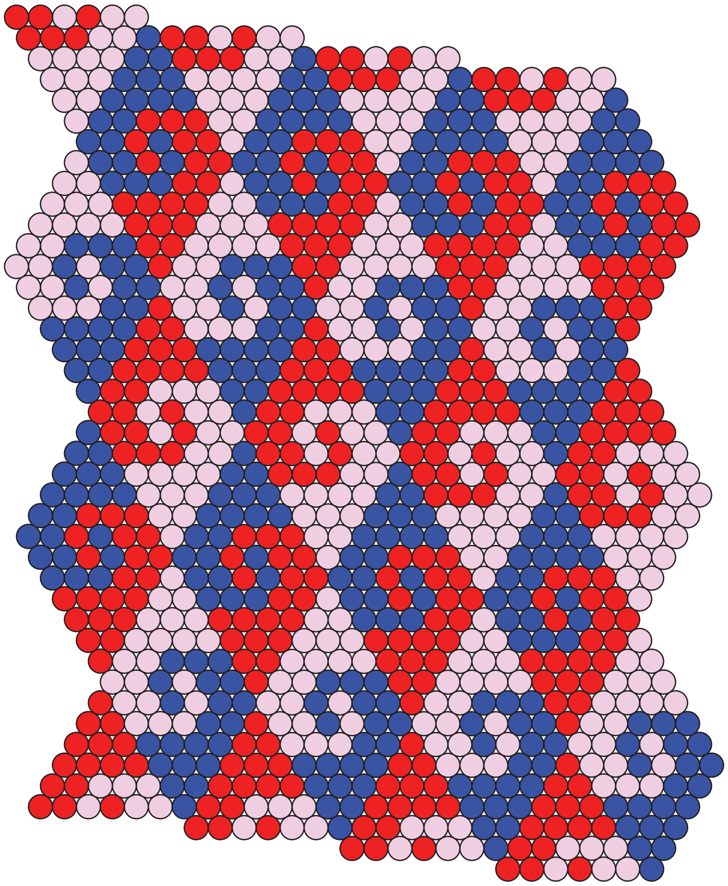
Tags: Escher, 10-around





7 [1 1 × 2] 5 [1 1 × 3] 4 1 2 [1 1 × 2]
3 [1 1 × 4] 2 [1 1 × 4] 2 [1 1 × 4]
3 [1 1 × 2] 1 2 4 [1 1 × 3] 5 [1 1 × 31]
5 [1 1 × 3] 4 2 1 [1 1 × 2] 3 [1 1 × 4]
2 [1 1 × 4] 2 [1 1 × 4] 3 [1 1 × 2]
2 1 4 [1 1 × 3] 5 [1 1 × 2] 7 [1 1 × 40]

Hooked Harlequin



2 1 1 2 1 2 3 2 4 3 3 4 2 3 2 1 2 1 1 2
2 1 1 2 1 2 3 2 4 3 3 4 2 3 2 1 2 1 1 2
2 1 1 2 1 2 3 2 4 3 3 4 2 3 2 1 2 1 1 2

Notes:

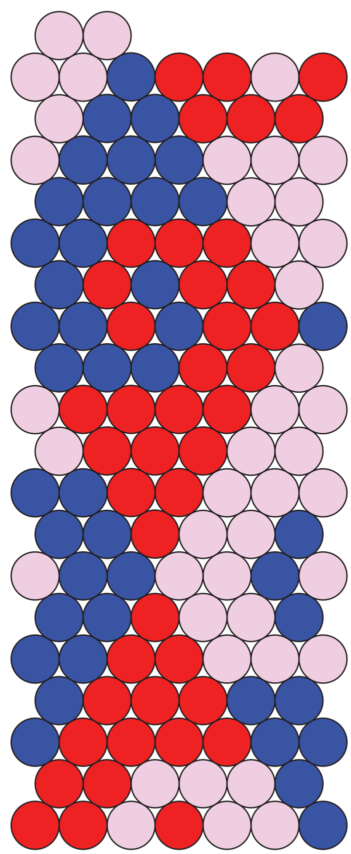
6-around

126 bead repeat

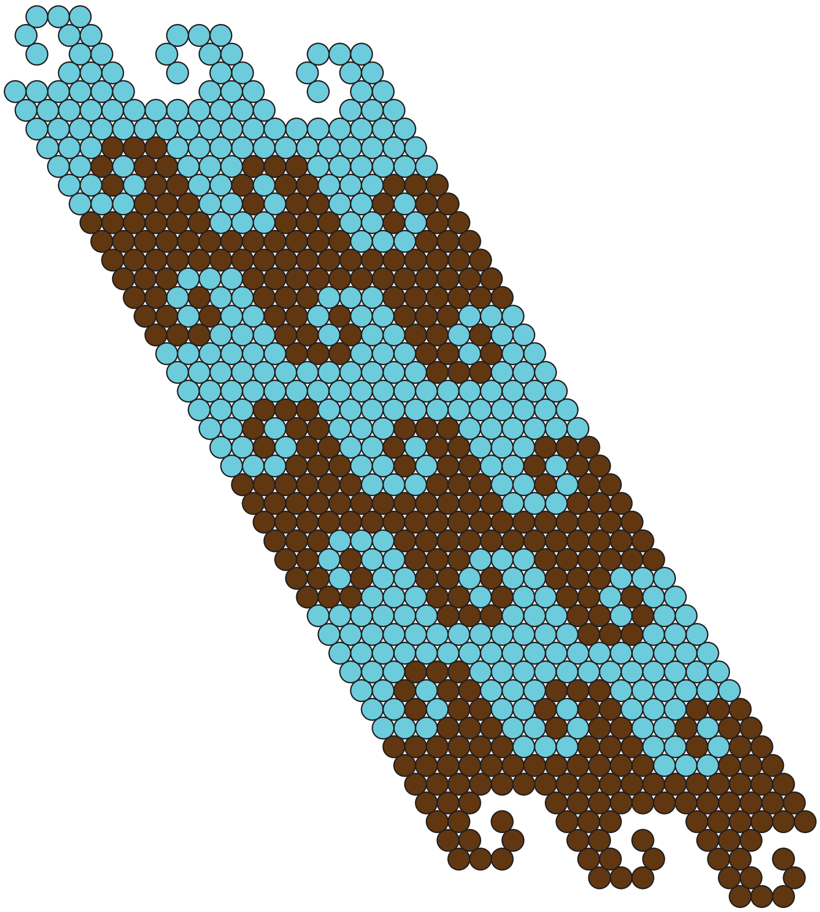
There are 6 repeats in a small sized bracelet using size 11 seed beads.

A twist of more than 360° lines up every repeat as in the bracelet shown (front and back). This design is based on the Harlequin pattern on p. 165.

Tags: Escher (tricolor), 6-around



Ocean Waves



Notes:

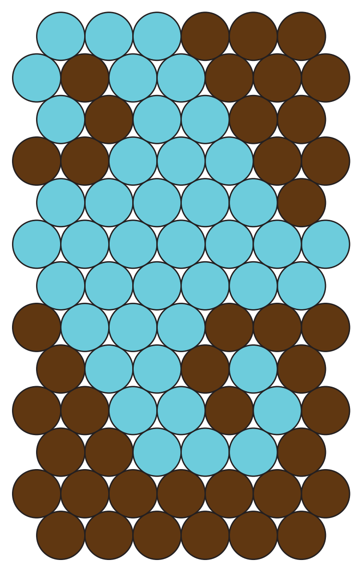
6-around

84 bead repeat

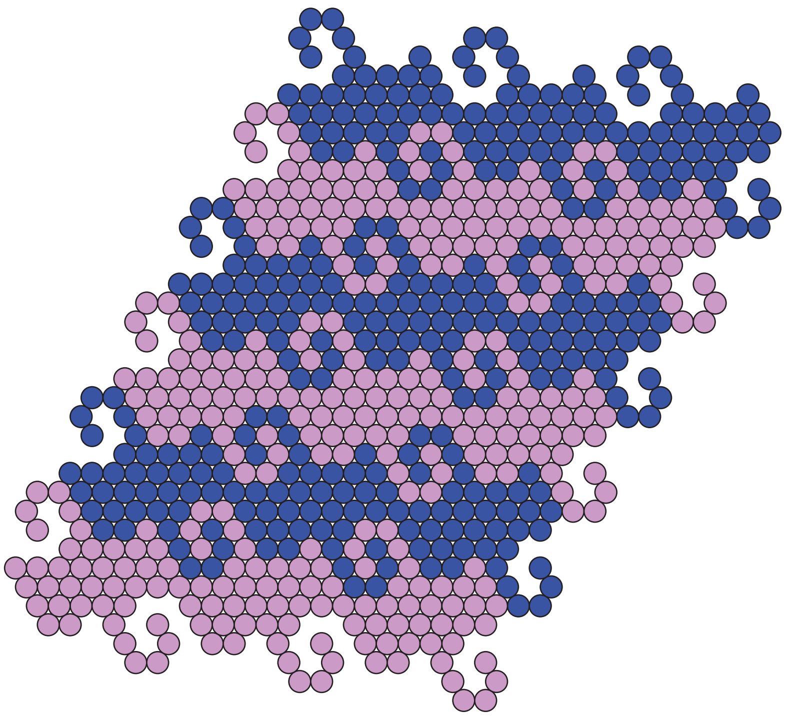
There are 8 repeats in a small sized bracelet using size 10 seed beads. Or try 9 repeats in size 11 seed beads for a larger bracelet. A small twist lines up every repeat, as in the bracelet shown.

Tags: Escher, Wallpaper Group P2, 6-around

18 3 3 2 1 1 2 2 1 1 2 3 3
18 3 3 2 1 1 2 2 1 1 2 3 3



Rough Sea Waves



15 2 5 1 1 1 1 2 1 1 1 1 1 5 2
15 2 5 1 1 1 1 2 1 1 1 1 1 5 2

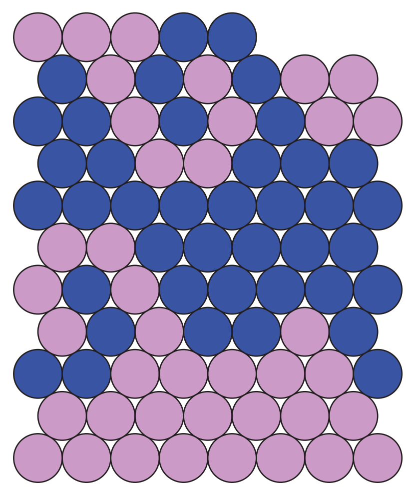
Notes:

7-around

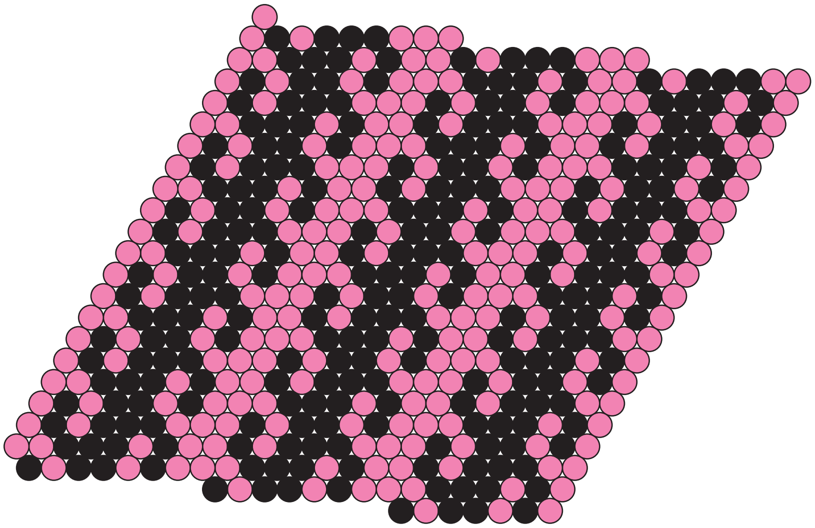
80 bead repeat

There are 12 repeats in a large sized bracelet using size 11 seed beads. Twist to line up every third repeat as in the bracelet shown. Or try using 12 repeats in size 11 Delicas for a smaller bracelet.

Tags: Escher, 7-around



Aztec Wave



Notes:

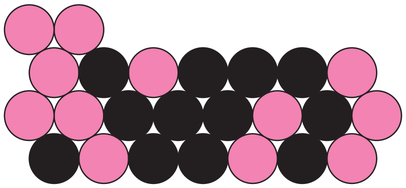
7-around

24 bead repeat

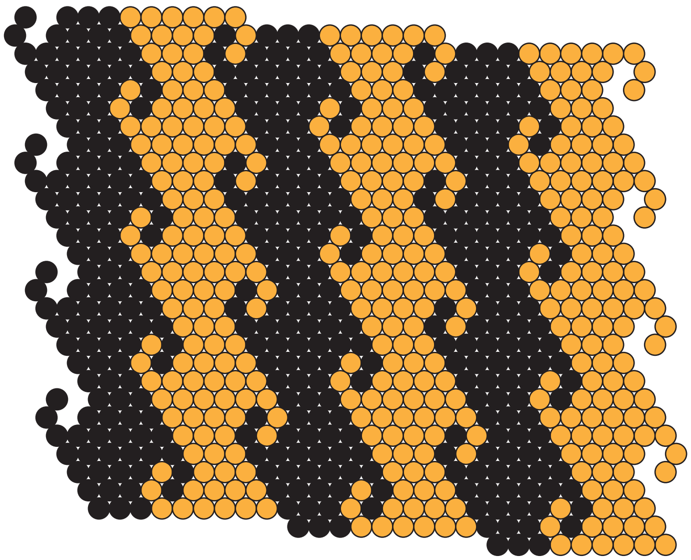
There are 33 to 35 repeats in an average sized bracelet using size 11 seed beads. Because of the short repeat, sizing is very flexible.

Tags: Escher, Wallpaper Group P2, 7-around

1 1 2 1 1 3 3
1 1 2 1 1 3 3



Zadie's Wave



Notes:

9-around

66 bead repeat

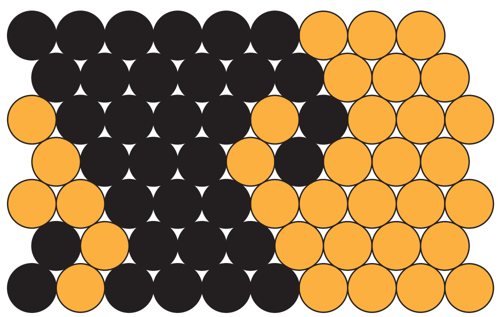
There are 17 or 18 repeats in an average sized bracelet using size 11 Delicas. The bracelet on the right uses 24K gold-lined crystal with matte black Delicas.

Consider the twist options before closing! Two of our favorites are shown below.

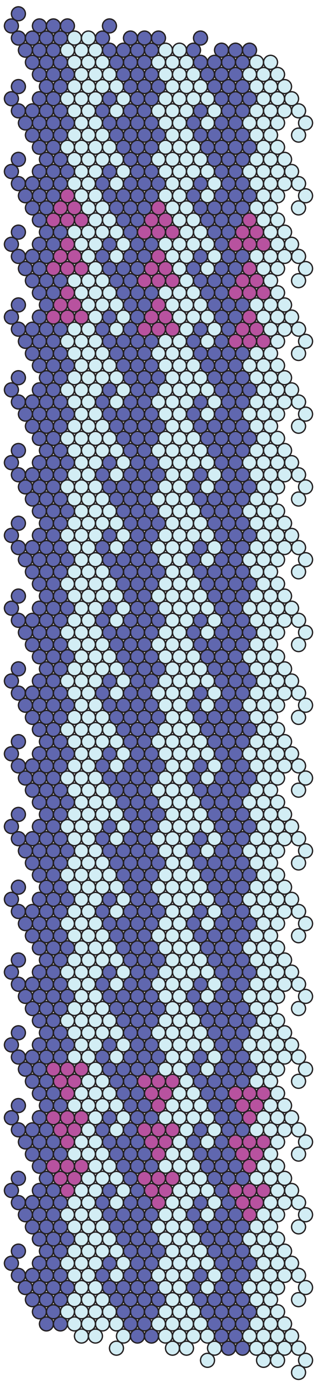
This design is dedicated to Adolph Baker, who always reminded us to "ride the waves."

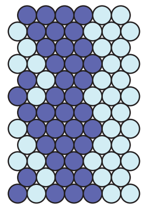
Tags: Escher, Wallpaper Group P2, 9-around

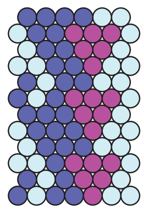
1 1 4 4 1 1 3 6 3 6 3
1 1 4 4 1 1 3 6 3 6 3

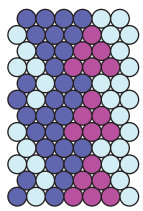


Tridelta Wave Rider









Notes:

6-around

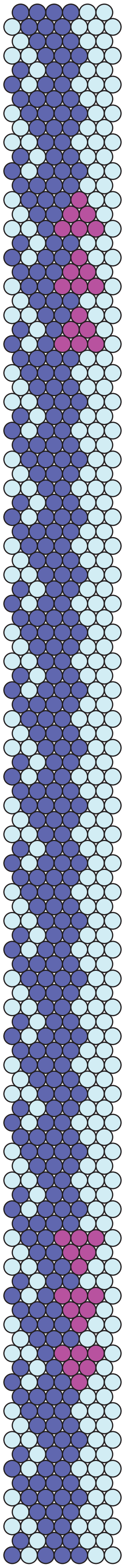
78 bead repeat, with three different versions shown at left. The numerical pattern is for the first version only.

There are 9 repeats in an average sized bracelet using size 11 seed beads. Distribute the repeats as desired. A suggested complete bracelet is charted and a slightly different version (with only one plain repeat separating the triangle segments) appears in the photo below.

The bracelet has three colors and a total number of beads and waves divisible by 3.

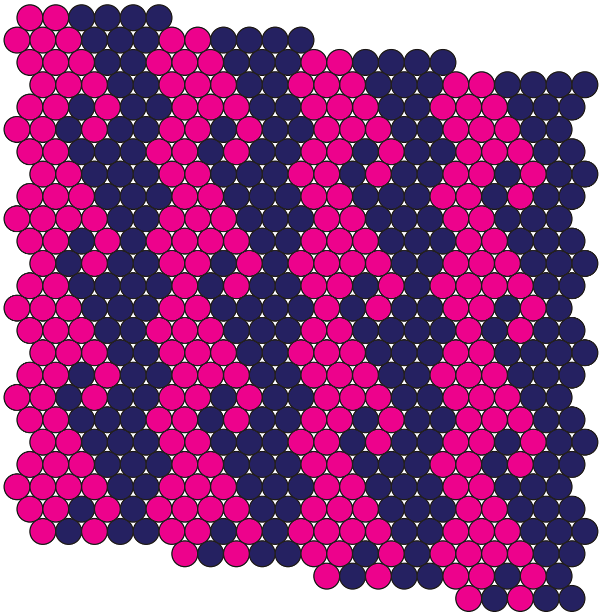
This design is dedicated to the Colgate University Tridelta Sorority.

Tags: Escher, 6-around



1 1 3 2 1 1 2 4 2 4 3 3 4 2 4 2
1 1 2 3 1 1 2 4 3 3 3 3 3 3 4 2

Wave Rider 5-around



1 1 2 2 1 1
1 4 2 3
3 2 3 2 3 2
1 1 2 2 1 1
2 3 2 3 2 3
3 2 4 1

Notes:

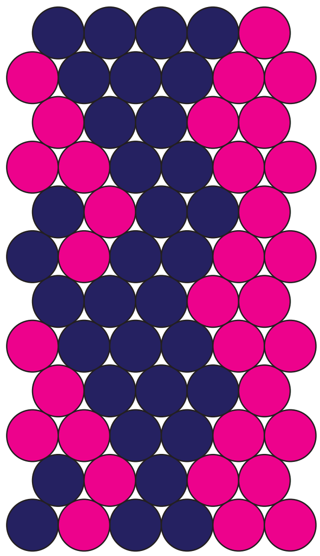
5-around

66 bead repeat

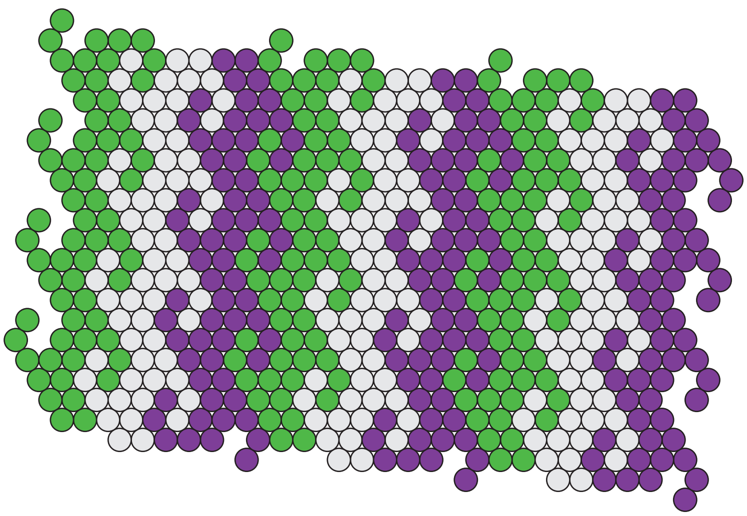
There are 6 repeats in a small to average sized bracelet using size 8 seed beads. This design is vertically aligned.

Crochet loosely or 6 repeats may not be big enough (and 7 repeats will likely be too big).

Tags: Escher, 5-around



Tricolor Wave



Notes:

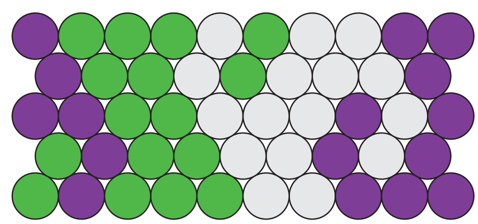
9-around

48 bead repeat

There are 23 or 24 repeats in an average sized bracelet using size 11 Delicas. The bracelet on the left uses 1.8mm sterling silver rounds for one wave. Two twist possibilites are shown: one that lines up every repeat and one that winds the waves once around the meridian of the bracelet. Consider your options before closing!

Tags: Escher (tricolor), 9-around

1 1 3 2 3 1 1 2 2
1 1 3 2 3 1 1 2 2
1 1 3 2 3 1 1 2 2



Crashing Waves

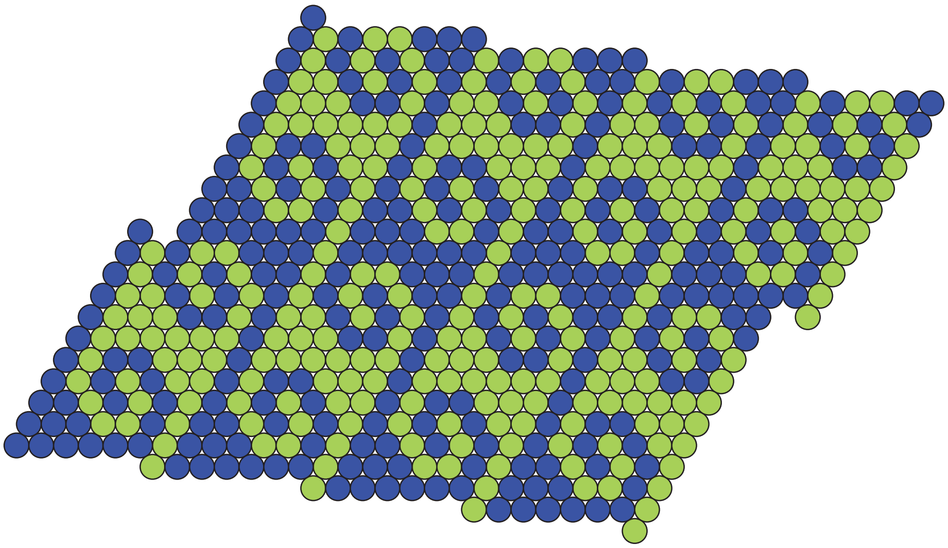
Notes:

6-around

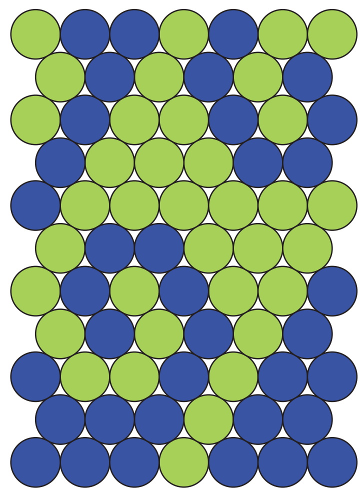
72 bead repeat

There are 10 repeats in an average sized bracelet using size 11 seed beads. Use a small twist to line up every repeat as in the bracelet shown (in two different views).

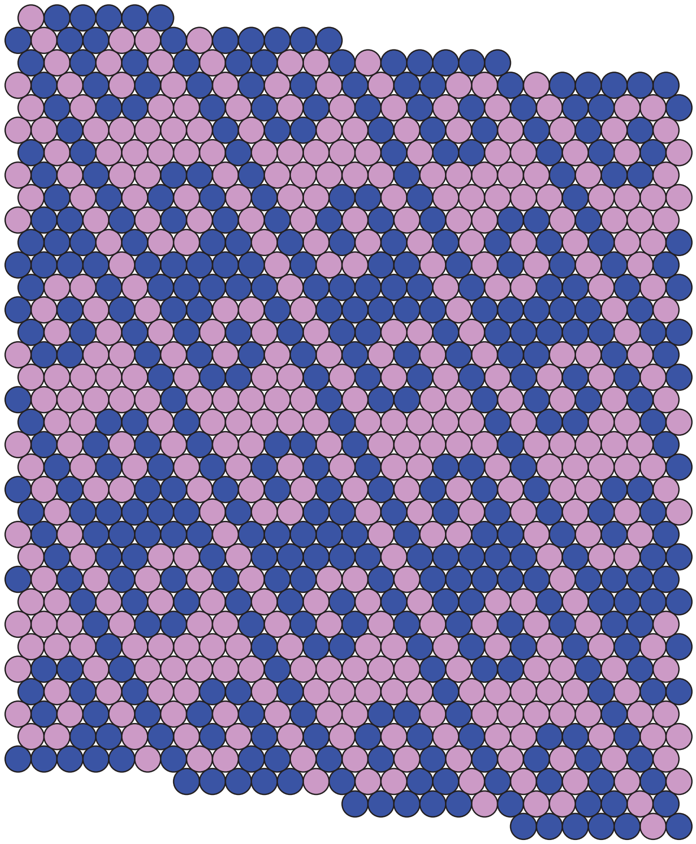
Tags: Escher, Wallpaper Group P2, 6-around



3 1 6 1 3
2 1 1 2
1 1 1 1 1 1 1 1 1 1
2 1 1 2
3 1 6 1 3
2 1 1 2
1 1 1 1 1 1 1 1 1 1
2 1 1 2



Greek Columns



Notes:

6-around

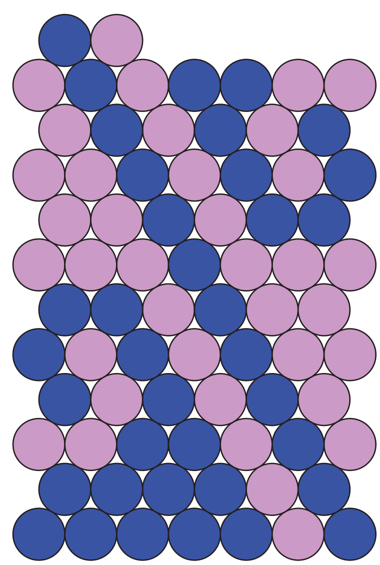
74 bead repeat

There are 10 repeats in an average sized bracelet using size 11 seed beads.

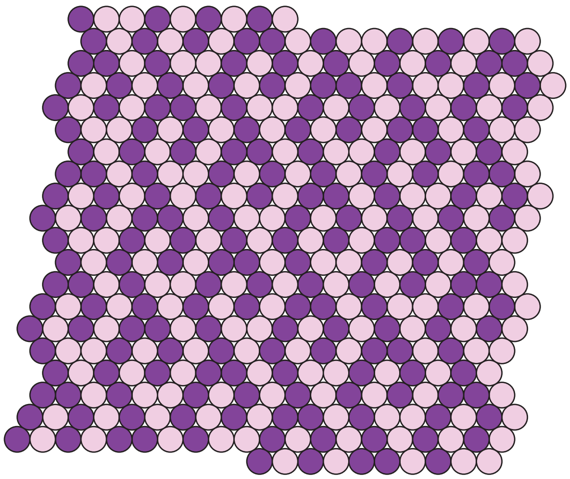
Use a multiple of 2 repeats to line up every other repeat as in the bracelet shown.

Tags: Escher, Wallpaper Group P2, 6-around

5 1 5 1 1 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 2 1 1
5 1 5 1 1 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 2 1 1



Viny Wave



2 1 1 2 1 1 1 1 1 1 1 1 1 1

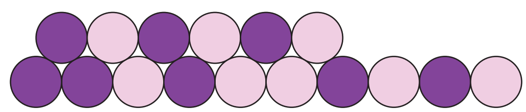
Notes:

9-around

16 bead repeat

There are 75 repeats in an average sized bracelet using size 11 Delicas. Use a multiple of 3 repeats and a twist to line up the vines if desired. This design can be modified to use three colors or six colors for the different vines with a 48 bead repeat.

Tags: Escher, 9-around



Interlocking Vines

Notes:

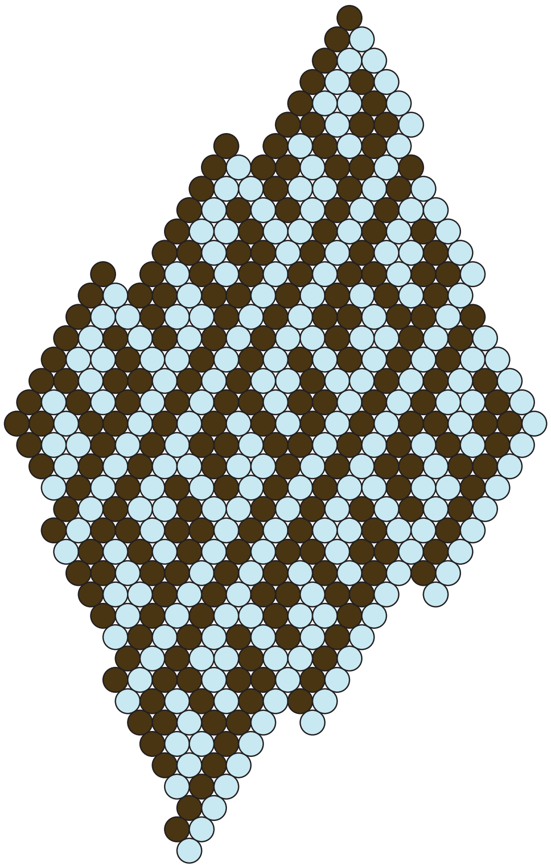
7-around

50 bead repeat

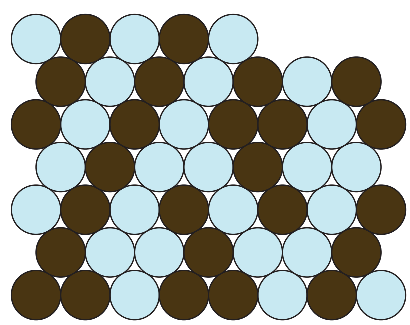
There are 18 repeats in an average to large sized bracelet using size 11 seed beads.

This design is vertically aligned by 3. Use a multiple of 3 repeats to line up every third repeat as in the bracelet shown.

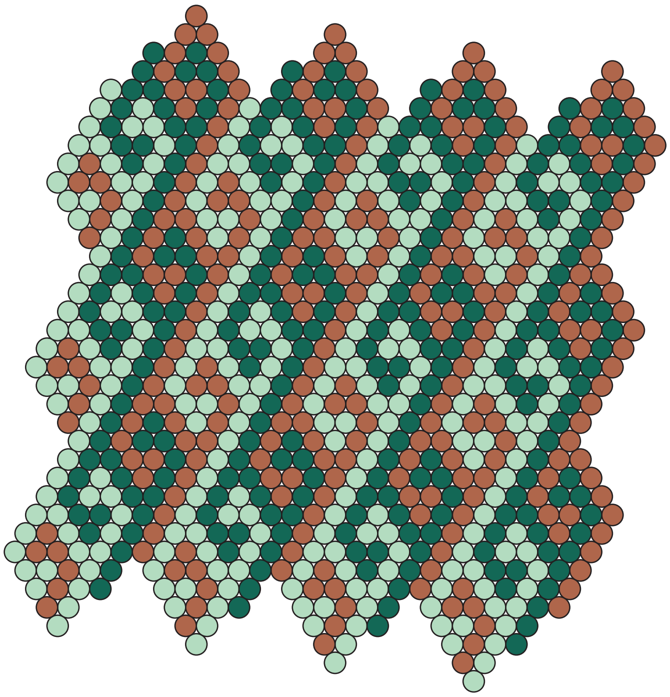
Tags: Escher, Wallpaper Group P2, 7-around



2 1 2 1 1 1 1
2 1 2 1 1 1 1 1 1 1 1 1 1 1
2 1 2 1 1 1 1
2 1 2 1 1 1 1 1 1 1 1 1 1 1



Tricolor Vines



Notes:

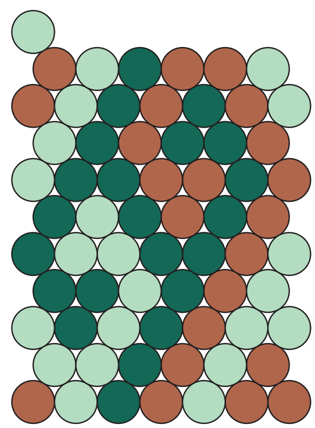
6-around

66 bead repeat

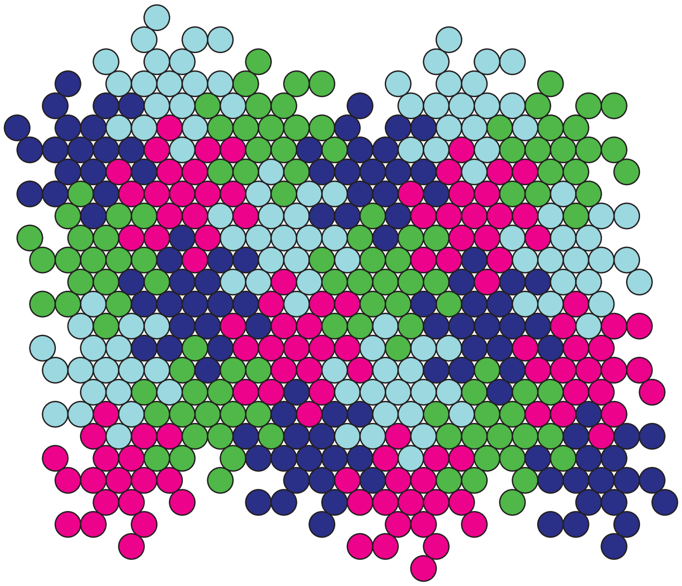
There are 11 repeats in an average sized bracelet using size 11 seed beads. A minor twist aligns every repeat.

Tags: Escher (tricolor), 6-around

1 1 1 1 1 2 2 1 1 1 1 1 1 1 1 1 2 2
1 1 1 1 1 2 2 1 1 1 1 1 1 1 1 1 2 2
1 1 1 1 1 2 2 1 1 1 1 1 1 1 1 1 2 2



Pinwheel



Notes:

11-around

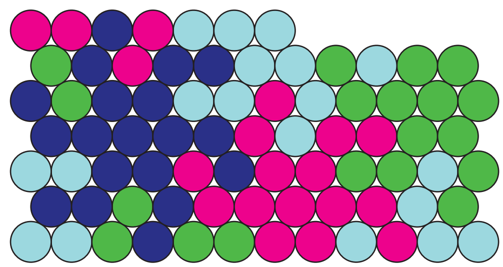
76 bead repeat

There are 19 repeats in an average sized bracelet using size 11 Delicas.

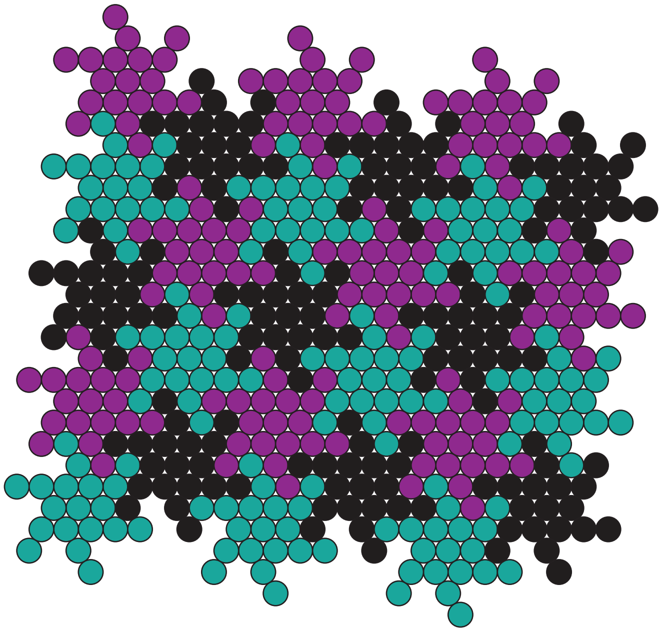
Choose four colors that all constrast one another well to display this pattern to full effect.

Tags: Escher (four-color), 11-around

1	1	2	2	1	1	2	2	1	1	5
1	1	2	2	1	1	2	2	1	1	5
1	1	2	2	1	1	2	2	1	1	5
1	1	2	2	1	1	2	2	1	1	5



Pinwheel Star



Notes:

7-around

57 bead repeat

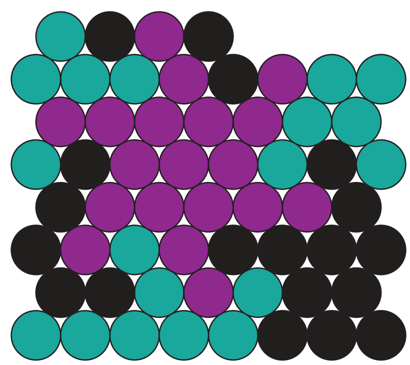
There are 14 repeats in an average sized bracelet using size 11 seed beads.

Use a multiple of 2 repeats and a twist when closing to line up every other repeat as in the bracelet shown.

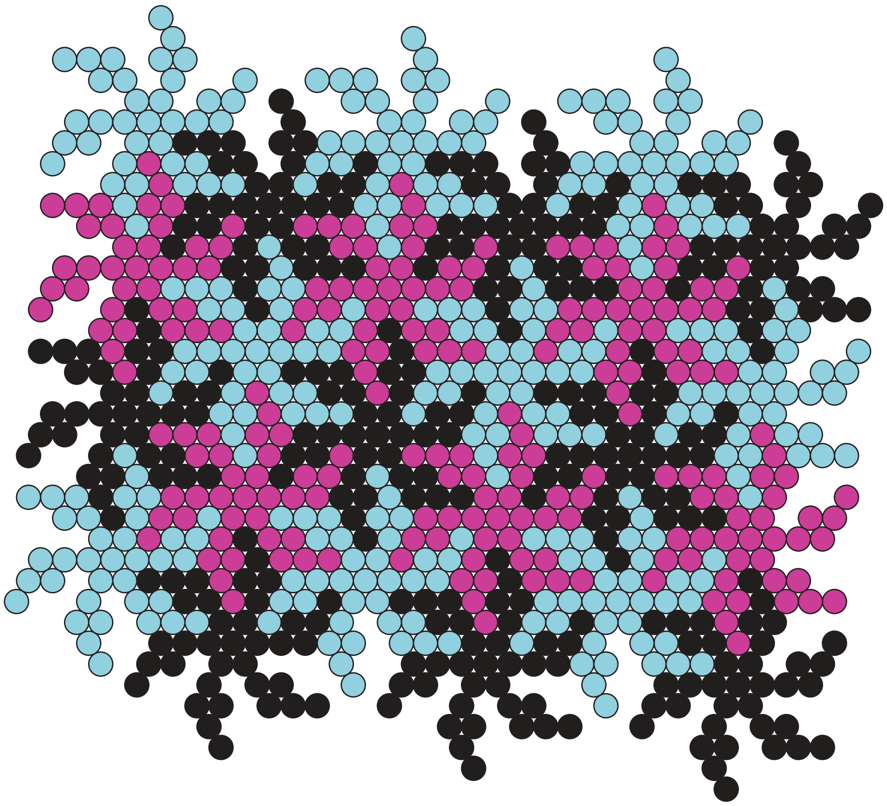
This design is based on the Hexagonal Grid pattern on p. 166.

Tags: Escher (tricolor), Wallpaper Group P3, 7-around

5 5 1 1 1 3 1 1 1
5 5 1 1 1 3 1 1 1
5 5 1 1 1 3 1 1 1



Pinwheel Snowflake



1 2 1 1 2 2 1 1 2 1 2 3 1 2 7 2 1 3 2
1 2 1 1 2 2 1 1 2 1 2 3 1 2 7 2 1 3 2
1 2 1 1 2 2 1 1 2 1 2 3 1 2 7 2 1 3 2

Notes:

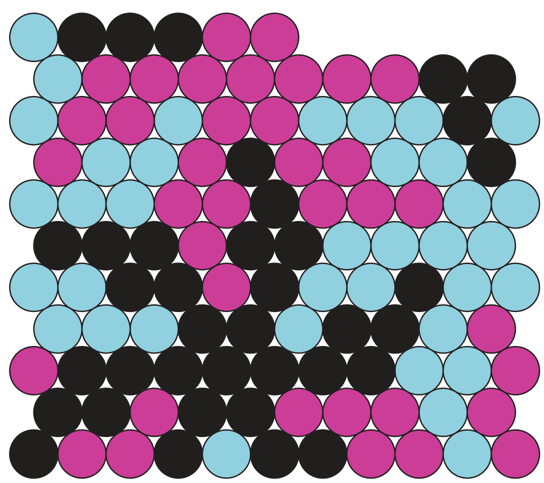
10-around

111 bead repeat

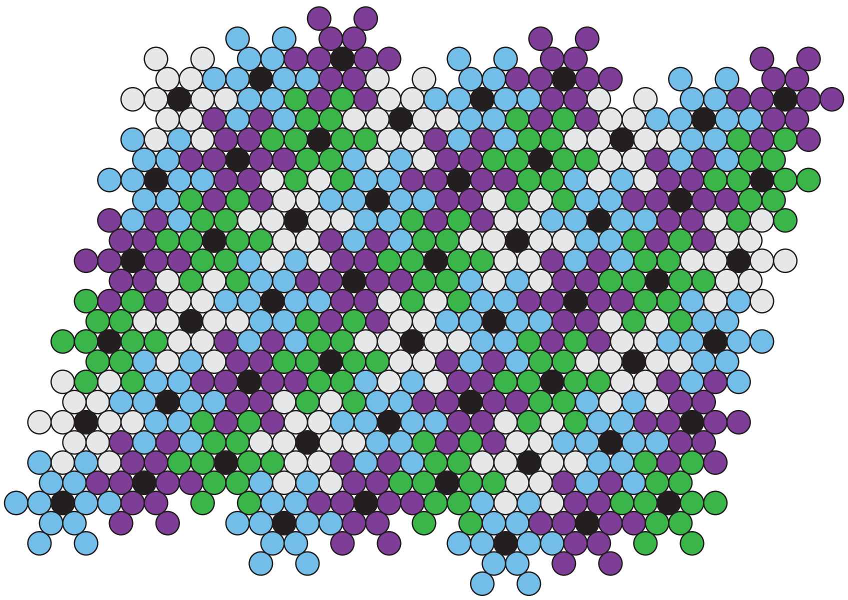
There are 12 repeats in an average sized bracelet using size 11 Delicas.

Use a multiple of 2 repeats to line up every other repeat as in the bracelet shown. Because of the large repeat length, sizing is not flexible.

Tags: Escher (tricolor), Wallpaper Group P3, 10-around



Snowflakes



Notes:

9-around

52 bead repeat

There are 22 or 23 repeats in an average sized bracelet using size 11 Delicas.

The numerical pattern is given in two variations, one with the snowflake centers painted black as in the bead plane chart shown, and one with solid snowflakes as in the bracelet shown, which uses 23 repeats. Use a multiple of 2 repeats instead if you prefer to line up every other snowflake with a twist.

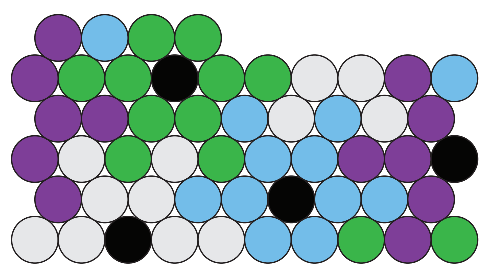
The variation in the chart works nicely with 2mm sterling silver or gold-filled rounds in the center of each snowflake.

Tags: Escher (four-color), 9-around

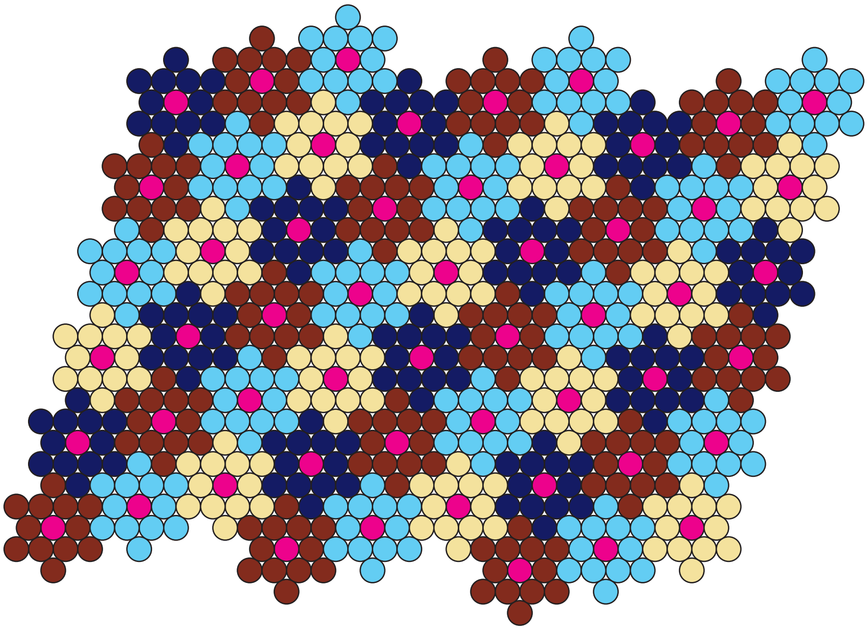
2	1	2	2	1	1	1	1	2
2	1	2	2	1	1	1	1	2
2	1	2	2	1	1	1	1	2
2	1	2	2	1	1	1	1	2

or

5	2	1	1	1	1	2
5	2	1	1	1	1	2
5	2	1	1	1	1	2
5	2	1	1	1	1	2



Four-Color Flowers



Notes:

9-around

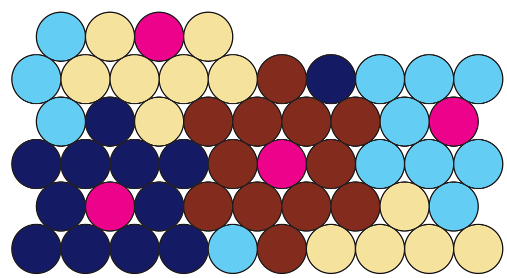
52 bead repeat

There are 22 or 23 repeats in an average sized bracelet using size 11 Delicas.

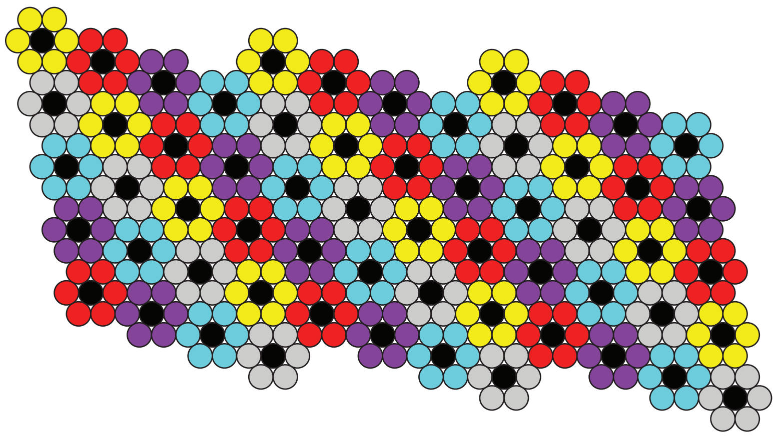
Use a multiple of 2 repeats to line up every other repeat as in the bracelet shown.

Tags: Escher (four-color), 9-around

4 1 1 4 1 1 1
4 1 1 4 1 1 1
4 1 1 4 1 1 1
4 1 1 4 1 1 1



Tiny Flowers



2	1	1	1	2
2	1	1	1	2
2	1	1	1	2
2	1	1	1	2
2	1	1	1	2

Notes:

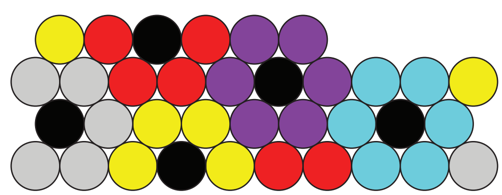
9-around

35 bead repeat

There are 33 repeats in an average sized bracelet using size 11 Delicas. The bracelet shown uses 2mm sterling silver rounds for the flower centers.

Use a multiple of 3 repeats to line up every third repeat as in the bracelet shown. Choose five colors that all contrast one another well to display the design to full effect.

Tags: Escher (five-color), 9-around



Jaguar

Notes:

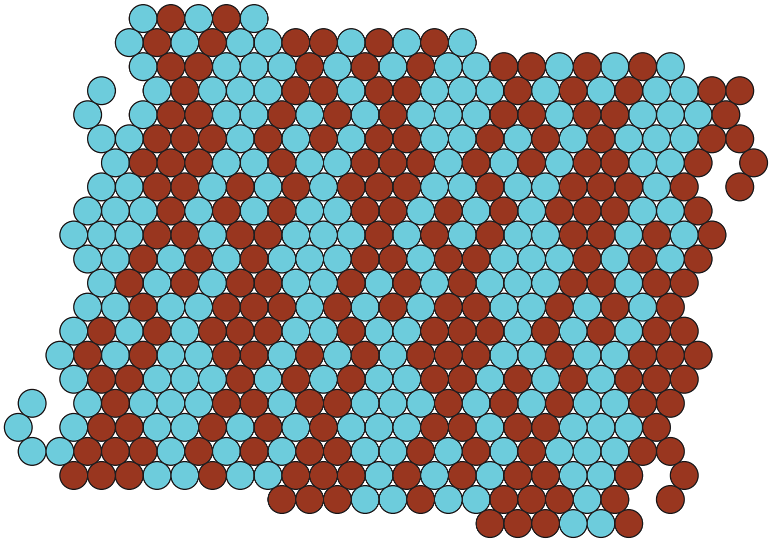
7-around

50 bead repeat

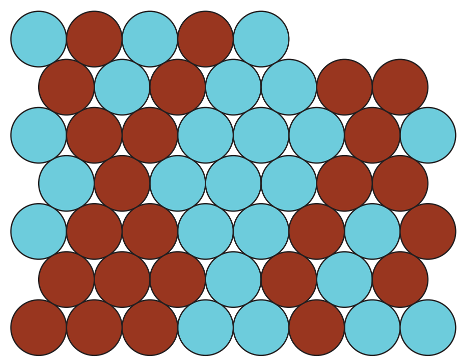
There are 18 repeats in an average to large sized bracelet using size 11 seed beads.

This design is vertically aligned by 3. Use a multiple of 3 repeats to line up every third repeat, as in the bracelet shown.

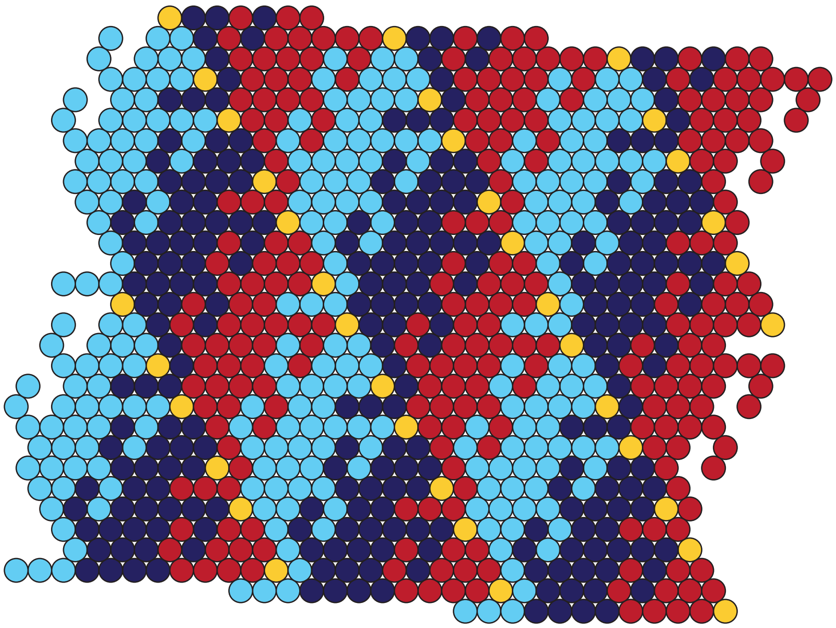
Tags: Escher, Wallpaper group P2, 7-around

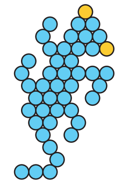


3 2 1 2 3 1 1 1 1 1 2 2 1 1 1 1 1
3 2 1 2 3 1 1 1 1 1 2 2 1 1 1 1 1



Escher Lizards





Notes:

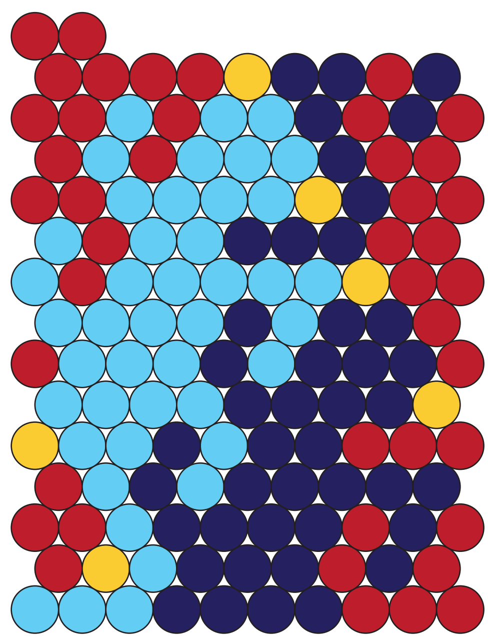
9-around

135 bead repeat

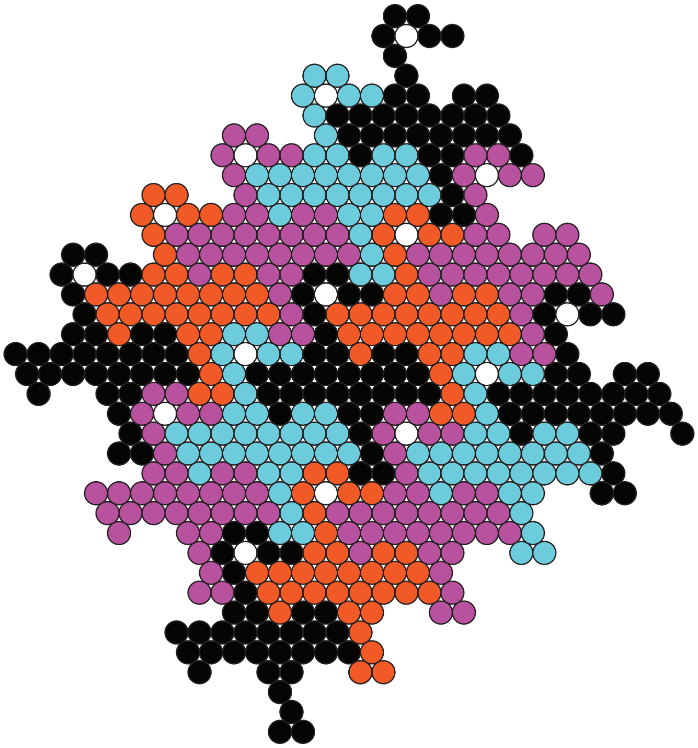
There are 9 repeats in an average sized bracelet using size 11 Delicas. Choose contrasting colors carefully to make the lizard forms pop out clearly. Use a minor twist before closing to line up every repeat as in the bracelet shown.

Tags: Escher (tricolor), 9-around

3 4 4 1 1 3 1 1 3 1 4 1 1 2 1 1 1 5 1 2 1 1 2
3 4 4 1 1 3 1 1 3 1 4 1 1 2 1 1 1 5 1 2 1 1 2
3 4 4 1 1 3 1 1 3 1 4 1 1 2 1 1 1 5 1 2 1 1 2



Four-Color Escher Birds



Notes:

10-around

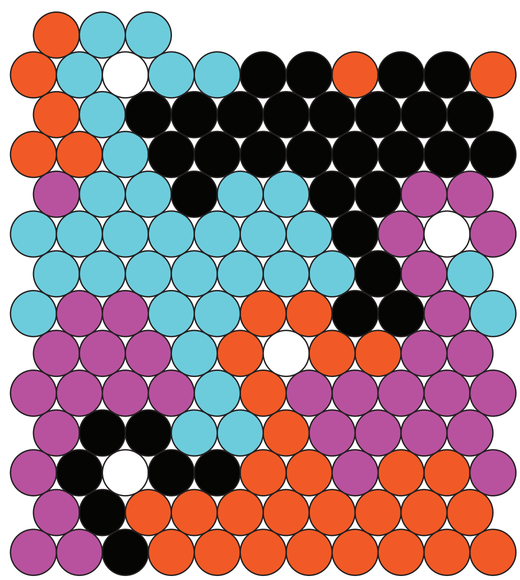
140 bead repeat

There are 9 repeats in an average sized bracelet using size 11 Delicas and 1.8mm sterling silver rounds for the eyes.

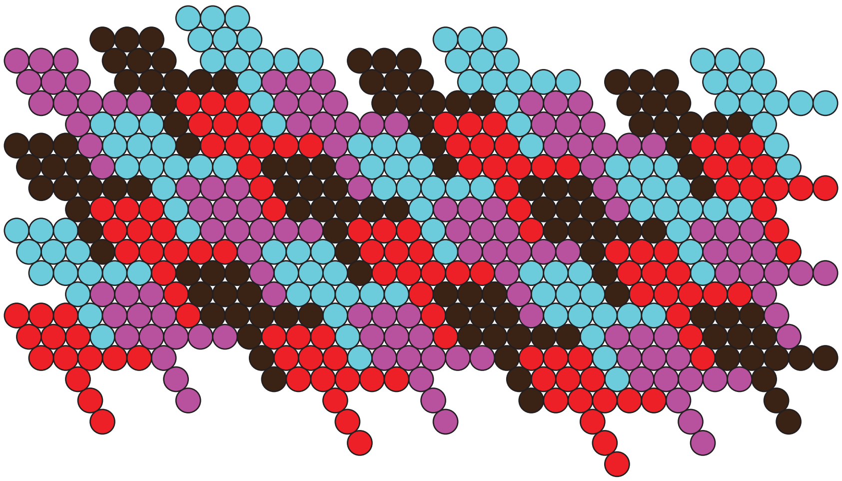
This design is vertically aligned by 3. Use a multiple of 3 repeats to line up every third repeat as in the bracelet shown. Restricting to a multiple of 3 repeats does make sizing fairly inflexible.

Tags: Escher, 10-around

2 1 8 1 1 8 1 1 1 2 2 1 2 2 2
2 1 8 1 1 8 1 1 1 2 2 1 2 2 2
2 1 8 1 1 8 1 1 1 2 2 1 2 2 2
2 1 8 1 1 8 1 1 1 2 2 1 2 2 2



Little Fishes



Notes:

10-around

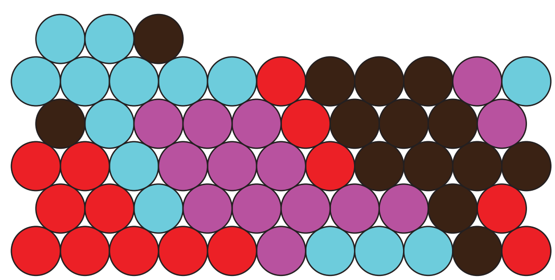
56 bead repeat

There are 24 repeats in a large sized bracelet using size 11 Delicas.

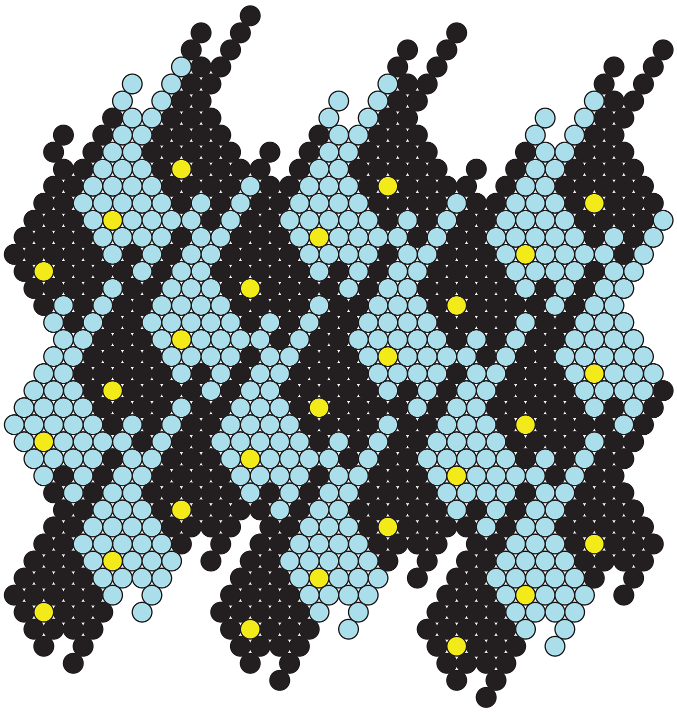
Use a multiple of 3 repeats to line up every third repeat as in the bracelet shown.

Tags: Escher (four-color), 10-around

5 1 3 1 3 1
5 1 3 1 3 1
5 1 3 1 3 1
5 1 3 1 3 1



Escher Fish



Notes:

10-around

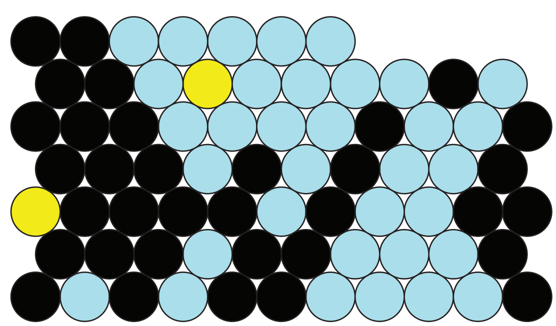
70 bead repeat

There are 18 repeats in an average sized bracelet using size 11 Delicas and 2mm sterling silver rounds for the eyes.

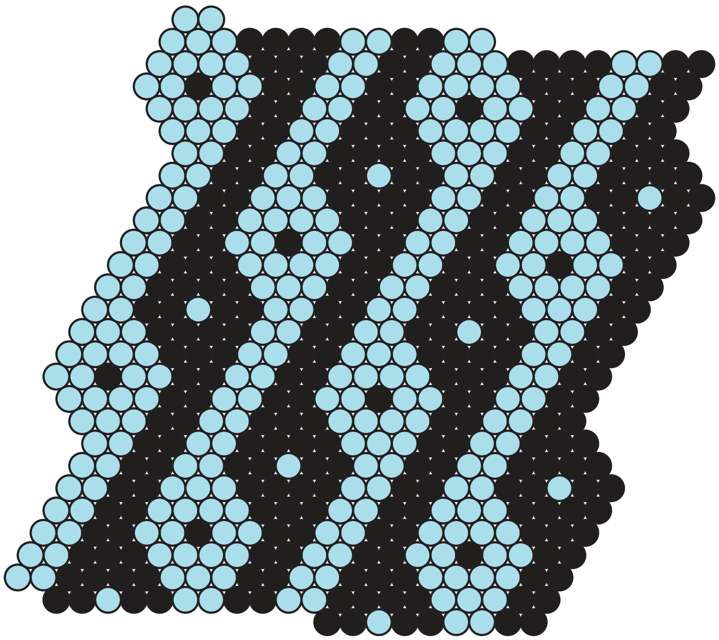
This design is vertically aligned by 3. Use a multiple of 3 repeats to line up every third repeat as in the bracelet shown. Restricting to a multiple of 3 repeats does make sizing fairly inflexible.

Tags: Escher, 10-around

1 1 1 1 2 4 4 1 2 3 1 1 4 1 1 2 5 1
1 1 1 2 4 4 1 2 3 1 1 4 1 1 2 5



Snake Eyes



Notes:

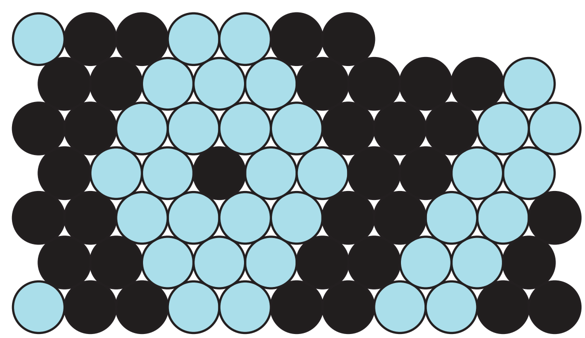
10-around

70 bead repeat

There are 18 repeats in an average sized bracelet using size 11 Delicas. This design is based on the Escher Fish pattern. Use a multiple of 3 repeats to line up every third repeat as in the bracelet shown.

Tags: Escher, Wallpaper Group P2, 10-around

1 2 2 2 2 4 3 2 2 3 4 2 2 2 2
1 2 2 2 2 4 3 2 2 3 4 2 2 2 2



Fish Transformation

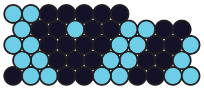
Notes:

10-around

70 bead repeat for the nontransitional segments A and C. General notes about making transformation patterns appear on p. 129.

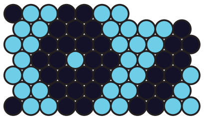
The numbers given make a small sized bracelet using size 11 Delicas and 2mm sterling silver rounds for the eyes. This pattern can be elongated into a necklace by cycling through the pattern more than once and/or by repeating segments A and C more than 8 times.

Tags: Escher, Transformation, 10-around



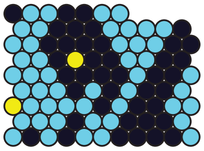
D once

1 2 2 2 2
4 3 2 2 3 4
2 2 2 2
1 2 2 2 2 5



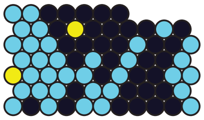
C times 8

1 2 2 2 2
4 3 2 2 3 4
2 2 2 2
1 2 2 2 2
4 3 2 2 3 4
2 2 2 2



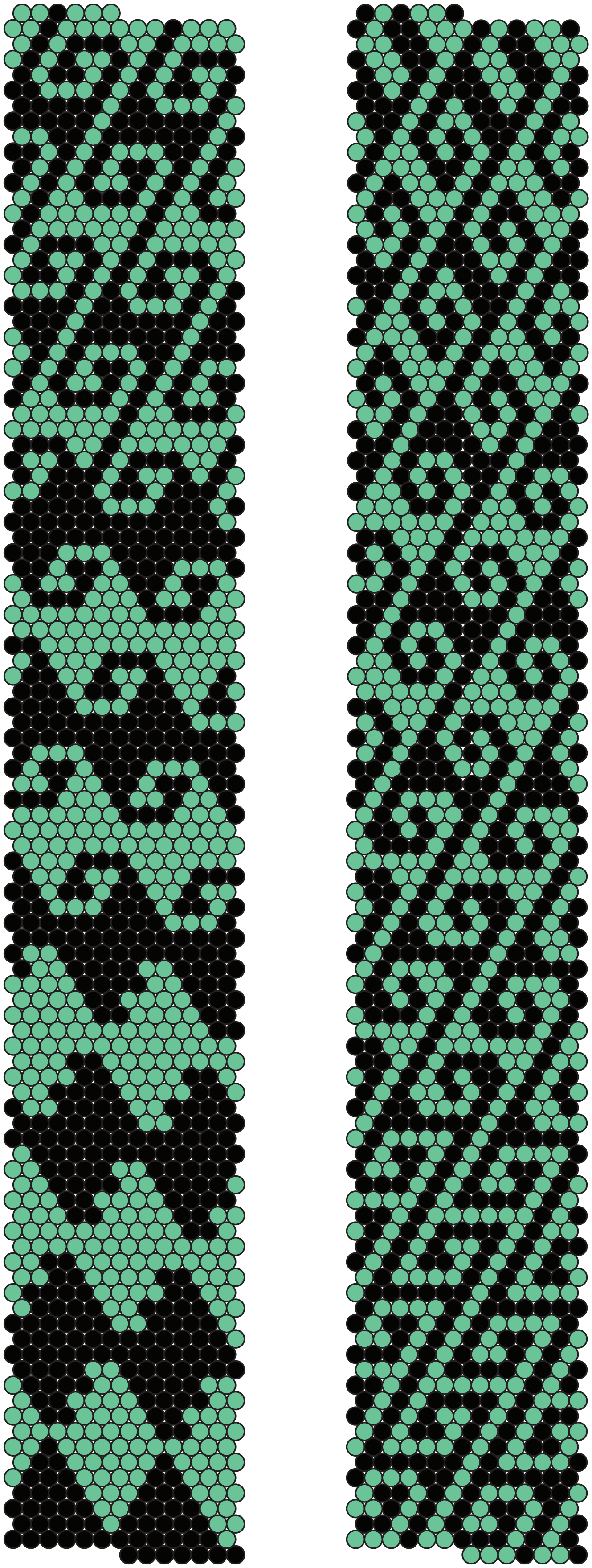
B once

1 1 1 1 2 4 4
1 2 3 1 1
4 1 1 2 5
1 1 1 1 2 4 4
1 2 3 1 1
2 2 2 2
4 3 2 2 3 4
2 2 2 2



A times 8

1 1 1 1 2 4 4
1 2 3 1 1
4 1 1 2 5
1 1 1 1 2 4 4
1 2 3 1 1
4 1 1 2 5



Waves and Diamonds Transformation Necklace

Notes:

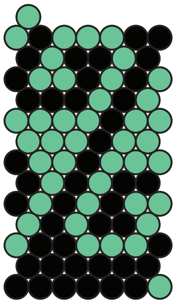
6-around

The repeat on the nontransitional segments varies between 72 and 86 beads. General notes about making transformation patterns appear on p. 129.

Sections that are repeated more than once (B, C, E, F, G, H, J, L, M, O, and Q) can be repeated more or fewer times to adjust the length of the necklace or the emphasis on the different designs. The necklace in size 11 seed beads photographed here is 28 inches long. The chart on the preceding page shows an abbreviated version of the necklace in which one repeat of each motif is removed; this is the version shown in size 8 seed beads at the end of Chapter 6.

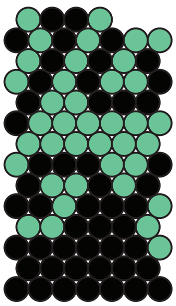
The three bracelets in the lower photograph are made with 10 repeats of section O, 8 repeats of section C, and 10 repeats of section H (also given as Crashing Waves on p. 189), respectively.

Tags: Escher, Transformation, Necklace, 6-around



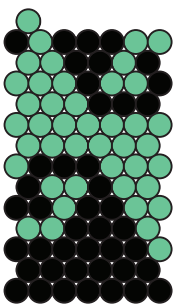
E times 3

6 1 6
1 2 3 1 1 1 1
2 1 1 1 1 1 1 2
1 1 1 1 3 2 1
6 1 6
1 2 3 1 1 1 1
2 1 1 1 1 1 1 2
1 1 1 1 3 2 1



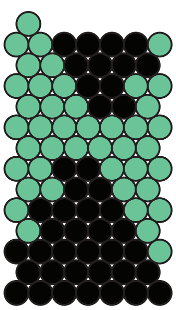
D once

19
3 3 1 2 1 3 1 1 2
1 1 1 1 3 2 1
6 1 6
1 2 3 1 1 1 1
2 1 1 1 1 1 1 2
1 1 1 1 3 2 1



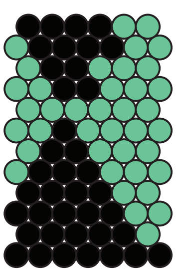
C times 3

19 3 3 1 2 1 2
2 1 2 1 3 3
19 3 3 1 2 1 2
2 1 2 1 3 3



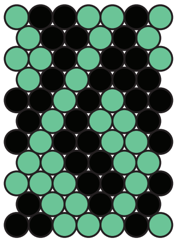
B times 3

19 2 4 2 4 4 2 4 2
19 2 4 2 4 4 2 4 2



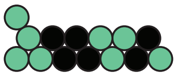
A once

12 1 5 2 4 3
3 4 2 5 1
12 2 4 2 4 4 2 4 2



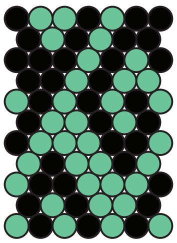
L times 3

2 3 1 1 1 2 1 1 1 3 2
1 1 2 2 2 1 1
2 2 2 1 1
2 3 1 1 1 2 1 1 1 3 2
1 1 2 2 2 1 1
2 2 2 1 1



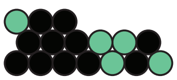
K once

2 2 1 1 2 2 2 1 1



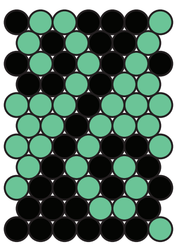
J times 3

1 1 1 1 1 1 1 1 1 1
2 1 1 2
3 1 1 1 2 1 1 1 3
2 1 1 2
1 1 1 1 1 1 1 1 1 1
2 1 1 2
3 1 1 1 2 1 1 1 3
2 1 1 2



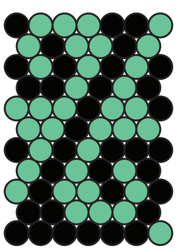
I once

4 1 1 1 3 2 1 1 2



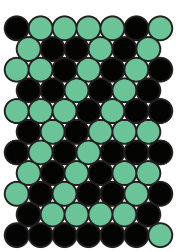
H times 3

6 1 3 2 1 1 2
1 1 1 1 1 1 1 1 1 1
2 1 1 2 3 1
6 1 3 2 1 1 2
1 1 1 1 1 1 1 1 1 1
2 1 1 2 3 1



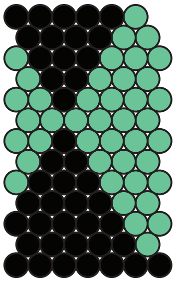
G times 3

6 1 2 3 1 1 1
2 1 1 1 1 1 1 1 1 2
1 1 1 3 2 1
6 1 2 3 1 1 1
2 1 1 1 1 1 1 1 1 2
1 1 1 3 2 1



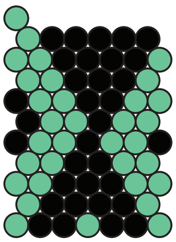
F times 3

6 1 1 4 1 1 1 1
2 1 1 1 1 1 1 2
1 1 1 1 4 1 1
6 1 1 4 1 1 1 1
2 1 1 1 1 1 1 2
1 1 1 1 4 1 1



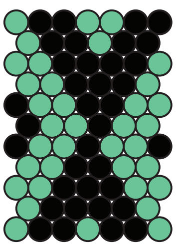
Q times 2

12 1 5 2 4 3
3 4 2 5 1
12 1 5 2 4 3
3 4 2 5 1



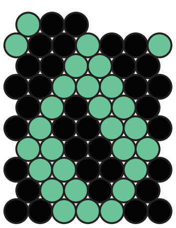
P once

1 2 1 2 2 4 3
3 4 2 2 1 2 1
2 2 2
1 2 1 2 2 4 3
3 4 2 5 1



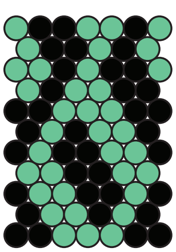
O times 3

1 2 1 2 2 4 3
3 4 2 2 1 2 1
2 2 2
1 2 1 2 2 4 3
3 4 2 2 1 2 1
2 2 2



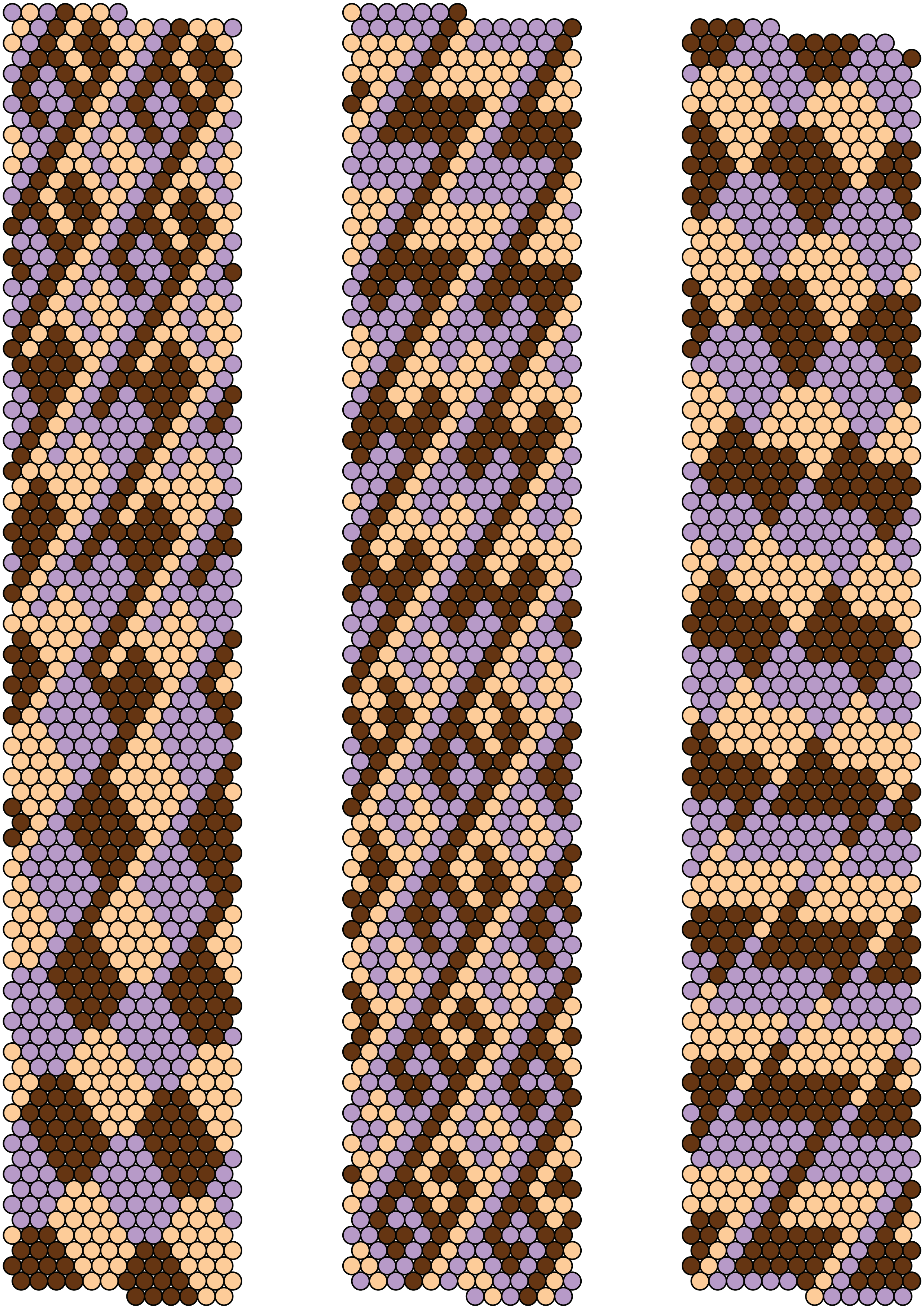
N once

2 3 3 2 1 1
2 2 2 1 1
2 2 2 1 1
2 2 2 1 1
2 3 3 4 2
2 1 2 1 2 2 2



M times 3

2 3 3 2 1 1
2 2 2 1 1
2 2 2 1 1
2 2 2 1 1
2 3 3 2 1 1
2 2 2 1 1
2 2 2 1 1
2 2 2 1 1



Tricolor Transformation Necklace

Notes:

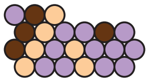
6-around

66 bead repeat for the nontransitional segments.
General notes about making transformation patterns appear on p. 129.

Sections that are repeated more than once (A, B, C, D, E, G, I, K, M, O, and P) can be repeated more or fewer times to adjust the length of the necklace or the emphasis on the different designs. The chart on the preceding page shows an abbreviated version of the necklace in which these sections are repeated twice instead of three times. The necklace in size 11 seed beads photographed here, which follows the numerical pattern as written, is 25.5 inches long.

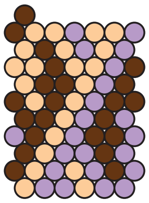
The five bracelets in the second photograph are made with 11 repeats each of sections A, B, E, K, and I. Section E is the same pattern as Tricolor Vines (p. 193).

Tags: Escher (tricolor), Transformation, Necklace, 6-around



H once

1 2 1 2 1 1 1 1 3 1 1 2 1 2 1 1

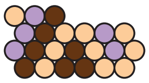


G times 3

1 2 1 2 1 1 1 1 1 1 1 1 1 2 1 2 1 1

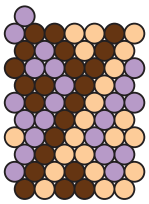
1 2 1 2 1 1 1 1 1 1 1 1 1 2 1 2 1 1

1 2 1 2 1 1 1 1 1 1 1 1 1 2 1 2 1 1



F once

1 1 2 2 1 1 1 1 1 1 1 1 1 2 1 2 1 1

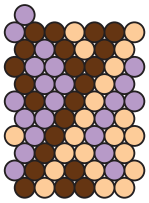


E times 3

1 1 2 2 1 1 1 1 1 1 1 1 1 2 2 1 1 1

1 1 2 2 1 1 1 1 1 1 1 1 1 2 2 1 1 1

1 1 2 2 1 1 1 1 1 1 1 1 1 2 2 1 1 1

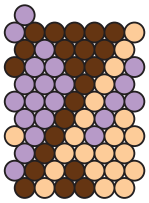


D times 3

1 1 2 2 1 1 1 1 1 1 1 1 1 2 1 2 1 1

1 1 2 2 1 1 1 1 1 1 1 1 1 2 1 2 1 1

1 1 2 2 1 1 1 1 1 1 1 1 1 2 1 2 1 1

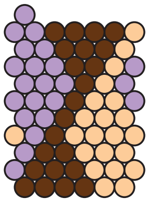


C times 3

1 1 2 2 1 1 1 1 3 1 1 5 1 1

1 1 2 2 1 1 1 1 3 1 1 5 1 1

1 1 2 2 1 1 1 1 3 1 1 5 1 1

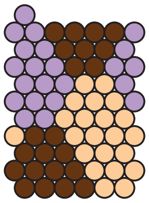


B times 3

4 2 1 3 3 1 2 4 1 1

4 2 1 3 3 1 2 4 1 1

4 2 1 3 3 1 2 4 1 1

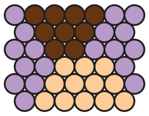


A times 3

4 2 4 3 3 4 2

4 2 4 3 3 4 2

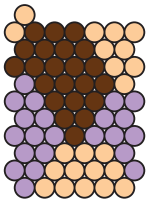
4 2 4 3 3 4 2



Q once

1 5 2 4 3 3 3

2 4 3 3 4 2

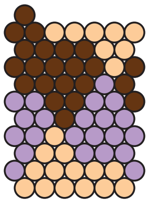


P times 3

1 5 2 4 3 3 4

1 5 2 4 3 3 4

1 5 2 4 3 3 4

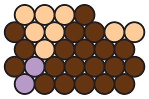


O times 3

6 1 5 2 2 5 1

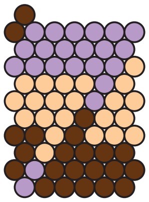
6 1 5 2 2 5 1

6 1 5 2 2 5 1



N once

1 6 1 6 1 5 2 2 5 1

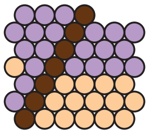


M times 3

1 6 1 6 1 6 1

1 6 1 6 1 6 1

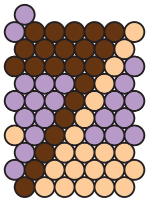
1 6 1 6 1 6 1



L once

1 5 1 1 5 1 1 5 1 1

6 1 6 1 6 1

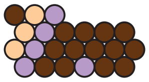


K times 3

1 5 1 1 5 1 1 5 1 1

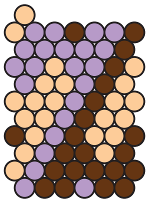
1 5 1 1 5 1 1 5 1 1

1 5 1 1 5 1 1 5 1 1



J once

1 2 1 2 1 1 5 1 1 5 1 1

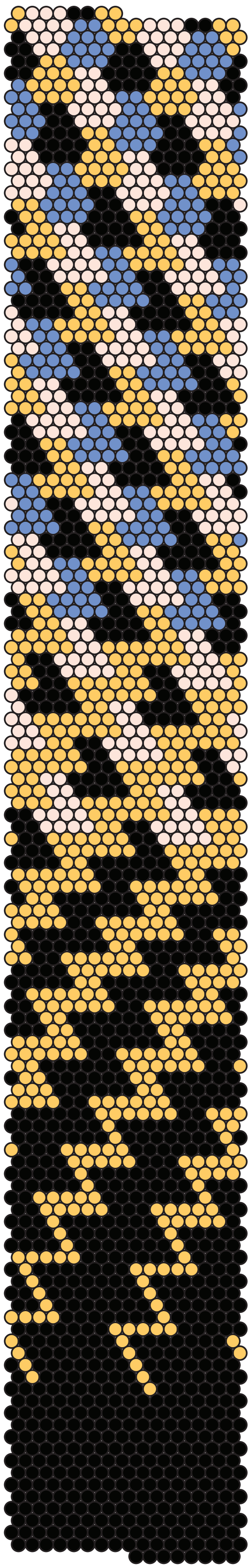
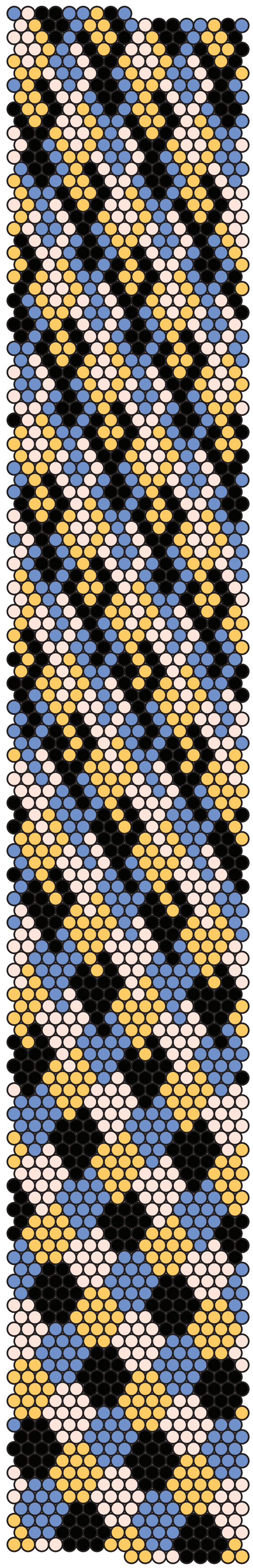
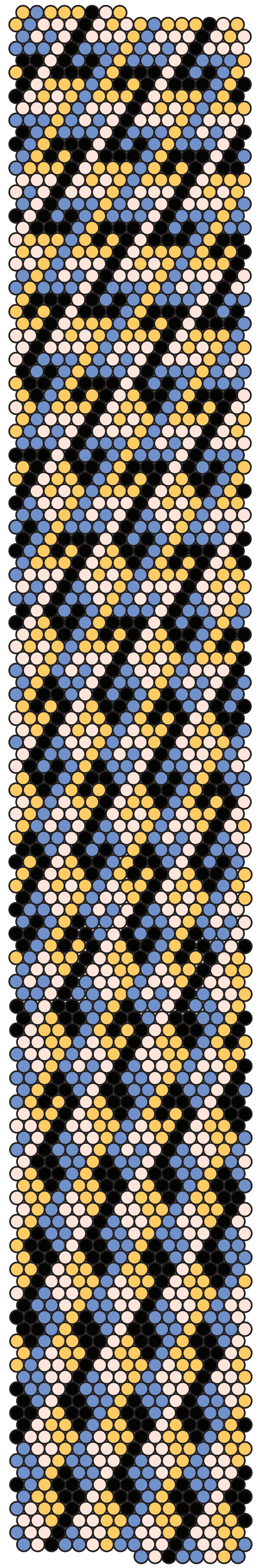
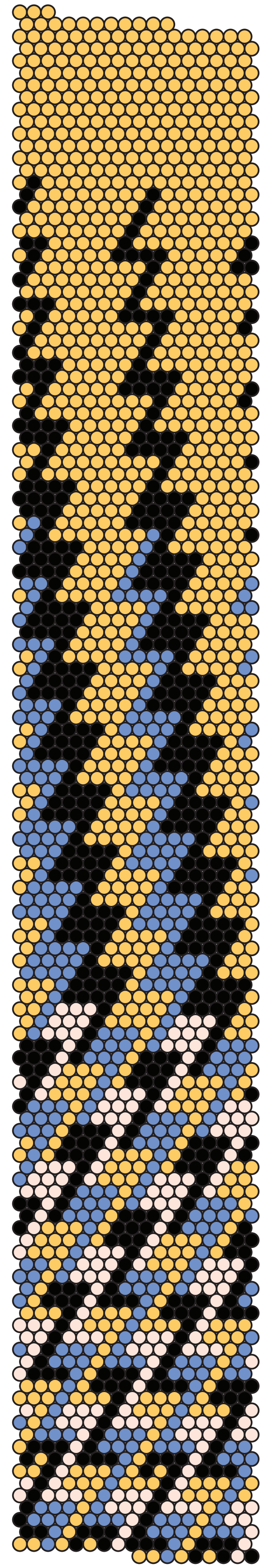


I times 3

1 2 1 2 1 1 5 1 1 2 1 2 1 1

1 2 1 2 1 1 5 1 1 2 1 2 1 1

1 2 1 2 1 1 5 1 1 2 1 2 1 1

Tessellation Evolution Necklace

Notes:

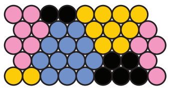
8-around

56 bead repeat for most nontransitional segments. General notes about making transformation patterns appear on p. 129.

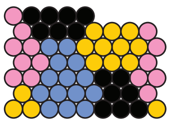
This necklace was made for the art exhibit at the national Joint Mathematics Meetings in January 2013. It contains 16 different four-color tilings, some of which are repeated more often than others for aesthetic reasons. The ends of the necklace are attached to the clasp with bead-woven caps to match the colors in the necklace. Another option is to use end cap beads, available from most bead suppliers. Not counting the clasp, the necklace, made in size 11 Delicas, is 28 inches long.

The top photograph shows the necklace as it was mounted for exhibit. The strips of beads at the top and bottom of the framed necklace were woven with brick stitch to show the shapes of the 16 different tiles. The bracelets on the lower right were made with sections E, M, U, and W of the necklace pattern.

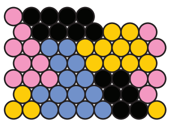
Tags: Escher (four-color), Transformation, Necklace, 8-around



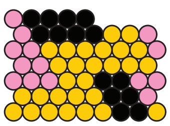
G once
2 3 3 2 4
2 3 3 2 4
2 3 3 2 4



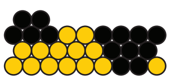
F times 2
2 3 3 2 4
2 3 3 2 4
2 3 3 2 4
2 3 3 2 4



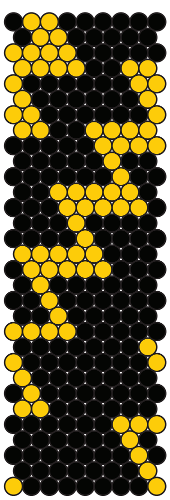
E times 5
2 4 2 2 4
2 4 2 2 4
2 4 2 2 4
2 4 2 2 4



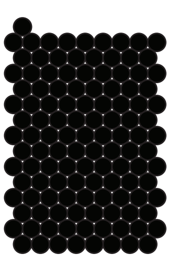
D times 3
6 2 6 2 4 2 2 4
6 2 6 2 4 2 2 4



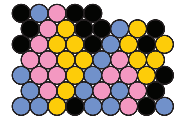
C times 4
6 2 6 6 2 6



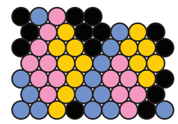
B once
1 7 1 7 1 7 1 7 1 7
5 7 1 7 1 7 1 7 1 7
5 7 1 7 1 7 1 7
5 3 5 7 1 7 1 7
5 3 5 7 1 7 1 7
6 2 6 6 2 6 2 6
6 2 6 6 2 6 2 6



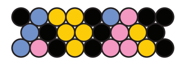
A once
120



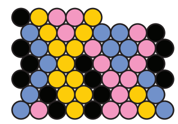
Q times 2
2 1 1 2 1 1 2 1 1 1 1
1 1 1 1 2 1 1 2 1 1 2
2 1 1 2 1 1 2 1 1 1 1
1 1 1 1 2 1 1 2 1 1 2



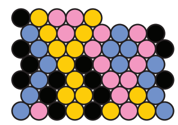
P times 5
2 1 1 2 1 1 2 1 1 2
2 1 1 2 1 1 2 1 1 2
2 1 1 2 1 1 2 1 1 2
2 1 1 2 1 1 2 1 1 2



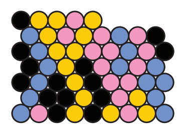
O once
1 1 1 1 1 2 1 1 1 1 2 1
1 1 2 1 2 1 1 1 2



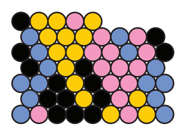
N times 2
1 1 1 1 1 2 1 2 1 2 1
1 1 1 1 1 2 1 2 1 2 1
1 1 1 1 1 2 1 2 1 2 1
1 1 1 1 1 2 1 2 1 2 1



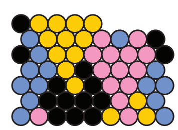
M times 5
1 1 1 1 1 1 1 2 1 2 1
1 1 1 1 1 1 1 2 1 2 1
1 1 1 1 1 1 1 2 1 2 1
1 1 1 1 1 1 1 2 1 2 1



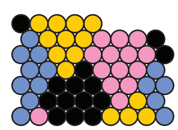
L times 2
1 1 1 1 1 1 1 2 2 1 1
1 1 1 1 1 1 1 2 2 1 1
1 1 1 1 1 1 1 2 2 1 1
1 1 1 1 1 1 1 2 2 1 1



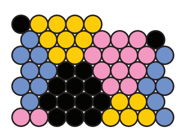
K times 2
1 1 3 1 1 1 2 2 1 1
1 1 3 1 1 1 2 2 1 1
1 1 3 1 1 1 2 2 1 1
1 1 3 1 1 1 2 2 1 1



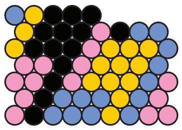
J times 2
1 1 3 1 1 1 2 4
1 1 3 1 1 1 2 4
1 1 3 1 1 1 2 4
1 1 3 1 1 1 2 4



I times 2
1 1 3 3 2 4
1 1 3 3 2 4
1 1 3 3 2 4
1 1 3 3 2 4

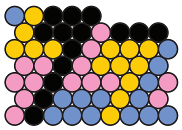


H times 4
2 3 3 2 4
2 3 3 2 4
2 3 3 2 4
2 3 3 2 4



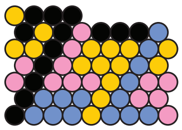
Z times 3

1 1 3 1 1 3 1 3
1 1 3 1 1 3 1 3
1 1 3 1 1 3 1 3
1 1 3 1 1 3 1 3



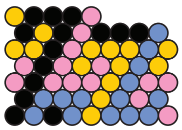
Y times 2

1 1 3 1 3 1 1 3
1 1 3 1 3 1 1 3
1 1 3 1 3 1 1 3
1 1 3 1 3 1 1 3



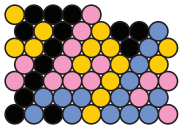
X once

1 3 1 3 1 1 3 1
1 3 1 3 1 1 3 1
1 3 1 3 1 1 3 1
1 1 1 1 1 3 1 1 3



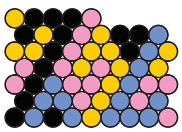
W times 5

1 1 1 1 1 3 1 1 3 1
1 1 1 1 1 3 1 1 3 1
1 1 1 1 1 3 1 1 3 1
1 1 1 1 1 3 1 1 3 1



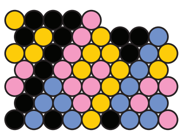
V once

1 1 1 1 1 3 1 1 3 1
1 1 1 1 1 3 1 1 3 1
1 1 1 1 1 3 1 1 2 1 1
1 1 1 1 1 1 2 1 1 3 1



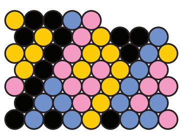
U times 2

1 1 1 1 1 3 1 1 2 1 1
1 1 1 1 1 1 2 1 1 3 1
1 1 1 1 1 3 1 1 2 1 1
1 1 1 1 1 1 2 1 1 3 1



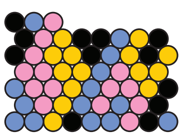
T once

1 1 1 1 1 1 2 1 1 2 1 1
1 1 1 1 1 1 2 1 1 3 1
1 1 1 1 1 3 1 1 2 1 1
1 1 1 1 1 1 2 1 1 3 1



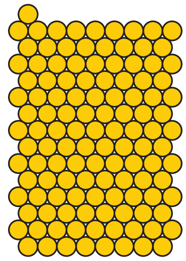
S times 3

1 1 1 1 1 1 2 1 1 2 1 1
1 1 1 1 1 1 2 1 1 2 1 1
1 1 1 1 1 1 2 1 1 2 1 1
1 1 1 1 1 1 2 1 1 2 1 1



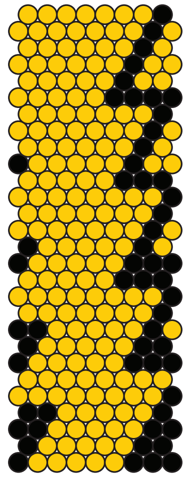
R once

2 1 1 2 1 1 2 1 1 1 1
1 1 1 1 2 1 1 2 1 1 2
2 1 1 2 1 1 2 1 1 1 1
1 1 1 1 2 1 1 2 1 1



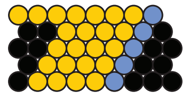
GG once

120



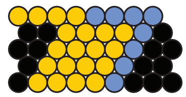
FF once

1 5 4 5 4 5 4 5
1 8 1 8 1 5 4 5 4 5
1 8 1 8 1 5 4 5 4 5
1 8 1 8 1 8 1 5 4 5
1 8 1 8 1 8 1 5 4 5
1 8 1 8 1 8 1 8 1



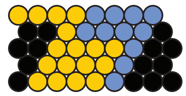
EE once

1 4 1 4 4
1 4 4 1 4
4 1 1 7 1



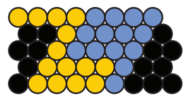
DD times 2

1 4 1 4 4
1 4 4 1 4
4 1 1 4 4



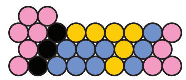
CC times 2

1 4 1 4 4
1 4 4 1 4
1 4 1 4 4



BB times 3

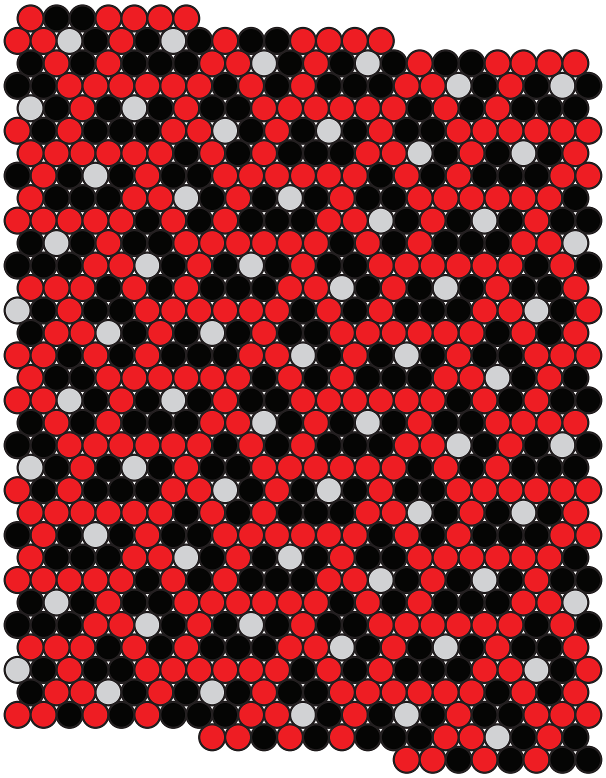
1 4 1 4 4
1 4 1 4 4
1 4 1 4 4



AA once

1 1 3 1 1 3 1 3
1 1 3 1 4 1 3

Googly Eyes



3 2 1 1 1 1 1 1 1 2 6 1 1 1 1

Notes:

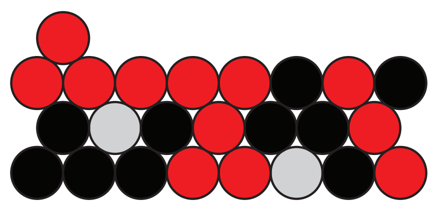
7-around

24 bead repeat

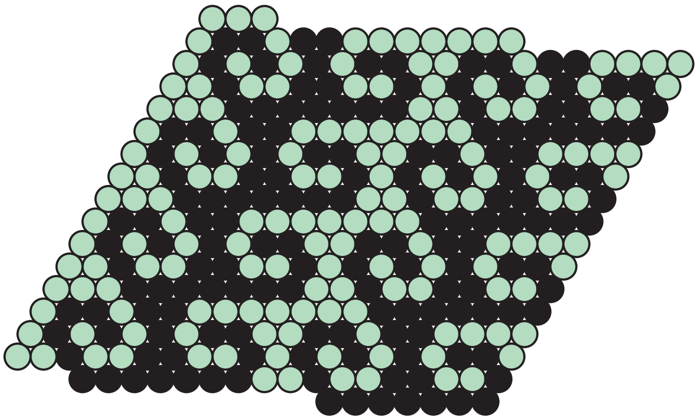
There are 33 to 35 repeats in an average sized bracelet using size 11 seed beads. Because of the short repeat, sizing is very flexible.

The bracelet shown has 33 repeats in size 11 seed beads and 2mm sterling silver rounds.

Tags: Wallpaper Group PG, 7-around



Gliding Vines



7 2 1 2 2 2
1 1 1 1 1 1 1 1
2 2 2 1 2 7

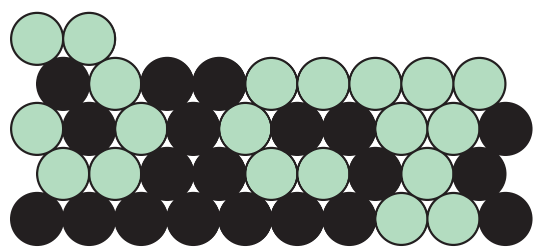
Notes:

9-around

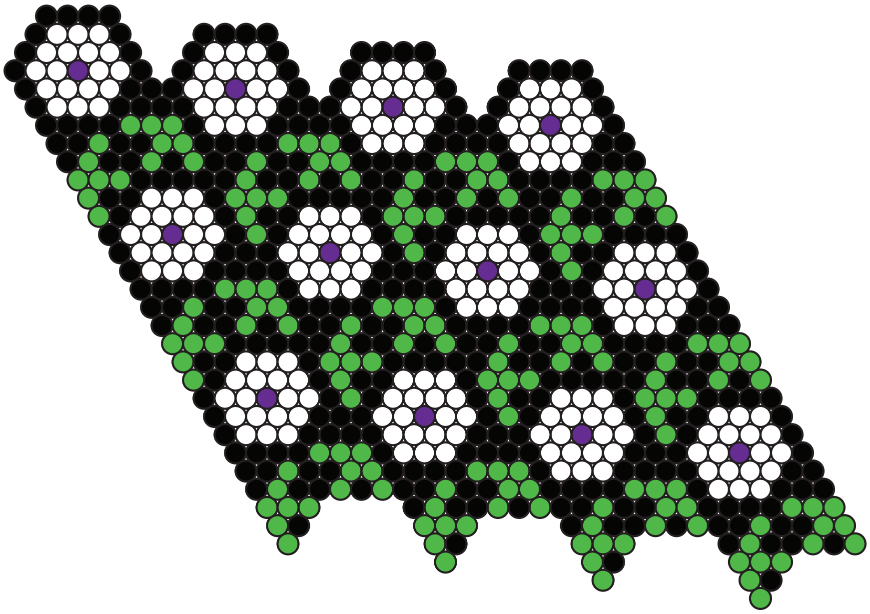
40 bead repeat

There are 30 repeats in an average sized bracelet using size 11 Delicas. Sizing is very flexible.

Tags: Escher, Wallpaper Group PG, 9-around



Springtime



1 3 5 1 2 1 1 1 2 1 2 2 4 3 1
3 4 4 3 2 1 2 1 1 1 4 1 1 2 3

Notes:

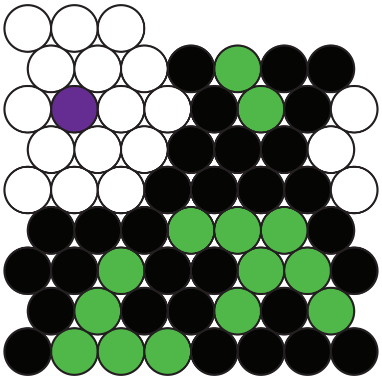
7-around

63 bead repeat

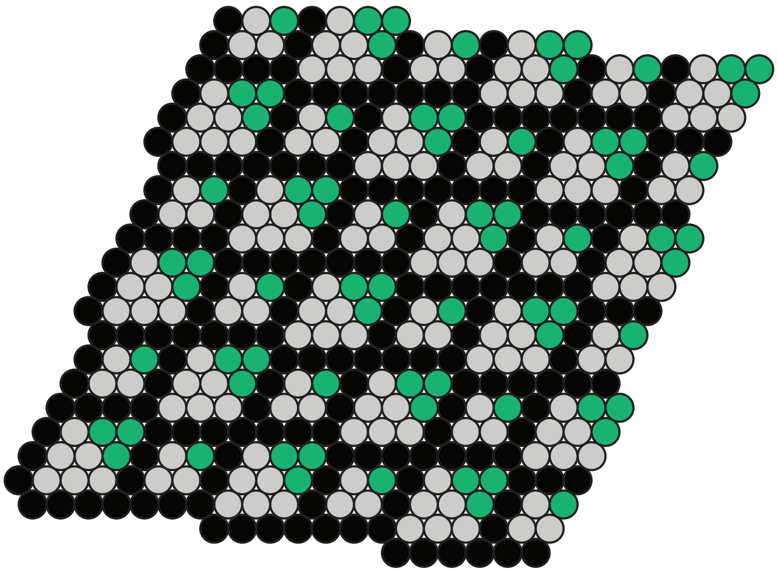
The large bracelet shown uses 12 repeats with 2.5mm sterling silver rounds, 2mm green glass rounds, 2mm black onyx rounds, and 3mm amethyst rounds. In size 11 seed beads, use 13 or 14 repeats for an average sized bracelet or 15 repeats for a large bracelet.

You can line up every second, third (as in the bracelet shown), or fifth repeat with multiples of 2, 3, or 5 repeats, respectively.

Tags: Wallpaper Group CM, 7-around



Shadow Boxes



7 3 1 2 1
2 1 1 1 1 1 1 2

Notes:

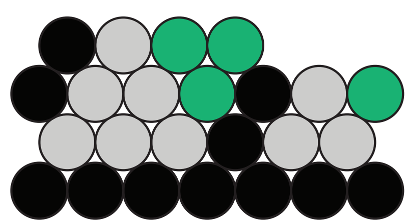
6-around

24 bead repeat

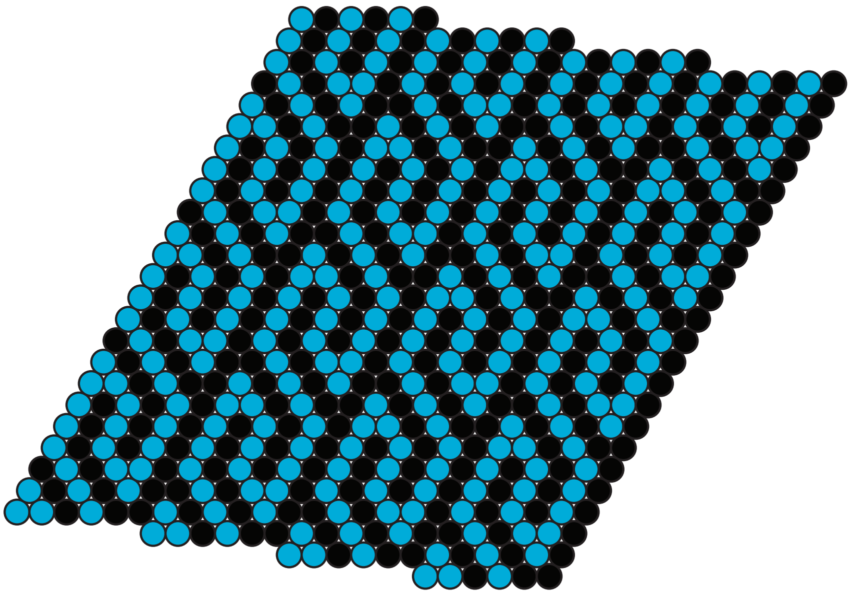
The bracelet in the photograph contains 28 repeats with 2mm rounds in black onyx, sterling silver, and adventurine.

In size 11 seed beads, an average sized bracelet takes 30 repeats. Use a multiple of 3 repeats and a small twist to line up every third repeat.

Tags: Wallpaper group PM, 6-around



Greek Key



Notes:

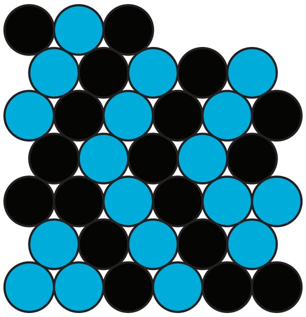
5-around

36 bead repeat

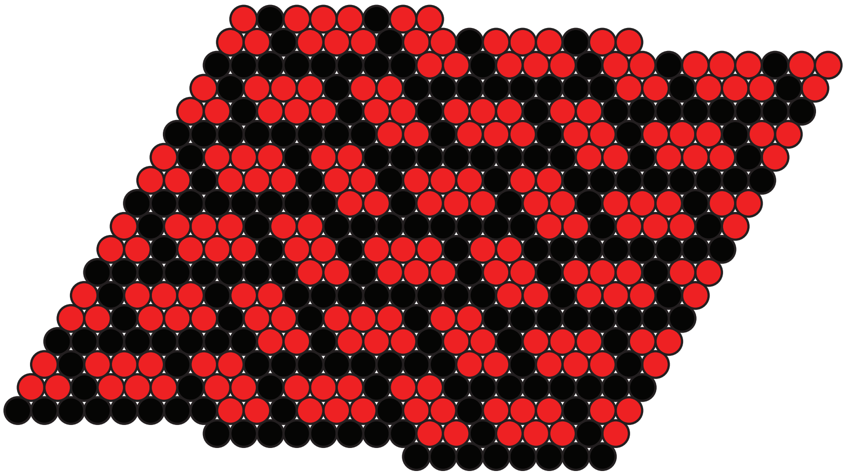
18 repeats in a large sized bracelet using size 10 seed beads, as in the bracelet shown. Or try 12 repeats in size 8 seed beads for an average sized bracelet. Use a multiple of 2 repeats to line up every other repeat for greater symmetry, as in the bracelet shown.

Tags: Wallpaper Group P2, 5-around

2 1 1 2 1 1 1 1 1 2 1 1 2
[1 1 × 9] 1



Brick Walkway



Notes:

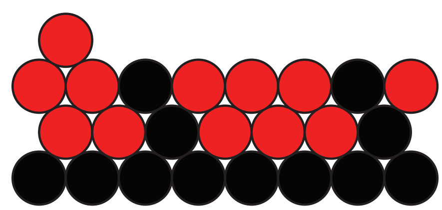
7-around

24 bead repeat

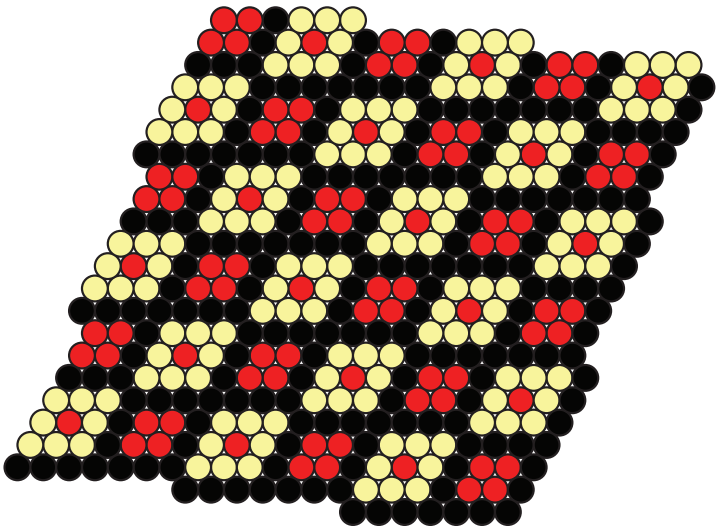
There are 31 repeats in an average sized bracelet using 2mm sterling silver and hemalyke rounds, as in the bracelet shown. In size 11 seed beads, use 33 to 35 repeats for an average sized bracelet.

Tags: Wallpaper Group PGG, 7-around

8 2 1 3 1 2 1 3 1 2



Bon Bon Checkerboard



7 3 1 2 1 1 1 1 1 2 1 3

Notes:

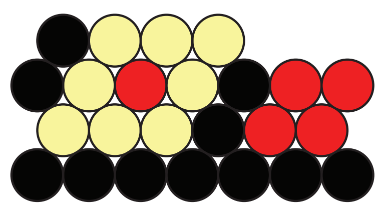
6-around

24 bead repeat

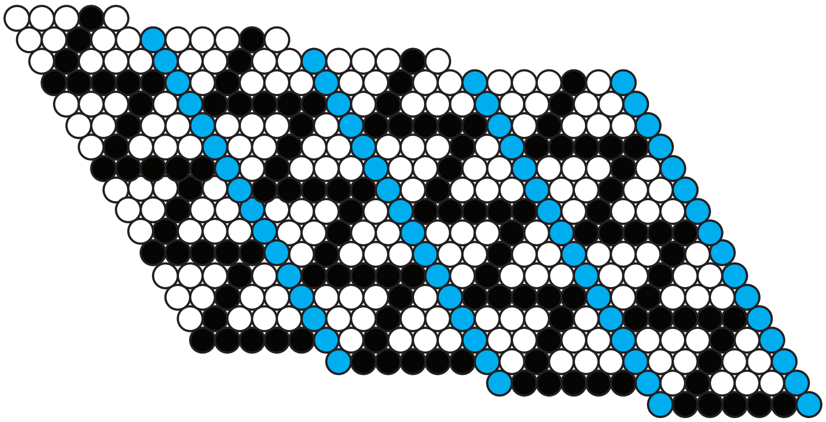
There are 30 repeats in an average sized bracelet using size 11 seed beads. Or try 24 repeats in 2mm semiprecious, glass, and/or metal rounds. The bracelet shown has 31 repeats in size 11 seed beads.

Use a multiple of 3 repeats to line up every third repeat, if desired.

Tags: Wallpaper Group PMM, 6-around



Caged Zigzag



1 5 1

1 1 3 1 2 1 2 1 3 1 1

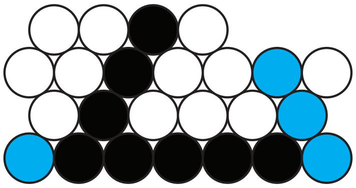
Notes:

6-around

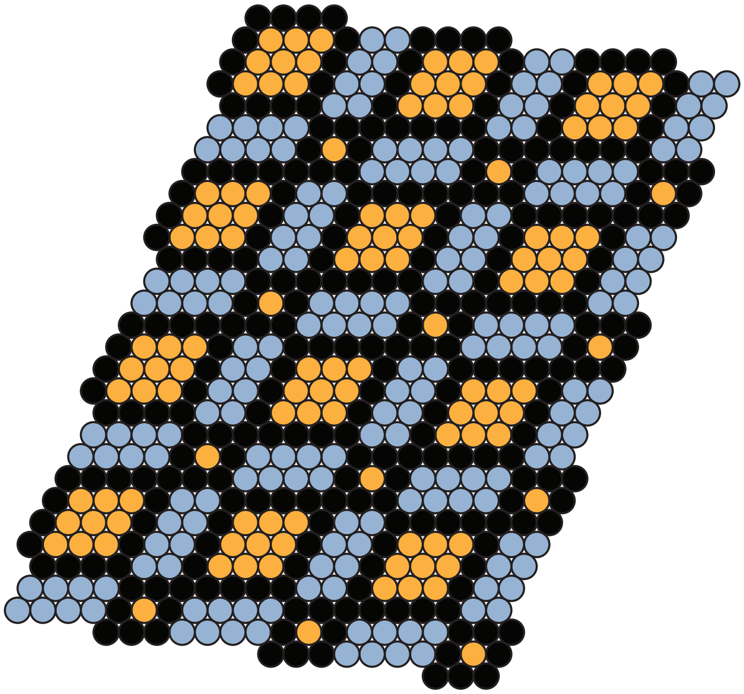
24 bead repeat

There are 30 repeats in an average sized bracelet using size 11 seed beads. Sizing is flexible.

Tags: Wallpaper Group PMG, 6-around



Woven Ribbons



Notes:

6-around

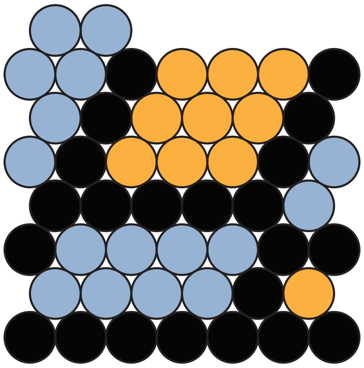
48 bead repeat

There are 14 to 15 repeats in an average sized bracelet using size 11 seed beads. The large bracelet in the photograph has 15 repeats in 2mm black onyx rounds, 2mm gold-filled rounds, and size 11 seed beads.

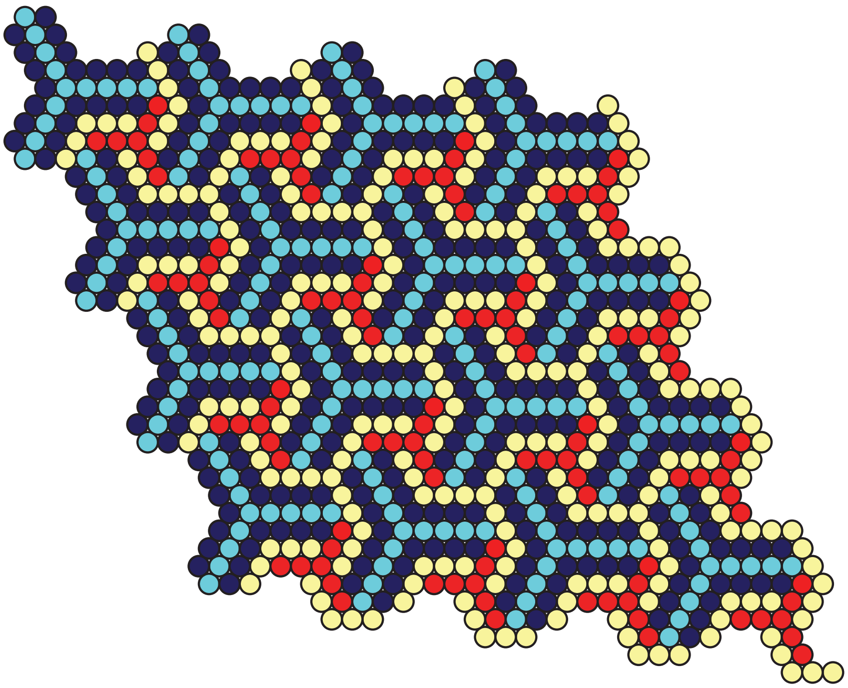
Use a multiple of 3 repeats to line up every third repeat, if desired.

Tags: Wallpaper Group CMM, 6-around

7 4 1 1 1 4 7
2 1 3 1 2 1 3 1 2 1 3 1 2



Woven P3



Notes:

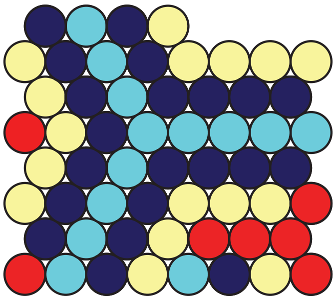
7-around

57 bead repeat

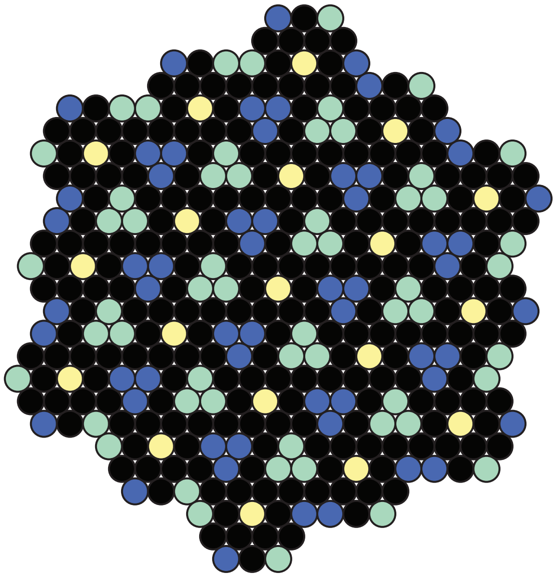
There are 14 or 15 repeats in an average sized bracelet using size 11 seed beads. Use a multiple of 2 repeats to align every other repeat as in the bracelet shown.

Tags: Wallpaper Group P3, 7-around

1 1 1 1 1 1 1 1 1 1 1 1
3 1 1 1 1 3 1 1 1 1
4 1 1 1 5 1 1 1
4 1 1 1 1 4 1 1 1 1



Triangle Twist



1 8 1
1 2 1 1 1 2 1

Notes:

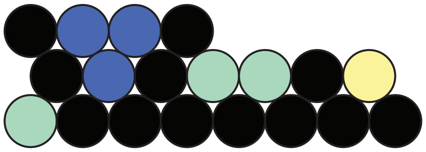
7-around

19 bead repeat

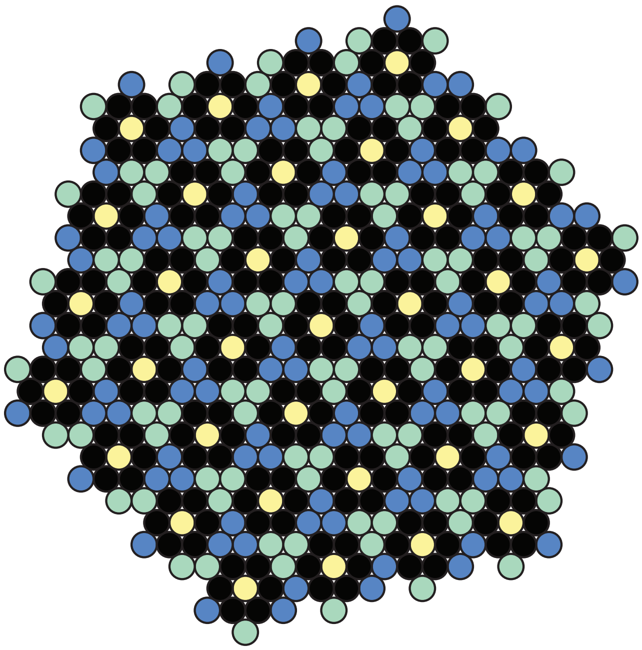
There are 43 to 45 repeats in an average sized bracelet using size 11 seed beads. The bracelet pictured contains 49 repeats in black size 11 seed beads, blue and green size 11 Delicas, and 2mm sterling silver rounds. Using the slightly smaller Delicas gives the design a subtle textured effect.

With a multiple of 2 repeats and essentially no twist, the design will stack the blue triangles and green triangles in vertical lines, yielding a fairly dull composition. This bracelet uses an odd number of repeats and is twisted so that the silver beads follow a (5,2) torus knot.

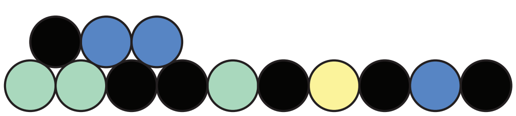
Tags: Wallpaper group P3, 7-around

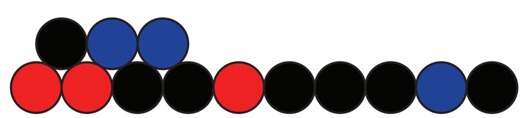


Star of David Twist



2 2 1 1 1 1 1 2 2
or
2 2 1 3 1 2 2





Notes:

9-around

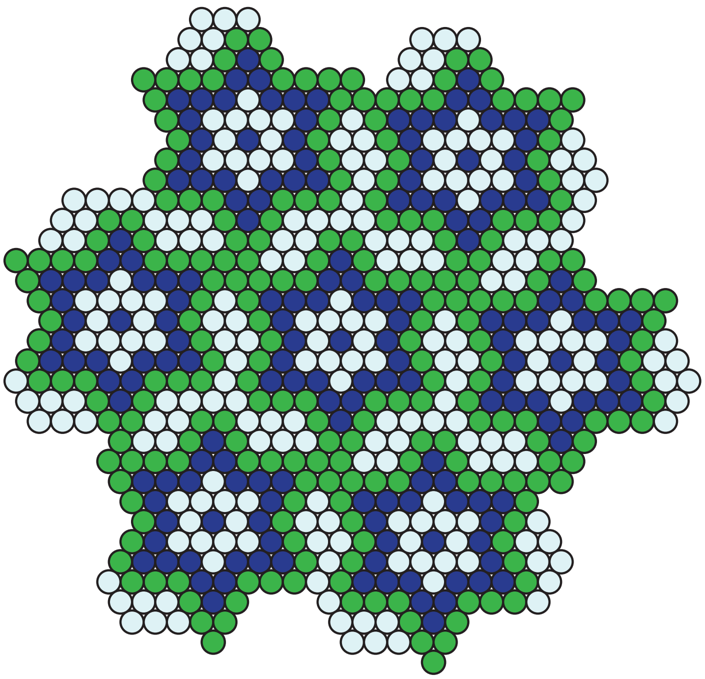
13 bead repeat

An average sized bracelet uses 91 repeats in size 11 Delicas. Because of the short repeat, sizing is extremely flexible.The bracelet on the left has 2mm sterling silver rounds in the centers of the hexagons and uses only 85 repeats. The bracelet on the right uses the second version of the pattern, in which the dots in the hexagon centers are omitted.

The hexagons and triangles are small enough compared to the circumference of the bracelet that no special twist is needed to make the design clear. The small motifs also make the 120˚ rotations in this pattern, centered at the silver beads, more apparent than in other patterns with the same symmetry.

Tags: Wallpaper Group P3, 9-around

Big Star of David



Notes:

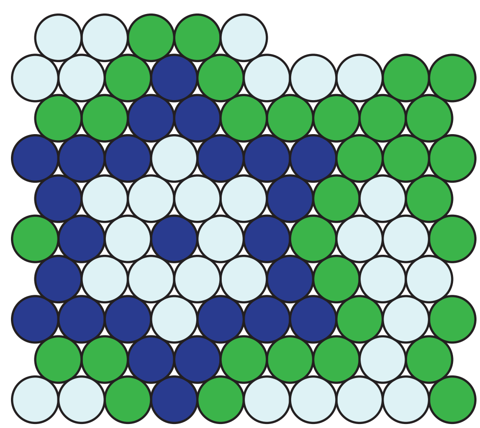
9-around

91 bead repeat

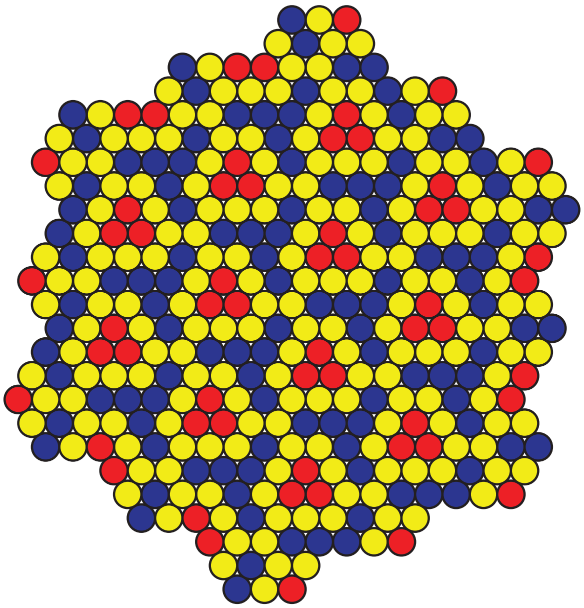
There are 12 repeats in a small sized bracelet, or 13 repeats in an average one, using size 11 Delicas. The long repeat and the need for a multiple of 2 repeats to line up every other repeat makes sizing trickier. The two bracelets shown demonstrate the dramatic impact of different color choices.
A version with 12 repeats with the stars lined up is shown on the left, and a version with 13 repeats, in which the stars cannot be lined up, is shown on the right.

Tags: Wallpaper Group P3, 9-around

3 1 1 1 4 3 2 3 1 1 3 1 3 1 1 1 1
4 1 1 2 1 1 1 1 1 1 1 2 1 1 4 1 1
1 1 3 1 3 5 2 5 2 1 1 1 3 2 2 2



Threefold Pinwheel



Notes:

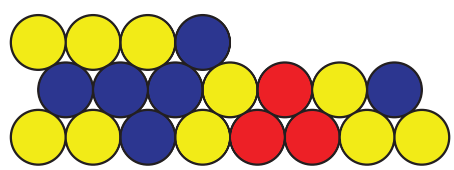
7-around

19 bead repeat

There are 36 repeats in an average to large sized bracelet using 2.5mm gold-filled rounds and 2mm glass rounds.

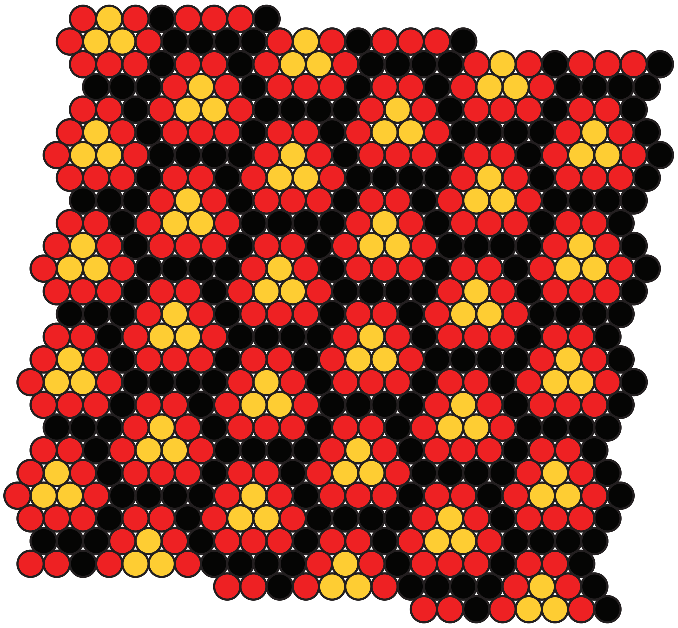
Use a multiple of 2 repeats and a twist to line up every other repeat as in the bracelet shown.

Tags: Wallpaper Group P3, 7-around



2 1 1 2 2 3 1 1 1 1 3 1

Pressed Berries



2 1 1 2 1
4 1 1 1
1 3 1

Notes:

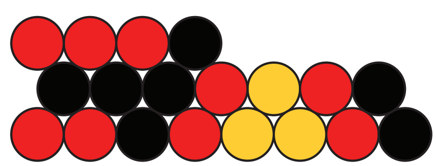
7-around

19 bead repeat

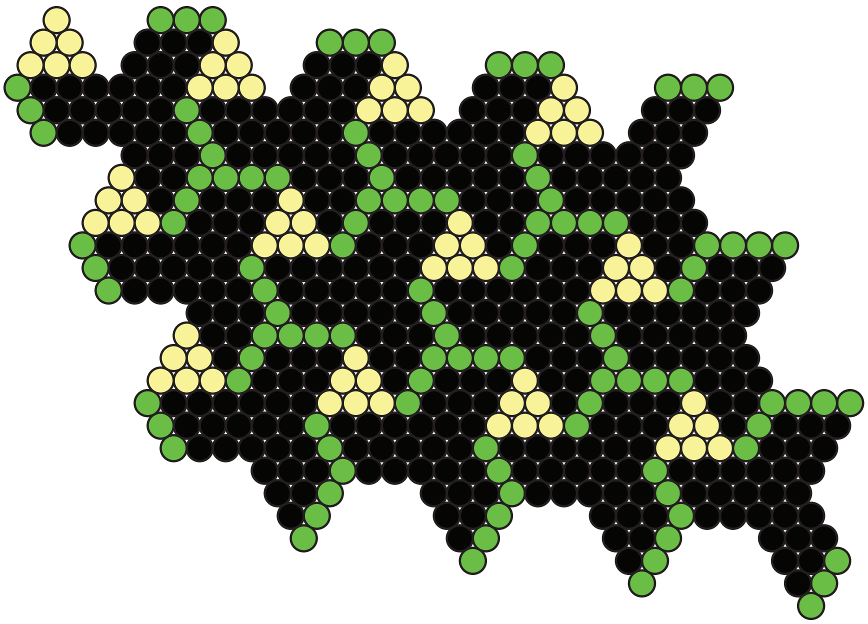
A comparison of the charts shows that this design differs from Threefold Pinwheel by a single bead. Its significantly altered appearance is created by using larger beads to outline the hexagons.

The bracelet on the left contains 42 repeats in 2mm gold-filled rounds, size 11 red seed beads, and size 8 black seed beads, which make the bracelet lie almost flat. The bracelet on the right has 2mm black onyx rounds in place of the size 8 seed beads, creating a more rounded effect. It has 49 repeats and is twisted so that the hexagons align in a (3,2) torus knot.

Tags: Faceted, Wallpaper Group P3, 7-around



Grapevine



3 1 3 2 1 1 3 1
2 4 3 1 5 1 5 1 6

Notes:

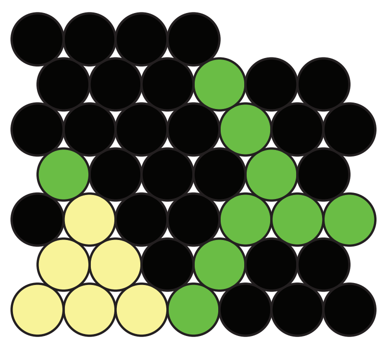
6-around

43 bead repeat

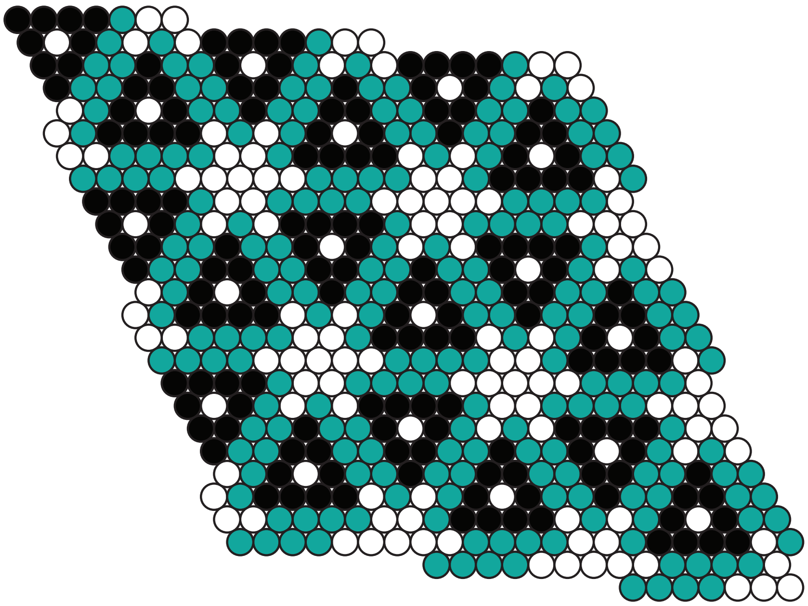
There are 14 repeats in an average sized bracelet using 2mm hemalyke and glass rounds with 3mm sterling rounds.

Use a multiple of 2 repeats to line up every other repeat, as in the bracelet shown.

Tags: Wallpaper Group P3, 6-around



Snowflake and Triangle



5 4 2 1 4 1 1 1 1 1 1 1
2 1 2 2 2 2 2 1 2
1 1 1 1 1 1 1 4 1 2 4

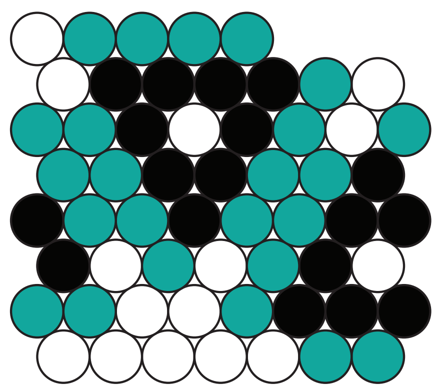
Notes:

7-around

57 bead repeat

There are 14 repeats in an average sized bracelet using size 11 seed beads with 2mm sterling rounds for the snowflakes and triangle centers. Use a multiple of 2 repeats and a twist to line up every other repeat as in the bracelet shown.

Tags: Wallpaper Group P6, 7-around



Rattan 6-around

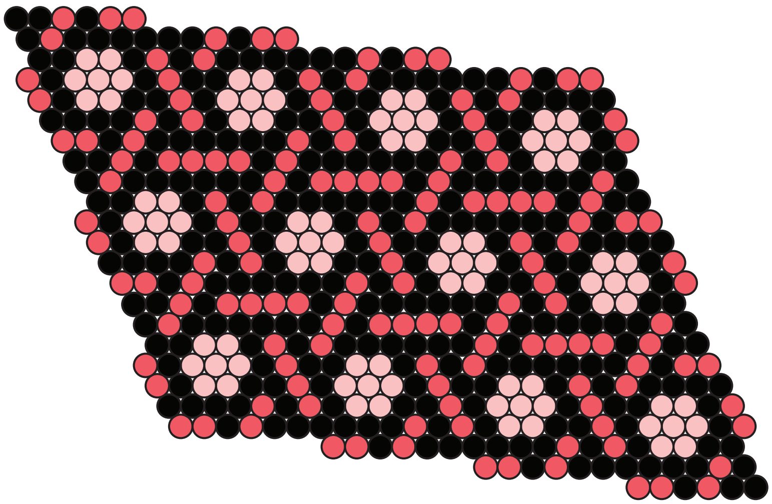
Notes:

6-around

43 bead repeat

There are 16 repeats in an average sized bracelet using size 11 seed beads. Use a multiple of 2 repeats and a twist to line up every other repeat as in the bracelets shown. The variation in the photograph on the right uses 2mm gold-filled and black onyx rounds with 2mm green glass rounds in the hexagon centers.

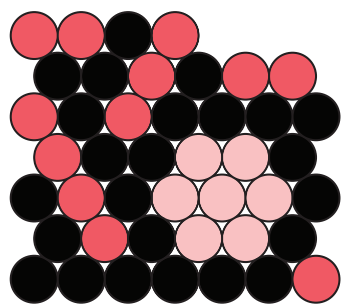
Tags: Wallpaper Group P6, 6-around



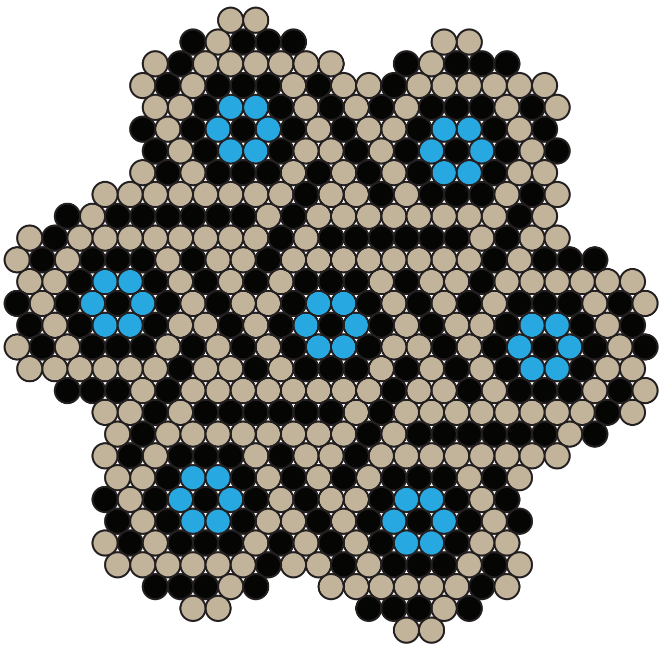
6 1 1 1 1 2 2
1 1 3 1 1
2 2 1 1 1 1 6
1 1 4 1 1

or

6 1 1 1 1 2 2
1 1 1 1 1 1 1
2 2 1 1 1 1 6
1 1 4 1 1



Rattan 8-around



Notes:

8-around

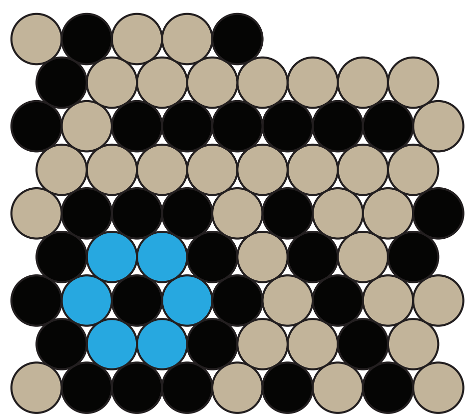
73 bead repeat

There are 12 repeats in an average sized bracelet using size 11 seed beads and size 20 crochet thread.

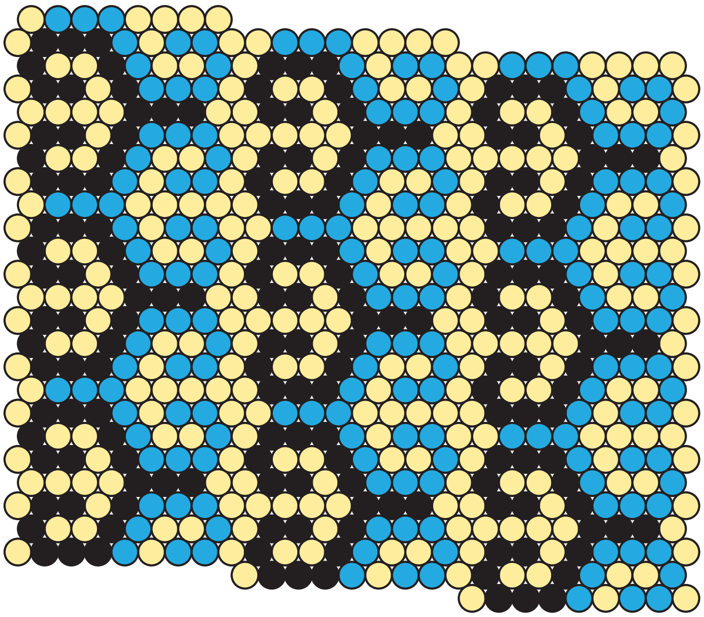
Use a multiple of 2 repeats and a twist to line up every other repeat as in the bracelet shown.

Tags: Wallpaper Group P6, 8-around

1 3 1 1 1 1 1 1 2 1 2
1 1 1 1 1 1 1 1 1
2 1 2 1 1 1 1 1 1 3 1
1 2 1 8 1 1 6 1 1 8 1 2 1



Skating Key



3 1 1 2 1 1 2 1 1 2
1 1 2 1 1 3 5 3 2
2 1 1 3 1 1 2 1 1 2
1 1 3 1 1 2 2 3 5

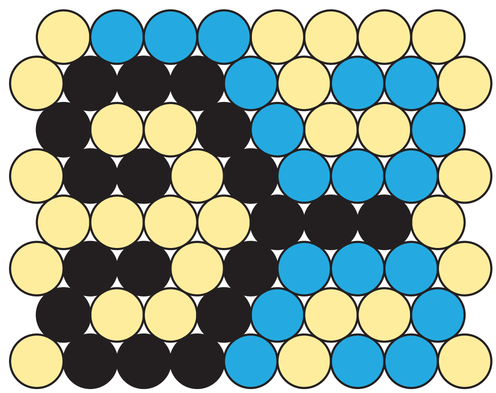
Notes:

8-around

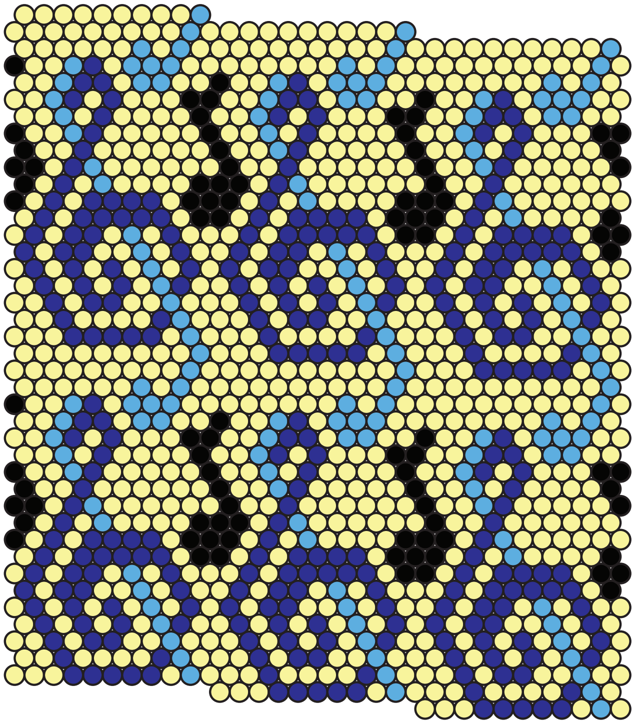
68 bead repeat

There are 15 repeats in an average sized bracelet using size 11 Delicas. Or try 13 repeats with size 11 seed beads and size 20 crochet thread. This design is vertically aligned; use care to avoid a twist when closing.

Tags: 8-around



Music



3 5 1 1 3 1 4 1 1 3 1 1 2 2 1 3 1
1 1 1 1 1 1 1 2 1 1 1 1 1 1 1 1 1
1 1 1 2 2 1 1 1 2 1 1 2 1 1 1 1 3
1 1 5 1 2 1 1 1 4 1 3 1 1 1 1 4 3
1 1 1 6 1 2 1 6 1 2 1 1 5 1 2 1 1
1 4 1 2 1 1 1 1 3 2 2 1 2 1 2 2 1
2 1 1 1 3 8 1 1 1 10 1 10 1

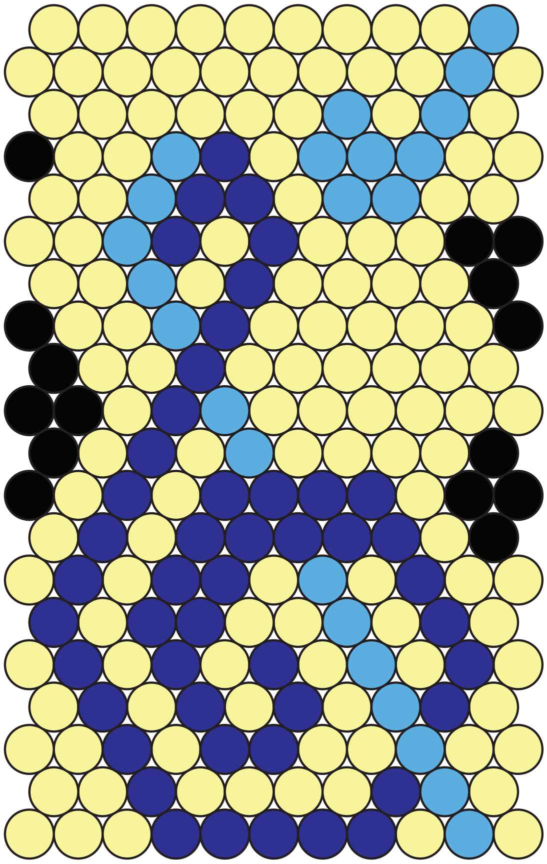
Notes:

10-around

210 bead repeat

There are 6 repeats in an average sized bracelet using size 11 Delicas, as in the bracelet shown (front and back). Since the repeat is so long, there is no sizing flexibility.

Tags: 10-around



Pattern Index

Italic page numbers indicate pages with illustrations.

Alphabetical

N-around

Other Tags

Repeat Length

Subject Index

Italic page numbers indicate pages with illustrations.

U

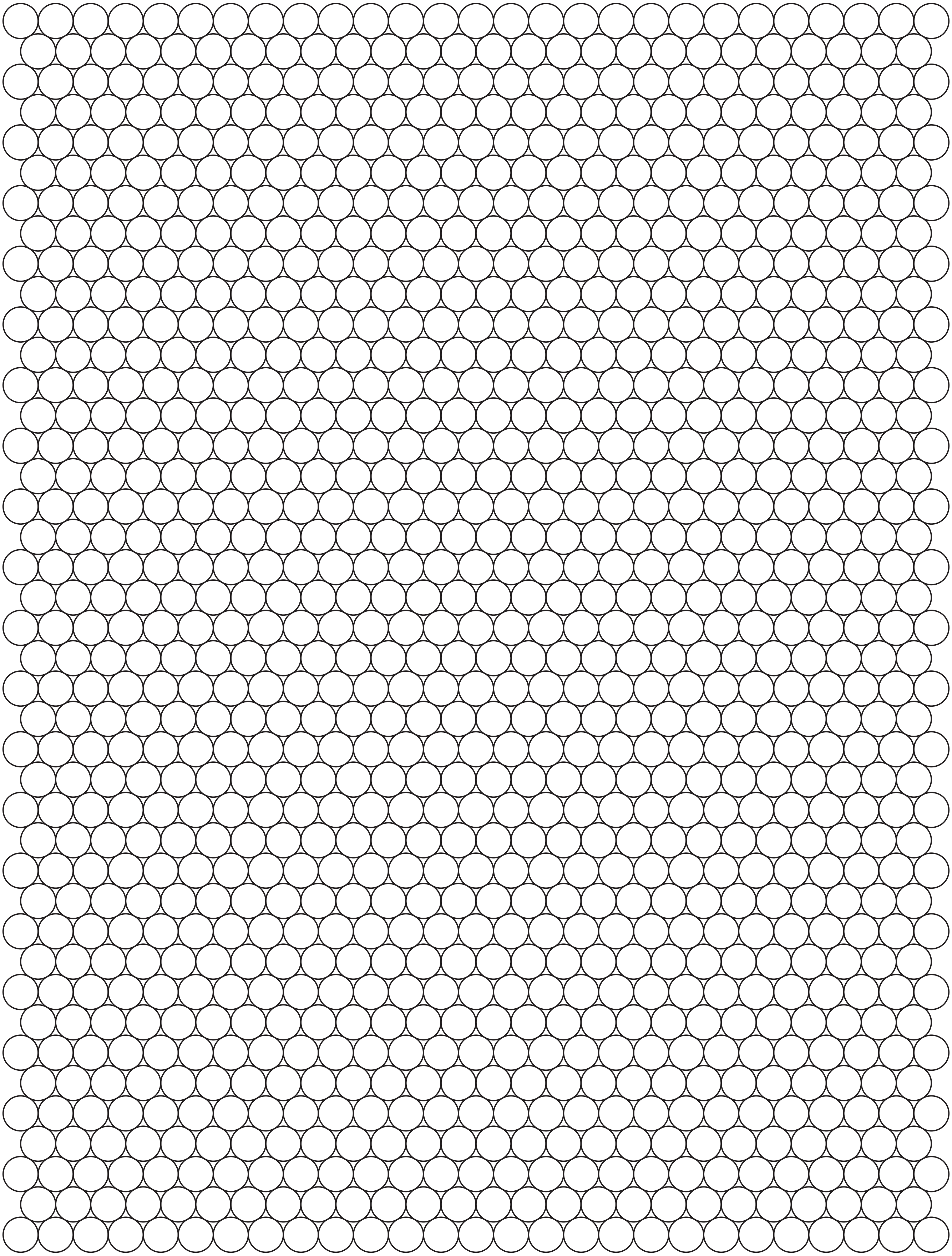
V

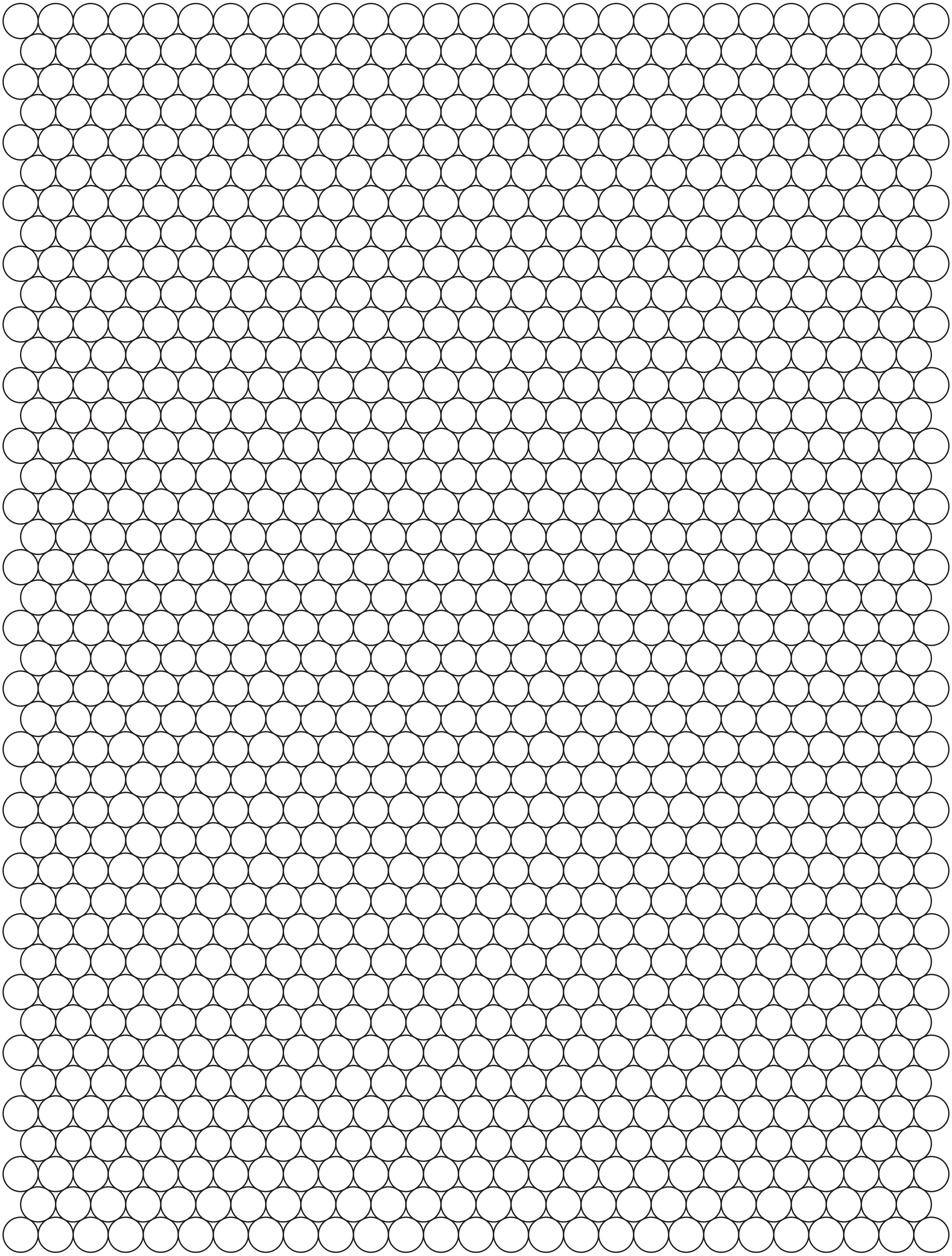
W

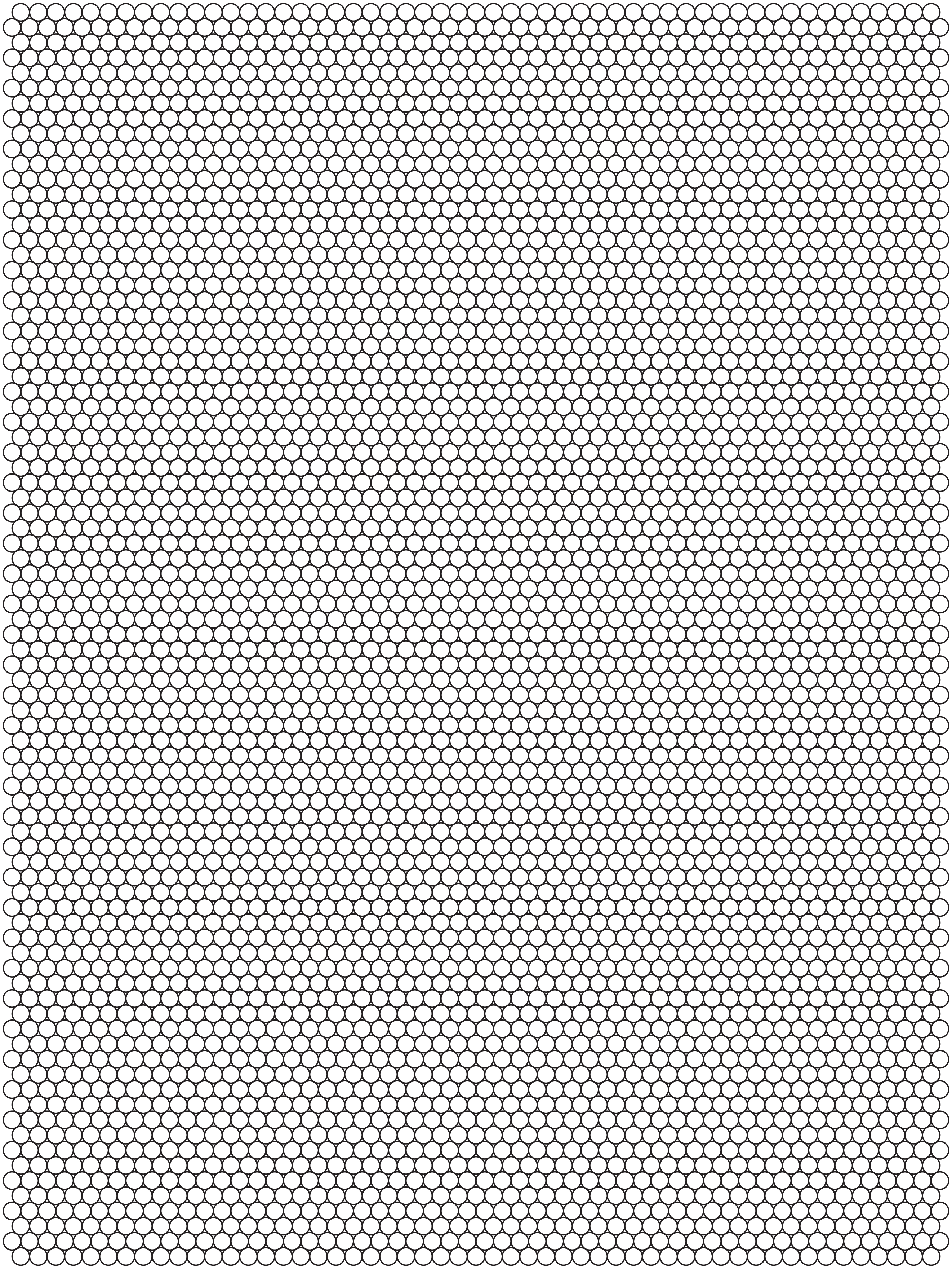
X

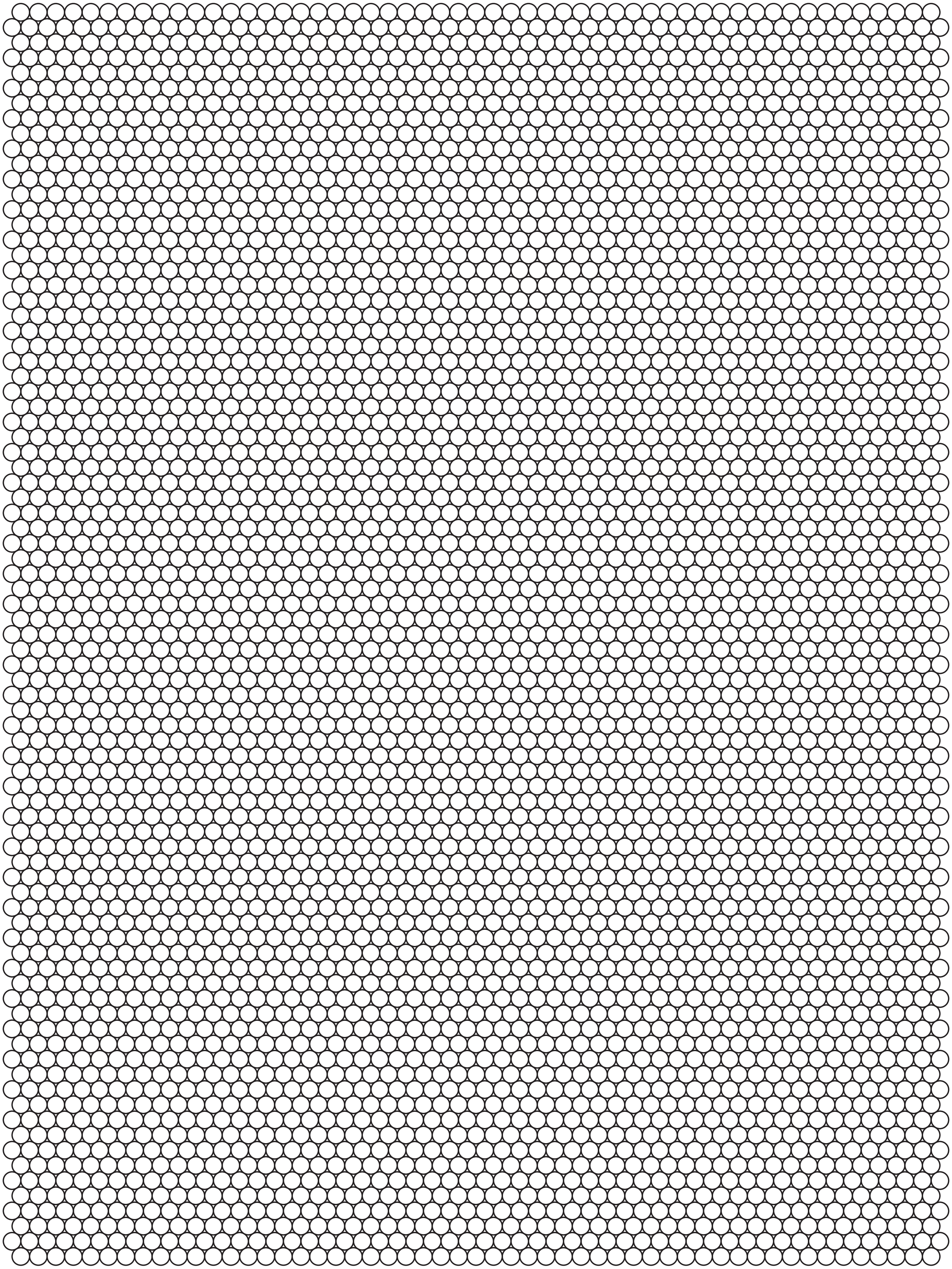
Y

Z









About the Authors

Ellie Baker holds a BFA in sculpture from the Rhode Island School of Design and a PhD in computer science from Harvard University. She has worked as a high-school fine arts teacher, a software engineer, and a computer science researcher, including positions at Harvard University and at computing pioneers Bolt, Beranek and Newman and Thinking Machines Corporation. In her research, Ellie's affinity with all things visual led her to study facial image database search strategies and to develop the Drawing Evolver, an interactive system for creating drawings of faces and other subjects using simulated evolution. Independently and in collaboration with Susan Goldstine, she has produced artworks exhibited in juried shows of mathematical art, such as the national Joint Mathematics Meetings, Bridges international mathematics and art interdisciplinary conference, and MoSAIC (Mathematics of Science, Art, Industry, and Culture). Ellie currently lives in Lexington, Massachusetts, where she enjoys drawing, writing, photography, jogging, and her mindfulness meditation group.

Susan Goldstine received her BA in mathematics and French from Amherst College in 1993 and her PhD in mathematics from Harvard University in 1998. She joined the faculty of St. Mary's College of Maryland in 2004, where she is currently Associate Professor of Mathematics and Chair of the Department of Mathematics and Computer Science. While her original research specialty is number theory and algebraic dynamics, Susan has become increasingly focused on the intersection of mathematics and the arts. She has constructed numerous tactile and visual mathematical models with yarn, fabric, thread, beads, paper, steel wire, copper tubes, pinecones, and pottery, though not all at once. Her joint and individual artworks have appeared in *Math Horizons*, the *Journal of Mathematics and the Arts*, and various art exhibits including the 2013 Exhibition of Mathematical Art at the national Joint Mathematics Meetings, where she received the Honorable Mention for *Tessellation Evolution*. Susan is also an avid cook when she has the time, and while she usually pursues non-mathematical cookery, she hopes one day to reproduce the interlocking Escher swan cookies she made for a lark as an undergraduate. Her guiding principle is that a professor's office can never have too many toys.

Mathematics

Crafting Conundrums

Puzzles and Patterns for the Bead Crochet Artist

"This book is a collection of wonderful tools for mastering geeky and beautiful projects that in a tactile and creative way explore notions like universal covering space, four color theorem, wallpaper groups, and seven color tori that unfairly seem to be reserved for mathematicians only. Crafters, puzzle lovers, and pattern designers will be delighted to find clear instructions on how to do the projects. I hope that non-crafting mathematicians will also peek in the book to see how mathematical concepts can be expressed in amazingly visual ways. It is indeed written with experience and love of both math and craft."
—Daina Taimina, Adjunct Associate Professor of Mathematics, Cornell University, and Author of *Crocheting Adventures with Hyperbolic Planes*

"This is a must-have book for anybody interested in bead crochet bracelets and cords. It provides a perfect balance between the design and construction of bead crochet, and the underlying mathematics that dictates what is and is not possible within this art form."
—Gwen Fisher, beAd Infinitum, www.beadinfinitum.com

Designed for crafters, puzzle lovers, and pattern designers alike, **Crafting Conundrums: Puzzles and Patterns for the Bead Crochet Artist** provides methods, design challenges, and patterns that offer a springboard for creative exploration. Experienced bead crochet crafters looking for a project may choose to skip ahead to the pattern pages and begin crocheting from an abundance of unique, mathematically inspired designs. Those wishing to design their own patterns will find many useful tools, template patterns, and a new methodology for understanding how to do so even without using math. Puzzle lovers without previous knowledge of bead crochet will also find ample inspiration for learning the craft. Supplementary materials, including demo videos, are available on the book's CRC Press web page.

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